EXPLICIT PLANCHEREL FORMULA FOR THE p-ADIC GROUP $\mathrm{GL}(n)$

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Abstract. We provide an explicit Plancherel formula for the p-adic group GL(n), using the Harish-Chandra product formula and the Langlands-Shahidi formula. Our formula involves the p-adic gamma function, and extends the classical formula of Macdonald from the unramified unitary principal series to the whole tempered dual. We determine explicitly the Bernstein decomposition of Plancherel measure, including all numerical constants. We also prove a transfer-of-measure formula for GL(n).

Keywords: Plancherel measure, product formula, Langlands-Shahidi formula, Bernstein decomposition

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1. Introduction

In this article we provide an explicit Plancherel formula for the p-adic group $\mathrm{GL}(n)$.

In fact, we achieve more than this. Plancherel measure admits a canonical decomposition, the *Bernstein decomposition*: we determine explicitly the Bernstein decomposition of Plancherel measure, including all numerical constants.

Harish-Chandra, in his search for the Plancherel measure on a reductive p-adic group, had a clear conception of the *support* of the Plancherel measure [15]. The support is $\bigcup \mathcal{E}_2(M)/W(M)$ where $\mathcal{E}_2(M)$ is the set of equivalence classes of the discrete series of M, $W(M)$ is the Weyl group of M (*i.e.,* the quotient by M of the normalizer of the maximal split torus in the center of M) and one M is chosen in each conjugacy class. The space $\mathcal{E}_2(M)$ is the disjoint union of compact tori of dimension equal to the parabolic rank of M. So the support is a disjoint union of countably many compact orbifolds, each the quotient of a compact torus by a finite group (product of symmetric groups). Each orbifold is the product of symmetric products of circles and is of dimension d with $1 \leq d \leq n$. At one extreme, the supercuspidal representations arrange themselves into circles; at the other extreme we have the unramified unitary principal series which, as a subspace of the tempered dual, is the symmetric product of n circles.

We depend on the Harish-Chandra product formula, see [15, p. 92–93], [15, Theorem 5, p.359]. This formula was published posthumously in 1984 and without proof [15]. The detailed proof has recently been given by Waldspurger [32].

We also depend on a formula of Shahidi [27] which resolves a conjecture of Langlands. The Langlands-Shahidi formula gives the Plancherel density as a ratio of certain local L-functions and root numbers [27]. In [28] Shahidi applies this formula to the special case of a maximal parabolic subgroup of $\mathrm{GL}(n)$. The resulting formula contains some very confusing typos [28, p. 292 –293]: we give a new derivation of the correct formula. One part of the Plancherel formula refers to the spherical part of the tempered dual: in this context we extend the classical formula of Macdonald [20], [21, Theorem 5.1.2] from $\mathrm{SL}(n)$ to $\mathrm{GL}(n)$ and then from the spherical component to the whole of the tempered dual.

Let F be a nonarchimedean local field, let $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$, and let $\mathfrak{A}$ be the reduced C^*-algebra of G. Let Ω be a component in the Bernstein variety $\Omega(G)$ (each point in $\Omega(G)$ is a conjugacy class of cuspidal pairs (M, σ), where M is a Levi subgroup of G, and σ is an irreducible supercuspidal representation of M), see [2]. The Bernstein decomposition

$$\mathfrak{A} = \bigoplus \mathfrak{A}(\Omega)$$

is a canonical decomposition of $\mathfrak{A}$ into C^*-ideals $\mathfrak{A}(\Omega)$, see [24].

Let $\mathrm{Irr}^{\mathrm{t}}(G)$ denote the tempered dual of G and let $\mathrm{Irr}^{\mathrm{t}}(G)_\Omega$ denote the subset of those tempered representations whose infinitesimal characters belong to Ω. The Bernstein decomposition of $\mathfrak{A}$ determines the following partition of $\mathrm{Irr}^{\mathrm{t}}(G)$:

$$\mathrm{Irr}^{\mathrm{t}}(G) = \bigsqcup \mathrm{Irr}^{\mathrm{t}}(G)_\Omega.$$

This determines the Bernstein decomposition of Plancherel measure:

$$\nu = \sum \nu_\Omega.$$

We can think of a component Ω in the Bernstein variety $\Omega(G)$ as a vector $(\sigma_1, \ldots, \sigma_k)$ of irreducible supercuspidal representations of smaller general linear groups: the entries of this vector are determined up to tensoring with unramified quasicharacters and permutation. If the vector is $(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_t, \ldots, \sigma_t)$ with σ_j repeated e_j times, $1 \le j \le t$, and $\sigma_1, \ldots, \sigma_t$ pairwise distinct (after unramified twist) then we say that Ω has *exponents* $e_1, \ldots, e_t$. Let

$$d(\Omega) = e_1 + \cdots + e_t$$

and let

$$W(\Omega) = S_{e_1} \times \cdots \times S_{e_t}$$

a product of symmetric groups. Let $\mathbb{T}^d$ denote the standard compact torus of dimension d. Let σ_j be an irreducible supercuspidal representation of the smaller general linear group $\mathrm{GL}(m_j)$. Each representation σ_j of $\mathrm{GL}(m_j)$ has a *torsion number*: the order of the cyclic group of all those unramified characters η for which $\sigma_j \otimes \eta \cong \sigma_j$. The torsion number of σ_j will be denoted $r_j = \mathrm{Tor}(\sigma_j)$.

Let ρ be a representation of the Weil group W_F, and let J be an open normal subgroup of the inertia group $I = I_F$. We now follow Deligne [14, (4.5.4),p. 538]. Denoting $\mathrm{A}_{I/J}$, $\mathrm{Sw}_{I/J}$ the Artin representation and the Swan representation, respectively, of I/J, the Artin conductor $a(\rho)$ is defined as

$$a(\rho) = \dim \mathrm{Hom}_{I/J}(\mathrm{A}_{I/J}, \rho) = \mathrm{sw}(\rho) + \dim(\rho - \rho^I),$$

where ρ^I denotes the subspace of I-invariant vectors, and $\mathrm{sw}(\rho)$ is the Swan conductor

$$\mathrm{sw}(\rho) = \dim \mathrm{Hom}_{I/J}(\mathrm{Sw}_{I/J}, \rho).$$

The above definitions do not depend on the choice of J (see [14, (4.5.2)]) and show that $a(\rho)$ and $\mathrm{sw}(\rho)$ are natural numbers.

Let $\mathrm{Cusp}(G)$ denote the set of isomorphism classes of irreducible supercuspidal representations of G and let $\mathrm{Irr}_n(W_F)$ denote the set of isomorphism classes of n-dimensional irreducible representations of the Weil group W_F of F. Let

$$\mathrm{rec}_F : \mathrm{Cusp}(G) \longrightarrow \mathrm{Irr}_n(W_F)$$

denote the local Langlands correspondence as in [16, p. 2–3]. We will write

$$a_j = a(\rho_j^\vee \otimes \rho_j)$$

where

$$\rho_j = \mathrm{rec}_F(\sigma_j).$$

In this way, the Bernstein component Ω creates the following natural number invariants:

- the sizes $m_1, m_2, \ldots, m_t$ of the smaller general linear groups,
- the exponents $e_1, e_2, \ldots, e_t$,
- the torsion numbers $r_1, r_2, \ldots, r_t$,
- the Artin conductors $a_1, a_2, \ldots, a_t$.

We have to refine these invariants by taking *partitions* of $e_1, \dots, e_t$. It is precisely these invariants, along with the formal degree, which enter into the Plancherel formula.

According to [24], the support of Plancherel measure is a disjoint union of extended quotients:

$$\mathrm{Irr}^{\mathrm{t}}(G) = \bigsqcup_{\Omega} \widetilde{\mathbb{T}^{d(\Omega)}}/W(\Omega).$$

We prove an explicit formula for Plancherel density on $\mathrm{Irr}^{\mathrm{t}}(G)$ in terms of the above invariants. We know in advance that *Plancherel measure is rotation-invariant* [24]. This implies that Plancherel density is a constant on each circle in the discrete series of $\mathrm{GL}(n)$.

In section 2, we give a précis of the Harish-Chandra Plancherel Theorem, including the Harish-Chandra product formula, following Waldspurger [32].

In section 3, we compute the Plancherel density μ by using the Langlands-Shahidi formula.

In section 4, we write the formula for the density μ in terms of the p-adic gamma function, and recover, as a special case, the classical formula of Macdonald [20, 21].

In section 5, we compute the constants $q^{n(\,)}$ in terms of Artin conductors.

In section 6, we compute the constants $\gamma(GL(n)/GL(n_1) \times GL(n_2))$ as explicit rational functions in q.

In section 7, we give a new formula for the density μ.

In section 8, we give the explicit Bernstein decomposition of Plancherel measure.

So far, our results are completely independent of the theory of types. In section 9, we discuss the connection with the theory of types and prove the transfer-of-measure formula for $GL(n)$. In this, we have been much influenced by the preprint of Bushnell–Henniart–Kutzko [9].

2. The Plancherel theorem after Harish-Chandra

Let A be the diagonal split torus, with F-points

$$A(F) = \left\{ \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \,:\, x_i \in F^{\times} \right\}.$$

We denote a typical element of $A(F)$ by $\mathrm{diag}(x_1, \dots, x_n)$. The root system $\Sigma(G)$ for $G = \mathrm{GL}(n)$ with respect to A is of type A_{n-1}, with roots $\alpha_{i,j}$, $i, j \in \{1, 2, \dots, n\}$, where

$$\alpha_{i,j}\left(\operatorname{diag}(x_1, \dots, x_n)\right) := \frac{x_i}{x_j}. \tag{1}$$

The positive roots are the $\alpha_{i,j}$ with $i < j$, $i, j \in \{1, \dots, n\}$, and their set will be denoted by $\Sigma(G)^+$. The number of positive roots $|\Sigma(G)^+|$ is $\frac{n(n-1)}{2}$. The simple roots are the $\alpha_{i,i+1}$ for $i = 1, 2, \dots, n-1$.

We will denote by $\mathrm{GL}(n_1, \dots, n_k)$ the block diagonal Levi subgroup $\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)\times\cdots\times\mathrm{GL}(n_k)$ of G. Let P be the upper block triangular parabolic subgroup of G with Levi subgroup $M = \mathrm{GL}(n_1, \dots, n_k)$ and unipotent radical denoted by N. The root system $\Sigma(M)$ of M is of type $A_{n_1-1}\times\cdots\times A_{n_k-1}$, with positive roots $\alpha_{n_1+\cdots+n_{i-1}+h, n_1+\cdots+n_{i-1}+l}$, where $1 \le i \le k$, $1 \le h \le n_i - 1$, $2 \le l \le n_i$, and $h < l$. Let A_M be the split component of M, that is, the maximal split torus in the center of M:

$$A_M = \left\{ \begin{pmatrix} \lambda_1 \mathrm{Id}_{n_1} & & & \\ & \lambda_2 \mathrm{Id}_{n_2} & & \\ & & \ddots & \\ & & & \lambda_k \mathrm{Id}_{n_k} \end{pmatrix} \;:\; \lambda_i \in F^\times \right\}. \tag{2}$$

Let $\Sigma(A_M)$ be the set of roots of A_M in G, that is,

$$\Sigma(A_M) = \left\{ a_{i,j} := \alpha_{n_1+\cdots+n_i, n_1+\cdots+n_j} \;:\; (i,j) \in \{1, \dots, k\},\ i \neq j \right\}.$$

We denote by $\Sigma(P)$ the subset of roots in $\Sigma(A_M)$ which are positive with respect to P, that is, the roots in $\Sigma(A_M)$ which occur in the action of A_M on the Lie algebra of N. Let $\Sigma_{\mathrm{nd}}(P)$ denote the subset of non-divisible roots in $\Sigma(P)$.

For α in $\Sigma_{\mathrm{nd}}(P)$, let A_α be the connected component of $A \cap \ker(\alpha)$. Let M_α be the centraliser of A_α in G and let $N_\alpha := M_\alpha \cap N$. Then MN_α is a (maximal) parabolic subgroup of M_α: $M_{a_{i,j}} \supset \mathrm{GL}(n_1, \dots, n_k)$ is the set of matrices

$$\begin{pmatrix} \mathrm{GL}(n_1, \dots, n_{i-1}) & & & & \\ & x & & y & \\ & & \mathrm{GL}(n_{i+1}, \dots, n_{j-1}) & & \\ & z & & t & \\ & & & & \mathrm{GL}(n_{j+1}, \dots, n_k) \end{pmatrix},$$

such that $x \in \mathrm{GL}(n_i)$, $t \in \mathrm{GL}(n_j)$ and $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{GL}(n_i + n_j)$, and $M = \mathrm{GL}(n_1, \dots, n_k)$ is a Levi subgroup of a maximal parabolic subgroup of $M_{a_{i,j}} \simeq \mathrm{GL}(n_i + n_j) \times \mathrm{GL}(n_1, \dots, \hat{n}_i, \dots, \hat{n}_j, \dots, n_k)$.

For every g in G, we choose $n_P(g) \in N$, $m_P(g) \in M$, $k_P(g) \in \mathrm{GL}(n, \mathfrak{o}_F)$ such that $g = u_P(g) m_P(g) k_P(g)$, where $\mathfrak{o}_F$ denote the ring of integers of F. Let $\overline{P}$ be the opposite parabolic subgroup of P and let $\overline{N}$ be its unipotent radical. We denote by $\overline{\mathfrak{n}}$ the Lie algebra of $\overline{N}$. Let $d\overline{n}$ be the Haar measure on $\overline{N}$ assigning volume 1 to $\overline{N} \cap \mathrm{GL}(n, \mathfrak{o}_F)$. We set

$$\gamma(G/P) := \int_{\overline{N}} \delta_P(m_P(\overline{n}))\, d\overline{n}. \tag{3}$$

Note that $\gamma(G/P)$ does not depend on the choice of the parabolic P (see for instance [32, I.1.(3)]). Hence from now we will write $\gamma(G/M)$ for $\gamma(G/P)$. It is clear that $\gamma(M_{-\alpha}/M) = \gamma(M_\alpha/M)$, for each $\alpha \in \Sigma_{\mathrm{nd}}(P)$. Then we set

$$c(G/P) = c(G/M) := \gamma(G/M)^{-1} \prod_{\alpha \in \Sigma_{\mathrm{nd}}(P)} \gamma(M_\alpha/M),$$

that is,

$$c(G/M) = \gamma(G/M)^{-1} \prod_{i<j} \gamma(M_{a_{i,j}}/M). \tag{4}$$

Remark 2.1. Let J_G be an open compact subgroup of G which admits an Iwahori decomposition $J = (J_G \cap N) \cdot (J_G \cap M) \cdot (J_G \cap \overline{N})$. We equip each closed algebraic subgroup H of $\mathrm{GL}(n)$ with the Haar measure such that $v(H \cap \mathrm{GL}(n, \mathfrak{o}_F)) = 1$. By the definition 3 of $\gamma(G/M)$, we have

$$\int_G f(g) dg = \gamma(G/M)^{-1} \int_{N \times M \times \overline{N}} \delta_P(m)^{-1} f(nm\overline{n})\, dn\, dm\, d\overline{n}.$$

Applying the above identity to the characteristic function of J_G (as in [32][preuve de (3)]), we obtain

$$\gamma(G/M) = \frac{v(J_G \cap N) \cdot v(J_G \cap M) \cdot v(J_G \cap \overline{N})}{v(J_G)}. \tag{5}$$

The formula (5) shows that the gamma function satisfy the following product formula

$$\gamma(G/M) = \gamma(G/M') \cdot \gamma(M'/M), \tag{6}$$

for any Levi subgroup $M' \supset M$.

Let $\mathrm{i}_{M,P}^G$ denote the normalized parabolic induction functor, *i.e.,* if (π_M, V_M) is a smooth representation of M, then $\mathrm{i}_{M,P}^G(V_M)$ will be the space of smooth (*i.e.,* locally constant and biinvariant with respect to some open subgroup of G) functions $f\colon G \to V$ such that

$$f(mng) = \delta_P(m)^{\frac{1}{2}}\, \pi(m)\, f(g), \;\; m \in M,\, n \in N,\, g \in G;$$

the action of G on $\mathrm{i}_{M,P}^G(V_M)$ via $\mathrm{i}_{L,P}^G(\pi_M)$ is that of right translation. Here, δ_P denotes the module of P, that is the character of P defined by $\delta(p) := |\det \mathrm{Ad}(p)_{|\overline{\mathfrak{n}}}|$. Note that, when $P = B$ is the upper Borel subgroup of $\mathrm{GL}(n)$, we have

$$(7)\quad \delta_B\left(\mathrm{diag}(x_1, \dots, x_n)\right) = \prod_{i=1}^{n} |x_i|^{n+1-2i} = |x_1|^{n-1}|x_2|^{n-3} \cdots |x_n|^{1-n}.$$

Let Q be a parabolic subgroup of G with Levi subgroup M, and unipotent radical denoted by N_Q. We assume that, for every $f \in \mathrm{i}_{M,P}^G(V_M)$ and every $g \in G$, there exists a $\xi \in V_M$ such that for any $\xi^\vee$ in the contragredient $V_M^\vee$ of V_M, the integral $\int_{N\cap N_Q\backslash N_Q} \langle f(n_Q g)\xi, \xi^\vee\rangle dn_Q$ is absolutely convergent, and equal to $\langle \xi, \xi^\vee\rangle$. Then we set

$$\int_{N\cap N_Q\backslash N_Q} f(n_Q g) dn_Q := \xi,$$

and we define a homomorphism $J_{Q|P}(\pi_M)\colon \mathrm{i}_{M,P}^G(V_M) \to \mathrm{i}_{M,Q}^G(V_M)$ as

$$J_{Q|P}(\pi_M) := \int_{N\cap N_Q\backslash N_Q} f(n_Q g)\, dn_Q,$$

for $f \in \mathrm{i}_{M,P}^G(V_M)$ (see [32, IV.1]). The operator $J_{Q|P}(\pi_M)$ is G-invariant. Note that in the special case where $Q = \overline{\mathrm{P}}$, we get the following simpler expressions:

$$(8)\qquad J_{P|\overline{\mathrm{P}}}(\pi_M) = \int_N f(n) dn \;\text{ and }\; J_{\overline{\mathrm{P}}|P}(\pi_M) = \int_{\overline{\mathrm{N}}} f(\overline{n}) d\overline{n}.$$

Let $X(A_G)$ denote the set of F-rational characters of A_G. We set $\mathfrak{a}_G^* := X(A_G) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_{G,\mathbb{C}}^* := X(A_G) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\Psi(G)$ be the set of unramified characters of G. Then, the map $\mathfrak{a}_{G,\mathbb{C}}^* \to \Psi(G)$ which sends $\chi \otimes s$ to $g \mapsto |\chi(g)|^s$ is surjective, with kernel of the form $\frac{2\pi i}{\log(q)}\Lambda$, for some lattice Λ in $\Psi(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that Λ is the kernel of the homomorphism $H_G\colon G \to \mathfrak{a}_G$ defined by

$$q^{-\langle \chi, H_G(g)\rangle} = |\chi(g)|.$$

All the elements in the fibre of $\chi \in \Psi(G)$ under this map have the same real part that we denote by $\mathrm{Re}(\chi)$. We denote by $\Psi^{\mathrm{t}}(G)$ the group of unitary unramified characters:

$$(9)\qquad \Psi^{\mathrm{t}}(G) = \{\chi \in \Psi(G) \mid \mathrm{Re}(\chi) = 0\}.$$

The restriction map $\Psi^{\mathrm{t}}(G) \to \Psi^{\mathrm{t}}(A_G)$ is surjective with finite kernel. We put on $\Psi^{\mathrm{t}}(A_G)$ the Haar measure with total mass 1, and on $\Psi^{\mathrm{t}}(G)$ the measure such that the restriction map preserves the measures locally. We set $\mathcal{O}_{\mathbb{C}}(\pi_M) := \{\pi_M \otimes \chi \mid \chi \in \Psi(M)\}$. Assume that π_M is irreducible. Then the set of $\pi' \in \mathcal{O}_{\mathbb{C}}(\pi_M)$ such that $\mathrm{i}_{M,P}^G(\pi'_M)$ is irreducible is a dense Zariski open set of $\mathcal{O}_{\mathbb{C}}(\pi_M)$ (see [32, IV.3]), and for such π'_M the operator

$$J_{\overline{P}|P}(\pi'_M) \circ J_{P|\overline{P}}(\pi'_M) \in \mathrm{End}_G(\mathrm{i}_{M,P}^G(V_M)) \tag{10}$$

is a homothety. Hence there is a rational function j_P on $\mathcal{O}_{\mathbb{C}}$ such that this operator is a homothety of ratio $j_P(\pi_M)$. The term $j_P(\pi_M)$ does not depend on the choice of the parabolic subgroup P with Levi subgroup M (see [32, IV.3]). We will write it simply $j(\pi)$. Note that $j(\pi_M^\vee) = j(\pi_M)$.

Attach discrete series representations $\omega_1, \ldots, \omega_k$ to the blocks in M and form the discrete series representation

$$\omega = \omega_1 \otimes \cdots \otimes \omega_k.$$

We denote by $\mathcal{E}_2(M)$ the set of equivalence classes of discrete series in M and by $\mathcal{O}(\omega_0) \subset \mathcal{E}_2(M)$ an orbit under the action of $\Psi^{\mathrm{t}}(M)$, that is,

$$\mathcal{O}(\omega_0) := \left\{\omega_0 \otimes \chi \mid \chi \in \Psi^{\mathrm{t}}(M)\right\}, \quad \text{where} \quad \omega_0 \in \mathcal{E}_2(M).$$

For every $\omega \in \mathcal{O}(\omega_0)$, we define

$$\mu(\omega) = \mu_G(\omega) := j(\omega)^{-1} \prod_{\alpha \in \Sigma_{\mathrm{nd}}(A_M)(\text{up to sign})} \gamma(M_\alpha|M)^2. \tag{11}$$

Remark 2.2. In the case where the parabolic P is maximal, it gives

$$\mu(\omega) = j(\omega)^{-1} \cdot \gamma(G/M)^2.$$

Now $\mathrm{GL}(n_i) \times \mathrm{GL}(n_j)$ is a standard Levi factor in a maximal parabolic subgroup of $\mathrm{GL}(n_i+n_j)$, and there is a Plancherel density $\mu_{M_{a_{i,j}}/M}$ attached to $M_{a_{i,j}}$.

$$\mu_{M_{a_{i,j}}/M}(\omega) = \mu_{\mathrm{GL}(n_i+n_j)/\mathrm{GL}(n_i)\times\mathrm{GL}(n_j)}(\omega_i \otimes \omega_j), \tag{12}$$

The Harish-Chandra product formula is used in the following way. The formula for the Plancherel density is (see [32, V 2.1])

$$\mu_{G/M}(\omega) = \prod_{1 \le j < i \le k} \mu_{M_{a_{i,j}}/M}(\omega).$$

In other words, we have:

Theorem 2.3. (Harish-Chandra)

$$\mu_{G/M}(\omega) = \prod_{1 \le j < i \le k} \mu_{\mathrm{GL}(n_i+n_j)/\mathrm{GL}(n_i)\times\mathrm{GL}(n_j)}(\omega_i \otimes \omega_j).$$

For $\omega \in \mathcal{E}_2(M)$, we denote by $d(\omega)$ its formal degree and by $\theta_M^G(\omega)$ the character of $\mathrm{i}_{M,P}^G(\omega)$. Let $r(g)$ be the right translation by g, and let $W(M) := \mathrm{N}_G(A_M)/M$. Let $d\omega$ be the canonical measure on $\mathcal{E}_2(M)$ defined by Harish-Chandra in [15, §2]. For f in the Schwartz space on G and $g \in G$, we define $f_M(g)$ as

(13)
$$c(G/M)^{-2}\,\gamma(G/M)^{-1}\,|W(M)|^{-1}\int_{\mathcal{E}_2(M)} \mu_{G/M}(\omega)\,d(\omega)\,\theta_\omega^G(r(g)f)\,d\omega.$$

Then Harish-Chandra's Plancherel Theorem [15, p.367], [32, VIII] states that

$$f = \sum_M f_M, \tag{14}$$

where M runs over the Levi subgroups $M \supset A$ of G up to conjugacy.

Remark 2.4. We observe that
$$c(G/M)^{-2}\,\gamma(G/M)^{-1}\,\mu_{G/M}(\omega) \;= \gamma(G/M)\,j(\omega)^{-1}.$$

3. Calculation of the Plancherel density

From (2.3) of the previous section, it is enough to consider *maximal* parabolic subgroups of GL(n). A standard Levi subgroup M of a maximal parabolic subgroup is of the form $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$. Attach the discrete series representation $\omega_1 \otimes \omega_2$ to this Levi subgroup. We have the Langlands-Shahidi formula for the Plancherel constant associated to a maximal parabolic subgroup of GL(n), see [28, §7] or [29, §6]:

$$\mu(\frac{s}{2}, \omega_1 \otimes \omega_2) = const. \cdot \frac{L(1+s, \omega_1 \times \omega_2^\vee)\,L(1-s, \omega_1^\vee \times \omega_2)}{L(s, \omega_1 \times \omega_2^\vee)\,L(-s, \omega_1^\vee \times \omega_2)}, \tag{15}$$

where

$$const. = \gamma(G/M)^2 \cdot q^{n(\omega_1^\vee \times \omega_2)}, \tag{16}$$

where the exponent $n(\;)$ is defined by formula (42).

Now we start the calculations. We shall make use of the identity

$$1 - x^r = \prod_{j=0}^{r-1}(1 - \zeta^j x)$$

where $\zeta = e^{2\pi i/r}$ is a primitive rth root of unity in $\mathbb{C}$.

For any smooth representation π of G and any quasicharacter χ, we denote by $\pi(\chi)$ the twist of π by χ

$$\pi(\chi)(g) := \chi(\det(g))\,\pi(g).$$

If σ_1 (resp. σ_2) is an irreducible supercuspidal representation of $\mathrm{GL}(m_1)$ (resp. $\mathrm{GL}(m_2)$), then we have $L(s, \sigma_1 \times \sigma_2^\vee) = 1$ unless $\sigma_1 \cong \sigma_2(\psi)$ with ψ an unramified character of $F^\times$ in which case we have

$$L(s, \sigma_1 \times \sigma_2^\vee) = \prod_\psi L(s, \psi) \tag{17}$$

the product taken over all *unramified* characters ψ of $F^\times$ such that $\sigma_1 \cong \sigma_2(\psi)$ (see [18, Prop. 8.1], [7, p.27, first paragraph]). Note that this formula is more precise than [19, (3.2.5)].

Let ϖ denote a fixed uniformizer parameter, and let q denote the order of the residue field of F. We quote the standard formula for the L-function $L(s, \psi)$, see [31, 3.1.3]:

$$L(s, \psi) = (1 - \psi(\varpi) q^{-s})^{-1} \tag{18}$$

if ψ is an unramified quasicharacter, and

$$L(s, \psi) = 1 \tag{19}$$

if ψ is ramified.

In the next lemma we recover a formula of Shahidi [28, p.269].

Lemma 3.1. *Let σ_2 have torsion number r and let $\sigma_1 \cong \sigma_2(\chi)$ with χ an unramified quasicharacter such that $\chi(\varpi) = \alpha$. Then we have*

$$L(s, \sigma_1 \times \sigma_2^\vee) = (1 - \alpha^r q^{-rs})^{-1}. \tag{20}$$

Proof. We assume that $\sigma_2 \cong \sigma_2(\eta)$, where η is unramified. Then $\eta^{n_2} = 1$. Since each η is determined by $\eta(\varpi)$ and since finite subgroups of $\mathbb{C}^\times$ are cyclic, the set of all unramified η such that $\sigma_2 \cong \sigma_2(\eta)$ is a cyclic group of order r, where r divides n_2. We will write $\zeta = \eta(\varpi)$. We have $\sigma_1 \cong \sigma_2(\psi)$ where the possibilities for ψ are χ, $\chi\eta$, $\chi\eta^2$, $\chi\eta^3$, $\ldots$, $\chi\eta^{r-1}$. Then we have

$$L(s, \sigma_1 \times \sigma_2^\vee) = \prod_{j=0}^{r-1} L(s, \chi\eta^j) \tag{21}$$

$$= \prod_{j=0}^{r-1} (1 - (\chi\eta^j)(\varpi) q^{-s})^{-1} \tag{22}$$

$$= \prod_{j=o}^{r-1} (1 - \zeta^j \alpha q^{-s})^{-1} \tag{23}$$

$$= (1 - \alpha^r q^{-rs})^{-1}, \tag{24}$$

□

Let $M = \mathrm{GL}(n_1)\times\mathrm{GL}(n_2)$ be the Levi subgroup of a maximal parabolic of $G = \mathrm{GL}(n_1+n_2)$ and let ω_1, ω_2 be discrete series representations of $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$. Let χ_1, χ_2 be unramified (unitary) characters of $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$ with $\chi_1(\varpi) = \alpha_1$, $\chi_2(\varpi) = \alpha_2$. Then the group of unramified (unitary) characters $\Psi^{\mathrm{t}}(M)$ of M has, via the map

$$(\chi_1,\chi_2) \mapsto (\alpha_1,\alpha_2),$$

the structure of the compact torus $\mathbb{T}^2$.

Consider now the *orbit* $\Psi^{\mathrm{t}}(M)\cdot(\omega_1\otimes\omega_2)$ in the Harish-Chandra parameter space $\Omega^{\mathrm{t}}(G)$. The action of $\Psi^{\mathrm{t}}(M)$ creates a short exact sequence

$$1 \to \mathcal{G} \to \mathbb{T}^2 \to \mathbb{T}^2 \to 1$$

with

$$\mathbb{T}^2 \to \mathbb{T}^2,\ (\alpha_1,\alpha_2) \mapsto (\alpha_2^r,\alpha_2^r).$$

The finite group $\mathcal{G}$ is precisely the group in [2, Section 3.1] and is the product of cyclic groups:

$$\mathcal{G} = \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}.$$

We will write $z_1 = \alpha_1^r, z_2 = \alpha_2^r$ so that z_1, z_2 are precisely the coordinates of a point in the orbit.

Remark 3.2. We recall the following facts about the discrete series of $\mathrm{GL}(n)$. Let ω_1 and ω_2 are two discrete series representations of $\mathrm{GL}(n_1)$ and $\mathrm{GL}(n_2)$, respectively. By [33] (see [19, Prop. 1.2.3]), there exist two pairs of integers $(m_1, 2g_1+1)$ and $(m_2, 2g_2+1)$ and two irreducible supercuspidal representations σ_1 and σ_2 of $\mathrm{GL}(m_1)$ and $\mathrm{GL}(m_2)$ respectively such that, for $i = 1, 2$, we have $m_i(2g_i+1) = n_i$ and the representation ω_i is the Langlands quotient $Q(\Delta_i)$ associated to the segment $\Delta_i = \{|\ |^{-g_i}\sigma_i, |\ |^{-g_i+1}\sigma_i, \dots, |\ |^{g_i-1}\sigma_i, |\ |^{g_i}\sigma_i\}$ (that is, ω_i is the unique irreducible quotient of the parabolically induced representation of $|\ |^{-g_i}\sigma_i\otimes\cdots\otimes|\ |^{g_i}\sigma_i$).

Theorem 3.3. *Let σ be an irreducible pre-unitary supercuspidal representation of* $\mathrm{GL}(m)$ *with torsion number r. Let π_1, π_2 be discrete series representations of* $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$ *such that π_i is the Langlands quotient associated to the segment*

$$\{|\ |^{-g_i}\sigma, \dots, |\ |^{g_i}\sigma\}.$$

Let χ_1, χ_2 be unramified characters with $\chi_1(\varpi) = \alpha_1$, $\chi_2(\varpi) = \alpha_2$. Let $z_1 = \alpha_1^r$, $z_2 = \alpha_2^r$. Then, as a function on the compact torus $\mathbb{T}^2$ with *co-ordinates* (z_1, z_2) *we have*

$$\mu(\chi_1\pi_1 \otimes \chi_2\pi_2) = const. \cdot \prod_{g=|g_1-g_2|}^{g_1+g_2} \left| \frac{1 - z_1 z_2^{-1}/q_K^g}{1 - z_1 z_2^{-1}/q_K^{g+1}} \right|^2$$

where

$$const. = \gamma(G/M)^2 \cdot q^{n(\omega_1^\vee \times \omega_2)}$$

and

$$q_K = q^r.$$

Proof. We will fix notation as follows. Let

$$\omega_i = \chi_i\pi_i = Q(\chi_i \cdot \Delta_i),\ \chi_i\Delta_i = \{|\ |^{-g_i}\chi_i\sigma, \ldots, |\ |^{g_i}\chi_i\sigma\},\ \sigma_i = \chi_i\sigma.$$

We note that

$$\sigma_1 \cong \chi_1\sigma \cong \chi_1\chi_2^{-1}\sigma_2 \cong \sigma_2(\chi_1/\chi_2)$$

$$\chi(\varpi) = (\chi_1/\chi_2)(\varpi) = \alpha_1/\alpha_2 = \alpha.$$

Remark 3.4. We also note that $\omega_2^\vee$ is the quotient $Q(\Delta_2^\vee)$ where

$$\Delta_2^\vee = \{|\ |^{-g_2}\sigma_2^\vee, \ldots, |\ |^{g_2}\sigma_2^\vee\}.$$

We can see this via Langlands parameters. If ω_2 has Langlands parameter $\rho_2 \otimes |\ |^{-g_2} \otimes \mathrm{sp}(2g_2+1)$ then $\omega_2^\vee$ has Langlands parameter

$$\rho_2^\vee \otimes |\ |^{g_2} \otimes \mathrm{sp}(2g_2+1) \otimes |\ |^{-2g_2} = \rho_2^\vee \otimes |\ |^{-g_2} \otimes \mathrm{sp}(2g_2+1),$$

see [16, p. 252].

It follows from [18, Th. 8.2] (see [19, (3.2.3)]) that

$$L(s, \omega_1 \times \omega_2^\vee) = \prod_{i=1}^{2\max(g_1,g_2)+1} L(s + 2g_1 + 2g_2 - i + 1, |\ |^{-g_1}\sigma_1 \times |\ |^{-g_2}\sigma_2^\vee),$$

that is,

$$L(s, \omega_1 \times \omega_2^\vee) = \prod_{i=1}^{2\max(g_1,g_2)+1} L(s + g_1 + g_2 - i + 1, \sigma_1 \times \sigma_2^\vee).$$

Finally we obtain

$$L(s, \omega_1 \times \omega_2^\vee) = \prod_{g=|g_1-g_2|}^{g_1+g_2} L(s+g, \sigma_1 \times \sigma_2^\vee) \tag{25}$$

$$= \prod_{g=|g_1-g_2|}^{g_1+g_2} (1 - \alpha^r q^{-r(s+g)})^{-1}, \tag{26}$$

by using (20).

Since $L(s, \omega_1^\vee \times \omega_2) = L(s, \omega_2 \times \omega_1^\vee)$, it follows

$$L(s, \omega_1^\vee \times \omega_2) = \prod_{g=|g_1-g_2|}^{g_1+g_2} (1 - \alpha^{-r} q^{-r(s+g)})^{-1}. \tag{27}$$

The we have

$$\frac{L(1+s, \omega_1 \times \omega_2^\vee)\, L(1-s, \omega_1^\vee \times \omega_2)}{L(s, \omega_1 \times \omega_2^\vee)\, L(-s, \omega_1^\vee \times \omega_2)} \tag{28}$$

$$= \prod_{g=|g_1-g_2|}^{g_1+g_2} \frac{(1 - \alpha^r q^{-r(s+g)})(1 - \alpha^{-r} q^{r(s-g)})}{(1 - \alpha^r q^{-r(s+1+g)})(1 - \alpha^{-r} q^{r(s-1-g)})} \tag{29}$$

$$= \prod_{g=|g_1-g_2|}^{g_1+g_2} \frac{(1 - \alpha^r q^{-rg} \cdot q^{-rs})(1 - \alpha^{-r} q^{-rg} \cdot q^{rs})}{(1 - \alpha^r q^{-rg-r} \cdot q^{-rs})(1 - \alpha^{-r} q^{-rg-r} \cdot q^{rs})} \quad . \tag{30}$$

At the point $s = 0$ we obtain

$$\frac{L(1, \omega_1 \times \omega_2^\vee)\, L(1, \omega_1^\vee \times \omega_2)}{L(0, \omega_1 \times \omega_2^\vee)\, L(0, \omega_1^\vee \times \omega_2)} \tag{31}$$

$$= \prod_{g=|g_1-g_2|}^{g_1+g_2} \frac{(1 - \alpha^r q^{-rg})(1 - \alpha^{-r} q^{-rg})}{(1 - \alpha^r q^{-rg-r})(1 - \alpha^{-r} q^{-rg-r})} \quad . \tag{32}$$

Since α is a complex number of modulus 1 and $\alpha^r = z_1/z_2$, this becomes

$$\frac{L(1, \omega_1 \times \omega_2^\vee)\, L(1, \omega_1^\vee \times \omega_2)}{L(0, \omega_1 \times \omega_2^\vee)\, L(0, \omega_1^\vee \times \omega_2)} \tag{33}$$

$$= \prod_{g=|g_1-g_2|}^{g_1+g_2} \left| \frac{1 - z_1 z_2^{-1} q^{-rg}}{1 - z_1 z_2^{-1} q^{-rg-r}} \right|^2 \quad . \tag{34}$$

Now let $q_K = q^r$. Then this equation becomes

$$\frac{L(1, \omega_1 \times \omega_2^\vee)\, L(1, \omega_1^\vee \times \omega_2)}{L(0, \omega_1 \times \omega_2^\vee)\, L(0, \omega_1^\vee \times \omega_2)} \tag{35}$$

$$= \prod_{g=|g_1-g_2|}^{g_1+g_2} \left| \frac{1 - z_1 z_2^{-1}/q_K^g}{1 - z_1 z_2^{-1}/q_K^{g+1}} \right|^2 \quad . \tag{36}$$

□

In section 9 we shall interpret q_K as the cardinality of the residue field of a canonical extension field K/F.

4. The gamma function

Let K be a nonarchimedean field such that the cardinality of its residue field is $q_K = q_F^r$. Now let $\Gamma_1(x)$ be the p-adic gamma function [30, p. 51] associated to the local field K. The p-adic gamma function attached to the local field K is the following meromorphic function of a single complex variable:

$$\Gamma_1(\alpha) = \frac{1 - q_K^\alpha / q_K}{1 - q_K^{-\alpha}}.$$

We will change the variable via $s = q_K^\alpha$ and write

$$\Gamma_K(s) = \frac{1 - s/q_K}{1 - s^{-1}}.$$

Let $\alpha \in i\mathbb{R}$ so that s has modulus 1. Then we have

$$1/|\Gamma_K(s)|^2 = \left| \frac{1 - s}{1 - q_K^{-1} s} \right|^2.$$

This leads immediately to the next result.

Theorem 4.1. *We have*

$$\mu(\chi_1\pi_1 \otimes \chi_2\pi_2) = const./ \prod_{g=|g_1-g_2|}^{g_1+g_2} |\Gamma_K(q_K^{-g} z_1/z_2)|^2,$$

where

$$const. = \gamma(G/M)^2 \cdot q^{n(\omega_1^\vee \times \omega_2)}.$$

This function is invariant under the rotation $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$ with λ a complex number of modulus 1, and under the $\mathbb{Z}/2\mathbb{Z}$-action on the torus $\mathbb{T}^2$ which is generated by the map $(z_1, z_2) \mapsto (z_2, z_1)$.

Theorem 4.2. *If π_1, π_2 are supercuspidal then $g = g_1 = g_2 = 0$ and we have*

$$\mu(\chi_1\pi_1 \otimes \chi_2\pi_2) = const./ \left|\Gamma_K(z_1/z_2)\right|^2. \tag{37}$$

If the torsion number is 1 then this formula further simplifies to

$$\mu(\chi_1\pi_1 \otimes \chi_2\pi_2) = const. \left| \frac{1 - z_1/z_2}{1 - q_F^{-1} z_1/z_2} \right|^2.$$

This formula brings us close to the classical formula of Macdonald [20].

Note that the irreducible supercuspidal representation ω has torsion number 1 if ω is *wildly ramified* and n is a power of $p = char(k_F)$, see [7, p.1].

Compare the classical formula for the Plancherel density attached to the unitary principal series of GL(2) in [1, §2.2 (1)].

Let T be the maximal torus in GL(n) and let $\widehat{T}$ denote the unitary dual of T. Then $\widehat{T}$ has the structure of a compact torus $\mathbb{T}^n$ (the space of Satake parameters) and the unramified unitary principal series of GL(n) is parametrized by the quotient $\mathbb{T}^n/S_n$. Let now $t = (z_1, \ldots, z_n) \in \mathbb{T}^n$. Applying the above formulas the Plancherel density $\mu_{G/T}$ is given by

$$\mu_{G/T} = \gamma(G/T)^{n(n-1)} \cdot \prod_{i<j} \left| \frac{1 - z_i z_j^{-1}}{1 - z_i z_j^{-1}/q} \right|^2 \tag{38}$$

$$= \gamma(G/T)^{n(n-1)} \cdot \prod_{0<\alpha} \left| \frac{1 - \alpha(t)}{1 - \alpha(t)/q} \right|^2 \tag{39}$$

$$= \gamma(G/T)^{n(n-1)} \cdot \prod_{\alpha} 1/\Gamma_1(\alpha(t)) \tag{40}$$

where α is a root of the Langlands dual group GL($n, \mathbb{C}$) so that $\alpha_{ij}(t) = z_i/z_j$. Note that this formula is invariant under the *rotation* $t \mapsto \lambda t$ where λ is a complex number of modulus 1, as predicted in [24].

For GL(n), one connected component in the tempered dual is the compact orbifold $\mathbb{T}^n/S_n$, the symmetric product of n circles. On this component we have the Macdonald formula [20]:

$$d\mu(\omega_\lambda) = const. \cdot d\lambda / \prod_{\alpha} \Gamma(i\lambda(\alpha^\vee))$$

the product over all roots α where $\alpha^\vee$ is the coroot. This formula is a very special case of our formula for GL(n).

5. The Artin conductor

We shall need the ϵ-factor $\epsilon(r, s, \Psi)$, see [16, p.243]. In the notation of Tate [31] we have

$$\epsilon(r, s, \Psi) = \epsilon(r\omega_s, \Psi, \mu_\Psi) = \epsilon(r\omega_s, \Psi, dx)$$

where μ_Ψ is the additive Haar measure on F which is self-dual with respect to Ψ. We will choose Ψ such that $\Psi(\mathcal{O}) = 1, \Psi(\varpi^{-1}\mathcal{O}) \neq 1$. (Note that Shahidi uses precisely this normalization in [29].) Then $n(\Psi) = 0$ and $\delta(\Psi) = 1$ in Tate's notation [31, 3.2.6, 3.4.5].

Let $f(V)$ denote the absolute norm of the Artin conductor of V. This f can be characterized as the unique function inductive in degree 0 such that $f(\chi) = q_F^{a(\chi)}$ for quasicharacters χ. The integer $a(\chi)$ is the (exponent of the) conductor of χ. So $a(\chi) = 0$ if χ is unramified, and is the smallest integer m such that χ is trivial on units $\equiv 1 \ mod\ \varpi^m$ if χ is ramified.

We then have the following crucial formulas [31, 3.4.5]:

- $\epsilon(V, s, \Psi) = \epsilon(V|\ |^s, 0, \Psi) = \epsilon(V, 0, \Psi) f(V)^{-s}$
- $f(V) = \epsilon(V, 0, \Psi)/\epsilon(V|\ |, 0, \Psi)$

Lemma 5.1. *Let* W *be an unramified quasicharacter of the Weil group* W_F*. Then we have*

$$f(V\chi) = f(V).$$

Proof. Let $\chi = |\ |^\alpha$, then we have

$$f(V\chi) = \frac{\epsilon(V\chi, 0, \Psi)}{\epsilon(V\chi \cdot |\ |, 0, \Psi)} = \frac{\epsilon(V, 0, \Psi) f(V)^{-\alpha}}{\epsilon(V, 0, \Psi) f(V)^{-\alpha-1}} = f(V).$$

□

Lemma 5.2. *Let* W *be an unramified* Fr*-semisimple representation of* W_F*, i.e. a direct sum of unramified quasicharacters. Then we have*

$$f(V \otimes W) = f(V)^{\dim W}.$$

Proof. We write $W = \sum \chi_j$ and use Lemma 5.1 and the fact that f is additive. We have

$$f(V \otimes W) = f(V(\oplus\chi_j)) = f(\oplus V\chi_j) = \prod f(V\chi_j) = f(V)^{\dim W}.$$

□

Let

$$\mathrm{rec}_F : \mathrm{Irr}(\mathrm{GL}(n, F)) \longrightarrow \mathrm{WDRep}_n(W_F)$$

denote the local Langlands correspondence as in [16, p.2] and let ω_1, ω_2 be in the discrete series of $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$. We keep the notation of remarks 3.2, 3.4, and set $l_i := 2g_i + 1$, $i = 1, 2$. We have

$$\mathrm{rec}_F(\omega_1) = \rho_1 \otimes \mathrm{sp}(l_1),\ \mathrm{rec}_F(\omega_2) = \rho_2 \otimes \mathrm{sp}(l_2)$$

where $\mathrm{sp}(l) = (\rho, N)$ as in [16, p.251] is the l-dimensional Weil-Deligne representation on a complex vector space with basis $e_0, \ldots, e_{l-1}$ where

- $\rho(w) = ||w||^i e_i$ for all $w \in W_F$ and all $i = 0, \ldots, l-1$
- $Ne_i = e_{i+1}$, $Ne_{l-1} = 0$.

We note that $\mathrm{sp}(l)^\vee = \mathrm{sp}(l) \otimes |\ |^{1-l}$ by [16, p.252] and we will write

$$\mathrm{sp}(l_1) = (\rho_1, N_1),\ \mathrm{sp}(l_2) = (\rho_2, N_2).$$

We now use properties (2) and (5) in [16, p.2] of the local Langlands correspondence rec_F (*i.e.,* preservation of local L-functions and ϵ-factors for pairs, and compatibility with taking the contragredient). We have

$$\begin{aligned}\epsilon(\omega_1^\vee \times \omega_2, s, \Psi) &= \epsilon((\rho_1 \otimes \mathrm{sp}(l_1))^\vee \otimes \rho_2 \otimes \mathrm{sp}(l_2), s.\Psi)\\ &= \epsilon(\rho_1^\vee \otimes \mathrm{sp}(l_1) \otimes |\ |^{1-l_1} \otimes \rho_2 \otimes \mathrm{sp}(l_2), s, \Psi)\\ &= \epsilon(\rho', s, \Psi)\\ &= \epsilon(\rho, s, \Psi) \cdot \det(-\mathrm{Fr}|V^{I_F}/V_N^{I_F})\end{aligned} \tag{41}$$

by [31, 4.1.6] where $\rho' = (\rho, N) \in \mathrm{WDRep}_n(W_F)$. To determine ρ we use the crucial formula in Tate [31, 4.1.5] and we obtain

$$\rho = V \otimes \lambda$$

where $V = \rho_1^\vee \otimes \rho_2$ and λ is a direct sum of $l_1 l_2$ unramified quasicharacters of $F^\times$:

$$\lambda(x) = |x|^{1-l_1} \otimes \rho_1(x) \otimes \rho_2(x) = \sum_k |x|^k$$

with $k = i + j + 1 - l_1$, $0 \le i \le l_1 - 1$, $0 \le j \le l_2 - 1$, $x \in F^\times$.

The exponent $n(\ )$ is determined by the formula in Shahidi [29]:

$$q^{n(\omega_1^\vee \times \omega_2)} = \frac{\epsilon(0, \omega_1^\vee \times \omega_2, \Psi_F)}{\epsilon(1, \omega_1^\vee \times \omega_2, \Psi_F)}. \tag{42}$$

Theorem 5.3. *We have the following formula for the exponent n:*

$$q^{n(\omega_1^\vee \times \omega_2)} = f(\rho_1^\vee \otimes \rho_2)^{l_1 l_2}$$

where $\mathrm{rec}_F(\omega_1) = \rho_1 \otimes \mathrm{sp}(l_1)$, $\mathrm{rec}_F(\omega_2) = \rho_2 \otimes \mathrm{sp}(l_2)$ *and* $f(\rho_1^\vee \otimes \rho_2)$ *is the Artin conductor of the representation* $\rho_1^\vee \otimes \rho_2$. *If* ρ_1, ρ_2 *are both unramified then we have*

$$q^{n(\omega_1^\vee \times \omega_2)} = 1.$$

Proof. We have

$$q^{n(\omega_1^\vee \times \omega_2)} = \frac{\epsilon(0, \omega_1^\vee \times \omega_2, \Psi_F)}{\epsilon(1, \omega_1^\vee \times \omega_2, \Psi_F)} = \frac{\epsilon(\rho, 0, \Psi)}{\epsilon(\rho, 1, \Psi)} = f(\rho) = f(V)^{l_1 l_2}$$

by Lemma 5.2. □

We observe that when π_1, π_2 are supercuspidal, we have

$$q^{n(\pi_1^\vee \times \pi_2)} = f(\rho_1^\vee \otimes \rho_2)$$

where $\mathrm{rec}_F(\omega_1) = \pi_1$, $\mathrm{rec}_F(\pi_2) = \rho_2$.

We also note, since l_1, l_2 are the segment-lengths of Δ_1, Δ_2 respectively, the first equality in the statement of Theorem 5.3, can be rewritten as

$$\frac{n(\omega_1^\vee \times \omega_2)}{n_1 n_2} = \frac{a(\rho_1^\vee \otimes \rho_2)}{\deg \rho_1 \, \deg \rho_2}, \tag{43}$$

for each discrete series representations ω_1, ω_2 of $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$ such that $\mathrm{rec}_F(\omega_1) = \rho_1 \otimes \mathrm{sp}(l_1)$, $\mathrm{rec}_F(\omega_2) = \rho_2 \otimes \mathrm{sp}(l_2)$.

Let $\mathrm{sw}(\rho)$ denote the Swan conductor of ρ, and define $\widetilde{\mathrm{sw}}$ by $\widetilde{\mathrm{sw}}(\rho) = \mathrm{sw}(\rho)$ if ρ has no unramified indecomposable component and $\widetilde{\mathrm{sw}}(\rho) = -\infty$ otherwise. Heiermann has proved in [17] that the map

$$(\rho_1, \rho_2) \mapsto q^{\frac{\widetilde{\mathrm{sw}}(\rho_1^\vee \otimes \rho_2)}{\deg \rho_1 \, \deg \rho_2}}$$

defines, by passing to the quotient, an ultrametric distance on the set of equivalence classes of irreducible representations of W_F, modulo unramified twist. This implies, thanks to the local Langlands correspondence, that one defines in the same way a distance on the set of essentially square integrable irreductible representations of the group G, modulo unramified twist. Note that if there is no unramified character χ such that $\rho_1 \simeq \chi \rho_2$, then

$$a(\rho_1^\vee \otimes \rho_2) - \deg(\rho_1) \deg(\rho_2) = \mathrm{sw}(\rho_1^\vee \otimes \rho_2).$$

6. Calculation of the γ factors

The longest element in the Weyl group of $\mathrm{GL}(n_1, \dots, n_k)$ is

$$w_{\mathrm{GL}(n_1,\dots,n_k)} = \begin{pmatrix} w_{\mathrm{GL}(n_1)} & & & \\ & w_{\mathrm{GL}(n_2)} & & \\ & & \ddots & \\ & & & w_{\mathrm{GL}(n_k)} \end{pmatrix}, \tag{44}$$

where

$$w_{\mathrm{GL}(n)} = \begin{pmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \tag{45}$$

denote the longest element in the Weyl group of $\mathrm{GL}(n_l)$. We put $w := w_{n_1,n_2,\dots,n_k} := w_{\mathrm{GL}(n_1,\dots,n_k)} w_{\mathrm{GL}(n)}$. We observe that, the element w_{n_1,n_2} (for $k = 2$) is the same as the one defined in [29]. We put $q_w := q^{\ell(w)}$, where $\ell(w)$ denotes the length of w.

We put $M_w := w^{-1}Mw$ and $P_w := M_w \cdot (w^{-1}\overline{N}w)$. Note that P_w is a standard parabolic subgroup of G and that $M_w = \mathrm{GL}(n_k, n_{k-1}, \cdots, n_1)$. It follows from the definition (3) of the γ factor, that

$$\gamma(G/M_w) = \int_{w^{-1}Nw} \delta_P(m_P(\dot{w}^{-1}n\dot{w}))\, d(w^{-1}nw), \tag{46}$$

where $\dot{w}$ denotes any representative of w chosen in $\mathrm{GL}(n, \mathfrak{o}_F)$. The above expression has been computed by Reeder in [26, proof of Proposition 11.1]. Using his result, we get

$$\gamma(G/M_w) = q_w^{-1}\, \zeta(\delta_B^{1/2}), \tag{47}$$

where

$$\zeta := \prod_{\alpha_{i,j} \in \Sigma(G)^+ - \Sigma(M)^+} \frac{q - q\, c_{\alpha_{i,j}}}{1 - q\, c_{\alpha_{i,j}}}, \tag{48}$$

with c_α is the usual c-function (see [13]). We have $c_{\alpha_{i,j}}(\delta_B^{\frac{1}{2}}) = q^{i-j}$. Since $w_{\mathrm{GL}(n)} = \begin{pmatrix} & & w_{\mathrm{GL}(n_1)} \\ & \cdot & \\ \cdot & & \\ w_{\mathrm{GL}(n_k)} & & \end{pmatrix}$, we have

$$w_{\mathrm{GL}(n_1,\dots,n_k)} = \begin{pmatrix} & & & \mathrm{Id}_{n_1} \\ & & \cdot & \\ & \cdot & & \\ & \mathrm{Id}_{n_{k-1}} & & \\ \mathrm{Id}_{n_k} & & & \end{pmatrix}. \tag{49}$$

Setting

$$\tilde{w}_{n_h, n_{h+1}+\cdots+n_k} := \begin{pmatrix} w_{n_h, n_{h+1}+\cdots+n_k} & 0 \\ 0 & \mathrm{Id}_{n_1+\cdots+n_{h-1}} \end{pmatrix},$$

we get

$$w = w_{n_1, n_2+\cdots+n_k} \cdot \tilde{w}_{n_2, n_3+\cdots+n_k} \cdots \tilde{w}_{n_h, n_{h+1}+\cdots+n_k} \cdots \tilde{w}_{n_{k-1}, n_k}. \tag{50}$$

Using (50), we have

$$\begin{aligned} \ell(w) = & \sum_{i=1}^{k} \ell(w_{n_i, n_{i+1}+\cdots+n_k}) = \sum_{i=1}^{k-1} n_i(n_{i+1} + \ldots + n_k) \\ = & \sum_{\substack{i,j \in \{1,\dots,k\}, \\ i<j}} n_i n_j = |\Sigma(G)^+ - \Sigma(M)^+|. \end{aligned} \tag{51}$$

Note that if $n_1 = n_2 = \ldots = n_k = m$ then we obtain $\ell(w_{m,m,\ldots m}) = \frac{m^2 k(k-1)}{2}$.

It then follows from (46) that

$$\gamma(G/M_w) = \prod_{\alpha_{i,j} \in \Sigma(G)^+ - \Sigma(M)^+} \frac{1 - q^{i-j-1}}{1 - q^{i-j}},$$

that is,

$$\gamma(G/M_w) = \prod_{\substack{i<j \\ 1 \le h \le n_i, 1 \le l \le n_j}} \frac{1 - q^{h-l-(n_i+\cdots+n_{j-1})-1}}{1 - q^{h-l-(n_i+\cdots+n_{j-1})}}. \tag{52}$$

The Poincaré polynomial P_W of a Coxeter group W is defined as

$$P_W(t) := \sum_{w \in W_L} t^{\ell(w)},$$

where $\ell(w)$ is the lenght of w. Using [22, Corollary 2.5], we have

$$P_{W_L}(t) := \prod_{\alpha \in \Sigma(L)^+} \frac{1 - t^{1+\mathrm{ht}(\alpha)}}{1 - t^{\mathrm{ht}(\alpha)}}, \tag{53}$$

where $\mathrm{ht}(\alpha)$ is the *height* of α. Since the height of $\alpha_{i,j}$ is $j - i$ (for $i < j$), we obtain

$$P_{W_L}(q^{-1}) = \prod_{\alpha_{i,j} \in \Sigma(L)^+} \frac{1 - q^{i-j-1}}{1 - q^{i-j}}. \tag{54}$$

From (52), we obtain

(55)

$$\gamma(G/M) = \gamma(G/M_w) = \frac{P_{S_n}(q^{-1})}{P_{S_{n_1} \times \cdots \times S_{n_k}}(q^{-1})} = \frac{P_{S_n}(q^{-1})}{P_{S_{n_1}}(q^{-1}) \times \cdots \times P_{S_{n_k}}(q^{-1})}.$$

It gives the following expression for the c-function defined in (4):

$$c(G/M) = \frac{\displaystyle\prod_{1 \le i < j \le k} P_{S_{n_i+n_j}}(q^{-1})}{P_{S_n}(q^{-1}) \cdot \displaystyle\prod_{i=1}^{k} (P_{S_{n_i}}(q^{-1}))^{k-2}}. \tag{56}$$

Since P_W is also the Poincaré polynomial $P(G/B)$ of the complex manifold G/B with cellular structure given by the Bruhat decomposition (the cell BwB has dimension $q^{\ell(w)}$), and using the fibration $P/B \to G/B \to G/P$, we get $P_W = P_{W_M} P(G/P)$, where $P(G/P)$ denotes the Poincaré polynomial of the flag manifold G/P. It follows that $\gamma(G/M)$ is equal to $P(G/P)(q^{-1})$, as noticed in [26, §11].

The Poincaré polynomial P_W of any finite Coxeter group W admits the following expression (see [22][(2.6)]):

$$P_W(q) = \prod_{j=1}^{l} \frac{1 - q^{m_j+1}}{1-q},$$

where $m_1, \ldots, m_l$ are the *exponents* of W (as defined in [5][p. 118]). For $W = S_n$, the exponents are 1, 2, ..., $n-1$, that is, $l = n-1$ and $m_j = j$. Hence we get:

$$P_{S_n}(q) = \prod_{j=1}^{n-1} \frac{1-q^{j+1}}{1-q} = \prod_{j=1}^{n-1} (1 + q + q^2 + \cdots + q^j). \tag{57}$$

Using the formula (55), we obtain

$$\gamma(G/M) = \frac{\prod_{j=1}^{n-1} (1 + q^{-1} + q^{-2} + \cdots + q^{-j})}{\prod_{i=1}^{k} \prod_{j_i=1}^{n_i-1} (1 + q^{-1} + q^{-2} + \cdots + q^{-j_i})}. \tag{58}$$

Example: Take $G = \mathrm{GL}(4)$ and $M = \mathrm{GL}(2) \times \mathrm{GL}(2)$ (that is, $n = 4$, $k = 2$, $n_1 = n_2 = 2$). Then

$$\gamma(G/M) = \frac{(1 + q^{-1} + q^{-2})(1 + q^{-1} + q^{-2} + q^{-3})}{1 + q^{-1}}.$$

When the parabolic P is maximal, we obtain from (55), the following result.

Theorem 6.1. *We have*

$$\gamma(\mathrm{GL}(n)/\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)) = \frac{\prod_{i=n_1}^{n-1} (1 + q^{-1} + q^{-2} + \cdots + q^{-i})}{\prod_{j=1}^{n_2-1} (1 + q^{-1} + q^{-2} + \cdots + q^{-j})}.$$

7. Local γ-factor

Let the local γ-factor be as defined in [16, p.243]:

$$\gamma(r, s, \psi) = \frac{L(r^\vee, 1-s)\epsilon(r, s, \psi)}{L(r, s)}.$$

Then we have the following formula for the Plancherel density. This appears to be a new formula.

Theorem 7.1. *Let π_1, π_2 be irreducible supercuspidal representations of* $\mathrm{GL}(n_1)$, $\mathrm{GL}(n_2)$. *Then we have*

$$\mu(\pi_1 \otimes \pi_2) = \gamma(G/M)^2 \cdot \frac{\gamma(\rho^\vee, 0, \psi)}{\gamma(\rho^\vee, 1, \psi)}$$

where

$$\rho = \mathrm{rec}_F(\pi_1) \otimes \mathrm{rec}_F(\pi_2)^\vee.$$

Proof. This proof depends on a brief computation, starting with the Langlands-Shahidi formula. We have

$$\begin{aligned}
\mu(\pi_1 \otimes \pi_2) &= \gamma(G/M)^2 \cdot \frac{\epsilon(0, \rho^\vee, \psi)}{\epsilon(1, \rho^\vee, \psi)} \cdot \frac{L(1,\rho)\, L(1, \rho^\vee)}{L(0,\rho)\, L(0,\rho^\vee)} \\
&= \gamma(G/M)^2 \cdot \frac{L(\rho,1)\, \epsilon(\rho^\vee, 0, \psi)}{L(\rho^\vee, 0)} \cdot \frac{L(\rho^\vee, 1)}{L(\rho, 0)\, \epsilon(1, \rho^\vee, \psi)} \\
&= \gamma(G/M)^2 \cdot \frac{\gamma(\rho^\vee, 0, \psi)}{\gamma(\rho^\vee, 1, \psi)}.
\end{aligned}$$

□

8. The Bernstein decomposition of Plancherel measure

Let X be a space on which the finite group Γ acts. The extended quotient associated to this action is the quotient space $\tilde{X}/\Gamma$ where

$$\tilde{X} = \{(\gamma, x) \in \Gamma \times X : \gamma x = x\}.$$

The group action on $\tilde{X}$ is $g.(\gamma, x) = (g\gamma g^{-1}, gx)$. Let $X^\gamma = \{x \in X : \gamma x = x\}$ and let $Z(\gamma)$ be the Γ-centralizer of γ. Then the extended quotient is given by:

$$\tilde{X}/\Gamma = \bigsqcup_\gamma X^\gamma / Z(\gamma)$$

where one γ is chosen in each Γ-conjugacy class. If $\gamma = 1$ then $X^\gamma/Z(\gamma) = X/\Gamma$ so *the extended quotient always contains the ordinary quotient*:

$$\tilde{X}/\Gamma = X/\Gamma \sqcup \ldots$$

We shall need only the special case in which X is the compact torus $\mathbb{T}^n$ of dimension n and Γ is the symmetric group S_n acting on $\mathbb{T}^n$ by permuting co-ordinates.

Let α be a partition of n, and let γ have cycle type α. Each cycle provides us with one circle, and cycles of equal length provide us with a symmetric product of circles. For example, the extended quotient

$\widetilde{\mathbb{T}^5}/S_5$ is the following disjoint union of compact orbifolds (one for each partition of 5):

$$\mathbb{T} \sqcup \mathbb{T}^2 \sqcup \mathbb{T}^2 \sqcup (\mathbb{T} \times \mathrm{Sym}^2\mathbb{T}) \sqcup (\mathbb{T} \times \mathrm{Sym}^2\mathbb{T}) \sqcup (\mathbb{T} \times \mathrm{Sym}^3\mathbb{T}) \sqcup \mathrm{Sym}^5\mathbb{T}$$

where $\mathrm{Sym}^n\mathbb{T}$ is the n-fold symmetric product of the circle $\mathbb{T}$. This extended quotient is a model of the arithmetically unramified tempered dual of GL(5).

Let $\Omega \subset \Omega\mathrm{GL}(n)$ have one exponent e. Then we have $e|n$ and so $em = n$.

There exists an irreducible unitary supercuspidal representation σ of $\mathrm{GL}(m)$ such that the conjugacy class of the cuspidal pair $(\mathrm{GL}(m) \times \cdots \times \mathrm{GL}(m), \sigma \otimes \cdots \otimes \sigma)$ is an element in Ω. We have $\Omega \cong \mathrm{Sym}^e\,\mathbb{C}^\times$ as complex affine algebraic varieties. Consider now a partition of e into k parts:

$$l_1 + \cdots + l_k = e$$

and write $2g_1 + 1 = l_1, \ldots, 2g_k + 1 = l_k$. Let

$$\pi_i = Q(\Delta_i),\ \Delta_i = \{|\,|^{-g_i}\sigma, \ldots, |\,|^{g_i}\sigma\}$$

as in Theorem 3.2. Then $\pi_1 \in \mathcal{E}_2(\mathrm{GL}(ml_1)), \ldots, \pi_k \in \mathcal{E}_2(\mathrm{GL}(ml_k))$. Note that $ml_1 + \ldots + ml_k = n$ so that $\mathrm{GL}(ml_1) \times \ldots \times \mathrm{GL}(ml_k)$ is a standard Levi subgroup M of $\mathrm{GL}(n)$. Now consider

$$\pi = \chi_1\pi_1 \otimes \ldots \otimes \chi_k\pi_k$$

with $\chi_1, \ldots, \chi_k$ unramified (unitary) characters. Then $\pi \in \mathcal{E}_2(M)$. We have

$$\omega = \mathrm{Ind}_{MN}^{\mathrm{GL}(n)}(\pi \otimes 1) \in \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)$$

and each element $\omega \in \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)$ for which $inf.ch.\omega \in \Omega$ is accounted for on this way. As explained in detail in [24], we have

$$\widetilde{X}/\Gamma \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega$$

where $X = \mathbb{T}^e$, $\Gamma = S_e$, *i.e.*,

$$\bigsqcup_\gamma X^\gamma/Z(\gamma) \cong \mathrm{Irr}^{\mathrm{t}}\,\mathrm{GL}(n)_\Omega.$$

The partition $l_1 + \cdots + l_k = e$ determines a permutation γ of the set $\{1, 2, \ldots, e\}$: γ is the product of the cycles $(1, \ldots, l_1)\cdots(1, \ldots, l_k)$. Then the fixed set X^γ is

$$\{(z_1, \ldots, z_1, \ldots, z_k, \ldots, z_k) \in \mathbb{T}^e : z_1, \ldots, z_k \in \mathbb{T}\}$$

and so $X^\gamma \cong \mathbb{T}^k$.

Explicitly, we have

$$X^\gamma \longrightarrow \operatorname{Irr}^{\mathrm{t}} \mathrm{GL}(n)_\Omega$$

$$(z_1, \dots, z_k) \mapsto \operatorname{Ind}_{MN}^{\mathrm{GL}(n)}(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k)$$

with $\chi_1(\varpi) = \alpha_1, \dots, \chi_k(\varpi) = \alpha_k, z_1 = \alpha_1^r, \dots, z_k = \alpha_k^r$ exactly as in Theorem 3.2. This map is constant on each $Z(\gamma)$-orbit and descends to an *injective* map

$$X^\gamma/Z(\gamma) \to \operatorname{Irr}^{\mathrm{t}} \mathrm{GL}(n)_\Omega.$$

Taking one γ in each Γ-conjugacy class we have the bijective map

$$\bigsqcup_\gamma X^\gamma/Z(\gamma) \cong \operatorname{Irr} \mathrm{GL}(n)_\Omega.$$

This bijection, by transport of structure, equips $\operatorname{Irr}^{\mathrm{t}} \mathrm{GL}(n)_\Omega$ with the structure of disjoint union of finitely many compact orbifolds.

We now describe the restriction μ_Ω of Plancherel density to the compact orbifold $X^\gamma/Z(\gamma)$.

Theorem 8.1. *Let σ be an irreducible pre-unitary supercuspidal representation of $\mathrm{GL}(m)$ with torsion number r. For $i = 1, \dots, k$, let π_i be the Langlands quotient associated to the segment*

$$\{|\ |^{-g_i}\sigma, \dots, |\ |^{g_i}\sigma\},$$

let χ_i be an unramified character with $\chi_i(\varpi) = \alpha_i$, and let $z_i = \alpha_i^r$.

Then, as a function on the compact torus $\mathbb{T}^k$ with co-ordinates $(z_1, \dots, z_k)$ we have

$$\mu(\chi_1\pi_1 \otimes \cdots \otimes \chi_k\pi_k) = const./\prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2,$$

where the product is taken over those i, j, g for which the following inequalities hold: $1 \le i < j \le k$, $|g_i - g_j| \le g \le g_i + g_j$, $2g_i + 1 = l_i$.

Proof. Apply Theorem 4.1 and the Harish-Chandra product formula, Theorem 2.3. Note that the function

$$(z_1, \dots, z_k) \mapsto const./\prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2$$

is a $Z(\gamma)$-invariant function on the γ-fixed set $X^\gamma = \mathbb{T}^k$, and descends to a non-negative function on the orbifold $X^\gamma/Z(\gamma)$:

$$X^\gamma/Z(\gamma) \longrightarrow \mathbb{R}_+.$$

□

Theorem 8.2. *We have the following numerical formula for const.*

$$const. = q^{\sum_{1\le i<j\le k} l_i l_j a(\rho^\vee \otimes \rho)} \cdot \gamma(G/M)^2 \cdot c(G/M)^2,$$

where $\operatorname{rec}_F(\rho) = \sigma$.

Proof. The numerical constant is determined by Theorem 4.1, Theorem 5.3, and Theorem 2.3. Explicitly, for $i, j \in \{1, \dots, k\}$, setting

$$\gamma_{i,j} := \gamma(\mathrm{GL}(n_i + n_j)/\mathrm{GL}(n_i) \times \mathrm{GL}(n_j)),$$

for the γ-factor of the Levi subgroup $GL(n_i) \times \mathrm{GL}(n_j)$ of the maximal standard parabolic subgroup in $\mathrm{GL}(n_i + n_j)$,

$$\begin{aligned} const. = & \quad q^{\sum_{1 \le i < j \le k} n(\omega_i^\vee \times \omega_j)} \cdot \prod_{1 \le i < j \le k} \gamma_{i,j}^2 \\ = & \quad q^{\sum_{1 \le i < j \le k} l_i l_j a(\rho^\vee \otimes \rho)} \cdot \gamma(G/M)^2 \cdot c(G/M)^2. \end{aligned}$$

□

The following result is an immediate consequence of Theorems 8.1, 8.2 and Remark (2.4).

Corollary 8.3. *We have*

$$j(\omega) = q^{-\sum_{1 \le i < j \le k} l_i l_j a(\rho^\vee \times \rho)} \cdot \prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2.$$

Given $G = \mathrm{GL}(n) = \mathrm{GL}(n, F)$ choose $e|n$ and let $m = n/e$. Let Ω be a Bernstein component in $\Omega(\mathrm{GL}(n))$ with one exponent e. The compact extended quotient attached to Ω has finitely many components, each component is a compact orbifold. We now have enough results to write down explicitly the component μ_Ω. Let $l_1 + \cdots + l_k = e$ be a partition of e, let $\gamma = (1, \dots, l_1) \cdots (1, \dots, l_k) \in S_e = \Gamma$, $g_1 = (l_1 - 1)/2, \dots, g_k = (l_k - 1)/2$. Then we have the fixed set $X^\gamma = \mathbb{T}^k$.

Let σ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$ and let the conjugacy class of the cuspidal pair $(\mathrm{GL}(m)^e, \sigma^{\otimes e})$ be a point in the Bernstein component Ω. Let $r = \mathrm{Tor}(\sigma)$ (the torsion number of σ) and choose a field K such that $q_K = q_F^r$. Let $\rho \in \mathrm{Irr}(W_F)$ such that $\mathrm{rec}_F(\rho) = \sigma$.

We have [24]

$$\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n, F)_\Omega \cong \widetilde{X}/\Gamma.$$

This compact Hausdorff space admits the Harish-Chandra *canonical measure* $d\omega$: on each connected component in the extended quotient $\widetilde{X}/\Gamma$, $d\omega$ restricts to the quotient by the centralizer $Z(\gamma)$ of the normalized Haar measure on the compact torus X^γ.

Let $d\nu$ denote Plancherel measure on the tempered dual of $\mathrm{GL}(n, F)$.

Theorem 8.4. *On each connected component of the extended quotient of* $\widetilde{X}/\Gamma$ *we have:*

$$d\nu(\omega) = q^{\sum_{1 \le i < j \le k} l_i l_j a(\rho^\vee \otimes \rho)} \cdot \gamma(G/M) \cdot \frac{d(\omega)}{\prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2} \cdot d\omega.$$

Proof. By (13), (14), the Plancherel measure on $\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n,F)_\Omega$ is given by

$$d\nu(\omega) = c(G/M)^{-2}\,\gamma(G/M)^{-1}\,\mu(\omega)\,d(\omega)\,d\omega$$

Then, the result follows from Theorem 8.1 and Theorem 8.2. □

We now pass to the general case of a component $\Omega \in \Omega\mathrm{GL}(n)$ with exponents $e_1, \ldots, e_t$. We first note that each component $\Omega \subset \Omega\mathrm{GL}(n)$ creates several natural-number invariants:

1. m_j, the sizes of the small general linear groups,
2. e_j, the exponents,
3. r_j, the torsion numbers,
4. a_j, the Artin conductors $a(\rho_j^\vee \otimes \rho_j)$,

with $1 \le j \le t$.

The Plancherel *density* μ_Ω depends on these invariants, and on no others. In other words, we have:

Theorem 8.5. *If Ω^1 and Ω^2 share the same invariants (1) – (4) then*

$$\mu_{\Omega^1} = \mu_{\Omega^2}.$$

In order to express our next result in a conceptual way, we will construct the disjoint union

$$E = \Omega\mathrm{GL}(\infty) = \{\bigsqcup \Omega\mathrm{GL}(n) : n = 0, 1, 2, 3, \ldots\}$$

with the convention that $\Omega\mathrm{GL}(0) = \mathbb{C}$.

We will say that two components $\Omega_1, \Omega_2 \in E$ are *disjoint* if the following is true: none of the irreducible unitary supercuspidals which occur in Ω_1 is equivalent (after unramified twist) to any of the supercuspidals which occur in Ω_2.

We now define a law of composition on *disjoint components* in E. With the cuspidal pair $(M_1, \sigma_1) \in \Omega_1$ and the cuspidal pair $(M_2, \sigma_2) \in \Omega_2$ we define $\Omega_1 \times \Omega_2$ as the unique component determined by

$$(M_1 \times M_2, \sigma_1 \otimes \sigma_2).$$

In the terminology of Bourbaki [4, 1.1] the set E admits a law of composition *not everywhere defined* on E. In Bourbaki terminology [4, 1.1 – 1.2], E is a commutative associative unital magma in which all elements are cancellable. In geology, the term *magma* refers to molten rock beneath the earth's crust: in mathematics, it refers, by elegant transfer of meaning, to a set with minimal structure. Rather surprisingly, this magma admits prime elements: the prime elements are precisely the components with a single exponent. Each element in E admits a unique factorization into prime elements:

$$\Omega = \Omega_1 \times \cdots \times \Omega_t.$$

Plancherel measure respects the unique factorization into prime elements, modulo constants. Quite specifically, we have

Theorem 8.6. *Let Ω have the unique factorization*

$$\Omega = \Omega_1 \times \cdots \times \Omega_t$$

so that Ω has exponents $e_1, \ldots, e_t$ and $\Omega_1, \ldots, \Omega_t$ are pairwise disjoint prime elements with the individual exponents $e_1, \ldots, e_t$. Let

$$\nu = \sum \nu_\Omega$$

denote the Bernstein decomposition of Plancherel measure. Then we have

$$\nu_\Omega = const.\, \nu_{\Omega_1} \cdots \nu_{\Omega_t}$$

where $\nu_{\Omega_1}, \ldots, \nu_{\Omega_t}$ are given by Theorem 8.1 and the constant is given by Theorem 8.2. Plancherel measure respects the law of composition in E, modulo computable constants.

Proof. In the Harish-Chandra product formula, all the cross-terms are constant, by Shahidi [28, Prop. 7.1, 1st assertion]. This depends precisely on the following fact: all the L-functions which occur in cross-terms are equal to 1:

$$L(s, \omega_1 \times \omega_2^\vee) = 1 = L(s, \omega_1^\vee \times \omega_2).$$

□

Our theorem that Plancherel measure respects the law of composition in E, modulo computable constants, blends the Harish-Chandra product formula with the Bernstein decomposition. This would appear to be the conceptual version of the Harish-Chandra product formula.

It is tempting to conjecture that a version of Theorem 8.6 will hold for reductive p-adic groups in general, certainly the classical groups: but we do not at present have a precise formulation of this conjecture.

9. Transfer-of-measure formula

The beginning of this section is completely general: here G denotes the group of F-rational points of some connected reductive algebraic group defined over F. We fix a Levi subgroup M of G and a pair $(P, \overline{P})$ of mutually opposite parabolic subgroups of G with Levi component

M. We write N, $\overline{N}$, respectively, for their unipotents radicals and fix Haar measures $v(N)$, $v(\overline{N})$, respectively.

Let $\mathrm{r}_{M,P}^G$ denote the normalized Jacquet restriction functor, which is the left adjoint functor of $\mathrm{i}_{M,P}^G$, *i.e.,* given smooth representations (π, V), (π_M, V_M) of G and M respectively, we have an isomorphism

$$\mathrm{Hom}_G(V, \mathrm{i}_{M,P}^G(V_M)) \simeq \mathrm{Hom}_M(\mathrm{r}_{M,P}^G(V), V_M), \tag{59}$$

which is natural in both V and V_M (see [3]).

Let Ω be a Bernstein component in G which admits an Ω-type (J_G, λ_G) in the terminology of [11, Definition 4.1], *i.e.,* the full subcategory of the category $\mathfrak{R}(G)$ of smooth representations of G whose objects are the representations which are generated by their λ_G-isotypic components is the subcategory $\mathfrak{R}(G)_\Omega$ defined by Ω. In particular J_G is a compact open subgroup of G and (λ_G, U) is an irreducible smooth representation of J_G. Let $\mathcal{H}(G, \lambda_G)$ denote the convolution algebra of $\mathrm{End}_{\mathbb{C}}(U^\vee)$-valued functions f on G which are compactly supported and which satisfy

$$f(j_1 g j_2) = \lambda_G^\vee(j_1) f(g) \lambda_G^\vee(j_2), \quad j_1, j_2 \in J_G,\ g \in G.$$

Let Ω_M be a Bernstein component in M which admits an Ω_M-type (J_M, λ_M). We assume that (J_G, λ_G) is a G-cover of (J_M, λ_M), as defined in [11, Definition 8.1]. It implies, [11, Corollary 8.4], that there is a canonical algebra homomorphism

$$t_P \colon \mathcal{H}(M, \lambda_M) \to \mathcal{H}(G, \lambda_G)$$

which *realizes* the parabolic induction and the Jacquet restriction functors $\mathrm{i}_{M,P}^G$, $\mathrm{r}_{M,P}^G$, *i.e.,* such that the diagrams

$$\begin{array}{ccc} \mathfrak{R}(M)_{\Omega_M} & \xrightarrow{\sim} & \mathcal{H}(M, \lambda_M)-\mathrm{Mod} \\ {\scriptstyle \mathrm{i}_{M,P}^G}\big\downarrow & & \big\downarrow{\scriptstyle (t_P)_*} \\ \mathfrak{R}(G)_\Omega & \underset{\sim}{\longrightarrow} & \mathcal{H}(G, \lambda_G)-\mathrm{Mod} \end{array} \tag{60}$$

and

$$\begin{array}{ccc} \mathfrak{R}(G)_\Omega & \xrightarrow{\sim} & \mathcal{H}(G, \lambda_G)-\mathrm{Mod} \\ {\scriptstyle \mathrm{r}_{M,P}^G}\big\downarrow & & \big\downarrow{\scriptstyle t_P^*} \\ \mathfrak{R}(M)_{\Omega_M} & \underset{\sim}{\longrightarrow} & \mathcal{H}(M, \lambda_M)-\mathrm{Mod} \end{array} \tag{61}$$

commute, where t_P^* denotes the restriction along t_P and $(t_P)_*$ denote the adjoint of t_P^*.

We will consider the following situation. Let K be a nonarchimedean field and let G_0 be the group of K-rational points of some connected reductive algebraic group defined over K. As for G, we fix a Levi subgroup M_0 of G_0 and a pair $(P_0, \overline{P}_0)$ of mutually opposite parabolic subgroups of G with Levi component M_0. We write N_0, $\overline{N}_0$, respectively, for their unipotents radicals and fix Haar measures $v(N_0)$, $v(\overline{N}_0)$, respectively. Let Ω_{M_0}, Ω_0 be two Bernstein components in M_0, G_0 respectively, such that Ω_{M_0} admits an Ω_{M_0}-type (J_{M_0}, λ_{M_0}) in M_0 and let (J_0, λ_0) be a G_0-cover of (J_{M_0}, λ_{M_0}). We will be especially interested in the case where J_{M_0}, J_0 are Iwahori subgroups and λ_{M_0}, λ_0 are the trivial representation.

We assume that the following assumption holds and proceed as in [8, §4-5].

Transfer assumption 9.1. *There exist two algebra isomorphisms*

$$\Phi_M\colon \mathcal{H}(M_0, \lambda_{M_0}) \to \mathcal{H}(M, \lambda_M) \quad \textit{and} \quad \Phi\colon \mathcal{H}(G_0, \lambda_0) \to \mathcal{H}(G, \lambda_G),$$

which preserve support of functions (in the following sense: for example, take $f \in \mathcal{H}(G_0, \lambda_0)$, then the function $\Phi(f)$ then has support $J_G \cdot \mathrm{supp}(f) \cdot J_G$), such that the diagram

$$\begin{array}{ccc} \mathcal{H}(M_0, \lambda_{M_0}) & \xrightarrow{\Phi_M} & \mathcal{H}(M, \lambda_M) \\ {\scriptstyle t_{P_0}}\big\downarrow & & \big\downarrow{\scriptstyle t_P} \\ \mathcal{H}(G_0, \lambda_0) & \xrightarrow[\Phi]{} & \mathcal{H}(G, \lambda_G) \end{array} \tag{62}$$

commutes.

The above commutative diagram joint with the commutative diagrams (60) and (61) give natural isomorphisms

$$\mathrm{i}_{M_0,P_0}^{G_0}(\Phi_M^*(\pi_M)) \simeq \Phi^*(\mathrm{i}_{M,P}^{G}(\pi_M)) \quad \text{and} \quad \mathrm{r}_{M_0,P_0}^{G_0}(\Phi^*(\pi)) \simeq \Phi_M^*(\mathrm{r}_{M,P}^{G}(\pi)),$$

for any $\pi_M \in \mathfrak{R}(M)_{\Omega_M}$ and any $\pi \in \mathfrak{R}(G)_{\Omega}$, where Φ^* denote the equivalence $\mathfrak{R}(G)_\Omega \simeq \mathfrak{R}(G_0)_{\Omega_0}$ induced by Φ.

On the other hand, by [2] or [6], the functor $\mathrm{r}_{M,P}^G$ has a co-adjoint, namely the induction functor $\mathrm{i}_{M,\overline{\mathrm{P}}}^G$. A co-adjoint is uniquely determined up to natural equivalence, so the second isomorphism above induces a natural isomorphism

$$\mathrm{i}_{M,\overline{\mathrm{P}}}^G(\Phi_M^*(\pi_M)) \simeq \Phi^*(\mathrm{i}_{M,\overline{\mathrm{P}}}^G(\pi_M)). \tag{63}$$

Lemma 9.2. *The diagram*

$$\begin{array}{ccc} \mathcal{H}(M_0, \lambda_{M_0}) & \xrightarrow{\Phi_M} & \mathcal{H}(M, \lambda_M) \\ {\scriptstyle t_{\overline{\mathrm{P}}_0}}\Big\downarrow & & \Big\downarrow{\scriptstyle t_{\overline{\mathrm{P}}}} \\ \mathcal{H}(G_0, \lambda_0) & \xrightarrow[\Phi]{} & \mathcal{H}(G, \lambda_G) \end{array}$$

commutes.

Proof. The algebra $\mathcal{H}(M, \lambda_M)$ has a sub-algebra $\mathcal{H}^-(M, \lambda_M)$ of functions supported on elements m of M which are $J_G \cap \overline{\mathrm{N}}$-*positive*, in the sense of [11, Definition 6.5], *i.e.,* which satisfy

$$m(J_G \cap \overline{\mathrm{N}})m^{-1} \subset J_G \cap \overline{\mathrm{N}} \quad \text{and} \quad m^{-1}(J_G \cap N)m \subset J_G \cap N.$$

For $f \in \mathcal{H}^-(M, \lambda_M)$, the map $t_{\overline{\mathrm{P}}}(f)$ has support $J_G \cdot \mathrm{supp}(f) \cdot J_G$. Thus, if we take $f \in \mathcal{H}^-(M_0, \lambda_{M_0})$, the functions $(t_P \circ \Phi_M)(f)$ and $(\Phi \circ t_{P_0})(f)$ both have support $J_G \cdot \mathrm{supp}(f) \cdot J_G$. A (J_G, J_G)-double coset in G supports at most a one-dimensional space of functions in $\mathcal{H}(G, \lambda_G)$ (see [10, 4.1.2]) and the three other algebras in the picture share also this property. Hence, if a function f is supported on, say $m(M_0 \cap J_0)$, we have a non-zero scalar $\phi(m)$ such that

$$(t_P \circ \Phi_M)(f) = \phi(m)(\Phi \circ t_{P_0})(f).$$

The quantity $\phi(m)$ depends only on $m(M_0 \cap J_0)$; by [11, Proposition 7.1], ϕ is multiplicative in the positive element m: in other words, ϕ extends uniquely to an unramified quasicharacter of M_0. Given an unramified quasicharacter χ of M_0 and $f \in \mathcal{H}^-(M_0, \lambda_{M_0})$, define $f\chi$ by $f\chi \colon m \mapsto f(m)\chi(m)$. We have just shown that

$$(t_P \circ \Phi_M)(f) = (\Phi \circ t_{P_0})(f\phi). \tag{64}$$

The relation (63) then forces $\phi = 1$. ☐

Let (π_M, V_M) denote a smooth representation of M. We have a natural map

$$\eta(\pi_M) \colon (\mathrm{r}^G_{M,\overline{\mathrm{P}}} \circ \mathrm{i}^G_{M,\overline{\mathrm{P}}})(V_M) \to V_M,$$

defined by sending $\mathrm{r}^G_{M,\overline{\mathrm{P}}}(f)$ to $f(1)$, for any $f \in \mathrm{i}^G_{M,\overline{\mathrm{P}}}(V_M)$. By taking contragredients, we obtain a map

$$\eta(\pi_M)^\vee \colon V_M^\vee \to ((\mathrm{r}^G_{M,\overline{\mathrm{P}}} \circ \mathrm{i}^G_{M,\overline{\mathrm{P}}})(V_M))^\vee.$$

By using the natural M-isomorphisms

$$\mathrm{r}^G_{M,\overline{\mathrm{P}}}(\pi^\vee, V^\vee) \simeq \mathrm{r}^G_{M,P}(\pi, V)^\vee,$$

([8, (3.3.2)] for admissible representations, [6, Theorem 5.1] or [2] for smooth representations), and

$$\mathrm{i}_{M,\overline{\mathrm{P}}}^{G}(\pi_M^\vee, V_M^\vee) \simeq \mathrm{i}_{M,\overline{\mathrm{P}}}^{G}(\pi_M, V_M^\vee)^\vee,$$

we get a map

$$\eta(\pi_M, v(N))^\vee \colon V_M^\vee \to (\mathrm{r}_{M,P}^G \circ \mathrm{i}_{M,\overline{\mathrm{P}}}^G)(V_M^\vee).$$

On the other hand, we choose a compact open subgroup K_N of N, and for $v \in V_M$, we define a function $f_{v,K_N} : G \to V_M$ as follows: the support of f_{v,K_N} is to be $\overline{\mathrm{P}}K_N$, and

$$f_{v,K_N}(\overline{p}k) = v(K_N)^{-1}\, \delta_{\overline{\mathrm{P}}}(p)^{\frac{1}{2}}\, \pi_M(p)\, v, \quad \overline{p} \in \overline{\mathrm{P}},\ k \in K_N.$$

Then $f_{v,K_N} \in \mathrm{i}_{M,\overline{\mathrm{P}}}^G(\pi_M)$. Let pr_N denote the canonical map

$$\mathrm{i}_{M,\overline{\mathrm{P}}}^G(V_M) \to (\mathrm{r}_{M,P}^G \circ \mathrm{i}_{M,\overline{\mathrm{P}}}^G)(V_M).$$

By [8, Proposition 1.2], the process $v \mapsto \mathrm{pr}_N(f_{v,K_N})$ gives an injective M-homorphism

$$\iota = \iota(\pi_M, v(N)) \colon \pi_M \to (\mathrm{r}_{M,P}^G \circ \mathrm{i}_{M,\overline{\mathrm{P}}}^G)(\pi_M),$$

which is independent of the choice of K_N, but depends on the choice of Haar measure on N in the following way:

$$\iota(\pi_M, c\, v(N)) = c^{-1}\iota(\pi_M, v(N)).$$

Theorem 9.3. *We have*

$$v(J_G \cap N) \cdot J_{P_0|\overline{\mathrm{P}}_0}(\Phi_M^*(\pi_M\chi)) = v(J_0 \cap N_0) \cdot \Phi^*(J_{P|\overline{\mathrm{P}}}(\pi_M\chi)),$$

$$v(J_G \cap \overline{\mathrm{N}}) \cdot J_{\overline{\mathrm{P}}_0|P_0}(\Phi_M^*(\pi_M\chi)) = v(J_0 \cap \overline{\mathrm{N}}_0) \cdot \Phi^*(J_{\overline{\mathrm{P}}|P}(\pi_M\chi)),$$

for any $\pi_M \in \mathrm{Irr}(M)$ *and for any* $\chi \in \Psi(M)$.

Proof. There is a non-empty Zariski-open set $\Psi(M, \pi_M)$ of $\Psi(M)$, on which $J_{P|\overline{\mathrm{P}}}(\pi_M)$ is defined and $\mathrm{i}_{M,P}^G(\chi\pi_M)$ and $\mathrm{i}_{M,\overline{\mathrm{P}}}^G(\chi\pi_M)$ are irreducible. Let $f^M \in \mathrm{Hom}_M((\mathrm{r}_{M,P}^G \circ \mathrm{i}_{M,\overline{\mathrm{P}}}^G)(\chi\pi_M), \chi\pi_M))$ denote the image by the isomorphism (59) of $f \in \mathrm{Hom}_G(\mathrm{i}_{M,\overline{\mathrm{P}}}^G(\chi\pi_M), \mathrm{i}_{M,P}^G(\chi\pi_M))$. We have the following characterization of the intertwining operator $J_{P|\overline{\mathrm{P}}}(\pi_M)$: if $\chi \in \Psi(M, \pi_M)$ and $f \in \mathrm{Hom}_G(\mathrm{i}_{M,\overline{\mathrm{P}}}^G(\chi\pi_M), \mathrm{i}_{M,P}^G(\chi\pi_M))$ satisfies

$$f^M \circ \iota(\pi_M, v(N)) = 1_{V_M},$$

then $f = J_{\overline{\mathrm{P}}|P}(\pi_M)$ (see [8, Proposition 1.3]).

This characterization of $J_{P|\overline{\mathrm{P}}}(\pi_M)$ and the analogous characterization for $J_{P_0|\overline{\mathrm{P}}_0}(\Phi_M^*(\pi_M\chi))$, joint with the fact that

$$\iota(\pi_M, v(N)) = \eta(\pi_M, v(N))^\vee$$

(see [8, Proposition 3.5]), imply, in the same way as in the proof of [8, Theorem 4.2], the first assertion of the Theorem. By Lemma (9.2), we can interchange the roles of P and $\overline{\mathrm{P}}$ without changing Φ. The second assertion follows. ☐

The following result is then an immediate consequence of the definition of the j-function. It extends [8, Theorem 4.2].

Corollary 9.4. *For any $\chi \in \Psi(M)$ and any $\pi_M \in \mathrm{Irr}(M)$, we have*

$$v(J_G \cap N)\, v(J_G \cap \overline{\mathrm{N}})\, j_0(\Phi_M^*(\pi_M\chi)) \;=\; v(J_0 \cap N_0)\, v(J_0 \cap \overline{\mathrm{N}}_0)\, j(\pi_M\chi),$$

where j_0 denote the j-function for the group G_0.

We keep the notation of the end of the previous section (right before Theorem 8.4). Then, the conjugacy class of γ determines a partition of e. This partition determines the standard Levi subgroup

$$M = \mathrm{GL}(ml_1) \times \cdots \times \mathrm{GL}(ml_k) \subset \mathrm{GL}(n, F) \tag{65}$$

and, at the same time, the standard Levi subgroup

$$M_0 = \mathrm{GL}(l_1) \times \cdots \times \mathrm{GL}(l_k) \subset \mathrm{GL}(e, K) = G_0. \tag{66}$$

Let T be the diagonal subgroup of G_0 and let Ω_0 be the Bernstein component in $\Omega(G_0)$ which contains the cuspidal pair $(T, 1)$. The component Ω_0 has (as Ω) the single exponent e and parametrizes those irreducible smooth representations of $\mathrm{GL}(e, K)$ which admit nonzero Iwahori fixed vectors. We have [24]

$$\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n, F)_\Omega \cong \widetilde{X}/\Gamma \cong \mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(e, K)_{\Omega_0}.$$

The theory of types of [10] produces a canonical extension K of F such that $q_K = q^r$. Indeed, let (J, λ) be a maximal simple type in $\mathrm{GL}(m)$ contained in σ, and let $\mathfrak{A}$ and $E = F[\beta]$ respectively denote the corresponding hereditary order in $A = M(m, F)$ and the corresponding field extension of F (see [10, Definition 5.5.10 (iii)]). It is proved in [10, Lemma 6.2.5] that

$$r = \frac{m}{e(E|F)}, \tag{67}$$

where $e(E|F)$ denotes the ramification index of E with respect to F. Let B denote the centraliser of E in A. We set $\mathfrak{B} := \mathfrak{A} \cap B$. Then $\mathfrak{B}$ is a maximal hereditary order in B, see [10, Theorem 6.2.1]. Let K be an unramified extension of E which normalises $\mathfrak{B}$ and is maximal with respect to that property, as in [10, Proposition 5.5.14]. Then $[K : F] = m$, and (67) gives that r is equal to the residue index $f(K|F)$ of K with respect to F. Thus $Q = q^r$ is equal to the order q_K of the residue field of K.

Also the number Q is the one which occurs for the Hecke algebra $\mathcal{H}(\mathrm{GL}(m),\lambda)$ associated to (J,λ), see [10, Theorem 5.6.6]. Indeed, since the order of the residue field of E is equal to $q^{f(E|F)}$, that number is $(q^{f(E|F)})^f$, with

$$f = \frac{m}{[E:F]\, e(\mathfrak{B})},$$

where $e(\mathfrak{B})$ denotes the period of a lattice chain attached to $\mathfrak{B}$ as in [10, (1.1)]. Since σ is supercuspidal, $e(\mathfrak{B}) = 1$, see [10, Corollary 6.2.3]). It follows that

$$f \,\cdot\, f(E|F) = \frac{m \,\cdot\, f(E|F)}{[E:F]} = \frac{m}{e(E|F)} = r.$$

For each $i \in \{1,\dots,k\}$, the pair $(J^{l_i},\lambda^{\otimes l_i})$ is a type in $\mathrm{GL}(m)^{l_i}$ which is contained in $\sigma^{\otimes l_i}$. By [10, (7.2.17)] and [12, prop. 1.4], it admits a $\mathrm{GL}(l_i m)$-cover (J_i,λ_i), which is contained in a simple type, say $(J_i^{\mathrm{s}},\lambda_i^{\mathrm{s}})$. The type (J_i,λ_i) is defined as follows. Let $\mathfrak{A}_i$ denote an hereditary $\mathfrak{o}_F$-order in $M(l_i m,F)$ and $\beta_i \in M(l_i m,F)$ such that $J_i^{\mathrm{s}} = J(\beta_i,\mathfrak{A}_i)$ in notation [10, (3.1.4)], then $J_i = (J_i^{\mathrm{s}} \cap P_i) H^1(\beta_i,\mathfrak{A}_i)$, in notation [10, (3.1.4)], where P_i denotes the upper-triangular parabolic subgroup of $\mathrm{GL}(lm_i)$ with Levi component $\mathrm{GL}(m)^{l_i}$, and unipotent radical denoted N_i. Let λ_i be the natural representation of J_i on the space of $(J^1(\beta_i,\mathfrak{A}_i)\cap N_i)$-fixed vectors in λ_i^{s}. The representation λ_i is irreducible and $\lambda_i \simeq \text{c–Ind}_{J_i}^{J_i^{\mathrm{s}}}(\lambda_i^{\mathrm{s}})$.

Let T_i denote the diagonal torus in $\mathrm{GL}(l_i,K)$ and let $B_i \supset T_i$ denote the Borel subgroup of upper-triangukar matrices in $\mathrm{GL}(l_i,K)$. We set $I_{M_0} = I_1 \times \cdots \times I_k$, where, for each i, I_i is the Iwahori subgroup of $\mathrm{GL}(l_i,K)$ attached to the Bernstein component in $\Omega(\mathrm{GL}(l_i,K))$ which contains the cuspidal pair $(T_i,1)$. Then I_{M_0} is an Iwahori subgroup of M_0. Let $P_0 = M_0 N_0$ is the standard parabolic subgroup of G_0 with Levi subgroup M_0 and let I_0 be the corresponding G_0-cover of $(I_{M_0},1)$: it is an Iwahori subgroup of G_0.

Then $(J_M,\lambda_M) = (\prod_{i=1}^k J_i, \bigotimes_{i=1}^k \lambda_i)$ is a M-cover of $(J^e,\lambda^{\otimes e})$ (see [12, Corollary 1.5]), and

$$\mathcal{H}(J_M,\lambda_M) \simeq \bigotimes_{i=1}^{k} \mathcal{H}(\mathrm{GL}(lm_i,\lambda_i)).$$

By [10, Theorem 7.6.20], there exists, for each $i \in \{1,\dots,k\}$, support-preserving algebra isomorphisms

$$\Phi_i \colon \mathcal{H}(\mathrm{GL}(l_i,K),1) \to \mathcal{H}(\mathrm{GL}(ml_i,F),\lambda_i),$$

$$\Phi_1^{\otimes l_i} \colon \mathcal{H}(T_i,1) \to \mathcal{H}(\mathrm{GL}(m,F)^{l_i},\lambda_i | J_i \cap \mathrm{GL}(m,F)^{l_i})$$

such that the diagram

$$
\begin{array}{ccc}
\mathcal{H}(T_i, 1) & \xrightarrow{\Phi_1^{\otimes l_i}} & \mathcal{H}(\mathrm{GL}(m,F)^{l_i}, \lambda_i | J_i \cap \mathrm{GL}(m,F)^{l_i}) \\
{\scriptstyle t_{B_i}}\big\downarrow & & \big\downarrow{\scriptstyle t_{P_i}} \\
\mathcal{H}(\mathrm{GL}(l_i, K), 1) & \xrightarrow[\Phi]{} & \mathcal{H}(\mathrm{GL}(ml_i, F), \lambda_i)
\end{array}
\tag{68}
$$

commute.

We have $T = \otimes_{i=1}^k T_i$. We set $B := \otimes_{i=1}^k B_i$, $\Phi_T := \otimes_{i=1}^k \Phi_1^{\otimes l_i}$, $\Phi_M = \otimes_{i=1}^k \Phi_i$, $t_P = \otimes_{i=1}^k t_{P_i}$, and $t_{P_0} = \otimes_{i=1}^k t_{B_i}$. Then, since $e = l_1 + \cdots + l_k$,

$$\Phi_T \colon \mathcal{H}(T, 1) \to \mathcal{H}(\mathrm{GL}(m,F)^e, \lambda^{\otimes e}) \quad \text{and} \quad \Phi_M \colon \mathcal{H}(M_0, 1) \to \mathcal{H}(M, \lambda_M)$$

define support-preserving algebra isomorphisms such that the diagram

$$
\begin{array}{ccc}
\mathcal{H}(T, 1) & \xrightarrow{\Phi_T} & \mathcal{H}(\mathrm{GL}(m,F)^e, \lambda^{\otimes e}) \\
{\scriptstyle t_B}\big\downarrow & & \big\downarrow{\scriptstyle t_{P\cap M}} \\
\mathcal{H}(M_0, 1) & \xrightarrow[\Phi_M]{} & \mathcal{H}(M, \lambda_M)
\end{array}
\tag{69}
$$

commutes.

Let (J_G, λ_G) be a G-cover of (J_M, λ_M) (it exists by [12, Main Theorem]). In particular, we have $J_M = J_G \cap M$ and $J_G = (J_G \cap N) \cdot J_M \cdot (J_G \cap \overline{N})$. It follows from [10, Theorem 7.6.20] and [25, (6.6)] that there exists a (uniquely determined) support-preserving algebra isomorphism

$$\Phi \colon \mathcal{H}(G_0, 1) \to \mathcal{H}(G, \lambda_G)$$

such that the diagram

$$
\begin{array}{ccc}
\mathcal{H}(M_0, 1) & \xrightarrow{\Phi_T} & \mathcal{H}(M, \lambda_M) \\
{\scriptstyle t_{P_0}}\big\downarrow & & \big\downarrow{\scriptstyle t_P} \\
\mathcal{H}(G_0, 1) & \xrightarrow[\Phi]{} & \mathcal{H}(G, \lambda_G)
\end{array}
\tag{70}
$$

commutes. Hence the assumption (9.1) is satisfied.

The Pontrjagin dual of N can be identified with $\overline{N}$, and from now we choose on $\overline{N}$ the measure dual to the one chosen on N. By [8, Theorem 5.3], this choice of measures is compatible with the one made by Shahidi in [27]. We set $\pi_M \chi := \pi_1 \chi_1 \otimes \cdots \otimes \pi_k \chi_k$ and $\chi = \chi_1 \otimes \cdots \otimes \chi_k$ in the notation of Theorem 8.1.

Lemma 9.5. *We have*

$$\frac{v(J_G \cap N) \cdot v(J_G \cap \overline{N})}{v(I_0 \cap N_0) \cdot v(I_0 \cap \overline{N}_0)} = \frac{j(\pi_M \chi)}{j_0(\chi|M_0)} = q^{-\sum_{1\le i<j\le k} l_i l_j a(\rho^\vee \otimes \rho)}.$$

Proof. We apply Corollary 9.4. It gives

$$v(J_G \cap N)\, v(J_G \cap \overline{\mathrm{N}})\, j_0(\chi|M_0)) \;=\; v(I_0 \cap N_0)\, v(I_0 \cap \overline{\mathrm{N}}_0)\, j(\pi_M \chi),$$

since under the equivalence Φ_M^*, the representation $\pi_M\chi$ corresponds to $\chi|M_0$ (see [10, (7.5.12)]). ¿From Corollary 8.3, we have

$$q^{\sum_{1\le i<j\le k} l_i l_j a(\rho^\vee \times \rho)} \cdot j(\pi_M \chi) = \prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2 = j_0(\chi|M_0).$$

□

Theorem 9.6. *Let $d\nu$ (resp. $d\nu_0$) denote Plancherel measure on the tempered dual of* $\mathrm{GL}(n,F)$ *(resp.* $\mathrm{GL}(e,K)$*). We have*

$$d\nu(\omega) = const.\, d\nu_0(\omega),$$

where

$$const. = \frac{v(I_0)}{v(J_G)} \cdot \dim(\lambda_G).$$

Proof. By Theorem 8.4, the Plancherel measures on $\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(n,F)_\Omega$ and $\mathrm{Irr}^{\mathrm{t}}\mathrm{GL}(e,K)_{\Omega_0}$ are given respectively by

$$d\nu(\omega) = q^{\sum_{1\le i<j\le k} l_i l_j a(\rho^\vee \otimes \rho)} \cdot \gamma(G/M) \cdot \frac{d(\omega)}{\prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2} \cdot d\omega$$

$$d\nu_0(\omega) = \gamma(G_0/M_0) \cdot \frac{d(\omega)_0}{\prod |\Gamma_K(q_K^{-g} z_i/z_j)|^2} \cdot d\omega,$$

where $d(\omega)_0$ denotes the formal degree of $\omega \in \mathcal{E}_2(M_0)$, and $q_K = q^r$.

By [10, (7.7.11)], we have

$$v(J_M)\, d(\omega) = v(I_{M_0})\, d(\omega)_0 \, \dim(\lambda_M).$$

Therefore we have

$$d\nu(\omega) = const.\, d\nu_0(\omega),$$

with *const.* equal to

$$q^{\sum_{1\le i<j\le k} l_i l_j (a(\rho^\vee \otimes \rho))} \cdot \frac{v(J_G \cap N) \cdot v(J_G \cap \overline{N})}{v(I_0 \cap N_0) \cdot v(I_0 \cap \overline{N}_0)} \cdot \frac{v(I_0)}{v(J_G)} \cdot \dim(\lambda_M),$$

by using the formula (5). Since by definition of covers (see [11, §8]), the representation λ_G is trivial on $J_G \cap N$ and $J_G \cap \overline{N}$, while $\lambda_G | J_M \simeq \lambda_M$, we have $\dim(\lambda_G) = \dim(\lambda_M)$. Then, it follows from Lemma 9.5 that

$$const. = \frac{v(I_0)}{v(J_G)} \cdot \dim(\lambda_G).$$

□

References

[1] A-M. Aubert, J.-L. Kim, A Plancherel formula on Sp_4, preprint 2001.

[2] J. Bernstein, Representations of p-adic groups, Notes by K.E. Rumelhart, Harvard University 1992.

[3] I.N. Bernstein, A.V. Zelevinsky, Induced representations of reductive p-adic groups, Ann. scient. Éc. Norm. Sup. 10 (1977) 441–472.

[4] N. Bourbaki, Algebra I, Springer-Verlag, 1989.

[5] N. Bourbaki, Groupes et Algèbres de Lie, chap. IV, V, VI, Paris, Hermann 1968.

[6] C.J. Bushnell, Representations of reductive p-adic groups: Localization of Hecke algebras and applications, J. Lond. Math. Soc., II. Ser. 63, No.2 (2001) 364–386.

[7] C.J. Bushnell, G. Henniart, Local tame lifting for GL(n) II: wildly ramified supercuspidals, Astérisque 254 (1999).

[8] C.J. Bushnell, G. Henniart, P.C. Kutzko, Local Rankin-Selberg convolutions for GL_n: explicit conductor formula, J. Amer. Math. Soc. 11 (1998) 703–730.

[9] C.J. Bushnell, G. Henniart, P.C. Kutzko, Towards an explicit Plancherel theorem for reductive p-adic groups, preprint 2001.

[10] C.J. Bushnell, P.C. Kutzko, The admissible dual of $GL(n)$ via compact open subgroups, Ann. Math. Study 129, Princeton Univ. Press 1993

[11] C.J. Bushnell, P.C. Kutzko, Smooth representations of reductive p–adic groups: Structure theory via types, Proc. London Math. Soc. 77 (1998) 582–634.

[12] C.J. Bushnell, P.C. Kutzko, Semisimple types in GL_n, Comp. Math. 119 (1999) 53–97.

[13] W. Casselman, The unramified principal series of p-adic groups I. The spherical function, Comp. Math. 40 (1980) 387-406.

[14] P. Deligne, Les constantes des équations fonctionnelles des fonctions L. Modular Functions one Variable II, Proc. internat. Summer School, Univ. Antwerp 1972, Lect. Notes Math. 349 (1973), 501–597.

[15] Harish-Chandra, Collected papers, volume 4, Springer, Berlin 1984.

[16] M. Harris, R. Taylor, On the geometry and cohomology of some simple Shimura varieties, Ann. Math. Study 151, Princeton University Press 2001.

[17] V. Heiermann, Sur l'espace des représentations irréductibles du groupe de Galois d'un corps local. C.R. Acad. Sci., Paris, Sér. I, 323, No. 6 (1996) 571–576.

[18] H. Jacquet, I. Piatetskii-Shapiro, J. Shalika, Rankin-Selberg convolutions, Amer.J.Math 105 (1983) 367–483.

[19] S.S. Kudla, The local Langlands correspondence, Proc. Symp. Pure Math. 55 (1994) 365–391.

[20] I.G. Macdonald, Harmonic analysis on semi-simple groups, Actes Congr. internat. Math. Nice 1970, Tome 2, (1971) 331–335.

[21] I.G. Macdonald, Spherical functions on a group of p-adic type, Publ. Ramanujan Inst. 2, 1971.

[22] I.G. Macdonald, The Poincaré series of a Coxeter group, Math. Ann. 199 (1972) 161–174.

[23] R.J. Plymen, Reduced C^*-algebra of the p-adic group GL(n), J. Functional Analysis 72 (1987) 1–12.

[24] R.J. Plymen, Reduced C^*-algebra of the p-adic group GL(n) II, J. Functional Analysis 196 (2002) 119 –134.

[25] K. Procter, Parabolic induction via Hecke algebras and the Zelevinsky duality conjecture, Proc. London Math. Soc. 77 (1998) 79–116.

[26] M. Reeder, Hecke algebras and harmonic analysis on p-adic groups, American J. Math. 119 (1997) 225–249.

[27] F. Shahidi, A proof of Langlands conjecture on Plancherel measure; complementary series for p-adic groups, Annals of Math. 132 (1990), 273–330.

[28] F. Shahidi, Langlands' conjecture on Plancherel measures for p-adic groups, in Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., 101, Birkhäuser Boston, Boston, MA, 277–295, 1991.

[29] F. Shahidi, Fourier transforms of intertwining operators and Plancherel measures for GL(n), Amer. J. Math. 106 (1984), 67–111.

[30] M.H. Taibleson, Fourier Analysis on local fields, Math. Notes 15, Princeton University Press.

[31] J. Tate, Number theoretic background, Proc. Symp. Pure Math 33 (1979) 3–26.

[32] J-L. Waldspurger. La formule de Plancherel d'après Harish-Chandra. Journal de l'Institut Math. Jussieu (to appear in 2003).

[33] A.V. Zelevinsky. Induced representations of reductive p-adic groups II, Ann. Sci. Ec. Norm. Sup. 4 13 (1980) 154–210.

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