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Infinite Dimensional Harmonic Analysis and Probability

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In these notes we will present some recent results about harmonic analysis on groups of the type

$$G = \bigcup_{n=1}^{\infty} G(n),$$

where $G(n)$ is a sequence of classical groups, with a sugbroup K of the same type

$$K = \bigcup_{n=1}^{\infty} K(n), \quad K(n) \subset G(n).$$

One of the main problem in this harmonic analysis is to decompose a continuous K-biinvariant function φ on G which is of positive type in a sum or an integral of indecomposable ones, *i.e.* to establish a Bochner type theorem. These indecomposable functions are called spherical. This problem has already been considered by Schoenberg [1938,1942] and Krein [1949]. In the first chapter we consider the case of $K(n) = O(n)$, the orthogonal group, and $G(n) = O(n) \ltimes \mathbb{R}^n$ the affine motion group. then $K = O(\infty)$ is the infinite dimensional orthogonal group and G the infinite dimensional motion group. We give a proof of the main result in [Schoenberg,1948]. In later chapters we present recent results about the space of infinite dimensional Hermitian matrices,

$$H(\infty) = \bigcup_{n=1}^{\infty} Herm(n, \mathbb{C}).$$

In that case $K(n) = U(n)$, the unitary group, and $G(n)$ is the semi-direct product, $G(n) = U(n) \ltimes Herm(n, \mathbb{C})$. Then $K = U(\infty)$ is the infinite dimensional unitary group, and $G = U(\infty) \ltimes H(\infty)$. These results are due to Pickrell, Olshanski, and Vershik. The main source is the beautiful paper

Ergodic unitarily invariant measures on the space of infinite Hermitian matrices,

by Olshanski and Vershik [1996].

Surprisingly these results involve the theory of totally positive functions which has been developped by Schoenberg ([1951]) without any connection with group theory. It is an interesting aspect of this infinite dimensional harmonic analysis that it involves many topics of classical analysis. One can say the same about the theory of random matrices. But in fact this infinite dimensional harmonic analysis involves probability measures on spaces of infinite dimensional matrices, and hence belongs to the theory of random matrices.

We will study a family of special functions in one variable depending on infinitely many parameters. They show up naturally in the harmonic analysis of $H(\infty)$. So far I know they don't have any name. We will call them Pólya functions because they appeared in several Pólya's papers.

A similar analysis has been studied on the infinite dimensional unitary group $U(\infty)$. In [1976], Voiculescu establish an integral representation for functions of positive type on $U(\infty)$ which are central. See also [Olshanski,2001], [Borodin-Olshanski,2001].

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CONTENTS

Chapter I

ROTATION INVARIANT PROBABILITY MEASURES ON $\mathbb{R}^\infty$

We give first some prelimminaries about probability measures on the infinite dimensional vector space $\mathbb{R}^\infty$, and convergence of sequences of such measures. Then, as a propotype for the infinite dimensional harmonic analysis, we present classical results about probability measures on $\mathbb{R}^\infty$ which are rotation invariant. These results are due to Schoenberg [1938]. We give a proof which differs from the original one, and ressembles to the proof of the theorem of Pickrell we will present in Chapter V.

1. Measures on $\mathbb{R}^\infty$. — The space $\mathbb{R}^\infty$ is the set of all sequences $x = (x_1, x_2, \ldots)$ of real numbers. One considers on $\mathbb{R}^\infty$ the product topology, *i.e.* the topology of pointwise convergence. This topology is metrizable: it can be defined by the following distance,

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \inf(|x_k - y_k|, 1).$$

The space $\mathbb{R}^\infty$ is complete, and separable. In fact the set

$$\mathbb{Q}^{(\infty)} = \bigcup_{n=1}^{\infty} \mathbb{Q}^n$$

of sequences of rationnal numbers with finite support is dense. The Borel σ-field will be denoted by $\mathcal{B}$. For every n we consider the projection

$$p_n : \mathbb{R}^\infty \to \mathbb{R}^n,\ x = (x_k) \mapsto (x_1, \ldots, x_n),$$

and, for $m < n$, the projection

$$p_{m,n} : \mathbb{R}^n \to \mathbb{R}^m.$$

Notice that $p_m = p_{m,n} \circ p_n$.

If μ is a positive measure on the measurable space $(\mathbb{R}^\infty, \mathcal{B})$, then $\mu_n = p_n(\mu)$ is a positive measure on $\mathbb{R}^n$. For $m < n$, clearly $\mu_m = p_{m,n}(\mu_n)$. The converse is the following

THEOREM I.1.1 (KOLMOGOROV CONSISTENCY THEOREM). — *Assume that $\{\mu_n\}$ is a family of positive measures, where μ_n is a measure on $\mathbb{R}^n$, such that*

$$p_{m,n}(\mu_n) = \mu_m.$$

One says that $\{\mu_n\}$ is a projective system or a consistent family. Then there exists a unique measure μ on $\mathbb{R}^\infty$ such that for all n,

$$p_n(\mu) = \mu_n.$$

[Parthasarathy,1967], Theorem 5.1, p.144.

One says that a sequence $\mu^{(\nu)}$ of probability measures on $V = \mathbb{R}^n$ or $\mathbb{R}^\infty$ converges *weakly* to a probability measure μ if, for any bounded continuous function f on V,

$$\lim_{\nu\to\infty} \int_V f(x)\mu^{(\nu)}(dx) = \int_V f(x)\mu(dx).$$

A sequence $\mu^{(\nu)}$ of probability measures on $\mathbb{R}^\infty$ converges weakly to a measure μ if and only if, for any n, the sequence $p_n\big(\mu^{(\nu)}\big)$ converges weakly to the measure $\mu_n = p_n(\mu)$. (See [Billingsley,1968].)

The infinite dimensional orthogonal group $O(\infty)$ is defined as

$$O(\infty) = \bigcup_{n=1}^{\infty} O(n).$$

We identify an orthogonal matrix $u \in O(n)$ with the infinite matrix

$$g = \begin{pmatrix} u & \vdots & & 0 \\ \cdots & & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix}.$$

An element $g \in O(\infty)$ defines a homeomorphism of $\mathbb{R}^\infty$. A measure μ on $\mathbb{R}^\infty$ is $O(\infty)$-invariant if and only if, for any n, the measure $\mu_n = p_n(\mu)$ is $O(n)$-invariant.

Example: Gaussian measure. For $t > 0$ fixed, the family of the measures $\{\gamma_{t,n}\}$ $(n \in \mathbb{N}^*)$,

$$\gamma_{t,n}(dx) = p_{t,n}(x)m_n(dx),$$

with

$$p_{t,n}(x) = \frac{1}{(\sqrt{2\pi t})^n} e^{-\frac{\|x\|^2}{2t}},$$

is consistent, hence defines a measure γ_t on $\mathbb{R}^\infty$ (m_n is the Lebesgue measure on $\mathbb{R}^n$). In fact

$$\int_{\mathbb{R}^{n-m}} p_{t,n}(x_1, \ldots, x_n, x_{m+1}, \ldots, x_n) dx_{m+1} \ldots dx_n = p_{t,m}(x_1, \ldots, x_m).$$

2. Fourier transform of measures. — Let μ be a bounded positive measure on $\mathbb{R}^n$. Its Fourier transform is defined by

$$\varphi(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \mu(dx).$$

It is a bounded continuous function,

$$|\varphi(\xi)| \leq \varphi(0) = \int_{\mathbb{R}^n} \mu(dx),$$

of positive type : for $\xi^1, \ldots, \xi^N \in \mathbb{R}^n$, $c_1, \ldots, c_N \in \mathbb{C}$,

$$\sum_{j,k=1}^{N} \varphi(\xi^j - \xi^k) c_j \bar{c}_k \geq 0.$$

In fact

$$\sum_{j,k=1}^{N} \varphi(\xi^j - \xi^k) c_j \bar{c}_k = \int_{\mathbb{R}^n} \Big| \sum_{j=1}^{N} c_j e^{-i\langle x,\xi^j\rangle} \Big|^2 \mu(dx).$$

BOCHNER THEOREM, I. — *Let φ be a continuous function on $\mathbb{R}^n$. The function φ is the Fourier transform of a bounded positive measure if and only if it is of positive type.*
([Bochner,1959], Theorem 23.)

The Fourier transform is a powerful tool for studying convergence of measures because of the following theorem.

LÉVY-CRAMÉR CONTINUITY THEOREM, I. — *Let $\mu^{(\nu)}$ be a sequence of bounded positive measures on $\mathbb{R}^n$, and let $\varphi^{(\nu)}$ be the Fourier transform of $\mu^{(\nu)}$. Assume that*

$$\forall \xi \in \mathbb{R}^n, \ \lim_{\nu\to\infty} \varphi^{(\nu)}(\xi) = \varphi(\xi),$$

and that φ is continuous at 0. *Then the sequence $\mu^{(\nu)}$ converges weakly to a measure μ whose φ is the Fourier transform.*

The dual space of the topological vector space $\mathbb{R}^\infty$ is the space

$$\mathbb{R}^{(\infty)} = \bigcup_{n=1}^{\infty} \mathbb{R}^n,$$

consisting in sequences with finitely many non zero elements. We consider on $\mathbb{R}^{(\infty)}$ the inductive limit topology. A function f defined on $\mathbb{R}^{(\infty)}$ is continuous if and only if, for every n, its restriction to $\mathbb{R}^n$ is continuous.

The Fourier transform of a bounded measure μ on $\mathbb{R}^\infty$ is the function φ defined on $\mathbb{R}^{(\infty)}$ by

$$\varphi(\xi) = \int_{\mathbb{R}^\infty} e^{-i\langle x,\xi\rangle} \mu(dx).$$

Notice that the restriction φ_n of φ to $\mathbb{R}^n$ is the Fourier transform of $\mu_n = p_n(\mu)$. From the theorems of Bochner (I) and Lévy-Cramér (I) for $\mathbb{R}^n$, and the theorem of Kolmogorov it follows :

Theorem I.2.1 (Bochner Theorem, II). — *Let φ a continuous function on $\mathbb{R}^{(\infty)}$. The function φ is the Fourier transform of a bounded positive measure μ on $\mathbb{R}^\infty$ if and only if it is of positive type.*

([Schwartz,1973], Proposition 2, p.187.)

Theorem I.2.2 (Lévy-Cramér continuity theorem, II). — *Let $\mu^{(\nu)}$ be a sequence of bounded positive measures on $\mathbb{R}^\infty$, and let $\varphi^{(\nu)}$ be the Fourier transform of $\mu^{(\nu)}$. Assume that*

$$\forall \xi \in \mathbb{R}^{(\infty)}, \ \lim_{\nu\to\infty} \varphi^{(\nu)}(\xi) = \varphi(\xi),$$

and that φ is continuous at 0. *Then the sequence $\mu^{(\nu)}$ converges weakly to a measure μ whose φ is the Fourier transform.*

3. Asymptotics of uniform measures on spheres. — As an illustration of the theorem of Lévy-Cramér we will present a classical example of convergence of probability measures on $\mathbb{R}^\infty$. Let us consider the uniform measure σ_r on the sphere of radius r and centre 0 in $\mathbb{R}^n$, with total measure equal to one. The Fourier transform φ_r of σ_r,

$$\varphi_r(\xi) = \int e^{-i\langle x,\xi\rangle} \sigma_r(d\xi),$$

is a radial function. It is essentialy a Bessel function:

$$\varphi_r(\xi) = \mathcal{J}_n(r\|\xi\|),$$

with

$$\mathcal{J}_n(z) = \Gamma\Big(\frac{n}{2}\Big)\Big(\frac{2}{z}\Big)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(z),$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{n(n+2)\dots(n+2m-2)} \frac{1}{m!}\Big(\frac{z^2}{2}\Big)^m.$$

Fix $t > 0$, and let $\sigma^{(k)}$ be the uniform measure on the sphere $S^{(k)}$ of radius $\sqrt{kt}$ and centre 0 in $\mathbb{R}^k$,

$$S^{(k)} = \{x = (x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1^2 + \dots + x_k^2 = kt\},$$

considered as a measure on $\mathbb{R}^\infty$.

Proposition I.3.1. — *As $k \to \infty$, the measure $\sigma^{(k)}$ converges weakly to the Gaussian measure γ_t whose Fourier transform is given by*

$$\psi_t(\xi) = e^{-\frac{t}{2}\rho^2}, \quad \rho^2 = \sum_{n=1}^{\infty} \xi_n^2 \quad (\xi \in \mathbb{R}^{(\infty)}).$$

Proof. Let $\psi^{(k)}$ be the Fourier transform of $\sigma^{(k)}$. For $\xi \in \mathbb{R}^n$, $n < k$

$$\psi^{(k)}(\xi) = \mathcal{J}_k(\rho\sqrt{k})$$
$$= \sum_{m=0}^{\infty} \frac{k^m}{k(k+2)\dots(k+2m-2)} \frac{(-1)^m}{m!}\Big(t\frac{\rho^2}{2}\Big)^m.$$

From

$$\lim_{k\to\infty} \frac{k^m}{k(k+2)\dots(k+2m-2)} = 1,$$
$$0 < \frac{k^m}{k(k+2)\dots(k+2m-2)} \le 1,$$

it follows that

$$\lim_{k\to\infty} \varphi^{(k)}(\xi) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Big(t\frac{\rho^2}{2}\Big)^m$$
$$= e^{-\frac{t}{2}\rho^2}.$$

∎

The statement means that, for every n, the projection $\sigma_n^{(k)} = p_n\big(\sigma^{(k)}\big)$ of the measure $\sigma^{(k)}$ on $\mathbb{R}^n$ converges as $k \to \infty$ to the Gaussian measure $\gamma_{t,n}$ on $\mathbb{R}^n$ given by

$$\gamma_{t,n}(dx) = \frac{1}{\big(\sqrt{2\pi t}\big)^n} e^{-\frac{1}{2t}\|x\|^2} m_n(dx),$$

where m_n is the Lebesgue measure on $\mathbb{R}^n$. For $n < k$, this projection $\sigma_n^{(k)} = p_n\big(\sigma^{(k)}\big)$ can be computed:

$$\int_{\mathbb{R}^n} f(x)\sigma_n^{(k)}(dx)$$
$$= \pi^{-\frac{n}{2}} \frac{\Gamma\big(\frac{k}{2}\big)}{\Gamma\big(\frac{k-n}{2}\big)}(kt)^{1-\frac{k}{2}} \int_{\|x\|\leq\sqrt{kt}} f(x)\big(kt - \|x\|^2\big)^{\frac{k-n-2}{2}} m_n(dx).$$

4. Invariant probability measures. — We come now to the determination of the probability measures on $\mathbb{R}^\infty$ which are invariant by $O(\infty)$. Let μ be an even probability measure on $\mathbb{R}$, and φ be its Fourier transform. Then define the function φ_n on $\mathbb{R}^n$ by

$$\varphi_n(\xi) = \varphi(\|\xi\|).$$

It is a bounded function, therefore is the Fourier transform of a tempered distribution T_n on $\mathbb{R}^n$: for every function f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$,

$$\langle T_n, f\rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\varphi(\|\xi\|)dm_n(\xi).$$

The distribution T_n, being radial, can be written as

$$\langle T_n, f\rangle = \langle \tau_n, \mathcal{M}f(r)\rangle,$$

where τ_n is an even distribution on $\mathbb{R}$, and, for $r \geq 0$, $\mathcal{M}f(r)$ is the mean of f on the sphere of radius r,

$$\mathcal{M}f(r) = \int f(u)\sigma_r(du),$$

where σ_r is the uniform measure on the sphere of radius r, the function $\mathcal{M}f$ being extended as an even function on $\mathbb{R}$.

PROPOSITION I.4.1. — *For $n = 2k+1$,*

$$\tau_n = \frac{(-1)^k}{1.3\ldots(2k-1)} r^{2k}\Big(\frac{1}{r}\frac{d}{dr}\Big)^k \mu.$$

Proof.

We saw that the Bessel function $\mathcal{J}_n$ has the following Taylor expansion:

$$\mathcal{J}_n(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{n(n+2)\ldots(n+2m-2)} \frac{1}{m!}\Big(\frac{r^2}{2}\Big)^m.$$

By a simple computation it follows that, for $n = 2k + 1$,

$$\mathcal{J}_{2k+1}(r) = (-1)^k 1.3 \dots (2k-1) \Big(\frac{1}{r}\frac{d}{dr}\Big)^k \cos r.$$

Let f be a radial function in $\mathcal{S}(\mathbb{R}^n)$: $f(x) = F(\|x\|)$, where F is an even Schwartz function on $\mathbb{R}$. Its Fourier transform $\hat{f}$,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x) m_n(dx),$$

is also radial: $\hat{f}(\xi) = \tilde{F}(\|\xi\|)$, with

$$\tilde{F}(\rho) = c_n \int_{\mathbb{R}} \mathcal{J}_n(r\rho) F(r) r^{n-1} dr,$$

where

$$c_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\big(\frac{n}{2}\big)}.$$

For $n = 2k + 1$,

$$\rho^{2k} \mathcal{J}_{2k+1}(r\rho) = (-1)^k a_k \Big(\frac{1}{r}\frac{d}{dr}\Big)^k \cos r\rho,$$

with $a_k = 1.3. \dots (2k-1)$. Therefore

$$\rho^{2k} \tilde{F}(\rho) = (-1)^k a_k c_{2k+1} \int_{\mathbb{R}} \Big(\frac{1}{r}\frac{d}{dr}\Big)^k \cos r\rho F(r) r^{2k} dr,$$

and, by integrating by part,

$$= a_k c_{2k+1} \int_{\mathbb{R}} \cos r\rho \Big(\frac{d}{dr}\frac{1}{r}\Big)^k \big(r^{2k} F(r)\big) dr.$$

This means that $\rho^{2k}\tilde{F}$ is the one dimensional Fourier transform of

$$a_k c_{2k+1} \Big(\frac{d}{dr}\frac{1}{r}\Big)^k \big(r^{2k} F(r)\big).$$

Let us now come back to the distributions τ_n:

$$\begin{aligned}
\langle T_n, f\rangle &= \langle \tau_n, F\rangle \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\|\xi\|) \hat{f}(\xi) dm_n(\xi) \\
&= \frac{c_n}{(2\pi)^n} \int_{\mathbb{R}} \varphi(\rho) \tilde{F}(\rho) \rho^{n-1} d\rho.
\end{aligned}$$

For $n = 2k+1$,

$$\begin{aligned}\langle \tau_{2k+1}, F\rangle &= \frac{c_{2k+1}}{(2\pi)^{2k+1}} \int_{\mathbb{R}} \varphi(\rho)\rho^{2k}\tilde{F}(\rho)d\rho \\ &= \frac{c_{2k+1}}{(2\pi)^{2k}} a_k c_{2k+1} \int_{\mathbb{R}} \Big(\frac{d}{dr}\frac{1}{r}\Big)^k \big(r^{2k}F(r)\big)\mu(dr) \\ &= (-1)^k a_k \frac{(c_{2k+1})^2}{(2\pi)^{2k}} \int_{\mathbb{R}} F(r) r^{2k} \Big(\frac{1}{r}\frac{d}{dr}\Big)^k \mu(dr).\end{aligned}$$

This is the statement, since

$$c_{2k+1} = \frac{(2\pi)^k}{1.3\ldots(2k-1)}.$$

∎

Let us recall that a function Φ defined on an open interval $]\alpha,\beta[$ $(-\infty \leq \alpha < \beta \leq \infty)$ is said to be *completely monotone* if f is $\mathcal{C}^\infty$ and, for all $n \geq 0$,

$$(-1)^n \Phi^{(n)}(u) \geq 0 \quad (\alpha < u < \beta).$$

THEOREM I.4.2 (THEOREM OF BERNSTEIN). — *A continuous function Φ on $[0,\infty[$ is completely monotone on $]0,\infty[$ if and only if it is the Laplace transform of a bounded positive measure ν on $[0,\infty[$,*

$$\Phi(u) = \int_{[0,\infty[} e^{-tu}\nu(dt).$$

([Widder,1946], Theorem 12a.)

We are now ready to state the main result:

THEOREM I.4.3 (SCHOENBERG). — *Let Φ be a continuous function on $[0,\infty[$, with $\Phi(0) = 1$. The following properties are equivalent:*

(i) *The function φ defined on $\mathbb{R}^{(\infty)}$ by*

$$\varphi(\xi) = \Phi\big(\tfrac{1}{2}\,\|\xi\|^2\big),$$

is of positive type.

(ii) *Φ is completely monotone on $]0,\infty[$,*

(iii) *There is a probability measure ν on $[0,\infty[$ such that*

$$\Phi(u) = \int_0^\infty e^{-tu}\nu(dt).$$

[Schoenberg,1938]

Proof.
By the theorem of Bernstein (I.4.2) properties (i) and (ii) are equivalent.

Assume (iii), then

$$\varphi(\xi) = \Phi(\tfrac{1}{2}\,\rho^2) = \int_{[0,\infty[} e^{-\frac{t}{2}\rho^2}\nu(dt),$$

where ν is a probability measure on $[0,\infty[$. It follows that the restriction φ_n of φ to $\mathbb{R}^n$ is the Fourier transform of the measure

$$\mu_n(dx) = f_n(\|x\|)m_n(dx) + a\delta,$$

with

$$f_n(r) = \int_{]0,\infty[} \frac{1}{(\sqrt{2\pi t})^n} e^{-\frac{r^2}{2t}}\nu(dt),$$

$$a = \nu(\{0\}).$$

Therefore the function φ is of positive type.

Assume now (i): the function φ,

$$\varphi(\xi) = \Phi\big(\tfrac{1}{2}\,\|\xi\|^2\big),$$

is of positive type. For every n the restriction φ_n of φ to $\mathbb{R}^n$ is the Fourier transform of a probability measure μ_n on $\mathbb{R}^n$. The measure μ_n, being radial, can be written as

$$\int_{\mathbb{R}^n} f(x)\mu_n(dx) = \int_{\mathbb{R}} \mathcal{M}f(r)\tilde{\mu}_n(dr),$$

where $\tilde{\mu}_n$ is an even probability measure on $\mathbb{R}$. By Proposition I.4.1, for $n = 2k+1$,

$$\tilde{\mu}_{2k+1} = \frac{(-1)^k}{1.3\cdots(2k-1)} r^{2k}\Big(\frac{1}{r}\frac{d}{dr}\Big)^k \mu_1.$$

Therefore, for every k, the restriction to $]0,\infty[$ of the distribution

$$\Big(\frac{1}{r}\frac{d}{dr}\Big)^k \mu_1$$

is a positive measure, and the restriction to $]0,\infty[$ of the measure μ_1 is of the form $h(r^2)dr$ where the function h is completely monotone. By the theorem of Bernstein, h is the Laplace transform of a positive measure α on $[0,\infty[$,

$$h(u) = \int_{[0,\infty[} e^{-su}\alpha(du),$$

or

$$\mu_1(dr) = f_1(r)dr + \mu_1(\{0\})\delta,$$

with

$$f_1(r) = \int_{[0,\infty[} e^{-sr^2}\alpha(ds).$$

Let $\tilde{\alpha}$ be the image of α through the map $s \mapsto \frac{1}{2t}$, and define

$$\nu(dt) = \frac{1}{\sqrt{2\pi t}}\tilde{\alpha}(dt) + \mu_1(\{0\})\delta.$$

Then

$$\varphi(\xi) = \int_{[0,\infty[} e^{-\frac{t}{2}\rho^2}\nu(dt).$$

Example: The Cauchy measure

For $s > 0$, we consider the following probability measure on $[0,\infty[$,

$$\nu_s(dt) = \frac{s}{\sqrt{2\pi}}e^{-\frac{s^2}{2t}}t^{-\frac{3}{2}}dt.$$

The measures μ_n have densities:

$$\mu_n(dx) = q_{s,n}(x)m_n(dx),$$

with

$$q_{s,n}(x) = \int_0^\infty \frac{1}{(\sqrt{2\pi t})^n}e^{-\frac{\|x\|^2}{2t}}\frac{s}{\sqrt{2\pi}}e^{-\frac{s}{2t}}t^{-\frac{3}{2}}dt.$$

By letting $t = \frac{1}{u}$ one computes easily the integral:

$$q_{s,n}(x) = \pi^{-\frac{n+1}{2}}\Gamma\big(\frac{n+1}{2}\big)\frac{s}{(s^2+\|x\|^2)^{-\frac{n+1}{2}}}.$$

In particular, for $n = 1$,

$$q_{s,1}(x) = \frac{1}{\pi}\frac{s}{s^2+x^2}.$$

The function φ is given by

$$\begin{aligned}\varphi(\xi) &= \int_0^\infty e^{-\frac{t}{2}\rho^2}\nu_s(dt) \\ &= \int_0^\infty e^{-\frac{t}{2}\rho^2}\frac{s}{\sqrt{2\pi}}e^{-\frac{s^2}{2t}}t^{-\frac{3}{2}}dt \\ &= e^{-s\rho} = e^{-s\|\xi\|}.\end{aligned}$$

According to Proposition I.4.1,

$$c_{2k+1}q_{s,2k+1}(r)r^{2k} = \frac{(-1)^k}{1.3\cdots(2k-1)}r^{2k}\Big(\frac{1}{r}\frac{d}{dr}\Big)^k q_{s,1}(r).$$

The function $q_{s,n}(x)$ is the Poisson kernel for the half-space $\mathbb{R}^n \times \mathbb{R}^+$: If f is a bounded continuous function on $\mathbb{R}^n$, then

$$u(x,s) = \int_{\mathbb{R}^n} q_{s,n}(x-y)f(y)m_n(dy)$$

is the unique bounded solution of the Dirichlet problem :

$$\begin{aligned}\Delta u + \frac{\partial^2 u}{\partial s^2} &= 0, \\ u(x,s) &= f(x).\end{aligned}$$

This explains why the family of the measures $\{\mu_n\}$ is consistent:

$$\int_{\mathbb{R}} q_{s,n}(x_1,\ldots,x_{n-1},x_n)dx_n = q_{s,n-1}(x_1,\ldots,x_{n-1}).$$

In fact, if the boundary function f does not depend on x_n, the Dirichlet problem reduces to a Dirichlet problem on $\mathbb{R}^{n-1}$.

Chapter II

HARMONIC ANALYSIS ON THE SPACE OF HERMITIAN MATRICES

Let $H_n = Herm(n, \mathbb{C})$ denote the space of $n \times n$ Hermitian matrices. We consider on H_n the Euclidean inner product given by

$$(x|y) = \mathrm{tr}(xy).$$

The unitary group $U(n)$ acts on H_n by the isometric transformations

$$T_u(x) = uxu^*.$$

Every Hermitian matrix $x \in H_n$ is diagonalizable in an orthonormal basis, and its eigenvalues are real; this means that x can be written

$$x = uau^*,$$

where u is a unitary matrix, $a = \mathrm{diag}(a_1, \ldots, a_n)$, and $a_1, \ldots, a_n$ are the eigenvalues of x.

1. Weyl integration formulas. — Let D_n denote the space of real diagonal matrices, $D_n \simeq \mathbb{R}^n$. For $x \in \mathbb{R}^n$ or $\mathbb{C}^n$, the Vandermonde polynomial is defined by

$$D(x) = \prod_{j<k} (x_j - x_k).$$

Let m be the Euclidean measure on H_n, and α the normalized Haar measure on the unitary group $U(n)$.

THEOREM II.1.1. — *There exists a constant $C_n > 0$ such that, if f is an integrable function on H_n,*

$$\int_{H_n} f(x) m(dx) = C_n \int_{D_n} \Bigl(\int_{U(n)} f(uau^*) \alpha(du) \Bigr) D(a)^2 da_1 \ldots da_n.$$

In particular, if f is $U(n)$-invariant :

$$f(uxu^*) = f(x) \qquad (u \in U(n)),$$

then

$$\int_{H_n} f(x)m(dx) = C_n \int_{D_n} f(a)D(a)^2 da_1 \dots da_n.$$

There is a similar integration formula for the group $U(n)$. If f is an integrable function on $U(n)$ which is central :

$$f(ugu^{-1}) = f(g) \qquad (u \in U(n))$$

(note that $u^{-1} = u^*$), then

$$\int_{U(n)} f(g)\alpha(dg) = \frac{1}{n!}\int_{T_n} f(t)|D(t)|^2\beta(dt),$$

where T_n is the set of unitary diagonal matrices, $T_n \simeq \mathbb{T}^n$, and β is the normalized Haar measure on T_n : if $t = \mathrm{diag}(t_1, \dots, t_n)$, with $t_j = e^{i\theta_j}$, then

$$\beta(dt) = \frac{1}{(2\pi)^n} d\theta_1 \dots d\theta_n.$$

2. Schur functions. — Let f be a function on the unitary group which is central :

$$f(ugu^{-1}) = f(g) \qquad (u \in U(n)),$$

then f is determined by its restriction to the subgroup T_n of unitary diagonal matrices. Define

$$F(t_1, \dots, t_n) = f\big(\mathrm{diag}(t_1, \dots, t_n)\big).$$

Then F is a symmetric function of $t_1, \dots, t_n$, i.e. is a function which is invariant under the group $\mathfrak{S}_n$ of permutations. Furthermore the map $f \mapsto F$ is a bijection from the space of central functions on $U(n)$ to the space of symmetric functions on $\mathbb{T}^n$.

For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, $t^{\mathbf{m}}$ denotes the corresponding momonial

$$t^{\mathbf{m}} = t_1^{m_1} \dots t_n^{m_n}.$$

Let $\mathcal{F}(\mathbb{T}^n)$ be the space of trigonometric polynomials, i.e. functions of the form

$$p(t) = \sum_{\mathbf{m}\in\mathbb{Z}^n} a_{\mathbf{m}} t^{\mathbf{m}},$$

where the coefficients $a_{\mathbf{m}}$ are complex numbers, only a finite number of them being non zero. The polynomial p is said to be symmetric if, for every permutation $\sigma \in \mathfrak{S}_n$,

$$p(\sigma \cdot t) = p(t),$$

where $\sigma \cdot t = (t_{\sigma(1)}, \ldots, t_{\sigma(n)})$, and skewsymmetric if

$$p(\sigma \cdot t) = \varepsilon(\sigma) p(t)$$

($\varepsilon(\sigma)$ is the signature of the permutation σ). Let $\mathcal{F}_0(\mathbb{T}^n)$ denote the space of symmetric trigonometric polynomials, and $\mathcal{F}_1(\mathbb{T}^n)$ the space of antisymmetric ones.

For $\mathbf{m} = (m_1, \ldots, m_n)$, $m_1 > \cdots > m_n$ the polynomial

$$A_{\mathbf{m}}(t) = \begin{vmatrix} t_1^{m_1} & \ldots & t_1^{m_n} \\ \vdots & & \vdots \\ t_n^{m_1} & \ldots & t_n^{m_n} \end{vmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) t^{\sigma \cdot \mathbf{m}}$$

is skewsymmetric. In particular, for $\mathbf{m} = \delta := (n-1, n-2, \ldots, 0)$, A_δ is the Vandermonde polynomial, $A_\delta = D$. The polynomials $A_{\mathbf{m}}$, for $m_1 > \cdots > m_n$ constitute an orthonormal basis of $\mathcal{F}_1(\mathbb{T}^n)$, and

$$\int_{\mathbb{T}^n} |A_{\mathbf{m}}(t)|^2 \beta(dt) = \#(\mathfrak{S}_n) = n!.$$

For $m_1 \geq \cdots \geq m_n$, the *Schur* function $s_{\mathbf{m}}$ is defined by

$$s_{\mathbf{m}}(t) = \frac{A_{\mathbf{m}+\delta}(t)}{D(t)}.$$

It is a symmetric trigonometric polynomial, and the Schur functions constitute a basis of $\mathcal{F}_0(\mathbb{T}^n)$.

Let $\chi_{\mathbf{m}}$ be the central function on $U(n)$ whose restriction to $T_n \simeq \mathbb{T}^n$ is equal to $s_{\mathbf{m}}$. The functions $\chi_{\mathbf{m}}$ constitute a Hilbert basis of the space of (classes of) square integrable central functions on $U(n)$. The function $\chi_{\mathbf{m}}$ extends as a holomorphic function on $GL(n, \mathbb{C})$, and, if $m_n \geq 0$, as a polynomial function on $M(n, \mathbb{C})$.

The function $\chi_{\mathbf{m}}$ is the character of an irreducible representation $(\pi_{\mathbf{m}}, \mathcal{H}_{\mathbf{m}})$ of $U(n)$,

$$\chi_{\mathbf{m}}(g) = \operatorname{tr} \pi_{\mathbf{m}}(g).$$

In particular

$$\chi_{\mathbf{m}}(e) = d_{\mathbf{m}} := \dim \mathcal{H}_{\mathbf{m}}.$$

The character $\chi_{\mathbf{m}}$ satisfies the following functional equation: for $x, y \in GL(n, \mathbb{C})$,

$$\int_{U(n)} \chi_{\mathbf{m}}(xuyu^{-1})\alpha(du) = \frac{1}{d_{\mathbf{m}}}\chi_{\mathbf{m}}(x)\chi_{\mathbf{m}}(y).$$

If $\mathbf{m} = (m, 0, \dots, 0)$ $(m \geq 0)$, then $s_{\mathbf{m}}(t) = h_m(t)$, the complete symmetric function:

$$h_m(t) = \sum_{|\alpha|=m} t^\alpha.$$

The generating function of the functions h_m is given by

$$H(t,z) = \sum_{m=0}^{\infty} h_m(t)z^m = \prod_{j=1}^{n} \frac{1}{1 - zt_j}.$$

In fact

$$\begin{aligned}\sum_{\alpha \in \mathbb{N}^n} (zt)^\alpha &= \sum_{m=0}^{\infty}\Big(\sum_{|\alpha|=m} t^\alpha\Big)z^m = \sum_{m=0}^{\infty} h_m(t)z^m \\ &= \prod_{j=1}^{n}\Big(\sum_{\alpha_j=0}^{\infty} (zt_j)^{\alpha_j}\Big) = \prod_{j=1}^{n} \frac{1}{1 - zt_j}.\end{aligned}$$

3. Schur functions expansions. — To a function F defined on the circle $\mathbb{T}$ with an absolute convergent Fourier series:

$$F(t) = \sum_{m=-\infty}^{\infty} c_m t^m, \qquad \sum_{m=-\infty}^{\infty} |c_m| < \infty,$$

we associate the function f on the unitary group $U(n)$ by

$$f(g) = \det F(g).$$

It means that f is a central function: $f(ugu^{-1}) = f(g)$, and

$$f\big(\mathrm{diag}(t_1, \dots, t_n)\big) = F(t_1)\dots F(t_n).$$

Proposition II.3.1. — *The Fourier expansion of f on $U(n)$ is given by*

$$f(g) = \sum_{m_1 \geq \dots \geq m_n} a_{\mathbf{m}} \chi_{\mathbf{m}}(g),$$

with

$$a_{\mathbf{m}} = \det\big((c_{m_i-i+j})_{1\le i,j\le n}\big).$$

Equivalently

$$\prod_{i=1}^n F(t_i) = \sum_{m_1\ge\cdots\ge m_n} a_{\mathbf{m}} s_{\mathbf{m}}(t_1,\ldots,t_n).$$

[Voiculescu,1976], Lemme 2.

If the Fourier series of F extends as a Laurent series in a crown $r_1 < |t| < r_2$, the Fourier expansion of f extends as a Fourier-Laurent series in the domain in $M(n,\mathbb{C})$ defined by

$$\{g = u_1 \mathrm{diag}(t_1,\ldots,t_n)u_2 | u_1,u_2 \in U(n),\ t_j\in\mathbb{C},\ r_1<|t_j|<r_2\}.$$

Proof.

We will give two proofs of this formula.

a) Let us expand the following product

$$\begin{aligned} &D(t)F(t_1)\ldots F(t_n)\\ &= \sum_{\sigma\in\mathfrak{S}_n}\sum_{p_1,\ldots,p_n=-\infty}^{\infty} \varepsilon(\sigma)c_{p_1}\ldots c_{p_n} t_1^{p_1+\delta_{\sigma(1)}}\ldots t_n^{p_n+\delta_{\sigma(n)}}.\end{aligned}$$

The number $a_{\mathbf{m}}$ is, in this sum, the coefficient of the monomial $t_1^{m_1+\delta_1}\ldots t_n^{m_n+\delta_n}$. It comes from the terms for which $p_i+\delta_{\sigma(i)} = m_i+\delta_i$ or

$$p_i = m_i + (n-i) - \big(n-\sigma(i)\big) = m_i - i + \sigma(i).$$

Therefore

$$a_{\mathbf{m}} = \sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\prod_{k=1}^n c_{m_i-i+\sigma(i)} = \det\big((c_{m_i-i+j})_{1\le i,j\le n}\big).$$

b) We can also start from the integral formula which gives the Fourier coefficient of a Fourier expansion for a central function on $U(n)$:

$$a_{\mathbf{m}} = \frac{1}{n!}\int_{\mathbb{T}^n}\prod_{i=1}^n F(t_i)\overline{s_{\mathbf{m}}(t)}|D(t)|^2\beta(dt).$$

Since

$$s_{\mathbf{m}}(t) = \frac{A_{\mathbf{m}+\delta}(t)}{D(t)},$$

this integral can be written

$$a_{\mathbf{m}} = \frac{1}{n!}\int_{\mathbb{T}^n}\prod_{i=1}^{n} F(t_i)\overline{A_{\mathbf{m}+\delta}(t)}D(t)\beta(dt).$$

Let us compute

$$\begin{aligned}&A_{\mathbf{m}+\delta}(t)\overline{D(t)}\\ &= \Big(\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\prod_{i=1}^{n} t_{\sigma(i)}^{m_i+\delta_i}\Big)\Big(\sum_{\sigma'\in\mathfrak{S}_n}\varepsilon(\sigma')\prod_{i=1}^{n} t_{\sigma'(j)}^{-\delta_j}\Big).\end{aligned}$$

By putting $\sigma' = \sigma\circ\tau$, and then $\tau^{-1}(j) = i$, we obtain

$$= \sum_{\sigma}\sum_{\tau}\varepsilon(\tau)\prod_{i=1}^{n} t_{\sigma(i)}^{m_i+\delta_i-\delta_{\tau(i)}}.$$

By integrating this gives

$$\int_{\mathbb{T}^n}\prod_{i=1}^{n} F(t_i)\overline{A_{\mathbf{m}+\delta}(t)}D(t)\beta(dt) = n!\det\big((c_{m_i-i+j})_{1\le i,j\le n}\big).\qquad\square$$

For the last part of this section we follow [Hua,1963], Chapter II, 1.2. Let us consider n Taylor series

$$f_i(z) = \sum_{m=0}^{\infty} c_m^{(i)} z^m \qquad (i = 1,\ldots,n),$$

which are convergent for $|z| < r$. For $z = (z_1,\ldots,z_n)$, with $|z_j| < r$,

$$\det\Big(\big(f_i(z_j)\big)_{1\le i,j\le n}\Big) = \sum_{m_1>\cdots>m_n\ge 0}\det\Big(\big(c_{m_j}^{(i)}\big)_{1\le i,j\le n}\Big)A_{\mathbf{m}}(z).$$

In fact

$$\begin{aligned}&\det\Big(\big(f_i(z_j)\big)_{1\le i,j\le n}\Big)\\ &= \sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)f_1\big(z_{\sigma(1)}\big)\ldots f_n\big(z_{\sigma(n)}\big)\\ &= \sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\Big(\sum_{m_1=0}^{\infty} c_{m_1}^{(1)} z_{\sigma(1)}^{m_1}\Big)\cdots\Big(\sum_{m_n=0}^{\infty} c_{m_n}^{(n)} z_{\sigma(n)}^{m_n}\Big)\\ &= \sum_{m_1,\ldots,m_n=0}^{\infty} c_{m_1}^{(1)}\ldots c_{m_n}^{(n)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)z_{\sigma(1)}^{m_1}\ldots z_{\sigma(n)}^{m_n}\\ &= \sum_{m_1>\cdots>m_n\ge 0}\Big(\sum_{\tau\in\mathfrak{S}_n}\varepsilon(\tau)c_{m_{\tau(1)}}^{(1)}\ldots c_{m_{\tau(n)}}^{(n)}\Big)A_{\mathbf{m}}(z_1,\ldots,z_n).\end{aligned}$$

This can be written

$$\det\Big(\big(f_i(z_j)\big)_{1\le i,j\le n}\Big) = D(z) \sum_{m_1\ge\cdots\ge m_n\ge 0} a_{\mathbf{m}} s_{\mathbf{m}}(z),$$

where

$$a_{\mathbf{m}} = \det\Big(\big(c^{(i)}_{m_j+\delta_j}\big)_{1\le i,j\le n}\Big).$$

Looking at the value at $z = 0$ of the series, one obtains

$$\lim_{z\to 0} \frac{\det\Big(\big(f_i(z_j)\big)_{1\le i,j\le n}\Big)}{D(z)} = a_{\mathbf{0}} = \det\Big(\big(c^{(i)}_{\delta_j}\big)_{1\le i,j\le n}\Big).$$

Since

$$c^{(i)}_m = \frac{1}{m!} f_i^{(m)}(0),$$

the coefficient $a_{\mathbf{0}}$ can be written

$$a_{\mathbf{0}} = \frac{1}{\delta!} \det\Big(\big(f_i^{(n-j)}(0)\big)_{1\le i,j\le n}\Big).$$

In general, for $\mathbf{m} = (m_1, \ldots, m_n)$ with $m_j \ge 0$, one defines

$$\mathbf{m}! = m_1! \ldots m_n!.$$

It follows that, for every a,

$$\lim_{z\to(a,\ldots,a)} \frac{\det\Big(\big(f_i(z_j)\big)_{1\le i,j\le n}\Big)}{D(z)} = \frac{1}{\delta!} \det\Big(\big(f_i^{(n-j)}(a)\big)_{1\le i,j\le n}\Big).$$

For instance, if

$$f_i(z) = z^{m_i},$$

then

$$\det\Big(\big(f_i(z_j)\big)_{1\le i,j\le n}\Big) = A_{\mathbf{m}}(z).$$

From the formula

$$f_i^{(n-j)}(1) = m_i(m_i-1)\ldots(m_i-n+j-1),$$

it follows that

$$\det\Big(\big(f_i^{(n-j)}(1)\big)_{1\le i,j\le n}\Big) = D(\mathbf{m}),$$

and one obtains the Weyl formula for the dimension:

$$d_{\mathbf{m}} = s_{\mathbf{m}}(1, \ldots, 1) = \frac{D(\mathbf{m} + \delta)}{D(\delta)}$$

(by observing that $D(\delta) = \delta!$).

Consider now one Taylor series

$$f(z) = \sum_{m=0}^{\infty} c_m z^m,$$

and n complex numbers $x_1, \ldots, x_n$. Form the n Taylor series

$$f_i(z) = f(x_i z) = \sum_{m=0}^{\infty} c_m x_i^m z^m.$$

Then $c_m^{(i)} = c_m x_i^m$, and

$$a_{\mathbf{m}} = \det\Big(\big(c_{m_j}^{(i)}\big)_{1 \le i,j \le n}\Big) = \det\Big(\big(c_{m_j} x_i^{m_i}\big)_{1 \le i,j \le n}\Big) = c_{m_1} \ldots c_{m_n} A_{\mathbf{m}}(x).$$

Therefore

$$\det\Big(\big(f(x_i y_j)\big)_{1 \le i,j \le n}\Big) = \sum_{m_1 > \cdots > m_n \ge 0} c_{m_1} \ldots c_{m_n} A_{\mathbf{m}}(x) A_{\mathbf{m}}(y),$$

or

$$\frac{\det\Big(\big(f(x_i y_j)\big)_{1 \le i,j \le n}\Big)}{D(x)D(y)} = \sum_{m_1 \ge \ldots \ge m_n \ge 0} c_{m_1+\delta_1} \ldots c_{m_n+\delta_n} s_{\mathbf{m}}(x) s_{\mathbf{m}}(y).$$

Consider the limit of the left handside as $y \to (a, \ldots, a)$. Since

$$f_i^{(n-j)}(a) = x_i^{n-j} f^{(n-j)}(x_i a),$$

we obtain

$$\lim_{y \to (a,\ldots,a)} \frac{\det\Big(\big(f(x_i y_j)\big)_{1 \le i,j \le n}\Big)}{D(x)D(y)} = \frac{1}{\delta!} \frac{\det\Big(\big(x_i^{n-j} f^{(n-j)}(x_i a)\big)_{1 \le i,j \le n}\Big)}{D(x)}.$$

By specializing now to the case

$$f(z) = e^z, \quad c_m = \frac{1}{m!},$$

we obtain

Proposition II.3.2.

$$\frac{\det\Big(\big(e^{x_i y_j}\big)_{1\le i,j\le n}\Big)}{D(x)D(y)} = \sum_{m_1\ge\cdots\ge m_n\ge 0} \frac{1}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(x)s_{\mathbf{m}}(y).$$

Furthermore

$$f_i^{(n-j)}(1) = x_i^{n-j}e^{x_i},$$

and

$$\det\Big(\big(x_i^{(n-j)}e^{x_i}\big)_{1\le i,j\le n}\Big) = D(x)\exp(x_1+\cdots+x_n).$$

Hence, as $y \to (1,\ldots,1)$, we obtain

$$\exp(x_1+\cdots+x_n) = \delta! \sum_{m_1\ge\cdots\ge m_n\ge 0} \frac{1}{(\mathbf{m}+\delta)!} d_{\mathbf{m}} s_{\mathbf{m}}(x).$$

Considering the corresponding central functions we obtain, for $x \in M(n,\mathbb{C})$, the following expansion:

Proposition II.3.3.

$$\exp(\operatorname{tr} x) = \delta! \sum_{m_1\ge\cdots\ge m_n\ge 0} \frac{1}{(\mathbf{m}+\delta)!} d_{\mathbf{m}} \chi_{\mathbf{m}}(x).$$

Consider now the case

$$f(z) = \frac{1}{1-z} = \sum_{m=0}^{\infty} z^m \quad (|z|<1).$$

Since $c_m = 1$ for all m, we obtained

$$\frac{1}{D(x)D(y)} \det\Big(\big(\frac{1}{1-x_i y_j}\big)_{1\le i,j\le n}\Big) = \sum_{m_1\ge\cdots\ge m_n\ge 0} s_{\mathbf{m}}(x)s_{\mathbf{m}}(y).$$

It is possible to evaluate this determinant; this is essentially the Cauchy determinant, and

$$\det\Big(\big(\frac{1}{1-x_i y_j}\big)_{1\le i,j\le n}\Big) = D(x)D(y) \prod_{i,j=1}^{n} \frac{1}{1-x_i y_j}.$$

(See for instance [Pólya-Szegö,1976] II, Part VII, No 3.)

Therefore:

Proposition II.3.4.

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} s_{\mathbf{m}}(x) s_{\mathbf{m}}(y).$$

If we apply Proposition II.3.1 to the function

$$F(x) = \prod_{j=1}^{n} \frac{1}{1 - x y_j} = \sum_{m=0}^{\infty} h_m(y) x^m,$$

we get

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \det\Big(\big(h_{m_i - i + j}(y)\big)_{1 \leq i,j \leq n}\Big) s_{\mathbf{m}}(x).$$

Comparing with the previous equality one obtains the so called Jacobi-Trudi identity:

$$s_{\mathbf{m}}(y) = \det\Big(\big(h_{m_i - i + j}(y)\big)_{1 \leq i,j \leq n}\Big).$$

4. Fourier transform of orbital measures. — For $x \in H_n$, the orbital measure μ_x is defined by

$$\int f(y) \mu_x(dy) = \int_{U(n)} f(uxu^*) \alpha(du),$$

where f is a continuous function on H_n. We will determine the Fourier transform of μ_x:

$$\begin{aligned} \widehat{\mu_x}(\xi) &= \int e^{-i \operatorname{tr}(y\xi)} \mu_x(dy) \\ &= \int_{U(n)} e^{-i \operatorname{tr}(uxu^* \xi)} \alpha(du). \end{aligned}$$

Note that $\widehat{\mu_x}(\xi)$ only depends on the eigenvalues of x and ξ. We will use the following notation. For a kernel $K(x,y)$ one defines

$$K\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} = \det\Big(\big(K(x_i, y_j)\big)_{1 \leq i,j \leq n}\Big).$$

Let E be the kernel defined on $\mathbb{R}^2$ by

$$E(x,y)=e^{xy}.$$

Theorem II.4.1. — *For $x,\xi\in H_n$,*

$$\widehat{\mu_x}(\xi)=\delta!\sum_{m_1\geq\cdots\geq m_n\geq 0}\frac{1}{(\mathbf{m}+\delta)!}\chi_{\mathbf{m}}(x)\chi_{\mathbf{m}}(-i\xi).$$

And, for $x=\mathrm{diag}(x_1,\ldots,x_n)$, $\xi=(\xi_1,\ldots,\xi_n)$,

$$\widehat{\mu_x}(\xi)=\delta!\frac{1}{D(x)D(-i\xi)}E\begin{pmatrix} x_1 & \ldots & x_n \\ -i\xi_1 & \ldots & -i\xi_n\end{pmatrix}.$$

This last formula is a special case of a formula proved by Harish-Chandra which gives the Fourier transform of an orbital measure for a compact semi-simple Lie group acting on its Lie algebra (see [Helgason,1984], Theorem 5.35).

Proof.
By Proposition II.3.3, for $z\in M(n,\mathbb{C})$,

$$e^{\operatorname{tr} z}=\sum_{m_1\geq\ldots\geq m_n\geq 0} d_{\mathbf{m}}\frac{\delta!}{(\mathbf{m}+\delta)!}\chi_{\mathbf{m}}(z),$$

and this series converges uniformly on compact sets. Therefore, for $x,y\in M(n,\mathbb{C})$,

$$\begin{aligned} f(x,y) :&= \int_{U(n)} e^{\operatorname{tr}(uxu^*y)}\alpha(du) \\ &= \sum_{m_1\geq\ldots\geq m_n\geq 0} d_{\mathbf{m}}\frac{\delta!}{(\mathbf{m}+\delta)!}\int_{U(n)}\chi_{\mathbf{m}}(uxu^*y)\alpha(du). \end{aligned}$$

Since

$$\int_{U(n)}\chi_{\mathbf{m}}(uxu^*y)\alpha(du)=\frac{1}{d_{\mathbf{m}}}\chi_{\mathbf{m}}(x)\chi_{\mathbf{m}}(y),$$

it follows that

$$f(x,y)=\sum_{m_1\geq\ldots\geq m_n\geq 0}\frac{\delta!}{(\mathbf{m}+\delta)!}\chi_{\mathbf{m}}(x)\chi_{\mathbf{m}}(y).$$

By Proposition II.3.2, if $x = \mathrm{diag}(x_1, \dots, x_n)$, $y = \mathrm{diag}(y_1, \dots, y_n)$,

$$f(x,y) = \delta! \frac{1}{D(x)D(y)} E \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix}. \qquad \square$$

As an application we will establish a formula for the Fourier transform of a $U(n)$-invariant function on H_n. Let $f \in \mathcal{S}(H_n)$ be $U(n)$-invariant, and define

$$F(a_1, \dots, a_n) = f\big(\mathrm{diag}(a_1, \dots, a_n)\big).$$

We denote by $\hat{f}$ the Fourier transform of f on H_n:

$$\hat{f}(\xi) = \int_{H_n} e^{-i\,\mathrm{tr}(x\xi)} f(x) m(dx),$$

and $\hat{F}$ the Fourier transform of F on $\mathbb{R}^n$,

$$\hat{F}(b) = \int_{\mathbb{R}^n} e^{-i(a|b)} F(a) da_1 \dots da_n.$$

Furthermore we write

$$\tilde{F}(b_1, \dots, b_n) = \hat{f}\big(\mathrm{diag}(b_1, \dots, b_n)\big).$$

Proposition II.4.2.

$$\tilde{F}(b) = C_n 1!2! \dots n! \frac{1}{D(b)} D\Big(\frac{\partial}{\partial b}\Big) \hat{F}(b).$$

Proof.

By using the Weyl integration formula (Theorem II.1.1)

$$\hat{f}(\xi) = C_n \int_{D_n} \Big(\int_{U(n)} e^{-i\,\mathrm{tr}(uau^*\xi)} \alpha(du) \Big) F(a) D(a)^2 da_1 \dots da_n.$$

By Theorem II.4.1, for $\xi = \mathrm{diag}(b_1, \dots, b_n)$,

$$\tilde{F}(b) = C_n \delta! \frac{1}{D(-ib)} \int_{D_n} E \begin{pmatrix} a_1 & \dots & a_n \\ -ib_1 & \dots & -ib_n \end{pmatrix} D(a) F(a) da_1 \dots da_n$$

$$= C_n \delta! \frac{1}{D(-ib)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \int_{D_n} e^{-i(a_1 b_{\sigma(1)} + \dots + a_n b_{\sigma(n)})} D(a) F(a) da_1 \dots da_n.$$

By classical properties of the Fourier transform,

$$G(b) := \int_{D_n} e^{-i(a_1 b_1 + \dots + a_n b_n)} D(a) F(a) da_1 \dots da_n$$

$$= D\Big(-\frac{1}{i} \frac{\partial}{\partial b} \Big) \hat{F}(b).$$

Observe further that G is skewsymmetric. Finally

$$\tilde{F}(b) = C_n \delta! n! \frac{1}{D(b)} D\Big(\frac{\partial}{\partial b}\Big) \hat{F}(b). \qquad \square$$

Chapter III

PÓLYA FUNCTIONS

1. Pólya functions, definition. — Let Φ be a continuous function on $\mathbb{R}$, with $\Phi(0) = 1$. For every n we associate to Φ a function φ_n on $H_n = Herm(n, \mathbb{C})$:

$$\varphi_n(x) = \det \Phi(x) \qquad (x \in H_n).$$

Note that $H_1 = \mathbb{R}$, $\varphi_1 = \Phi$, and that φ_n is the restriction of φ_{n+1} to H_{n+1} (with the natural embedding $H_n \subset H_{n+1}$). The function φ_n is $U(n)$-invariant, and, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x,

$$\varphi_n(x) = \Phi(\lambda_1) \ldots \Phi(\lambda_n).$$

We say that Φ is a *Pólya function* if, for all n, φ_n is of positive type. The set of Pólya functions is stable under multiplication and closed for the topology of uniform convergence on compact sets. Even more: if Φ_k is a sequence of Pólya functions such that, for all $\lambda \in \mathbb{R}$,

$$\lim_{k \to \infty} \Phi_k(\lambda) = \Phi(\lambda),$$

and if Φ is continuous at 0, then Φ is a Pólya function.

For $\beta \in \mathbb{R}$, the exponential function

$$\Phi(\lambda) = e^{i\beta\lambda}$$

is a Pólya function. In fact

$$\varphi_n(x) = e^{i\beta \operatorname{tr}(x)}$$

is of positive type on H_n.

For $\gamma > 0$, the Gauss function

$$\Phi(\lambda) = e^{-\frac{1}{2}\gamma\lambda^2}$$

is a Pólya function. In fact

$$\varphi_n(x) = e^{-\frac{1}{2}\gamma \operatorname{tr}(x^2)} = e^{-\frac{1}{2}\gamma\|x\|^2}$$

is of positive type on H_n.

Let Φ be a continuous function of positive type on $\mathbb{R}$, Fourier transform of a probability measure μ,

$$\Phi(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t}\mu(dt).$$

We associate to Φ the function φ_n on H_n by

$$\varphi_n(x) = \det \Phi(x).$$

Recall that Φ is said to be a Pólya function if, for all n, the function φ_n is of positive type.

Proposition III.1.1. — *The function Φ is a Pólya function if and only if, for all n, the distribution on $\mathbb{R}^n$*

$$D(t)D\Big(\frac{\partial}{\partial t}\Big)\mu \otimes \cdots \otimes \mu$$

is a positive measure.

Recall that D denotes the Vandermonde polynomial.

Proof.

The function φ_n is bounded, hence defines a tempered distribution. Let $T_n \in \mathcal{S}'(H_n)$ be its Fourier transform: for $f \in \mathcal{S}(H_n)$,

$$\langle T_n, f\rangle = \int_{H_n} \varphi_n(x)\hat{f}(x)m(dx).$$

By the theorem of Bochner, the function Φ is a Pólya function if and only if, for all n, the distribution T_n is a positive measure. Assume that f is $U(n)$-invariant. Then $\hat{f}$ is $U(n)$-invariant too. By using Proposition II.4.2, with the same notation,

$$\begin{aligned}
&\langle T_n, f\rangle \\
&= C_n \int_{D_n} \Phi(b_1)\dots\Phi(b_n)\tilde{F}(b_1,\dots,b_n)D(b)^2 db_1\dots db_n \\
&= C_n' \int_{D_n} \Phi(b_1)\dots\Phi(b_n)\frac{1}{D(b)}D\Big(\frac{\partial}{\partial b}\Big)\hat{F}(b_1,\dots,b_n)D(b)^2 db_1\dots db_n \\
&= C_n' \int_{D_n} \Phi(b_1)\dots\Phi(b_n)D(b)D\Big(\frac{\partial}{\partial b}\Big)\hat{F}(b_1,\dots,b_n)db_1\dots db_n \\
&= C_n' \langle D(b)D\Big(\frac{\partial}{\partial b}\Big)\mu\otimes\cdots\otimes\mu, F\rangle.
\end{aligned}$$

Therefore the $U(n)$-invariant distribution T_n is a positive measure if and only if

$$D(b)D\Big(\frac{\partial}{\partial b}\Big)\mu \otimes \cdots \otimes \mu$$

is a positive measure. ∎

Corollary III.1.2. — *If the measure μ has a $\mathcal{C}^\infty$ density f:*

$$\mu(dt) = f(t)dt,$$

then the function Φ is a Pólya function if and only if, for $t_1 < \cdots < t_n$,

$$\det\Big(\big(f^{(n-j)}(t_i)\big)_{1\le i,j\le n}\Big) \ge 0.$$

Proof. For a $\mathcal{C}^{n-1}$-function f,

$$D\big(\frac{\partial}{\partial t}\big)f(t_1)\dots f(t_n) = \det\Big(\big(f^{(n-j)}(t_i)\big)_{1\le i,j\le n}\Big).$$

∎

Example

The function

$$\Phi(\lambda) = e^{-\frac{1}{2}\gamma\lambda^2} \quad (\gamma > 0)$$

is the Fourier transform of the function

$$f(t) = \frac{1}{\sqrt{2\pi\gamma}}e^{-\frac{1}{2\gamma}t^2}.$$

We saw that it is a Pólya function. This can be checked by using Corollary III.1.2. In fact, if

$$F(t) = e^{-\frac{1}{2}t^2},$$

then

$$F^{(k)}(t) = (-1)^k e^{-\frac{1}{2}t^2}(t^k + \cdots),$$

and

$$\det\Big(F^{(n-j)}(t_i)\Big) = e^{-\frac{1}{2}(t_1^2+\cdots+t_n^2)}\prod_{i<j}(t_j - t_i).$$

2. Generalized Wishart measures and Pólya functions. — We consider the quadratic map

$$Q : M(n,k;\mathbb{C}) \to H_n$$

given by

$$Q(\xi) = \xi\xi^*.$$

The Wishart measure $W_{n,k}$ is the image by Q of the Gauss measure on the space $E = M(n,k;\mathbb{C})$,

$$\pi^{-nk} e^{-\|\xi\|^2} m(d\xi).$$

The space E is equipped with the Euclidean inner product

$$(\xi|\eta) = \Re\,\mathrm{tr}(\xi\eta^*),$$

and m is the corresponding Euclidean measure. The Fourier transform of the Wishart measure $W_{n,k}$ is

$$\begin{aligned}
\widehat{W_{n,k}}(x) &= \int_{H_n} e^{-i(x|y)} W_{n,k}(dy) \\
&= \pi^{-nk} \int_E e^{-i\left(x|Q(\xi)\right)} e^{-\|\xi\|^2} m(d\xi) \\
&= \pi^{-nk} \int_E e^{-\left((I+ix)\xi|\xi\right)} m(d\xi) \\
&= \det(I+ix)^{-k}.
\end{aligned}$$

Therefore

$$\widehat{W_{n,k}}(x) = \det\Phi(x),$$

with

$$\Phi(\lambda) = (1+i\lambda)^{-k}.$$

This shows that Φ is a Pólya function, for which

$$\varphi_n(x) = \widehat{W_{n,k}}(x), \quad \mu_n = W_{n,k}.$$

With $k=1$, and replacing Q by αQ, $\alpha \in \mathbb{R}$, we obtain the Pólya function

$$\Phi(\lambda) = \frac{1}{1+i\alpha\lambda}.$$

Consider now a selfadjoint operator A on $\mathbb{C}^k$, and let W_A be the image of the Gauss measure on E by the quadradic map

$$Q_A(\xi) = \xi A\xi^*.$$

Its Fourier transform is given by

$$\widehat{W_A}(x) = \pi^{nk} \int_E e^{-i(x|\xi A\xi^*)} e^{-\|\xi\|^2} m(d\xi).$$

PROPOSITION III.2.1. — *Let $\alpha_1, \ldots, \alpha_k$ be the eigenvalues of A.*

$$\widehat{W_A}(x) = \prod_{j=1}^{n} \prod_{\ell=1}^{k} \frac{1}{1 + i\alpha_\ell \lambda_j},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x.

Proof.

We consider an orthonormal basis $\{f_\ell\}$ consisting of eigenvectors of A,

$$Af_\ell = \alpha_\ell f_\ell,$$

and the canonical basis $\{e_i\}$ of $\mathbb{C}^n$. If $\xi \in M(n, k; \mathbb{C})$, $v \in \mathbb{C}^k$,

$$\xi v = \sum_{i=1}^{n} \sum_{\ell=1}^{k} \xi_{i\ell} (v|f_\ell) e_i,$$

and, if $u \in \mathbb{C}^n$,

$$\xi^* u = \sum_{i=1}^{n} \sum_{\ell=1}^{k} \overline{\xi_{i\ell}} (u|e_i) f_\ell.$$

It follows that

$$(\xi A \xi^*)_{ij} = \sum_{\ell=1}^{k} \alpha_\ell \xi_{i\ell} \overline{\xi_{j\ell}}.$$

After integrating, one obtains

$$\widehat{W_A}(x) = \prod_{j=1}^{n} \prod_{\ell=1}^{k} (1 + i\lambda_j \alpha_\ell)^{-\frac{d}{2}}.$$ ∎

It follows that

$$\Phi(\lambda) = \det(I + i\lambda A)^{-1} = \prod_{\ell=1}^{k} \frac{1}{1 + i\alpha_\ell \lambda}$$

is a Pólya function for which

$$\varphi_n(x) = \widehat{W_A}(x), \quad \mu_n = W_A.$$

The mean M_A of W_A is given by

$$\begin{aligned} M_A &= \int_{H_n} y W_A(dy) \\ &= \int_E \xi A \xi^* e^{-\|\xi\|^2} m(d\xi) = \mathrm{tr}(A) I. \end{aligned}$$

In fact

$$(M_A)_{ij} = \pi^{-nk} \int_E \sum_{\ell=1}^{k} \alpha_\ell \xi_{i\ell} \overline{\xi_{j\ell}} e^{-\|\xi\|^2} m(d\xi).$$

Therefore $(M_A)_{ij} = 0$ if $i \neq j$, and $(M_A)_{ii} = \mathrm{tr}(A)$. By shifting the Wishart measure by $-\mathrm{tr}(A)I$ one obtains a probability measure W_A^0 with mean 0. Its Fourier transform is given by

$$\widehat{W_A^0}(x) = \det \Phi^0(x),$$

where

$$\Phi^0(\lambda) = e^{i\lambda\,\mathrm{tr}(A)} \det(I + i\lambda A)^{-1}.$$

It can be written

$$\Phi^0(\lambda) = \det{}_2(I + i\lambda A)^{-1},$$

where $\det_2$ denotes the regularized determinant. This formula still makes sense when A is a Hilbert-Schmidt selfadjoint operator, and one cheks that for every Hilbert-Schmidt selfadjoint operator the function

$$\Phi(\lambda) = \det{}_2(I + i\lambda A)^{-1}$$

is a Pólya function. Recall that, if A is a Hilbert-Schmidt selfadjoint operator with eigenvalues α_k, then

$$\det{}_2(I + A) = \prod_k e^{-\alpha_k}(1 + \alpha_k).$$

3. The Pólya-Laguerre class of entire functions. — Let us consider first the infinite product

$$F(s) = \prod_{k=1}^{\infty} e^{\alpha_k s}(1 - \alpha_k s),$$

where α_k is a sequence of complex numbers such that

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.$$

Proposition III.3.1. — *The infinite product is uniformly convergent on compact sets, and F is an entire function. Its zeros are the numbers*

$$\frac{1}{\alpha_k} \qquad (\alpha_k \neq 0).$$

Lemma III.3.2. — *For $|z| \leq \frac{1}{2}$,*

$$|e^z(1-z)-1| \leq 2|z|^2.$$

Proof.
For $|z|<1$,

$$e^z(1-z) = \exp\bigl(-\sum_{m=2}^{\infty}\frac{z^m}{m}\bigr),$$

$$|\sum_{m=2}^{\infty}\frac{z^m}{m}| \leq \sum_{m=2}^{\infty}\frac{|z|^m}{m} \leq \tfrac{1}{2}\sum_{m=2}^{\infty}|z|^m \leq \tfrac{1}{2}\frac{|z|^2}{1-|z|},$$

and, if $|z| \leq \frac{1}{2}$,

$$|\sum_{m=2}^{\infty}\frac{z^m}{m}| \leq |z|^2.$$

Furthermore, for $w \in \mathbb{C}$,

$$|e^w - 1| \leq e^{|w|} - 1 \leq |w|e^{|w|}.$$

Therefore

$$|e^z(1-z)-1| \leq |z|^2 e^{|z|^2},$$

and $e^{\frac{1}{4}} < 2$. ∎

Proof of Proposition III.3.1.
For $R>0$, there exists N such that, if $k \geq N$, then $|\alpha_k| \leq \frac{1}{2R}$. Therefore, if $|s| \leq R$, $|\alpha_k s| \leq \frac{1}{2}$, and by Lemma III.3.2,

$$\sum_{k=N}^{\infty}|e^{\alpha_k s}(1-\alpha_k s)-1| \leq 2R^2\sum_{k=N}^{\infty}|\alpha_k|^2.$$ ∎

Proposition III.3.3. — *The entire function F is of order two at most. More precisely, for every $\varepsilon > 0$, there exists $C > 0$ such that*

$$|F(s)| \leq Ce^{\varepsilon|s|^2}.$$

Lemma III.3.4. — *For $z \in \mathbb{C}$,*

$$|e^z(1-z)| \le e^{4|z|^2}.$$

Proof.

For $|z| \le \frac{1}{2}$, we saw in the proof of Lemma III.3.2 that

$$|\sum_{m=2}^{\infty} \frac{z^m}{m}| \le |z|^2,$$

therefore $|e^z(1-z)| \le e^{|z|^2}$. For $|z| \ge \frac{1}{2}$,

$$|e^z(1-z)| \le e^{2|z|} \le e^{4|z|^2}.$$

(one used $|1-z| \le e^{|z|}$.) ∎

Proof of Proposition III.3.3.

Let $\varepsilon > 0$. There exists ℓ such that

$$\sum_{k=\ell+1}^{\infty} |\alpha_k|^2 \le \varepsilon,$$

and

$$|\prod_{k=\ell+1}^{\infty} e^{\alpha_k s}(1-\alpha_k s)| \le e^{4|s|^2 \sum_{k=\ell}^{\infty} |\alpha_k|^2} \le e^{4\varepsilon|\lambda|^2}.$$

On the other hand, by using the inequality

$$|e^z(1-z)| \le e^{2|z|},$$

one obtains

$$|\prod_{k=1}^{\ell} e^{\alpha_k s}(1-\alpha_k s)| \le e^{2|s| \sum_{k=1}^{\ell} |\alpha_k|},$$

and there exists a constant C such that

$$e^{2|s| \sum_{k=1}^{\ell} |\alpha_k|} \le C e^{\varepsilon|\lambda|^2}.$$ ∎

We will also need the Taylor expansion of the logarithmic derivative of F:

$$\begin{aligned} \frac{F'(s)}{F(s)} &= \sum_{k=1}^{\infty}\Big(\alpha_k - \frac{\alpha_k}{1-\alpha_k s}\Big) \\ &= -\sum_{k=1}^{\infty} \frac{\alpha_k^2 s}{1-\alpha_k s} \\ &= -\sum_{m=1}^{\infty} p_{m+1}(\alpha) s^m, \end{aligned}$$

where $p_m(\alpha)$ is the m-th power sum

$$p_m(\alpha) = \sum_{k=1}^{\infty} \alpha_k^m,$$

which is well defined for $m \geq 2$. The Taylor series converges for $|s| < a$, $\frac{1}{a} = \sup |\alpha_k|$.

Around 1910, motivated by the Riemann hypothesis, there has been an intense research activity about entire functions with only real zeros. For instance the following striking result has been obtained by Pólya and Schur.

THEOREM III.3.5. — *Let Ψ be an entire function with $\Psi(0) = 1$. Then Ψ is a uniform limit on compact sets of polynomials with only real zeros if and only if Ψ has the following form*

$$\Psi(s) = e^{-\beta s} e^{-\frac{1}{2}\gamma s^2} \prod_{k=1}^{\infty} e^{\alpha_k s}(1 - \alpha_k s),$$

with $\beta \in \mathbb{R}$, $\gamma \geq 0$, $\alpha_k \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

[Pólya,1913], [Pólya-Schur,1913], see also [Karlin,1968], Theorem 2.2, Chapter 7, p.338. Among other papers about the same topic: [Jensen,1912-13], [Pólya,1915] (see also [Pólya-Szegö,1976], II, Part V, No 165 and following numbers).

One says that the entire function Ψ belongs to the *Pólya-Laguerre class.*

Let us consider its inverse, more precisely

$$\Phi(\lambda) = \frac{1}{\Psi(-i\lambda)} = e^{-i\beta\lambda} e^{-\frac{1}{2}\lambda^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k \lambda}}{1 + i\alpha_k \lambda}.$$

It is a meromorphic function whose poles are the numbers $\frac{i}{\alpha_k}$. It is holomorphic in the strip

$$|\Im\lambda| < a, \quad \frac{1}{a} = \sup |\alpha_k|.$$

Since

$$\Phi(\lambda) = \lim_{\ell \to \infty} \Phi_\ell(\lambda),$$

with

$$\Phi_\ell(\lambda) = e^{-i\beta\lambda} e^{-\frac{1}{2}\lambda^2} \prod_{k=1}^{\ell} \frac{e^{i\alpha_k \lambda}}{1 + i\alpha_k \lambda},$$

it is clear that Φ is a Pólya function by what we saw in Section III.2.

In fact one obtains in that way all Pólya functions. This is the fundamental result by Pickrell [1991] that we will present in next chapter (see also [Olshanski-Vershik,1996]).

Exemples

In the following examples the Pólya function Φ is the Fourier transform of a positive integrable function f:

$$\Phi(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} f(t)dt.$$

$\Phi(\lambda) = e^{-\frac{1}{2}\gamma\lambda^2}$	$f(t) = \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{1}{2\gamma}t^2}$
$\phi(\lambda) = \frac{1}{1+i\lambda}$	$f(t) = \begin{cases} e^{-t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$
$\Phi(\lambda) = \frac{1}{(1+i\lambda)^2}$	$f(t) = \begin{cases} te^{-t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$
$\Phi(\lambda) = \frac{1}{1+\lambda^2}$	$f(t) = \frac{1}{2} e^{-\|t\|}$
$\Phi(\lambda) = \Gamma(1+i\lambda) = e^{-iC\lambda} \prod_{k=1}^{\infty} e^{i\frac{\lambda}{k}} \left(1 + i\frac{\lambda}{k}\right)^{-1}$	$f(t) = e^{-e^{-t}} e^{-t}$
$\Phi(\lambda) = \frac{\pi\lambda}{\operatorname{sh} \pi\lambda} = \prod_{k=1}^{\infty} \left(1 + \frac{\lambda^2}{k^2}\right)^{-1}$	$f(t) = \frac{e^t}{(1+e^t)^2} = \frac{4}{\operatorname{ch}^2 \frac{t}{2}}$
$\Phi(\lambda) = \frac{1}{\operatorname{ch} \pi\lambda} = \prod_{k=1}^{\infty} \left(1 + \frac{2\lambda^2}{(2k+1)^2}\right)^{-1}$	$f(t) = \frac{1}{2\pi} \frac{1}{\operatorname{ch} \frac{t}{2}}$

Chapter IV

TOTALLY POSITIVE FUNCTIONS AND THE THEOREM OF PICKRELL

There is a surprising connection between infinite dimensional harmonic analysis and the classical theory of totally positive functions.

1. Totally positive kernels. — A kernel $K(s,t)$ defined over an interval $I \subset \mathbb{R}$ is said to be *totally positive* if, for all numbers $s_1 < \cdots < s_n$, $t_1 < \cdots < t_n$ in I,

$$K\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} \geq 0.$$

and *strictly totally positive* if these inequalities are strict. Let us recall the notation

$$K\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} = \det\Big(\big(K(s_i,t_j)\big)_{1\leq i,j\leq n}\Big).$$

Example. — The kernel defined by

$$K(s,t) = \begin{cases} 1 & \text{if } s \geq t \\ 0 & \text{otherwise} \end{cases}$$

is totally positive. This can be seen as follows. The entries of the matrix $\big(K(s_i,t_j)\big)$, are equal to 0 or 1, increasing in each column, decreasing in each line. Therefore its determinant is 0 unless $K(s_i,t_j) = 1$ for $i \geq j$, $K(s_i,t_j) = 0$ for $i < j$, which means that

$$t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq \cdots \leq s_{n-1} < t_n \leq s_n,$$

and is then equal to one.

As in Section II.3 let us consider a Taylor series

$$F(z) = \sum_{m=0}^{\infty} c_m z^m,$$

converging for $|z| < r$, and define the $U(n)$-invariant function f_n on $H_n = Herm(n,\mathbb{C})$ by

$$f_n\big(\mathrm{diag}(t_1,\dots,t_n)\big) = \sum_{m_1\geq\cdots\geq m_n\geq 0} d_{\mathbf{m}}\tilde{a}_{\mathbf{m}} s_{\mathbf{m}}(t),$$

where

$$\tilde{a}_{\mathbf{m}} = c_{m_1+\delta_1} \dots c_{m_n+\delta_n},$$

or

$$f_n(x) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} d_{\mathbf{m}} \tilde{a}_{\mathbf{m}} \chi_{\mathbf{m}}(x).$$

We consider the kernel K on $]-r,r[\times]-r,r[$ given by

$$K(s,t) = F(st).$$

Proposition IV.1.1. — *If, for all n, $f_n \geq 0$, then the kernel K is totally positive. And, if $f_n > 0$, then K is strictly totally positive.* [Gross-Richards,1989]

Proof.

We saw in Section II.3 that

$$K\begin{pmatrix} s_1 \dots s_n \\ t_1 \dots t_n \end{pmatrix} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \tilde{a}_{\mathbf{m}} s_{\mathbf{m}}(s) s_{\mathbf{m}}(t).$$

By using the functional equation

$$\int_{U(n)} \chi_{\mathbf{m}}(xuyu^{-1}) \alpha(du) = \frac{1}{d_{\mathbf{m}}} \chi_{\mathbf{m}}(x) \chi_{\mathbf{m}}(y),$$

one obtains

$$\begin{aligned} K\begin{pmatrix} s_1 \dots s_n \\ t_1 \dots t_n \end{pmatrix} &= \sum_{m_1 \geq \cdots \geq m_n \geq 0} d_{\mathbf{m}} \tilde{a}_{\mathbf{m}} \int_{U(n)} \chi_{\mathbf{m}}(xuyu^{-1} \alpha(du) \\ &= \int_{U(n)} f_n(xuyu^{-1}) \alpha(du), \end{aligned}$$

with $x = \operatorname{diag}(s_1, \dots, s_n)$, $y = \operatorname{diag}(t_1, \dots, t_n)$. The statement follows from that formula. ∎

Corollary IV.1.2. — *The kernel*

$$K(s,t) = e^{st}$$

is strictly totally positive.

Proof.

It is a special case of Proposition IV.1.1:

$$F(z) = e^z = \sum_{m=1}^{\infty} \frac{z^m}{m!},$$
$$f(x) = \frac{1}{\delta!} e^{\operatorname{tr} x} > 0. \qquad \square$$

In fact it can be shown directly (see [Polyá-Szegö,1976] II, Part V, No 76). First show inductively on n that, for distinct real numbers $\alpha_1, \ldots, \alpha_n$, and real numbers $a_1, \ldots, a_n$, the function

$$f(t) = a_1 e^{\alpha_1 t} + \cdots a_n e^{\alpha_n t},$$

has at most $n-1$ real zeros, if it does not vanish identically. Then show inductively on n that, for $s_1 < \cdots < s_n$, $t_1 < \cdots < t_n$,

$$K \begin{pmatrix} s_1 & \ldots & s_n \\ t_1 & \ldots & t_n \end{pmatrix} > 0.$$

For that consider the function

$$f(t) = \begin{vmatrix} e^{s_1 t_1} & \ldots & e^{s_n t_1} \\ \vdots & & \vdots \\ e^{s_1 t_{n-1}} & \ldots & e^{s_n t_{n-1}} \\ e^{s_n t} & \ldots & e^{s_n t} \end{vmatrix} = a_1 e^{s_1 t} + \cdots + a_n e^{s_n t}.$$

The function f does not vanish identically since

$$f(t) \sim a_n e^{s_n t} \quad (t \to \infty),$$

and

$$a_n = K \begin{pmatrix} s_1 \ldots s_{n-1} \\ t_1 \ldots t_{n-1} \end{pmatrix} > 0.$$

Furthermore

$$f(t_1) = 0, \ldots, f(t_{n-1}) = 0.$$

Hence $f(t_n) > 0$.

Proposition IV.1.3. — *Let K and L be totally positive kernels. If the composition kernel*

$$M(s,t) = \int_I K(s,u) L(u,t) du$$

is well defined, it is totally positive.

Proof. It follows from the formula

$$M\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}$$
$$= \int_{\{u_i\in I | u_1<\dots<u_n\}} K\begin{pmatrix} s_1 & \dots & s_n \\ u_1 & \dots & u_n \end{pmatrix} L\begin{pmatrix} u_1 & \dots & u_n \\ t_1 & \dots & t_n \end{pmatrix} du_1 \dots du_n,$$

which follows from

$$M\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}$$
$$= \frac{1}{n!}\int_{I\times\dots\times I} K\begin{pmatrix} s_1 & \dots & s_n \\ u_1 & \dots & u_n \end{pmatrix} L\begin{pmatrix} u_1 & \dots & u_n \\ s_1 & \dots & s_n \end{pmatrix} du_1 \dots du_n.$$

In fact

$$K\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix} = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\prod_{i=1}^n K\big(s_i, t_{\sigma(i)}\big).$$

Therefore we can write

$$\int_{I\times\dots\times I} K\begin{pmatrix} s_1 & \dots & s_n \\ u_1 & \dots & u_n \end{pmatrix} L\begin{pmatrix} u_1 & \dots & u_n \\ t_1 & \dots & t_n \end{pmatrix} du_1 \dots du_n$$
$$= \int_{I\times\ldots\times I} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\prod_{i=1}^n K\big(s_{\sigma(i)}, u_i\big) \sum_{\tau\in\mathfrak{S}_n} \varepsilon(\tau)\prod_{i=1}^n L\big((u_i, t_{\tau(i)}\big) du_1 \dots du_n$$
$$= \sum_{\sigma,\tau\in\mathfrak{S}_n} \varepsilon(\sigma\tau)\prod_{i=1}^n M\big(s_{\sigma(i)}, t_{\tau(i)}\big)$$
$$= n! M\begin{pmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{pmatrix}.$$

∎

Let K be a totally positive kernel, and f, g two positive functions. Then the kernel $L(s,t) = f(s)g(t)K(s,t)$ is totally positive.

Let K be a totally positive kernel, and φ, ψ two increasing functions, then the kernel

$$L(s,t) = K\big(\varphi(s), \psi(t)\big)$$

is totally positive.

For instance, if φ, ψ are increasing functions, then the kernel

$$K(s,t) = e^{\varphi(s)\psi(t)}$$

is totally positive.

2. Totally positive functions. — A measurable function f defined on $\mathbb{R}$ is said to be *totally positive* if the kernel

$$K(s,t) = f(s-t)$$

is totally positive.

Examples

a) Exponential function

$$f(t) = e^{\alpha t}, \qquad K(s,t) = e^{\alpha s} e^{-\alpha t}.$$

b) Gauss function

$$f(t) = e^{-\frac{1}{2}t^2}, \qquad K(s,t) = e^{-\frac{1}{2}s^2} e^{-\frac{1}{2}t^2} e^{st}.$$

c)

$$f(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

If f is totally positive then the functions $e^{\alpha t} f(t)$, $f(at+b)$ are totally positive too. For instance, for $\alpha \in \mathbb{R}$,

$$f(t) = \begin{cases} e^{\alpha t} & \text{if } t \geq b \\ 0 & \text{otherwise} \end{cases}$$

is totally positive.

Proposition IV.2.1. — *If f and g are totally positive and integrable, then $f * g$ is totally positive too.*

Proof.

It follows from Proposition IV.1.3. ∎

In the particular case of the convolution product, the formula which was used in the proof of Proposition III.1.3 specializes as

$$\det\big(f * g(t_i - s_j)\big) = \frac{1}{n!} \int_{\mathbb{R}^n} \det\big(f(t_i - u_k)\big) \cdot \det\big(g(u_k - s_j)\big) du_1 \dots du_n.$$

Proposition IV.2.2. — *If f is totally positive and of class $\mathcal{C}^\infty$, then, for $t_1 < \cdots < t_n$,*

$$\det\Big(\big(f^{(n-j)}(t_i)\big)_{1 \leq i,j \leq n}\Big) \geq 0.$$

LEMMA IV.2.3 . — *If f is of class $\mathcal{C}^\infty$, and if $D(h)\neq 0$, then*

$$\lim_{\varepsilon\to 0}\frac{1}{D(\varepsilon h)}\det\Big(\big(f(t_i+\varepsilon h_j)\big)_{1\le i,j\le n}\Big)$$
$$=\frac{1}{1!2!\dots(n-1)!}\det\Big(\big(f^{(n-j)}(t_i)\big)_{1\le i,j\le n}\Big).$$

[Hua,1963], Theorem I.2.4.
Proof.
Consider first the case of a polynomial f of degree $\le n-1$:

$$f(t+h)=f(t)+hf'(t)+\dots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(t).$$

Then the matrix $\big(f(t_i+h_j)\big)$ can be written as a product

$$\big(f(t_i+h_j)\big)$$
$$=\begin{pmatrix} f(t_1) & f'(t_1) & \dots & f^{(n-1)}(t_1)\\ f(t_2) & f'(t_2) & \dots & f^{(n-1)}(t_2)\\ \vdots & \vdots & & \vdots\\ f(t_n) & f'(t_n) & \dots & f^{(n-1)}(t_n)\end{pmatrix}.$$
$$\cdot\begin{pmatrix} 1 & 1 & \dots & 1\\ h_1 & h_2 & \dots & h_n\\ \vdots & \vdots & & \vdots\\ \frac{h_1^{n-1}}{(n-1)!} & \frac{h_2^{n-1}}{(n-1)!} & \dots & \frac{h_n^{n-1}}{(n-1)!}\end{pmatrix}$$

Therefore

$$\det\big(f(t_i+h_j)\big)=(-1)^{\frac{n(n-1)}{2}}\frac{1}{1!2!\dots(n-1)!}D(h)\det\big(f^{(j-1)}(t_i)\big).$$

For a $\mathcal{C}^\infty$ function f, one uses a Taylor expansion:

$$f(t+h)=f(t)+hf'(t)+\dots+\frac{h^k}{k!}f^{(k)}(t)+h^k\varepsilon(h),$$

with

$$\lim_{h\to 0}\varepsilon(h)=0.$$

By expanding the determinant

$$\det\Big(\big(f(t_i+h_j)\big)_{1\le i,j\le n}\Big),$$

we obtain

$$\det\Big(\big(f(t_i+h_j)\big)_{1\le i,j\le n}\Big)=\sum a_{\mathbf{m}}(t)A_{\mathbf{m}}(h)+R(h),$$

with

$$a_{\mathbf{0}}(t)=\frac{1}{1!2!\dots(n-1)!}\det\Big(\big(f^{(n-j)}(t_i)\big)_{1\le i,j\le n}\Big),$$

and $R(h)$ is a finite sum of the form

$$R(h)=\sum h^{\mathbf{m}}\varepsilon_{\mathbf{m}}(h),$$

with

$$m_1+\dots+m_n\ge k,\qquad \lim_{h\to 0}\varepsilon_{\mathbf{m}}(h)=0.$$

The polynomial $A_{\mathbf{m}}(h)$ is divisible by $D(h)$,

$$\frac{1}{D(h)}\det\big(f(t_i+\varepsilon h_j)\big)=\sum a_{\mathbf{m}}(t)\frac{A_{\mathbf{m}}(\varepsilon h)}{D(\varepsilon h)}+\frac{R(\varepsilon h)}{D(\varepsilon h)},$$

and

$$\frac{R(\varepsilon h)}{D(\varepsilon h)}=\varepsilon^{\frac{n(n-1)}{2}}\frac{R(\varepsilon h)}{D(h)}.$$

By taking $k>\frac{n(n-1)}{2}$, one obtains the statement. ∎

The following converse of Proposition IV.2.2 holds:

PROPOSITION IV.2.3. — *Let f be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ which is integrable, with all its derivatives. Assume that, for all $t_1<\dots<t_n$,*

$$\det\Big(\big(f^{(n-j)}(t_i)\big)_{1\le i,j\le n}\Big)\ge 0,$$

then f is totally positive.

Proof.

Define

$$f_\gamma=f*g_\gamma,$$

where g_γ is the Gauss function

$$g_\gamma(t)=\frac{1}{\sqrt{2\pi\gamma}}e^{-\frac{t^2}{2\gamma}}\quad(\gamma>0).$$

Then, for $t_1<\dots<t_n$, $s_1<\dots<s_n$,

$$\det\big(f_\gamma(t_i-s_j)\big)$$
$$=\int_{\{u_1<\dots<u_n\}}\det\big(f(t_i-u_k)\big)\det\big(g_\gamma(u_k-s_j)\big)du_1\dots du_n.$$

From Lemma III.2.3 it follows that

$$\det\bigl(f_\gamma^{(i-1)}(-s_j)\bigr)$$
$$=\int_{\{u_1<\cdots<u_n\}} \det\bigl(f^{(i-1)}(-u_k)\bigr)\det\bigl(g_\gamma(u_k-s_j)\bigr)du_1\dots du_n.$$

If the function

$$\det\bigl(f^{(i-1)}(-u_k)\bigr),$$

which is ≥ 0 on $\{u_1<\cdots<u_n\}$, would identically vanish, the functions $f, f', \dots, f^{(n-1)}$ were linearly dependent, *i.e.* f were solution of a constant coefficient differential equation. But this is not possible since f is integrable.

Since g_γ is strictly totally positive, the function

$$\det\bigl(g_\gamma(u_k-s_j)\bigr)$$

is strictly positive. Therefore

$$\det\bigl(f_\gamma^{(i-1)}(t_i)\bigr)$$

is strictly positive too. By Theorem 2.4 of Chapter 2, p.55, in [Karlin,1968], f_γ is totally positive. Since, for all t,

$$\lim_{\gamma\to 0} f_\gamma(t)=f(t),$$

the function f is totally positive too. ∎

([Olshanski-Vershik], Proposition 7.8.)

3. Theorem of Schoenberg. — Recall that one says that an entire function Ψ, with $\Psi(0)=1$, belongs to the Pólya-Laguerre class if it is a uniform limit on compact sets of polynomials with only real zeros (see Section III.3). Such a function has a representation as an infinite product:

$$\Phi(s)=e^{\beta s}e^{-\frac{1}{2}\gamma s^2}\prod_{k=1}^{\infty} e^{-\alpha_k s}(1+\alpha_k s),$$

with $\beta\in\mathbb{R}$ $\gamma\geq 0$, $\alpha_k\in\mathbb{R}$ with

$$\sum_{k=1}^{\infty}\alpha_k^2<\infty.$$

The following theorem is one of the key result for the topic of these notes.

THEOREM IV.3.1 (THEOREM OF SCHOENBERG). — *a) Let f be a totally positive function on $\mathbb{R}$ which is integrable, with*

$$\int_{\mathbb{R}} f(t)dt = 1.$$

Then its Fourier transform Φ,

$$\Phi(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} f(t)dt,$$

is of the form

$$\Phi(\lambda) = \frac{1}{\Psi(i\lambda)},$$

where Ψ is an entire function of the Pólya-Laguerre class:

$$\Phi(\lambda) = e^{-i\beta\lambda} e^{-\frac{1}{2}\gamma\lambda^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda},$$

with

$$\gamma + \sum_{k=1}^{\infty} \alpha_k^2 > 0.$$

b) Conversely, if

$$\Phi(\lambda) = \frac{1}{\Psi(i\lambda)},$$

where Ψ is an entire function of the Pólya-Laguerre class, and if

$$\gamma + \sum_{k=1}^{\infty} \alpha_k^2 > 0,$$

then Φ is the Fourier transform of a totally positive function which is integrable.

[Schoenberg,1951], see also [Karlin,1968] Chapter 7, Theorem 3.2.

4. Theorem of Pickrell. — Let Φ be a continuous function of positive type on $\mathbb{R}$, Fourier transform of a probability measure μ,

$$\Phi(\lambda) = \int_{\mathbb{R}} e^{-i\lambda t} \mu(dt).$$

We associate to Φ the function φ_n on H_n by

$$\varphi_n(x) = \det \Phi(x).$$

Recall that Φ is said to be a Pólya function if, for all n, the function φ_n is of positive type.

THEOREM IV.4.1 (THEOREM OF PICKRELL). — *Every Pólya function Φ is given as an infinite product*

$$\Phi(\lambda) = e^{-i\beta\lambda} e^{-\frac{1}{2}\gamma\lambda^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda},$$

with $\beta \in \mathbb{R}$, $\gamma \geq 0$, $\alpha_k \in \mathbb{R}$, and

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

[Pickrell,1991], Proposition 5.9, and also [Olshanski-Vershik,1996],

Proof.
Let us regularizes the measure μ: for $\gamma_1 > 0$ define

$$f_{\gamma_1} = g_{\gamma_1} * \mu.$$

The Fourier transform of f_{γ_1} is the product

$$\Phi_0(\lambda) = e^{-\frac{1}{2}\gamma_1\lambda^2} \Phi(\lambda).$$

By Corollary III.1.2, for $t_1 < \cdots < t_n$,

$$\det\Big(\big(f_{\gamma_1}^{(n-j)}(t_i) \big)_{1 \leq i,j \leq n} \Big) \geq 0.$$

By Proposition IV.2.3, the function f_{γ_1} is totally positive. By the Theorem of Schoenberg (Theorem IV.3.1), the function Φ_0 admits a representation as an infinite product

$$\Phi_0(\lambda) = e^{-i\beta\lambda} e^{-\frac{1}{2}\gamma_0\lambda^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda}.$$

Therefore

$$\Phi(\lambda) = e^{-i\beta\lambda} e^{-\frac{1}{2}(\gamma_0 - \gamma_1)\lambda^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda}.$$

It remains to check that $\gamma_0 - \gamma_1 \geq 0$. Since

$$|\Phi(\lambda)| \leq 1 \qquad (\lambda \in \mathbb{R}),$$

this follows from Proposition III.3.3: for every $\varepsilon > 0$, there exists $C > 0$ such that

$$\Big| \prod_{k=1}^{\infty} e^{-i\alpha_k\lambda}(1 + i\alpha_k\lambda)\Big| \leq Ce^{\varepsilon|\lambda|^2}.$$

Therefore

$$e^{-\frac{1}{2}(\gamma_0-\gamma_1)\lambda^2} \leq Ce^{\varepsilon|\lambda|^2},$$

and $\gamma_0 - \gamma_1 \geq -2\varepsilon$. ∎

The theorem of Pickrell holds also in the case of real symmetric matrices with a slight change. Let Φ be a continuous function on $\mathbb{R}$, with $\Phi(0) = 1$. For every n one defines the function φ_n on $Sym(n, \mathbb{R})$ by

$$\varphi_n(x) = \det \Phi(x),$$

or equivalently, φ_n is invariant under $O(n)$ and

$$\varphi_n\big(\mathrm{diag}(a_1, \ldots, a_n)\big) = \Phi(a_1) \ldots \Phi(a_n) \quad (a_j \in \mathbb{R}).$$

Then the function φ_n is of positive type for all n if and only if it can be written

$$\Phi(\lambda) = e^{-i\beta\lambda}e^{-\frac{1}{2}\gamma\lambda^2} \prod_{k=1}^{\infty} \Big(\frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda}\Big)^{\frac{1}{2}},$$

with

$$\beta \geq 0,\ \gamma \geq 0,\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

By using the inclusions

$$Sym(n, \mathbb{R}) \subset Herm(n, \mathbb{C}) \subset Sym(2n, \mathbb{R}),$$

and the fact that the restriction of a function of positive type to a subgroup is of positive type too, it follows from Theorem IV.4.1. ([Pickrell,1991], Proposition 5.12.)

Proposition IV.4.2. — *The support of the measure μ is contained in $[0, \infty[$ if and only if the Pólya function Φ can be written*

$$\Phi(\lambda) = e^{-i\beta_0\lambda} \prod_{k=1}^{\infty} \frac{1}{1 + i\alpha_k\lambda},$$

with

$$\beta_0 \geq 0, \quad \alpha_k \geq 0, \quad \sum_{k=1}^{\infty} \alpha_k < \infty.$$

([Olshanski-Vershik,1996], Remark 2.11.)

Proof.

First, if $\mathrm{supp}(\mu) \subset [0, \infty[$, then $\beta \geq 0$ since β is the mean of μ. The support of μ is contained in $[0, \infty[$ if and only if Φ is holomorphic in the lower halfspace $\Im\lambda < 0$, and

$$|\Phi(\lambda)| \leq 1 \quad (\Im\lambda \leq 0).$$

One cheks easily that it holds if Φ has the given form.

Assume that Φ is holomorphic for $\Im\lambda < 0$, and

$$|\Phi(\lambda)| \leq 1 \quad (\Im\lambda \leq 0).$$

Then $\alpha_k \geq 0$. For $\lambda = -i\nu$, $\nu \geq 0$,

$$\Phi(-i\nu) = e^{-\beta\nu} e^{\frac{1}{2}\gamma\nu^2} \prod_{k=1}^{\infty} \frac{e^{\alpha_k \nu}}{1 + \alpha_k \nu} \leq 1.$$

By Proposition III.3.3, for every $\varepsilon > 0$, there exists $C > 0$ such that

$$\prod_{k=1}^{\infty} \frac{e^{\alpha_k \nu}}{1 + \alpha_k \nu} \geq C e^{-\varepsilon\nu^2}.$$

It follows that $\gamma = 0$. For every $\delta > 0$, $0 < \delta < 1$, there exists $D > 0$ such that

$$\frac{e^x}{1 + x} \geq D e^{\delta x} \quad (x \geq 0).$$

Therefore, for every N, and $\nu \geq 0$,

$$D^N \exp\Big((\delta \sum_{k=1}^{N} \alpha_k - \beta)\nu \Big) \leq e^{-\beta\nu} \prod_{k=1}^{N} \frac{e^{\alpha_k \nu}}{1 + \alpha_k \nu} \leq 1.$$

Hence

$$\delta \sum_{k=1}^{N} \alpha_k \leq \beta,$$

and, since $\delta < 1$ and N are arbitrary,

$$\sum_{k=1}^{\infty} \alpha_k \leq \beta.$$

By putting

$$\beta_0 = \beta - \sum_{k=1}^{\infty} \alpha_k,$$

we obtain

$$\Phi(\lambda) = e^{-i\beta_0\lambda} \prod_{k=1}^{\infty} \frac{1}{1 + i\alpha_k\lambda}. \qquad \square$$

Chapter V

THE OLSHANSKI THEORY OF SPHERICAL PAIRS

This chapter follows closely [Olshanski,1990], 23.

1 Spherical pairs. — We consider a topological group G, and a closed subgroup K. We denote by $\mathfrak{P}$ the set of continuous functions φ on G which are of positive type with $\varphi(e) = 1$, and which are K-biinvariant. It is a convex set. If $\varphi \in \mathfrak{P}$, there exists a unitary representation π of G on a Hilbert space $\mathcal{H}$, and a unit vector u which is K-invariant and cyclic such that

$$\varphi(g) = \bigl(\pi(g)u|u\bigr).$$

The following properties are equivalent

(1) φ is extremal in $\mathfrak{P}$,

(2) the representation π is irreducible.

Let $\mathcal{H}^K$ be the space of K-invariant vectors. If $\dim \mathcal{H}^K = 1$, then the representation π is irreducible.

The pair (G, K) is said to be *spherical* if, for every irreducible unitary representation π of G on a Hilbert space $\mathcal{H}$,

$$\dim \mathcal{H}^K \leq 1.$$

If $\dim \mathcal{H}^K = 1$, the irreducible representation π is said to be *spherical.* A function φ on G is said to be *spherical* if it can be written

$$\varphi(g) = \bigl(\pi(g)u|u\bigr),$$

where π is a spherical representation, $u \in \mathcal{H}^K$, with $\|u\| = 1$. Hence the spherical functions are the extremal points in the convex set $\mathfrak{P}$.

If G is locally compact, and K compact, then the pair (G, K) is spherical if and only if it is a Gelfand pair, *i.e.* if the algebra of K-biinvariant integrable functions is commutative. In that case a function $\varphi \in \mathfrak{P}$ is spherical if and only if it satisfies the functional equation

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y),$$

where α is the normalized Haar measure of K.

Let $\big(G(n), K(n)\big)$ be a sequence of Gelfand pairs. One assume that $G(n)$ is a closed subgroup of $G(n+1)$, $K(n)$ is a closed subgroup of $K(n+1)$, and $K(n) = G(n) \cap K(n+1)$. Let

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

THEOREM V.1.1. — (i) *The pair (G, K) is spherical.*
(ii) *Let $\varphi \in \mathfrak{P}$. The function φ is spherical if and only if*

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y),$$

where α_n is the normalized Haar measure on $K(n)$.

Proof.

a) Let π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$ with $\mathcal{H}^K \neq \{0\}$. We will show that $\dim \mathcal{H}^K = 1$. The orthogonal projection onto $\mathcal{H}^{K(n)}$ can be written

$$P_n = \int_{K(n)} \pi(k)\alpha_n(dk).$$

Note that $P_{n+1} = P_{n+1}P_n = P_nP_{n+1}$, since $K(n) \subset K(n+1)$, and therefore $\mathcal{H}^{K(n+1)} \subset \mathcal{H}^{K(n)}$. The projections P_n strongly converge to the orthogonal projection P onto $\mathcal{H}^K$. Since $\big(G(n), K(n)\big)$ is a Gelfand pair, for $x, y \in G(n)$,

$$P_n\pi(x)P_n\pi(y)P_n = P_n\pi(y)P_n\pi(x)P_n.$$

As $n \to \infty$ one obtains

$$P\pi(x)P\pi(y)P = P\pi(y)P\pi(x)P.$$

In fact, using $P_{n+m} = P_{n+m}P_n = P_nP_{n+m}$, one obtains

$$P_{n+m}\pi(x)P_n\pi(y)P_{n+m'} = P_{n+m}\pi(y)P_n\pi(x)P_{n+m'}.$$

Let $m \to \infty$, $m' \to \infty$, and then $n \to \infty$.

Let $\mathcal{A}$ be the algebra generated by the operators $P\pi(x)P$, for $x \in G$. This algebra is commutative. The subspace $\mathcal{H}^K$ is invariant under $\mathcal{A}$. We will show that $\mathcal{H}^K$ is irreducible under $\mathcal{A}$. Since an irreducible representation of a commutative Banach algebra is one dimensional, it will follow that $\dim \mathcal{H}^K = 1$.

Assume that

$$\mathcal{H}^K = \mathcal{Y}_1 \oplus \mathcal{Y}_2,$$

where $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two orthogonal subspaces of $\mathcal{H}^K$ which are $\mathcal{A}$-invariant. Let $u \in \mathcal{Y}_1$, $u \neq 0$. For $v \in \mathcal{Y}_2$, $x \in G$,

$$\big(P\pi(x)Pu|v\big) = 0.$$

This means that, for all x,

$$\big(\pi(x)u|v\big) = 0.$$

Since π is irreducible, this implies that $v = 0$. Therefore $\mathcal{Y}_2 = \{0\}$.

b) Let φ be a spherical function:

$$\varphi(g) = \big(\pi(g)u|u\big),$$

where π is an irreducible unitary representation, $u \in \mathcal{H}^K$, $\|u\| = 1$. For $v \in \mathcal{H}$,

$$(P\pi(g)Pu|v) = (\pi(g)u|Pv),$$

and, since $Pv = (v|u)u$,

$$= (u|v)\big(\pi(g)u|u\big) = \varphi(g)(u|v).$$

Therefore

$$\begin{aligned} &P\pi(g)Pu = \varphi(g)u, \\ &P\pi(x)P\pi(y)Pu = \varphi(x)\varphi(y)u, \\ &\big(\pi(x)P\pi(y)u|u\big) = \varphi(x)\varphi(y). \end{aligned}$$

Since $P_n \to P$ strongly,

$$\begin{aligned} \varphi(x)\varphi(y) &= \lim_{n\to\infty} \big(\pi(x)P_n\pi(y)u|u\big) \\ &= \lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk). \end{aligned}$$

c) Let $\varphi \in \mathfrak{P}$, and assume that

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y).$$

The function φ can be written as

$$\varphi(g) = \big(\pi(g)u|u\big),$$

where π is a unitary representation of G on a Hilbert space $\mathcal{H}$, $u \in \mathcal{H}^K$ and is cyclic. We will show that $\mathcal{H}^K = \mathbb{C}u$, and this implies that π is irreducible. In fact we will show that

$$P\pi(g)u = \varphi(g)u.$$

By assumption

$$\begin{aligned}\varphi(x)\varphi(y) &= \lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) \\ &= \lim_{n\to\infty} \big(\pi(x)P_n\pi(y)u|u\big) \\ &= \big(\pi(x)P\pi(y)u|u\big).\end{aligned}$$

This can be written

$$\big(P\pi(y)u|\pi(x^{-1})u\big) = \varphi(y)\big(u|\pi(x^{-1})u\big),$$

and, since u is cyclic, this implies that

$$P\pi(y)u = \varphi(y)u. \qquad \square$$

Examples

1) $G(n) = O(n) \ltimes \mathbb{R}^n$ is the motion group, $K(n) = O(n)$. Then

$$G = O(\infty) \ltimes \mathbb{R}^{(\infty)}, \quad K = O(\infty).$$

A K-biinvariant continuous function φ on G can be seen as a radial function on $\mathbb{R}^{(\infty)}$, and can be written

$$\varphi(x) = \Phi(\|x\|^2),$$

where Φ is a continuous function on $[0,\infty[$, and then, if $a = \|x \cdot 0\|$, $b = \|y \cdot 0\|$ $(x, y \in G)$,

$$\begin{aligned}&\int_{K(n)} \varphi(xky)\alpha_n(dk) \\ &= \frac{\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\Gamma\big(\frac{n}{2}\big)} \int_0^\pi \Phi(a^2 + b^2 + 2ab\cos\theta)\sin^{n-1}(\theta)d\theta.\end{aligned}$$

For every continuous function f on $[0,\pi]$,

$$\lim_{n\to\infty} \frac{\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\Gamma\big(\frac{n}{2}\big)} \int_0^\pi f(\theta)\sin^{n-1} d\theta = f\big(\frac{\pi}{2}\big).$$

Therefore, if φ is a spherical function,

$$\Phi(a^2+b^2)=\Phi(a^2)\Phi(b^2).$$

It follows that there exists $\lambda\in\mathbb{C}$ such that

$$\Phi(u)=e^{-\lambda u},\quad \varphi(x)=e^{-\lambda\|x\|^2}.$$

Since φ is of positive type, necessarily λ is real, and $\lambda\geq 0$. Hence the spherical functions of the spherical pair (G,K) are the following

$$\varphi(x)=e^{-\lambda\|x\|^2}\quad(\lambda\geq 0).$$

Is it remarkable that such a function, which is defined on $\mathbb{R}^{(\infty)}$, extends as a continuous function on $\ell^2(\mathbb{N})$.

2) $G(n)=SO(n+1)$, $K(n)\simeq SO(n)$ is the subgroup of the $g\in G(n)$ such $g\cdot e_0=e_0$, $\{e_0,\ldots,e_n\}$ being the canonical basis of $\mathbb{R}^{n+1}$. A K-biinvariant continuous function φ on G can be written

$$\varphi(g)=\Phi\big((g\cdot e_0|e_0)\big),$$

where Φ is a continuous function on $[-1,1]$. For such a function, if $(x\cdot e_0|e_0)=\cos a$, $(y\cdot e_0|e_0)=\cos b$ $(x,y\in G)$,

$$\begin{aligned}&\int_{K(n)}\varphi(xky)\alpha_n(dk)\\&=\frac{\Gamma\big(\frac{n+1}{2}\big)}{\sqrt{\pi}\Gamma\big(\frac{n}{2}\big)}\int_0^{\pi}\Phi(\cos a\cos b+\sin a\sin b\cos\theta)\sin^{n-1}\theta d\theta.\end{aligned}$$

If φ is spherical, it follows similarly that

$$\Phi(\cos a\cos b)=\Phi(\cos a)\Phi(\cos b).$$

This implies that there exists an integer $m\geq 0$ such that

$$\varphi(g)=(g\cdot e_0|e_0)^m.$$

This function φ is of positive type. Therefore the spherical functions are the following:

$$\varphi(x)=(g\cdot e_0|e_0)^m\quad(m\in\mathbb{N}).$$

3) $G(n) = SO_0(1,n)$, $K(n) = SO(n)$. A K-biinvariant continuous function on G can be seen as a continuous function on the hyperboloid with one sheet

$$x_0^2 - x_1^2 - \cdots - x_n^2 = 1,\ x_0 > 0,$$

and can be written

$$\varphi(x) = \Phi\big([g \cdot e_0, e_0]\big),$$

where, for $x, y \in \mathbb{R}^{(\infty)}$,

$$[x, y] = x_0 y_0 - \sum_{n=1}^{\infty} x_n y_n,$$

and Φ is a continuous function on $[1, \infty[$. Furthermore, if $[x \cdot e_0, e_0] = \cosh a$, $[y \cdot e_0, e_0] = \cosh b$ $(x, y \in G)$,

$$\int_{K(n)} \varphi(xky)\alpha_n(dk)$$
$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \int_0^\pi \Phi(\cosh a \cosh b + \sinh a \sinh b \cos\theta) \sin^{n-1}\theta d\theta.$$

If φ is spherical, then

$$\Phi(\cosh a \cosh b) = \Phi(\cosh a)\Phi(\cosh b).$$

Hence there exists λ such that

$$\varphi(x) = x_0^{-\lambda}.$$

This function is of positive type if and only if λ is real and ≥ 0. Therefore the spherical functions are the following

$$\varphi(x) = x_0^{-\lambda} \quad (\lambda \geq 0).$$

V.2 The multiplicative property of the spherical functions. We assume that $\big(G(n), K(n)\big)$ is one of the examples given in Section IV.1.

THEOREM V.2.1. — *Let $\varphi \in \mathfrak{P}$. The function φ is spherical if and only if there exists a function Φ defined on*
$\mathbb{R}$ *in case (a),*
$\mathbb{T}$ *in case (b),*
$\mathbb{R}_+$ *in case (c),*

such that

$$\varphi\big(\mathrm{diag}(a_1,\dots,a_n)\big) = \Phi(a_1)\dots\Phi(a_n).$$

We will need some preliminaries about the asymptotic behaviour of the normalized Haar measure α_n of $K(n)$.

For $n \geq m$ let $K_m(n)$ be the following subgroup of $K(n)$:

$$K_m(n) = \left\{ \begin{pmatrix} I_m & 0 \\ 0 & v \end{pmatrix} \big| v \in U(n-m;\mathbb{F}) \right\} \simeq U(n-m;\mathbb{F}),$$

and

$$K_m = \bigcup_{n=m}^{\infty} K_m(n) \subset K.$$

Define also

$$K(m,n) = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \big| u \in U(m;\mathbb{F}), v \in U(n-m;\mathbb{F}) \right\}$$
$$\simeq U(m;\mathbb{F}) \times U(n-m;\mathbb{F}).$$

The coset space $K(n)/K(m,n)$ is a compact symmetric space, of rank m if $n \geq 2m$. For $\theta = (\theta_1,\dots,\theta_m) \in \mathbb{R}^m$ define

$$a(\theta) = \begin{pmatrix} \cos\theta_1 & & & -\sin\theta_1 & & & \\ & \ddots & & & \ddots & & \\ & & \cos\theta_m & & & -\sin\theta_m & \\ \sin\theta_1 & & & \cos\theta_1 & & & \\ & \ddots & & & \ddots & & \\ & & \sin\theta_m & & & \cos\theta_m & \\ & & & & & & I_{n-2m} \end{pmatrix}.$$

The set A_m of the matrices $a(\theta)$ is a Cartan subgroup for the symmetric pair $\big(K(n),K(m,n)\big)$. By the Cartan decomposition, every element k of $K(n)$ can be written

$$k = h_1 a(\theta) h_2,$$

where $h_1,h_2 \in K(m,n)$. Let $\mathfrak{a}_m = Lie(A_m) \subset \mathfrak{k}(n) = Lie\big(K(n)\big)$. It is the space of the matrices

$$H(\theta) = \begin{pmatrix} 0 & & & -\theta_1 & & & \\ & \ddots & & & \ddots & & \\ & & 0 & & & -\theta_m & \\ \theta_1 & & & 0 & & & \\ & \ddots & & & \ddots & & \\ & & \theta_m & & & 0 & \\ & & & & & & 0_{n-2m} \end{pmatrix}.$$

For the pair $\big(\mathfrak{k}(n), \mathfrak{a}_m\big)$, the roots and the multiplicities are the following

$$\begin{array}{ll} \sqrt{-1}\ (\pm\theta_i \pm \theta_j) \quad (i \neq j) & d \\ \pm\sqrt{-1}\ (2\theta_i) & d-1 \\ \pm\sqrt{-1}\ \theta_i & (n-2m)d \end{array}$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$ $(d = 1, 2, 4)$. The Weyl integration formula corresponding to the Cartan decomposition is given as

$$\int_{K(n)} f(k)\alpha_n(dk)$$
$$= \int_{[0,\frac{\pi}{2}]^m} \int_{K(m,n)\times K(m,n)} f\big(h_1 a(\theta) h_2\big) dh_1 dh_2 D_{m,n}(\theta) d\theta_1 \ldots d\theta_m,$$

with

$$D_{m,n}(\theta) = a_{m,n} \cdot \Big| \prod_{1\leq i<j\leq m} \big(\sin(\theta_i+\theta_j)\big)^d \big(\sin(\theta_i-\theta_j)\big)^d \prod_{i=1}^{m} (\sin 2\theta_i)^{d-1} (\sin\theta_i)^{(n-2m)d} \Big|.$$

(See [Helgason,1984], Ch. I, Theorem 5.10.) The constant $a_{m,n}$ is determined by the condition

$$\int_{[0,\frac{\pi}{2}]^m} D_{m,n}(\theta) d\theta_1 \ldots d\theta_m = 1$$

THEOREM V.2.2. — *Let f be a continuous function on K which is K_m-biinvariant, then*

$$\lim_{n\to\infty} \int_{K(n)} f(k)\alpha_n(dk) = \int_{K(m)\times K(m)} f(h_1 w_m h_2)\alpha_m(dh_1)\alpha_m(dh_2),$$

where

$$w_m = \begin{pmatrix} 0 & -I_m & & \\ I_m & 0 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

LEMMA IV.2.3. — *Let X be a compact space, and μ a positive measure such that every non empty open set has a positive measure.*

Let $\delta \geq 0$ be a continuous function on X which attains its maximum at only one point x_0. Define

$$\frac{1}{a_n} = \int_X \delta(x)^n \mu(dx),$$

and, for a continuous function f on X,

$$L_n(f) = a_n \int_X f(x)\delta(x)^n \mu(dx).$$

then

$$\lim_{n\to\infty} L_n(f) = f(x_0).$$

Proof.

For $0 < \alpha < M = \max \delta$, there exists a constant $C_\alpha > 0$ such that

$$a_n \leq C_\alpha \alpha^{-n}.$$

In fact there is a neighborhood V of x_0 such that $\delta(x) \geq \alpha$ for $x \in V$, and

$$\frac{1}{a_n} \geq \mu(V)\alpha^n.$$

Let W be a neighborhood of x_0. For $x \in X \setminus W$, $\delta(x) \leq \beta < M$. Choose α such that $\beta < \alpha < M$. Then

$$a_n \int_{X\setminus W} \delta(x)^n \mu(dx) \leq C_\alpha \mu(X)\Big(\frac{\beta}{\alpha}\Big)^n,$$

and

$$\lim_{n\to\infty} a_n \int_{X\setminus W} \delta(x)^n \mu(dx) = 0. \qquad \square$$

From Lemma V.2.3 it follows that, if f is a continuous function on $[0, \frac{\pi}{2}]^m$, then

$$\lim_{n\to\infty} \int_{[0,\frac{\pi}{2}]^m} f(\theta) D_{m,n}(\theta) d\theta_1 \dots d\theta_m = f\Big(\frac{\pi}{2}, \dots, \frac{\pi}{2}\Big).$$

Proof of Theorem V.2.2

If f is K_m-biinvariant, then

$$\int_{K(n)} f(k)\alpha_n(dk) = \int_{[0,\frac{\pi}{2}]^m} \int_{K(m)\times K(m)}$$
$$f\big(h_1 a(\theta) h_2\big)\alpha_m(dh_1)\alpha_m(dh_2) D_{m,n}(\theta) d\theta_1 \dots d\theta_m.$$

By using Lemma IV.2.3, and noticing that

$$w_n = a\Big(\frac{\pi}{2}, \dots, \frac{\pi}{2}\Big),$$

we obtain

$$\lim_{n\to\infty} \int_{K(n)} f(k)\alpha_n(dk) = \int_{K(m)\times K(m)} f(h_1 w_m h_2)\alpha_m(dh_1)\alpha_m(dh_2).$$

∎

Corollary V.2.4. — *Let φ be a continuous function on G which is K-biinvariant. For* $x = \operatorname{diag}(a_1, \dots, a_m)$, $y = \operatorname{diag}(b_1, \dots, b_m)$,

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi\big(\operatorname{diag}(a_1, \dots, a_m, b_1, \cdots, b_m)\big).$$

Proof.

The function $k \mapsto \varphi(xky)$ is K_m-biinvariant. Hence we can apply Theorem V.2.2 :

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk)$$
$$= \int_{K(m)\times K(m)} \varphi(xh_1 w_m h_2 y)\alpha_m(dh_1)\alpha_m(dh_2).$$

The statement follows since, for $h_1, h_2 \in K(m)$,

$$xh_1 w_m h_2 y w_m^{-1} \in K\operatorname{diag}(a_1, \dots, a_m, b_1, \dots, b_m)K.$$

∎

Proof of Theorem V.2.1

Let $\varphi \in \mathfrak{P}$. If φ is spherical, then, for $x = \operatorname{diag}(a_1, \dots, a_m)$, $y = \operatorname{diag}(b_1, \dots, b_m)$,

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y).$$

By Corollary V.2.4 it follows that

$$\varphi\big(\operatorname{diag}(a_1, \dots, a_m)\big)\varphi\big(\operatorname{diag}(b_1, \dots, b_m)\big)$$
$$= \varphi\big(\operatorname{diag}(a_1, \dots, a_m, b_1, \dots, b_m)\big),$$

and that

$$\varphi\big(\operatorname{diag}(a_1, \dots, a_m)\big) = \Phi(a_1) \dots \Phi(a_m),$$

where Φ is the restriction of φ to $G(1)$.

Conversely, if

$$\varphi\big(\operatorname{diag}(a_1, \dots, a_m)\big) = \Phi(a_1) \dots \Phi(a_m),$$

it follows from Corollary V.2.4 that

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y),$$

and that φ is spherical. ∎

Chapter VI

HARMONIC ANALYSIS ON THE SPACE OF INFINITE DIMENSIONAL HERMITIAN MATRICES

1 Spherical functions and ergodic measures. — Let $(X, \mathcal{B})$ be a measurable space, and K a group of measurable transformations of X. Let $\mathfrak{M}$ be the set of K-invariant probability measures on X. The set $\mathfrak{M}$ is convex. Fix $\mu \in \mathfrak{M}$. A measurable set $E \subset X$ is said to be *K-invariant relatively to μ* if, for every $g \in K$,

$$\mu\big((gE)\Delta E\big) = 0.$$

The measure μ is said to be *ergodic* if, for every measurable set E which is K-invariant relatively to μ,

$$\mu(E) = 0 \text{ or } 1.$$

PROPOSITION VI.3.1. — *Let $\mu \in \mathfrak{M}$. The following properties are equivalent:*
(a) μ is ergodic.
(b) μ is extremal in $\mathfrak{M}$.
(c) The subspace of K-invariant vectors in $L^2(X, \mu)$ reduces to constant functions.
([Phelps,1966], Proposition 10.4.)

If X is a locally compact topological space, and K a compact group acting on X by homeomorphisms, then the ergodic measures are exactly the orbital measures:

$$\mu_a(f) = \int_G f(g \cdot x)\alpha(dg) \quad (a \in X),$$

where α is the normalized Haar measure of K.

Let us consider the following special case: $X = V$ is a finite dimensional Euclidean space, and K is a closed subgroup of the orthogonal group $O(V)$. In that case the ergodic measures are the orbital measures. Let $\mathfrak{P}$ be the set of continuous functions of positive type on V which are K-invariant, with $\varphi(0) = 1$. The Fourier transform maps bijectively $\mathfrak{M}$ onto $\mathfrak{P}$, and $\text{ext}(\mathfrak{M})$ onto $\text{ext}(\mathfrak{P})$.

Let $G = K \ltimes V$ be the associated affine motion group. A K-biinvariant function φ on G can be seen as a K-invariant function on V. If φ is continuous of positive type, with $\varphi(0) = 1$, it is the Fourier transform of a probability measure μ on V. The function φ is spherical if and only if the measure μ is an orbital measure for the group K.

Assume now that $V(n)$ is an increasing sequence of Euclidean spaces, $V(n) \subset V(n+1)$, $K(n)$ is a closed subgroup of the orthogonal group $O\big(V(n)\big)$, and

$$K(n) = K(n+1) \cap O\big(V(n)\big).$$

Define

$$X = V = \bigcup_{n=1}^{\infty} V(n),$$

equipped with the inductive limit topology,

$$K = \bigcup_{n=1}^{\infty} K(n).$$

Here $X = V^*$ is the dual space of V. It is the projective limit of the sequence $V(n)$ relatively to the orthogonal projections

$$p_{m,n} : V(n) \to V(m) \quad (n > m).$$

For each n consider an orthonormal basis $\{e_1^{(n)}, \ldots, e_{d_n}^{(n)}\}$ of the orthogonal of $V(n)$ in $V(n+1)$. Then

$$\{e_i^{(n)} \mid 1 \leq i \leq d_n,\ n \in \mathbb{N}^*\}$$

is an orthonormal basis of V, hence V can be identified with $\mathbb{R}^{(\infty)}$ and V^* with $\mathbb{R}^{\infty}$. The group K acts on V^*. Let $\mathcal{M}$ be the set of K-invariant probability measures on V^*. In this case also the Fourier transform maps bijectively $\mathcal{M}$ onto $\mathcal{P}$, and $\text{ext}(\mathfrak{M})$ onto $\text{ext}(\mathfrak{P})$.

Define

$$G = \bigcup_{n=1}^{\infty} G(n) = K \ltimes V.$$

Then (G, K) is a spherical pair. A K-biinvariant continuous function φ on G, which is of positive type, with $\varphi(0) = 1$, seen as a K-invariant function on V, is the Fourier transform of a probability measure μ on V^*. The restriction φ_n of φ to $V(n)$ is the Fourier transform of a probability measure μ_n on $V(n)$, and $\{\mu_n\}$ is a consistent family of measures. In fact $\mu_n = p_n(\mu)$, where

$$p_n : V^* \to V(n)$$

is the projection of V^* onto $V(n)$. The function φ is spherical if and only if the measure μ is ergodic with respect to K. The spherical representation π of G associated to the spherical function φ can be realized on the Hilbert space $\mathcal{H} = L^2(V^*, \mu)$. For $(k, a) \in G$, $f \in \mathcal{H}$,

$$\big(\pi(k, a)f\big)(\xi) = e^{-\langle a, \xi \rangle} f(k \cdot \xi).$$

By using the following formula for the product in G:

$$(k_1, a_1)(k_1, a_2) = (k_1 k_2, k_1 \cdot a_2 + a_1),$$

one cheks that π is a representation. It is clearly unitary. The K-invariant vectors are the constants, and

$$\big(\pi(k, a)1|1\big) = \int_{V^*} e^{-\langle a, \xi \rangle} \mu(d\xi) = \varphi(a).$$

Example 1

In Chapter I we saw the case of

$$V(n) = \mathbb{R}^n,\ K(n) = O(n),$$
$$V = \mathbb{R}^{(\infty)},\ K = O(\infty),\ V^* = \mathbb{R}^\infty.$$

The spherical functions of positive type for the pair $\big(G(n), K(n)\big)$ are the Bessel functions:

$$\varphi_r(x) = \mathcal{J}_n(r\|x\|) \quad (r \geq 0),$$

and the orbital measures are the uniform spherical measures σ_r. The spherical functions for the pair (G, K) are the Gaussian functions

$$\varphi_t(x) = e^{-\frac{t}{2}\|x\|^2},$$

and the ergodic measures are the Gaussian measures γ_t $(t > 0)$, and the Dirac measure δ at 0.

Example 2

The main topic of these notes is the case $V(n) = Herm(n,\mathbb{C})$, $K(n) \simeq U(n)$ acting on $V(n)$ by the transformations

$$k : x \mapsto k \cdot x = uxu^* \quad \big(u \in U(n)\big),$$

and

$$V = H(\infty),\ K \simeq U(\infty).$$

A spherical function of positive type for the pair $\big(G(n), K(n)\big)$ is the Fourier transform of an orbital measure. In Chapter II (Theorem II.4.1) we established a formula for such a function: if μ is the orbital measure associated to the orbit of $a = \operatorname{diag}(a_1, \ldots, a_n)$, then, for $x = \operatorname{diag}(x_1, \ldots, x_n)$,

$$\varphi_a(x) = \delta! \frac{1}{D(a)D(-ix)} \det\Big((e^{-ia_j x_k})_{1 \le j,k \le n}\Big).$$

The spherical functions of the pair (G, K) are the functions of the following form:

$$\varphi\big(\operatorname{diag}(x_1, \ldots, x_n)\big) = \Phi(x_1) \ldots \Phi(x_n),$$

where Φ is a Pólya function.

We saw in Section 3 of Chapter I that in Example 1 an ergodic measure on $V^* = \mathbb{R}^\infty$ with respect to the action of $K = O(\infty)$ is the limit of a sequence of orbital measures on $V(n) = \mathbb{R}^n$ for the action of $O(n)$ (Proposition I.3.1), or, equivalently, that a spherical function φ for the pair (G, K) is the limit of a sequence φ_n, where φ_n is a spherical function for the pair $\big(G(n), K(n)\big)$. It turns out that it is a general fact. In the last section of this chapter we will present a proof of this fact in the case of Example 2, following closely [Olshanski-Vershik,1996]. We will need first some preparation about convergence of Pólya functions.

2. Convergence of Pólya functions. — By Theorem IV.4.1, every Pólya function has a representation as an infinite product

$$\Phi(\lambda) = \Phi(\lambda; \alpha, \beta, \gamma) = e^{-i\beta\lambda} e^{-\frac{1}{2}\gamma\lambda^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda},$$

with $\beta \in \mathbb{R}$, $\gamma \ge 0$, $\alpha_k \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

In the following it will be useful to consider the logarithmic derivative of the Pólya function Φ:

$$\frac{\Phi'(\lambda)}{\Phi(\lambda)} = -i\beta - \gamma\lambda + \sum_{k=1}^{\infty}\Big(i\alpha_k - \frac{i\alpha_k}{1+i\alpha_k}\Big)$$
$$= -i\beta - \big(\gamma + p_2(\alpha)\big)\lambda + \sum_{m=2}^{\infty} p_{m+1}(-i\alpha)\lambda^m.$$

We consider on the set Ω of Pólya functions the topology of uniform convergence on compact sets of $\mathbb{R}$. The topological space Ω is metrizable and complete. In fact if Φ is a limit of Pólya functions, and is continuous, then Φ is a Pólya function. We would like to express this topology in terms of the parameters α, β, γ.

First let us observe that, if a sequence Φ_n of Pólya functions converges to Φ, then, in general, the corresponding sequence $\gamma^{(n)}$ does not converge to γ. In fact, let us consider the following example. For $\alpha \neq 0$, define

$$\Phi_n(\lambda) = \Big(\frac{e^{i\frac{\alpha}{\sqrt{n}}\lambda}}{1 + i\frac{\alpha}{\sqrt{n}}\lambda}\Big)^n,$$

i.e. $\beta^{(n)} = 0$, $\gamma^{(n)} = 0$, and

$$\alpha_k^{(n)} = \begin{cases} \frac{\alpha}{\sqrt{n}} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}.$$

Then one checks easily that

$$\lim_{n\to\infty} \Phi_n(\lambda) = e^{-\frac{1}{2}\alpha^2\lambda^2} = \Phi(\lambda).$$

Notice that $\gamma^{(n)} = 0$, $\gamma = \alpha^2$.

To a Pólya function $\Phi(\lambda; \alpha, \beta, \gamma)$ we associate a bounded positive measure σ on $\mathbb{R}$ defined by:

$$\int_{\mathbb{R}} f(t)\sigma(dt) = \sum_{k=1}^{\infty} \alpha_k^2 f(\alpha_k) + \gamma f(0).$$

THEOREM VI.2.1. — *Let*

$$\Phi_n(\lambda) = \Phi(\lambda; \alpha^{(n)}, \beta^{(n)}, \gamma^{(n)}),$$
$$\Phi(\lambda) = \Phi(\lambda; \alpha, \beta, \gamma),$$

be Pólya functions, and let $\sigma^{(n)}$, σ be the associated measures as above. Assume that

$$\lim_{n\to\infty} \beta^{(n)} = \beta,$$

and, for every bounded continuous function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\sigma^{(n)}(dt) = \int_{\mathbb{R}} f(t)\sigma(dt).$$

Then

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda)$$

uniformly on compact sets of $\mathbb{R}$.

LEMMA VI.2.2. — *Let μ be a probability measure on $\mathbb{R}$, and φ its Fourier transform:*

$$\varphi(\lambda) = \int_{-\infty}^{\infty} e^{-it\lambda}\mu(dt).$$

Assume that φ admits a power series expansion for $|\lambda| < R$:

$$\varphi(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m.$$

Then φ has a holomorphic extension to the strip

$$\Sigma_R = \{\zeta = \lambda + i\eta \mid |\eta| < R\}.$$

For $|\eta| < R$,

$$\int_{-\infty}^{\infty} e^{t\eta}\mu(dt) < \infty,$$

and, for $\zeta \in \Sigma_R$,

$$\varphi(\zeta) = \int_{\infty}^{\infty} e^{-it\zeta}\mu(dt).$$

This a special case of a general result by Graczyk and Loeb in [1994] (Theorem 1).

Proof.

a) We show first that, since φ is $\mathcal{C}^\infty$ on $]-R, R[$, μ has moments of all orders: for all k,

$$\int_{-\infty}^{\infty} |t|^k \mu(dt) < \infty.$$

For $\varepsilon > 0$,

$$\frac{2\varphi(0) - \varphi(\varepsilon) - \varphi(-\varepsilon)}{\varepsilon^2} = 2\int_{-\infty}^{\infty} \frac{1 - \cos \varepsilon t}{\varepsilon^2} \mu(dt).$$

By Fatou's Lemma,

$$\int_{-\infty}^{\infty} t^2 \mu(dt) \leq \lim_{\varepsilon \to 0} \frac{2\varphi(0) - \varphi(\varepsilon) - \varphi(-\varepsilon)}{\varepsilon^2} = -\varphi''(0).$$

It follows that

$$-\varphi''(\lambda) = \int_{-\infty}^{\infty} e^{-it\lambda} t^2 \mu(dt).$$

By repeating the argument we obtain, for all ℓ,

$$\int_{-\infty}^{\infty} t^{2\ell} \mu(dt) < \infty,$$

and also

$$a_m = \frac{1}{m!}\varphi^{(m)}(0) = \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} t^m \mu(dt).$$

b) We show now that, for $|\eta| < R$,

$$\int_{-\infty}^{\infty} \cosh(t\eta)\mu(dt) < \infty.$$

For $|\eta| < R$,

$$\begin{aligned}
\tfrac{1}{2}\big(\varphi(i\eta) + \varphi(-i\eta)\big) &= \tfrac{1}{2} \sum_{m=0}^{\infty} a_m \big((i\eta)^m + (-i\eta)^m\big) \\
&= \sum_{\ell=0}^{\infty} a_{2\ell} (-1)^\ell \eta^{2\ell} \\
&= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \int_{-\infty}^{\infty} (t\eta)^{2\ell} \mu(dt) \\
&= \int_{-\infty}^{\infty} \Big(\sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} (t\eta)^{2\ell}\Big) \mu(dt) \\
&= \int_{-\infty}^{\infty} \cosh(t\eta)\mu(dt).
\end{aligned}$$

□

Lemma VI.2.3. — *Let μ_k be a sequence of probability measures on $\mathbb{R}$, and φ_k the Fourier transform of μ_k. One assumes that each function φ_k has a power series expansion for $|\lambda| < R$, hence a holomorphic extension to the disc*

$$D_R = \{\zeta \in \mathbb{C} \mid |\zeta| < R\}.$$

Assume moreover that the sequence φ_k converges uniformly on the disc D_r for all $r < R$. Then the sequence μ_k converges weakly to a probability measure μ. The functions φ_k and φ, the Fourier transform of μ, have holomorphic extensions to the strip

$$\Sigma_R = \{\zeta = \lambda + i\eta \mid |\eta| < R\},$$

and φ_k converges to φ uniformly on every compact $Q \subset \Sigma_R$.

In order to apply this Lemma one usually proves that

$$\varphi_k(\lambda) = \sum_{m=0}^{\infty} a_{k,m}\lambda^m \quad (|\lambda| < R),$$
$$\lim_{k\to\infty} a_{k,m} = a_m,$$

and

$$|a_{k,m}| \leq u_m,$$

with

$$\sum_{m=0}^{\infty} u_m r^m < \infty,$$

for $r < R$.

Proof.

By Lemma VI.2.2, the functions φ_k have holomorphic extensions to the strip Σ_R. For $0 \leq \eta < R$,

$$\int_{-\infty}^{\infty} \cosh(t\eta)\mu_k(dt) = \tfrac{1}{2}\big(\varphi_k(i\eta) + \varphi_k(-i\eta)\big).$$

Since

$$\lim_{k\to\infty} \tfrac{1}{2}\big(\varphi_k(i\eta) + \varphi_k(-i\eta)\big) = \tfrac{1}{2}\big(\varphi(i\eta) + \varphi(-i\eta)\big),$$

there exists a constant $M(\eta) > 0$ such that, for all k,

$$\int_{-\infty}^{\infty} \cosh(t\eta)\mu_k(dt) \leq M(\eta).$$

For $\zeta \in \Sigma_R$,

$$\varphi_k(\zeta) = \int_{-\infty}^{\infty} e^{-it\zeta} \mu_k(dt),$$

and, for $\zeta \in \Sigma_r$, $r < R$,

$$|\varphi_k(\zeta)| \leq 2M(r).$$

By the theorem of Montel, a subsequence converges uniformly on every compact $Q \subset \Sigma_R$. Since φ_k converges on D_R, it follows that the sequence φ_k itself converges uniformly on every compact $Q \subset \Sigma_R$. ∎

Proof of Theorem VI.2.1

Since the sequence

$$p_2\big(\alpha^{(n)}\big) + \gamma^{(n)}$$

is convergent, there exists $R > 0$ such

$$p_2\big(\alpha^{(n)}\big) + \gamma^{(n)} \leq R^2.$$

It follows that the measures $\sigma^{(n)}$ and σ are supported by $[-R, R]$. Furthermore, for $m \geq 3$,

$$\lim_{n\to\infty} p_m\big(\alpha^{(n)}\big) = \lim_{n\to\infty} \int_{\mathbb{R}} t^{m-2} \sigma^{(n)}(dt) = \int_{\mathbb{R}} t^{m-2} \sigma(dt) = p_m(\alpha),$$

and

$$\begin{aligned} p_m\big(\alpha^{(n)}\big) &\leq \big(\sup|\alpha_k^{(n)}|\big)^{m-2} p_2\big(\alpha^{(n)}\big) \\ &\leq p_2\big(\alpha^{(n)}\big)^{\frac{m}{2}-1} p_2\big(\alpha^{(n)}\big) \\ &= p_2\big(\alpha^{(n)}\big)^{\frac{m}{2}} \leq R^m. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{\Phi_n'(\lambda)}{\Phi_n(\lambda)} = \frac{\Phi'(\lambda)}{\Phi(\lambda)},$$

uniformly on $\{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$ for $r < R$, and hence

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda)$$

also. We obtained the statement by applying Lemma VI.2.3. ∎

For $R > 0$ we define

$$\Omega_R = \{\Phi(.;\alpha,\beta,\gamma) \in \Omega \mid -\Phi''(0) = p_2(\alpha) + \beta^2 + \gamma \leq R^2\}.$$

Proposition VI.2.5. — *The set Ω_R is compact.*

Proof.

a) *The set Ω_R is relatively compact.* In fact, if Φ is the Fourier transform of a probability measure μ, then

$$-\Phi''(0) = \int_{\mathbb{R}} t^2 \mu(dt),$$

and the set of probability measures μ such that

$$\int_{\mathbb{R}} t^2 \mu(dt) \leq R^2$$

is relatively compact.

b) *Let $Q \subset \Omega$ be compact. There exists $R > 0$ such that $Q \subset \Omega_R$.* Observe that a Pólya function does not vanish. Let $\lambda_0 > 0$. There exists $A > 0$ such that, for $\Phi \in Q$,

$$|\Phi(\lambda_0)| \geq A,$$

therefore

$$|\Phi(\lambda_0)|^2 = e^{\gamma \lambda_0^2} \prod_{k=1}^{\infty} \frac{1}{1+\alpha_k^2 \lambda_0^2} \geq A^2,$$

and

$$\gamma \lambda_0^2 + \sum_{k=1}^{\infty} \alpha_k^2 \lambda_0^2 \leq e^{\gamma \lambda_0^2} \prod_{k=1}^{\infty} (1+\alpha_k^2 \lambda_0^2) \leq \frac{1}{A^2},$$

or

$$\gamma + p_2(\alpha) \leq \frac{1}{A^2 \lambda_0^2}.$$

It follows that the set

$$Q_0 = \{\Phi_0(\lambda) = e^{i\beta\lambda}\Phi(\lambda) = \Phi(\lambda; \alpha, 0, \gamma) \mid \Phi \in Q\}$$

is relatively compact. Since Q and Q_0 are relatively compacts, the set of the numbers β is bounded:

$$|\beta| \leq B.$$

Therefore $Q \subset \Omega_R$, with

$$R^2 = \frac{1}{A^2 \lambda_0^2} + B^2.$$

c) *If* $\Phi_n = \Phi(\cdot;\alpha^{(n)},\beta^{(n)},\gamma^{(n)})$ *converges to* $\Phi(\cdot;\alpha,\beta,\gamma)$*, then*

$$\lim_{n\to\infty} p_2\big(\alpha^{(n)}\big) + \gamma^{(n)} = p_2(\alpha) + \gamma,$$

$$\lim_{n\to\infty} p_m\big(\alpha^{(n)}\big) = p_m(\alpha) \text{ for } m \geq 3,$$

$$\lim_{n\to\infty} \beta^{(n)} = \beta.$$

By b) there exists $R > 0$ such that, for all n,

$$p_2\big(\alpha^{(n)}\big) + \gamma^{(n)} + \beta^2 \leq R^2.$$

It follows that the functions Φ_n are holomorphic for $|\lambda| < R$, and, by Lemma V.2.2, in Σ_R. Furthermore, for $r < R$, there exists a constant $M(r) > 0$ such that

$$|\Phi_n(\lambda)| \leq M(r) \text{ for } \lambda \in \Sigma_r.$$

From the theorem of Montel, it follows that there is a subsequence Φ_{n_j} which converges uniformly on compact sets in Σ_R. Since the sequence itself converges to Φ on $\mathbb{R}$, it follows that the sequence Φ_n converges to Φ uniformly on compact sets in Σ_R. Therefore the logarithmic derivatives

$$\frac{\Phi_n'}{\Phi_n}$$

converge uniformly in a neighborhood of 0, and the coefficients of their Taylor expansions at 0 also.

d) The map

$$\Phi(\cdot;\alpha,\beta,\gamma) \mapsto p_2(\alpha) + \gamma, \quad \Omega \to \mathbb{R}$$

is continuous. It follows that Ω_R is closed. ∎

VI.3 Ergodic measures are limits of orbital measures. — Let μ be a probability measure on H_∞, which is ergodic with respect to the action of $U(\infty)$. Its Fourier transform φ is a spherical function for the spherical pair (G,K) (see Example 2 in Section 1), and has the form

$$\varphi\big(\mathrm{diag}(x_1,\ldots,x_n,0,\ldots)\big) = \Phi(x_1)\ldots\Phi(x_n),$$

where Φ is a Pólya function. This Pólya function can be written as an infinite product

$$\Phi(\lambda) = \Phi(\lambda;\alpha,\beta,\gamma) = e^{-i\beta\lambda}e^{-\frac{1}{2}\gamma\lambda^2}\prod_{k=1}^{\infty}\frac{e^{i\alpha_k\lambda}}{1+i\alpha_k\lambda}.$$

We will see that there exists a sequence μ_n of orbital measures which converges to μ: there exists a sequence $a^{(n)}$ of diagonal matrices

$$a^{(n)} = \mathrm{diag}(a_1^{(n)}, \ldots, a_n^{(n)}, 0, \ldots),$$

such that μ is the weak limit of the sequence μ_n of orbital measures defined by

$$\int f(x)\mu_n(dx) = \int_{U(n)} f(ua^{(n)}u^*)\alpha_n(du).$$

As we will see this makes possible to relate the asymptotic behaviour of the sequence $a^{(n)}$ to the parameters α, β, γ of the Pólya function Φ.

To the diagonal matrix $a^{(n)}$ one associates the measure $\sigma^{(n)}$ on $\mathbb{R}$ by

$$\int_{\mathbb{R}} f(t)\sigma^{(n)}(dt) = \sum_{k=1}^{n} \frac{1}{n^2}\big(a_k^{(n)}\big)^2 f\Big(\frac{1}{n}a_k^{(n)}\Big).$$

Here is the fundamental theorem of Olshanski and Vershik in [1996] (Theorem IV.1), in a slightly different formulation.

Theorem VI.3.1. — *Assume that*

$$\lim_{n\to\infty} \frac{1}{n}\,\mathrm{tr}\big(a^{(n)}\big) = \beta,$$

and that

$$\lim_{n\to\infty} \sigma^{(n)} = \sigma$$

weakly, with

$$\int_{\mathbb{R}} f(t)\sigma(dt) = \sum_{k=1}^{\infty} \alpha_k^2 f(\alpha_k) + \gamma f(0).$$

Then the measure μ_n converges weakly to an ergodic measure μ, whose Fourier transform is given by

$$\varphi\big(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\big) = \Phi(\lambda_1)\ldots\Phi(\lambda_n),$$

where Φ is the following Pólya function

$$\Phi(\lambda) = e^{-i\beta\lambda}e^{-\frac{1}{2}\gamma\lambda^2}\prod_{k=1}^{\infty}\frac{e^{i\alpha_k\lambda}}{1+i\alpha_k\lambda}.$$

Proof.

(a) Let us consider the following sequence of Pólya functions

$$\Phi^{(n)}(\lambda) = \prod_{k=1}^{n} \frac{1}{1 + i\frac{1}{n}a_k^{(n)}\lambda}.$$

The function $\Phi^{(n)}$ corresponds to the parameters $\left(\alpha^{(n)}, \beta^{(n)}, \gamma^{(n)}\right)$, with

$$\begin{aligned}
\alpha^{(n)} &= \left(\frac{1}{n}a_1^{(n)}, \ldots, \frac{1}{n}a_n^{(n)}, 0, \ldots\right), \\
\beta^{(n)} &= \frac{1}{n}\sum_{k=1}^{n} a_k^{(n)}, \\
\gamma^{(n)} &= 0.
\end{aligned}$$

By Theorem V.2.1, $\Phi^{(n)}$ converges to the Pólya function Φ,

$$\Phi(\lambda) = e^{-i\beta\lambda}e^{-\frac{1}{2}\gamma\lambda^2}\prod_{k=1}^{\infty}\frac{e^{i\alpha_k\lambda}}{1 + i\alpha_k\lambda}.$$

The function $\Phi^{(n)}$ has the following power series expansion near 0:

$$\begin{aligned}
\Phi^{(n)}(\lambda) &= \sum_{m=0}^{\infty} h_m\left(\frac{1}{n}a^{(n)}\right)(-i\lambda)^m \\
&= \sum_{m=0}^{\infty} c_{n,m}\lambda^m,
\end{aligned}$$

where h_m is the complete symmetric function (see the end of Section II.2).

The function Φ has also a power series expansion:

$$\Phi(\lambda) = \sum_{m=0}^{\infty} c_m\lambda^m.$$

It follows from the proof of Theorem VI.2.1 that

$$c_m = \lim_{n\to\infty} c_{n,m}.$$

Furthermore, there exists $R > 0$ such that

$$|c_{n,m}| \leq R^m.$$

This estimate can be obtained by using the Cauchy inequalities, or the following lemma:

Lemma VI.3.2. — *Let*

$$f(z)=\sum_{m=0}^{\infty}a_m z^m, \quad a_0\neq 0.$$

Assume that

$$\frac{f'(z)}{f(z)}=\sum_{m=0}^{\infty}b_m z^m,$$

with $|b_m|\leq R^{m+1}$*. Then* $|a_m|\leq |a_0|R^m$*.*

Proof.

From

$$f(z)=f(0)\exp\Bigl(\int_0^z \frac{f'(t)}{f(t)}dt\Bigr),$$

it follows that

$$a_m=a_0P_m(b_0,b_1,\ldots,b_{m-1}),$$

where P_m is a polynomial with positive coefficients. If $b_m=R^{m+1}$, then $a_m=a_0R^m$. In fact

$$\frac{f'(z)}{f(z)}=\frac{R}{1-Rz},\quad f(z)=\frac{a_0}{1-Rz}. \qquad \square$$

(b) By Theorem II.4.1 the Fourier transform φ_n of μ_n is given by

$$\varphi_n\bigl(\mathrm{diag}(\lambda_1,\ldots,\lambda_n)\bigr)=\delta!\sum_{m_1\geq\cdots\geq m_n\geq 0}\frac{1}{(\mathbf{m}+\delta)!}s_{\mathbf{m}}\bigl(a^{(n)}\bigr)s_{\mathbf{m}}(-i\lambda).$$

Since

$$s_{\mathbf{m}}(\lambda,0,\ldots,0)=\begin{cases}\lambda^m & \text{if } \mathbf{m}=(m,0,\ldots,0),\\ 0 & \text{otherwise,}\end{cases}$$

and, if $\mathbf{m}=(m,0,\ldots,0)$,

$$s_{\mathbf{m}}(x)=h_m(x)=\sum_{|\mathbf{m}|=m}x^{\mathbf{m}}.$$

It follows that

$$\begin{aligned}&\varphi_n\bigl(\mathrm{diag}(\lambda,0,\ldots,0)\bigr)\\ &=(n-1)!\sum_{m=0}^{\infty}\frac{1}{(m+n-1)!}h_m\bigl(a^{(n)}\bigr)(-i\lambda)^m\\ &=\sum_{m=0}^{\infty}\frac{n^m}{n(n+1)\cdots(n+m-1)}h_m\Bigl(\frac{1}{n}a^{(n)}\Bigr)(-i\lambda)^m.\end{aligned}$$

This can be written

$$\varphi_n\big(\mathrm{diag}(\lambda, 0, \dots, 0)\big) = \sum_{m=0}^{\infty} \frac{n^m}{n(n+1)\dots(n+m-1)} c_{n,m}\lambda^m.$$

Since

$$\lim_{n\to\infty} \frac{n^m}{n(n+1)\dots(n+m-1)} = 1, \quad \frac{n^m}{(n(n+1)\dots(n+m-1)} \leq 1,$$

it follows from Lemma VI.2.3 that

$$\lim_{n\to\infty} \varphi_n\big(\mathrm{diag}(\lambda, 0, \dots, 0)\big) = \sum_{m=0}^{\infty} c_m\lambda^m = \Phi(\lambda).$$

(c) Let us show now that, for k fixed,

$$\lim_{n\to\infty} \varphi_n\big(\mathrm{diag}(\lambda_1, \dots, \lambda_k, 0 \dots)\big) = \Phi(\lambda_1)\dots\Phi(\lambda_k).$$

If $m_{k+1} > 0$, then

$$s_{\mathbf{m}}(\lambda_1, \dots, \lambda_k, 0, \dots) = 0.$$

Hence

$$\begin{aligned}
&\varphi_n\big(\mathrm{diag}(\lambda_1, \dots, \lambda_k, 0, \dots)\big) \\
&= \delta! \sum_{m_1\geq\dots\geq m_k\geq 0} \frac{1}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}\big(a^{(n)}\big) s_{\mathbf{m}}(-i\lambda_1, \dots, -i\lambda_k) \\
&= \delta! \sum_{m_1\geq\dots\geq m_k\geq 0} \frac{n^{|\mathbf{m}|}}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}\Big(\frac{1}{n}a^{(n)}\Big) s_{\mathbf{m}}(-i\lambda_1, \dots, -i\lambda_k).
\end{aligned}$$

On the other hand, by using the identity

$$\prod_{i,j=1}^{k} \frac{1}{1 - x_i y_j} = \sum_{m_1\geq\dots\geq m_k\geq 0} s_{\mathbf{m}}(x) s_{\mathbf{m}}(y)$$

(Proposition II.3.4), one obtains

$$\Phi_n(\lambda_1)\dots\Phi_n(\lambda_k) = \sum_{m_1\geq\dots\geq m_k\geq 0} s_{\mathbf{m}}\Big(\frac{1}{n}a^{(n)}\Big) s_{\mathbf{m}}(-i\lambda_1, \dots, -i\lambda_n).$$

By generalizing the proof in (c) one obtains

$$\lim_{n\to\infty} \varphi_n\big(\mathrm{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots)\big) = \Phi(\lambda_1)\cdots\Phi(\lambda_k).$$

By a multivariate analogue of Lemma VI.2.3 this implies the statement of Theorem VI.3.1. ∎

4. Is there a Bochner type theorem for spherical pairs ?. Let (G, K) be a spherical pair (see the definition in Section V.1). We denoted by $\mathfrak{P}$ the set of K-biinvariant continuous functions φ on G which are of positive type, with $\varphi(0) = 1$. Let Ω be the set of spherical functions of positive type. We saw that the extremal function in $\mathfrak{P}$ are the spherical ones: $\mathrm{ext}(\mathfrak{P}) = \Omega$.

Assume first that (G, K) is a Gelfand pair. For the topology of uniform convergence on compact sets of G, the set Ω is locally compact. The Bochner theorem has been extended to this setting by Godement:

THEOREM (BOCHNER-GODEMENT). — *If $\varphi \in \mathfrak{P}$, then there exists a unique probability measure ν on Ω such that*

$$\varphi(x) = \int_\Omega \omega(x)\nu(d\omega).$$

In Section V.1, we considered an increasing sequence $\big(G(n), K(n)\big)$ of Gelfand pairs: $G(n)$ is a closed subgroup of $G(n+1)$, $K(n)$ is a closed subgroup of $K(n+1)$, $K(n) = K(n+1) \cap G(n)$,

$$G = \bigcup_{n=1}^\infty G(n), \quad K = \bigcup_{n=1}^\infty K(n).$$

Then the pair (G, K) is spherical (Theorem V.1.1). It should be natural to look for an extension of the Bochner-Godement theorem to this general setting. In several cases the set Ω has been determined and such a statement has been established.

1. $V(n) = \mathbb{R}^n$, $K(n) = O(n)$, $G(n) = O(n) \ltimes \mathbb{R}^n$,

$$G = O(\infty) \ltimes \mathbb{R}^{(\infty)}, \quad K = O(\infty).$$

A K-biinvariant function φ on G can be seen as a radial function on $\mathbb{R}^{(\infty)}$. As we saw in Section V.1, the spherical functions are the Gaussian functions

$$\varphi_t(x) = e^{-\frac{t}{2}\|x\|^2} \quad (t \geq 0).$$

Therefore $\Omega \simeq [0, \infty[$. And as we saw in Section I.4, every function $\varphi \in \mathfrak{P}$ can be written

$$\varphi(x) = \int_{[0,\infty[} e^{-\frac{t}{2}\|x\|^2} \nu(t),$$

with a unique probability measure ν on $[0, \infty[$.

2. $G(n) = SO(n+1)$, $K(n) = SO(n)$. A biinvariant continuous function φ can be written

$$\varphi(g) = \Phi\big((g \cdot e_0 | e_0)\big),$$

where φ is a continuous function on $[-1, 1]$. We saw in Section V.1 that the spherical functions are the functions

$$\varphi(g) = (g \cdot e_0 | e_0)^m \quad (m \in \mathbb{N}).$$

Therefore $\Omega \simeq \mathbb{N}$. Every function $\varphi \in \mathfrak{P}$ can be uniquely written

$$\varphi(g) = \Phi\big((g \cdot e_0 | e_0)\big),$$

where

$$\Phi(u) = \sum_{m=0}^{\infty} \nu_m u^m,$$

with

$$\nu_m \geq 0, \quad \sum_{m=0}^{\infty} \nu_m = 1.$$

([Schoenberg,1942])

3. $G(n) = SO_0(1, n)$, $K(n) = SO(n)$. A K-biinvariant continuous function φ on G can be written

$$\varphi(g) = \Phi\big([g \cdot e_0, e_0]\big),$$

where, for $x, y \in \mathbb{R}^{(\infty)}$,

$$[x, y] = x_0 y_0 - \sum_{n=1}^{\infty} x_n y_n,$$

and Φ is a continuous function on $[1, \infty[$. We saw in Section V.1 that the spherical functions are the functions

$$\varphi(g) = [g \cdot e_0, e_0]^{-\lambda} \quad (\lambda \geq 0).$$

Therefore $\Omega \simeq [0,\infty[$. Every function $\varphi \in \mathfrak{P}$ can be written

$$\varphi(g) = \Phi\big([g \cdot e_0, e_0]\big),$$

where

$$\Phi(u) = \int_{[0,\infty[} u^{-\lambda}\nu(\lambda),$$

with a unique probability measure on $[0,\infty[$.

([Krein,1949], [Faraut-Harzallah,1974])

For the examples 1,2,3 and related questions see also [Berg-Christensen-Ressel,1984].

4. $V(n) = Herm(n,\mathbb{C})$, $K(n) = U(n)$ acting on $V(n)$ by the transformations

$$x \mapsto uxu^* \quad \big(u \in U(n)\big),$$

and $G(n) = U(n) \ltimes Herm(n,\mathbb{C})$

We saw that the spherical functions are $U(\infty)$-invariant functions on $H(\infty)$ for which

$$\varphi\big(\mathrm{diag}(a_1,\ldots,a_n)\big) = \Phi(a_1)\ldots\Phi(a_n),$$

where Φ is a Pólya function. Therefore $\mathrm{ext}(\mathfrak{P})$ is parametrized by the set Ω of Pólya functions. From the work of Borodin and Olshanski ([2001], Theorem 9.1) one can deduce that, for every $\varphi \in \mathfrak{P}$, there is a unique probability measure ν on Ω such that

$$\varphi(x) = \int_\Omega \varphi_\omega(x)\nu(d\omega).$$

5. Similar results have been obtained about central functions of positive type on the infinite dimensional unitary group $U(\infty)$. Let $\mathfrak{P}$ be the set of continuous functions φ of positive type on $U(\infty)$ which are central: $\varphi(gxg^{-1}) = \varphi(x)$ $(g,x \in U(\infty))$, with $\varphi(e) = 1$. Let us consider the pair

$$G = U(\infty) \times U(\infty), \quad K = \mathrm{diag}\big(U(\infty) \times U(\infty)\big) \simeq U(\infty).$$

It is a spherical pair, and a central function on $U(\infty)$ can be seen as a K-biinvariant function on G.

By Theorem V.2.1, if φ is a spherical function there is a continuous function Φ defined on the torus $\mathbb{T}$ such that

$$\varphi\big(\mathrm{diag}(t_1,\ldots,t_n)\big)=\Phi(t_1)\ldots\Phi(t_n)\quad (t_j\in\mathbb{T}).$$

This has been proved in another way by Voiculescu ([1976], Proposition 1). The function Φ is of positive type, and has a Fourier expansion:

$$\Phi(t)=\sum_{m=-\infty}^{\infty}c_m t^m,$$

with $c_m\geq 0$, $\sum_{m=-\infty}^{\infty}c_m=1$. The restriction of φ to $U(n)$ has a Schur expansion (Proposition II.3.1):

$$\varphi\big(\mathrm{diag}(t_1,\ldots,t_n)\big)=\Phi(t_1)\ldots\Phi(t_n)=\sum_{m_1\geq\cdots\geq m_n}a_{\mathbf{m}}^{(n)}s_{\mathbf{m}}(t),$$

where

$$a_{\mathbf{m}}^{(n)}=\det\big((c_{m_i-i+j})_{1\leq i,j\leq n}\big).$$

The function φ is of positive type if and only if, for all n and $\mathbf{m}$,

$$a_{\mathbf{m}}^{(n)}\geq 0.$$

It has been shown by Voiculescu ([1976], Proposition 2) that it holds for the following functions

$$\Phi(t)=t^m e^{\lambda(t-1)}e^{\mu(t^{-1}-1)}$$
$$\prod_{k=1}^{\infty}\Big(\frac{1-\alpha_k}{1-\alpha_k t}\Big)\Big(\frac{1-\beta_k t}{1-\beta_k}\Big)\Big(\frac{1-\gamma_k}{1-\gamma_k t^{-1}}\Big)\Big(\frac{1+\delta_k t^{-1}}{1+\delta_k}\Big),$$

with

$$m\in\mathbb{Z},\ \lambda\geq 0,\ \mu\geq 0,$$
$$0\leq\alpha_k<1,\ 0\leq\beta_k<1,\ 0\leq\gamma_k<1,\ 0\leq\delta_k<1,$$
$$\sum_{k=1}^{\infty}(\alpha_k+\beta_k+\gamma_k+\delta_k)<\infty.$$

These functions ressemble very much to Pólya functions. Later it was noticed by Vershik and Kerov [1981], and by Boyer [1983], that it holds if and only if the sequence $\{c_m\}$ is totally positive, i.e., for $k_1<\cdots<k_n$, $\ell_1<\cdots<\ell_n$,

$$\det\big((c_{k_i-\ell_j})_{1\leq i,j\leq n}\big)\geq 0.$$

Then, by a theorem of Edrei [1953], the sequence $\{c_m\}$ is totally positive if and only if the function Φ can be written as an infinite product as above (as it was conjectured by Schoenberg [1948]).

This parametrization of Ω has been obtained in another way by Vershik and Kerov [1981], by studying the asymptotics of the characters of $U(n)$ as $n \to \infty$ (see also [Okunkov-Olshanski,1998]). This approach is more informative since it gives a geometric meaning to this parametrization.

Here also the analogue of the Bochner-Godement holds ([Voiculescu,1976], Théorème 2, see also [Olshanski,2001], Theorem 9.1). The set Ω of spherical functions (extremal functions in $\mathfrak{P}$) is locally compact (for the topology of uniform convergence on compact sets). For every function $\varphi \in \mathfrak{P}$ there is a unique probability measure ν on Ω such that

$$\varphi(x) = \int_\Omega \omega(x)\nu(d\omega).$$

6. The infinite symmetric group

$$\mathfrak{S}_\infty = \bigcup_{n=1}^\infty \mathfrak{S}_n$$

is the group of the bijections $g : \mathbb{N}^* \to \mathbb{N}^*$ whose support $\{k \in \mathbb{N}^* \mid g(k) \neq k\}$ is finite. For studying central functions on $\mathfrak{S}_\infty$ it is equivalent to consider K-biinvariant functions for the following spherical pair

$$G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty, \quad K = \mathrm{diag}(\mathfrak{S}_\infty \times \mathfrak{S}_\infty) \simeq \mathfrak{S}_\infty,$$

The set Ω of spherical functions, or extremal functions in $\mathfrak{P}$, is compact for the pointwise convergence. For every function $\varphi \in \mathfrak{P}$ there is a unique probability measure ν on Ω such that

$$\varphi(x) = \int_\Omega \omega(x)\nu(dx).$$

The description of the spherical functions involves functions in one complex variable with an infinite product product representation. Here also there functions ressemble very much to the Pólya functions. An element $g \in \mathfrak{S}_\infty$ is a product of cycles. For $m \geq 2$ let $\gamma_m = \gamma_m(g)$ be the number of cycles of length m in the decomposition of g. The sequence $\{\gamma_2, \gamma_3, \ldots\}$ determines the conjugacy class of g. If φ is spherical, then it is multiplicative in the following sense:

$$\varphi(g) = \prod_{m=2}^\infty s_m^{\gamma_m(g)},$$

where $\{s_m\}$ is a sequence of real numbers, $-1 \leq s_m \leq 1$. Let us consider the generating function of the sequence $\{s_m\}$:

$$F(z) = \sum_{m=0}^{\infty} s_{m+1} z^m,$$

and also the Taylor expansion

$$\Phi(z) = \sum_{m=0}^{\infty} c_m z^m,$$

of the function Φ defined by

$$\frac{\Phi'(z)}{\Phi(z)} = F(z), \ \Phi(0) = 1.$$

The function φ is of positive type if and only

$$\Phi(z) = e^{\delta z} \prod_{k=1}^{\infty} \frac{1 + \beta_k z}{1 - \alpha_k z},$$

with

$$\alpha_k \geq 0, \ \beta_k \geq 0, \ \sum_{k=1}^{\infty} (\alpha_k + \beta_k) \leq 1.$$

Since

$$\frac{\Phi'(z)}{\Phi(z)} = \delta + \sum_{k=1}^{\infty} \frac{\beta_k}{1 + \beta_k z} + \sum_{k=1}^{\infty} \frac{\alpha_k}{1 - \alpha_k z},$$

this means that

$$s_m = \sum_{k=1}^{\infty} \alpha_k^m + (-1)^{m+1} \sum_{k=1}^{\infty} \beta_k^m.$$

These results have been proved by Thoma [1964a,1964b]. It turns out that the function φ is of positive type if and only if the sequence $\{c_m\}$ is of totally positive, but Thoma does not refer explicitely to total positivity. In fact his paper [1964b] contains essentially a proof of Schoenberg's theorem about one sided totally positive sequences ([1948]).

These results have been obtained in another way by Vershik and Kerov ([1981]) by studying asymtotics of the characters of $\mathfrak{S}_n$ as $n \to \infty$.

The symmetric group $\mathfrak{S}_n$ is the Weyl group of the root system of type A_n, and one can say that $\mathfrak{S}_\infty$ is the Weyl group of type A_∞. Similarly one can consider the infinite Weyl groups of type B_∞, C_∞ and D_∞. For these groups the spherical functions (or characters) have been determined in [Hirai-Hirai,2002].

In [Kerov-Olshanski-Vershik,1993], for $z \in \mathbb{C}$, a family T_z of unitary representations of the spherical pair (G, K) is introduced. The representation T_z decomposes as a direct integral of spherical unitary representations. This decomposition is analysed by studying the integral representation of the function of positive type

$$\varphi_z(g) = \big(T_z(g)f_0 | f_0\big),$$

where f_0 is a cyclic vector in the representation space.

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