On 4-flattening Theorems and the Curves of Charatheodory, Barner and Segre

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Abstract. We discuss three classes of closed curves in the Euclidean space $\mathbb{R}^3$ which have non–vanishing curvature and at least 4 *flattenings* (points at which the torsion vanishes). Calling these classes (defined below) *Barner*, *Segre* and *Charatheodory*, we prove that Barner⊂(Segre∩Charatheodory). We also prove that (Segre)\(Segre∩Charatheodory) and (Charatheodory)\(Segre∩Charatheodory) are open sets in the space of closed smooth curves with the C^∞–topology. Finally, we define a class of closed curves containing the class of Segre curves, and we establish the conjecture that any curve of our class has at least 4 flattenings. [1]

1. Introduction and Main Results.

By a *closed curve* in the Euclidean space $\mathbb{R}^n$ (projective space $\mathbb{R}P^n$) we mean a C^∞ immersion $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ ($\gamma : \mathbb{S}^1 \to \mathbb{R}P^n$, respectively). We consider the space of all closed curves in the Euclidean space (or in the projective space) equipped with the C^∞–topology.

We recall that a closed curve embedded in $\mathbb{R}^n$ (in $\mathbb{R}P^n$) is called *convex* if it intersects no hyperplane at more than n points, counting multiplicities.

Definition – A *flattening* of a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a point where the derivatives of γ of order $1, \ldots, (n-1)$, are linearly independent and those of order $1, \ldots, n$, are linearly dependent. For curves in $\mathbb{R}P^n$ it is possible to take an affine chart. The linear independence (or dependence) of the derivatives does not depend on the choise of the chart.

If a closed curve γ in $\mathbb{R}P^n$ (in $\mathbb{R}^n$) can be projected, from a point exterior to it, onto a convex curve of $\mathbb{R}P^{n-1}$ (of $\mathbb{R}^{n-1}$) then γ has at least $n+1$ flattenings [4].

Finally, we recall that a closed curve in $\mathbb{R}^n$ (in $\mathbb{R}P^n$) is called *Barner curve* if for every $(n-1)$–tuple of (not necesarily geometrically different) points of the curve there exists a hyperplane through these points that does not intersect the curve elsewhere. More precisely:

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DEFINITION – A curve embedded in the Euclidean space $\mathbb{R}^n$ (or in $\mathbb{R}P^n$) is called a *Barner curve* if it is closed and if for each k–tuple of points of the curve, $k \leq n-1$, with positive multiplicities satisfying $m_1+\cdots+m_k = n-1$, there exists at least one hyperplane of $\mathbb{R}^n$ (of $\mathbb{R}P^n$) intersecting the curve at these points, with corresponding multiplicities, that does not intersect the curve elsewhere.

In the Euclidean case, Barner curves exist only in odd dimensions, while in the projective case there exist convex curves for any positive dimension. *Any Barner curve in $\mathbb{R}^n$ (in $\mathbb{R}P^n$) has at least $n+1$ flattenings* [6].

A closed curve γ in $\mathbb{R}P^n$ ($\mathbb{R}^n$) which can be projected, from a point exterior to it, into a convex curve of $\mathbb{R}P^{n-1}$ ($\mathbb{R}^{n-1}$) is a Barner curve. Answering the question about the relation between these two classes of curves (V.I. Arnol'd 1996) V.D. Sedykh ([13]) proved: *There is a non–empty open set of embedded closed curves in $\mathbb{R}P^n$ which are Barner curves and have no convex projection into any hyperplane.*

The conditions defining classes of closed curves in $\mathbb{R}^n$ that guarantee the number of flattenings (or vertices) on each curve of that class is greater or equal to some positive lower bound $c(n)$ has been a classical object of study. The interest on this subject was revived by the recent progres in simplectic and contact geometries and by the relations of these problems with Sturm theory (see [4], [2], [3], [5], [9], [1], [13], [15], [17], [18]). We consider three classes of closed curves in the three–dimensional Euclidean space $\mathbb{R}^3$ all whose elements have at least four flattenings. In particular, any Barner curve of $\mathbb{R}^3$ has at least 4 flattenings.

We call *Charatheodory curve* any C^3–smooth closed curve in $\mathbb{R}^3$ with non–vanishing curvature and lying on the boundary of its convex hull.

C. Romero–Fuster proved that *a generic Charatheodory curve has at least 4 flattenings* ([10]); in [7], Blaschke attributes an equivalent result to Charatheodory.

SEDYKH'S THEOREM: *Any Charatheodory curve* (without genericity conditions) *has at least 4 flattenings* ([12]).

We call *Segre curve* any closed curve in $\mathbb{R}^3$ with non–vanishing curvature and no parallel tangents with the same orientation ([17]).

SEGRE'S THEOREM: *Any Segre curve has at least 4 flattenings* ([14]).

The natural problems arises:

a) Are there Charatheodory curves which are not Segre curves? (C. Romero–Fuster [11]).

b) Are there Segre curves which are not Charatheodory curves?

c) How are the Barner curves in $\mathbb{R}^3$ related to Charatheodory curves and to Segre curves?

The answer to these questions is given by the following three theorems:

THEOREM A– *There is a non–empty open set of Charatheodory curves in* $\mathbb{R}^3$ *which are not Segre curves.*

THEOREM B– *There is a non–empty open set of Segre curves in* $\mathbb{R}^3$ *which are not Charatheodory curves.*

THEOREM C– *Any Barner curve in* $\mathbb{R}^3$ *is a Charatheodory curve and is also a Segre curve.*

Sedykh's Theorem is considered, in most part of the concerned literature, as the more general $4-$flattening theorem for curves in $\mathbb{R}^3$. However, Theorem B shows that this is not true.

To prove theorems A and B we give methods to construct generic examples.

A Four–Flattening Conjecture. When the unit tangent vector $\mathbf{t}$ of a curve γ in the Euclidean Space $\mathbb{R}^3$ is translated to a fixed point O, the end points of the translated vectors describe a curve $\mathbf{T}$ on the unit sphere $\mathbb{S}^2$, called the *tangent indicatrix* of γ. The points of γ at which the curvature vanishes correspond to the cusps of the tangent indicatrix. So Segre's Theorem can be reformulated in the following way:

Any closed curve in $\mathbb{R}^3$ *whose tangent indicatrix is embedded in* $\mathbb{S}^2$ *has at least four flattenings.*

We say that a curve on $\mathbb{S}^2$ has *direct self–tangency* if it has self–tangency and the tangent branches have the same orientation at the point of tangency. Let $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^3$, $0 \le t \le 1$, be a one–parameter family of immersed curves. Suppose that γ_0 is a Segre curve (for instance a plane convex curve in $\mathbb{R}^3$) and that for all $t \in [0, 1]$ the tangent indicatrix $\mathbf{T}_t$ of γ_t is an immersed curve of $\mathbb{S}^2$ having no direct self–tangencies.

CONJECTURE 1.– *The curve* γ_1 *(and each curve* γ_t*) has at least* 4 *flattenings.*

We stated conjecture 1 in terms of the curves $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^3$ and its tangent indicatrices in $\mathbb{S}^2$, but it comes from a conjecture about some class of Legendrian knots in $ST^*\mathbb{S}^2$ and the number of spherical inflections of the fronts in $\mathbb{S}^2$ of these Legendrian knots (see [3] and [19] for more information about the Legendrian knots associated to curves in $\mathbb{S}^2$). More precisely, to each smoothly immersed co–oriented curve $\alpha : \mathbb{S}^1 \to \mathbb{S}^2$ a (may be singular) Legendrian knot $L_\alpha \subset ST^*\mathbb{S}^2$ is associated, consisting of the co–oriented contact elements of $\mathbb{S}^2$ tangent to α with corresponding co-orientation. Conversely, to each closed Legendrian knot in $ST^*\mathbb{S}^2$ there corresponds a co–oriented curve in $\mathbb{S}^2$ which is not necessarily smooth and is called the front of the Legendrian knot. We recall (see [8]) that *a curve* $\alpha : \mathbb{S}^1 \to \mathbb{S}^2$ *is the tangent indicatrix of some smoothly immersed curve* $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ *if and only if it intersects each great circle of* $\mathbb{S}^2$. So we formulate the

CONJECTURE 1'.– *Let* L_0 *be the Legendrian knot associated to a co–oriented great circle of* $\mathbb{S}^2$*. Let* L_1 *be any Legendrian knot which can be joined to* L_0 *by a Legendrian isotopy* L_t *(i.e. a homotopy of Legendrian knots for which the knot type does not change) satisfying the following condition: The front* α_t *of each Legendrian knot* L_t *is a smooth immersed curve of* $\mathbb{S}^2$ *which intersects every great circle of* $\mathbb{S}^2$*. Then the front* γ_1 *(and each front* γ_t*,* $0 \leq t \leq 1$*) has at least four spherical inflections (counted geometrically).*

The relation between both conjectures comes from the fact that the spherical inflections of the tangent indicatrix of a curve in $\mathbb{R}^3$ correspond to the flattenings of the original curve in $\mathbb{R}^3$. Moreover, for any Legendrian isotopy L_t of the Legendrian knots associated to the tangent indicatrices of a family of closed curves $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^3$, the corresponding curves cannot have points with zero curvature. More precisely, if for a parameter value t_0 of a generic family of smooth closed curves $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^3$ the curve γ_{t_0} has one point of curvature zero then the corresponding Legendrian knot changes its knot type at $t = t_0$. It is known (see [3]) that $ST^*\mathbb{S}^2 \simeq \mathbb{R}P^3$. In particular, the Legendrian knot $L_{t_0} \subset \mathbb{R}P^3$ is reducible, one of its components being a projective line. If for $t < t_0$ the Legendrian knot L_t is contractible (or not contractible) then for $t > t_0$ the Legendrian knot L_t is not contractible (resp. contractible). The pictures showing the bifurcation of the legendrian knot can be found in [17], citeUribetdv and citeUriberm.

In [5], V. Arnol'd gave the first step towards a Legendrian Sturm theory of space curves. He imposed some conditions on the curves in terms of the 2–dimensional Legendrian knot, of the space $PT^*\mathbb{R}^3$ of contact elements of $\mathbb{R}^3$, associated to each curve in $\mathbb{R}^3$ (or in $\mathbb{R}P^3$). This Legendrian 2–dimensional knot consists of the contact elements of $\mathbb{R}^3$ (or in $\mathbb{R}P^3$) tangent to the curve.

However, even considering curves only inside the class of Barner curves, it is easy to go outside the class of curves considered in [5], (in [5] there is one example). With our conjecture we try to work in the same spirit of Arnol'd and Chekanov. But instead of considering the 2–dimensional Legendrian knot in $PT^*\mathbb{R}^3$ (or in $PT^*\mathbb{R}P^3$) associated to a curve in $\mathbb{R}^3$ (or in $\mathbb{R}P^3$), we consider the 1–dimensional Legendrian knot in $ST^*\mathbb{S}^2$ associated to the tangent indicatrix of a curve in $\mathbb{R}^3$. The class of curves considered in our conjecture contains the whole class of Segre curves which, by Theorem C, contains the whole classe of Barner curves.

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2. Proof of theorem C:

A Barner Curve is a Charatheodory Curve. The Barner curve have no points with vanishing curvature. Let γ be a Barner curve in $\mathbb{R}^3$. The definition of Barner curves implies that for any point $p \in \gamma$ there is a plane tangent to the curve at p not intersecting the curve elsewhere. This plane determines a closed half–space H_p containing the curve. The convex hull of γ is contained in H_p and the point p lies on the boundary of the convex set H_p. A point lying on the convex hull of γ which lies on the the baudary of H_p must also lie on the boundary of the convex hull of γ. So p lies on the boundary of the convex hull of γ. Thus γ is a Charatheodory curve. $\square$

A Barner Curve is a Segre Curve. We will prove that any curve which is not a Segre curve cannot be a Barner curve. Let γ be a closed curve with non–vanishing curvature. Suppose that γ has two parallel tangents with the same orientation at the points p_1 and p_2 of γ. We will prove that any plane containing the points p_1 and p_2 must intersect the curve at least at 4 points, taking multiplicities into account. Consider the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ parallel to the line p_1p_2. The projection π sends the curve γ onto a plane curve $\hat{\gamma} = \pi(\gamma)$. The point $p = \pi(p_1) = \pi(p_2)$ is a point of self–tangency with the same orientation as the curve $\hat{\gamma}$. So the curve $\hat{\gamma}$ can be decomposed into two closed curves having a tangency at the point p. Any line of $\mathbb{R}^2$ containing p intersects each one of these curves at least at two points, taking multiplicities into account. Each line of $\mathbb{R}^2$ containing p is the image (by the projection π) of a plane of $\mathbb{R}^3$ containing the points p_1 and p_2. So any plane containing p_1 and p_2 intersects γ at least at 4 points, taking multiplicities into account. $\square$

3. Proof of theorem A:

Charatheodory Curves which are not Segre Curves. Consider a smooth, closed and strictly convex smooth surface S (for instance an ellipsoid) in the Euclidean space $\mathbb{R}^3$. Consider a bundle of parallel lines of $\mathbb{R}^3$. Let Γ denote the set of points of S at which a line of the bundle is tangent to S. The set Γ is a closed curve of S, which separates S in two parts S_1 and S_2. Any embedded curve of S is a Charatheodory curve.

Proposition 1– *Let $\gamma : \theta \mapsto \gamma(\theta)$ be a closed embedded curve of S crossing the curve Γ transversally at $2k > 2$ points $\theta_1, \ldots, \theta_{2k}$. If the tangent lines of γ at two crossings θ_i and θ_j with the same parity ($i = j$ (mod 2)) are lines of the bundle then γ is not a Segre curve.*

Proof– The tangents of the curve γ at θ_i and θ_j are parallel. We must only prove that the tangents at these points have the same orientation. Suppose that at θ_1 the curve γ traverses from S_1 to S_2. Then, at the odd (even) crossings the curve γ traverses from S_1 to S_2 (from S_2 to S_1). So if both i and j are odd (even) then the crossing from S_1 to S_2 (from S_2 to S_1) gives to the tangents at the points θ_i and θ_j the same orientation (See Fig. 1). □

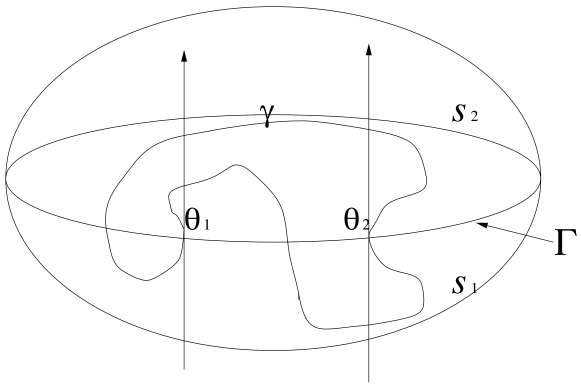

Figure 1: A Charatheodory curve which is not a Segre curve.

A closed curve in $\mathbb{R}^3$ is a Segre curve if and only if its tangent indicatrix on $\mathbb{S}^2$ has no double points (self–intersections). If the tangent indicatrix of a closed curve γ has transversal self–intersections, then any small enough perturbation of γ (taking the derivatives into account) is not a Segre curve: transversality is an open condition. The tangent indicatrix $\mathbf{T} : \mathbb{S}^1 \to \mathbb{S}^2$ of a closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ has a transversal self–intersection at the point $\mathbf{T}(\theta_1) = \mathbf{T}(\theta_2)$ if the tangents to γ at θ_1 and at θ_2 are parallel with the same orientation, but the osculating planes at these points are not parallel.

The curves of Proposition 1 can be constructed in such a way that the osculating planes at the points θ_i and θ_j are not parallel. This proves theorem A.

4. Proof of theorem B:

Segre Curves which are not Charatheodory Curves. We give a method of constructing Segre curves. Let γ be an oriented convex curve, with two orthogonal axes of symmetry l_1, l_2, in the Euclidean plane $\mathbb{R}^2 \subset \mathbb{R}^3$

(for instance an ellipse). Deform the plane $\mathbb{R}^2$ in $\mathbb{R}^3$ on a right cylinder C with the following conditions:

a) The base of the cylinder C can be any smooth immersed plane curve.

b) The lines of $\mathbb{R}^2$ parallel to one of the axes of symmetry of γ, say l_1, must become the generatrices of C.

c) The image, under the deformation, of the lines of $\mathbb{R}^2$ parallel to the axes of symmetry l_2 must be orthogonal to the generatrices of C.

Write $\tilde{\gamma}$ for the image of γ by this deformation, and $\tilde{p} \in \tilde{\gamma}$ for the image of $p \in \gamma$ by this deformation.

Proposition 2– *The deformed curve $\tilde{\gamma}$ is a Segre curve.*

Proof – We will use the fact that two unit tangent vectors are parallel and have the same orientation if and only if for any orthogonal projection on a plane (or on a line) their images are parallel with the same orientation and the same length. Let $\mathbf{t}(\tilde{p})$ be the unit tangent vector of $\tilde{\gamma}$ (given by the orientation of $\tilde{\gamma}$) at the point $\tilde{p} \in \tilde{\gamma}$. Let P_2 be the plane orthogonal to the generatrices of C and containing the image of the axis of symmetry l_2 of γ. By construction, the curve $\tilde{\gamma}$ is symmetric with respect to the plane P_2. Let π_1 (or π_2) be the orthogonal projection of the unit tangent vectors of $\tilde{\gamma}$ on a plane orthogonal to the generatrices (respectively on a line parallel to the generatrices). The projections by π_1 (respectively by π_2) of two unit tangent vectors of $\tilde{\gamma}$ have the same length if and only if the corresponding points of the plane curve γ are symmetric with respect to any one of the axes of symmetry l_1, l_2 of γ. If two points p and q of γ are symmetric with respect to l_1 or l_2 and lie on one of these axis of symmetry of γ, then $\mathbf{t}(\tilde{p})$ and $\mathbf{t}(\tilde{q})$ have opposite orientation. Let $\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \tilde{p_4}$ be four points of $\tilde{\gamma}$, such that the corresponding points p_1, p_2, p_3, p_4 of γ are symmetric and don't lie in the axes of symmetry of γ. We will prove that $\mathbf{t}(\tilde{p_1}) \neq \mathbf{t}(\tilde{p_i})$, $i = 2, 3, 4$. Suppose that $\tilde{p_1}$ and $\tilde{p_2}$ (and consequently $\tilde{p_3}$ and $\tilde{p_4}$) are symmetric with respect to P_2. The projections $\pi_1(\mathbf{t}(\tilde{p_1}))$ and $\pi_1(\mathbf{t}(\tilde{p_2}))$ have the same length but different orientation. So $\mathbf{t}(\tilde{p_1}) \neq \mathbf{t}(\tilde{p_2})$. The projections $\pi_2(\mathbf{t}(\tilde{p_3}))$ and $\pi_2(\mathbf{t}(\tilde{p_4}))$ are oriented in opposite direction with respect to $\pi_2(\mathbf{t}(\tilde{p_1}))$, Thus $\mathbf{t}(\tilde{p_3}) \neq \mathbf{t}(\tilde{p_1}) \neq \mathbf{t}(\tilde{p_4})$. This proves proposition 2.

The curves of proposition 2 can be constructed in such a way that the curve $\tilde{\gamma}$ does not lie on the boundary of its convex hull. This proves theorem B.

Realizations of this construction are given by the families of curves of the following examples.

Example 1 – The curves of the family $\tilde{\gamma}_\varepsilon : \mathbb{S}^1 \to \mathbb{R}^3$ given by the

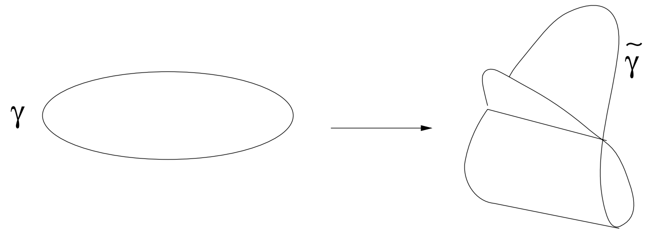

Figure 2: A Segre curve which is not a Charatheodory curve.

parametrization

$$\tilde{\gamma}_\varepsilon(\theta) = ((2\cos\theta + \varepsilon)^3 - (2\cos\theta + \varepsilon),\ \sin\theta,\ (2\cos\theta + \varepsilon)^2)$$

are Segre curves for any value of ε but are not Charatheodory curves for any small enough ε. The curve $\tilde{\gamma}_0$ is not embedded (it has two points of self–intersection), so it is not a Charatheodory curve. For any small enough $\varepsilon \neq 0$, the curve $\tilde{\gamma}_\varepsilon$ is embedded and does not lie in the boundary of its convex hull. This family of curves lies on the cylinder given by the following parametrization: $(s,t) \mapsto (t^3 - t, s, t^2 - 1)$. In Figure 2 we have considered the plane curve γ as the boundary of a disc, and the spatial curve $\tilde{\gamma}$ as the image of γ by the deformation of the disc.

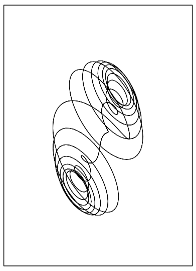

Figure 3: A Segre curve which is not a Charatheodory curve.

Example 2 – The curves of the family $\tilde{\gamma}_\lambda : \mathbb{S}^1 \to \mathbb{R}^3$ given by the parametrization

$$\tilde{\gamma}_\lambda(\theta) = (e^{\cos\theta}\sin(\lambda\pi\cos\theta),\ \sin\theta,\ -e^{\cos\theta}\cos(\lambda\pi\cos\theta))$$

are Segre curves and for any $\lambda \geq 0.7$ they are not Charatheodory curves. In Fig.3 we have the curve $\tilde{\gamma}_\lambda$ for $\lambda = 10$. The curve $\tilde{\gamma}_\lambda$ of this family lies on the cylinder C_λ given by the following parametrization: $(s,t) \mapsto (e^{2t/\lambda}\sin(2\pi t),\ s,\ -e^{2t/\lambda}\cos(2\pi t))$.

References

[1] **Anisov S.S.**, *Convex Curves in* $\mathbb{R}P^n$, Proc. of Steklov Math. Institut, Vol. 221, (1998) p. 3–39.

[2] **Arnol'd V.I.**, *Sur les propriétés des projections Lagrangiennes en géométrie symplectique des caustiques*, (Preprint Cahiers de Math. de la Decision. Vol. 9320. CEREMADE, 1993, pp. 1–9.) Rev. mat. Univ. Complut. Madrid, 1995, **8**:1, p.109–119.

[3] **Arnol'd V.I.**, *The Geometry of Spherical Curves and the Algebra of Quaternions*, Russian Math. Surveys **50**:1, 1–68 (1995).

[4] **Arnol'd V.I.**, *On the Number of Flattening Points of Space Curves*, Amer. Math. Soc. Trans. Ser. **171**, 1995, p. 11–22.

[5] **Arnol'd V.I.**, *Towards the Legendrian Sturm Theory of Space Curves*, Funct. Anal. and Appl. Vol. 32 No.2. (1998) p.75–80.

[6] **Barner M.**, *Über die Mindestanzahl stationärer Schmiegeebenen bei geschlossenen Streng-Konvexen Raumkurven*, Abh. Math. Sem. Univ. Hamburg, **20**, 1956, p. 196–215.

[7] **Blaschke W.**, *Vorlesungen über Differential–Geometrie* I, 3th ed. Spriger–Verlag, Berlin, 1930.

[8] **Fenchel W.**, *The Differential Geometry of Closed Space Curves*, Bull. Am. Math. Soc. 57, 44–54 (1951).

[9] **Kazarian M.**, *Nonlinear Version of Arnol'd's Theorem on Flattening Points*, C.R. Acad. Sci. Paris, t.**323**, Série I, no.1, 1996, p. 63–68.

[10] **Romero–Fuster M.C.**, *Convexly-generic curves in* $\mathbb{R}^3$, Geometriae Dedicata 28 (1988), 7–29.

[11] **Romero–Fuster M.C.**, *Personal comunication at Universitat de València, June 1999.*

[12] **Sedykh V.D.**, *The Theorem About Four Vertices of a Convex Space Curve*, Functional Anal. Appl. **26**:1 (1992), 28–32.

[13] **Sedykh V.D.**, *On Some Class of Curves in a Projective Space*, Geometry and Topology of Caustics, Banach Center Publications, **50**, (1999) 237–266.

[14] **Segre B.**, *Alcune proprietà differenziali delle curve chiuse sghembe*, Rend. Mat. 6 (1) (1968), 237–297.

[15] **Uribe–Vargas R.**, *Four–Vertex Theorems in Higher Dimensional Spaces for a Larger Class of Curves than the Convex Ones*, C.R. Acad. Sci. Paris, t.**330**, Série I, 2000, p. 1085–1090.

[16] **Uribe-Vargas R.**, *Twistings and Darboux Vertices of Curves in* $\mathbb{R}^n$ *and Spherical Indicatrices of Curves in* $\mathbb{R}^3$, To appear.

[17] **Uribe–Vargas R.**, *Singularités symplectiques et de contact en Géométrie différentielle des courbes et des surfaces*, Ph. D. Thesis, Université Paris 7, 2001. Ch.3. (In English).

[18] **Uribe–Vargas R.**, *4-Vertex Theorems and Sturm Theory*, Preprint Inst. de Math. de Jussieu 2002. To appear.

[19] **Uribe–Vargas R.**, *Rigid Body Motions and Arnold's Theory of Fronts on* $\mathbb{S}^2 \subset \mathbb{R}^3$, Preprint Inst. de Math. de Jussieu 2002. To appear.