Families of the characters of the cyclotomic Hecke algebras of $G(de, e, r)$

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1. Abstract

Motivated by the work of Lusztig on the families of $Irr(W)$ for the Weyl group W of any type, especially for the types B_r and D_r, we described in the article [BrKim] the families of the characters of $G(e, 1, r)$ and $G(e, e, r)$, which are the blocks of the cyclotomic Hecke algebras over an appropriate ring called "Rouquier Ring". The results coincide with the results of Lusztig on the Weyl groups of types B_r and D_r when specializing the parameters of the Hecke algebras.

In this paper, generalizing this work to the whole series of $G(de, e, r)$ which is a normal subgroup of $G(de, 1, r)$ of index e, we extend the results of [BrKim].

2. Introduction

In the representation theory of finite reductive groups, over a finite field, the notion of "families" in $Irr(W)$ of Weyl groups plays a fundamental role (as an example in the classification of characters, or see the Main Theorem 4.23 of [Lus]).

In the case of Weyl groups and their usual Hecke algebras, the families of irreducible characters have been defined by using the Kazhdan–Lusztig basis. Lusztig determined the families for all types of the Coxeter groups.

In generalizing to the case of the complex reflection groups, the first obstacle was that there is no notion of Kazhdan–Lusztig basis and thus we could not use the same definition of families as Lusztig defined for the Weyl groups.

From the results of Gyoja (cf. [Gy] 1996), extended by Rouquier for all types of Weyl groups (cf. [Rou] 1999), we now have a notion of families without using the Kazhdan–Lusztig basis. This is found to be the blocks of the irreducible characters of the Hecke algebra over an appropriate ring which is called *Rouquier ring* in [BrKim] and in the present paper. Rouquier also proved that this new notion of families coincides with that of Lusztig, giving the same results on deciding families of characters for the case of Weyl groups.

The complex reflection groups of type $G(e, e, r)$ can be viewed as a generalization of the Weyl group of type D_r (which are the groups $G(2, 2, r)$), and the dihedral groups ($G(e, e, 2)$ is the dihedral group of order $2e$). In the last section of the article [BrKim], we determined the "families" of the cyclotomic algebras of the groups $G(e, e, r)$. We also obtained in particular (considering the case of the algebras called "spetsiales") a generalization of the classification of Lusztig on the families of the groups of type D_r (which appears here as a simple application of

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the "Clifford theory" of the blocks), as well as a new proof of the classification of the families of the dihedral groups which was obtained only experimentally by Lusztig ([Lu2]), then by Malle and Rouquier by direct study of the "Rouquier blocks" (cf. [MaRo]).

In this paper, we generalize the results on the complex reflection groups of types $G(e,e,r)$ and $G(e,1,r)$ to the infinite series $G(de,e,r)$. By noting that $G(de,e,r)$ is a normal subgroup of $G(de,1,r)$ of index e, we generalize the method of [BrKim] for the case $G(e,e,r)$. In [BrKim] we used the fact that $G(e,e,r)$ is a normal subgroup of $G(e,1,r)$ of index e and then by Clifford theory we could obtain the families of the cyclotomic Hecke algebras of $G(e,e,r)$. In section 4, we explain how we generalize this method to the case $G(de,e,r)$. In our case, instead of considering simply a cyclic permutation action σ on the e-partition of r, we consider the action σ^d on the de-partition of r and we apply Clifford theory (using §11 of [CuRei]) to obtain the property we had in [BrKim] 1.42 – 1.45. In the Theorem 4.3 which is the main result of this paper, we get the analogous results as in [BrKim] to determine the families of the cyclotomic algebras of the groups $G(de,e,r)$, *i.e.*, the blocks of the cyclotomic Hecke algebras over the Rouquier ring defined in the section 3.D. This coincides with the results of [BrKim] when $d=1$. Let $\overline{\mathcal{H}}_{de,r}$ be the cyclotomic Hecke algebra of $G(de,e,r)$. In our case, the form of the symbols corresponding to each character $\chi \in Irr(\overline{\mathcal{H}}_{de,r})$ is supposed to have the first d parts being repeated e times. And our main result (see the Theorem 4.3) goes as follows.

Main result : 1. If the de-partitions λ, μ are not d-stuttering (see the Definition 4.2), then the irreducible characters χ_λ, χ_μ of $Irr(\overline{\mathcal{H}}_{de,r})$ are in the same family if and only if their corresponding symbols have the same contents, *i.e.*, same entries with same multiplicities, but possibly arranged in different rows.

2. If the de-partition is d-stuttering, then there are e corresponding different irreducible characters of $\mathrm{Irr}(\overline{\mathcal{H}}_{de,r})$ and each of these characters gives a singleton family.

Also at the end of the present paper, we show that the values of a_χ and A_χ of the Schur element S_χ corresponding to each $\chi \in Irr(\overline{\mathcal{H}}_{de,r})$ are constant within each family. In fact, this property was proved by Lusztig for the Weyl groups (see for example [Lu1], 3.3 and 3.4 (e)) : *i.e.*, for each irreducible character χ Lustig computed the integers a_χ and A_χ of the Schur elements S_χ, for which he shows that they are constant on the families. In [BrKim] 4.5 also, we obtained the same results for the complex reflection groups $G(e,1,r)$ and $G(e,e,r)$. Finally, we give some examples of families for $G(de,e,r)$, explaining at the same time a difference from the case of $G(de,de,r)$.

For the spetsial exceptional cases, *i.e.*, for the 18 irreducible reflection groups

$$G_4, G_6, G_8, G_{14}, G_{23,\dots,30}, G_{32,\dots,37}$$

following the Shephard-Todd notation, the author refers the article of G. Malle and R. Rouquier "Familles de caractères de groupes de réflexions complexes" (cf. [MaRou]). They determined the families for all exceptional cases listed above with spetsial parameters mainly.

3. **Reminder on basic definitions :**

3.A. Some reminder on combinatorial notations.

a) A partition of n and a multipartition of n.

- Let $\lambda = \lambda_1\lambda_2\dots\lambda_h$ be a partition, *i.e.*, a finite sequence of positive integers in non-decreasing order : $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h \geq 1$.
  The integer $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_h$ is called *the rank* of λ. The integer h is called *the height* of λ and we put $h_\lambda := h$.
- *Multipartition of n :*

A d–partition of n is defined as the list $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ where each $\lambda^{(j)}$ denotes a partition such that $\sum_{d-1}^{j=0} \mid \lambda^{(j)} \mid = n$.

- *Stuttering partition :*

  We call a multipartition with all $\lambda^{(j)}$s equal "Stuttering partition", *i.e.*, the case

$$\lambda = (\lambda^{(0)}, \lambda^{(0)}, \ldots, \lambda^{(0)}).$$

b) β–numbers and shifting β–numbers.

We associate to the partition λ

- its Young diagram

$$Y_\lambda := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid (1 \leq i \leq h)(1 \leq j \leq \lambda_i)\},$$

- its β–number

$$\beta_\lambda := (\beta_1, \beta_2, \ldots, \beta_h)$$

  defined by

$$\beta_1 := h + \lambda_1 - 1\,,\; \beta_2 := h + \lambda_2 - 2\,,\; \ldots\;,\; \beta_h := h + \lambda_h - h\,,$$

  i.e., from the Young diagram of λ, by looking at only the first column, read the hook lengths of the boxes on the first column starting from the bottom box to the top one. The list of hook lengths in this order is the β–numbers of the given λ and these numbers are strictly increasing.

- its residue $\mathrm{Res}_\lambda(x)$, element of $\mathbb{Z}[x, x^{-1}]$ defined by

$$\mathrm{Res}_\lambda(x) := \sum_{\substack{1 \leq i \leq h \\ 1 \leq j \leq \lambda_i}} x^{j-i}\,,$$

  i.e., $\mathrm{Res}_\lambda(x)$ is the sum of monomials filling out the Young diagram as follows (an example above corresponding to the partition $\lambda = (4, 3, 2)$) :

1	x	x^2	x^3
x^{-1}	1	x	
x^{-2}	x^{-1}		

- *Shifting β–numbers of λ by one :*

  Let $(\beta_1, \beta_2, \ldots, \beta_h)$ be the β–number of λ. By the list of numbers $(0, \beta_1 + 1, \beta_2 + 1, \ldots, \beta_h + 1)$, we say that the β–number of λ is shifted by one. By repeating m times, we get the shift of β–number of λ by m as follows :

$$(0, 1, \ldots, m - 1, \beta_1 + m, \beta_2 + m, \ldots, \beta_h + m)$$

c) Symbols

"The symbol of λ is defined as the ordered list of β–numbers $\beta^{(j)}$ of each $\lambda^{(j)}$ from a multipartition $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$. We put this list row by row as follows :

$$\Lambda_\lambda := \begin{pmatrix} \beta^{(0)} \\ \beta^{(1)} \\ \dots \\ \beta^{(d-1)} \end{pmatrix}$$

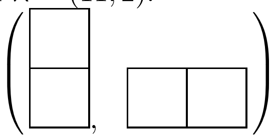
We can shift the β–numbers in the symbol Λ_λ corresponding to λ so as to make it into the form of symbols we want :

- If we shift the β–numbers in the symbol so that the topmost β–number has just one more element than the others, then we say that the symbol (after the shift) is in the shape of "spetsial symbol".
- If we shift the β–numbers in the symbol so that all the β–numbers have the same number of elements, then we say that the symbol (after the shift) is in the shape of "ordinary (or rectangular) symbol".
- For more general type of symbol is in form $\underline{\mathbf{m}}$ with each m_j integer, which is called "the symbol of type $(m_0, m_1, \dots, m_{d-1})$."

The content $\mathrm{Cont}_{\overline{\lambda}}(x)$ of a symbol Λ_λ is a polynomial in x where each term $k_i x^i$ signifies the multiplicity k_i of the number i in the collection of all β–numbers in the symbol Λ_λ.

Example.

Let's take an example with a multipartition $\lambda = (11, 2)$.

Then its associated Young diagram will be  .

Now the symbol corresponding to λ becomes

$$\Lambda_\lambda := \begin{pmatrix} 1 & 2 \\ 2 & \end{pmatrix}.$$

This is a spetsial symbol and its content is $1 \cdot x^1 + 2 \cdot x^2 = x + 2x^2$.

By shifting by one on the second β-number, we can get an ordinary symbol as follows :

$$\Lambda_\lambda := \begin{pmatrix} 1 & 2 \\ 0 & 2+1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

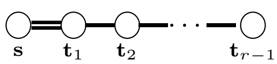
The content of $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is $1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 = 1 + x + x^2 + x^3$.

3.B.The Groups $G(de, 1, r), G(de, e, r), and\ G(de, de, r)$**.**

We recall that $W_{de,r} := G(de, 1, r)$, following the notation of Shephard and Todd, is defined as the group $\boldsymbol{\mu}_{de} \wr \mathfrak{S}_r \cong (\mathbb{Z}/de\mathbb{Z})^r \rtimes \mathfrak{S}_r$.

There is a presentation with set of generators $s, t_1, t_2, \dots, t_{r-1}$ for $W_{de,r}$ and relations :

- Order relations : $s^{de} = t_j^2 = 1$ $(j = 1, 2 \dots, r-1)$
- Braid relations symbolized by the diagram



(figure 1)

i.e., $st_1st_1 = t_1st_1s$, $t_{i-1}t_it_{i-1} = t_it_{i-1}t_i$, $st_i = t_is$ $(i = 2, \dots, r-1)$, and $t_it_j = t_jt_i$ $(|i-j| > 1)$.

Now for the group $\overline{W}_{de,r} := G(de, e, r)$, by composing the natural surjective morphism $W_{de,r} \twoheadrightarrow \mathfrak{S}_r$ with the signature morphism, we obtain a morphism which will be denoted again by $\mathrm{sgn}\colon W_{de,r} \longrightarrow \boldsymbol{\mu}_2 = \{-1, +1\}$.

We write $\det\colon W_{de,r} \twoheadrightarrow \boldsymbol{\mu}_{de}$ for the determinant linear map, and we denote

$$\mathrm{sgn}_{de}\colon W_{de,r} \twoheadrightarrow \boldsymbol{\mu}_{de}$$

for the homomorphism defined by the multiplication $\det = \mathrm{sgn}\cdot \mathrm{sgn}_{de}$.

The group $G(de, e, r)$ is defined by the group $\overline{W}_{de,r} := \ker(\mathrm{sgn}_{de}^d) \lhd W_{de,r}$, a normal subgroup of $W_{de,r}$ of index e. In other words, $\overline{W}_{de,r}$ is the group of monomial $n\times n$ matrices (that is, matrices with precisely one non-zero entry in each row and column) with non-zero entries in the set of de–th roots of unity, such that the product over these entries is a d–th root of unity. As a remark, the group $G(de, de, r)$ defined by $G(de, de, r) := \ker(\mathrm{sgn}_{de})$ which is a normal subgroup of $W_{de,r}$ of index de and at the same time becomes a normal subgroup of $\overline{W}_{de,r}$ of index d.

3.C. The cyclotomic Hecke algebras $\mathcal{H}^{\mathbf{m}}_{de,r}, \overline{\mathcal{H}}_{de,r}$.

1. The generic Ariki–Koike algebra $\mathcal{H}_{de,r}$.

The generic Ariki–Koike Algebra is the algebra $\mathcal{H}(W_{de,r})$ generated by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1}$ over the ring of Laurent polynomials $\mathbb{Z}[u_0, u_1, \ldots, u_{de-1}, u_0^{-1}, u_1^{-1}, \ldots, u_{de-1}^{-1}, v_0, v_1, v_0^{-1}, v_1^{-1}]$, satisfying

- the Braid relations symbolized by the diagram (figure 1) for the group $W_{de,r}$

and

- the Order relations

$$(\mathbf{s}-u_0)(\mathbf{s}-u_1)\cdots(\mathbf{s}-u_{de-1}) = (\mathbf{t}_j - v_0)(\mathbf{t}_j - v_1) = 0 \quad (\text{for } 0\le j < r)\,.$$

2. the cyclotomic Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$.

We denote by $\mathcal{H}^{\mathbf{m}}_{de,r}$ the algebra obtained from the generic Ariki–Koike algebra $\mathcal{H}(W_{de,r})$ by specializing the parameters $u_j = q^{m_i}\zeta_{de}^i$ where $\zeta_{de} = exp(2\pi i/de)$ and $m_i\in\mathbb{Z}$. Thus $\mathcal{H}^{\mathbf{m}}_{de,r}$ is the algebra over the ring $\mathbb{Z}[\zeta_{de}][q, q^{-1}]$ generated by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1}$ satisfying the order relations

$$(\mathbf{t}_j - q)(\mathbf{t}_j + 1) = 0 \quad (\text{pour } 1\le j < r)\,.$$

$$(\mathbf{s}-q^{m_0})(\mathbf{s}-\zeta_{de}^1 q^{m_1})(\mathbf{s}-\zeta_{de}^2 q^{m_2})\cdots(\mathbf{s}-\zeta_{de}^{de-1} q^{m_{de-1}}) = 0 \quad (\text{where } m_i\in\mathbb{Z})\,.$$

and the Braid relations symbolized by the diagram (figure 1).

This Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$ is called the *cyclotomic* Hecke algebra of $W_{de,r}$ and $\underline{\mathbf{m}} := (m_1, m_2, \ldots, m_{de-1})$ is called the *weight* of the cyclotomic Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$. One should note here that depending on the choice of $\underline{\mathbf{m}}$, we get different cyclotomic Hecke algebra and this choice will determine the shapes of symbols.

3. The Hecke algebra $\overline{\mathcal{H}}_{de,r}$.

The Hecke algebra $\overline{\mathcal{H}}_{de,r}$ of $\overline{W}_{de,r}$ is an associative algebra over the ring $\mathcal{A} := \mathbb{Z}[q, q^{-1}, x_0, x_1^{-1}, \ldots, x_{d-1}, x_d^{-1}]$, generated by $\{a_0, a_1, \ldots, a_r\}$ satisfying the following relations (cf. [Ar2]) :

- Order relations :
  $(a_0 - x_0)(a_0 - x_2)\cdots(a_0 - x_{d-1}) = 0,$
  $(a_i - q)(a_i + 1) = 0,$
- Braid relations :
  $a_1a_3a_1 = a_1a_3a_1$, $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1} \quad (2\le i\le n-1)$
  $(a_1a_2a_3)^2 = (a_3a_1a_2)^2$, $a_0a_j = a_ja_0 \quad (3\le j\le n)$,
  $a_1a_j = a_ja_1 \quad (4\le j\le n)$ and $a_ia_j = a_ja_i \quad (2\le i< j\le n, j\ge i+2)$
- supplementary relation :
  $a_0a_1a_2 = (q^{-1}a_1a_2)^{2-e}a_2a_0a_1 + (q-1)\sum_{k=1}^{e-2}(q^{-1}a_1a_2)^{1-k}a_0a_1 = a_1a_2a_0.$

3.D. Embedding of $\overline{\mathcal{H}}_{de,r}$ into $\mathcal{H}^{\mathbf{m}}_{de,r}$ after Ariki.

We suppose that K is the sub–field of the field $\mathbb{Q}(\mu_{de})$ generated by the de-th roots of unity. Let $\mathbb{Z}_K$ be the ring of integers of K : the ring $\mathbb{Z}_K$ is a Dedekind ring.

Definition. We call "Rouquier ring" of K and we denote by $\mathcal{R}_K(q)$ the $\mathbb{Z}_K$–subalgebra of the quotient field $K(q)$ generated by q, q^{-1} and all the $(q^n-1)^{-1}$ for $n \geq 1$.

Thus we have
$$\mathcal{R}_K(q) = \mathbb{Z}_K \left[q, q^{-1}, (q^n-1)^{-1}_{n\geq 1}\right]$$
and $\mathcal{R}_K(q)$ is the set of all the rational fractions of the form $\dfrac{P(q)}{Q(q)}$ where $P(q), Q(q) \in \mathbb{Z}_K[q]$ and where $Q(q) = q^n \prod_{\Phi\in \mathrm{Cycl}(K)} \Phi(q)^{n_\Phi}$ with $n, n_\Phi \in \mathbb{N}$ (and $n_\Phi = 0$ for almost all Φ).

Consider the cyclotomic Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$ of $W_{de,r}$ over the Rouquier-ring with the fixed weight $\underline{\mathbf{m}} = (m_0, m_1, \cdots, m_{d-1}, m_0, m_1, \cdots, m_{d-1}, \ldots, m_0, m_1, \cdots, m_{d-1})$ (note that the first d m_i's $m_0, m_1, \cdots, m_{d-1}$ are repeated e times).

By the results of Ariki (Prop. 1.6 of [Ar2]) and of [BMR] (Prop. 1.17), we know that the algebra $\overline{\mathcal{H}}_{de,r}$ embeds nicely into the particular sub-algebra of $\mathcal{H}^{\mathbf{m}}_{de,r}$, which we will use the same notation $\overline{\mathcal{H}}_{de,r}$ however for this isomorphic subalgebra of $\mathcal{H}^{\mathbf{m}}_{de,r}$.

It is done by the specialization of parameters as follows :

Ariki defines an injective algebra homomorphism $\phi : \overline{\mathcal{H}}_{de,r} \hookrightarrow \mathcal{H}^{\mathbf{m}}_{de,r}$ given by : $a_0 \mapsto \mathbf{s}^e, a_1 \mapsto \tilde{\mathbf{t}}_1 = \mathbf{s}^{-1}\mathbf{t}_1\mathbf{s}, a_i \mapsto \mathbf{t}_{i-1} (i = 2, \ldots, r)$.

So, by identifying the corresponding elements in the above homomorphism, we adjust the parameters of the Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$ so as to see clearly how the parameters for $\overline{\mathcal{H}}_{de,r}$ are determined from the given parameters for $\mathcal{H}^{\mathbf{m}}_{de,r}$. Thus $\overline{\mathcal{H}}_{de,r}$ can be restated simply as follows.

Let $v_0, v_1, \ldots, v_{d-1}$ be indeterminates. The Hecke algebra of $\overline{W}_{de,r}$ over the ring $\mathcal{O} := \mathbb{Z}[\zeta_{de}, v_0, v_1, \ldots, v_{d-1}]$ is the subalgebra of $\mathcal{H}^{\mathbf{m}}_{de,r}$:

$$\overline{\mathcal{H}}_{de,r} =< \mathbf{s}^e = \mathbf{t}_0, \mathbf{t}_1, \tilde{\mathbf{t}}_1 = \mathbf{s}^{-1}\mathbf{t}_1\mathbf{s}, t_2, \ldots, \mathbf{t}_{r-1} >$$
where the order relation for $\mathbf{s}^e$ is given by
$(\mathbf{s}^e - v_0^e)(\mathbf{s}^e - v_1^e)\cdots(\mathbf{s}^e - v_{d-1}^e) = (\mathbf{s}^e - x_0)(\mathbf{s}^e - x_1)\cdots(\mathbf{s}^e - x_{d-1}) = 0 \quad (\text{with} x_i = v_i^e)$.
The Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r} =< \mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1} >$ containing this $\overline{\mathcal{H}}_{de,r}$ has the following order relation for $\mathbf{s}$ with the corresponding parameters :
$$(\mathbf{s} - v_0)(\mathbf{s} - v_1)\cdots(\mathbf{s} - v_{d-1})\cdot(\mathbf{s} - v_0\zeta)(\mathbf{s} - v_1\zeta)\cdots(\mathbf{s} - v_{d-1}\zeta)\cdots\cdot \\ \cdots(\mathbf{s} - v_0\zeta^{e-1})(\mathbf{s} - v_1\zeta^{e-1})\cdots(\mathbf{s} - v_{d-1}\zeta^{e-1}) = 0,$$
where $\zeta = exp(2\pi i/e)$.

Remark. We note here that since the blocks of $\mathcal{H}^{\mathbf{m}}_{de,r}$ with the fixed weight

$$\underline{\mathbf{m}} = (m_0, m_1, \cdots, m_{d-1}, m_0, m_1, \cdots, m_{d-1}, \ldots, m_0, m_1, \cdots, m_{d-1})$$
are determined by the contents of the symbols of the corresponding form $(m_0, m_1, \cdots, m_{d-1}, m_0, m_1, \cdots, m_{d-1}, \ldots, m_0, m_1, \cdots, m_{d-1})$, thus there are relevant combinatorial datum for $\overline{\mathcal{H}}_{de,r}$.

The author refers the article [BrKim] to the reader for some more detailed combinatorial notations , especially the paragraph 1.D in [BrKim], in the following arguments. Also about the Rouquier ring in our case, we note that the same Rouquier ring $\mathcal{R}_K(q)$ mentioned in [BrKim] (cf. section 2.B, page 31, for the cyclotomic Hecke algebra $\mathcal{H}_{d,r}$) will be applied because we just work on the same type of Hecke algebra $\mathcal{H}_{de,r}$ (we have here de instead of d). Here's the definition of Rouquier ring (as in [BrKim]) :

4. **Main Result :**

As the weight $\underline{\mathbf{m}}$ is fixed for the cyclotomic Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$, we will omit the weight notation in $\mathcal{H}^{\mathbf{m}}_{de,r}$ throughout this section.

From the embedding process in the previous section, we get the following proposition.

4.1. Proposition. *With $\mathcal{H}_{de,r}$ and $\overline{\mathcal{H}}_{de,r}$ as above, we have*

$$\mathcal{H}_{de,r} = \overline{\mathcal{H}}_{de,r} \oplus \overline{\mathcal{H}}_{de,r}\mathbf{s} \oplus \cdots \oplus \overline{\mathcal{H}}_{de,r}\mathbf{s}^{e-1}.$$

That is, $\mathcal{H}_{de,r}$ is a free module over $\overline{\mathcal{H}}_{de,r}$ of rank e with the basis $\{1, \mathbf{s}, \dots, \mathbf{s}^{e-1}\}$.

Proof of 4.1.
After Ariki in his article [Ar2], $\mathcal{H}_{de,r}$ over the ring $\mathbb{Z}[\zeta_{de}][q, q^{-1}]$ has a basis

$$\mathfrak{B} := \{\boldsymbol{\mu}_1^{k_1}\boldsymbol{\mu}_2^{k_2}\boldsymbol{\mu}_3^{k_3}\cdots\boldsymbol{\mu}_r^{k_r}\mathbf{t}_w \mid w \in \mathfrak{S}_r, 0 \le k_1, k_2, \dots, k_r \le de-1\}$$

where $\boldsymbol{\mu}_1 = \mathbf{s}, \boldsymbol{\mu}_{i+1} = \mathbf{t}_i\boldsymbol{\mu}_i\mathbf{t}_i (i = 1, \dots, r-1)$ (the Murphy elements). Let's denote the elements of $\mathfrak{B}$ simply by $\{b_1, b_2, \dots, b_m\}$ where $m = |\mathfrak{B}|$. The subalgebra $\overline{\mathcal{H}}_{de,r}$ embedded in $\mathcal{H}_{de,r}$ has a basis
$\overline{\mathfrak{B}} := \{\boldsymbol{\mu}_1^{k_1}\boldsymbol{\mu}_2^{k_2}\boldsymbol{\mu}_3^{k_3}\cdots\boldsymbol{\mu}_r^{k_r}\mathbf{t}_w \mid w \in \mathfrak{S}_r, 0 \le k_1, k_2, \dots, k_r \le de-1, k_1+k_2+\cdots+k_r \equiv 0 \pmod e\}$

and we denote these basis elements simply by $\{\overline{b}_1, \overline{b}_2, \cdots, \overline{b}_n\}$ with $n = |\overline{\mathfrak{B}}|$.

We first prove the following lemma :

Lemma. *With the basis $\mathfrak{B}$, $\overline{\mathfrak{B}}$ above for $\mathcal{H}_{de,r}$ and $\overline{\mathcal{H}}_{de,r}$ respectively, we have*
1) For all j, $\mathbf{s}^j\overline{b}_i$ can be written as a linear combination of the basis elements of $\mathfrak{B}$ of $\mathcal{H}_{de,r}$.
2)For all b in $\mathfrak{B}$, b can be written as a linear combination of the elements

$$\{\mathbf{s}^j\overline{b} \mid \overline{b} \in \mathfrak{B}, j = 0, 1, \dots, e-1\}.$$

That is to say, the set $\{\overline{b}_1, \overline{b}_2, \cdots, \overline{b}_n, \mathbf{s}\overline{b}_1, \mathbf{s}\overline{b}_2, \cdots, \mathbf{s}\overline{b}_n, \cdots, \mathbf{s}^{e-1}\overline{b}_1, \mathbf{s}^{e-1}\overline{b}_2, \cdots, \mathbf{s}^{e-1}\overline{b}_n\}$ and $\mathfrak{B}$ generate the same module and thus we have

$$\mathcal{H}_{de,r} = \overline{\mathcal{H}}_{de,r} \oplus \overline{\mathcal{H}}_{de,r}s \oplus \cdots \oplus \overline{\mathcal{H}}_{de,r}\mathbf{s}^{e-1}.$$

Proof of Lemma.
1) First notice that from the relation $(\mathbf{s}^e - v_0^e)(\mathbf{s}^e - v_1^e)\cdots(\mathbf{s}^e - v_{d-1}^e) = (\mathbf{s}^e - x_0)(\mathbf{s}^e - x_1)\cdots(\mathbf{s}^e - x_{d-1}) = 0$ (with$x_i = v_i^e$), we can prove that $\forall j$, $\mathbf{s}^j \in \sum_{0 \le k \le de-1} r_k\mathbf{s}^k$, where r_k some constants in the Rouquier ring $\in \mathcal{R}_K(q)$.

More precisely, from $(\mathbf{s}-v_0)(\mathbf{s}-v_1)\cdots(\mathbf{s}-v_{d-1})\cdot(\mathbf{s}-v_0\zeta)(\mathbf{s}-v_1\zeta)\cdots(\mathbf{s}-v_{d-1}\zeta)\cdot\cdots(\mathbf{s}-v_0\zeta^{e-1})(\mathbf{s}-v_1\zeta^{e-1})\cdots(\mathbf{s}-v_{d-1}\zeta^{e-1}) = 0$, by developing the factors we get $\mathbf{s}^{de} - c_1\mathbf{s}^{de-1} + \cdots + (-1)^{de}x_0x_1\cdots x_{de-1} = 0$. Here, the c_i are the elements (as functions in ζ^j, v^j and v^{-j}) in the Rouquier ring $\mathcal{R}_K(q)$. Now passing the constant term to the other side of the equality, we get $\mathbf{s}^{de} - c_1\mathbf{s}^{de-1} + c_2\mathbf{s}^{de-2} + \cdots + (-1)^{de-1}c_{de-1}\mathbf{s} = (-1)^{de+1}x_0x_1\cdots x_{de-1}$. By factoring out by $\mathbf{s}$, we get

$$\mathbf{s}\left(\frac{\mathbf{s}^{de-1} - c_1\mathbf{s}^{de-2} + c_2\mathbf{s}^{de-2} + \cdots + (-1)^{de-1}c_{de-1}}{(-1)^{de+1}x_0x_1\cdots x_{de-1}}\right) = 1.$$

Note that the element $(-1)^{de+1}\lambda_1\lambda_2\cdots\lambda_{de}$ is invertible in the Rouquier ring $\mathcal{R}_K(q)$, so the inverse of $\mathbf{s}$, denoted by $\mathbf{s}^{-1}$, exists and is

$$\mathbf{s}^{-1} = \frac{\mathbf{s}^{de-1} - c_1\mathbf{s}^{de-2} + c_2\mathbf{s}^{de-2} + \cdots + (-1)^{de-1}c_{de-1}}{(-1)^{de+1}\lambda_1\lambda_2\cdots\lambda_m},$$

which is a $\mathcal{R}_K(q)$ − linear combination of $\mathbf{s}^k$, for $0 \le k \le de-1$.

We write $\mathbf{s}^j\overline{b}^i = \boldsymbol{\mu}_1^{j+k_1}\boldsymbol{\mu}_2^{k_2}\cdots\boldsymbol{\mu}_r^{k_r}\mathbf{t}_w,\ (w \in \mathfrak{S}_r)$.
• if $0 \le j+k_1 \le de-1$, then it is already an element of $\mathfrak{B}$.
• if $j+k_1 \ge de$, then since $\mathbf{s} = \boldsymbol{\mu}_1$ and $\forall j,\ \mathbf{s}^j \in \sum_{0\le k\le de-1} r_k\mathbf{s}^k$, where $r_k \in \mathcal{R}_K(q)$ as we saw above, we can replace any $\mathbf{s}^k$ (for $k \ge de$) by this linear combinations to develop in the variable $\mathbf{s}$ and in that way we can reduce the exponent of $\boldsymbol{\mu}_1$ in each terms. Thus we can write $\overline{b}$ as a $\mathcal{R}_K(q)$-linear combinations of b in $\mathfrak{B}$.

2) We want to prove that $\forall b \in \mathfrak{B}$, $b = \sum_{\overline{b}\in\overline{\mathfrak{B}},0\le j\le e-1} d_j\mathbf{s}^j\overline{b}$, *i.e.*, a $\mathcal{R}_K(q)$-linear combination of the $\mathbf{s}^j\overline{b}$ $(j = 0,\dots,e-1)$. The d_j are some coefficients in $\mathcal{R}_K(q)$.

Let $b = \boldsymbol{\mu}^{k_1}\boldsymbol{\mu}^{k_2}\boldsymbol{\mu}^{k_3}\cdots\boldsymbol{\mu}^{k_r}\mathbf{t}_w$, where $0 \le k_1, k_2, \dots, k_r \le de-1$, an element of $\mathfrak{B}$.
• If $k_1 + k_2 + \cdots + k_r \equiv 0(\text{mod } e)$, then it is already an element of $\overline{\mathfrak{B}}$.
• If $k_1 + k_2 + \cdots + k_r \equiv l(\text{mod} e)$ $(0 \le l \le e-1)$, then use the following relations between $\boldsymbol{\mu}_i$ and $\mathbf{t}_j$:

$$\boldsymbol{\mu}_i\boldsymbol{\mu}_j = \boldsymbol{\mu}_j\boldsymbol{\mu}_i,\ \mathbf{t}_i\boldsymbol{\mu}_k = \boldsymbol{\mu}_k\mathbf{t}_i\ (k \ne i-1, i),$$

to get an expression of b as a $\mathcal{R}_K(q)$-linear combination of the elements $\mathbf{s}^j\overline{b}$ where $j = 0,\dots,e-1$. □

The claim of the proposition is immediate from this Lemma. Thus , with G:=< $\mathbf{s}$ >, according to the difinition in [BrKim, page 80], $\mathcal{H}^{\mathbf{m}}_{de,r}$ is the symmetric algebra of the group G over the subalgebra $\overline{\mathcal{H}}_{de,r}$ of $\mathcal{H}^{\mathbf{m}}_{de,r}$, *i.e.*, $\mathcal{H}^{\mathbf{m}}_{de,r} = \overline{\mathcal{H}}_{de,r}$G. □

Remark.
bul One thing to notice here is that in the case of $\overline{\mathcal{H}}_{e,r} = \mathcal{H}(G(e,e,r))$ and $\mathcal{H}_{e,r} = \mathcal{H}(G(e,1,r))$ in [BrKim], *i.e.*, with $d = 1$, we had

$$\mathcal{H}_{e,r} = \overline{\mathcal{H}}_{e,r} \oplus \overline{\mathcal{H}}_{e,r}\mathbf{s} \oplus \cdots \oplus \overline{\mathcal{H}}_{e,r}\mathbf{s}^{e-1}.$$

with $\mathbf{s}^e = 1$, so the group $G = \langle\mathbf{s}\rangle$ was a cyclic group of order e. But in general case for $\overline{\mathcal{H}}_{de,r}$ and $\mathcal{H}(G(de,1,r))$ *i.e.*, with $d > 1$, we have

$$\mathcal{H}_{de,r} = \overline{\mathcal{H}}_{de,r} \oplus \overline{\mathcal{H}}_{de,r}\mathbf{s} \oplus \cdots \oplus \overline{\mathcal{H}}_{de,r}\mathbf{s}^{e-1}.$$

where $\mathbf{s}^e \ne 1$. But since $\mathbf{s}^e$ belongs to the smaller algebra $\overline{\mathcal{H}}_{de,r}$, by applying the proposition 11.14 in [CuRei], the same Clifford theory works as we had for the case in [BrKim] .
• We also have $\overline{\mathcal{H}}_{de,r}\mathbf{s}^i \cdot \overline{\mathcal{H}}_{de,r}\mathbf{s}^j \subset \overline{\mathcal{H}}_{de,r}\mathbf{s}^k$ where $k \equiv i+j$ (mod e).

4.B. Reminder on Clifford Theory.

We recall that $Irr(W_{de,r})$ is indexed by the de-partitions of r.
Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)}, \lambda^{(d)}, \dots, \lambda^{(2d-1)}, \dots, \lambda^{(e-1)d}, \dots, \lambda^{(ed-1)})$ be a de–partition of r and consider the following cyclic permutation action by d packages on this de–partition (cf. also by example [Ma2], §4.A) :

$$\sigma^d \colon (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)}, \lambda^{(d)} \dots \lambda^{(2d-1)}, \dots, \lambda^{(e-1)d} \dots \lambda^{(de-1)})$$
$$\mapsto (\lambda^{(e-1)d} \dots \lambda^{(de-1)}, \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)}, \lambda^{(d)} \dots \lambda^{(2d-1)}, \dots).$$

In particular, for the cases of $G(e, e, r)$ treated in the article [BrKim], the cyclic permutation concerned would be "1"-cyclic permutation action.

Let Ω be the orbit of λ under this action and let $\overline{\Omega}$ be the set of the corresponding irreducible characters of $\overline{W}_{de,r}$ by restriction. By Clifford theory, (cf. [CuRei] or [Is]), we have the following assertion (the Prop. 1.42 [BrKim]): Let $\mathcal{O}$ be a commutative integral domain, F its quotient field, K an extension field of F so that KA is split semi-simple. Let A be a $\mathcal{O}$–symetric algebra endowed with the symmetrizing form t and let $\overline{A}$ be a sub–algebra of A which are free of finite rank as $\mathcal{O}$–module. More precisely, $A = \overline{A} \oplus \overline{A}g \oplus \cdots \oplus \overline{A}g^{k-1} = \overline{A}G$ where $G =< g >$, a subgroup of $A^\times$ (cf. the definition of the symmetric algebra A of a finite group over the subalgebra $\overline{A}$ in [BrKim], page 80). Then we have

$$|\Omega||\overline{\Omega}| = |G| \quad , \quad \text{and} \quad \begin{cases} \forall \chi \in \Omega, \ \operatorname{Res}^{KA}_{K\overline{A}}(\chi) = \sum_{\overline{\chi} \in \overline{\Omega}} \overline{\chi}\,, \\ \forall \overline{\chi} \in \overline{\Omega}, \ \operatorname{Ind}^{KA}_{K\overline{A}}(\overline{\chi}) = \sum_{\chi \in \Omega} \chi\,. \end{cases}$$

where $G \cong W_{de,r}/\overline{W}_{de,r}$. Moreover, for all $\chi \in \Omega$ and $\overline{\chi} \in \overline{\Omega}$, we have

$$s_\chi = |\Omega| s_{\overline{\chi}}\,.$$

where s_χ is the Schur element corresponding to χ (cf. [BrKim] or [GePf]).

The operation of the group $G^\vee := Hom(G, K^\times)$ on $\operatorname{Irr}(\mathcal{H}_{de,r})$ corresponds to the action generated by the d-cyclic permutation σ^d described above.

So, we identify the group $G^\vee$ to the cyclic group C_e of order e generated by the d–cyclic permutation group described above.

As we had in [BrKim] for the case of $G(d, 1, r)$ and $G(d, d, r)$, from the above assertion above and the proposition 3.1, we obtain the following partition of $\operatorname{Irr}(\overline{\mathcal{H}}_{de,r})$.

$$\operatorname{Irr}(\overline{\mathcal{H}}_{de,r}) = \dot{\bigcup_{\overline{\lambda}}} \operatorname{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$$

indexed by the set of orbits of C_e on the de–partitions, characterized by the condition

$$\operatorname{Res}^{\mathcal{H}_{de,r}}_{\overline{\mathcal{H}}_{de,r}} \chi_\lambda = \sum_{\overline{\chi} \in \operatorname{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}} \overline{\chi}\,,$$

and $\operatorname{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$ is an orbit of G in its action on $\operatorname{Irr}(\overline{\mathcal{H}}_{de,r})$, satisfying the condition,

$$|\Omega||\overline{\Omega}| = e$$

where Ω (resp. $\overline{\Omega}$) denotes the orbit of χ by the d-cyclic permutation action (resp. the corresponding irreducible characters of $G(de, e, r)$ by restriction, *i.e.*, $\operatorname{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$).

4.C. Rouquier Ring and Blocks.

4.2. Definition. Generalized stuttering partition *:*
We call d-stuttering partition for the partition of the form

$$(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}, \lambda^{(0)} \ldots \lambda^{(d-1)}, \ldots, \lambda^{(0)} \ldots \lambda^{(d-1)}),$$

i.e., the first d partitions are repeated by packages.

So for a d-stuttering partition of $G(de,1,r)$, the first d partitions will be repeated e times. The stuttering partition (called "partition bégayante") we called in the Définition 3.20 of the article [BrKim] is thus a 1-stuttering partition here.

The theorem below describes the blocks of the algebra $\mathcal{R}_K(q)\overline{\mathcal{H}}_{de,r}$. where $\mathcal{R}_K(q)$ is a Rouquier ring defined in the section 3.D. The claim for the particular case with $d=1, e=2$ gives the description given by Lusztig ([Lu1])of the families of characters of the Weyl groups of type D_r. The claim for $d=1 (e=1$ resp.) furnishes the description given in the article [BrKim] of the families of $Irr(G(e,e,r))$ $(Irr(G(d,1,r))$ resp.).

4.3.Main Theorem. *For the blocks of the cyclotomic Hecke algebras of $G(de,e,r)$:*

- *If a de–partition λ of r is a d-stuttering partition, repeated e times, and so its orbit under the σ^d action is a singleton set, then there are corresponding e irreducible characters $\{\overline{\chi}\}_{\overline{\chi}\in \mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}}$ in $G(de,e,r)$ and each of these singletons gives a block of the Rouquier ring $\mathcal{R}_K(q)\mathcal{H}_{de,r}$ and there are e of them.*
- *The other blocks of $\mathcal{R}_K(q)\mathcal{H}_{de,r}$ are in bijection with the contents of the de-partitions not "d–stuttering" (i.e., not repeated e times) : those are the set of charcters of $\overline{\mathcal{H}}_{de,r}$ of the form*

$$\dot{\bigcup_{\overline{\lambda}}} \mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$$

where $\overline{\lambda}$ runs over the set of the orbits of de-partitions (not stuttering) with the same fixed contents $\mathrm{Cont}_{\overline{\lambda}}(x)$*,*

i.e., same entries with the same multiplicities and possibly arranged in different rows.

Here, since the weight of the cyclotomic Hecke algebra $\mathcal{H}^{\mathbf{m}}_{de,r}$ is already fixed, through the construction of the embedding above, the corresponding shape of symbols in $\overline{\mathcal{H}}_{de,r}$ which came from the choice of the weight for $\mathcal{H}^{\mathbf{m}}_{de,r}$ has already been determined, *i.e.*, having the first d rows are repeated e times.

Proof of the main Theorem.

(1) It results from the reminder on Clifford theory in the section 3.B, that for $\overline{\chi} \in \mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$, its Schur element $s_{\overline{\chi}}$ is equal to the Schur element s_{χ_λ} of the corresponding character of $\mathcal{H}_{de,r}$. Since we know from the lemma 3.21 in [BrKim] that the Schur element is invertible in the ring $\mathcal{R}_K(q)$, we can see that each singleton $\{\overline{\chi}\}$ (for $\overline{\chi} \in \mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$) is a Rouquier block.

(2) The proof of this part goes exactly in the same way as we had for the main theorem 4.3 for $G(e,e,r)$ in the article [BrKim].

Consider now the Rouquier block–idempotent $\overline{b}$ of $\overline{\mathcal{H}}_{de,r}$ containing an element of $\mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}}$ where λ is a partition non d–stuttering. Let b be the Rouquier block-idempotent of $\mathcal{H}_{de,r}$ containing χ_λ. We want to prove that $\overline{b} = b$, which yields the claim. (See the claims 1.41–1.45 in [BrKim]).

For this, let's prove first the following lemma.

4.4. Lemma. *Let λ be a de–partition non d–stuttering of r. For all prime divisor p of e, there exists a de–partition $\lambda(p)$ of r, having the same contents as λ, such that the order of the d–cyclic permutations in C_e fixing $\lambda(p)$ is not divisible by p.*

Proof of the lemma. We denote $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}, \lambda^{(d)} \ldots \lambda^{(2d-1)}, \ldots, \lambda^{(e-1)d} \ldots \lambda^{(ed-1)})$ $=: (\overline{\lambda}_0, \ldots, \overline{\lambda}_{e-1})$ and let $\overline{\lambda}_i := (\lambda^{(i)}, \lambda^{(i+1)}, \ldots, \lambda^{(i+d-1)})$. That is, we cut the given de–partition by grouping the d partitions from left to right and each package of d partitions is be denoted by

$\overline{\lambda}_i$. Here, the d-cyclic permutation action on packages means the 1– cyclic permutation on the set of $\overline{\lambda}_i$ where $i = 0, \dots, (e-1)d$. So we end up with the same situation as we had in [BrKim]. Since λ is not d–stuttering, there exists an integer $m \geq 1$ such that $\overline{\lambda}^{(0)} \neq \overline{\lambda}^{(m)}$. We denote by $\lambda(p)$ the de–partition obtained from λ by exchanging $\overline{\lambda}^{(m)}$ and$\overline{\lambda}^{((e/p)-1)}$. By construction, we see that the de–partition $\lambda(p)$ is not fixed by the generator of the unique sub–group of order p of C_e, which proves that its stblizer is of order prime to p. $\square$

Now let's prove that the lemma above yields the second claim of the theorem. Before that we need the proposition 1.45 of the article [BrKim] (For more details, see the pages 25, 26 of the article [BrKim]) : In the following theorem, $\mathrm{BlId}(A)$ denotes for the set of block-idempotents of A where Υ ($\overline{\Upsilon}$ resp.) is an orbit of $G^\vee$ (G resp.) on the set $\mathrm{BlId}(A)$ ($\mathrm{BlId}(\overline{A})$ resp.) and $b(\Upsilon)$ is the block-idempotent of $(ZA)^{G^\vee}$ defined by

$$b(\Upsilon) := \sum_{b \in \Upsilon} b\,.$$

Proposition 1.45 [BrKim]. *Let $\mathcal{O}$ be a commutative integral domain, F its quotient field, K an extension field of F, and A be a $\mathcal{O}$–algebra which is free of finite rank as $\mathcal{O}$–module. Suppose that G is cyclic. There exists a bijection*

$$\mathrm{BlId}(A)/G^\vee \xleftrightarrow{\sim} \mathrm{BlId}(\overline{A})/G \quad , \quad \Upsilon \xleftrightarrow{\sim} \overline{\Upsilon}$$

such that

$$b(\Upsilon) = \overline{b}(\overline{\Upsilon})\,,$$

or, in other words,

$$\mathrm{Tr}(G, \overline{b}) = \mathrm{Tr}(G^\vee, b) \quad \text{for all } \overline{b} \in \overline{\Upsilon} \text{ and } b \in \Upsilon\,.$$

In particular, the algebras $(ZA)^{G^\vee}$ and $(Z\overline{A})^G$ have the same idempotents.

So,

• We know that $\overline{\chi} \in \mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda(p)}}$, and we have $|G^\vee_{\chi_{\lambda(p)}}||G_{\overline{\chi}}| = e$. From this, we can see that $|G_{\overline{\chi}}|$ is divisible by the biggest power of p dividing e.

• Since (cf. 1.45 in [BrKim]) we have $b = \mathrm{Tr}(G, \overline{b})$, we can see that the elements of $\mathrm{Irr}(\overline{\mathcal{H}}_{de,r})_{\overline{\lambda}(p)}$ belong to the Rouquier block of $\overline{\mathcal{H}}_{de,r}$ conjugated of $\overline{b}$ by G, thus the stabilizer in G is equal to $G_{\overline{b}}$.

• After the claim of the lemma 1.43 [BrKim], we can thus see that, for all prime number p, $|G_{\overline{b}}|$ is divisible by the biggest power of p dividing e, thus is equal to e. Hence we get $G_{\overline{b}} = G$

Then now, using the proposition 1.45 in [BrKim], we get the claim 2) of the main Theorem. $\square$

Remark : On the invariance of the integers $a_{\overline{\chi}}$ and $A_{\overline{\chi}}$.

1. From the theorem 2.9 – (2) in [BrKim], the sum $a_{\overline{\chi}} + A_{\overline{\chi}}$ was already constant on families of $G(de, e, r)$.

2. With exactly the same reasoning as we had in [BrKim], *i.e.*, from the following three facts :

- the Main Result 4.3 on the Rouquier blocks of $\overline{\mathcal{H}}_{de,r}$
- the relation between the Schur elements S_χ of $Irr(\mathcal{H}^{\mathbf{m}}_{de,r})$ and the Schur elements $S_{\overline{\chi}}$ of $Irr(\overline{\mathcal{H}}_{de,r})$ given in the section 4.B or in the proposition 1.42 in [BrKim]
- the invariance of the integers $a_{\overline{\chi}}$ and $A_{\overline{\chi}}$ on the blocks of $\mathcal{H}_{d,r}$ which works also for $\mathcal{H}^{\mathbf{m}}_{de,r}$ with the same calulations,

the values $a_{\overline{\chi}}$ and $A_{\overline{\chi}}$ of the Schur elements of $Irr(\overline{\mathcal{H}}_{de,r})$ are constant on the Rouquier blocks.

EXAMPLES :

Consider the group $G(6,2,6)$ which is a normal subgroup of $G(6,1,6)$ of index 2. (In this case, we have $d=3, e=2, r=6$).

Here, in the following table, we give some 6–partitions of 6 belonging in the same family following the criteria in the main theorem.

One thing to remark here is that our cyclic permutation action on the 6-partitions of 6 is "3"-cyclic permutation, that is by packages by packages of 3 . For the former case in [BrKim], the cyclic permutation there was the "1"–cyclic permutation. One extreme case is the partition $(1,1,1,1,1,1)$. This partition corresponds to 6 distinct irreducible characters (by Clifford theory) in $G(6,6,6)$ which is a normal subgroup of $G(6,1,6)$ of index 6. And each of these 6 irreducible characters gives a singleton family and there are 6 of them. For the group $G(6,2,6)$, since 3-cyclic permutation action concerns here, by Clifford theory, we would have only 2 corresponding irreducible characters in $G(6,2,6)$ and each of these gives a singleton family. And there are 2 of them.

Let's take another example : for $G(4,4,4)$ which is a normal subgroup of $G(4,1,4)$ of index 4, with the chosen weight $(1,0,1,0)$, the 4-partition $(1,1,1,1)$ gives 4 irreducible characters in $Irr(G(4,4,4)$ and thus 4 families. And the partition $(2,\emptyset,2,\emptyset)$ gives one family since its orbit has two partitions $(2,\emptyset,2,\emptyset)$ and $(\emptyset,2,\emptyset,2)$ and thus there will be two corresponding irreducible characters in the group $G(4,4,4)$ by Clifford theory.

Now by checking the corresponding symbols with the shape $(1,0,1,0)$, we get two symbols $\begin{pmatrix} 2 \\ \emptyset \\ 2 \\ \emptyset \end{pmatrix}$, $\begin{pmatrix} \emptyset \\ 2 \\ \emptyset \\ 2 \end{pmatrix}$ having the same contents. So the two 4–partitions are in the same family of $Irr(G(4,4,4))$.

On the other hand, for the case of $G(4,2,4)$ which is a normal subgroup of index 2 of $G(4,1,4)$, the 4–partition $(1,1,1,1)$ is 2–stuttering. Through the σ^2 action on this partition, we get two corresponding irreducible characters for $G(4,2,4)$ and each of these gives a singleton family of $Irr(G(4,2,4))$.

The table on the next page shows some families for $G(6,2,6)$ with the chosen weight $(1,0,0,1,0,0)$.

$\mathrm{Irr}(G(6,1,6))$	$\mathrm{Irr}(G(6,2,6))$	Families	Symbols	Contents
$\chi_{(6,\emptyset,\emptyset,\,\vert\,\emptyset,\emptyset,\emptyset)}$	$\overline{\chi_{(6,\emptyset,\emptyset,\,\vert\,\emptyset,\emptyset,\emptyset)}}$	$\mathfrak{F}_1$	$\begin{pmatrix} 6 \\ \emptyset \\ \emptyset \\ 0 \\ \emptyset \\ \emptyset \end{pmatrix}$	$1{+}x^6$
$\chi_{(\emptyset,\emptyset,\emptyset,\,\vert\,6,\emptyset,\emptyset)}$	$\overline{\chi_{(6,\emptyset,\emptyset,\,\vert\,\emptyset,\emptyset,\emptyset)}}$	$\mathfrak{F}_1$	$\begin{pmatrix} 0 \\ \emptyset \\ \emptyset \\ 6 \\ \emptyset \\ \emptyset \end{pmatrix}$	$1{+}x^6$
$\chi_{(3,\emptyset,\emptyset,\,\vert\,3,\emptyset,\emptyset)}$	$\overline{\chi_{(3,\emptyset,\emptyset,\,\vert\,3,\emptyset,\emptyset)}}$	$\mathfrak{F}_2$singleton		
	$\overline{\chi'_{(3,\emptyset,\emptyset,\,\vert\,3,\emptyset,\emptyset)}}$	$\mathfrak{F}_3$singleton		
$\overline{\chi_{(1,1,1,1,1,1)}}$	$\overline{\chi_{(1,1,1,1,1,1)}}$	$\mathfrak{F}_4$singleton		
	$\overline{\chi'_{(1,1,1,1,1,1)}}$	$\mathfrak{F}_5$singleton		
$\chi_{(21,21,\emptyset,\,\vert\,\emptyset,\emptyset,\emptyset)}$	$\overline{\chi_{(21,21,\emptyset,\,\vert\,\emptyset,\emptyset,\emptyset)}}$	$\mathfrak{F}_6$	$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & \\ 0 & 1 & \\ 0 & 1 & 2 \\ 0 & 1 & \\ 0 & 1 & \end{pmatrix}$	$5{+}5x{+}2x^2{+}x^3{+}x^4$
$\chi_{(\emptyset,\emptyset,\emptyset,\,\vert\,21,21,\emptyset)}$	$\overline{\chi_{(21,21,\emptyset,\,\vert\,\emptyset,\emptyset,\emptyset)}}$	$\mathfrak{F}_6$	$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & \\ 0 & 1 & \\ 0 & 2 & 4 \\ 1 & 3 & \\ 0 & 1 & \end{pmatrix}$	$5{+}5x{+}2x^2{+}x^3{+}x^4$

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