Non-commutative Entropy Computations for continuous fields and cross-products

Emmanuel Germain
Institut de Mathématique de Jussieu
germain@math.jussieu.fr

Abstract

We present here two non-commutative situations where dynamical entropy estimates are possible. The first result is concerned with automorphisms of cross-products by an exact group that commute with the group action and generalizes the result known for amenable group. The second is about continuous fields of C*-algebras and $C(X)$-automorphisms and shows that one need more than the knowledge of entropy in each fiber. Each result relies on explicit factorization via matrices.

Although exact C*-algebras have a weaker form of factorization through finite dimensional matrix algebras than nuclear C*-algebras, N. Brown [6] was able to show that Voiculescu's initial definition of non-commutative entropy for automorphisms [15] can be extended to this situation and produced the first computations, notably for automorphisms of cross-products of an exact C*-algebra by the group of the integers. Other results, like a striking formula for free product automorphisms of reduced free product C*-algebras [7], were obtained proving that it was the correct setting for this kind of entropy.

For group C*-algebra, Ozawa [12] showed about the same time that the reduced C*-algebra of a discrete group is exact if and only if it has an "amenable" action on a compact space. It was long known that this condition was a sufficient one [1]. Indeed since the groupoid associated to the group action on the compact space is amenable, hence its reduced C*-algebra is nuclear, the reduced C*-algebra of the group is a subalgebra of a nuclear one. Therefore Ozawa result is a geometric caracterization of exactness parallel to Kirchberg's caracterization of exact C*-algebras as subalgebras of nuclear C*-algebras.

It was then obvious that in the situation of a reduced cross product of an exact C*-algebra by an exact group a formula for the entropy of an automorphism that commutes with the group action should be available, extending the previous result obtained for amenable groups (cf. [13]) and fullfilling C. Anantharaman's remark in her article [2]. Explicitely, we prove in the first section that if an exact C*-algebra A is endowed with an action of a group Γ commuting with an automorphism β of A, then the entropy of β in A is the same as the entropy of its unique extension $\bar{\beta}$ to the cross-product $A \rtimes_r \Gamma$ defined as $\bar{\beta}(a)(g) = \beta(a(g))$ for any element a in the convolution algebra $L^1(\Gamma; A)$.

The crucial ingredient used in the first section to get this result is the existence of explicit matrix factorizations due to the fact that analogues of Folner functions exist for amenable groupoids. Extending the results of [2] section 8, one shows that if A is an exact C*-algebra and Γ has an amenable action on a compact space X then $C(X) \otimes A$ is an exact $\Gamma - C(X)$-algebra whose cross-product by Γ has factorizations through finite dimensional matrix algebras which can be made out of factorizations of A. We show that the ranks of these factorizations are linearly related which yields the entropy comparison we want.

If the algebra A is the algebra of continuous functions on a compact space E with an action of Γ such that $B = E/\Gamma$ is again compact then $A \rtimes_r \Gamma$ is fibered over B (it is actually a $C(B)$-algebra) hence the result above can be reinterpreted as computing the entropy of an automorphism of a fibered space whose action factors through the base space B (a "transverse" automorphism as there are transverse differential operators). The second part of this article investigates then the "longitudinal" case, i.e entropy of an automorphism that would act only in the fibers.

It is not clear what is the correct setting for such an appraoch. Of course when the base is discrete, we are dealing with direct sums and it is known that one should take the supremum of the entropy of the automorphisms in each summands (i.e. fibers). But for continuous base space, it must be trickier. First there are two notions of a C*-algebra fibered over a compact space: $C(X)$-algebras and continuous fields, the latter asking for a strictier continuity condition for sections. Then a subtlety arises as it is not true that the whole algebra is exact whenever all fiber algebras are (even for continuous fields see [5]). Therefore we turned our attention to lipschitz continuous fields introduced by Kirchberg and Phillips [11] because, for such continuous fields, explicit matrix factorizations can be realized via the knowledge of factorizations of the fibers. We then found an upper bound for the entropy of an automorphism of such fields that has an extra term which incorporates geometric data (dimension of the base space, lipschitz exponent of the field) and a symbolic dynamics entropy term.

This second part is organized as follows: we defined an entropy for linear endomorphisms of the non-commutative polynomials using as a gauge the norm of the non-commutative gradient of a polynomial, we then describe the factorization of lipschitz fields and compute entropy. At last we apply our result to the C*-algebra of the Heisenberg group (of unipotent upper-triangular 3×3 matrices with integer coefficients) since it can be seen as a lipschitz continuous fields over the unit circle of the non-commutative tori with exponent 1/2 as it has been proved by Haagerup and Rordam in [10].

1 Cross-product by exact groups

An exact discrete group Γ is a group such that the reduced cross-product of any exact sequence of Γ-algebras (i.e. C*-algebras with an action of Γ via automorphisms) is again exact. In particular if E is an exact C*-algebra with an action of an exact discrete group Γ then the reduced cross-product $E \rtimes_r \Gamma$ is again exact. Indeed let $0 \to I \to A \to B \to 0$ be an exact sequence. Then by exactness of E, on gets that $0 \to I \otimes_{min} E \to A \otimes_{min} E \to B \otimes_{min} E \to 0$ is again exact. Endowing I, A, B with a trivial action, it is also a sequence of Γ-algebras. By definition, its reduced cross product by Γ is again exact. Now observing that $(A \otimes_{min} E) \rtimes_r \Gamma$ is $A \otimes_{min} (E \rtimes_r \Gamma)$ for any A ensures that tensoring (for the minimal norm) the original sequence by $E \rtimes_r \Gamma$ leaves it exact.

For discrete groups, exactness need only be checked for the trivial action (see [1]), therefore the reduced C*-algebra $C^*_r(\Gamma)$ is an exact C*-algebra if and only if Γ is exact. It has recently been proved by Ozawa ([12]) and independantly by Anantharaman ([2]) that it is equivalent to amenability at infinity i.e. the existence of a compact Hausdorff space X with an action of Γ such that the action is amenable, a term defined for general groupoids in [1]. Using this amenable action, C. Anatharaman proved that there exists explicit matrix factorizations for the algebra $E \rtimes_r \Gamma$ when E is nuclear (see section 8 of [2]). We extend here this construction to the exact case and use the notations found therein to prove:

Theorem 1.1 *Let E be an unital exact C*-algebra with an action α of an exact countable discrete group Γ. Let β be an automorphism of E such that for all $g \in \Gamma$, β and α_g commutes, then β extends to $\bar{\beta}$ on $E \rtimes_r \Gamma$ and*

$$ht_E(\beta) = ht_{E \rtimes_r \Gamma}(\bar{\beta}).$$

Since $E \subset E \rtimes_r \Gamma$, one already has $ht_E(\beta) \leq ht_{E \rtimes_r \Gamma}(\bar{\beta})$. For the reverse inequality, we will consider an amenable action of Γ on a compact Hausdorff set X. Now $A = C(X) \otimes E$ is a $\Gamma - C(X)$-algebra meaning that A is a

Γ-algebra (with the diagonal action), a $C(X)$-algebra (actually it is a trivial continuous field) and has the compatibility condition: $g.(fa) = (g.f)(g.a)$ with $g \in \Gamma$, $f \in C(X)$ and $a \in A$. Because $C(X)$ is unital, one has that $E \rtimes_r \Gamma \subset A \rtimes_r \Gamma$.

Let π_0 be a faithful representation of A in $B(H)$ such that the action of Γ is implemented by a unitary representation. Hence $B(H)$ is endowed with an action of Γ which we will still call α and $A \rtimes_r \Gamma \subset B(H) \rtimes_r \Gamma$. The covariant pair of representations $(\pi, \lambda \otimes 1_H)$ of $B(H) \rtimes_r \Gamma$ in $B(\ell^2(\Gamma) \otimes H)$ is defined as $\pi(a)\xi(t) = \alpha_{t^{-1}}(a)\xi(t)$ with λ the regular representation of Γ. Now lemma 8.1 of [2] can be identically reformulated with $\Phi : A \to B(H)$ a completely positive (or completely bounded) map instead of $\Phi : A \to A$.

As a consequence Proposition 8.2 obviously becomes

Proposition 1.2 *Let X be a compact space with an amenable action of the discrete group Γ, and let A be an exact $\Gamma - C(X)$-algebra. Then $A \rtimes_r \Gamma$ is exact.*

Recall now the definition

Definition 1.3 *Let $\epsilon > 0$ and $\omega \subset A$ finite. $rcp_A(\pi, \epsilon, \omega)$ is the smallest integer p such that there exists a completely positive contractive (ϵ, ω)-factorization $A \xrightarrow{\sigma} M_p(\mathbb{C}) \xrightarrow{\tau} B(H)$ of the faithful morphism $A \xrightarrow{\pi} B(H)$, i.e. such that for all x in ω, $||\pi(x) - \tau \circ \sigma(x)|| \leq \epsilon$.*

Call U_g the unitary in $A \rtimes_r \Gamma$ that implements the action of Γ on A. With this notation, we prove:

Lemma 1.4 *Let $\omega \subset \Gamma$ be a finite set and $O \subset E$ be a finite set of norm 1 element. Let Ω be the set $\{aU_g, g \in \omega, a \in O\}$ in $E \rtimes_r \Gamma$. Then there exists a finite set F in Γ such that*

$$rcp_{E \rtimes_r \Gamma}((\pi, \lambda \otimes 1_H), \epsilon, \Omega) \leq |F| rcp_E(\pi_0, \epsilon/2, \cup_{t \in F} \alpha_{t^{-1}}(O))$$

Indeed this is just a reformulation of the proof of the above mentionned proposition 8.2 of [2].

If one chooses a $(\epsilon/2, \cup_{t \in F} \alpha_{t^{-1}}(O))$ factorization (σ, τ) of E through $M_n(\mathbb{C})$ then it can be extended by Areveson's extension theorem for completely positive maps to a factorization $(\bar{\sigma}, \tau)$ of A through the same $M_n(\mathbb{C})$. With the help of a function $f : \Gamma \to C_c(X)$ with finite support C such that $||f||_2^2 = 1$ and

$$|\sum_{t \in \Gamma} f(t)(x)\overline{f(s^{-1}t)(s^{-1}x)} - 1| < \epsilon/2$$

for $s \in \omega$ which exists by the amenability of the action on X, we define the set $F = \cup_{s \in \omega \cup \{e\}} s^{-1}C$ and the two completely positive maps

$$\tilde{\sigma}(aU_g) = I \otimes \bar{\sigma}(P_F \otimes I(\pi(a)\lambda(g))P_F \otimes I)$$

with P_F the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(F)$ and

$$\tilde{\tau}(x) = T_f(I \otimes \tau(x))$$

with T_f from $B(\ell^2(\Gamma) \otimes H)$ onto itself as defined is [2].

The composition of the two produces a map Ψ such that $||\Psi(aU_g) - \pi(a)\lambda(g)|| < \epsilon$ for all $a \in O$ and $g \in \omega$ because the completely positive map $\Phi = \tau \circ \bar{\sigma}$ has the property that $||\Phi(\alpha_{t^{-1}}(a)) - a|| < \epsilon/2$ for all $a \in 0$.

Now the rank of $\tilde{\sigma}$ is $|F|$ multiplied by the rank of σ which is what we seek.

Corollary 1.5 *Let $\omega \subset \Gamma$ be a finite set and $O \subset E$ be a finite set of norm 1 element. Let Ω be the set $\{aU_g, g \in \omega, a \in O\}$ in $E \rtimes_r \Gamma$. Then there exists a finite set F in Γ such that*

$$rcp_{E\rtimes_r\Gamma}((\pi, \lambda \otimes 1_H), \epsilon, \Omega \cup \bar{\beta}(\Omega) \cup \cdots \cup \bar{\beta}^k(\Omega)) \leq$$
$$\leq |F| rcp_E(\pi_0, \epsilon/2, \cup_{t\in F}\alpha_{t^{-1}}(O) \cup \beta(\cup_{t\in F}\alpha_{t^{-1}}(O)) \cup \cdots \cup \beta(\cup_{t\in F}\alpha_{t^{-1}}(O))).$$

Indeed β commutes with the action of Γ, hence $\bar{\beta}(\Omega) = \{aU_g,\ g \in \omega, a \in \beta(O)\}$ and $\cup_{t\in F}\alpha_{t^{-1}}(\beta(O)) = \beta(\cup_{t\in F}\alpha_{t^{-1}}(O))$.

Since the entropy is then defined as

$$ht_{E\rtimes_r\Gamma}(\bar{\beta}) = \sup_{\epsilon>0} \sup_{\Omega\in\mathcal{F}(E)} \lim_{n\to\infty} \frac{1}{n} \log(rcp_E((\pi, \lambda \otimes 1_H), \epsilon, \Omega \cup \bar{\beta}(\Omega) \cup \cdots \cup \bar{\beta}^n(\Omega))$$

with $\mathcal{F}(E)$ the set of all finite subsets of the linear span of elements of the form aU_g with $a \in E$, $g \in \Gamma$ by Kolmogorov density property, we have that

$$ht_{E\rtimes_r\Gamma}(\bar{\beta}) \leq ht_E(\beta)$$

keeping in mind that entropy can be computed via the rcp function of any faithfull representation.

2 Entropy for continuous fields of C*-algebras

A unital continuous field A of C*-algebras over a compact Hausdorff space X is caracterized by two properties. First it is a $C(X)$-algebra, meaning there is a unital morphism of $C(X)$ into the center of A. There is thus an action of $C(X)$ on A that we denote as $f.a$ for a function f and an element a of A.

Note that the norm in A is given as a supremum. Indeed, for any $x \in X$, let's call $C_x(X)$ the ideal of functions vanishing at x. Then A_x is the quotient algebra $A/(C_x(X).A)$ and note a_x the image of $a \in A$ in this quotient. We have the embedding $A \hookrightarrow \Pi_{x\in X} A_x$. (see Blanchard [4])

A $C(X)$-algebra is a continous field if and only if the map $x \mapsto ||a_x||_{A_x}$ is continuous.

We are interested in a $C(X)$-automorphism α of a continuous field A, meaning an automorphism such that for any function $f \in C(X)$ and $a \in A$ we have that $f.\alpha(a) = \alpha(f.a)$. Note that α factorizes through all the algebras A_x. Let's call α_x the induced automorphism.

For the moment we will study entropy of linear endomorphisms on non-commutative polynomials and propose a definition of symbolic entropy for automorphisms of C*-algebras having a dense finitely generated subalgebra.

2.1 Symbolic entropy

Let $\mathbb{C} < X_1 \cdots X_n >$ denotes the set of non-commutative polynomials in n variables.

Definition 2.1 *If $P \in \mathbb{C} < X_1 \cdots X_n >$ then $\mathcal{J}P \in \mathbb{C} < X_1 \cdots X_n > \otimes \mathbb{C} < X_1 \cdots X_n >$ will denote the non-commutative gradient of P with respect to the variable $X_1, ..., X_n$ and is defined by linearity on generators as follows*

$$\mathcal{J}X_i = 1 \otimes 1, \forall i = 1 \cdots n$$

$$\begin{aligned}\mathcal{J}X_{i_1} \cdots X_{i_n} = \; & 1 \otimes X_{i_2} \cdots X_{i_n} + \textstyle\sum_{k=2}^{n-1} X_{i_1} \cdots X_{i_{k-1}} \otimes X_{i_{k+1}} \cdots X_{i_n} \\ & + X_{i_1} \cdots X_{i_{n-1}} \otimes 1\end{aligned}$$

Because the tensors $1\otimes 1$, $1 \otimes X_{j_1} \cdots X_{j_n}$, $X_{i_1} \cdots X_{i_n} \otimes 1$ and $X_{i_1} \cdots X_{i_n} \otimes X_{j_1} \cdots X_{j_n}$ form a basis of $\mathbb{C} < X_1 \cdots X_n > \otimes \mathbb{C} < X_1 \cdots X_n >$, there is an associated ℓ_1-norm (for which the base elements have norm 1), we will call it $||.||_1$.

The total variation of $P \in \mathbb{C} < X_1 \cdots X_n >$ will then be $||\mathcal{J}P||_1$. Note that on monomials, this gives the total degree of P with respect to $X_1 \cdots X_n$. This name is approriate because of:

Proposition 2.2 *If $\mathcal{A}$ is a complex normed algebra and Σ_i for $i = 1, 2$ are two algebra homomorphims from $\mathbb{C} < X_1 \cdots X_n >$ to $\mathcal{A}$ such that $\Sigma_i(X_j)$ is* *a norm* 1 *element in* $\mathcal{A}$ *for* $i = 1, 2$ *and* $j = 1, ..., n$*, then for all* $P \in \mathbb{C} < X_1 \cdots X_n >$*,*

$$||\Sigma_1(P) - \Sigma_2(P)||_{\mathcal{A}} \leq ||\mathcal{J}P||_1 \sup_{i \in \{1,2,..,n\}} ||\Sigma_1(X_i) - \Sigma_2(X_i)||_{\mathcal{A}}$$

The proof is obvious with the remarks that

$$||\Sigma_2(X_{i_1}...X_{i_{k-1}})\Sigma_1(X_{i_k}...X_{i_p}) - \Sigma_2(X_{i_1}...X_{i_k})\Sigma_1(X_{i_{k+1}}...X_{i_p})||_{\mathcal{A}} \leq \\ \leq ||\Sigma_1(X_{i_k}) - \Sigma_2(X_{i_k})||_{\mathcal{A}}$$

and $||\Sigma_j(X_{i_1}...X_{i_n})||_{\mathcal{A}} = 1$.

Now if θ is a linear homomorphism of $\mathbb{C} < X_1 \cdots X_n >$, we will define its symbolic entropy as

$$se(\theta) = \sup_{P \in \mathbb{C} < X_1 \cdots X_n >} \limsup_{n \to \infty} \frac{1}{n} \log ||\mathcal{J}(\theta^n P)||_1$$

The above quantity behaves almost as an entropy for we have

Proposition 2.3 *1.* $se(\theta^k) = k\, se(\theta), \forall k \geq 0$*.*

2. If $\theta(P) = QP$ *or* PQ *for some* $Q \in \mathbb{C} < X_1 \cdots X_n >$*, then* $se(\theta) \leq \log ||Q||_1$*.*

For 1., one just need to remark that $se(\theta)$ is the infimum of the constants σ such that for all polynomial P there exists a constant C_P such that $||\mathcal{J}(\theta^n(P))||_1 \leq C_P \exp(n\sigma)$. Hence $se(\theta^k) \leq k se(\theta)$ and by considering the maximum of $\{C_P, C_{\theta(P)}..., C_{\theta^{k-1}(P)}\}$ one gets the reverse inequality.

For 2., we of course endow $\mathbb{C} < X_1...X_n >$ with the ℓ_1-norm for which the monomials have norm 1 which is an algebra norm. Then for the bimodule structure of $\mathbb{C} < X_1...X_n > \otimes \mathbb{C} < X_1...X_n >$ we have that $||Q_1.P.Q_2||_1 \leq ||Q_1||_1||P||_1||Q_2||_1$ with $Q_i \in \mathbb{C} < X_1...X_n >$ and $P \in \mathbb{C} < X_1...X_n > \otimes \mathbb{C} < X_1...X_n >$. Finally note that $\mathcal{J}(QP) = \mathcal{J}(Q).P + Q.\mathcal{J}(P)$ and $\mathcal{J}(Q^n) = \sum_{i=0}^{n-1} Q^i \mathcal{J}(Q).Q^{n-i-1}$. Therefore the result follows from the inequality

$$||\mathcal{J}(Q^n P)||_1 \leq n||Q||_1^{n-1}||\mathcal{J}(Q)||_1||P||_1 + ||Q||_1^n||\mathcal{J}(P)||_1.$$

Note that if Q is a monomial then $se(\theta) = 0$.

Finally we propose this definition for C^*-algebra automorphisms:

If A is a unital C^*-algebra and α an automorphism, let F be the set of all dense finitely generated subalgebras $\mathcal{A}$ of A such that α induces an automorphism of $\mathcal{A}$.

Now take G as the set of all linear extensions of α i.e. the set of linear endomorphisms θ of $\mathbb{C} < X_1 \cdots X_n >$ such that there exists an epimorphism π from $\mathbb{C} < X_1 \cdots X_n >$ to $\mathcal{A} \in F$ with $\pi(\theta(P)) = \alpha(\pi(P))$ for all polynomials P.

Definition 2.4 *The symbolic entropy of the automorphism α is*

$$se(A, \alpha) = \inf_{\theta \in G} se(\theta)$$

The infimum is taken to be $+\infty$ if F is empty.

2.2 Exact lipschitz continuous fields over a compact metric space

Suppose A is a unital continuous field over a compact metric space X. Let's assume that A is exact (in particular all the A_x are exact since they are quotients). It is then known that A admits a $C(X)$-embedding in some $C(X) \otimes B(\mathcal{H})$. Consider the following definition:

Definition 2.5 *Suppose A is an exact continuous field on some compact metric space X with metric d, we say that A is lipschitz of exponent L if there exists a $C(X)$-linear embedding π of A in some $C(X) \otimes B(\mathcal{H})$ such that for all $a \in A$ the map $x \mapsto \pi_x(a_x)$ from X to $B(\mathcal{H})$ is lipschitz with exponent L i.e. for all $a \in A$ there exists a constant C such that*

$$||\pi_x(a_x) - \pi_y(a_y)|| \leq C d(x, y)^L.$$

In [5] Blanchard showed that exact continuous fields over a compact space X have $C(X)$-embeddings but in [11] the authors proved the existence of lipschitz embeddings when an intrinsecally defined metric function is itself lipschitz (see theorem 2.10). It is the case for example of the continuous fields of the non-commutative tori (reproving a theorem of Haagerup-Rordam, see[10]).

This section is now aimed at establishing the following statements.

Theorem 2.6 *Suppose A is an continuous field over a compact subset X, and α is a $C(X)$-automorphism of A. Then*

$$ht_A(\alpha) \geq sup_{x \in X'} ht_{A_x}(\alpha_x)$$

where X' is the set of all such $x \in X$ with A_x commutative.

Indeed we know topological entropy dominates CNT-entropy [9], therefore $ht_A(\alpha) \geq ht_A^{CNT}(\alpha)$. Since CNT-entropy decreases in quotient, one gets $ht_A(\alpha) \geq sup_{x\in X} ht_{A_x}^{CNT}(\alpha_x)$. Since all entropy definitions coincide in the commutative case, one gets the result.

Now to get an upper bound is a bit more difficult:

Theorem 2.7 *Suppose A is an lipschitz continuous field of exponent L over a compact metric space X of Hausdorff dimension N. If A is exact and if α is an $C(X)$-automorphism then*

$$ht_A(\alpha) \leq sup_{x\in X} ht_{A_x}(\alpha_x) + \frac{N}{L} se(A,\alpha)$$

Corollary 2.8 *With the hypothesis of the theorem, if α is inner, then*

$$ht_A(\alpha) \leq sup_{x\in X} ht_{A_x}(\alpha_x).$$

It is clear when A admits a dense finitely generated subalgebra $\mathcal{A}$ because if one considers the sublagebra generated by the unitary implementing the automorphism α and $\mathcal{A}$ then $se(A,\alpha) = 0$ by prop 2.3 2). For the general case, it is just a slight modification of the proof of theorem 2.7.

Proof of theorem 2.7:

We assume A is faithfully represented in $C(X) \otimes B(\mathcal{H})$ via a lipschitz $C(X)$-homomorphism so that we identify any element of A with a function with value in $B(\mathcal{H})$. Note that A_x embedds then in $B(\mathcal{H})$ since a_x is the evaluation at x of $a \in A$.

Since X is of Hausdorff dimension N, there exists a constant C_1 such that when X is covered by balls of radius η, the smallest number of such balls is bounded by $C_1\eta^{-N}$.

Let δ be positive and $\mathcal{A}$ be a dense finitely generated algebra in A with an epimorphism p from $\mathbb{C} < X_1 ... X_n >$ onto $\mathcal{A}$ such that α induces a map θ of $\mathbb{C} < X_1 ... X_n >$ for which $se(\theta) \leq se(A,\alpha) + \delta$ and choose $\epsilon > 0$ and a finite set ω in $\mathcal{A}$. There exists then a constant C_3 such that for all integer k, $||\mathcal{J}\theta^k(P)|| \leq C_3 \exp(k\, se(\theta) + \delta)$ with P in a finite set $\bar{\omega}$ with $p(\bar{\omega}) = \omega$.

By Lipschitz continuity, there exists a constant C_2 such that for all $a \in \omega$ or in the generating set of $\mathcal{A}$, $||a_x - a_y|| \leq C_2 d(x,y)^L$.

Consider $\Omega = \omega \cup \alpha(\omega) \cup \cdots \cup \alpha^n(\omega)$, we are going to construct now a factorization for Ω with error bounded by ϵ of the embedding of A in $C(X) \otimes B(\mathcal{H})$.

Take $\eta = [\frac{\epsilon}{2C_2} \min(1, \frac{1}{C_3} \exp(-nse(\theta) - \delta))]^{1/L}$ and cover X with balls of radius η: $X \subset \cup_{j\in J} B(x_j, \eta)$. Then there exists σ_j from A_{x_j} to $M_{p_j}(\mathbb{C})$

completely contractive and τ_j from $M_{p_j}(\mathbb{C})$ to $B(\mathcal{H})$ completely contractive such that $\forall a \in \Omega$, $||\tau_j \circ \sigma_j(a_{x_j}) - a_{x_j}|| \leq \epsilon/2$ and $p_j \leq e^{nH_j}$ for n large enough with $H_j = ht_{A_{x_j}}(\alpha_{x_j}) + \delta$.

Supppose $(\varphi_j)_{j\in J}$ is a partition of unity associated to the covering and consider σ from A to $\oplus_{j\in J} M_{p_j}(\mathbb{C})$ defined as

$$\sigma(a) = \bigoplus_{j\in J} \sigma_j(a_{x_j})$$

and τ from $\oplus_{j\in J} M_{p_j}(\mathbb{C})$ to $C(X) \otimes B(\mathcal{H})$ defined as

$$\tau(\bigoplus_{j\in J} x_j) = \sum_{j\in J} \varphi_j(x)\tau_j(x_j)$$

Then for all $a \in \Omega$, $||\tau \circ \sigma(a) - a|| \leq \epsilon/2 + \max_{j\in J} \sup_{x\in B(x_j,\eta)} ||a_x - a_{x_j}||$

Now if $a \in \omega$, $\sup_{x\in B(x_j,\eta)} ||a_x - a_{x_j}|| \leq \epsilon/2$ by definition of η and the lipschitz continuity.

Let P in $\mathbb{C} < X_1...X_q >$ be such that $p(P) = a$ and define Σ_1 and Σ_2 the homomorphisms of $\mathbb{C} < X_1...X_q >$ in $B(\mathcal{H})$ obtained by composition of p with the evaluation at x or at x_j.

Then for all integer k,

$$\begin{aligned} ||\alpha_x^k(a_x) - \alpha_{x_j}^k(a_{x_j})|| = \; & ||\Sigma_1(\theta^k(P)) - \Sigma_2(\theta(P))|| \\ & \leq ||\mathcal{J}\theta^k(P)||_1 ||\Sigma_1(P) - \Sigma_2(P)|| \\ & \leq C_3 \exp(k\, se(\theta) + \delta)||_1 ||a_x - a_{x_j}|| \end{aligned}$$

Thus $||\alpha_x^k(a_x) - \alpha_{x_j}^k(a_{x_j})|| \leq \epsilon/2$ for $k \leq n$ and $a \in \omega$, so $||\tau \circ \sigma(a) - a|| \leq \epsilon$ for all $a \in \Omega$.

Now the rank of the matrix algebra $\oplus_{j\in J} M_{p_j}(\mathbb{C})$ is bounded by

$$C_1(\frac{2C_2C_3}{\epsilon})^{N/L} \exp(n[\frac{N}{L}(se(A,\alpha) + \delta) + H] + \delta\frac{N}{L})$$

for n large enough with $H = \max_{j\in J} H_j$.

For any faithfull representation $C(X)$ in $B(K)$, we have a faithfull representation π of $C(X) \otimes B(H)$ hence A in $B(K \otimes H)$.

Hence $\limsup_{n\to\infty} \frac{1}{n} \log rcp(\pi, \epsilon, \Omega) \leq H + \frac{N}{L}[se(A,\alpha) + \delta]$.

Since $ht_A(\alpha)$ is computed by taking the sup over all finite set of $\mathcal{A}$ since it is dense, we have that, for all δ positive,

$$ht_A(\alpha) \leq \sup ht_{A_x}(\alpha_x) + \frac{N}{L} se(A,\alpha) + \delta(\frac{N}{L} + 1)$$

which proves our theorem.

In the case of the continuous field of the non-commutative tori, one gets the result:

Proposition 2.9 *Let M be a matrix in $SL_2(\mathbb{Z})$ with non negative entries and α_M the induced automorphism on the continuous field of the non-commutative tori $A = (A_\theta)_{\theta\in\mathbb{T}}$, then*

$$\sup_{\lambda\in Sp(M)} \log|\lambda| \leq ht_A(\alpha_M) \leq 3. \sup_{\lambda\in Sp(M)} \log|\lambda|.$$

Note that it actually gives a computation for an automorphism of the C^*-algebra of the Heisenberg group in $M_3(\mathbb{C})$. Indeed this group is generated by the three matrices $u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now call U, V, W the corresponding three unitaries in the group C*-algebra A (the group is amenable so there is no need to specify a norm). Then since w commutes with u and v and is the commutator of the two, we have that W is in the center of A and $UV = WVU$. So A is a $C(\mathbb{T})$-algebra with $\mathbb{T} = Spec(W)$. But Haagerup and Rordam proved that A is actually a lipschitz continuous fields of exponent $1/2$.

Now take a matrix $M \in SL_2(\mathbb{N})$, then it induces an automorphism α_A of this field as follows:

$$\alpha_M(U) = U^a V^c, \quad \alpha_M(V) = U^b V^d \quad \alpha_M(W) = W$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

One can check that $\alpha_M(U)\alpha_M(V) = W^{\det M}\alpha_M(V)\alpha_M(U)$ and $\alpha_M^{-1} = \alpha_{M^{-1}}$ so that $\det M = 1$ is the only requirement to get an automorphism of the continuous field as well as of all the A_θ.

Now the lower bound comes from the computation for entropy in $A_0 = C^*(\mathbb{Z}^2)$. For the upper bound first recall that in each A_θ the entropy is bounded by $\sup_{\lambda\in Sp(M)} \log|\lambda|$ (cf. [15]). It remains to compute the symbolic entropy of the automorphism. Consider the dense algebra generated by the six unitaries U, V, W, U^{-1}, V^{-1}, W^{-1}. Since the automorphism leaves W invariant, we only need to concentrate on iterates of polynomials in U,V,U^{-1},V^{-1}. Since the image of monomials are monomials and we have an algebra homomorphism, we just have to bound the total degree of iterates of each of the unitaries. Because the coefficients of M are all positive (hence no cancellation need to occur between U and U^{-1} or V and V^{-1} hence no commutativity is required) the degree of the n-th iterate is given by the matrix product

$$(1,1,1,1).\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}^n .E$$

where E is a vector of the canonical basis of $\mathbb{N}^4$; $(1,0,0,0)$ representing U, $(0,1,0,0)$ representing V, and so on. But these quantities are bounded by $C.|\lambda|^n$ where λ is the eigenvalue of M of maximal modulus. Hence the result.

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