4-Vertex Theorems, Sturm Theory and Lagrangian Singularities

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Abstract. We prove that the vertices of a curve $\gamma \subset \mathbb{R}^n$ are critical points of the radius of the osculating hypersphere. Using Sturm Theory, we give a new proof of the $(2k+2)$-Vertex Theorem ([10], [19]) for convex curves in the Euclidean space $\mathbb{R}^{2k}$. We obtain a very practical formula to calculate the vertices of a curve in $\mathbb{R}^n$. We apply our formula and Sturm theory to calculate the number of vertices of the *generalized ellipses* in $\mathbb{R}^{2k}$. Moreover, we explain the relations between vertices of curves in Euclidean n-space, singularities of caustics and Sturm theory (for the fundamental systems of solutions of *disconjugate* homogeneous linear differential operators $L : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$).

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Introduction

The geometry of curves is a classical subject which relates geometrical intuition with analysis and topology.

Sturm theory and oscillatory properties of solutions have clear geometrical interpretation in terms of geometry of the curve.

In particular, to estimate the possible number of special points, e.g. *flattenings* (points at which the last torsion vanishes) for different types of closed curves is an important problem involving topological, symplectic and analytic methods.

In [2], V.I. Arnol'd pointed out that "most of the facts of the differential geometry of submanifolds of Euclidean or of Riemannian space may be translated into the language of contact (or symplectic) geometry and may be proved in this more general setting. Thus we can use the intuition of Euclidean or Riemannian geometry to guess general results of contact (or symplectic) geometry, whose applications to the problem of ordinary differential geometry provide new information in this classical domain." The validity of the four-vertex theorem, in the particular case of a Riemannian metric, may be re-established if the vertices of a curve are defined in the following way:

For each point of our curve, consider the geodesic issuing from that point perpendicularly to the curve and in the direction of the inward normal. The point of intersection of such a geodesic with an infinitely close geodesic normal is said to be a *conjugate* point (along the original normal). All these conjugate points form the caustic of the original curve (the envelope of the family of geodesic normals). The points of our curve corresponding to the singular points of the caustic (the generic singular points of the caustic are semi-cubical cusps) are called the *vertices* of the curve.

So the four-vertex theorem in the Riemannian case asserts that: *The caustic of a generic closed convex curve has at least four cusps (counted geometrically).* If the curve is non-generic, then multiplicities must be counted. For instance, the caustic of a circle is a single point.

We give a formula to calculate the vertices of a curve in $\mathbb{R}^n$ and a new proof a higher dimensional 4-vertex theorem ([19] Theorem 1, below) applying Sturm theory and the theory of Lagrangian singularities. We show the relations between the vertices of curves, the singularities of caustics and Sturm theory of fundamental systems of solutions of *disconjugate* homogeneous linear differential operators $L : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$.

1. Statement of Results on Vertices

In the sequel $\mathbb{R}^n$ will denote a Euclidean space and we assume that the derivatives of order $1, \ldots, n-1$, of our curves are linearly independent at any point (this is true for generic curves).

DEFINITION 1. Let M be a d-dimensional submanifold of $\mathbb{R}^n$, considered as a complete intersection: $M = \{x \in \mathbb{R}^n : g_1(x) = \cdots = g_{n-d}(x) = 0\}$. We say that k is the *order of contact* of a curve $\gamma : t \mapsto \gamma(t) \in \mathbb{R}^n$ with the submanifold M, or that γ and M have *k-point contact*, at a point $\gamma(t_0)$, if each function $g_1 \circ \gamma, \ldots, g_{n-d} \circ \gamma$ has a zero of multiplicity at least k at $t = t_0$, and at least one of them has a zero of multiplicity k at $t = t_0$.

Remark. If one needs to make this definition more invariant, one could denote the image of γ by Γ and then write that the *order of contact* at a point is the minimum of the multiplicity of zero among the functions of the form $g_{|\Gamma} : \Gamma \to \mathbb{R}$, at that point, where g belongs to the generating ideal of M.

Example. A smooth curve in $\mathbb{R}^n$ has 2-point contact with its tangent line (at the point of tangency) for the generic points of the curve. The curve $y = x^3$ has 3-point contact with the line $y = 0$, at the origin: the equation $x^3 = 0$ has a root of multiplicity 3.

By convention, the k-dimensional affine subspaces of the Euclidean space $\mathbb{R}^{m+1}$ will be considered as k-dimensional spheres of infinite radius.

DEFINITION 2. For $k = 1, \ldots, n-1$, a *k-osculating sphere* at a point of a curve in the Euclidean space $\mathbb{R}^n$ is a k-dimensional sphere having at least $(k+2)$-point contact with the curve at that point. For $k = n-1$ we will simply write *osculating hypersphere.*

Example. A generic plane curve and its osculating circle have 3-point contact at an ordinary point of the curve.

DEFINITION 3. A *vertex* of a curve in $\mathbb{R}^n$ is a point at which the curve has at least $(n+2)$-point contact with its osculating hypersphere.

Example. A non-circular ellipse in the plane $\mathbb{R}^2$ has 4 vertices. They are the points at which the ellipse intersects its principal axes.

DEFINITION 4. An embedded closed curve in $\mathbb{R}^n$ (or in $\mathbb{R}P^n$) is called *convex* if it intersects any hyperplane (or projective hyperplane, respectively) at no more than n points, taking multiplicities into account.

Example. A closed plane curve is convex if it intersects any straight line in at most two points, taking multiplicities into account.

Example . For $n = 2k$, the *generalized ellipse*, defined as the image of the embedding $t \mapsto (\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt)$, is convex in $\mathbb{R}^{2k}$.

The following theorem was proved in [10] and [19].

THEOREM 1. *Any closed convex curve in $\mathbb{R}^{2k}$ has at least $2k + 2$ vertices (counted geometrically).*

Vertices of curves in Euclidean spaces and flattenings of curves in projective (or affine) spaces are related to Sturm Theory. In section 4, we give a new proof of this theorem based on Sturm theory. This new proof allows us to give a formula to calculate the vertices of a curve in $\mathbb{R}^n$ as the zeroes of a determinant:

THEOREM 2. *The vertices of any curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ (or $\gamma : \mathbb{R} \to \mathbb{R}^n$), $\gamma : s \mapsto (\varphi_1(s), \ldots, \varphi_n(s))$ are given by the solutions $s \in \mathbb{S}^1$ (or $s \in \mathbb{R}$) of the equation*

$$\det(R_1, \ldots, R_n, G) = 0,$$

where R_i (G) is the column vector defined by the first $n+1$ derivatives of φ_i (of $g = \frac{\gamma^2}{2}$, respectively).

COROLLARY 1. (see also [23]) *The vertices of any curve* $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ *(or* $\gamma : \mathbb{R} \to \mathbb{R}^n$*),* $\gamma : s \mapsto (\varphi_1(s), \ldots, \varphi_n(s))$ *correspond to the flattenings of the curve* $\Gamma : \mathbb{S}^1 \to \mathbb{R}^{n+1}$ *(or* $\Gamma : \mathbb{R} \to \mathbb{R}^{n+1}$*),*

$$\Gamma : s \mapsto \left(\varphi_1(s), \ldots, \varphi_n(s), \frac{\gamma^2(s)}{2} \right).$$

Remark. This means that the vertical projection of a curve $\gamma \subset \mathbb{R}^n$ to the paraboloid 'of revolution' $z = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$ sends the vertices of the curve γ onto the flattenings of its image.

Remark. Another formula for calculate the vertices of a curve (unfortunately not very practical) was found in [11]. For curves in $\mathbb{R}^3$, such formula appears in [15], exercise 6.8.

A curve in the Euclidean plane, has a vertex if and only if the radius of its osculating circle is critical. In higher dimensional spaces we have the following theorem (announced in [19] and proved in section 3)

THEOREM 3. *The vertices of a smoothly immersed curve in the Euclidean space* $\mathbb{R}^n$ *are critical points of the radius of the osculating hypersphere.*

Remark. For $n > 2$, the converse is not always true. For example, all the points of the circular helix $t \mapsto (\cos t, \sin t, t)$ are critical points of the radius of the osculating hypersphere. However it has no vertex. A more generic example is given by the curve $t \mapsto (a\cos t, b\sin t, t)$ which has no vertex for any $a, b \in \mathbb{R} \setminus \{0\}$ such that $|a^2 - b^2| < 1/3$.

Proof of remark. Apply our formula of theorem 1 to obtain

$$\begin{vmatrix} -a\sin t & b\cos t & 1 & 1/2(b^2-a^2)\sin 2t + t \\ -a\cos t & -b\sin t & 0 & (b^2-a^2)\cos 2t + 1 \\ a\sin t & -b\cos t & 0 & -2(b^2-a^2)\sin 2t \\ a\cos t & b\sin t & 0 & -4(b^2-a^2)\cos 2t \end{vmatrix} = 0,$$

which gives $ab(1 - 3(b^2 - a^2)\cos 2t) = 0$. For $|a^2 - b^2| < 1/3$, this equation has no solution $t \in \mathbb{R}$. □

The (non-circular) ellipse is the simplest closed convex curve in the plane having the minimum number of vertices: 4.

DEFINITION 5. A *generalized ellipse* in $\mathbb{R}^{2k}$ is a convex curve given by the following parametrization ([7]):

$$\theta \mapsto (a_1\cos\theta, b_1\sin\theta, a_2\cos 2\theta, b_2\sin 2\theta, \ldots, a_k\cos k\theta, b_k\sin k\theta).$$

One can expect that generalized ellipses are convex curves in $\mathbb{R}^{2k}$ having the minimum number of vertices, i.e. $2k+2$. However, the following example shows that the generalized ellipses in $\mathbb{R}^{2k}$ can have more than $2k+2$ vertices.

Example. *The generalized ellipse in* $\mathbb{R}^4$,

$$\gamma(\theta) = (a_1 \cos\theta, b_1 \sin\theta, a_2 \cos 2\theta, b_2 \sin 2\theta),$$

with $a_2^2 \neq b_2^2$ *and* $a_1 b_1 a_2 b_2 \neq 0$ *has* 8 *vertices. If* $a_2^2 = b_2^2$ *then* γ *is a spherical curve and all its points are thus vertices.*

Denote $C_k = \cos k\theta$ and $S_k = \sin k\theta$.

THEOREM 4. *Consider the generalized ellipse in* $\mathbb{R}^{2k}$ *given by*

$$\gamma(\theta) = (a_1 C_1, b_1 S_1, a_2 C_2, b_2 S_2, \dots, a_k C_k, b_k S_k),$$

with $a_1 b_1 a_2 b_2 \cdots a_k b_k \neq 0$. *Then, for even* k, γ *can have* $2k+4$, $2k+8$,$\dots$,$4k$ *or an infinity of vertices depending on the values of the parameters* a_j *and* b_j, *for* $j \geq \frac{k}{2}+1$. *For odd* k, γ *can have* $2k+2$, $2k+6$,$\dots$,$4k$ *or an infinity of vertices depending on the values of the parameters* a_j *and* b_j, *for* $j \geq \frac{k+1}{2}$.

Remark. In the space of parameters a_j and b_j, there is a hypersurface which separates the domains with different number of vertices. The points of this hypersurface correspond to generalized ellipses having vertices with multiplicity > 1.

To prove that Theorem 1 is sharp, we will construct a convex curve in $\mathbb{R}^{2k}$ having the minimum number of vertices, i.e. $2k+2$.

Consider the generalized ellipse of Theorem 3 with coefficients

$$a_1 = b_1 = \dots = a_k = b_k = 1$$

and denote it by γ_0. Obviously γ_0 is a spherical curve and all its points are vertices. In order to obtain the desired convex curve, we will perturb γ_0 in the "radial direction":

THEOREM 5. *For* $\varepsilon \neq 0$ *sufficiently small, the curve*

$$\gamma_\varepsilon = (1 + \varepsilon \cos(k+1)\theta)\gamma_0(\theta)$$

has exactly $2k+2$ *vertices.*

Theorems 4 and 5 are proved in section 5.

2. Recall on Singularities in Symplectic Geometry

We recall some basic facts from the theory of Lagrangian singularities (see [3] or [4]) for the particular case of the Normal map (introduced below).

Lagrangian Singularities

A *symplectic structure* on a manifold M is a closed differentiable 2-form ω, non-degenerate on M, also called *symplectic form*. A manifold equipped with a symplectic structure is called a *symplectic manifold*.

Example. The total space of the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ is a symplectic manifold.

A submanifold of a symplectic manifold (M^{2n}, ω) is called *Lagrangian* if it has dimension n and the restriction to it of the symplectic form ω is equal to 0.

Example. Let N be any submanifold in the Euclidean space $\mathbb{R}^n$ and let L be the n-dimensional manifold formed by the covectors $(v, \cdot)$ at the end-points of the normal vectors v to N. Then L is a Lagrangian submanifold of the symplectic space $T^*\mathbb{R}^n$.

A fibration of a symplectic manifold is called *Lagrangian fibration* if all the fibers are Lagrangian submanifolds.

Example. The cotangent bundle $T^*V \to V$ of any manifold V is a Lagrangian fibration. The standard 1-form $\lambda = pdq$ vanishes along the fibers. Thus its differential $\omega = d\lambda$ also vanishes.

Consider the inclusion $i : L \to E$ of an immersed Lagrangian submanifold L in the total space of a Lagrangian fibration $\pi : E \to B$. The restriction of the projection π to L, that is $\pi \circ i : L \to B$ is called a *Lagrangian map*. Thus, a *Lagrangian map* is a triple $L \to E \to B$, where the left arrow is a Lagrangian immersion and the right arrow a Lagrangian fibration.

The Normal map. Consider the set of all vectors normal to a submanifold N in the Euclidean space $\mathbb{R}^n$. Associate to each vector its end point. To the vector v based at the point q associate the point $q + v$. This Lagrangian map of the n-dimensional manifold of normal vectors to N into the n-dimensional Euclidean space $\mathbb{R}^n$ is called *normal map*. The Lagrangian submanifold L in $T^*\mathbb{R}^n$ is formed by the covectors $(v, \cdot)$ at the end points of the normal vectors v.

The set of critical values of a Lagrangian map is called its *caustic*.

Example. The set of centers of curvature of a hypersurface N in the Euclidean space $\mathbb{R}^n$ is called the *focal set* or *evolute* of the hypersurface. The caustic of the normal map of the hypersurface N is its focal set.

Remark. The focal set of a submanifold in the Euclidean space $\mathbb{R}^n$ of codimension greater than 1 (for instance, of a curve in $\mathbb{R}^3$) is defined as the envelope of the family of normals to the submanifold. It is the caustic of the normal map associated to that submanifold.

A *Lagrangian equivalence* of two Lagrangian maps is a symplectomorphism of the total space transforming the first Lagrangian fibration to the second, and the first Lagrangian immersion to the second. Caustics of equivalent Lagrangian maps are diffeomorphic.

There is a very important example of Lagrangian manifold. Consider the total space of the standard Lagrangian fibration $\mathbb{R}^{2n} \to \mathbb{R}^n$, $(p,q) \mapsto q$ with the form $dp \wedge dq$. Let $F(x,q)$ be the germ, at the point (x_0, q_0), of a family of smooth functions of k variables $x = (x_1, \ldots, x_k)$ which depends smoothly on the parameters q. Suppose

a) $\frac{\partial F}{\partial x}(x_0, q_0) = 0$ and
b) the map $(x,q) \mapsto \frac{\partial F}{\partial x}$ has rank k at (x_0, q_0).

Then the germ at the point $(\frac{\partial F}{\partial q}(x_0, q_0), q_0)$ of the set

$$L_F = \left\{ (p,q) : \exists x : \frac{\partial F}{\partial x} = 0, p = \frac{\partial F}{\partial q} \right\},$$

is the germ of a smoothly immersed Lagrangian submanifold of $\mathbb{R}^{2n}$. The family germ F is said to be a *generating family* of L_F and of its Lagrangian map $\pi_F : (q,p) \mapsto q$.

It turns out that the germ of each Lagrangian map is equivalent to the germ of the Lagrangian map π_F for a suitable family F.

The equivalence classes of germs of Lagrangian maps are called *Lagrangian singularities*. A classification of singularities of Lagrangian maps of manifolds in general position of dimension $n \leq 10$ is given in [3].

The germ of each Lagrangian map is Lagrange equivalent to the germ of the Lagrangian map π_F for a suitable Lagrangian generating family F. Such Lagrangian generating family is said to be a *Lagrangian representative family* of the corresponding germ of Lagrangian map.

3. Proof of Theorem 3 and Study of the Focal Set of a Curve in terms of its Normal Map

Proof of theorem 3. We assume that the derivatives of order $1, \ldots, n-1$, of our curve are linearly independent at any point (which is true for generic curves). The generating family $F : \mathbb{R}^n \times S^1 \to \mathbb{R}$ of the normal map associated to the curve is given by

$$F(q,s) = \frac{1}{2} \parallel q - \gamma(s) \parallel^2 .$$

The caustic of this map is the focal set of the curve. We shall write

$$\Sigma(i) = \{(q,s)/\partial_s F(q,s) = 0, ..., \partial_s^i F(q,s) = 0\}.$$

Thus $\Sigma(1)$ is the set of pairs (q,s) such that q is the center of some hypersphere of $\mathbb{R}^n$ whose order of contact with γ at s is at least 2 (this means that q is in the normal hyperplane to γ at s). So $\Sigma(2)$ is the set of pairs (q,s) such that q is the center of some hypersphere of $\mathbb{R}^n$ whose order of contact with γ at s is at least 3. One can see from the equations that these points generate a plane of dimension $n-2$ contained in the normal hyperplane to γ at s. If $\gamma(s)$ is not a flattening of γ then the curve γ has only one osculating hypersphere at $\gamma(s)$. So $\Sigma(n)$ is the set of pairs $(q(s),s)$ such that $q(s)$ is the centre of an osculating hypersphere at $\gamma(s)$. Hence, if $\gamma(s)$ is not a flattening of γ then the value of F at the point $(q(s),s)$ in $\Sigma(n)$ is one half of the square of the radius of the osculating hypersphere at $\gamma(s)$. The condition for a point $p = \gamma(s)$ to be a vertex is equivalent to the fact that the first $n+1$ derivatives of F with respect to s vanish at s. Hence $\Sigma(n+1)$ is the set of vertices of the curve. It is a well-known fact of singularity theory [3] that a point belonging to $\Sigma(n+1)$ is a critical point of the restriction of F to $\Sigma(n)$. So a vertex is a critical point of the radius of the osculating hypersphere. □

Remark. The centers of the osculating hyperspheres at the vertices of γ are given by the $q \in \mathbb{R}^n$ for which there exists a solution s of the $(n+1)$-system of equations

$$\begin{aligned} F_q'(s) &= 0 \\ F_q''(s) &= 0 \\ &\vdots \\ F_q^{(n+1)}(s) &= 0. \end{aligned}$$

For a fixed s, the first equation gives the normal hyperplane to the curve at the point $\gamma(s)$. The first two equations give a codimension 1 subspace of the normal hyperplane to the curve at the point $\gamma(s)$. Following this process we obtain (for a generic curve) a complete flag at each non-flattening point of the curve. The *focal curve* $q(s)$, formed by the centers of the osculating hyperspheres, is determined by the n

first equations. The complete flag is the osculating flag of the focal curve. In particular, the osculating hyperplane of the focal curve at the point $q(s)$ is the normal hyperplane to the curve γ at the point $\gamma(s)$. As the point moves along the curve γ, the corresponding flag (starting with the codimension 2 subspace) generates a hypersurface which is stratified in a natural way by the components of the flag. This stratified hypersurface is a component of the focal set of the curve γ. The other component of the focal set is the curve itself. The stratum of dimension 1 (generated by the 0-dimensional subspace of the flag, i.e. generated by center of the osculating hypersphere at the moving point) is the focal curve of γ. The equation $F_q^{(n+1)}(s) = 0$ gives a finite number of isolated points on the focal curve. These points correspond to the vertices.

As we explained above, the focal set is the caustic of the Normal map defined by the generating family $F(q, s)$. Thus —according to Arnold's classification of singularities of caustics (see [3] or [4])— the vertices of a curve in $\mathbb{R}^n$ correspond to a Lagrangian singularity A_{n+1} of the Normal map.

4. Proof of the $(2k + 2)$-Vertex Theorem by Sturm Theory

We begin this paragraph with some definitions and results of Sturm theory, taken from [7] and [12].

A set of functions $\{\varphi_1, \ldots, \varphi_{2k+1}\}$ with $\varphi_i : \mathbb{S}^1 \to \mathbb{R}$ is a *Chebishev system* if any linear combination $a_1\varphi_1 + \cdots + a_{2k+1}\varphi_{2k+1},\ a_i \in \mathbb{R}$, with $a_1^2 + \ldots + a_{2k+1}^2 \neq 0$ has at most $2k$ zeros on $\mathbb{S}^1$.

Example 1. The set of functions $\{1, \cos\theta, \sin\theta\}$ is a Chebishev system.

Remark. Any convex closed curve $\theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{2k}(\theta))$ in $\mathbb{R}^{2k}$ defines a Chebishev system: $\{1, \varphi_1, \ldots, \varphi_{2k}\}$.

DEFINITION 6. A linear homogeneous differential operator

$$L : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$$

is called *disconjugate* if it has a fundamental system of solutions for the equation $Lg = 0$ which are defined on the circle and form a Chebishev system.

Example 2. The operator $L = \partial(\partial^2 + 1)$ is disconjugate. The Chebishev system $\{1, \cos\theta, \sin\theta\}$ is a fundamental system of solutions for it.

Example 3. Any convex curve $\gamma : \theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{2k}(\theta))$ in $\mathbb{R}^{2k}$ defines a $(2k+1)$-order disconjugate operator L_γ defined by

$$L_\gamma g = \det(R_1, \ldots, R_{2k}, G),$$

where R_i (G) is the column vector defined by the first $2k+1$ derivatives of φ_i (of g, respectively). The Chebishev system $\{1, \varphi_1, \ldots, \varphi_{2k}\}$ is a fundamental system of solutions of the equation $L_\gamma g = 0$.

Example 4. The generalized ellipse ([7])

$$\gamma : \theta \mapsto (a_1 \cos\theta, b_1 \sin\theta, a_2 \cos 2\theta, b_2 \sin 2\theta, \ldots, a_k \cos k\theta, b_k \sin k\theta),$$

defines, up to a constant factor, the $(2k+1)$-order disconjugate operator

$$L_\gamma = \partial(\partial^2 + 1) \cdots (\partial^2 + n^2).$$

Some proofs of 4-vertex type theorems are (implicitly) based on the following theorem due to Hurwitz ([13]) which generalize a theorem of Sturm ([18]) :

Hurwitz Theorem. *Any function $f \in C^\infty(\mathbb{S}^1)$ whose Fourier series begins with the harmonics of order N, $f = \sum_{k \geq N} a_k \cos k\theta + b_k \sin k\theta$, has at least $2N$ zeroes.*

In fact any function $f \in C^\infty(\mathbb{S}^1)$ without harmonics up to order n is orthogonal to the solutions of the equation $\partial(\partial^2+1)\cdots(\partial^2+n^2)\varphi = 0$, and such solutions form a Chebishev system.

The following theorem generalizes Hurwitz's theorem.

Sturm-Hurwitz Theorem ([7],[12]). *Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a C^∞ function such that $\int_{\mathbb{S}^1} f(\theta)\varphi_i(\theta)d\theta = 0$, $\{\varphi_i\}_{i=1,\ldots,2k+1}$ being a Chebishev system. Then f has at least $2k+2$ sign changes.*

COROLLARY 2. ([12]) *Any function in the image of a disconjugate operator ($f = Lg$, where $g \in C^\infty(\mathbb{S}^1)$ is any function) of order $2k+1$ has at least $2k+2$ sign changes.*

Proof of the $(2k+2)$-vertex theorem in $\mathbb{R}^{2k}$. Let

$$\gamma : \theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{2k}(\theta))$$

be a convex curve in $\mathbb{R}^{2k}$. Consider the family of functions on the circle $F : \mathbb{S}^1 \times \mathbb{R}^{2k} \to \mathbb{R}$ defined by

$$F_q(\theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2 .$$

In the proof of theorem 2 we saw that the centers of the osculating hyperspheres at the vertices of γ are the points $q \in \mathbb{R}^n$ for which there exists a solution θ of the following system of $2k+1$ equations:

$$\begin{aligned} F_q'(\theta) &= 0, \\ F_q''(\theta) &= 0, \\ &\vdots \\ F_q^{(2k+1)}(\theta) &= 0. \end{aligned}$$

The *focal curve* $q(\theta)$ of centers of the osculating hyperspheres is determined by the first $2k$ equations. The last equation is the condition on this curve determining the vertices. Write $g = \frac{\gamma^2}{2}$. Using the fact that

$$-F = \gamma \cdot q - \frac{\gamma^2}{2} - \frac{q^2}{2},$$

the preceding system of equations can be written as

$$\begin{aligned} \gamma' \cdot q - g' &= 0, \\ \gamma'' \cdot q - g'' &= 0, \\ &\vdots \\ \gamma^{(2k+1)} \cdot q - g^{(2k+1)} &= 0. \end{aligned}$$

This means that the vector $(q, -1)$ in $\mathbb{R}^{2k+1}$ is orthogonal to the $2k+1$ vectors $(\gamma', g'),\ (\gamma'', g''), \ldots, (\gamma^{(2k+1)}, g^{(2k+1)})$. So the vertices of γ are given by the zeros of the determinant of the matrix whose lines are these $2k+1$ vectors. This determinant is equal to $\det(R_1, \ldots, R_{2k}, G)$ where R_i (G) is the column vector defined by the first $2k+1$ derivatives of φ_i (of $g = \frac{\gamma^2}{2}$, respectively). This determinant is the image of the function $g = \frac{\gamma^2}{2}$ under the operator L_γ (see example 3). So Corollary 2 implies that this determinant has at least $2k+2$ sign changes. This proves the theorem. □

Proof of Theorem 2. In the above proof of the $(2k+2)$-vertex theorem for convex curves in $\mathbb{R}^{2k}$, the convexity of the curve and the parity of the dimension were used only in the last step. So the determinant obtained in the proof gives a formula to calculate the vertices of a curve in $\mathbb{R}^n$. This proves theorem 2.

5. On the Number of Vertices of Generalized Ellipses

We will give two examples and then we will prove Theorem 4.

Example 1. *The generalized ellipse in* $\mathbb{R}^4$,

$$\gamma(\theta) = (a_1 \cos\theta, b_1 \sin\theta, a_2 \cos 2\theta, b_2 \sin 2\theta),$$

with $a_2^2 \neq b_2^2$ *and* $a_1 b_1 a_2 b_2 \neq 0$ *has* 8 *vertices. If* $a_2^2 = b_2^2$ *then* γ *is a spherical curve and all its points are thus vertices.*

Proof. Denote $C_k = \cos k\theta$, $S_k = \sin k\theta$ and

$$g = 2(a_1^2 C_1^2 + b_1^2 S_1^2 + a_2^2 C_2^2 + b_2^2 S_2^2).$$

Theorem 2 and Examples 3 and 4 of section 4 imply that the vertices of γ correspond to the roots of the equation $\partial(\partial^2+1)(\partial^2+2^2)g = 0$. The trigonometric identity

$$a^2\cos^2\theta + b^2\sin^2\theta = \frac{1}{2}(a^2+b^2+(a^2-b^2)\cos 2\theta)$$

allows us to write

$$g = (a_1^2 - b_1^2)C_2 + (a_2^2 - b_2^2)C_4 + a_1^2 + b_1^2 + a_2^2 + b_2^2.$$

The operator ∂ kills the constant terms (i.e. the harmonics of order zero), and the operator $(\partial^2 + 2^2)$ kills the second order harmonics. Moreover $\partial C_4 = -4S_4$, $\partial S_4 = 4C_4$ and $(\partial^2 + k^2)C_4 = (k^2 - 4^2)C_4$. Thus

$$\begin{aligned}\partial(\partial^2+1)(\partial^2+2^2)g &= \partial(\partial^2+1)(\partial^2+2^2)(a_2^2-b_2^2)C_4\\ &= K(a_2^2-b_2^2)S_4,\end{aligned}$$

where $K = -4(1-4^2)(2^2-4^2)$ is a non zero constant. Thus the vertices of γ correspond to the solutions of the equation $K(a_2^2 - b_2^2)S_4 = 0$, i.e. γ has 8 vertices for $a_2^2 \neq b_2^2$ and all its points are vertices for $a_2^2 = b_2^2$. □

We keep the notation $C_k = \cos k\theta$ and $S_k = \sin k\theta$.

Example 2. *The generalized ellipse in* $\mathbb{R}^6$,

$$\gamma(\theta) = (a_1\cos\theta, b_1\sin\theta, a_2\cos 2\theta, b_2\sin 2\theta, a_3\cos 3\theta, b_3\sin 3\theta),$$

with $\prod_{i=1}^3 a_i b_i \neq 0$ *may have* $8, 12$ *or an infinity of vertices, depending on the values of the parameters* a_2, b_2, a_3, b_3. *In particular, if* $a_2^2 = b_2^2$ *and* $a_3^2 \neq b_3^2$ *then* γ *has* 12 *vertices, and if* $a_2^2 \neq b_2^2$ *and* $a_3^2 = b_3^2$ *then* γ *has* 8 *vertices. If* $a_2^2 = b_2^2$ *and* $a_3^2 = b_3^2$ *then* γ *is a spherical curve and all its points are thus vertices.*

Proof. As in Example 1, the vertices of γ are the roots of the equation given by $\partial(\partial^2+1)(\partial^2+2^2)(\partial^2+3^2)g = 0$ where

$$g = (a_1^2 - b_1^2)C_2 + (a_2^2 - b_2^2)C_4 + (a_3^2 - b_3^2)C_6 + \sum_{i=1}^{3}(a_i^2 + b_i^2).$$

The operator $\partial(\partial^2+1)(\partial^2+2^2)(\partial^2+3^2)$ kills the harmonics of orders zero, one, two and three. Thus

$$\partial(\partial^2+1)(\partial^2+2^2)(\partial^2+3^2)g = K_2(a_2^2-b_2^2)S_4 + K_3(a_3^2-b_3^2)S_6,$$

where K_2 and K_3 are non zero constants. □

We recall Theorem 4:

THEOREM 4. *Consider the generalized ellipse in* $\mathbb{R}^{2k}$

$$\gamma(\theta) = (a_1C_1, b_1S_1, a_2C_2, b_2S_2, \ldots, a_kC_k, b_kS_k),$$

with $a_1b_1a_2b_2\cdots a_kb_k \neq 0$. *Then, for even* k, γ *can have* $2k+4$, $2k+8,\ldots,4k$ *or an infinity of vertices depending on the values of the parameters* a_j *and* b_j, *for* $j \geq \frac{k}{2}+1$. *For odd* k, γ *can have* $2k+2$, $2k+6,\ldots,4k$ *or an infinity of vertices depending on the values of the parameters* a_j *and* b_j, *for* $j \geq \frac{k+1}{2}$.

Proof of Theorem 4. As in examples 1 and 2, the vertices of γ are the roots of the equation given by

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)g = 0,$$

where $g = \sum_{i=1}^{k}(a_i^2-b_i^2)C_{2i} + \sum_{i=1}^{k}(a_i^2+b_i^2)$. The operator

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)$$

kills the harmonics from the order zero until order k. Thus, for even k,

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)g = \sum_{i\geq\frac{k}{2}+1} K_i(a_i^2-b_i^2)S_{2i},$$

where K_i is a non zero constant, for $i \geq \frac{k}{2}+1$. For odd k

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)g = \sum_{i\geq\frac{k+1}{2}} K_i(a_i^2-b_i^2)S_{2i},$$

where K_i is a non zero constant, for $i \geq \frac{k+1}{2}$. This proves Theorem 4.

Proof of Theorem 5. Applying our formula of Theorem 2 we obtain that the number of vertices of the curve $\gamma_\varepsilon(\theta) = (1+\varepsilon\cos(k+1)\theta)\gamma_0(\theta)$ is given by the number of solutions $\theta \in \mathbb{S}^1$ of an equation of the form

$$\varepsilon\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)\cos(k+1)\theta + \varepsilon^2 f(\theta,\varepsilon) = 0, \text{ i.e.}$$

$$\varepsilon K\sin(k+1)\theta + \varepsilon^2 f(\theta,\varepsilon) = 0,$$

where $K = -(k+1)\prod_{i=1}^{k}(-(k+1)^2 + i^2) \neq 0$ is a constant and the function $f(\theta, \varepsilon)$ is bounded for $|\varepsilon| < 1$. Thus for $\varepsilon \neq 0$ small enough this equation has exactly $2k+2$ solutions. □

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References

1. **Anisov S.S.**, *Convex Curves in* $\mathbb{R}P^n$, Proc. of Steklov Math. Institut, Vol. 221, (1998) p. 3–39.
2. **Arnol'd V.I.**, *Contact Geometry and Wave Propagation*, Monographie N°34 de L'Enseignement Mathématique, Genève, 1989.
3. **Arnol'd V.I., Varchenko A.N., Gussein-Zade S.M.**, *Singularities of Differentiable Maps*, Vol. 1, Birkhäser (1986). (French version: *Singularités des applications différentiables*, Vol. 1, Ed. Mir, Moscou, 1986. Russian version: Nauka, 1982)
4. **Arnol'd V.I.**, *Singularities of caustics and wave fronts*, Kluwer, Maths. and its Appls., Soviet series, vol.62., 1991.
5. **Arnol'd V.I.**, *Sur les propriétés des projections Lagrangiennes en géométrie symplectique des caustiques*, (Preprint Cahiers de Math. de la Decision. Vol. 9320. CEREMADE, 1993, pp. 1–9.) Rev. mat. Univ. Complut. Madrid, 1995, **8**:1, p.109–119.
6. **Arnol'd V.I.**, *The Geometry of Spherical Curves and the Algebra of Quaternions*, Russian Math. Surveys **50**:1, 1–68 (1995).
7. **Arnol'd V.I.**, *On the Number of Flattening Points of Space Curves*, Amer. Math. Soc. Trans. Ser. **171**, 1995, p. 11–22.
8. **Arnol'd V.I.**, *Topological problems of the theory of wave propagation*, Russian Math. Surveys **51**:1 (1996) pp. 1–47.
9. **Arnol'd V.I.**, *Towards the Legendrian Sturm Theory of Space Curves*, Funct. Anal. and Appl. Vol. 32 No.2. (1998) p.75–80.
10. **Barner M.**, *Über die Mindestanzahl stationärer Schmiegeebenen bei geschlossenen Streng-Konvexen Raumkurven*, Abh. Math. Sem. Univ. Hamburg, **20**, 1956, p. 196–215.
11. **Fukui T., Nuño-Ballesteros J.J.**, *Isolated roundings and flattenings of submanifolds in Euclidean Spaces*, Preprint.
12. **Guieu L., Mourre E., Ovsienko V.Yu.**, *Theorem on six vertices of a plane curve via the Sturm theory*, Arnol'd–Gelfand mathmatical seminars, V. I. Arnold, I.M. Gelfand, M. Smirnov, V.S. Retakh Editors, Birkhäuser Boston, (1997) p.257–266.
13. **Hurwitz, A.**, *Über die Fourierschen Konstanten integrierbarer Funktionen*, Math. Ann. 57, (1903) p.425–446.
14. **Kazarian M.**, *Nonlinear Version of Arnol'd's Theorem on Flattening Points*, C.R. Acad. Sci. Paris, t.**323**, Série I, no.1, 1996, p. 63–68.
15. **Porteous I.R.**, *Geometric Differentiation*, Cambridge Univ. Press. 1994.

16. **Romero-Fuster M.C.**, *Convexly-generic curves in* $\mathbb{R}^3$, Geometriae Dedicata 28 (1988), 7–29.
17. **Sedykh V.D.**, *The Theorem About Four Vertices of a Convex Space Curve*, Functional Anal. Appl. **26**:1 (1992), 28–32.
18. **Sturm J.C.F.**, *Mémoire sur les équations différentielles du second ordre*, J. Math. Pures Appl. 1, (1836), 106–186.
19. **Uribe-Vargas R.**, *On the Higher Dimensional Four-Vertex Theorem*, C.R. Acad. Sci. Paris, t.**321**, Série I, 1995, p. 1353–1358.
20. **Uribe-Vargas R.**, *On the* $(2k+2)$*-Vertex and* $(2k+2)$*-Flattening Theorems in Higher Dimensional Lobatchevskian Space*, C.R. Acad. Sci. Paris, t.**325**, Série I, 1997, p. 505–510.
21. **Uribe-Vargas R.**, *Four-Vertex Theorems in Higher Dimensional Spaces for a Larger Class of Curves than the Convex Ones*, C.R. Acad. Sci. Paris, t.**330**, Série I, 2000, p. 1085–1090.
22. **Uribe-Vargas R.**, *Singularités symplectiques et de contact en Géométrie différentielle des courbes et des surfaces*, Ph.D. Thesis, Univ. Paris 7, 2001. Ch.1. (In English).
23. **Uribe-Vargas R.**, *On Polar Duality, Lagrange and Legendre Singularities and Stereographic Projection to Quadrics*, 28p. To appear in Proc. of London Math. Soc.