Schur algebras of reductive p-adic groups I

Marie-France Vigneras

(Final Version)

Abstract *We give a link - through the affine Schur algebra - between the representations of the p-affine Schur algebra of $GL(n)$ over R, and the smooth R-representations of the p-adic group $GL(n,\mathbf{Q}_p)$ over any algebraically closed field R of characteristic $\neq p$.*

Plan

Introduction

There are relations between the representations of the following "groups": the p-adic linear groups, the p-adic Galois groups, the p-affine Schur algebras, the linear p-affine quantum groups. We insist to consider representations on any algebraically closed field R of characteristic $\neq p$, called R-representations or modular representations. The case where the characteristic of R is p, remains mysterious.

The R-representations of the p-affine Schur algebras are the rational representations of the linear p-affine quantum groups over R. They appear as the nilpotent part of the Jordan decomposition of the R-representations of the p-adic linear groups. The semi-simple part is given by the semi-simple R-representations of the p-adic Galois groups. The complete local Langlands R-correspondence gives a parametrization of the irreducible R-representations for the linear p-adic groups [Vig5], but here we consider the category of all R-representations, not only the irreducible ones. The main new result is the theorem 3, extending to R a theorem of Borel-Casselman in the complex case. The main tool is the parahoric induction and restriction for a general reductive p-adic group and for the corresponding Iwahori Hecke algebra.

1 Let
F a local non archimedean field of residual characteristic p, q is the order of the residual field,
$\overline{F}$ an algebraic separable closure of F,
R an algebraically closed field of characteristic 0 or $\ell\neq p$,
n an integer ≥ 1,
$\operatorname{Mod}_R G$ the category of smooth R-representations of $G := GL(n,F)$, equivalent to the category of right modules of the global Hecke R-algebra $\mathcal{H}_R(G)$ of G isomorphic to its opposite algebra [Vig1 I.4.4],

$$\operatorname{Mod}_R G \simeq \operatorname{Mod} \mathcal{H}_R(G).$$

The following theorem is a formulation of the local Langlands R-correspondence.

Theorem *a) The category* $\operatorname{Mod}_R G = \Pi_\tau \mathcal{B}_{R,\tau}$ *is a direct sum of blocks.*

b) The blocks $\mathcal{B}_{R,\tau}$ are parametrized by the semi-simple R-representations τ of dimension n of the inertia subgroup of the Galois group of $\overline{F}/F$, stable by the Frobenius of $\operatorname{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$.

c) The irreducible R-representations of the unipotent block $\mathcal{B}_{R,1}$ are parametrized by the pairs (s,N) where $s\in GL(n,R)$ is semi-simple, $N\in M(n,R)$ is nilpotent, and $sN = Nsq$, modulo isomorphism.

d) For any τ, the irreducible R-representations of the block $\mathcal{B}_{R,\tau}$ are parametrized by the irreducible R-representations of the unipotent block of a product of linear p-adic groups

$$G_\tau = \prod_\sigma GL(m_{\tau,\sigma}, F_{n(\sigma)})$$

for all orbits σ of the Frobenius in the irreducible R-representations of I_F occurring in τ, where

$m_{\tau,\sigma}$ is the multiplicity of σ in τ,

$F_{n(\sigma)} \subset \overline{F}$ is the unramified extension of F of degree the number $n(\sigma)$ of elements of σ.

The block $\mathcal{B}_{R,1}$ containing the trivial representation and the representations of $\mathcal{B}_{R,1}$ will be called unipotent. This is coherent with the terminology of Lusztig in the complex case.

Example: We deduce from c) the number $m(s)$ of irreducible unipotent R-representations with a given semi-simple $s \in GL(n, R)$.

a) If $s = 1_n$ is the identity, we haver $m(1_n) = 1$ corresponding to $N = 0$, when $q \neq 1$ in R. But if $q = 1$ in R, that is the characteristic of R is $\ell \neq p$ and $q = 1 \bmod \ell$, then $m(1_n)$ is the number $p(n)$ of partitions of n.

b) If s is the diagonal $(1, q, q^2, q^3)$, the number $m(s)$ depends on the multiplicative order e of q in R. When $e = 1, 2, 3, 4, > 4$ we have respectively $m(s) = 5, 10, 9, 15, 8$ [Vig2].

2 Let I be an Iwahori subgroup of G, let $\mathcal{J}_R$ be the annihilator in $\mathcal{H}_R(G)$ of the R-representation of G

$$R[I\backslash G].$$

The category $\mathcal{B}'_{R,1}$ of R-representations of G annihilated by $\mathcal{J}_R$ is an abelian subcategory of the unipotent block $\mathcal{B}_{R,1}$.

Main Theorem *There exists an integer N such that $\mathcal{J}_R^N$ annihilates the unipotent block. The category $\mathcal{B}'_{R,1}$ is equivalent to the category of modules of the affine Schur algebra $\mathcal{S}_R$ of G.*

One can give an explicit value for N. One can replace the ideal $\mathcal{J}_R$ by its component $\mathcal{J}_{R,1}$ in the unipotent part of the global Hecke algebra $\mathcal{H}_R(G)$. The ideal $\mathcal{J}_{R,1}$ is nilpotent, $\mathcal{J}_{R,1}^N = \{0\}$. We define the affine Schur algebra $\mathcal{S}_R$ of G in the paragraph 3.

The Iwahori-Hecke algebra $\mathcal{H}_R$ of G is the ring of endomorphisms of $R[I\backslash G]$

$$\mathcal{H}_R := \mathrm{End}_R\, R[I\backslash G].$$

It would be a natural candidate for the affine Schur algebra, modulo Morita equivalence. But it does not work, already there are not enough simple $\mathcal{H}_R$-modules. The irreducible *quotients* of $R[I\backslash G]$ are in bijection with the simple $\mathcal{H}_R$-modules. But a subquotient is not always a quotient. Example: $GL(2, \mathbf{Q}_5)$ when the characteristic of R is 3. The irreducible *subquotients* of $R[I\backslash G]$ are the irreducible unipotent R-representations of G [Vig3 II.11.2], they are also the irreducible R-representations of G annihilated by $\mathcal{J}_R$, they are in bijection with the simple $\mathcal{S}_R$-modules.

3 Definition *The Schur algebra $\mathcal{S}_R$ of G is the ring of endomorphisms of the right $\mathcal{H}_R$-module V_R^I*

$$\mathcal{S}_R := \mathrm{End}_{\mathcal{H}_R}\, V_R^I \tag{1}$$

where V_R is the R-representation of G

$$V_R := \oplus_P R[P\backslash G] \tag{2}$$

for all parahoric subgroups P of G containing I (called standard).

We have a natural inclusion $R[P\backslash G] \subset R[I\backslash G]$ in $\mathrm{Mod}_R\, G$, hence the annihilator of V_R is $\mathcal{J}_R$, the irreducible subquotients of V_R are the irreducible unipotent representations of G. We have

$$\mathrm{Hom}_{\mathcal{H}_R}(R[I\backslash G/I], R[P\backslash G/I]) \simeq R[P\backslash G/I] \tag{3}$$

hence

4 Proposition *The $\mathcal{S}_R$-module V_R^I is cyclic generated by the projector $e : V_R^I \to R[I\backslash G/I]$*

$$V_R^I \simeq \mathcal{S}_R e = \mathrm{Hom}_{\mathcal{H}_R}(R[I\backslash G/I], V_R^I), \tag{4}$$

and the $\mathcal{S}_R - \mathcal{H}_R$-module V_R^I satisfies the double centralizer property (see (1))

$$e\mathcal{S}_R e = \mathcal{H}_R = \mathrm{End}_{\mathcal{S}_R} V^I. \tag{5}$$

The simple $\mathcal{H}_R$-modules are in bijection with the simple $\mathcal{S}_R$-modules W such that $eW \neq 0$.

There is another property which is not so evident : the Schur algebra is isomorphic to the ring of RG-endomorphisms of V_R (theorem A.4.3 proved in C.2.14)

$$\mathrm{End}_{RG} V_R \simeq \mathrm{End}_{\mathcal{H}_R} V_R^I. \tag{6}$$

This is valid when $GL(n,F)$ is replaced by any reductive connected group over F (the group of F-points), and R is replaced by any commutative ring.

The Iwahori-Hecke R-algebra and the Schur R-algebra are naturally defined over $\mathbf{Z}$. They are free $\mathbf{Z}$-modules with natural basis and $\mathcal{S}_R = \mathcal{S}_{\mathbf{Z}} \otimes_{\mathbf{Z}} R$, and the same is well known for $\mathcal{H}_R$.

5 The theorem is interesting because it creates a bridge between the p-adic groups and the affine quantum groups. Indeed, in [Green R.M.] is introduced an affine quantum group $U_{\mathbf{C}}(n,q)$ for $gl(n)$ over $\mathbf{C}$ (an Hopf algebra) and a natural $U_{\mathbf{C}}(n,q)$-module W of infinite countable dimension such that the natural action of $U_{\mathbf{C}}(n,q)$ on the tensor space $W^{\otimes n}$, has the properties

- the image $\mathcal{S}_{\mathbf{C}}(n,q)$ of $U_{\mathbf{C}}(n,q)$ in $\mathrm{End}_R W^{\otimes n}$ is isomorphic to the Schur algebra $\mathcal{S}_{\mathbf{C}}$.
- the centralizer of the action of $U_{\mathbf{C}}(n,q)$ on $W^{\otimes n}$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_{\mathbf{C}}$.
- the tensor space $W^{\otimes n}$ is isomorphic to V^I as an $\mathcal{S}_{\mathbf{C}} - \mathcal{H}_{\mathbf{C}}$-module.

One hopes that there are natural integral structures over $A = \mathbf{Z}[v, v^{-1}]$, such that the image of $U_A(n,q)$ in $\mathrm{End}_A W^{\otimes n}$ is isomorphic to $\mathcal{S}_A$. When the characteristic of R is > 0 and $q \neq 1$ in R, if ζ is a root of 1 in $\mathbf{C}^*$ of the same order than q in R^*, one hopes to understand better problems as decomposition numbers or cohomology of modular representations of G with the help of the complex affine quantum group $U_{\mathbf{C}}(n,\zeta)$.

There are similar theorems and applications for finite linear groups $GL(n,\mathbf{F}_q)$ by Dipper and James [DJ1] [DJ2], Takeuchi [T], Cline, Parshall and Scott [CPS]. The proof of the main theorem does not rely on the theorem in the finite case.

6 The proof of the main theorem is a generalization of the proof of Takeuchi [T] for $GL(n,\mathbf{F}_q)$ and is given in the chapter D. It is more "conceptual" and gives a new proof also in the finite case. The idea is the following. As in Takeuchi, one considers the group $U^*(\mathbf{F}_q)$ of $(a_{ij}) \in GL(n,\mathbf{F}_q)$ with $a_{ij} = 0$ if $i > j$ or $j = i+1$ and $a_{ii} = 1$ for all i,j such that this has a sense. Let U^* be the inverse image of $U^*(\mathbf{F}_q)$ by the reduction modulo p

$$GL(n, O_F) \to GL(n, \mathbf{F}_q)$$

where O_F is the ring of integers of F. Set $G := GL(n,F)$. One considers the right $\mathcal{H}_R$-module $R[U^*\backslash G/I]$. Its ring of endomorphisms is Morita equivalent to the Schur algebra (D.5)

$$\mathrm{End}_{\mathcal{H}_R} R[U^*\backslash G/I] \simeq_{\mathrm{Morita}} \mathcal{S}_R,$$

and $R[U^*\backslash G/I]$ is the I-invariants of the R-representation of G

$$R[U^*\backslash G] = e^*\mathcal{H}_R(G)$$

which is cyclic and projective, generated by the idempotent e^* defined by the pro-p-group U^*. One shows that the algebra homomorphism of I-invariants

$$\mathrm{End}_{RG}\, e^*\mathcal{H}_R(G) \to \mathrm{End}_{\mathcal{H}_R}\, e^*\mathcal{H}_R(G)^I$$

is surjective in D.6, from which one deduces in D.8 that

$$e^*(\mathcal{H}_R(G)/\mathcal{J}_{\mathcal{R}})e^* \simeq_{\mathrm{Morita}} \mathcal{S}_R^o.$$

It remains to prove that the functor

$$\mathrm{Hom}(e^*\mathcal{H}_R(G)/\mathcal{J}_{\mathcal{R}}, -) : \mathrm{Mod}\,\mathcal{H}_R(G)/\mathcal{J}_{\mathcal{R}} \to \mathrm{Mod}\,\mathcal{S}_R$$

defines an equivalence of categories (D.10).

At this point, the property that we are working with a linear group is crucial. The theory of the Whittaker model for $GL(n, \mathbf{F}_q)$ shows that any irreducible R-representation of G annihilated by $\mathcal{J}_{\mathcal{R}}$ has a non zero vector invariant by U^*. This implies the equivalence of categories, and also that the unipotent part $e^*\mathcal{H}_R(G)_1$ of $e^*\mathcal{H}_R(G)$ is a progenerator of the unipotent block.

The composition of the parabolic induction functor in the finite group $GL(n, \mathbf{F}_q)$ with the inflation to $GL(n, O_F)$ and the compact induction to $G = GL(n, F)$ is a functor called parahoric induction. The parahoric induction commutes with the functor "unipotent part" (D.14). From this one deduces in D.13 that

$$e^*\mathcal{H}_R(G)_1 \simeq \oplus_J i_{M_J}^G \Gamma_{R,J,1}$$

is a direct sum of representations parahorically induced from the standard Levi subgroups M_J of $GL(n, \mathbf{F}_q)$, where $\Gamma_{R,J,1}$ is the unipotent part of the Gelfand-Graev representation of M_J (with some multiplicity). When M_J is the diagonal torus $T(\mathbf{F}_q)$

$$R[I\backslash G] = i_{T(\mathbf{F}_q)}^G 1_R.$$

The transitivity the parahoric induction shows that $e^*\mathcal{H}_R(G)_1$, hence any unipotent representation of G, has a finite filtration with quotients isomorphic to subquotients of $R[G/I]$. The length N of the filtration is bounded by the maximum of the analogous filtrations of $\Gamma_{R,J,1}$ with I replaced by a Borel subgroup of M_J, for all standard Levi subgroups M_J of $GL(n, \mathbf{F}_q)$.

7 One can compare $\mathcal{J}_R$ with the intersection J_R in $\mathcal{H}_R(G)$ of the annihilators of the irreducible unipotent R-representations of G. The unipotent part $J_{R,1}$ of J_R is the Jacobson radical of the unipotent part of $\mathcal{H}_R(G)$. The representation $R[I\backslash G]$ is isomorphic to

$$R[I\backslash G] \simeq i_T^G R[T(O_F)\backslash T]$$

where $T = T(F)$ is the diagonal group with the universal action on $R[T(O_F)\backslash T]$ and i_T^G is the nonnormalized parabolic induction from T to G [Dat]. The theory of R-types for $GL(n, F)$ shows that the lengths of the induced representations $i_T^G\chi$ for all R-characters χ of T trivial on $T(O_F)$ is bounded by the length of $i_T^G 1_R$ [V3]. This properties and the main theorem imply:

Proposition *Let M be the length of $i_{T(F)}^G 1_R$. Then*

$$J_R^M \subset \mathcal{J}_R \subset J_R.$$

A finite power of the Jacobson radical of the unipotent part of $\mathcal{H}_R(G)$ is zero.

8 Although the main theorem concerns only p-adic general linear groups, and algebraically closed fields of characteristic different from p, we develop a theory for a general reductive p-adic group G and for any commutative ring R (p can be 0 in R).

The main theorem is proved for a general reductive p-adic group G in the "linear case", when the finite reductive quotients of the parahoric subgroups of G are product of linear groups. In particular, when $G = GL(n, D)$ where D/F is a division algebra over F.

The chapter A presents the classical Schur algebra and introduces the Schur algebra of a reductive p-adic group, the chapter B recalls some properties of the functor $\mathrm{Hom}_A(Q, -)$ when Q is a quasi-projective module over an algebra A, no finiteness is required. In the chapter C one studies R-representations induced from the finite reductive quotient of a parahoric subgroup and their endomorphism rings. The Mackey decomposition of the parahoric restriction-induction plays a fundamental role. It is given in C.1.4 for the group and in C.2.1 for the Iwahori-Hecke algebra. One deduces a basis and relations for the Schur algebra of a reductive p-adic group in C.2.15 with some restrictions on R. A systematic use of the parahoric induction allows us to reduce problems concerning representations of a reductive p-adic group G to problems concerning representations of the finite reductive quotients of the parahoric subgroups. It is particularly useful to study the functor of I-invariants, linking R-representations of G and right modules for the Iwahori-Hecke R-algebra, as in C.3.

9 The main theorem can be generalized to other blocks using the theory of Bushnell-Kutzko types. One can also show that it is compatible with the reduction modulo ℓ. In more concrete terms, the decomposition numbers of $GL(n, F)$ are decomposition numbers for the affine Schur algebras, and conversely. This will be explained in the part II.

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A The Schur algebra and its deformations

A.1 The classical Schur algebra

Schur invented many important objects in his thesis (Berlin, 27 November 1901) and his later paper of 1927 [Schur]. Their natural generalizations are used all the time in the to-day representation theory of algebraic or finite or p-adic reductive groups (see the excellent book [Green J.A]). They are

1) the Schur algebra, the tensor space, the double centralizer theorem,

2) the Hecke algebra, the functor $\mathrm{Hom}_A(Ae, -)$ for an idempotent e of an algebra A.

A.1.1 Let R be a commutative ring and let N, n be two integers ≥ 1. We denote by V_N a free R-module of rank N, by S_n the symmetric group on n elements and by $R[S_n]$ the group algebra of S_n over R. The classical **Schur R-algebra**

$$\mathcal{S}_R(N, n) := \mathrm{End}_{RS_n} V_R(N, n)$$

is the endomorphism ring of the **tensor space**

$$V_R(N, n) := V_N^{\otimes n}$$

for the natural right action of the symmetric group S_n:

$$(v_1 \otimes \ldots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}, \qquad \sigma \in S_n, v_1, \ldots, v_n \in V_n.$$

The natural left action of the general linear group $GL(N, R)$ on the tensor space $V_R(N, n)$

$$g(v_1 \otimes \ldots \otimes v_n) = gv_1 \otimes \ldots \otimes gv_n$$

commutes with the action of the symmetric group S_n. It is polynomial and homogeneous of degree n. The image of $R[GL(N, R)]$ in $\mathrm{End}_R V_R(N, n)$ is $\mathcal{S}_R(N, n)$ when R is an infinite field [Green J.A. 2.6.c]. One can also describe $\mathcal{S}_R(N, n)$ as the dual of the ring of polynomial functions on $GL(N, R)$ which are homogeneous of degree n.

A.1.2 Let R be a commutative ring and let $H \subset G$ be two finite groups (or an open compact subgroup H of a locally profinite group G). The **Hecke R-algebra** $\mathcal{H}_R(G, H)$ is the endomorphism algebra of the "regular" module $R[H\backslash G]$ (with the left action of G by right translation)

$$\mathcal{H}_R(G, H) := \mathrm{End}_{RG} R[H\backslash G].$$

The value at the trivial class H of $H\backslash G$ gives an R-module isomorphism with the free R-module generated by the double (H, H) cosets of G

$$\mathcal{H}_R(G, H) \simeq_R R[H\backslash G/H]$$

(the index R recalls that this is only an R-module isomorphism). When $R = \mathbf{Z}$ one omits R.

A.1.3 Let A be a ring of finite dimension over a field and let $e = e^2$ be an idempotent of A. Denote by $\mathrm{Mod}\, A$ the category of left A-modules V. Then the **functor**

$$V \to Ve : \mathrm{Mod}\, A \to \mathrm{Mod}\, eAe$$

induces a bijection between the simple A-modules V with $Ve \neq 0$ (that is V is a quotient of eA), and the simple eAe-modules. Moreover if $e \neq 0$ for any simple $V \in \mathrm{Mod}\, A$, then $V \to Ve$ is an equivalence of categories, see B.5.

A.1.4 We deduce from the three paragraphs A.1.1, A.1.2 and A.1.3 that the R-representations of the symmetric groups S_n are related with the representations of the Schur algebras $\mathcal{S}_R(N, n)$.

Suppose that R is an infinite field. The theory of the polynomial representations of general linear groups is equivalent to the theory of representations of Schur algebras (see [Green J.A]), and does not depend on R. This contrasts with the classification of the irreducible R-representations of the symmetric groups, which depends on R.

The $S_R(N,n)$-modules identify with the polynomial representations of $GL(N,R)$ which are homogeneous of degree n. The theory of weights and characters shows that the simple $S_R(N,n)$-modules are classified by the set $\Lambda^+(N,n)$ of partitions of n of length $t \leq N$,

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_t > 0, \quad \sum_i \lambda_i = n.$$

When $n \leq N$, $\Lambda^+(N,n)$ is the set of all partitions of n.

A.1.5 The tensor space $V_R(N,n) = V_N^{\otimes n}$ as an $R[S_n]$-module is described as follows.

Let
$(\varepsilon_i)_{1\leq i \leq N}$ a basis of V_N.
$I(N,n) := \{1,2,\ldots,N\}^n$
$\Lambda(N,n)$ the set of $\alpha = (\alpha_1,\ldots,\alpha_N) \in \{0,1,\ldots,n\}^N$ with sum $\alpha_1 + \ldots + \alpha_N = n$.

A basis of $V_R(N,n)$ is $(\varepsilon_{\underline{i}} := \varepsilon_{i_1} \otimes \ldots \varepsilon_{i_n})$ for $\underline{i} = (i_1,\ldots,i_n) \in I(N,n)$. The action of S_n on $V_R(N,n)$ comes from a natural action of S_n on $I(N,n)$. The orbits of S_n in $I(N,n)$ are parametrized by $\Lambda(N,n)$. Each orbit contains a unique element $\underline{i}(\alpha) := (1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n})$ with $\alpha \in \Lambda(N,n)$ (a^m means that a appears with multiplicity m). Denote by S_n^α the stabilizer of $\underline{i}(\alpha)$ in S_n. The group S_n^α is isomorphic to a product of symmetric groups $S_{\alpha_1} \times \ldots \times S_{\alpha_n}$ with the convention $S_0 = \{1\}$.

$V_R(N,n)$ is a direct sum of representations of S_n

$$V_R(N,n) \simeq_{R[S_n]} \oplus_{\alpha \in \Lambda(N,n)} R[S_n^\alpha \backslash S_n].$$

Note the formula for the dimensions

$$N^n = \sum_{\alpha_1+\ldots+\alpha_N = n} \frac{n!}{\alpha_1! \ldots \alpha_N!}$$

A.1.6 For $\alpha \in \Lambda(N,n)$, the projection

$$e_\alpha : V_R(N,n) \to R[S_n^\alpha \backslash S_n]$$

is an idempotent in the Schur algebra $\mathcal{S}_R(N,n)$. The idempotents e_α for $\alpha \in \Lambda(N,n)$ are orthogonal with sum 1. For $\alpha, \beta \in \Lambda(N,n)$, we have

$$e_\beta \mathcal{S}_R(N,n) e_\alpha \simeq_R \mathrm{Hom}_{RS_n}(R[S_n^\alpha \backslash S_n], R[S_n^\beta \backslash S_n]).$$

When $\alpha = \beta$ this is an isomorphism of algebra :

$$e_\alpha \mathcal{S}_R(N,n) e_\alpha \simeq \mathcal{H}_R(S_n, S_n^\alpha).$$

The group S_n^α is trivial and $e_\alpha \mathcal{S}_R(N,n) e_\alpha \simeq R[S_n]$ if α_i are 0 or 1 for all $1 \leq i \leq N$. As $\sum_i \alpha_i = n$ this is possible if and only if $n \leq N$.

A.1.7 Suppose $n \leq N$. Choose any α with S_n^α trivial, for instance $\alpha = (1,\ldots,1,0,\ldots,0)$ and set $e := e_\alpha$. Then

$$e_\beta \mathcal{S}_R(N,n) e \simeq_R \mathrm{Hom}_{RS_n}(R[S_n], R[S_n^\beta \backslash S_n]) \simeq_R R[S_n^\beta \backslash S_n]$$

and the tensor space $V_R(N,n)$ is a **cyclic $\mathcal{S}_R(N,n)$-module** generated by e

$$V_R(N,n) \simeq \mathcal{S}_R(N,n)e.$$

An $\mathcal{S}_R(N,n)$-endomorphism of $\mathcal{S}_R(N,n)e$ is defined by the image of e which belongs to $e\mathcal{S}_R(N,n)e \simeq R[S_n]$. This is the **double centralizer theorem**

$$\operatorname{End}_{\mathcal{S}_R(N,n)} V_R(N,n) \simeq R[S_n].$$

A.1.8 Suppose that R is an infinite field and $n \leq N$. The functor

$$V \to Ve : \operatorname{Mod} \mathcal{S}_R(N,n) \to \operatorname{Mod} R[S_n]$$

induces a bijection between the irreducible polynomial R-representations of $GL(N,R)$ which are quotients of $V_R(N,n)$ and the irreducible R-representations of S_n.

If the characteristic of R is 0 or $p > n$, the algebra $\mathcal{S}_R(N,n)$ is semi-simple because $R[S_n]$ is semi-simple, and the endomorphism algebra of a semi-simple module is semi-simple. As $V_R(N,n)$ is a faithful $\mathcal{S}_R(N,n)$-module, any simple $\mathcal{S}_R(N,n)$-module is isomorphic to a quotient of $V_R(N,n)$. Hence by A.1.3, the functor $V \to Ve$ is an equivalence of categories.

If the characteristic of R is $p < n$, by reduction modulo p, any simple $\mathcal{S}_R(N,n)$-module is isomorphic to a *subquotient* of $V_R(N,n)$. The simple *quotients* of $V_R(N,n)$ correspond only to the partitions $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_t > 0$ of n which are column p-regular [Green J.A (6.4b) page 94], i.e. for which all the integers $\lambda_1 - \lambda_2, \ldots, \lambda_t$ lie between 0 and $p-1$.

A.1.9 Suppose that R is an infinite field and $M \leq N$ (so that we treat also the case $M \leq n$). Naturally $\Lambda(M,n) \subset \Lambda(N,n)$ and $e_M = \sum_{\alpha \in \Lambda(M,n)} e_\alpha$ is an idempotent of $\mathcal{S}_R(M,n)$ such that

$$e_M \mathcal{S}_R(N,n) e_M \simeq \mathcal{S}_R(M,n).$$

Again by A.1.3, the functor

$$V \to Ve_M : \operatorname{Mod} \mathcal{S}_R(N,n) \to \operatorname{Mod} \mathcal{S}_R(M,n)$$

induces a bijection between the irreducible R-representations of $GL(N,R)$ which are polynomial homogeneous of degree n and such that $e_M V \neq 0$ and the irreducible R-representations of $GL(M,R)$ which are polynomial homogeneous of degree n. It is an equivalence of categories when $\mathcal{S}_R(N,n)$ and $\mathcal{S}_R(M,n)$ have the same number of simple modules. In particular the Schur algebra $\mathcal{S}_R(N,n)$ is Morita equivalent to the Schur algebra $\mathcal{S}_R(n,n)$ when $n \leq N$.

A.2 q-deformation

Suppose that R is a commutative ring. Dipper and James [DJ1] [DJ2] constructed q-deformations of the group algebra $R[S_n]$, of the tensor space $V_R(N,n)$, and of the Schur algebra $\mathcal{S}_R(N,n)$ and generalized the theory given in A.1 to the q-deformations.

Let $\mathbf{F}_q$ be a finite field with q elements and let $G := GL(n, \mathbf{F}_q)$.

A.2.1 The q-deformation of $R[S_n]$ is based on the natural bijection [Carter 2.8.1 (iii)]

$$B \backslash G / B \simeq S_n$$

where B is the upper triangular subgroup of G.

A **q-deformation $R_q[S_n]$ of $R[S_n]$** is the Hecke R-algebra (A.1.2)

$$R_q[S_n] := \mathcal{H}(G, B).$$

isomorphic to $R[S_n]$ as an R-module, but with a different product. When $q = 1$ one recovers the algebra $R[S_n]$.

One deforms the tensor space $V_R(N, n)$ using the the more general bijection (loc. cit.)

$$P^\alpha \backslash G / P^\beta \simeq S_n^\alpha \backslash S_n / S_n^\beta$$

where $\alpha, \beta \in \Lambda(N, n)$, P^α is the upper parabolic subgroup of $GL(n, F)$ with blocs of size $(\alpha_i \times \alpha_j)$ for the α_i which are not zero.

A **q-deformation $V_{q,R}(N, n)$ of the tensor space $V_N^{\otimes n}$** is

$$V_{q,R}(N, n) := \oplus_{\alpha \in \Lambda(N,n)} R[P^\alpha \backslash G/B].$$

This is the space of B-invariants of a R-representation of G

$$\oplus_{\alpha \in \Lambda(N,n)} R[P^\alpha \backslash G].$$

The natural right action of $\mathcal{H}_R(G, B)$ on $V_{q,R}(N, n)$ is a deformation of the action of $R[S_n]$ on $V_R(N, n)$.

A q-deformation $S_{q,R}(N, n)$ of the Schur R-algebra $\mathcal{S}_R(N, n)$ is the endomorphism ring

$$\mathcal{S}_{q,R}(N, n) := \mathrm{End}_{R_q[S_n]} V_{q,R}(N, n).$$

When "$q = 1$" one recovers the classical Schur algebra $\mathcal{S}_R(N, n)$.

Suppose $N \geq n$. The argument given in the classical case A.1.7 shows that the $\mathcal{S}_{q,R}(N, n)$-module $V_{q,R}(N, n)$ is cyclic

$$V_{q,R}(N, n) = \mathcal{S}_{q,R}(N, n)e$$

where e is the projection on the $\alpha := (1, \ldots, 1, 0, \ldots, 0)$-component ($\sum 1 = n$) equal to $R_q[S_n] = \mathcal{H}_R(G, B)$ and that the $\mathcal{S}_{q,R}(N, n) - R_q[S_n]$-module $V_{q,R}(N, n)$ satisfies the double centralizer property [DJ2 6.6]

$$R_q[S_n] = \mathrm{End}_{\mathcal{S}_{q,R}(N,n)} V_{q,R}(N, n).$$

A.2.2 Suppose that R is an algebraically closed field. Dipper and James [DJ1] [DJ2], Dipper [D] and Dipper and Donkin [DD] explained how representations of quantum linear groups and finite general linear groups in the non-describing case are related, via the q-Schur algebras. Dipper and James [DJ1] explained how the decomposition numbers of $GL(n, \mathbf{F}_q)$ are determined by the decomposition numbers of various q^k-Schur algebras. Some part of the theory of Dipper and James has been simplified by Takeuchi [T].

It is remarkable that the simple modules of the Schur algebra $S_{q,R}(N, n)$ do not depend on (q, R) as long as $q \in R$ non zero [D 4.7] [DJ2 8.8].

A.3 q-affine deformation

There is a big jump from finite dimension to infinite dimension when one considers the affine case, as there is a big jump from a finite field to a p-adic field. The q-affine deformations of $R[S_n], V_R(N, n), S_R(N, n)$ are the q-deformations of the corresponding objects for the affine symmetric group $S_{n,aff}$.

Let F be a local non archimedean field with residual field $\mathbf{F}_q$ and ring of integers O_F. Let $G := GL(n, F)$.

The group S_n is isomorphic to the quotient N/T where N is the normalizer in G of the diagonal torus T of G. The affine symmetric group is the quotient

$$S_{n,aff} := N/T(O_F) \simeq \mathbf{Z}^n.S_n$$

where $T(O_F)$ is the maximal compact subgroup of T and $\mathbf{Z}^n.S_n$ is the naive semi-direct product. The "affine parabolic" or parahoric subgroups $\mathcal{P}^\alpha$ of G are the inverse images of the parabolic subgroups P^α of $GL(n, \mathbf{F}_q)$ by the reduction modulo p

$$GL(n, O_F) \to GL(n, \mathbf{F}_q).$$

The inverse image of the standard minimal parabolic subgroup B of $GL(n, \mathbf{F}_q)$ is the standard Iwahori subgroup I. The q-affine deformations are based on the natural bijections for $\alpha, \beta \in \Lambda(N, n)$

$$\mathcal{P}^\alpha \backslash G / \mathcal{P}^\beta \simeq S_n^\alpha \backslash S_{n,aff} / S_n^\beta$$

The Hecke algebra $\mathcal{H}(G, I)$ is called the **Iwahori-Hecke algebra of** G.

The q-affine deformation $\hat{R}_q[S_n]$ of the group R-algebra $R[S_n]$ is the Iwahori-Hecke R-algebra

$$\hat{R}_q[S_n] := \mathcal{H}_R(G, I).$$

The q-affine deformation of the tensor space $V_N^{\otimes n}$ is

$$\hat{V}_{q,R}(N, n) := \oplus_{\alpha \in \Lambda(N,n)} R[\mathcal{P}^\alpha \backslash G/I]$$

The q-affine deformation of the Schur R-algebra $S_R(N, n)$ is the endomorphism ring of $\hat{V}_{q,R}(N, n)$

$$\hat{S}_{q,R}(N, n) := \operatorname{End}_{\hat{R}_q[S_n]} \hat{V}_{q,R}(N, n).$$

$\hat{R}_q[S_n]$ is a q-deformation of $R[S_{r,aff}]$: they are isomorphic as R-modules and when "$q = 1''$ one recovers $\hat{R}_q[S_n] \simeq R[S_{r,aff}]$. The R-module $\hat{V}_{q,R}(N, n)$ is the module of I-invariants of a direct sum of regular representations of G

$$\oplus_{\alpha \in \Lambda(N,n)} R[\mathcal{P}^\alpha \backslash G],$$

and has a natural right action of $\mathcal{H}_R(G, I)$. The action of $\hat{R}_q[S_n]$ on $\hat{V}_{q,R}(N, n)$ is an affine deformation of the action of $R[S_n]$ on the tensor space $V_R(N, n)$.

Suppose $N \geq n$. Then the argument in the classical case A.1.7 shows that the $\hat{S}_{q,R}(N, n)$-module $\hat{V}_{q,R}(N, n)$ is cyclic

$$\hat{V}_{q,R}(N, n) = \hat{S}_{q,R}(N, n)e$$

where e is the projector on the $(1, \ldots, 1, 0, \ldots, 0)$-component $R[I\backslash G/I]$ $(\sum 1 = n)$ and that the $\hat{S}_{q,R}(N, n)-$ $\mathcal{H}_R(G, I)$-module $\hat{V}_{q,R}(N, n)$ satisfies the double centralizer property

$$\operatorname{End}_{\hat{S}_{q,R}(N,n)} \hat{V}_{q,R}(N, n) = \mathcal{H}_R(G, I).$$

A.4 The Schur algebra of a reductive p-adic group

The q-affine deformations of $R[S_n], V_R(n, n), \mathcal{S}_R(n, n)$ admit obvious analogues for a reductive connected p-adic (or finite) group G. We consider the p-adic case. The finite case can be treated in the same way, by exchanging the words parabolic and parahoric.

A.4.1 Let
F a local non archimedean field, with residual field $\mathbf{F}_q$ with $q = p^f$.
G the group of F points of a connected reductive group $\mathbf{G}$ over F.
T the group of F points of a maximal F-split torus $\mathbf{T}$ of $\mathbf{G}$
Z the group of F points of the centralizer of $\mathbf{T}$ in $\mathbf{G}$
N the group of F points of the normalizer $\mathbf{N}$ of $\mathbf{T}$ in $\mathbf{G}$
$W := \mathbf{N}/\mathbf{Z} \simeq N/Z$ the Weyl group of (G,T)
Π a basis of the affine simple roots defined by (G,T)

To any proper subset $J \subset \Pi$ and different from Π, Bruhat and Tits attached a "standard" parahoric subgroup P_J, which is an Iwahori subgroup $P_\emptyset = I$ when J is the emptyset. A parahoric subgroup is conjugate to a standard parahoric subgroup, but we will consider only standard parahoric subgroup, and we will omit "standard". The description of P_J [BTII 5.2.4] shows that $J \subset J'$ implies $P_J \subset P_{J'}$. The parahoric subgroup P_J has a pro-p-radical U_J of quotient $M_J := P_J/U_J$ the group of $\mathbf{F}_q$ points of a connected reductive group $\mathbf{M_J}$ over $\mathbf{F}_q$, a torus when $J = \emptyset$.

$W_{aff}(G) = N/(I \cap N)$ the generalized affine Weyl group of G, and $I \cap N = I \cap Z$ is contained in Z. The group $W_{aff}(G)$ is the semi-direct product of W and of $Z/(I \cap Z)$. The map $n \to InI$ induces a bijection from $W_{aff}(G)$ and the double (I,I)-cosets of G [M 3.22]. One denotes also by $IwI := InI$ the coset associated to $w \in W_{aff}(G)$ image of $n \in N$.

$s_i, i \in \Pi$, the element of order 2 in $W_{aff}(G)$ such that the $P_i := I \cup Is_iI$.

W_J the subgroup of $W_{aff}(G)$ generated by the affine reflections $s_j, j \in J$. The subgroup of W given by the "vector parts" of W_J is isomorphic to W_J and to the Weyl group of M_J. We will identify them. The group $W_\emptyset$ is trivial. We have $P_J = IW_JI$. For J, K two proper subsets of Π, the map $n \to P_J n P_K$ induces a bijection [M 3.22]

$$W_J \backslash W_{aff}(G)/W_K \simeq P_J \backslash G/P_K.$$

R a commutative ring, $\mathrm{Mod}_R\, G$ the category of smooth R-representations of G, $\mathrm{Irr}_R\, G$ the irreducible ones.

A.4.2 We consider the R-representation of G

$$V_R(G) := \oplus_J \, V_{R,J}, \quad V_{R,J} := R[P_J \backslash G].$$

The q-deformation of $R[W_{aff}(G)]$ is the Iwahori-Hecke R-algebra of G

$$\mathcal{H}_R(G,I) := \mathrm{End}_{RG}\, R[I \backslash G].$$

The tensor R-space of G is the right $\mathcal{H}_R(G,I)$-module

$$V_R(G)^I \simeq_R \oplus_J R[P_J \backslash G/I].$$

The Schur R-algebra $\mathcal{S}_R(G)$ of G is the endomorphism algebra of the tensor R-space

$$\mathcal{S}_R(G) := \mathrm{End}_{\mathcal{H}_R(G,I)}\, V_R(G)^I.$$

The $\mathcal{H}_R(G,I)$-module $V_R(G)^I$ is a deformation of the $R[W_{aff}(G)]$-module $\oplus_J R[W_J \backslash W_{aff}(G)]$. The argument of the classical case A.1.7 shows that the $\mathcal{S}_R(G,I)$-module $V_R(G)^I$ is cyclic

$$V_R(G)^I \simeq \mathcal{S}_R(G,I)e$$

where e is the projector of $R[I \backslash G/I] \simeq_R \mathcal{H}_R(G,I)$, and that the $\mathcal{S}_R(G,I) - \mathcal{H}_R(G,I)$-module $V_R(G)^I$ satisfies the double centralizer property

$$\mathcal{H}_R(G,I) = \mathrm{End}_{\mathcal{S}_R(G,I)}\, V_R(G)^I.$$

We will show in C.2.14 that the "Schur algebra" associated to $V_R(G)^I$ is isomorphic to the "Schur algebra" associated to $V_R(G)$.

A.4.3 Theorem *For any commutative ring R, and any reductive connected p-adic group G, the functor of I-invariants induces an algebra isomorphism*

$$\mathrm{End}_{RG}\, V_R(G) \simeq \mathrm{End}_{\mathcal{H}_R(G,I)}\, V_R(G)^I.$$

The proof consists in exhibiting two natural basis of $\mathrm{End}_{RG}\, V_R(G)$ and of $\mathrm{End}_{\mathcal{H}_R(G,I)}\, V_R(G)^I$ in correspondence by the functor of I-invariants.

It is clear that the Iwahori-Hecke algebra, the tensor space, the Schur algebra, and their various deformations are defined over $\mathbf{Z}$. For any of these objects X_R one has

$$X_R = X_{\mathbf{Z}} \otimes_{\mathbf{Z}} R.$$

B Morita Equivalences

The properties of the functor $\mathrm{Hom}(Ae,-)$ for an idempotent e of a finite dimensional algebra A described in A.1.3 can be generalized to algebras which are *infinite dimensional and without unit*. One can also replace Ae by a *quasi-projective* finitely generated left A module Q, although this will not be used for the proof of the main theorem. From Arabia [A], we know the conditions on Q which ensure that the functor $\mathrm{Hom}(Q,-)$ induces an equivalence of categories between the category of left A modules annihilated by the annihilator $\mathcal{J}$ of Q in A i.e. of left $A/\mathcal{J}$-modules, and the category of right $\mathrm{End}_A Q$ modules.

B.1 Let A be an algebra. We change the notation from A.1.3. We denote now by $\mathrm{Mod}\, A$ the category of non degenerate **right** A-modules $M = MA$ (we do not suppose that A contains a unit) and by ${}_A\,\mathrm{Mod}$ the non degenerate left A-modules, i.e. the non degenerate left modules for the opposite algebra A^o.

Let B be another algebra. One says that A, B are **Morita right equivalent** if the categories $\mathrm{Mod}\, A$ and $\mathrm{Mod}\, B$ are equivalent, **Morita left equivalent** if the categories ${}_A\,\mathrm{Mod}$ and ${}_B\,\mathrm{Mod}$ are equivalent and **Morita equivalent** if A, B are both left and right Morita equivalent.

$Q \in \mathrm{Mod}\, A$ is called **quasi-projective** when for each pair of homomorphisms in $\mathrm{Mod}\, A$

$$f, \pi : Q \to V$$

with π surjective, there exists $\phi \in \mathrm{End}_A Q$ with $f = \phi \circ \pi$.

$Q \in \mathrm{Mod}\, A$ is called **almost projective** when there exists a surjective homomorphism in $\mathrm{Mod}\, A$

$$\pi : P \to Q$$

with P projective and finitely generated, and kernel $\mathrm{Ker}\, \pi$ stable by $\mathrm{End}_A P$. Almost projective implies quasi-projective.

$Q \in \mathrm{Mod}\, A$ is called a **progenerator** when it is projective, finitely generated and any A-module is the quotient of a direct sum of representations isomorphic to Q.

The same definitions can be given for $Q \in {}_A\,\mathrm{Mod}$.

B.2 Example : G is a reductive p-adic group, R is a commutative ring where p isninvertible, A is the global Hecke algebra $\mathcal{H}_R(G)$. Then the cyclic R-representation of G

$$Q = R[I\backslash G]$$

where I is an Iwahori subgroup is almost projective [Vig3 I.3 proposition]. This is false in general when I is replaced by another parahoric subgroup P, for instance for $G = GL(n, \mathbf{Q}_p)$ and $P = GL(n, \mathbf{Z}_p)$.

The R-representation Q of G is projective when the pro-order of I is not 0 in R.

B.3 Theorem *Let $Q \in {}_A\,\mathrm{Mod}$ quasi-projective and finitely generated. The functor*

$$\mathrm{Hom}_A(Q,-) : {}_A\,\mathrm{Mod} \to \mathrm{Mod}\,\mathrm{End}_A Q$$

induces

1) a bijection between the simple quotients of Q and the simple right $\mathrm{End}_A Q$-modules,

2) an equivalence of categories of inverse the functor $- \otimes_A Q$, when Q is projective and $\mathrm{Hom}_A(Q,V) \neq 0$ for any simple $V \in {}_A\,\mathrm{Mod}$. Then Q is a progenerator of ${}_A\,\mathrm{Mod}$.

See [A] Th.4 2) (b-2). For the property 2), one uses the following lemma.

B.4 Lemma *When* $Q \in {}_A\mathrm{Mod}$ *is projective and finitely generated and* $\mathrm{Hom}_A(Q,V)\neq 0$ *for any simple* $V \in {}_A\mathrm{Mod}$, *then any* $V \in {}_A\mathrm{Mod}$ *is the sum* $\sum_f f(Q)$ *for all* $f \in \mathrm{Hom}_R(Q,V)$.

Proof a) Let $X \in {}_A\mathrm{Mod}$ non zero. Let W be a non zero finitely generated subrepresentation of X, let W_1 be an irreducible quotient of W. By hypothesis $\mathrm{Hom}_A(Q,W_1)\neq 0$. By hypothesis Q is projective, hence $\mathrm{Hom}_A(Q,W)\neq 0$. Hence $\mathrm{Hom}_A(Q,X)\neq 0$.

b) Let $V \in {}_A\mathrm{Mod}$ non zero. Let $V' := \sum_f f(Q)$ for all $f \in \mathrm{Hom}_R(Q,V)$. We have $\mathrm{Hom}_A(Q,V) = \mathrm{Hom}_A(Q,V')$. The functor $\mathrm{Hom}_A(Q,-)$ is exact hence $\mathrm{Hom}_A(Q,X) = 0$ for $X = V/V'$. By a) $V = V'$. $\diamond$

B.5 Example Suppose $Q = Ae$ generated by an idempotent $e = e^2$ of A. The "value at e" gives an isomorphism of abelian groups

$$\mathrm{Hom}_A(Ae,V) \simeq eV$$

for all $V \in {}_A\mathrm{Mod}$. In the case where $V = Ae$ the "value at e" is an isomorphism of algebra

$$(\mathrm{End}_A\, Ae)^o \simeq eAe.$$

The functor $V \to eV : {}_A\mathrm{Mod} \to {}_{eAe}\mathrm{Mod}$ is an equivalence of categories if $eV\neq 0$ for any non zero simple $V \in {}_A\mathrm{Mod}$. This is A.1.3 (the notation has been changed).

Let $M_1,\ldots,M_n \in {}_A\mathrm{Mod}$, there can be redundancy. We suppose that $M_1,\ldots M_k$ are not isomorphic and that for each $1 \le i \le n$, there is some $1 \le j \le k$ such that M_i is isomorphic to M_j. We set

$$B := \mathrm{End}_A\, M, \qquad M := M_1 \oplus \ldots \oplus M_n.$$

$$C := \mathrm{End}_A\, N, \qquad N := M_1 \oplus \ldots \oplus M_k.$$

The natural projection $e : M \to N$ is an idempotent in B and $eBe = C$.

B.6 Corollary *The functor* $V \to eV : {}_B\mathrm{Mod} \to {}_C\mathrm{Mod}$ *and the functor* $V \to Ve : \mathrm{Mod}\,B \to \mathrm{Mod}\,C$ *are equivalences of categories.*

Hence the algebra B and C are Morita equivalent.

Proof The proofs are symmetrical. Let us consider the functor $V \to eV$. By B.5 it is enough to prove that $eV\neq 0$ for any non zero $V \in {}_B\mathrm{Mod}$. The projections $e_i : M \to M_i$ for $1 \le i \le n$, are orthogonal idempotents in B with sum the unit 1 of B, and $e_1 + \ldots + e_k = e$. There exists some $1 \le i \le n$ with $e_iV\neq 0$. We will prove that $e_jV\neq 0$ for all j such that $M_j \simeq M_i$. This implies $eV\neq 0$.

One reduces easily to the case where all M_i are isomorphic i.e. $k = 1$. Then $B \simeq M(n,C)$ and the idempotent e_i identifies to a diagonal matrix with the single non zero entry equal to 1 at (i,i). The idempotents are permuted by the symmetric group naturally embedded in the group B^* of invertible elements of B, hence for any $1 \le i \le n$ there exist a unit $b \in B^*$ such that $e_1 = be_ib^{-1}$. If V is a non zero B-module then $e_iV\neq 0$ for some i and $e_1V = be_ib^{-1}V\neq 0$. $\diamond$

Application. *We may get rid of the redundancies in the tensor spaces defining the Schur algebra and its deformations* in A.4.2.

B.7 Corollary *Let* $Q \in \mathrm{Mod}\,A$ *with annihilator* $\mathcal{J}$ *in* A. *When* Q *is projective in* $\mathrm{Mod}\,A/\mathcal{J}$ *and* $\mathrm{Hom}_A(Q,V)\neq 0$ *for any simple* $V \in \mathrm{Mod}\,A/\mathcal{J}$, *the functor*

$$\mathrm{Hom}_A(Q,-) : \mathrm{Mod}\,A/\mathcal{J} \simeq \mathrm{Mod}(\mathrm{End}_A\, Q)^o$$

is an equivalence of categories, Q *is a progenerator of* $\mathrm{Mod}\,A/\mathcal{J}$. *The inverse of the functor is the functor* $- \otimes_{A/\mathcal{J}} Q$.

C Decomposition of the parahoric restriction-induction functor

We suppose that R is any commutative ring. We do not suppose that p is invertible in R, in view of future applications to the case where the characteristic of R is p. The parahoric restriction and induction functors for groups (C.1) are defined in [Vig6] where they play a fundamental role in the classification of the irreducible R-representations of G of level 0 when R is algebraically closed. They appear naturally with the Schur algebra. The decomposition of the restriction-induction functor is very useful to study the functor I-invariants or to get a natural basis for the affine Schur algebra. The analogue for the Hecke algebras will be given in (C.2), and the comparison via the invariant functors will be done in (C.3).

C.1 Groups

Notation. We need even more than in A.4. The situation is complicated by the fact that the generalized affine Weil group $W_{aff}(G)$ is not a Coxeter group (and we insist for working on a general reductive p-adic group).

Let

G' the normal subgroup of G generated by the parahoric subgroups P_J for all proper subsets $J \subset \Pi$.

$N' := G' \cap N$, the double (I, I)-cosets of G contained in G' are $In'I$ for $n' \in N'$,

$W_{aff} := N'/(I \cap N')$ the affine Weyl group of affine BN-pair (G', I, N'). It is a Coxeter group for $(s_i)_{i \in \Pi}$.

$\Omega \in W_{aff}(G)$ the subgroup such that the double (I, I)-cosets of G contained in the normalizer of I in G are $IwI, w \in \Omega$. The group Ω is abelian, isomorphic to $N/N' \simeq G/G'$, normalizes I and $(s_i)_{i \in \Pi}$. The generalized affine Weyl group $W_{aff}(G)$ is a semi direct product of the normal subgroup W_{aff} and of Ω.

ℓ the length on W_{aff} extended to a length on $W_{aff}(G)$ such that the set of elements of $W_{aff}(G)$ of length 0 is equal to Ω.

A set of representatives $n_w, w \in W_{aff}(G)$ in N of $W_{aff}(G)$ can be chosen such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$ when $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ and $n_w \in N'$ if $w \in W_{aff}$ [M 5.2]. We fix such a set, and we often identify n_w with w.

C.1.1 Distinguished elements

We will always denote by J or K a proper subset of Π. Any double (W_K, W_J) coset in $W_K \backslash W_{aff}(G)/W_J$ contains a unique element of minimal length, called distinguished by (K, J). The set of these elements is denoted by $D_{K,J}(G)$. It is identified with its representatives in N. Clearly $D_{J,K}(G)$ (in $W_{aff}(G)$) is the set $D_{K,J}(G)^{-1}$ of inverse elements of $D_{K,J}(G)$. The same of true for the representatives of $D_{J,K}(G)$ in N. When L is another proper subset of Π which contains $K \cup J$, it is clear that $D_{K,J}(G) \cap W_L$ is the set of (K, J)-distinguished elements of W_L. It will be denoted by $D^L_{K,J}$.

Let $d \in D_{K,J}(G)$. We denote by $\mathrm{Int}(d)$ the conjugation by d, when it has a sense. The important properties of d are :

a) For all $w \in W_J$ and $d' \in D_{J,\emptyset}$ we have [M 3.9]

$$\ell(w_J d') = \ell(w_J) + \ell(d').$$

For $w_K \in W_K, w_J \in W_J$, the relation $\ell(w_K d w_J) = \ell(w_K) + \ell(d) + \ell(w_J)$ is not always true (there is no "addition on the lengths in $W_K d W_J$"), but one has (there is no reference in Morris, but the proof is as in [Carter 2.7.5]) using [M 3.16, 3.17])

$$W_K d W_J = D^K_{\emptyset, K \cap dJ} d W_J$$

with unique decomposition $w = a d w_J, a \in D^K_{\emptyset, K \cap dJ}, w_J \in W_J, w \in W_K d W_J$, and there is "addition of the lengths in $D^K_{\emptyset, K \cap dJ} d W_J$" $\ell(a d w_J) = \ell(a) + \ell(d) + \ell(w_J)$, i.e. $ad \in D_{\emptyset, J}$.

b) [Carter 2.7.4] [M 3.17]

$$W_K \cap d W_J d^{-1} = W_{K \cap dJ}$$

The map $v \to dvd^{-1} : W_{d^{-1}K\cap J} \to W_{K\cap dJ}$ is an isomorphism.

c) [Carter 2.8.7] [M 3.19, 3.20, 3.21] The group $P_K \cap \mathrm{Int}(d)P_J$ modulo U_K, is a standard parabolic subgroup $P^K_{K\cap dJ}$ of M_K with unipotent radical $U^K_{K\cap dJ}$ equal to $P_K \cap \mathrm{Int}(d)U_J$ modulo U_K. The finite reductive quotient of $P^K_{K\cap dJ}$ is denoted $M^K_{K\cap dJ}$.

d) Suppose $K \subset J$. In the Bruhat-Tits semi-simple building, the facet $\mathcal{F}_J$ associated to J is contained in the closure of the facet $\mathcal{F}_K$ associated to K. This implies that $J \to P_J$ respects the inclusions and that $J \to U_J$ reverses the inclusions [BTII 5.2.4, BTI 6.4.9, 7.1.1] and [SS I.2.11],

$$U_J \subset U_K \subset P_K \subset P_J.$$

The groups P_K, U_K are the inverse images of P^J_K, U^J_K in P_J by the reduction modulo U_J, and we have a canonical isomorphism

$$M_K = P_K/U_K \simeq (P_K/U_J)/(U_K/U_J) = P^J_K/U^J_K = M^J_K.$$

In particular $M^K_{K\cap dJ} \simeq M_{K\cap dJ}$ will be identified.

e) The conjugation by d gives an isomorphism $\mathrm{Int}(d) : \mathrm{Int}(d^{-1})P_J \to P_J$ respecting the pro-p-radicals and by quotient an isomorphism between the reductive quotients

$$\mathrm{Int}(d) : \ M_{d^{-1}J} \to M_J.$$

f) The group $B_J := P^J_\emptyset$ is a minimal parabolic subgroup of M_J with unipotent radical $U^J_\emptyset$.

C.1.2 Parahoric induction and restriction

Let R be any commutative ring and let $\mathrm{Mod}_R\, G$ be the category of representations of G. We consider the functors :

a) **Inflation** Representations of the finite reductive group M_J identify with representations of the parahoric group P_J trivial on U_J by inflation

$$\mathrm{infl}^{P_J}_{M_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, P_J.$$

b) The **right adjoint** of the inflation is the U_J**-invariant** functor

$$\mathrm{inv}^{P_J}_{M_J} : \mathrm{Mod}_R\, P_J \to \mathrm{Mod}_R\, M_J.$$

c) The **compact induction** from representations of the parahoric group P_J to representations of the p-adic reductive group G

$$\mathrm{ind}^G_J : \mathrm{Mod}_R\, P_J \to \mathrm{Mod}_R\, G$$

d) The **right adjoint** of the compact induction is the **restriction**

$$\mathrm{res}^G_J : \mathrm{Mod}_R\, G \to \mathrm{Mod}_R\, P_J.$$

e) The **parahoric induction** i^G_J and its **right adjoint** the **parahoric restriction** r^G_J are:

$$i^G_J := \mathrm{ind}^G_{P_J} \circ \mathrm{infl}^{P_J}_{M_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, P_J \to \mathrm{Mod}_R\, G,$$

$$r^G_J := \mathrm{inv}^{P_J}_{M_J} \circ \mathrm{res}^G_{P_J} : \mathrm{Mod}_R\, G \to \mathrm{Mod}_R\, P_J \to \mathrm{Mod}_R\, M_J.$$

f) Suppose $K \subset J$. Then M_K is the Levi subgroup of the parabolic subgroup P_K^J of M_J. We have the functors of **parabolic induction** and its **right adjoint** the **parabolic restriction** between the finite reductive groups

$$i_K^J := \mathrm{ind}_{P_K^J}^{M_J} \circ \mathrm{infl}_{M_K}^{P_K^J} : \mathrm{Mod}_R\, M_K \to \mathrm{Mod}_R\, P_K^J \to \mathrm{Mod}_R\, M_J$$

$$r_K^J := \mathrm{inv}_{M_K}^{P_K^J} \circ \mathrm{res}_{P_K^J}^{M_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, P_K^J \to \mathrm{Mod}_R\, M_K.$$

It is well known that the parabolic induction and restriction functors in the finite or p-adic case are transitive. The parahoric functors relate p-adic groups and finite groups. We consider the composition of a parahoric functor with a parabolic functor between finite reductive subgroups.

C.1.3 Transitivity *When $K \subset J$, then*

$$i_K^G = i_J^G \circ i_K^J, \quad r_K^G = r_K^J \circ r_J^G.$$

Proof By adjunction, it is enough to prove one of the two equalities. They are true because we have an exact sequence

$$1 \to U_J \to U_K \to U_K^J \to 1.$$

C.1.4 Theorem [Vig6] *For any commutative ring R, the parahoric restriction-induction functor*

$$T_{K,J}^G = r_K^G \circ i_J^G : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, G \to \mathrm{Mod}_R\, M_K$$

is a direct sum

$$T_{K,J}^G = \oplus_{d \in D_{K,J}(G)}\ F_{K,J}^d$$

of functors $F_{K,J}^d$ isomorphic to

$$i_{K\cap dJ}^K \circ \mathrm{Int}(d) \circ r_{J\cap d^{-1}K}^J : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, M_{J\cap d^{-1}K} \to \mathrm{Mod}_R\, M_{K\cap dJ} \to \mathrm{Mod}_R\, M_K.$$

The decomposition is given by the restriction to the U_K-invariant functions with support $P_J d^{-1} P_K$. The isomorphism for $F_{K,J}^d$ is obtained via the map $f \to \phi(m) = f(d^{-1}m)$ modulo $U_{J\cap d^{-1}K}^J$ for $m \in M_K$.

The sum is infinite, but only finite reductive groups appear in the definition of the functors $F_{K,J}^d$.

The are three basic cases :
a) The parabolic restriction $r_K^J \simeq F_{K,J}^1$ when $K \subset J$
b) The parabolic induction $i_K^J \simeq F_{J,K}^1$ when $K \subset J$
c) The conjugation $\mathrm{Int}(d) \simeq F_{dJ,J}^d$ when $dJ \subset \Pi$.

C.1.5 Basic Example (for $K = J = \emptyset$). For any R-representation V of the Iwahori subgroup $I = P_\emptyset$ trivial on its pro-p-radical $I_p = U_\emptyset$, we have an isomorphism of I/I_p-modules

$$(\mathrm{ind}_I^G V)^{I_p} = \oplus_{w \in W_{aff}^G} \mathrm{Int}\, w\ V.$$

As r_K^G is the right adjoint of i_K^G, we have $\mathrm{Hom}_{RG}(R[P_K\backslash G], R[P_J\backslash G]) \simeq_R \mathrm{Hom}_{RP_K}(1, T_{K,J}^G 1)$, and the theorem gives a basis of this R-module.

C.1.6 Basis

$$\mathrm{Hom}_{RG}(R[P_J\backslash G], R[P_K\backslash G]) \simeq_R \oplus_{d \in D_{K,J}(G)} R\psi_{K,J}^d$$

where $\psi^d_{K,J}$ sends the characteristic function of P_J to the characteristic function of $P_K dP_J$.

C.2 Hecke algebras

Let R be any commutative ring. The affine Iwahori-Hecke algebra $\mathcal{H}_R(G,I)$ is denoted by $\mathcal{H}_R$. The double (I,I) coset of $w \in W_{aff}(G)$ considered in $\mathcal{H}_R$ is denoted by T_w. The product is the "convolution product for the Haar measure normalized by I". This "non sense" valid when $R = \mathbf{Q}$ is the field of rational numbers allows to compute the product when $R = \mathbf{Z}$ and by scalar extension for any R. As in C.1, J, K denote always proper subsets of Π.

The Hecke algebra $\mathcal{H}_R(P_J, I)$ is denoted by $\mathcal{H}_{R,J}$. It is a subalgebra of $\mathcal{H}_R$

$$\mathcal{H}_{R,J} = \oplus_{w\in W_J} RT_w$$

and is naturally isomorphic to the Hecke algebra $\mathcal{H}_R(M_J, B_J)$. For $K \subset J$, we have $\mathcal{H}_{R,K} \subset \mathcal{H}_{R,J}$.

Notations as in Chapter B for modules. We consider the functors:

a) The **induction**

$$i^{\mathcal{H}}_J : \operatorname{Mod} \mathcal{H}_{R,J} \to \operatorname{Mod} \mathcal{H}_R, \quad V \to V \otimes_{\mathcal{H}_{R,J}} \mathcal{H}_R.$$

b) The **right adjoint**[1] of the induction is the **restriction**

$$r^{\mathcal{H}}_J : \operatorname{Mod} \mathcal{H}_R \to \operatorname{Mod} \mathcal{H}_{R,J}.$$

C.2.1 Theorem *For any commutative ring R, the parahoric restriction-induction functor*

$$T^{\mathcal{H}}_{K,J} := r^{\mathcal{H}}_K \circ i^{\mathcal{H}}_J : \operatorname{Mod} \mathcal{H}_{R,J} \to \operatorname{Mod} \mathcal{H}_R \to \operatorname{Mod} \mathcal{H}_{R,K}$$

is a direct sum

$$T^{\mathcal{H}}_{K,J} = \oplus_{d\in D_{K,J}} \mathcal{F}^d_{K,J}$$

of functors $\mathcal{F}^d_{K,J}$ isomorphic to

$$i^{\mathcal{H},K}_{K\cap dJ} \circ \operatorname{Int}(d) \circ r^{\mathcal{H},J}_{J\cap d^{-1}K} : \operatorname{Mod} \mathcal{H}_{R,J} \to \operatorname{Mod} \mathcal{H}_{R,J\cap d^{-1}K} \to \operatorname{Mod} \mathcal{H}_{R,dJ\cap K} \to \operatorname{Mod} \mathcal{H}_{R,K}.$$

The three basic functors are
a) the restriction $r^{\mathcal{H},J}_K : \operatorname{Mod} \mathcal{H}_{R,J} \to \operatorname{Mod} \mathcal{H}_{R,K}$ when $K \subset J$
b) the induction $i^{\mathcal{H},J}_K : \operatorname{Mod} \mathcal{H}_{R,K} \to \operatorname{Mod} \mathcal{H}_{R,J}$ when $K \subset J$
c) the conjugation $\operatorname{Int}(d) : T_w \mapsto T_{dwd^{-1}}, \operatorname{Mod} \mathcal{H}_{R,J} \to \operatorname{Mod} \mathcal{H}_{R,dJ}$ when $dJ \subset \Pi$.

We need some preparation for the proof of the theorem given in C.2.12.

C.2.2 Relations in Iwahori-Hecke algebras

Let G as in A.4 and C.1. Change of notation: we identify the simple affine roots in Π with the associated reflexions. The Iwahori-Hecke algebras are described when G is a Chevalley group by Iwahori-Matsumoto

[1] the right adjoint of the restriction is $\operatorname{Hom}_{\mathcal{H}_M}(\mathcal{H}_M, -)$.

[IM] and when G is semi-simple in Borel [B]. The Iwahori-Hecke algebra $\mathcal{H}_R(G', I)$ of G' is the algebra associated to the Coxeter system (W_{aff}, Π), and the **constants**

$$q_s := [IsI/I] = q^{d_s}$$

for all $s \in \Pi$ [Bki GAL Ch.4, Ex.23, 24], where d_s is the integer attached to s as in [Tits 1.8.1, 2.4, 3.5.4]. When s, s' are conjugate in $W_{aff}(G)$ the constants are equal $q_s = q_{s'}$. When the torus $I/I_p = M_\emptyset$ is split, i.e. G is residually split, all the integers d_s are equal to 1. The order of a maximal unipotent subgroup of M_s is q_s. For $w \in W_{aff}(G)$ we set

$$q_w := [IwI : I] = q^d.$$

If $w = \sigma s_1 \dots s_r$ where $\sigma \in \Omega$, and $s_1 \dots s_r$ is a reduced word in W_{aff} then $d = \sum_{t=1}^r d_{s_t}$. When $w, w' \in W_{aff}(G)$ are conjugate then $q_w = q_{w'}$. In particular $q_w = q_{w^{-1}}$.

The relations between the T_w for $w \in W_{aff}$ are

$$(C.2.3) \qquad T_{ww'} = T_w T_{w'} \ \text{ if } \ \ell(ww') = \ell(w) + \ell(w') \ \text{ for all } w, w' \in W_{aff}$$

$$(C.2.4) \qquad T_s^2 = (q_s - 1)T_s + q_s T_1. \ \text{ for all } s \in \Pi$$

The Iwahori-Hecke algebra $\mathcal{H}_R = \mathcal{H}_R(G, I)$ of G is a generalized affine Hecke algebra. As Ω normalizes I, the R-submodule $\oplus_{\sigma \in \Omega} R T_\sigma$ is a subalgebra isomorphic to the group algebra $R[\Omega]$, and it is known that for all $w \in W_{aff}(G)$ and $\sigma \in \Omega$

$$(C.2.5) \qquad T_{w\sigma} = T_w T_\sigma, \quad T_{\sigma w} = T_\sigma T_w.$$

Hence we have

$$\mathcal{H}_R \simeq R[\Omega].\mathcal{H}_R(G', I)$$

The definition of the length on $W_{aff}(G)$ and the relation (C.2.5) show that the relation (C.2.3) is valid for all $w, w' \in W_{aff}(G)$. We have $T_{ww'} = T_w T_{w'}$ if there is "addition of the lengths in ww'".

When p is invertible in R, T_s is invertible in $\mathcal{H}_R$ by C.2.4

$$T_s^{-1} = (1 - q_s^{-1})T_1 + q_s^{-1} T_s$$

and by C.2.3 and C.2.5, T_w is invertible in $\mathcal{H}_R$ for all $w \in W_{aff}(G)$.

6.2.6 Characters The map $w \to q_w$ extends to a character of $\mathcal{H}_R$ called the "index" or "trivial" character. The map $w \to (-1)^{\ell(w)}$ extends to a character of $\mathcal{H}_R$ called the "sign" character. Same for the finite Hecke algebra $\mathcal{H}_{R,J}$. Set

$$x_J = \sum_{w \in W_J} T_w, \quad y_J = \sum_{w \in W_J} (-1)^{-\ell(w_{oJ} w)} q_{w_{oJ} w} T_w$$

where w_{oJ} is the longest element of W_J [Carter 2.211]. When $J = \emptyset$, then $x_\emptyset = y_\emptyset = T_1$.

Note that by additivity of the lengths $q_{w_{oJ}} = q_{w_{oJ} w} q_{w^{-1}} = q_w q_{w_{oJ} w}$ and by symmetry $q_{w_{oJ}} = q_w q_{w w_{oJ}}$. If p is invertible in R we have

$$y_J = (-1)^{-\ell(w_{oJ})} q_{w_{oJ}} \sum_{w \in W_J} (-1)^{-\ell(w)} q_w^{-1} T_w.$$

C.2.7 Lemma *Modulo multiplication by an element of R, x_J and y_J are the unique elements in $\mathcal{H}_{R,J}$ which satisfy the relations*

$$x_J T_s = T_s x_J = q_s x_J, \quad y_J T_s = T_s y_J = -y_J$$

for all $s \in J$.

Proof Let $a_w \in R$ for $w \in W_J$. Let $s \in J$. Then

$$(\sum_{w \in W_J} a_w T_w) T_s = \sum_{w \in W_J, \ell(w) < \ell(ws)} (a_w T_w + a_{ws} T_{ws}) T_s.$$

We have for $w \in W_J, \ell(w) < \ell(ws)$

$$(a_w T_w + a_{ws} T_{ws}) T_s = a_w T_{ws} + a_{ws} T_w (q_s T_1 + (q_s - 1) T_s) = q_s a_{ws} T_w + (q_s a_{ws} + a_w - a_{ws}) T_{ws}.$$

The equations

$$q_s a_w = q_s a_{ws}, \quad q_s a_{ws} = q_s a_{ws} + (a_w - a_{ws})$$

are equivalent to $a_w = a_{ws}$. They are satisfied for all $s \in J$ if and only if a_w is constant for $w \in W_J$. The equations

$$-a_w = q_s a_{ws}, \quad -a_{ws} = q_s a_{ws} + (a_w - a_{ws})$$

are equivalent to $a_w = -q_s a_{ws}$. They are satisfied for all $s \in J$ if and only if $a_w = (-1)^{\ell(w^{-1} w_{oJ})} q_{w^{-1} w_{oJ}} a_{w_{oJ}}$ for all $w \in W_J$. We can replace $w^{-1} w_{oJ}$ by its inverse $w_{oJ} w$ in the term before $a_{w_{oJ}}$.

Same proof for the other side.

C.2.8 Lemma
a) When $K \subset J$ we have

$$x_K x_J = x_J x_K = P_K(q) x_J, \quad \text{where} \ \ P_K(q) := \sum_{w \in W_K} q_w.$$

b) For $d \in D_{K,J}(G)$ we have $P_{d^{-1}K \cap J}(q) = P_{K \cap dJ}(q)$.
c) For $d \in D_{K,J}(G)$ we have

$$x_K T_d x_J = P_{K \cap dJ}(q) \sum_{w \in W_K d W_J} T_w.$$

Proof a) results from C.2.7.
b) results from C.1.1 b), and $q_w = q_{w'}$ when $w, w' \in W_{aff}(G)$ are conjugate.
We prove c). We have $W_K d W_J = D^K_{\emptyset, K \cap dJ} d W_J$ with addition of the lengths on the right side from C.1.1 a), hence

$$\sum_{w \in W_K d W_J} T_w = \sum_{w \in D^K_{\emptyset, K \cap dJ}} T_w T_d x_J$$

We have $W_K = D^K_{\emptyset, K \cap dJ} W_{K \cap dJ}$ with addition of the lengths from C.1.1 a). Hence

$$x_K = \sum_{w \in D^K_{\emptyset, K \cap dJ}} T_w \ x_{K \cap dJ}.$$

We have $W_{K \cap dJ} d = d W_{d^{-1}K \cap J}$ with addition of the lengths, hence

$$x_K T_d = \sum_{w \in D^K_{\emptyset, K \cap dJ}} T_w \ x_{K \cap dJ} T_d = \sum_{w \in D^K_{\emptyset, K \cap dJ}} T_w \ T_d x_{d^{-1}K \cap J}.$$

By right multiplication by x_J using a) and b) we get the formula. $\diamond$

C.2.9 Iwahori-Matsumoto automorphism *Suppose p invertible in R.*

a) There is an automorphism j of order 2 of the Iwahori-Hecke algebra $\mathcal{H}_R$

$$j(T_s) = -q_s T_s^{-1} = (q_s - 1)T_1 - T_s$$

for all $s \in \Pi$ and $j(T_\sigma) = T_\sigma$ for all $\sigma \in \Omega$.

b) This automorphism exchanges x_J and y_J modulo a unit

$$j(x_J) = c_J y_J$$

where $c_J \in R$ is invertible, for $J \subset \Pi$.

Proof a) $j(T_s)$ satisfies the equation $(X+1)(X-q) = 0$. We admit that j is an automorphism [IM 3.2], [DJ1 §2].

b) [DJ1 page 25]. $\diamond$.

C.2.10 Lemma *$\mathcal{H}_R$ is a free left (resp. right) $\mathcal{H}_{R,J}$ module with basis $(T_d)_{d \in D_{J,\emptyset}(G)}$ (resp. $(T_d)_{d \in D_{\emptyset,J}(G)}$).*

Proof We have $W_{aff}(G) = W_J D_{J,\emptyset}(G) = D_{\emptyset,J}(G) W_J$ with addition of the lengths by C.1.1.a). $\diamond$

C.2.11 Lemma *Let $d \in D_{J,K}(G)$. The R-submodule $R(I \backslash P_J d P_K / I) \simeq \oplus_{w \in W_J d W_K} R T_w$ of $\mathcal{H}_R$ is a free right $\mathcal{H}_{R,J}$-module (resp. left $\mathcal{H}_{R,K}$-module)*

$$\oplus_{w \in W_J d W_K} R T_w = \oplus_{a \in D^J_{\emptyset, K \cap dJ}} T_a T_d \mathcal{H}_{R,K} = \oplus_{b \in D^K_{d^{-1}K \cap J, \emptyset}} \mathcal{H}_{R,J} T_d T_b$$

with basis $T_a T_d$ for $a \in D^J_{\emptyset, K \cap dJ}$ (resp. $T_d T_b$ for $b \in D^K_{d^{-1}K \cap J, \emptyset}$).

Proof We have $I \backslash P_J d P_K / I \simeq W_J d W_K = D^J_{\emptyset, K \cap dJ} d W_K = W_J d D^K_{d^{-1}K \cap J, \emptyset}$ with addition of the lengths for the last two expressions by C.1.1 a). $\diamond$

C.2.12 *Proof of the theorem C.2.1* Let $V \in \operatorname{Mod} \mathcal{H}_{R,J}$. We want to describe $V \otimes_{\mathcal{H}_{R,J}} \mathcal{H}_R$ as a right $\mathcal{H}_{R,K}$-module. We have a decomposition of $(\mathcal{H}_{R,K}, \mathcal{H}_{R,J})$-module

$$\mathcal{H}_R \simeq \oplus_{d \in D_{J,K}(G)} R[I \backslash P_J d P_K / I].$$

For $d \in D_{J,K}(G)$ we want to understand the $\mathcal{H}_{R,K}$-module

$$V \otimes_{\mathcal{H}_{R,J}} R[I \backslash P_J d P_K / I].$$

We apply C.2.11.

$$V \otimes_{\mathcal{H}_{R,J}} R[I \backslash P_J d P_K / I] \simeq_R V \otimes_{\mathcal{H}_{R,J}} (\oplus_{a \in D^J_{\emptyset, dK \cap J}} T_a T_d \mathcal{H}_{R,K}).$$

We have $\sum_a V T_a = V$ because $1 \in D^J_{\emptyset, dK \cap J}$ and we can forget the "a" in the right side. We have an isomorphism of $\mathcal{H}_{R,K}$-module

$$\mathcal{F}^d_{K,J}(V) \simeq (V \otimes T_d) \otimes_{\mathcal{H}_{R, K \cap d^{-1}J}} \mathcal{H}_{R,K}.$$

C.2.13 We deduce from the theorem C.2.1 a basis for $\operatorname{Hom}_{\mathcal{H}_R}(x_J \mathcal{H}_R, x_K \mathcal{H}_R)$. The right module $x_J \mathcal{H}_R$ is induced from the "trivial" or "index" character of $\mathcal{H}_{R,J}$,

$$x_J \mathcal{H}_R \simeq x_J \mathcal{H}_{R,J} \otimes_{\mathcal{H}_{R,J}} \mathcal{H}_R = \{h \in \mathcal{H}_R \mid T_w h = q_w h \text{ for all } w \in W_J\}.$$

Same proof as in C.2.7 using C.2.10. The restriction is the right adjoint of the induction and with the theorem C.2.1 we get

$$\mathrm{Hom}_{\mathcal{H}_R}(x_J\mathcal{H}_R, x_K\mathcal{H}_R) = \oplus_{d\in D_{K,J}(G)} \mathrm{Hom}_{\mathcal{H}_{R,J}}(x_J\mathcal{H}_{R,J}, R[P_K\backslash P_K dP_J/I]).$$

Let $a_w \in R$ for $w \in W_K dW_J$ then

$$T_v(\sum_{w\in W_K dW_J} a_w T_w)T_u = q_w q_u(\sum_{w\in W_K dW_J} a_w T_w)$$

for all $v \in W_K,\ u \in W_J$ is equivalent to a_w constant for all $w \in W_K dW_J$ (same proof as in C.2.7). We deduce:

Basis *$\mathrm{Hom}_{\mathcal{H}_R}(x_J\mathcal{H}_R, x_K\mathcal{H}_R)$ has a basis $(\Phi^d_{K,J})$ parametrized by $D_{K,J}(G)$,*

$$\Phi^d_{K,J} x_J = \sum_{w\in W_K dW_J} T_w.$$

In particular
$\Phi^1_{\emptyset,J} : x_J\mathcal{H}_R \to \mathcal{H}_R$ is the inclusion
$\Phi^1_{J,\emptyset} : \mathcal{H}_R \to x_J\mathcal{H}_R$ is the left multiplication by x_J
$\Phi^w_{\emptyset,\emptyset} = T_w$ for all $w \in W_{aff}(G)$.

C.2.14 *Proof of the theorem A.3.4* As a right $\mathcal{H}_R$-module,

$$R[P_J\backslash G/I] = x_J\mathcal{H}_R.$$

From C.1.6 and C.2.13 *the functor of I-invariants gives an isomorphism*

$$\mathrm{Hom}_{RG}(R[P_J\backslash G], R[P_K\backslash G]) \simeq \mathrm{Hom}_{\mathcal{H}_R}(x_J\mathcal{H}_R, x_K\mathcal{H}_R)$$

sending $\Psi^d_{K,J}$ on $\Phi^d_{K,J}$ for all $d \in D_{K,J}(G)$.

Hence the Schur algebra $\mathcal{S}_R$ of G is also the ring of endomorphisms of a R-representation of G

$$\mathcal{S}_R = \mathrm{End}_{\mathcal{H}_R} \oplus_J x_J\mathcal{H}_R \simeq \mathrm{End}_{RG} \oplus_J R[P_J\backslash G].$$

C.2.15 The Schur algebra, basis, relations, generators From C.2.13, a basis of the Schur algebra $\mathcal{S}_R$ is $(\Phi^d_{K,J})$ where K, J are proper subsets of Π and $d \in D_{K,J}(G)$,

$$\Phi^d_{K,J} x_{J'} = \delta_{J,J'} \sum_{w\in W_K dW_J} T_w$$

and $\delta_{J,J'} = 1$ if $J = J'$ and 0 otherwise. The product

(0) $$\Phi^d_{K,J}\Phi^{d'}_{K',J'} = 0$$

of two elements of the basis is 0 when $J \neq K'$. The product $\Phi^d_{K,L}\Phi^{d'}_{L,J}$ is a finite sum $\sum a_{d''}\Phi^{d''}_{K,J}$.

The basis elements $\Phi^1_{\emptyset,J}, \Phi^1_{J,\emptyset}, T_w$ satisfy the relations:

(a) $T_s\Phi^1_{\emptyset,J} = q_s\Phi^1_{\emptyset,J}$ for $s \in J$

(b) $\Phi^1_{J,\emptyset}T_s = q_s\Phi^1_{J,\emptyset}$ for $s \in J$

(c) $\Phi^1_{\emptyset,J}\Phi^1_{J,\emptyset} = x_J$

(d) $\Phi^1_{K,\emptyset}T_d\Phi^1_{\emptyset,J} = P_{dJ\cap K}(q)\Phi^d_{K,J}$

They are all evident, the last one (d) is equivalent to the basic formula C.2.8

$$x_K T_d x_J = P_{K\cap dJ}(q)x_J.$$

We get a presentation of the Schur algebra $\mathcal{S}_R$ by generators $\Phi^1_{\emptyset,J}, \Phi^1_{J,\emptyset}, T_s, T_\sigma$ for J proper subset of Π $s\in\Pi, \sigma\in\Omega$ and relations (0), (a), (b), (c), (d), and the relations in $\mathcal{H}_R$ when $P_J(q)$ is invertible in R for all $J\subset\Pi$.

C.3 Going between groups and Hecke algebras

Let R be any commutative ring. The functor of invariants by the Iwahori subgroup I, or the Borel subgroup B_J,

$$\mathrm{inv}_I : \mathrm{Mod}_R\, G \to \mathrm{Mod}\,\mathcal{H}_R, \quad \mathrm{inv}_{B_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}\,\mathcal{H}_{R,J}$$

relates the R-representations of G, or M_J, with the right modules for the Iwahori-Hecke algebra $\mathcal{H}_R$, or the finite Hecke algebra $\mathcal{H}_{R,J}$.

We will show that the parahoric induction or restriction on the group side correspond via $\mathrm{inv}_I, \mathrm{inv}_{B_J}$ to the analogous functors on the Hecke algebra side.

C.3.1 Theorem *The functors*

$$i_J^{\mathcal{H}}\circ\ \mathrm{inv}_{B_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}\,\mathcal{H}_{R,J} \to \mathrm{Mod}\,\mathcal{H}_R$$
$$\mathrm{inv}_I\circ i_J^G : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, G \to \mathrm{Mod}\,\mathcal{H}_R$$

are equal.

Proof Let $V\in\mathrm{Mod}_R\, M_J$. By C.1.4, $\mathrm{inv}_I\circ i_J^G(V)$ is

$$(C.3.2) \qquad (i_J^G V)^I = \oplus_{d\in D_{J,\emptyset}(G)} V_d^I$$

where V_d^I is the R-module of functions of $i_J^G(V)$ with support $P_J dI$ and right invariant by I, and the value at d gives an isomorphism

$$V_d^I \simeq_R V^{B_J}.$$

We denote by $f_{v,d}$ the function in V_d^I with value $v\in V^{B_J}$ at d. As $\mathcal{H}_R$ is a free left $\mathcal{H}_{R,J}$-module of basis (T_d) for $d\in D_{J,\emptyset}(G)$ by C.2.10, $i_J^{\mathcal{H}}\circ\ \mathrm{inv}_{B_J}(V)$ is

$$(C.3.3) \qquad V^{B_J}\otimes_{\mathcal{H}_J}\mathcal{H}_R = \oplus_{d\in D_{J,\emptyset}(G)} V^{B_J}\otimes T_d.$$

We compare now the right actions of $\mathcal{H}_R$ on (C.3.2) and (C.3.3). Let $w\in W_{aff}(G)$ and let $f = f_{v,1} : P_J\to R$ be a right I-invariant function. By definition

$$fT_w(-) = \sum_g f(-g^{-1}) \quad \text{if} \quad IwI = \cup_g Ig \ \ \text{(disjoint union)}.$$

The support of fT_w is $P_J dI$. When $w\in W_J$ then $T_w\in\mathcal{H}_{R,J} = \mathcal{H}_R(P_J, I)$. We have $f(1)\in V^{B_J}$, and

$$fT_w(1) = f(1)T_w.$$

When $d\in D_{J,\emptyset}(G)$ we have

$$fT_d(d) = f(1).$$

because the terms in $\sum_g f(dg^{-1})$, $IdI = \cup_g Ig$ vanish if $Ig\neq Id$. Indeed $g = dx, x\in I$ and $dx^{-1}d^{-1} = y\in P_J$ implies $T_d = T_yT_d$ by "additivity of the lengths" (C.1.1 a), hence $y\in I$.

Let $\phi \in \mathcal{H}_R$. We write $\phi = hT_d$ for $h \in \mathcal{H}_{R,J}$ and $d \in D_{J,\emptyset}(G)$. We have clearly $(v \otimes T_1)hT_d = vh \otimes T_d$ and by the above computation $f_{v,1}hT_d = f_{vh,1}T_d = f_{vh,d}$. We deduce that the map

$$v \otimes T_d \to f_v T_d : V^{B_J} \otimes T_d \to V_d^I$$

is an isomorphism of right $\mathcal{H}_R$-modules. $\diamond$

C.3.4 Theorem *The functors*

$$\mathrm{inv}^{B_J} \circ r_J^G : \mathrm{Mod}_R\, G \to \mathrm{Mod}_R\, P_J \to \mathrm{Mod}\, \mathcal{H}_{R,J}$$

and

$$r_J^{\mathcal{H}} \circ \mathrm{inv}^I : \mathrm{Mod}_R\, G \to \mathrm{Mod}\, \mathcal{H}_R \to \mathrm{Mod}\, \mathcal{H}_{R,J}$$

are equal.

Proof U_J is a normal subgroup of I with quotient $I/U_J = B_J$. $\diamond$

C.3.5 Corollary *Let $d \in D_{K,J}(G)$. Then the functors*

$$\mathcal{F}_{K,J}^d \circ\ \mathrm{inv}_{B_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}\, \mathcal{H}_{R,J} \to \mathrm{Mod}\, \mathcal{H}_{R,K}$$

and

$$\mathrm{inv}_{B_K} \circ\ F_{K,J}^d : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, M_K \to \mathrm{Mod}\, \mathcal{H}_{R,K}$$

are equal. The functors

$$T_{K,J}^{\mathcal{H}} \circ\ \mathrm{inv}_{B_J} : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}\, \mathcal{H}_{R,J} \to \mathrm{Mod}\, \mathcal{H}_{R,K}$$

$$\mathrm{inv}_{B_K} \circ\ T_{K,J}^G : \mathrm{Mod}_R\, M_J \to \mathrm{Mod}_R\, M_K \to \mathrm{Mod}\, \mathcal{H}_{R,K}$$

are equal.

Our motivation is to study the homomorphism of I-invariants, when we restrict to parahorically induced representations. The above results allows to reduce from the p-adic case to the finite case. Let $V_J \in \mathrm{Mod}_R\, M_J, V_K \in \mathrm{Mod}_R\, M_K$ for two proper subsets J, K of Π.

C.3.6 Proposition *The homomorphism of the I-invariants*

$$\mathrm{Hom}_{RG}(i_K^G V_K, i_J^G V_J) \to \mathrm{Hom}_{\mathcal{H}_R}((i_K^{\mathcal{H}} V_K)^I, (i_J^{\mathcal{H}} V_J)^I)$$

is injective (resp. surjective, bijective) if and only if the homomorphisms of B_K-invariants

$$\mathrm{Hom}_{RM_K}(V_K, F_{K,J}^d V_J) \to \mathrm{Hom}_{\mathcal{H}_{R,K}}(V_K^{B_K}, (F_{K,J}^d V_J)^{B_K}) \simeq \mathrm{Hom}_{\mathcal{H}_{R,K}}(V_K^{B_K}, \mathcal{F}_{K,J}^d(V_J^{B_J}))$$

are injective (resp. surjective, bijective) for all $d \in D_{K,J}(G)$.

Proof We have by adjunction and decomposition of $T_{K,J}^G$

$$(a) \qquad \mathrm{Hom}_{RG}(i_K^G V_K, i_J^G V_J) \simeq \oplus_{d \in D_{K,J}(G)} \mathrm{Hom}_{RM_K}(V_K, F_{K,J}^d V_J)$$

Using that the parahoric induction commutes with the invariants, adjunction, decomposition of $T_{K,J}^{\mathcal{H}}$, and finally that the parahoric restriction-induction commutes with the invariants, one gets

$$(b) \qquad \mathrm{Hom}_{\mathcal{H}_R}((i_K^G V_K)^I, (i_J^G V_J)^I) \simeq \oplus_{d \in D_{K,J}(G)} \mathrm{Hom}_{\mathcal{H}_{R,K}}(V_K^{B_K}, (F_{K,J}^d V_J)^{B_K}).$$

It is easily seen that the homomorphism of the I-invariants from (a) to (b) on the left side respects the decomposition, and corresponds to the homomorphism of the B_K-invariants on each term of the right side. We apply (C.3.5) to replace $(F_{K,J}^d V_J)^{B_K}$ by $\mathcal{F}_{K,J}^d(V_J^{B_J})$. $\diamond$

C.3.7 Remark When V_K is generated by its B_K-invariants, $i_K^G V_K$ is generated by its I-invariants and the homomorphism of I-invariants in C.3.6 is injective. If moreover R is a field of characteristic $\neq p$, V_K, V_J are finite dimensional, the equality of the dimensions

$$\dim \mathrm{Hom}_{RM_K}(V_K, F_{K,J}^d V_J) = \dim \mathrm{Hom}_{\mathcal{H}_{R,K}}(V_K^{B_K}, \mathcal{F}_{K,J}^d(V_J^{B_J}))$$

for all $d \in D_{K,J}(G)$ is equivalent to the bijectivity of the homomorphisms of I-invariants in C.3.6.

D Proof of the main theorem

We will prove the main theorem of the introduction. As most of the arguments are valid for a general reductive group, we restrict to $GL(n,F)$ only at the end.

D.1 Let G be a general reductive p-adic group as in A.4 and C.1. The group of $\mathbf{F}_q$-points of the maximal split torus $\mathbf{T}$ of G is a maximal split torus of M_J and the set Φ_J of roots of M_J with respect to this torus are described in [Tits 3.5.1]. We denote by $\Phi_J^+, \Delta_J \simeq J$ the set of positive, simple roots of Φ_J with respect to the Borel subgroup B_J of M_J. The order of the unipotent subgroup U_α of M_J attached to $\alpha \in \Delta_J$ is q^{d_s} where s is the reflexion associated to a simple affine root in J identified with α. One considers the following subset $U_\emptyset^{*,J}$ of the unipotent radical $U_\emptyset^J$ of B_J

$$(1) \qquad U_\emptyset^{*,J} := \prod_{\alpha \in \Phi_J^+ - \Delta_J} U_\alpha$$

The commutator relations show that $U_\emptyset^{*,J}$ is a normal subgroup of $U_\emptyset^J$ and that the quotient $U_\emptyset^J / U_\emptyset^{*,J}$ is abelian. When $K \subset J$, we have the exact sequence (C.1.1)

$$(2)_{K\subset J} \qquad 1 \to U_J \to U_K \to U_J^K \to 1$$

and

$$U_J^K = \prod_{\alpha \in \Phi_J^+ - \Phi_K^+} U_\alpha$$

$$(2)'_{K\subset J} \qquad U_\emptyset^{*,J} \subset U_\emptyset^{*,K} U_J^K = \prod_{\alpha \in \Phi_J^+ - \Delta_K} U_\alpha.$$

We have $U_\emptyset^J = U_\emptyset^K U_J^K$. The group $U_\emptyset$ is the pro-p-radical of the Iwahori subgroup $I = P_\emptyset$. The group $U_\emptyset^{*,J}$ lifts to an open compact subgroup $V_\emptyset^{*,J}$ of $U_\emptyset$ using the exact sequence $(2)_{\emptyset \subset J}$.

Lemma *Suppose $K \subset J$. Then $V_\emptyset^{*,J} \subset V_\emptyset^{*,K}$.*

Proof The inverse image by $(2)_{\emptyset,J}$ of the inclusion $(2)'_{K\subset J}$ is the inclusion of the lemma. $\diamond$

D.2 Let G be the group of rational points of a reductive connected group over the finite field $\mathbf{F}_q$. We use the same notations as above (with $G = M_J$), T a maximal split torus, W the Weyl group, $B(= B_J) = ZU$ a minimal parabolic subgroup with unipotent radical $U(= U_\emptyset^J)$ and Levi subgroup a torus Z, Φ the roots of (G,T), Δ the simple roots, $U^*(= U_\emptyset^{*,J})$. A parabolic subgroup $P = MV$ with unipotent radical V and Levi M is standard if $B \subset P, Z \subset M$.

Definition *Let R be any commutative ring. A R-character $\chi : U \to R^*$ of U is called non degenerate if χ is not trivial on U_α for any simple root $\alpha \in \Delta$, and generic if χ is non degenerate and trivial on U^* . The R-representation of G*

$$\Gamma_{R,\chi} := \operatorname{ind}_U^G \chi$$

induced by a generic R-character χ of U is called a Gelfand-Graev R-representation of G.

When the generic characters are conjugate in G there is a unique Gelfand-Graev R-representation of G modulo isomorphism. When G is a torus Z, then $U = \{1\}$ and the (unique) Gelfand Graev representation of Z is the regular representation

$$\Gamma_R = i_1^Z 1 \simeq R[Z].$$

Note that 0 is the only element G-invariant of $\Gamma_{R,\chi}$ when G is not a torus. If $G = Z$ is a torus then $\Gamma_R^Z \simeq R$.

Let ℓ be a prime number different from p. We decompose $Z = Z_\ell Z^\ell$ where Z_ℓ is of order a power of ℓ and Z^ℓ of order prime to ℓ. Set $B = Z_\ell B^\ell$ with $B^\ell = Z^\ell U$ (the prime-to-ℓ part of B). When ℓ does not divide $|Z|$ we have $Z_\ell = \{1\}$ and $B = B^\ell$.

We collect facts on a Gelfand-Graev representation which will be used later. We insist on keeping a commutative ring R as general as possible and a general reductive connected finite group G.

D.3 Lemma *Let R be a commutative ring such that U has a generic R-character χ. Then the Gelfand-Graev R-representation $\Gamma_{R,\chi}$ has the following properties:*
1) The U-invariants of $\Gamma_{R,\chi}$ is

$$\Gamma_{R,\chi}^U \simeq R[Z].$$

2) $\mathrm{Hom}_{RG}(\mathrm{ind}_{B^\ell}^G 1, \Gamma_{R,\chi}) \simeq R[Z_\ell]$.
3) The B-invariants of $\Gamma_{R,\chi}$ is isomorphic to the sign character as a right $\mathcal{H}_R(G,B)$-module

$$\Gamma_{R,\chi}^B \simeq \mathrm{sign}\,.$$

4) Let $P = MV$ be a standard parabolic subgroup of G. The V-invariants of $\Gamma_{R,\chi}$ is a Gelfand-Graev representation of M

$$\Gamma_{R,\chi}^V \simeq \Gamma_{R,w_{Mo}w_o(\chi)|_M}$$

where w_o, resp. w_{Mo} is the longest element of W, resp. W_M and $w_{Mo}w_o(\chi)|_M : U\cap M \to R^*$ is the generic character $u \mapsto \chi(w_o w_{Mo} u w_{Mo} w_o)$. The P-invariants of $\Gamma_{R,\chi}$ is 0 if $P \neq B$.

Proof Note that 1) is a particular case of 4). We prove 4). The objects relative to M are denoted with an index M. By the Mackey decomposition C.1.4, $\Gamma_{R,\chi}^V$ is a direct sum indexed by W/W_M. Let $w \in W/W_M$ distinguished. We have

$$wVw^{-1} \cap U = \prod_{\alpha \in \Phi^+ - \Phi_M^+, w(\alpha)\in\Phi^+} U_{w(\alpha)}.$$

This group contains no U_β with $\beta \in \Delta$ if and only if $w^{-1}(\beta) \in \Phi_M^+$ or $w^{-1}(\beta) < 0$ for any $\beta \in \Delta$. This is equivalent to [Carter page 262] $w \in w_o w_{Mo}$. One deduces by C.1.4 that $\Gamma_{R,\chi}^V$ is the set of functions $f : Uw_o w_{Mo} MV \to R$ such that $f(uw_o w_{Mo} mv) = \chi(u) f(w_o w_{Mo} m)$ for all $u \in U, m \in M, v \in V$ with the natural action of M. By restriction to M we deduce the first part of 4). The second part of 4) comes from the fact that $\Gamma^M_{R,w_{Mo}w_o(\chi)|_M} = 0$ if M is not a torus.

We prove 2) using adjunction and 1). We get $\mathrm{Hom}_{RG}(\mathrm{ind}_{B^\ell}^G 1, \mathrm{ind}_U^M \chi) \simeq R[Z_\ell Z^\ell]^{Z^\ell} \simeq R[Z_\ell]$.

We prove 3). From 1) $\Gamma_{R,\chi}^B$ is a free R-module of rank 1, the set of functions $f : Uw_oB \to R$ such that $f(uw_ob) = \chi(u) f(w_o)$ for all $u \in U, b \in B$. Let $s \in W$ be a reflexion associated to a simple root $\alpha \in \Delta$. Then $B \cup BsB$ is a standard parabolic subgroup and $\Gamma_{R,\chi}^{B\cup BsB} = 0$ by 4). Hence $f[B] + f[BsB] = 0$ for any $f \in \Gamma_{R,\chi}^B$. Hence $\Gamma_{R,\chi}^B$ is the sign representation. $\diamond$

When R is a field of characteristic $\neq p$, the representation $\mathrm{ind}_B^G 1$ is quasi-projective [Vig3, I.3 proposition] and the B-invariant functor

$$V \to V^B : \mathrm{Mod}_R(G) \to \mathrm{Mod}_R \mathcal{H}_R(G,B)$$

induces a bijection between the irreducible representations of G generated by their B-invariant vectors and the simple modules for $\mathcal{H}_R(G,B)$.

D.4 Definition *When R is a field of characteristic $\neq p$, we denote by St_R the unique irreducible representation of G corresponding to the character* sign *of $\mathcal{H}_R(G,B)$ by the B-invariant functor.*

By D.1 (1), we have an isomorphism of locally profinite groups

$$U/U^* \simeq \prod_{\alpha\in\Delta} U_\alpha.$$

A R-character χ of U trivial on U^* defines a subset of Δ, the simple roots α such that χ is not trivial on U_α, hence a standard parabolic subgroup $P = MV$ containing B such that χ is the character of U obtained by inflation of a generic character of $U_M = U \cap M$. In this way, the R-characters of U which are trivial on U^* are in bijection with the disjoint union of the generic characters of U_M for all standard M.

We go back to the p-adic case. We use the notations introduced in D.1, D.2, D.3 introducing an index J for objects associated to the finite reductive group M_J. *Let R be any commutative ring where p is invertible and such that the characteristic function of the finite group U_α is the sum of the R-characters of U_α, for all simple roots $\alpha \in \Pi$.* We introduce the R-representation of G:

$$\mathbf{\Gamma}_R := \oplus_{i\in\Pi}\ R[U^*_{\Pi-i}\backslash G] = \oplus_{i\in\Pi}\ e_i\mathcal{H}_R(G) \tag{3}$$

where $U^*_{\Pi-i} := V_\emptyset^{*,\Pi-i}$, and e_i is the idempotent of $\mathcal{H}_R(G)$ associated to $U^*_{\Pi-i}$ for all $i \in \Pi$. The representation $\mathbf{\Gamma}_R \in \mathrm{Mod}_R\, G$ is projective and finitely generated.

Recall that J denotes always a proper subset of Π.

D.5 Proposition *1) The R-representation $\mathbf{\Gamma}_R$ of G is isomorphic to*

$$\mathbf{\Gamma}_R \simeq \oplus_J \oplus^{|\Pi|-|J|} \oplus_{\chi\in Y_J} i_J^G \Gamma_{R,\chi,J}$$

where Y_J is the set of generic R-characters of the unipotent radical of B_J and $\Gamma_{R,\chi,J}$ is the Gelfand-Graev R-representation of M_J associated to $\chi \in Y_J$.

2) The I-invariants of $\mathbf{\Gamma}_R$ is an $\mathcal{H}_R$-module isomorphic to

$$\mathbf{\Gamma}_R^I \simeq \oplus_J \oplus^{|Y_J|(|\Pi|-|J|)} y_J\mathcal{H}_R$$

where $\mathcal{H}_{R,J}y_J = \mathrm{sign}_J$ is the sign character of $\mathcal{H}_{R,J}$ (C.2.4).

3) The ring of $\mathcal{H}_R$-endomorphisms of $\mathbf{\Gamma}_R^I$ is Morita equivalent to the Schur algebra $\mathcal{S}_R$

$$\mathrm{End}_{\mathcal{H}_R}(\mathbf{\Gamma}_R^I) \simeq_{\mathrm{Morita}} \mathcal{S}_R.$$

Proof We prove the property 1) of the proposition. Let us fix $i \in \Pi$. Our hypothesis on R allows to write e_i as an orthogonal sum of idempotents e_χ for all R-characters χ of $U_\emptyset^{\Pi-i}$ trivial on $U_\emptyset^{\Pi-i,*}$. Such a character χ is the inflation of a generic character of the unipotent radical $U_\emptyset^J$ of B_J for some $J \subset \Pi - i$. We identify the characters of the unipotent radical $U_\emptyset^J$ of B_J to characters of $U_\emptyset$ trivial on U_J via the exact sequence $(2)_{\emptyset\subset J}$ in D.1. The idempotent e_χ identifies with an idempotent of the group R-algebra of M_J

$$R[M_J] = \mathcal{H}_R(P_J, U_J) \subset \mathcal{H}_R(G).$$

We have

$$e_i = \oplus_{J\subset\Pi-i} \oplus_{\chi\in Y_J} e_\chi \tag{4}$$

$$e_\chi R[M_J] \simeq \Gamma_{R,\chi,J} \tag*{(4)$'$}$$

hence

$$e_\chi\mathcal{H}_R(G) = i_J^G\Gamma_{R,\chi,J} \tag*{(4)$''$}$$

from which we deduce the proposition 1.

We prove the property 2) of the proposition. By the lemma D.3 we have in $\mathrm{Mod}\,\mathcal{H}_{R,J}$

$$\Gamma^{B_J}_{R,\chi,J} \simeq y_J\mathcal{H}_{R,J} \simeq \mathrm{sign}_J\,.$$

Hence by C.3.1 we have in $\mathrm{Mod}\,\mathcal{H}_R$

$$(i_J^G\Gamma_{R,\chi,J})^I \simeq y_J\mathcal{H}_R. \tag{5}$$

We prove the property 3) of the proposition. The algebra of $\mathcal{H}_R$-endomorphisms of $\oplus_{J\subset\Pi}\oplus^{m_J}\, y_J\mathcal{H}_R$ is Morita equivalent to the algebra of $\mathcal{H}_R$-endomorphisms of $\oplus_{J\subset\Pi}\oplus y_J\mathcal{H}_R$ because all the m_J are ≥ 1 and we can replace them by 1 by B.6. The algebra of $\mathcal{H}_R$-endomorphisms of $\oplus_{J\subset\Pi}\oplus y_J\mathcal{H}_R$ is isomorphic to the Schur algebra

$$\mathcal{S}_R := \mathrm{End}_{\mathcal{H}_R}\oplus_J\oplus x_J\mathcal{H}_R$$

because $\mathcal{H}_R$ has an automorphism of order 2 permuting x_J, y_J modulo a unit in R for all J by C.2.9.

We introduce the following property H_o for the Gelfand-Graev representations.

Property H_o *The B_J-invariants homomorphism*

$$\mathrm{Hom}_{RM_J}(\Gamma_{R,\chi,J},\Gamma_{R,\chi',J}) \to \mathrm{End}_{\mathcal{H}_{R,J}}\mathrm{sign}_J \tag*{(6)$_J$}$$

is surjective for all generic characters $\chi,\chi'\in Y_J$ for all J.

The surjectivity $(6)_J$ is clearly satisfied in the usual case where the Gelfand-Graev representations of M_J are isomorphic.

D.6 Lemma *The I-invariants algebra homomorphism*

$$\mathrm{End}_{RG}\,\mathbf{\Gamma}_R \to \mathrm{End}_{\mathcal{H}_R}(\mathbf{\Gamma}_R^I) \tag{6}$$

is surjective, when the property H_o is true.

In particular (6) is surjective when $GL(n,D)$ when D is a division algebra over F.

Proof By (4) and (5), we have to prove that the homomorphism of I-invariants

$$\mathrm{Hom}_{\mathcal{H}_R(G)}(e_\chi\mathcal{H}_R(G), e_\mu\mathcal{H}_R(G)) \to \mathrm{Hom}_{\mathcal{H}_R}(y_K\mathcal{H}_R, y_J\mathcal{H}_R)$$

is surjective for any pair of characters $(\chi,\mu)\in Y_K\times Y_J$ and for all proper subsets K,J of Π. By C.3.6 we have to prove that

$$\mathrm{Hom}_{RM_K}(\Gamma_{R,\chi,K}, F^d_{K,J}\Gamma_{R,\mu,J}) \to \mathrm{Hom}_{\mathcal{H}_{R,K}}(\mathrm{sign}_K, \mathcal{F}^d_{K,J}\,\mathrm{sign}_J)$$

is surjective for all $d\in D_{K,J}(G)$.

By definition $F^d_{K,J} = i^K_{K\cap dJ}\circ\mathrm{Int}(d)\circ r^J_{d^{-1}K\cap J}$ with a similar formula for $\mathcal{F}^d_{K,J}$. The image of a Gelfand-Graev representation by *parahoric restriction* is a Gelfand-Graev representation by D.3 4), the same is true for the conjugate by $\mathrm{Int}(d)$, and on the Hecke algebra side for the sign. The surjectivity is reduced to the surjectivity of $(6)_J$ for all J. $\diamond$

D.7 Let $\mathcal{J}_R$ be the annihilator of $R[I\backslash G]$ in the global Hecke algebra $\mathcal{H}_R(G)$. Set $\mathcal{J}_R^* := \{f^* \mid f\in\mathcal{J}_R\}$ where $f^*(g) := f(g^{-1})$ for all $g\in G$. A RG-endomorphism of $\mathbf{\Gamma}_R = \oplus_{i\in\Pi}\, e_i\mathcal{H}_R(G)$ is given by left multiplication by a matrix (a_{ji}) where $a_{ji}\in e_j\mathcal{H}_R(G)e_i$,

$$\oplus_{i\in\Pi}e_ix_i \to \oplus_{j\in\Pi}\sum_{i\in\Pi}a_{ji}e_ix_i = \oplus_{j\in\Pi}\sum_{i\in\Pi}a_{ji}x_i, \tag{7}$$

for $x_i \in \mathcal{H}_R(G)$.

Lemma *The kernel of the I-invariants algebra homomorphism*

$$\mathrm{End}_{RG}\,\mathbf{\Gamma}_R \to \mathrm{End}_{\mathcal{H}_R}(\mathbf{\Gamma}_R^I)$$

is the set of matrices (a_{ji}) with $a_{ji} \in e_j \mathcal{J}_R^* e_i$ for all $i, j \in \Pi$.

Proof The RG-endomorphism of $\mathbf{\Gamma}_R$ given by (a_{ji}) has a zero restriction to $\mathbf{\Gamma}_R^I = \oplus_{i\in\Pi}\, e_i R[G/I]$ if and only if $a_{ji}R[G/I] = 0$ for all $i, j \in \Pi$. The map $f \to f^*$ is an isomorphism from $\mathcal{H}_R(G)$ to its opposite algebra $\mathcal{H}_R(G)^o$ and sends $R[G/I]$ to $R[I\backslash G]$. Hence $a_{ji}R[G/I] = 0$ is equivalent to $a_{ji} \in \mathcal{J}_R^*$. We have $\mathcal{J}_R^* \cap e_j \mathcal{H}_R(G) e_i = e_j \mathcal{J}_R^* e_i$. $\diamond$

We deduce from D.6 that $\mathrm{End}_{\mathcal{H}_R}(\mathbf{\Gamma}_R^I)$ is isomorphic to the algebra of matrices (a_{ji}) with $a_{ji} \in e_j(\mathcal{H}_R(G)/\mathcal{J}_R^*)e_i$ for all $i, j \in \Pi$. It is clear that e_i is not zero in the quotient $\mathcal{H}_R(G)/\mathcal{J}_R^*$ because $e_i R[G/I] \neq 0$ and we do not change the notation for its image.

D.8 We introduce the R-representation of G

$$\mathbf{Q}_R := \mathbf{\Gamma}_R/\mathbf{\Gamma}_R\mathcal{J}_R = \oplus_{i\in\Pi}\, e_i(\mathcal{H}_R(G)/\mathcal{J}_R). \tag{8}$$

It is clear that $\mathbf{Q}_R$ is finitely generated and projective in $\mathrm{Mod}\,\mathcal{H}_R(G)/\mathcal{J}_R$. We have

$$\mathrm{End}_{RG}\,\mathbf{Q}_R \simeq \mathrm{End}_{RG}\,\mathbf{\Gamma}_R/\,\mathrm{Hom}_{RG}(\mathbf{\Gamma}_R, \mathbf{\Gamma}_R\mathcal{J}_R) \tag{9}$$

because the kernel $\mathbf{\Gamma}_R\mathcal{J}_R$ of the surjective homomorphism $\mathbf{\Gamma}_R \to \mathbf{Q}_R$ is stable by $\mathrm{End}_{RG}\,\mathbf{\Gamma}_R$. Hence the ring $\mathrm{End}_{RG}\,\mathbf{Q}_R$ is isomorphic to the algebra of matrices (a_{ji}) with $a_{ji} \in e_j(\mathcal{H}_R(G)/\mathcal{J}_R)e_i$ for all $i, j \in \Pi$. We deduce from D.5, D.6 and D.7 :

Proposition *When the property H_o is true, we have a Morita isomorphism*

$$(\mathrm{End}_{RG}\,\mathbf{Q}_R)^o \simeq_{\mathrm{Morita}} \mathcal{S}_R.$$

D.9 From B.7, the functor

$$\mathrm{Hom}_{\mathcal{H}_R(G)}(\mathbf{Q}_R, -) : \mathrm{Mod}\,\mathcal{H}_R(G)/\mathcal{J}_R \to \mathrm{Mod}(\mathrm{End}_{RG}\,\mathbf{Q}_R)^o \tag{10}$$

is an equivalence of categories, $\mathbf{Q}_R$ is a progenerator of $\mathrm{Mod}\,\mathcal{H}_R(G)/\mathcal{J}_R$, if and only if $\mathrm{Hom}_{RG}(\mathbf{Q}_R, V) \neq 0$ for any $V \in \mathrm{Irr}_R\, G$ annihilated by $\mathcal{J}_R$. From by (3) and (8) $\mathrm{Hom}_{RG}(\mathbf{Q}_R, V) \neq 0$ if and only if V has a non zero vector invariant by $U^*_{\Pi - i}$ for some $i \in \Pi$.

D.10 We suppose now, until the end of this chapter, that R is an algebraically closed field of characteristic $\neq p$, in order to have references for the following properties:

- a) The R-representations of level 0 of G form a sum of blocks and $\mathrm{ind}_I^G\, 1_R$ is of level 0 [Vig1]. Hence the R-representations of G annihilated by $\mathcal{J}_R$ are of level 0.

- b) For $GL(n, F)$ the irreducible R-representations annihilated by $\mathcal{J}_R$ are the irreducible unipotent representations (subquotients of $\mathrm{ind}_I^G\, 1_R$) [Vig3].

- c) For each irreducible R-representation V of level 0 of G, there exists a proper subset J of Π and an irreducible cuspidal R-representation σ_J of M_J such that V is a quotient of

$$i_J^G \sigma_J \tag{11}$$

i.e. the inflation to P_J of σ_J is contained in the restriction to P_J of V [Vig6].

We say that σ_J is *G-relevant* when $i_J^G \sigma_J$ has an irreducible quotient annihilated by $\mathcal{J}_R$.

- d) For $GL(n,F)$ the irreducible cuspidal G-relevant R-representations of M_J are the irreducible cuspidal subquotients of $\mathrm{ind}_{B_J}^{M_J} 1_R$.

We consider the following property (true when $G = GL(n,F)$):

Property H_1: *For any proper subset $J \subset \Pi$, any irreducible cuspidal G-relevant R-representation σ_J of M_J has a non zero vector invariant by $U_\emptyset^{*,J}$ (C.1.1).*

H_1 is trivially true, when there are no irreducible cuspidal G-relevant R-representation σ_J of M_J for all proper subsets J of Π.

Proposition $\mathrm{Mod}\,\mathcal{H}_R(G)/\mathcal{J}_R \simeq \mathrm{Mod}\,\mathcal{S}_R$ *and* $\mathbf{Q}_R$ *is a progenerator of* $\mathrm{Mod}\,\mathcal{H}_R(G)/\mathcal{J}_R$, *if the properties H_o, H_1 are true.*

Proof Let $V \in \mathrm{Irr}_R\, G$ annihilated by $\mathcal{J}_R$. Then V is of level 0 and contains a relevant type σ_J. There exists $i \in \Pi$ such that $J \subset \Pi - i$. By H_1, V has a non zero vector invariant by $V_\emptyset^{*,J}$ and by D.1, $V_\emptyset^{*,J}$ contains $V_\emptyset^{*,\Pi-i}$. By definition (3), $U_{\Pi-i}^* = V_\emptyset^{*,\Pi-i}$ hence $\mathrm{Hom}_{RG}(\mathbf{Q}_R, V) \neq 0$. We apply D.9. $\diamond$

D.11 *Remarks on the property H_1*

a) H_1 is not true in general. The property H_1 means that a cuspidal irreducible G-relevant representation σ_J is generic, i.e. contains a generic character of $U_\emptyset^J$ (the unipotent radical of B_J). Let $G = GU_3(\mathbf{F}_q)$, Over $\overline{\mathbf{Q}}_\ell$, by [G] the group G has three irreducible unipotent (in the sense of Lusztig) representations, trivial 1, Steinberg St, cuspidal π. Let $\ell \neq 2, 3, p$ and ℓ divides $q+1$. Then $r_\ell \pi$ is irreducible, and

$$r_\ell\,\mathrm{St} = 1 + \alpha r_\ell \pi + \rho \tag{12}$$

where $\rho \in \mathrm{Irr}_{\overline{\mathbf{F}}_\ell}\, G$, $2 \leq \alpha \leq (\ell^d + 1)/3$ and ℓ^d is the highest power of ℓ dividing $q+1$. The representation ρ is irreducible cuspidal and generic and $r_\ell \pi$ is irreducible cuspidal and not generic. But $r_\ell \pi$ is relevant.

b) The irreducible cuspidal R-representations of a finite linear group are generic [Vig1]. We say that we are in the *linear case when the groups M_J are isomorphic to product of finite linear groups* $\prod_i GL(n_i, \mathbf{F}_q)$. In the linear case, the properties H_o, H_1 are true. In particular when $G = GL(n,D)$ where D/F is a division algebra of finite dimension ($D = F$ in particular).

D.12 The *abelian subcategory of* $\mathrm{Mod}_R\, G$ *generated by* $V \in \mathrm{Mod}_R\, G$ is the full subcategory of R-representations of G with irreducible *subquotients* isomorphic to *subquotients* of V.

We introduce the abelian subcategory $\mathcal{B}_{R,1}(G)$ of $\mathrm{Mod}_R\, G$ generated by $R[I\backslash G]$. It is the unipotent block when $G = GL(n,F)$ for all R, or for G general when $R = \mathbf{C}$. We call the representations of $\mathcal{B}_{R,1}(G)$ "unipotent". When G is replaced by a finite reductive group, we give the same definition with a Borel subgroup B instead of I. A type of level 0 in an irreducible unipotent R-representation is relevant (D.9).

We consider the following properties H_2 and H_3 (true when $G = GL(n,F)$).

Property H_2 : *$\mathcal{B}_{R,1}(G)$ is a direct factor of* $\mathrm{Mod}_R(G)$. Then we can define the unipotent part V_1 of $V \in \mathrm{Mod}_R\, G$, for instance $\mathbf{\Gamma}_{R,1}$.

Property $H_{2,J}$: *$\mathcal{B}_{R,1}(M_J)$ is a direct factor of* $\mathrm{Mod}_R(M_J)$ *for all J*. Then we can define the unipotent part V_1 of $V \in \mathrm{Mod}_R\, M_J$, for instance $\Gamma_{R,\chi_J,1}$ for $\chi \in Y_J$.

Let N be the integer equal to the maximum length of $\Gamma_{R,\chi_J,1}$ for all J all $\chi \in Y_J$. Recall that $\mathcal{J}_R$ is the annihilator of $R[I\backslash G]$ in the global Hecke algebra $\mathcal{H}_R(G)$.

D.13 Theorem *1) When H_1, H_2 are true, the unipotent part $\mathbf{\Gamma}_{R,1}$ of $\mathbf{\Gamma}_R$ is a progenerator of $\mathcal{B}_{R,1}(G)$.*

2) When $H_2, H_{2,J}$ are true ,

$$\boldsymbol{\Gamma}_{R,1} = \oplus_{J\subset\Pi} \oplus^{|\Pi|-|J|} \oplus_{\chi_J} i_J^G \Gamma_{R,\chi_J,1}. \tag{13}$$

3) When $H_1, H_2, H_{2,J}$ are true, $\mathcal{J}_R^N$ annihilates the unipotent block $\mathcal{B}_{R,1}(G)$.

Proof The proof of the property 1) is as in D.10. The unipotent representation $\boldsymbol{\Gamma}_{R,1}$ is finitely generated and projective, by (3). Any simple unipotent R-representation V of G contains a relevant type of level 0, and H_1 implies $\mathrm{Hom}_{RG}(\boldsymbol{\Gamma}_{R,1}, V)\neq 0$. We apply B.7.

The property 2) results from D.14 below.

We prove the property 3) admitting 2). Each $i_J^G\Gamma_{R,\chi_J,1}$ has a finite filtration of length $\leq N$ with quotients of the form $i_J^G\rho$ where ρ is an irreducible subquotient of $R[B_J\backslash M_J]$. Hence $i_J^G\rho$ is isomorphic to a subquotient of $i_\emptyset^G 1 = R[I\backslash G]$ and is annihilated by $\mathcal{J}_R$. We deduce that $\mathcal{J}_R^N$ annihilates $i_J^G\Gamma_{R,\chi_J,1}$ for all J,χ_J. By the property 2) $\mathcal{J}_R^N$ annihilates $\boldsymbol{\Gamma}_{R,1}$. By the property 1) $\mathcal{J}_R^N$ annihilates $\mathcal{B}_{R,1}(G)$. $\diamond$

D.14 We show that the property of being unipotent is compatible with the functors of parahoric induction or restriction. Once this is proved, we get the part 2) of D.13. The *strong compatibility* is related to a weak form H_3 of the *conjecture of the unicity of the supercuspidal support*, known for $G = GL(n,F)$ [Vig3]. A representation $\pi \in \mathrm{Irr}_R G$ is called *supercuspidal* when it is not a subquotient of a proper parabolically induced representation. Same definition in the finite case.

Conjecture H_3 : *For any irreducible supercuspidal $\sigma \in \mathrm{Irr}_R M_J$ with (J,σ) different from $(\emptyset, 1)$, the representations $i_J^G\sigma$ and $R[I\backslash G]$ have no isomorphic irreducible subquotients.*

Lemma *Let $\pi \in \mathrm{Irr}_R M_J$. Then*
a_1) $i_J^G\pi$ is unipotent if π is unipotent.
a_2) If π is not unipotent, no subquotient of $i_J^G\pi$ is unipotent if H_3 is true.

Let $\pi \in \mathrm{Irr}_R G$. Then
b_1) $r_J^G\pi$ is unipotent if π is unipotent.
b_2) If π is not unipotent, no non zero subquotient of $r_J^G\pi$ is unipotent if H_3 is true.

As usual, the proof for the p-adic case is valid for the finite case.

Proof a_1) Let $\pi \in \mathrm{Irr}_R M_J$. If π is a subquotient of $\mathrm{ind}_{B_J}^{M_J} 1 = i_\emptyset^J 1$, then $i_J^G\pi$ is a subquotient of

$$i_J^G i_\emptyset^J 1 = i_\emptyset^G 1. \tag{14}$$

b_1) Let $\pi \in \mathrm{Irr}_R G$. If π is a subquotient of $R[I\backslash G] = i_\emptyset^G 1$, then $r_J^G\pi$ is a subquotient of (C.1.4)

$$r_J^G i_\emptyset^J 1 = \oplus_{w\in W_{aff}(G)/W_J} i_\emptyset^J 1. \tag{15}$$

a_2) Let $\pi \in \mathrm{Irr}_R M_J$ not isomorphic to a subquotient of $i_\emptyset^J 1$. There exists $K \subset J$ and $\sigma \in \mathrm{Irr}_R M_K$ supercuspidal with $(K,\sigma)\neq(\emptyset,1)$ such that π is isomorphic to a subquotient of $i_K^J\sigma$. Any subquotient of $i_J^G\pi$ is a subquotient of $i_K^G\sigma$ by transitivity of the induction. If H_3 is true for σ, then $i_J^G\pi$ and $R[I\backslash G]$ have no isomorphic irreducible subquotients.

b_2) Let $\pi \in \mathrm{Irr}_R G$ not isomorphic to a subquotient of $i_\emptyset^G 1$, and let J a proper subset of Π such that $r_J^G\pi\neq 0$. If π is not of level 0 then $r_J^G\pi$ has no subquotient of level 0 hence no unipotent subquotient. We may suppose π of level 0. There exists $K \subset J$ and $\sigma \in \mathrm{Irr}_R M_K$ supercuspidal with $(K,\sigma)\neq(\emptyset,1)$ such that π is isomorphic to a subquotient of $i_K^G\sigma$. Hence $r_J^G\pi$ is a subquotient of (C.1.4)

$$r_J^G i_K^G\sigma = \oplus_{w\in W_K\backslash W_{aff}(G)/W_J, wK=K}\ i_K^J \,\mathrm{Int}\, w.\sigma. \tag{16}$$

σ is not a subquotient of $i_\emptyset^K 1$ because π is isomorphic to a subquotient of $i_K^G\sigma$ but not of $i_\emptyset^G 1$ and (14). The same is true for $\mathrm{Int}\, w\sigma$ because $\mathrm{Int}\, w$ permutes the subquotients of $i_\emptyset^K 1$. We apply a_2) to M_J instead of G to finish the proof. $\diamond$

This ends the proof of the main theorem.

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Marie-France Vigneras
Institut de Mathématiques de Jussieu
UMR 7586
Université de Paris 7 Denis Diderot
Case 7012 - 2, place Jussieu
75251 Paris cedex 05