BANACH SPACES ADAPTED TO ANOSOV SYSTEMS

SÉBASTIEN GOUËZEL AND CARLANGELO LIVERANI

ABSTRACT. We study the spectral properties of the Ruelle-Perron-Frobenius operator associated to an Anosov map on classes of functions with high smoothness. To this end we construct anisotropic Banach spaces of distributions on which the transfer operator has a large spectral gap. In the $\mathcal{C}^\infty$ case, the spectral radius is arbitrarily small, which yields a description of the correlations with arbitrary precision. Moreover, we obtain sharp spectral stability results for deterministic and random perturbations. In particular, we obtain differentiability results for spectral data (which imply differentiability of the SRB measure, the variance for the CLT, the rates of decay for smooth observable, etc.).

1. INTRODUCTION

The study of the statistical properties of Anosov systems dates back almost half a century ([1]) and many approaches have been developed to investigate various aspects of the field (the most historically relevant one being based on the introduction of Markov partitions [2, 25, 5, 19]). At the same time the type of questions and the precision of the results have progressed through the years. In the last years the emphasis has been on strong stability properties with respect to various types of perturbations [3], dynamical zeta functions and related smoothness issue (see [11, 20, 6]). In the present paper we present a new approach, improving on a previous partial and still unsatisfactory one [4], that allows to obtain easily a manifold of results (many of which new) and we hope will reveal an even larger field of applicability. Indeed, the ideas in [4] have already been applied with success to some partially hyperbolic situations (flows) [16] and we expect them to be applicable to the study of dynamical zeta functions.

The basic idea is inspired by the work on piecewise expanding maps, starting with [12, 9] and the many others that contributed subsequently (see [3] for a nice review on the subject). That is to study directly the transfer operator (often called the Ruelle-Perron-Frobenius operator) on appropriate functional spaces.[1] For the case of smooth expanding maps, the Sobolev spaces $W^{n,1}$ turn out to be proper spaces where the transfer operator acts as a *smoothing* operator, [16]. In turn,

Date: May 10, 2004.

We wish to thank V.Baladi, G.Keller and D.Ruelle for helpful discussions. Last, but most of all, it is a pleasure to acknowledge our debit to D.Dolgopyat. In fact, a couple of years ago, while in Rome, he told the second named author that one should "project along the unstable direction" and a few other words. It is a fact that the above sentence is the key idea of the present work. It is only to our detriment that it took us so long to figure out the meaning and to understand how to implement it. C.Liverani acknowledges the support of M.I.U.R. and the hospitality of Paris seven and Georgia Institute of Technology where part of the paper was written.

[1] This, to our knowledge, has been the first Markov-partition–free approach to the study of the statistical properties of systems with sensitive dependence on initial conditions.

this implies that, on such spaces, the operator is quasi-compact with an essential spectral radius exponentially decreasing in n. The existence of a spectral gap and all kind of statistical properties (exponential decay of correlations, central limit theorem, holomorphic zeta functions, etc.) readily follow.

Unfortunately, for Anosov systems it is not helpful to consider spaces of smooth functions – on such spaces the spectral radius of the transfer operator is larger than one –, it is necessary to consider spaces of distributions. This was recognized in [22, 23, 24, 7] limited to the analytic case, and in [13] (only implicitly) and systematically in [4] for the $\mathcal{C}^{1+\alpha}$ case. Nevertheless, the latter setting had still several shortcomings. First of all, the Banach space was precisely patterned on the invariant distributions of the systems, which implied that transfer operators – even of close maps – where studied on different spaces. This was a serious obstacle to obtaining sharp perturbation results. Secondly, since in general the invariant distributions are only Hölder, it was not possible to have a scale of Banach spaces on which to study the influence of the smoothness of the map on the spectrum.

Both such shortcomings are overcome in the present approach. The spaces we introduce (partially inspired by [13]) are still related to the map one wishes to study, but in a much loser way so that the operators associated to nearby maps can be studied on the same space. In addition, we have a scale of spaces that can be used to investigate smoothness related issues (typically the dependence of the essential spectrum on the smoothness of the map). In particular, if the map is $\mathcal{C}^\infty$, we obtain a description of the correlations of $\mathcal{C}^\infty$ functions with an arbitrarily small error term.

In addition, the present norms allow easier estimates of the size of perturbations. This provides a very direct way of obtaining sharp perturbations results which substantially generalize the existing ones, e.g. [4, 18, 20, 21]. For example, in the $\mathcal{C}^\infty$ case all the simple eigenvalues and all the eigenspaces depend $\mathcal{C}^\infty$ on the map. The same holds for the variance in the CLT for a smooth zero average observable.

A further remarkable feature of the present approach is that, unlike all the previous ones, its implementation does not depend directly on subtle regularity properties of the foliations and of the holonomies. This makes possible to have a much simpler and *self contained* treatment of the statistical properties of the system and may lead to interesting generalizations in the partially hyperbolic setting.

The paper is organized as follows. In the second section, we introduce Banach spaces $\mathcal{B}^{p,q}$, explain why the transfer operator acting on $\mathcal{B}^{p,q}$ has a spectral gap and illustrate the stability results: the main ingredients are a compactness statement (Lemma 2.1), a Lasota-Yorke type inequality (Lemma 2.2) and the estimates on perturbations Lemmas 7.1 and 7.2. In Sections 3 and 4, we describe more precisely the spaces $\mathcal{B}^{p,q}$ and prove in particular that they are spaces of distributions. In Sections 5 and 6, which are the main parts of this article, we prove respectively the aforementioned compactness statement and Lasota-Yorke type inequality. In Section 7, we show how this framework implies very precise stability results on the spectrum, for deterministic and random perturbations. Section 8 contains an abstract perturbation result generalizing the setting of [10], along the direction adumbrated in [16], to cases where a control on the smoothness is available. Finally, Section 9 shows that smooth deterministic perturbations fit in the setting developed in Section 8.

2. The Banach spaces and the results

Let X be a d dimensional $\mathcal{C}^\infty$ compact connected Riemannian manifold and consider an Anosov map $T \in \mathcal{C}^{r+1}(X, X)$. Write d_s and d_u for the stable and unstable dimensions. Let $\lambda > 1$ be less than the minimal expansion along the unstable directions, $\nu < 1$ greater than the minimal contraction along the stable directions. In addition, let $\bar\zeta_0 > 0$ be such that there exists $C > 0$ such that, for any $x \in X$, any $v, w \in E^s(x)\backslash\{0\}$ and any $n \in \mathbb{N}$,

$$\frac{|DT^{-n}(x)v|}{|v|} \leq C\left(\frac{|DT^{-n}(x)w|}{|w|}\right)^{\bar\zeta_0}.$$

Hence, $\bar\zeta_0 \geq 1$ measures the nonconformality of T in the stable direction; it is a pinching constant of the map T. Take any $\bar\zeta > \bar\zeta_0$. We will express the spectral properties of T using the constants $\lambda, \nu, \bar\zeta$.

In Section 3, we will define a set Ω of admissible leaves. The elements of Ω are small $\mathcal{C}^r$ embedded compact manifolds with boundary, of dimension d_s, close to local stable manifolds.[2]

We will use the following notation: if $\alpha \in \mathbb{N}^d$ is a multi-index, write $\partial^\alpha := \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$ and $|\alpha| := \sum \alpha_i$. Denote by $\mathcal{P}(p)$ the set of $\mathcal{C}^r$-differential operators of order at most p: in coordinate charts, such an operator is given by $Ph(x) = \sum_{|\alpha|\leq p}\varphi_\alpha(x)\partial^\alpha h(x)$, where the functions φ_α are of class $\mathcal{C}^r$. The norm of P is given by $\|P\| = \sup_{|h|_{\mathcal{C}^{r+p}}\leq 1}|Ph|_{\mathcal{C}^r}$. In coordinate charts, this norm is equivalent to the norm $\sum|\varphi_\alpha|_{\mathcal{C}^r}$.[3]

We are now ready to introduce the relevant norms. When $W \in \Omega$, we will denote by $\mathcal{C}_0^q(W,\mathbb{R})$ the set of functions from W to $\mathbb{R}$ which are q times continuously differentiable on W and vanish on a neighborhood of the boundary of W. For each $h \in \mathcal{C}^r(X,\mathbb{R})$ and $q \geq p \in \mathbb{N}^*$ with $p+q \leq r$ (recall, T is $\mathcal{C}^{r+1}$ by definition), let[4]

$$\|h\|_{p,q} := \sup_{\substack{P\in\mathcal{P}(p)\\ \|P\|\leq 1}} \sup_{W\in\Omega} \sup_{\substack{\varphi\in\mathcal{C}_0^q(W,\mathbb{R})\\ |\varphi|_{\mathcal{C}^q}\leq 1}} \int_W Ph\cdot\varphi. \tag{2.1}$$

For example, if X is the torus, then this norm is equivalent to the norm given by

$$\|h\|_{p,q}^\sim := \sup_{|\alpha|\leq p} \sup_{W\in\Omega} \sup_{\substack{\varphi\in\mathcal{C}_0^q(W,\mathbb{R})\\ |\varphi|_{\mathcal{C}^q}\leq 1}} \int_W \partial^\alpha h\cdot\varphi.$$

Later on, in Section 3, we will give an explicit description of the norm (2.1) in coordinate charts. It will sometimes be easier to work with the coordinate-free definition given in (2.1) and sometimes with the explicit definition, depending on what we are trying to prove.

It is easy to see that $\|\cdot\|_{p,q}$ is a norm on $\mathcal{C}^r(X,\mathbb{R})$ (we will prove a more general result in Proposition 4.1). Hence, we can consider the completion $\mathcal{B}^{p,q}$ of $\mathcal{C}^r(X,\mathbb{R})$ with respect to this norm. Section 4 will be devoted to a description of this space. We will see in particular that it is canonically a space of distributions.

---

[2] The precise definition of the set Ω is given by (3.2).

[3] To fix notation, in this paper we choose, for each $s \in \mathbb{N}$, $|\varphi|_{\mathcal{C}^s} := |\varphi|_{\mathcal{C}^0} + C_s \sup_{1\leq|\beta|\leq s}|\partial^\beta\varphi|_{\mathcal{C}^0}$ where C_s is large enough so that $|\varphi_1\varphi_2|_{\mathcal{C}^s} \leq |\varphi_1|_{\mathcal{C}^s}|\varphi_2|_{\mathcal{C}^s}$.

[4] All integrals are taken with respect to Lebesgue or Riemannian measure except when another measure is explicitly mentioned.

Since $\|h\|_{p-1,q} \leq \|h\|_{p,q}$, the embedding of $\mathcal{C}^r(X,\mathbb{R})$ into $\mathcal{B}^{p,q}$ gives rise to a canonical map $\mathcal{B}^{p,q} \to \mathcal{B}^{p-1,q}$, which is in fact compact:

Lemma 2.1. *The unit ball of $\mathcal{B}^{p,q}$ is relatively compact in $\mathcal{B}^{p-1,q}$.*

The proof of Lemma 2.1 is the content of section 5.

The rest of the paper consists in the investigation of the properties of the transfer operator $\mathcal{L}$ seen as an operator acting on the spaces $\mathcal{B}^{p,q}$. As is well known, for each $h \in \mathcal{C}^r(X,\mathbb{R})$, the transfer operator $\mathcal{L} : \mathcal{C}^r(X,\mathbb{R}) \to \mathcal{C}^r(X,\mathbb{R})$, defined by duality by

$$\int h \cdot u \circ T =: \int \mathcal{L}h \cdot u,$$

is also given by

$$\mathcal{L}h = (h|\det(DT)|^{-1}) \circ T^{-1}.$$

The key information on the action of $\mathcal{L}$ on $\mathcal{B}^{p,q}$ is contained in the next lemma.

Lemma 2.2. *For each $\zeta > \bar{\zeta}$ there exists $\sigma \in (0,1)$ such that,*[5] *for each $p,q \in \mathbb{N}^*$ satisfying $q \geq \zeta p$ and $p+q \leq r$, $\mathcal{L}$ is a bounded operator on $\mathcal{B}^{p,q}$.*[6] *In addition, there exist $A_{p,q}, B_{p,q} > 0$ such that, for each $n \in \mathbb{N}^*$,*

$$\|\mathcal{L}^n h\|_{0,q} \leq A_{0,q}\|h\|_{0,q}; \tag{2.2}$$

$$\|\mathcal{L}^n h\|_{p,q} \leq A_{p,q}\sigma^{pn}\|h\|_{p,q} + B_{p,q}\|h\|_{p-1,q}. \tag{2.3}$$

The above Lemma is proven in section 6.

Lemmas 2.1 and 2.2 readily imply the basic result of the paper:

Theorem 2.3. *If $p,q \in \mathbb{N}^*$ satisfy $q \geq \zeta p$ and $p+q \leq r$, then the operator $\mathcal{L} : \mathcal{B}^{p,q} \to \mathcal{B}^{p,q}$ has spectral radius one. In addition, $\mathcal{L}$ is quasicompact with essential spectrum $\sigma_{ess}(\mathcal{L}) \subset \{z \in \mathbb{C} : |z| \leq \sigma^p\}$.*

Moreover, the eigenfunctions corresponding to eigenvalues of modulus 1 are distributions of order 0, i.e., measures. If the map is topologically transitive, then one is a simple eigenvalue, and no other eigenvalues of modulus one are present.

Proof. The first assertion follows from (2.3) since $\|h\|_{p-1,q} \leq \|h\|_{p,q}$. The proof of the second is completely standard and can be based, for example, on an argument by Hennion after a spectral formula due to Nussbaum (see [4, Theorem 1] for details).

The third is a consequence of the ergodic decomposition (see [4, Propositions 2.3.1 and 2.3.2] for details). □

Remark 2.4. *To choose $\sigma = \max\{\lambda^{-1}, \nu\}$, we have to take $\zeta > \bar{\zeta}+1$, which implies $p < \frac{r}{2+\zeta}$. Accordingly, if T is conformal in the stable direction (it is in particular satisfied if the stable direction is one dimensional) we can choose, at best, $p = \frac{r}{3}$. Such a condition is more restrictive than Kitaev's requirement* [11] *that, in our language, reads $p = \frac{r}{2}$. Yet, it should be noted that in Kitaev the operators are only formal entities as there is no Banach space on which they act. It is unclear if the present limitation is due to a substantial obstruction or is an artifact of the proof. Note however that the condition $q > \bar{\zeta}p$ depends exclusively on equation* (6.9) *and an improvement in that estimate would improve such a limitation.*

---

[5] It is possible to take any σ larger than $\max(\lambda^{-1}, \nu^{\zeta-\bar{\zeta}})$.

[6] That is, $\mathcal{L}$ can be extended to a bounded operator on $\mathcal{B}^{p,q}$ that, with a mild abuse of notation, we still call $\mathcal{L}$.

In the $\mathcal{C}^\infty$ case, Theorem 2.3 immediately implies the following description of the correlations of $\mathcal{C}^\infty$ functions:

Corollary 2.5. *Assume that T is $\mathcal{C}^\infty$. Then there exist a sequence of complex numbers λ_k such that $|\lambda_k|$ decreases to 0, and integers r_k such that: for any $f, g : X \to \mathbb{R}$ of class $\mathcal{C}^\infty$, there exist numbers $a_k(f,g)$ with*

$$\int f \cdot g \circ T^n \sim \sum_{k=0}^{\infty} a_k(f,g) n^{r_k} \lambda_k^n$$

in the following sense: for any $\varepsilon > 0$, let K be such that $|\lambda_K| < \varepsilon$. Then

$$\int f \cdot g \circ T^n = \sum_{k=0}^{K-1} a_k(f,g) n^{r_k} \lambda_k^n + o(\varepsilon^n).$$

The second part of the paper focuses on the spectral stability for a wide class of deterministic and random perturbations.

Let U be a small enough neighborhood of T in the $\mathcal{C}^{r+1}$ topology. Consider a probability measure μ on a probability space Ω and, for $\omega \in \Omega$, take $T_\omega \in U$ and $g(\omega, \cdot) \in \mathcal{C}^q(X, \mathbb{R}_+)$. Assume also that, for all $x \in X$, $\int g(\omega, x)\, \mathrm{d}\mu(\omega) = 1$, and that $\int |g(\omega,\cdot)|_{\mathcal{C}^q(X,\mathbb{R})}\, \mathrm{d}\mu(\omega) < \infty$. It is then possible to define a random walk in the following way: starting from a point x, choose a diffeomorphism T_ω randomly with respect to the measure $g(\omega, x)\, \mathrm{d}\mu(\omega)$, and go to $T_\omega(x)$. Then iterate this process independently.

When Ω is a singleton and $g(\omega, x) = 1$, then this is a deterministic perturbation T_ω of T. Random perturbations of the type discussed in [4] can also be described in this way.[7] Hence, this setting encompasses at the same time very general deterministic and random perturbations of T. Define the *size* of the perturbation by

$$\Delta(\mu, g) := \int |g(\omega,\cdot)|_{\mathcal{C}^q(X,\mathbb{R})} d_{\mathcal{C}^{p+q}}(T_\omega, T)\, \mathrm{d}\mu(\omega). \tag{2.4}$$

For definiteness, we will fix a large constant A and assume that, until the end of this paragraph, all the perturbations we consider satisfy $\int |g(\omega,\cdot)|_{\mathcal{C}^q(X,\mathbb{R})} \leq A$.

The transfer operator $\mathcal{L}_{\mu,g}$ associated to the previous random walk is given by

$$\mathcal{L}_{\mu,g} h(x) = \int_\Omega g(\omega, T_\omega^{-1}(x)) \mathcal{L}_{T_\omega} h(x)\, \mathrm{d}\mu(\omega)$$

where $\mathcal{L}_{T_\omega}$ is the transfer operator associated to T_ω.

In Lemma 7.1 we show that $\mathcal{L}_T$ and $\mathcal{L}_{\widetilde{T}}$ are $\|\cdot\|_{\mathcal{B}^{p,q} \to \mathcal{B}^{p-1,q}}$ close if T and $\widetilde{T}$ are close in the $\mathcal{C}^{p+q}$ topology. In turn, this implies that $\mathcal{L}_T$ and $\mathcal{L}_{\mu,g}$ are close if $\Delta(\mu, g)$ is small, see (7.4). In addition, it is possible to show that the operators $\mathcal{L}_{\mu,g}$ satisfy a uniform Lasota-Yorke type inequality (Lemma 7.2). These facts suffice to apply [10] to the present context, yielding immediately the strong perturbation results described below greatly generalizing the results in [4].

Fix any $\varrho \in (\sigma^p, 1)$ and denote by $\mathrm{sp}(\mathcal{L})$ the spectrum of $\mathcal{L} : \mathcal{B}^{p,q} \to \mathcal{B}^{p,q}$. Since the essential spectral radius of $\mathcal{L}$ does not exceed σ^p, the set $\mathrm{sp}(\mathcal{L}) \cap \{z \in \mathbb{C} : |z| \geq \varrho\}$ consists of a finite number of eigenvalues $\lambda_1, \dots, \lambda_p$ of finite multiplicity. Changing

[7] To obtain the latter case set $\Omega = \mathbb{T}^d$, μ is Lebesgue, $T_\omega(x) = Tx + \omega \mod 1$ and $g(\omega, x) = q_\varepsilon(\omega, Tx)$.

ϱ slightly we may assume that $\mathrm{sp}(\mathcal{L}) \cap \{z \in \mathbb{C} : |z| = \varrho\} = \emptyset$. Hence there exists $\delta_* < \varrho - \sigma^p$ such that

$$|\lambda_i - \lambda_j| > \delta_* \quad (i \neq j) \;;$$
$$\mathrm{dist}(\mathrm{sp}(\mathcal{L}), \{|z| = \varrho\}) > \delta_*.$$

Theorem 2.6. *For each $\delta \in (0, \delta_*]$ and $\eta < 1 - \frac{\log \varrho}{p \log \sigma}$, there exists ε_0 such that for any perturbation (μ, g) of T satisfying $\Delta(\mu, g) \le \varepsilon_0$,*

a) The spectral projectors

$$\begin{aligned} \Pi^{(j)}_{\mu,g} &:= \frac{1}{2\pi \imath} \int_{\{|z-\lambda_j|=\delta\}} (z - \mathcal{L}_{\mu,g})^{-1}\, dz \\ \Pi^{(\varrho)}_{\mu,g} &:= \frac{1}{2\pi \imath} \int_{\{|z|=\varrho\}} (z - \mathcal{L}_{\mu,g})^{-1}\, dz \end{aligned} \tag{2.5}$$

are well defined.

b) There is $K_1 > 0$ such that $\|\Pi^{(j)}_{\mu,g} - \Pi^{(j)}_0\|_{\mathcal{B}^{p,q} \to \mathcal{B}^{p-1,q}} \le K_1\, \Delta(\mu, g)^\eta$ and $\|\Pi^{(\varrho)}_{\mu,g} - \Pi^{(\varrho)}_0\|_{\mathcal{B}^{p,q} \to \mathcal{B}^{p-1,q}} \le K_1\, \Delta(\mu, g)^\eta$.

c) $\mathrm{rank}(\Pi^{(j)}_{\mu,g}) = \mathrm{rank}(\Pi^{(j)}_0)$.

d) There is $K_2 > 0$ such that $\|\mathcal{L}^n_{\mu,g} \Pi^{(\varrho)}_{\mu,g}\|_{p,q} \le K_2\, \varrho^n$ for all $n \in \mathbb{N}$.

If the perturbation enjoys stronger regularity properties, then much sharper results can be obtained. Such results follow from a generalization of [10], along the lines of [16], that can be found in Section 8.

To keep the exposition simple let us restrict ourselves to deterministic perturbations. Since $\mathcal{C}^{r+1}(X, X)$ has naturally the structure of a $\mathcal{C}^{r+1}$ Banach manifold, it makes sense to consider perturbations belonging to $\mathcal{C}^s([-1, 1], \mathcal{C}^{r+1}(X, X))$, that is curves T_t of $\mathcal{C}^{r+1}$ maps from X to X such that, when viewed in coordinates, their firsts s derivatives with respect to t are $\mathcal{C}^{r+1}$ functions.

Theorem 2.7. *Let $T_t \in \mathcal{C}^s([-1, 1], \mathcal{C}^{r+1}(X, X))$ and T_0 be an Anosov diffeomorphism. Let p_0, q be integers such that $q \ge \zeta(p_0 + s - 1)$ and $p_0 + s - 1 + q \le r$. Then there exists $\delta_* > 0$ such that, for all $t \in [-\delta_*, \delta_*]$, the eigenvalues and eigenprojectors $\lambda_i(t)$, $\Pi_i(t)$ associated to $\mathcal{L}_{T_t}$ with $|\lambda_i(0)| > \sigma^{p_0}$ satisfy:*

(1) if $\lambda_i(0)$ is simple, then $\lambda_i(t) \in \mathcal{C}^{s-1}$;
(2) $\Pi_i(t) \in \mathcal{C}^{s-1}(\mathcal{B}^{p_0+s-1,q}, \mathcal{B}^{p_0-1,q})$.

The above theorem is proven in Section 9 by showing that the hypotheses of Theorem 8.1 hold in the present context.

Remark 2.8. *Notice that, in Theorem 2.7, there is some limitation to the differentiability coming from $(1 + \zeta)s \le r$ (see also Remark 2.4). In certain cases this can yield a weaker result than [18] where, in the case $s = r + 1$, it is proven that the eigenvalues are $\mathcal{C}^{r-1}$. Yet, such a result is limited to the peripheral eigenvalues (larger that σ) and gives much less information on the eigenspaces.*

Remark 2.9. *Note that, although not explicitly stated, all the constants in Theorems 2.6 and 2.7 are constructive and can actually be computed in specific examples (see [15] for a discussion of such issues).*

Remark 2.10. *Beside the eigenvalues and the eigenprojectors, the above theory implies results also for other physically relevant quantities. For example, if $f \in \mathcal{C}^r$,* *let $f_t = f - \int f \,\mathrm{d}\mu_{SRB}(T_t)$, then it is well known that $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f_t \circ T_t^k$ converges in law to a Gaussian with zero mean and variance*

$$\sigma(t)^2 = -\mu_{SRB}(f_t^2) + 2\sum_{n=0}^{\infty}\mu_{SRB}(f_t \circ T_t^n f_t) = -\mu_{SRB}(f_t^2) + 2[(\mathbf{Id} - \mathcal{L}_{T_t})^{-1}\mu_t](f_t)$$

where $\mu_t(\varphi) := \mu_{SRB}(f_t\varphi)$. By the results of Sections 8 and 9 it follows then that $\sigma \in \mathcal{C}^{s-1}$ and one can actually compute formulae for its Taylor expansion up to order $s-1$.

We conclude this section with a warning to the reader.

Remark 2.11. *Through the paper we will use C to designate a generic constant depending only on the map and the Banach spaces. Its actual numerical value can thus change from one occurrence to the next.*

3. Definition and properties of the admissible leaves

Replacing the metric by an adapted metric *à la Mather* [17], we can assume that the expansion of $DT(x)$ along the unstable directions is stronger than λ, the contraction along the stable directions is stronger than ν, and the angle between the stable and unstable directions is everywhere arbitrarily close to $\pi/2$. For small enough κ, we define the stable cone at $x \in X$ by

$$\mathcal{C}(x) = \{u + v \in T_xX \mid u \in E^s(x), v \perp E^s(x), \|v\| \le \kappa \|u\|\}.$$

If κ is small enough, $DT^{-1}(x)(\mathcal{C}(x)\backslash\{0\})$ is included in the interior of $\mathcal{C}(T^{-1}x)$, and $DT^{-1}(x)$ expands the vectors in $\mathcal{C}(x)$ by ν^{-1}.

There exists a finite number of $\mathcal{C}^\infty$ coordinate charts $\psi_1, \dots, \psi_N$ such that ψ_i is defined on a subset $(-r_i, r_i)^d$ of $\mathbb{R}^d$ (with its standard euclidian norm), such that

1. $D\psi_i(0)$ is an isometry.
2. $D\psi_i(0) \cdot \left(\mathbb{R}^{d_s} \times \{0\}\right) = E^s(\psi_i(0))$.
3. The $\mathcal{C}^r$-norms of ψ_i and its inverse are bounded by $1+\kappa$.
4. There exists $c_i \in (\kappa, 2\kappa)$ such that the cone $\mathcal{C}_i = \{u + v \in \mathbb{R}^d \mid u \in \mathbb{R}^{d_s} \times \{0\}, v \in \{0\} \times \mathbb{R}^{d_u}, \|v\| \le c_i \|u\|\}$ satisfies the following property: for any $x \in (-r_i, r_i)^d$, $D\psi_i(x)\mathcal{C}_i \supset \mathcal{C}(\psi_i x)$ and $DT^{-1}(D\psi_i(x)\mathcal{C}_i) \subset \mathcal{C}(T^{-1} \circ \psi_i(x))$.
5. The manifold X is covered by the open sets $\left(\psi_i((-r_i/2, r_i/2)^d)\right)_{i=1\dots N}$.

It is easy to construct such a chart around any point of X, hence a finite number of them is sufficient to cover the whole manifold by compactness.

Let $G_i(K)$ be the set of graphs of functions χ defined on a subset of $(-r_i, r_i)^{d_s}$ and taking values in $(-r_i, r_i)^{d_u}$, with $|D\chi| \le c_i$ (i.e., the tangent space to the graph of χ belongs to the cone $\mathcal{C}_i$) and with $\|\chi\|_{\mathcal{C}^r} \le K$.[8]

The following is a classical consequence of the uniform hyperbolicity of T.

Lemma 3.1. *If K is large enough, then there exists $K' < K$ such that, for any $W \in G_i(K)$ and for any $1 \le j \le N$, the set $\psi_j^{-1} \circ T^{-1} \circ \psi_i(W)$ belongs to $G_j(K')$.*

If κ is small enough, then $\nu^{-1} > (1+\kappa)^2\sqrt{1+4\kappa^2}$. Hence, there exists $A > 0$ such that

$$\frac{\nu^{-1}}{(1+\kappa)^2\sqrt{1+4\kappa^2}}(A-1) > A. \tag{3.1}$$

[8] A function defined on an arbitrary subset A of $\mathbb{R}^d$ is of class $\mathcal{C}^r$ if there exists a $\mathcal{C}^r$ extension to an open neighborhood of A. Its norm is the infimum of the norms of such extensions.

Take $\delta > 0$ small enough so that $A\delta < \min(r_i)/6$.

We define an *admissible graph* as a map χ defined on some ball $\overline{B}(x, A\delta)$ included in $(-2r_i/3, 2r_i/3)^{d_s}$, taking its values in $(-2r_i/3, 2r_i/3)^{d_u}$, with range$(\mathbf{Id}, \chi) \in G_i(K)$. Denote by Ξ_i the set of admissible graphs on $(-2r_i/3, 2r_i/3)^{d_s}$.

Given an admissible graph $\chi \in \Xi_i$, we will call $\widetilde{W} := \psi_i \circ (\mathbf{Id}, \chi)(\overline{B}(x, A\delta))$ the associated *full admissible leaf* and $W := \psi_i \circ (\mathbf{Id}, \chi)(\overline{B}(x, \delta))$ the *admissible leaf.*[9]

Let

$$\Omega = \left\{\psi_i \circ (\mathbf{Id}, \chi)(\overline{B}(x, \delta)) \mid \chi : \overline{B}(x, A\delta) \to \mathbb{R}^{d_u} \text{ belongs to } \Xi_i\right\}. \tag{3.2}$$

This is the set of admissible leaves.

We can use these admissible leaves to give another expression of the norm (2.1) in coordinates. Set

$$\|h\|_{p,q}^{-} = \sup_{\substack{|\alpha|=p \\ 1\leq i\leq N}} \sup_{\substack{\chi:\overline{B}(x,A\delta)\to\mathbb{R}^{d_u} \\ \chi\in\Xi_i}} \sup_{\substack{\varphi\in\mathcal{C}_0^q(\overline{B}(x,\delta),\mathbb{R}) \\ |\varphi|_{\mathcal{C}^q}\leq 1}} \int_{B(x,\delta)} [\partial^\alpha (h \circ \psi_i)] \circ (\mathbf{Id}, \chi) \cdot \varphi. \tag{3.3}$$

This is a seminorm on $\mathcal{C}^r(X, \mathbb{R})$. Since the norm of χ is bounded, it is easy to check that $\|h\|_{p,q}$ is equivalent to the norm given by $\sup_{0\leq k\leq p} \|h\|_{k,q}^{-}$. In the following, we will work indifferently with one expression of the norm or the other.

The reason for integrating in (3.3) only on admissible leaves, rather than on full admissible leaves, is that the preimage of an admissible leaf can be covered by a finite number of admissible leaves. We will in fact need a slightly more precise result, conveniently expressed in terms of the following notion. For $\gamma > 1$, a γ*-admissible graph* is a map defined on a ball $\overline{B}(x, \gamma A\delta) \subset (-\frac{2r_i}{3\gamma}, \frac{2r_i}{3\gamma})^{d_s}$, taking its values in $(-\frac{2r_i}{3\gamma}, \frac{2r_i}{3\gamma})^{d_u}$, whose graph belongs to $G_i(K)$. The corresponding γ*-admissible leaf* is $\psi_i \circ (\mathbf{Id}, \chi)(\overline{B}(x, \delta/\gamma))$.

Lemma 3.2. *There exists $\gamma_0 > 1$ satisfying the following property: for any full admissible leaf $\widetilde{W}$ and $n \in \mathbb{N}^*$, for any $1 \leq \gamma \leq \gamma_0$, there exist γ-admissible graphs $W_1, \dots, W_\ell$, whose number ℓ is bounded by a constant depending only on n, such that*

(1) $T^{-n}(W) \subset \bigcup_{j=1}^{\ell} W_j$.
(2) $T^{-n}(\widetilde{W}) \supset \bigcup_{j=1}^{\ell} W_j$.
(3) There exists a constant C (independent of W and n) such that a point of $T^{-n}\widetilde{W}$ is contained in at most C sets W_j.
(4) There exist functions $\rho_1, \dots, \rho_\ell$ of class $\mathcal{C}^r$ and compactly supported on W_j such that $\sum \rho_j = 1$ on $T^{-n}(W)$, and $|\rho_j|_{\mathcal{C}^r} \leq C$.

Proof. Let $\chi : \overline{B}(x, A\delta) \to (-2r_i/3, 2r_i/3)^{d_u}$ be an admissible graph. Let $W = \psi_i \circ (\mathbf{Id}, \chi)(\overline{B}(x, \delta))$ be the admissible leaf corresponding to χ, and $\widetilde{W}$ the corresponding full admissible leaf.

Take $y \in \overline{B}(x, \delta)$ (so that $\overline{B}(y, (A-1)\delta) \subset \overline{B}(x, A\delta)$) and j such that $T^{-n} \circ \psi_i(y, \chi(y)) \in \psi_j\big((-r_j/2, r_j/2)^d\big)$. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d_s}$ be the projection on the first components. The map T^{-n} expands the distances by at least ν^{-n} along $\widetilde{W}$. The maps ψ_i^{-1} and ψ_j are $(1+\kappa)$-Lipschitz and $|\pi(v)| \geq \frac{1}{\sqrt{1+4\kappa^2}}|v|$ when the vector v points in a stable cone $\mathcal{C}_j$. Hence, the map $F := \pi \circ \psi_j^{-1} \circ T^{-n} \circ \psi_i \circ (\mathbf{Id}, \chi)$ expands

[9] Note that one can talk about an admissible leaf only if it is given by $\chi \in \Xi_i$, that is if there exists an associated full admissible leaf.

the distances by at least $\frac{\nu^{-n}}{(1+\kappa)^2\sqrt{1+4\kappa^2}}$. If γ is close enough to 1, then (3.1) implies that the image by F of the ball $\overline{B}(y,(A-1)\delta)$ contains the ball $\overline{B}(F(y),\gamma A\delta)$. We can then define a map $\chi_{F(y)}:\overline{B}(F(y),\gamma A\delta)\to(-2r_j/(3\gamma),2r_j/(3\gamma))^{d_u}$ such that its graph is contained in $\psi_j^{-1}(T^{-n}\widetilde{W})$. In particular, $\chi_{F(y)}$ is a γ-admissible graph, by Lemma 3.1.

We have shown that $T^{-n}W$ can be covered by γ-admissible leaves. The lemma is then a consequence of [8, Theorem 1.4.10]. □

Remark 3.3. *We will mostly use this lemma with $\gamma=1$, to get a covering of $T^{-n}W$ by admissible leaves. However, in the study of perturbations of T, we will need to use some $\gamma>1$.*

4. Description of the space $\mathcal{B}^{p,q}$

Take a covering of X by sets of diameter at most δ and a partition of unity subordinated to this covering. Using admissible leaves supported in each of these sets, we easily check that there exists a constant C such that, for all $h\in\mathcal{C}^r(X,\mathbb{R})$ and for all $\varphi\in\mathcal{C}^q(X,\mathbb{R})$,

$$\left|\int_X h\cdot\varphi\right|\leq C\|h\|_{p,q}|\varphi|_{\mathcal{C}^q}.$$

Passing to the completion, we obtain that any $h\in\mathcal{B}^{p,q}$ gives a distribution on X of order at most q. Denote by $\mathcal{D}'_q$ the set of distributions of order at most q with its canonical norm.

Proposition 4.1. *The map $\mathcal{B}^{p,q}\to\mathcal{D}'_q$ is a continuous injection.*

Proof. The continuity is trivial from the previous remarks.

Take $h\in\mathcal{C}^r(X,\mathbb{R})$. Let $\chi:\overline{B}(x,A\delta)\to(-2r_i/3,2r_i/3)^{d_u}$ be an admissible graph and $|\alpha|\leq p$. We can define a distribution $D_{\alpha,\chi}(h)$, on the ball $B(0,\delta)$, setting $\langle D_{\alpha,\chi}(h),\varphi\rangle=\int_{B(0,\delta)}\partial^\alpha(h\circ\psi_i)(x+\eta,\chi(x+\eta))\cdot\varphi(\eta)\,\mathrm{d}\eta$. The map $h\mapsto D_{\alpha,\chi}(h)$ is continuous for the $\|\cdot\|_{p,q}$-norm, whence it can be extended to the space $\mathcal{B}^{p,q}$. The norm of an element h of $\mathcal{B}^{p,q}$ is by definition equal to the maximum of the norms of the corresponding distributions $D_{\alpha,\chi}(h)$.

Assume that $\partial^\alpha=\partial_j\partial^\beta$. Let χ_ε be the admissible graph obtained by translating the graph of χ of ε in the direction x_j. For $h\in\mathcal{C}^r$, the map $\varepsilon\mapsto D_{\alpha,\chi_\varepsilon}(h)$ is continuous. By density, it is continuous for any $h\in\mathcal{B}^{p,q}$. Moreover,

$$D_{\beta,\chi_\varepsilon}(h)-D_{\beta,\chi}(h)=\varepsilon\int_0^1 D_{\alpha,\chi_{t\varepsilon}}(h)\,\mathrm{d}t.$$

Since $\varepsilon\mapsto D_{\alpha,\chi_\varepsilon}(h)$ is continuous, we obtain that, for any $h\in\mathcal{B}^{p,q}$,

$$D_{\alpha,\chi}(h)=\lim_{\varepsilon\to0}\frac{D_{\beta,\chi_\varepsilon}(h)-D_{\beta,\chi}(h)}{\varepsilon}. \tag{4.1}$$

Take $h\in\mathcal{B}^{p,q}$ different from 0. Then there exists an admissible graph χ such that $D_{\emptyset,\chi}(h)\neq0$: otherwise, (4.1) would imply that all the distributions $D_{\alpha,\chi}(h)$ vanish, which means that $h=0$ in $\mathcal{B}^{p,q}$. Take φ such that $\langle D_{\emptyset,\chi}(h),\varphi\rangle\neq0$. Then, for any χ' close enough to χ, we still have $\langle D_{\emptyset,\chi'}(h),\varphi\rangle\neq0$ by continuity. Hence, we can construct a $\mathcal{C}^\infty$ function $\widetilde{\varphi}$ defined on a neighborhood of the graph of χ and such that $\langle h,\widetilde{\varphi}\rangle\neq0$. Hence, the distribution given by h is nonzero. □

Remark 4.2. *There exist canonically defined maps $\mathcal{B}^{p,q} \to \mathcal{B}^{p-1,q}$ and $\mathcal{B}^{p,q} \to \mathcal{B}^{p,q+1}$, obtained by extending continuously the canonical embedding of $\mathcal{C}^r$ functions. Proposition 4.1 implies in particular that these maps are injective.*

Remark 4.3. *When h is $\mathcal{C}^r$, then $\|h\|_{p,q} \leq C\|h\|_{\mathcal{C}^r}$. Hence, the embedding of $\mathcal{C}^r(X,\mathbb{R})$ in $\mathcal{B}^{p,q}$ is continuous. Since $\mathcal{C}^\infty(X,\mathbb{R})$ is dense in $\mathcal{C}^r(X,\mathbb{R})$ for the $\mathcal{C}^r$-norm, it implies that $\mathcal{C}^\infty(X,\mathbb{R})$ is dense in $\mathcal{B}^{p,q}$. Hence, we could also have obtained $\mathcal{B}^{p,q}$ by completing $\mathcal{C}^\infty(X,\mathbb{R})$.*

It is interesting to give explicit examples of nontrivial elements of $\mathcal{B}^{p,q}$:

Proposition 4.4. *Let W be a $\mathcal{C}^{p+1}$-submanifold of dimension d_u everywhere transverse to the cones $D\psi_i(\mathcal{C}_i)$ (e.g. a piece of unstable manifold) and μ a $\mathcal{C}^p$-density on W, with compact support. Then the distribution $\ell(\varphi) := \int_W \varphi \,\mathrm{d}\mu$ belongs to $\mathcal{B}^{p,q}$.*

Proof. Without loss of generality we can assume that the manifold W belongs to one chart $((-r_i,r_i)^d,\psi_i)$. We will work only in such a chart, and omit the coordinate change ψ_i.

The manifold W is given by the graph of a $\mathcal{C}^{p+1}$ function $\zeta : \mathbb{R}^{d_u} \to \mathbb{R}^{d_s}$. The density of μ is then given by a $\mathcal{C}^p$ function $f : \mathbb{R}^{d_u} \to \mathbb{R}$ with compact support.

Let $f_\varepsilon \in \mathcal{C}^\infty$ be such that $|f - f_\varepsilon|_{\mathcal{C}^p} \leq \varepsilon$. Take also $\zeta_\varepsilon \in \mathcal{C}^\infty$ with $|\zeta - \zeta_\varepsilon|_{\mathcal{C}^{p+1}} \leq \varepsilon$. Let $\vartheta : \mathbb{R}^{d_s} \to \mathbb{R}_+$ be a $\mathcal{C}^\infty$ function supported in $B(0,1)$ and with $\int \vartheta = 1$. Set $\vartheta_\varepsilon(\eta) = \frac{1}{\varepsilon^{d_s}}\vartheta(\eta/\varepsilon)$: it is supported in $B(0,\varepsilon)$ and has integral 1. Let finally $h_\varepsilon(\eta,\xi) = \vartheta_\varepsilon(\eta - \zeta_\varepsilon(\xi)) f_\varepsilon(\xi)$: it is a $\mathcal{C}^\infty$ function, and h_ε converges to ℓ in the sense of distributions when $\varepsilon \to 0$. Hence, the result will be proved if we show that $\{h_\varepsilon\}$ is a Cauchy sequence in $\mathcal{B}^{p,q}$.

Take α with $|\alpha| \leq p$. Then one has

$$\partial^\alpha h_\varepsilon(\eta,\xi) = \sum_{|\beta|\leq p} (\partial^\beta \vartheta_\varepsilon)(\eta - \zeta_\varepsilon(\xi)) g_{\beta,\varepsilon}(\xi)$$

where the function $g_{\beta,\varepsilon}$ is in $\mathcal{C}^\infty$ and converges in $\mathcal{C}^{|\beta|}$ to $g_{\beta,0}$ when $\varepsilon \to 0$.

Let χ be an admissible graph and φ a $\mathcal{C}^q$ test function with $|\varphi|_{\mathcal{C}^q} \leq 1$. Then

$$\int \varphi(\eta)\partial^\alpha h_\varepsilon(\eta,\chi(\eta)) = \sum_\beta \int \varphi(\eta)(\partial^\beta\vartheta_\varepsilon)(\eta - \zeta_\varepsilon(\chi(\eta))g_{\beta,\varepsilon}(\chi(\eta)). \tag{4.2}$$

Since W is everywhere transverse to the cone $\mathcal{C}_i$, the map $\theta_\varepsilon : \eta \mapsto \eta - \zeta_\varepsilon(\chi(\eta))$ is a $\mathcal{C}^{p+1}$ diffeomorphism, and it converges when $\varepsilon \to 0$ to θ_0. Using this change of coordinates in (4.2), integrating by parts and given the fact that ϑ_ε is a $\mathcal{C}^\infty$ mollifier, we obtain that (4.2) converges when $\varepsilon \to 0$. Moreover, the speed of convergence is independent of the graph χ or the test function φ, since all norms are uniformly bounded. Hence, $\|h_\varepsilon - h_{\varepsilon'}\|_{p,q} \to 0$ when $\varepsilon, \varepsilon' \to 0$, i.e., h_ε is a Cauchy sequence in $\mathcal{B}^{p,q}$. □

5. Compactness

This paragraph is devoted to the proof of Lemma 2.1. We will work only in coordinate charts, using in an essential way the linear structure to interpolate between admissible leaves.

First of all, we need a technical lemma showing that the $\|\cdot\|_{p,q}$-norm gives a control on balls of radius slightly larger than δ.

Lemma 5.1. *There exist $E>0$ and $b_0>0$ such that, for any $b<b_0$, for any ball $\overline{B}(x, A\delta+Eb)$ contained in a set $(-2r_i/3, 2r_i/3)^{d_s}$, for any function $\chi : \overline{B}(x, A\delta+Eb) \to (-2r_i/3, 2r_i/3)^{d_u}$ such that $|D\chi| \leq c_i$ and $\|\chi\|_{\mathcal{C}^r} \leq K$, for any $\varphi : B(x,\delta+b) \to \mathbb{R}$ compactly supported, for any $h \in \mathcal{C}^r(X, \mathbb{R})$, for any α with $|\alpha| = p$,*

$$\left| \int_{B(x,\delta+b)} \partial^\alpha (h\circ\psi_i) \circ (\mathbf{Id}, \chi) \cdot \varphi \right| \leq C |\varphi|_{\mathcal{C}^q} \|h\|^-_{p,q}$$

where the constant C is independent of b.

Proof. We can assume that $x=0$.

Let $a>0$ be such that $[-a,a]^{d_s} \subset B(0,\delta)$. Let $E>0$ be such that $2\delta < \frac{Ea}{2\sqrt{d_s}}$. Let b_0 be such that $\frac{Eb_0}{\sqrt{d_s}} \leq a/2$ and $2\delta + b_0 < \frac{Ea}{2\sqrt{d_s}}$.

Let $\zeta_{-1}, \zeta_0, \zeta_1$ be a $\mathcal{C}^\infty$ partition of unity on $\mathbb{R}$ subordinated to the sets $(-\infty, -a/2)$, $(-a,a)$ and $(a/2, +\infty)$. For $\omega \in \{-1,0,1\}^{d_s}$, set $\zeta_\omega(x_1, \dots, x_{d_s}) = \prod_{i=1}^{d_s} \zeta_{\omega_i}(x_i)$. The functions ζ_ω form a partition of unity of $\mathbb{R}^{d_s}$.

Take $b<b_0$. For $\omega \in \{-1,0,1\}^{d_s}$, let $x_\omega(b) = \frac{Eb}{\sqrt{d_s}}(\omega_1, \dots, \omega_{d_s})$. We will show that ζ_ω vanishes on $\overline{B}(0,\delta+b) \backslash \overline{B}(x_\omega(b), \delta)$. This is trivial if $\omega = (0,\dots,0)$, so we can assume that $k := \#\{i \mid \omega_i \neq 0\}$ is nonzero. Take $x \in \overline{B}(0,\delta+b)$ such that $\zeta_\omega(x) \neq 0$. When $\omega_i \neq 0$, this implies that $|x_i| \geq a/2$ and that x_i and ω_i have the same sign. Hence,

$$\|x - x_\omega(b)\|^2 = \sum_i x_i^2 - 2x_i \frac{Eb}{\sqrt{d_s}} \omega_i + \frac{E^2 b^2}{d_s} \omega_i^2 \leq \|x\|^2 - 2\frac{a}{2} \frac{Eb}{\sqrt{d_s}} k + \frac{E^2 b^2}{d_s} k.$$

Since $\frac{Eb}{\sqrt{d_s}} \leq a/2$, $k \geq 1$ and $\|x\| \leq \delta + b$, we get

$$\|x - x_\omega(b)\|^2 \leq \delta^2 + 2\delta b + b^2 - \frac{a}{2} \frac{Eb}{\sqrt{d_s}} \leq \delta^2,$$

i.e., $x \in \overline{B}(x_\omega(b), \delta)$.

Since $\sum_\omega \zeta_\omega = 1$, it follows

$$\int_{B(x,\delta+b)} \partial^\alpha (h\circ\psi_i) \circ (\mathbf{Id}, \chi) \cdot \varphi = \sum_\omega \int_{B(x,\delta+b)} \partial^\alpha (h\circ\psi_i) \circ (\mathbf{Id}, \chi) \cdot \varphi\zeta_\omega.$$

But the function $\varphi\zeta_\omega$ is vanishing outside of the ball $B(x_\omega(b), \delta)$. Moreover, the ball $\overline{B}(x_\omega(b), A\delta)$ is contained in $\overline{B}(0, A\delta+Eb)$, whence χ defines an admissible graph on $\overline{B}(x_\omega(b), A\delta)$. Therefore,

$$\left| \int_{B(x,\delta+b)} \partial^\alpha (h\circ\psi_i) \circ (\mathbf{Id}, \chi) \cdot \varphi \right| \leq \sum_\omega |\varphi\zeta_\omega|_{\mathcal{C}^q} \|h\|^-_{p,q} \leq \sum_\omega |\varphi|_{\mathcal{C}^q} |\zeta_\omega|_{\mathcal{C}^q} \|h\|^-_{p,q}.$$

This concludes the proof since $\sum_\omega |\zeta_\omega|_{\mathcal{C}^q} \leq C$ independently of b. □

Proof of Lemma 2.1. Fix $\varepsilon > 0$.

Take $1 \leq i \leq N$. Since $q<r$, the injection $\mathcal{C}^r \to \mathcal{C}^q$ is compact. Therefore, there exists a finite number of admissible graphs $\chi_1, \dots, \chi_s$ defined on balls $\overline{B}(x_1, A\delta), \dots, \overline{B}(x_s, A\delta)$ such that any other admissible graph χ defined on a ball $\overline{B}(x, A\delta)$ is at a distance at most ε of some χ_j, in the sense that $|x - x_j| \leq \varepsilon$ and $|\eta \mapsto \chi(x+\eta) - \chi_j(x_j+\eta)|_{\mathcal{C}^q(\overline{B}(0,\delta), \mathbb{R}^{d_u})} \leq \varepsilon$. We can also assume that these admissible graphs satisfy $|D\chi_j| < c_i$ and $\|\chi_j\|_{\mathcal{C}^r} < K$.

Take α with $|\alpha| = p-1$ and $\varphi \in \mathcal{C}_0^q(\overline{B}(0,\delta),\mathbb{R})$ with $|\varphi|_{\mathcal{C}^q} \leq 1$. Write

$$f_t(\eta) = (x_j + \eta + t(x-x_j), \chi_j(x_j+\eta) + t(\chi(x+\eta) - \chi_j(x_j+\eta))).$$

Write also $F(z) = \partial^\alpha(h\circ\psi_i)(z)$. Then, for $\eta \in B(0,\delta)$,

$$\begin{aligned}\partial^\alpha(h\circ\psi_i)(x+\eta,\chi(x+\eta)) - \partial^\alpha(h\circ\psi_i)(x_j+\eta,\chi_j(x_j+\eta)) &= F(f_1(\eta)) - F(f_0(\eta)) \\ &= \int_0^1 DF(f_t(\eta))\cdot(x-x_j,\chi(x+\eta)-\chi_j(x_j+\eta))\,\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}\int \partial^\alpha(h\circ\psi_i)(x+\eta,\chi(x+\eta))\varphi(\eta)\,\mathrm{d}\eta - \int \partial^\alpha(h\circ\psi_i)(x_j+\eta,\chi_j(x_j+\eta))\varphi(\eta)\,\mathrm{d}\eta \\ = \int_0^1 \left(\int DF(f_t(\eta))(x-x_j,\chi(x+\eta)-\chi_j(x_j+\eta))\varphi(\eta)\,\mathrm{d}\eta\right)\mathrm{d}t.\end{aligned}$$

When t is fixed, the last integral is an integral along the graph given by f_t. This graph is admissible since it is an interpolation between two admissible graphs (here, the fact that the cone $\mathcal{C}_i$ is constant is essential). Since $|x-x_j| \leq \varepsilon$ and $|\chi(x+\eta) - \chi_j(x_j+\eta)|_{\mathcal{C}^q} \leq \varepsilon$, this term can be estimated by $C\varepsilon \|h\|^-_{p,q}$. We have proved that

$$\begin{aligned}\sup_{|\alpha|=p-1}\sup_{\chi\in\Xi_i}\sup_{\substack{\varphi\in\mathcal{C}_0^q(\overline{B}(x,\delta),\mathbb{R})\\|\varphi|_{\mathcal{C}^q}\leq 1}} \int_{B(x,\delta)} \partial^\alpha(h\circ\psi_i)\circ(\mathbf{Id},\chi)\cdot\varphi \\ \leq C\varepsilon\|h\|^-_{p,q} + \sup_{|\alpha|=p-1}\sup_{1\leq k\leq s}\sup_{\substack{\varphi\in\mathcal{C}_0^q(\overline{B}(x_k,\delta),\mathbb{R})\\|\varphi|_{\mathcal{C}^q}\leq 1}} \int_{B(x_k,\delta)} \partial^\alpha(h\circ\psi_i)\circ(\mathbf{Id},\chi_k)\cdot\varphi,\end{aligned}$$

i.e., we have only a finite number of admissible graphs to consider.

The graph χ_k is $\mathcal{C}^r$ on the closed ball $\overline{B}(x_k, A\delta)$. By definition, there exists an extension of χ_k to a slightly larger ball $B(x_k, A\delta + Eb)$ (where E is as in Lemma 5.1). We will still denote this extension by χ_k. Reducing b if necessary, we can assume that it satisfies $|D\chi_k| < c_i$ and $\|\chi_k\|_{\mathcal{C}^r} < K$. We can even take the same value of b for all the graphs χ_k and assume that $b \leq \varepsilon$ and $b \leq b_0$ given by Lemma 5.1.

In the following, we work with one graph $\chi = \chi_k$, and we can assume without loss of generality that $x_k = 0$.

Let $\vartheta : \mathbb{R}^{d_s} \to [0,\infty)$ be of class $\mathcal{C}^\infty$, supported in $B(0,1)$, and with integral 1, and write $\vartheta_b(v) = \frac{1}{b^{d_s}}\vartheta(v/b)$. For $\varphi \in \mathcal{C}_0^q(\overline{B}(0,\delta),\mathbb{R})$, set $\widetilde{\varphi}(x) = \int \varphi(x-v)\vartheta_b(v)$: this function belongs to $\mathcal{C}^{q+1}$, with $|\widetilde{\varphi}|_{\mathcal{C}^{q+1}} \leq C/b$, and it is supported in $B(0,\delta+b)$. Write also $F(\eta) = \partial^\alpha(h\circ\psi_i)\circ(\mathbf{Id},\chi)(\eta)$. Then

$$\begin{aligned}\int F(\eta)\widetilde{\varphi}(\eta) - \int F(\eta)\varphi(\eta) &= \int \varphi(\eta)\left(\int \big(F(\eta+v)-F(\eta)\big)\vartheta_b(v)\,\mathrm{d}v\right)\mathrm{d}\eta \\ &= \int \varphi(\eta)\left(\int_0^1\int DF(\eta+tv)\cdot v\,\vartheta_b(v)\,\mathrm{d}v\,\mathrm{d}t\right)\mathrm{d}\eta \\ &= \int_0^1\int\left(\int \varphi(\eta-tv)DF(\eta)\cdot v\,\mathrm{d}\eta\right)\vartheta_b(v)\,\mathrm{d}v\,\mathrm{d}t.\end{aligned}$$

When v and t are fixed, the integral $\int \varphi(\eta-tv)DF(\eta)\cdot v\,\mathrm{d}\eta$ is an integral of the form $\sum_{j=1}^d \int_{B(0,\delta+b)} \partial_j\partial^\alpha(h\circ\psi_i)\circ(\mathbf{Id},\chi)\varphi_j$, where the $\mathcal{C}^q$-norm of φ_j is bounded by $C|v| \leq Cb$. By Lemma 5.1, this integral can be estimated as follows:

$$\left|\int \varphi(\eta - tv) DF(\eta) \cdot v \,\mathrm{d}\eta\right| \leq Cb\|h\|^-_{p,q} \leq C\varepsilon\|h\|^-_{p,q}.$$

Integrating over t and v (and using the fact that $\int \vartheta_b(v) = 1$), we get

$$\int \partial^\alpha (h \circ \psi_i) \circ (\mathbf{Id}, \chi) \cdot \varphi \leq C\varepsilon \|h\|^-_{p,q} + \int \partial^\alpha (h \circ \psi_i) \circ (\mathbf{Id}, \chi) \cdot \widetilde{\varphi}$$

where $|\widetilde{\varphi}|_{\mathcal{C}^{q+1}} \leq C/b$ and $\widetilde{\varphi}$ is supported in the ball $B(0, \delta + b)$.

The set of functions satisfying these two last properties is relatively compact for the $\mathcal{C}^q$-topology. In particular, there exists a finite set of functions $\varphi_1, \dots, \varphi_k$ which are ε-dense. For any $\widetilde{\varphi}$ as above, there exists j such that $|\widetilde{\varphi} - \varphi_j|_{\mathcal{C}^q} \leq \varepsilon$, and we get (once again using Lemma 5.1)

$$\int \partial^\alpha (h \circ \psi_i) \circ (\mathbf{Id}, \chi) \cdot \widetilde{\varphi} \leq \int \partial^\alpha (h \circ \psi_i) \circ (\mathbf{Id}, \chi) \cdot \varphi_j + C\varepsilon \|h\|_{p,q}.$$

To summarize: we have proved the existence of a finite number of continuous linear forms $\nu_1, \dots, \nu_\ell$ on $\mathcal{B}^{p,q}$ such that, for any $h \in \mathcal{B}^{p,q}$,

$$\|h\|^-_{p-1,q} \leq C\varepsilon \|h\|_{p,q} + \sup |\nu_i(h)|.$$

This immediately implies the compactness we are looking for. □

6. Lasota-Yorke type inequality

This section is devoted to the proof of Lemma 2.2.

6.1. Proof of (2.2). By density it suffices to prove it for $h \in \mathcal{C}^r$.

Take $W \in \Omega$ and $n \in \mathbb{N}^*$. Let $\varphi \in \mathcal{C}^q_0(W, \mathbb{R})$ satisfy $|\varphi|_{\mathcal{C}^q} \leq 1$. Let $\rho_1, \dots, \rho_\ell$ be the partition of unity on $T^{-n}W$ given by Lemma 3.2 (for $\gamma = 1$), and $W_1, \dots, W_\ell$ the corresponding admissible leaves. Then

$$\int_W \mathcal{L}^n h \cdot \varphi = \int_{T^{-n}W} h|\det DT^n|^{-1} \varphi \circ T^n J_W T^n$$

where $J_W T^n$ is the jacobian of $T^n : T^{-n}W \to W$. Using the partition of unity,

$$\int_W \mathcal{L}^n h \cdot \varphi = \sum_{k=1}^{\ell} \int_{W_k} h|\det DT^n|^{-1} \varphi \circ T^n J_W T^n \rho_k. \tag{6.1}$$

The function $\varphi_k := \varphi \circ T^n \cdot \rho_k$ is compactly supported on the admissible leaf W_k. Using the definition of the $\|\cdot\|_{0,q}$ norm along this leaf yields

$$\left|\int_{W_k} h|\det DT^n|^{-1} \varphi_k J_W T^n\right| \leq C \|h\|_{0,q} \left\| |\det DT^n|^{-1} \varphi_k J_W T^n \right\|_{\mathcal{C}^q(W_k)}. \tag{6.2}$$

We will use repeatedly the following distortion lemma:

Lemma 6.1. *Let $\widetilde{W}$ be a full admissible leaf and W' an admissible leaf contained in $T^{-n}\widetilde{W}$. Let $g_0, \dots, g_{n-1}$ be strictly positive $\mathcal{C}^q$ functions on $W', \dots, T^{n-1}(W')$ and $L > 0$ be such that, for any $x \in T^i(W')$, for $1 \leq k \leq q$, $|D^k g_i(x)| \leq L g_i(x)$. Then*

$$\forall x \in W', \quad \left\| \prod_{i=0}^{n-1} g_i \circ T^i \right\|_{\mathcal{C}^q(W')} \leq Ce^{CL} \prod_{i=0}^{n-1} g_i \circ T^i(x)$$

for some constant C depending only on the map T.

Proof. Differentiating q times the function $\prod_{i=0}^{n-1} g_i \circ T^i$ and using the inequality $|D^k g_i(x)| \le L|g_i(x)|$ and the uniform contraction of T along W', it is easy to check that $\left\|\prod_{i=0}^{n-1} g_i \circ T^i\right\|_{\mathcal{C}^q(W')} \le CL^q \left\|\prod_{i=0}^{n-1} g_i \circ T^i\right\|_{\mathcal{C}^0(W')}$.

The differential of the function $\log\left(\prod_{i=0}^{n-1} g_i \circ T^i\right)$ is also bounded by CL, whence, for any $x, y \in W'$,

$$\prod_{i=0}^{n-1} g_i \circ T^i(x) \le Ce^{CL} \prod_{i=0}^{n-1} g_i \circ T^i(y). \qquad \square$$

The lemma applies to estimate $\left\||\det DT^n|^{-1} J_W T^n\right\|_{\mathcal{C}^q(W_k)}$. Since T^n is uniformly contracting along $T^{-n}(W)$, we have $\|\varphi \circ T^n\|_{\mathcal{C}^q(W_k)} \le C\|\varphi\|_{\mathcal{C}^q} \le C$, and $\|\rho_k\|_{\mathcal{C}^q(W_k)}$ is uniformly bounded by Lemma 3.2. We get that, for any $x \in W_k$,

$$\left\||\det DT^n|^{-1}\varphi_k J_W T^n\right\|_{\mathcal{C}^q(W_k)} \le C|\det DT^n|^{-1}(x) J_W T^n(x).$$

In particular,

$$\begin{aligned}\left\||\det DT^n|^{-1}\varphi_k J_W T^n\right\|_{\mathcal{C}^q(W_k)} &\le C\int_{W_k} |\det DT^n|^{-1}(x) J_W T^n(x)\\ &= C\int_{T^n(W_k)} |\det(DT^{-n})|.\end{aligned}$$

By Lemma 3.2, the sets $T^n(W_k)$ are contained in $\widetilde{W}$ and have a bounded number of overlaps. Together with (6.1) and (6.2), this implies

$$\left|\int_W \mathcal{L}^n h \cdot \varphi\right| \le C\|h\|_{0,q} \int_{\widetilde{W}} |\det(DT^{-n})|.$$

To estimate the last integral, let us consider the thickening $Z := \bigcup_{x\in\widetilde{W}} W^u_\rho(x)$, where $W^u_\rho(x)$ is the ball of size ρ in the unstable manifold through x. By usual distortion estimates,

$$\int_{\widetilde{W}} |\det(DT^{-n})| \le C\rho^{-d_u} \int_Z |\det DT^{-n}| = C\operatorname{Vol}(T^{-n}Z) \le C.$$

This concludes the proof of the first inequality.

6.2. **Proof of** (2.3)**.** Let $\chi \in \Xi_i$ be an admissible graph and let W, $\widetilde{W}$ be the corresponding admissible leaf and full admissible leaf. As before, we will use Lemma 3.2 (with $\gamma = 1$) to write $T^{-n}W \subset \bigcup_j W_j$ and denote by ρ_j the corresponding partition of unity given by Lemma 3.2.

Let $\widetilde{h} := h \circ \psi_i$. With an innocuous abuse of notations we will use T and $\mathcal{L}$ to designate the map and the transfer operator also when expressed in the various coordinate charts we will employ, and we will identify W_j with the corresponding submanifold in the chart. To each $\bar\alpha \in \{1,\dots,d\}^p$ we associate the multi-index α defined by $\alpha_i := \#\{j \in \{1,\dots,p\} : \bar\alpha_j = i\}$. Hence, for each $\bar\alpha$ with $|\alpha| = p$,

$$\begin{aligned}\partial^\alpha \mathcal{L}^n \widetilde{h} &= (\det DT^n)^{-1} \circ T^{-n} \cdot \partial^\alpha(\widetilde{h} \circ T^{-n}) + \sum_{|\beta|<p} \hat\Psi_\beta \partial^\beta(\widetilde{h} \circ T^{-n})\\ &= \sum_{\bar\beta\in\{1,\dots,d\}^p} (\Phi_{\bar\beta}\partial^\beta \widetilde{h}) \circ T^{-n} + \sum_{|\beta|<p} (\Psi_\beta \partial^\beta \widetilde{h}) \circ T^{-n},\end{aligned}$$

where[10] $\Psi_\beta \in \mathcal{C}^q$, and

$$\Phi_{\bar\beta} := \prod_{j=1}^{p}([DT^n]^{-1})_{\bar\beta_j,\bar\alpha_j} \ \ |\det DT^n|^{-1}. \tag{6.3}$$

Hence, for each $\varphi \in \mathcal{C}_0^q(W,\mathbb{R})$,

$$\int_W \varphi(\partial^\alpha \mathcal{L}^n \widetilde{h}) = \sum_{\substack{i \\ \bar\beta\in\{1,\dots,d\}^p}} \int_{W_i} \Phi_{i,\bar\beta}\partial^\beta \widetilde{h} + \sum_{i;\,|\beta|<p}\int_{W_i}\Psi_{i,\beta}\partial^\beta\widetilde{h}, \tag{6.4}$$

where $\Psi_{i,\beta} \in \mathcal{C}_0^q(B(x_i,\delta))$ and

$$\Phi_{i,\bar\beta} := \prod_{j=1}^{p}([DT^n]^{-1})_{\bar\beta_j,\bar\alpha_j} \cdot |\det DT^n|^{-1} \cdot J_W T^n \cdot \varphi\circ T^n \cdot \rho_i. \tag{6.5}$$

Clearly the last terms of equation (6.4) can be immediately estimated in the $\|\cdot\|_{p-1,q}$ norm, we have thus only to deal with the $\Phi_{i,\bar\beta}$ terms. To this end it is necessary to smoothen the test function, in order to perform integrations by parts. For $\varepsilon \le \delta$, let $\mathbb{A}_\varepsilon\varphi \in \mathcal{C}_0^{q+1}(\widetilde{W},\mathbb{R})$ be obtained by convolving φ with a $\mathcal{C}^\infty$ mollifier whose support is of size ε.

Lemma 6.2. *For each $\varphi \in \mathcal{C}^q$,*

$$\begin{aligned}&|\mathbb{A}_\varepsilon\varphi|_{\mathcal{C}^q} \le C|\varphi|_{\mathcal{C}^q}; \qquad |\mathbb{A}_\varepsilon\varphi|_{\mathcal{C}^{q+1}} \le C\varepsilon^{-1}|\varphi|_{\mathcal{C}^q}\\ &|\mathbb{A}_\varepsilon\varphi-\varphi|_{\mathcal{C}^{q-1}} \le C\varepsilon|\varphi|_{\mathcal{C}^q}\end{aligned}$$

The proof of the above lemma is standard and is left to the reader.

By definition of $\bar\zeta$, for large enough N one has

$$\frac{|DT^{-N}(x)v|}{|v|} < \left(\frac{|DT^{-N}(x)w|}{|w|}\right)^{\bar\zeta} \tag{6.6}$$

for any vectors v, w in the stable cone at x. Fix such an N. There exist smooth strictly positive functions $\nu_+(x) \le \nu_-(x) \le \nu^N$ such that $\nu_-(x)^{-1} \le \frac{|DT^{-N}(x)v|}{|v|} \le \nu_+(x)^{-1}$ and $\nu_+(x) \ge \nu_-(x)^{\bar\zeta}$. Lemma 6.2 yields

$$|(\mathbb{A}_\varepsilon\varphi-\varphi)\circ T^n|_{\mathcal{C}^q(W_i)} \le C\max\left\{\varepsilon, \left|\prod_{j=1}^{\lfloor n/N\rfloor}\nu_-\circ T^{jN}\right|_{\mathcal{C}^0(W_i)}^q\right\}|\varphi|_{\mathcal{C}^q}. \tag{6.7}$$

Moreover, Lemma 6.1 applied to T^N proves that, for any distinguished point y_i in W_i,

$$\left|\prod_{j=1}^{\lfloor n/N\rfloor}\nu_-\circ T^{jN}\right|_{\mathcal{C}^0(W_i)} \le C\prod_{j=1}^{\lfloor n/N\rfloor}\nu_-\circ T^{jN}(y_i).$$

It is then natural to define

$$\Phi^\varepsilon_{i,\bar\beta} := \prod_{j=1}^{p}([DT^n]^{-1})_{\bar\beta_j,\bar\alpha_j} \cdot |\det DT^n|^{-1} \cdot J_W T^n \cdot (\mathbb{A}_\varepsilon\varphi)\circ T^n \cdot \rho_i. \tag{6.8}$$

The same distortion arguments as in the end of the proof of (2.2) apply to estimate the norm of $\Phi_{i,\bar\beta} - \Phi^\varepsilon_{i,\bar\beta}$, except for the term $\prod_{j=1}^p([DT^n]^{-1})_{\bar\beta_j,\bar\alpha_j}$ whose $\mathcal{C}^q$-norm

[10] Here we are using that $T \in \mathcal{C}^{p+q}$.

is bounded by $C[\prod_{j=1}^{\lfloor n/N\rfloor} \nu_+ \circ T^{jN}(y_i)]^{-p}$, again by Lemma 6.1. Choosing $\varepsilon_i = \left[\prod_{j=1}^{\lfloor n/N\rfloor} \nu_- \circ T^{jN}(y_i)\right]^q$, it follows

$$\left|\sum_i \int_{W_i} [\Phi_{i,\bar\beta} - \Phi^{\varepsilon_i}_{i,\bar\beta}]\partial^\beta \widetilde{h}\right| \le C\nu^{n(\zeta-\bar\zeta)p}|\varphi|_{\mathcal{C}^q}\|h\|_{p,q}. \tag{6.9}$$

Thus only the terms involving $\Phi^{\varepsilon_i}_{i,\bar\beta}$ are left to investigate.

To this end for each manifold W, point $x \in W$ and linear space F transversal to W define $\mathbf{P}_{F,W}(x) : \mathbb{R}^d \to \mathbb{R}^d$ to be the projection on F along $\mathcal{T}W$. Hence, $\mathbf{P}^2_{F,W} = \mathbf{P}_{F,W}$ and, for each $v \in \mathcal{T}_xW$, it holds $\mathbf{P}_{F,W}\, v = 0$. Finally define the other projector $Q_{F,W} := \mathbf{Id} - P_{F,W}$.

Let $E = \{0\} \times \mathbb{R}^{d_u}$ then, for $y \in W_i$,

$$\mathbf{P}_{E,W_i}(y) DT^{-n}(T^n y) = DT^{-n}(T^n y)\,\mathbf{P}_{DT^nE,W}(T^n y).$$

Since E is uniformly transversal to the stable cone, DT^n is expanding by at least $C\lambda^n$ along E, i.e., DT^{-n} is contracting by at least $C\lambda^{-n}$ in the direction DT^nE. Hence

$$|\,\mathbf{P}_{E,W_i}[DT^n]^{-1}|_{\mathcal{C}^0(W_i)} \le C\lambda^{-n}.$$

By uniform contraction of T^n along W_i, this implies that

$$|\,\mathbf{P}_{E,W_i}[DT^n]^{-1}|_{\mathcal{C}^q(W_i)} \le C\lambda^{-n}. \tag{6.10}$$

We are now ready to conclude our argument. The key fact is that one can write

$$\begin{aligned}([DT^n]^{-1})_{\bar\beta_1,\bar\alpha_1} &= (\mathbf{P}_{E,W_i}[DT^n]^{-1})_{\bar\beta_1,\bar\alpha_1} + (\mathbf{Q}_{E,W_i}[DT^n]^{-1})_{\bar\beta_1,\bar\alpha_1}\\ &= (\mathbf{P}_{E,W_i}[DT^n]^{-1})_{\bar\beta_1,\bar\alpha_1} + \sum_{\tau=1}^d (\mathbf{Q}_{E,W_i})_{\bar\beta_1,\tau}([DT^n]^{-1})_{\tau,\bar\alpha_1},\end{aligned}$$

which gives rise to a decomposition

$$\Phi^{\varepsilon_i}_{i,\bar\beta} = \Phi^{\varepsilon_i,1}_{i,\bar\beta} + \sum_{\tau=1}^d (\mathbf{Q}_{E,W_i})_{\bar\beta_1,\tau}\cdot \Phi^{\varepsilon_i}_{i,\tau,(\bar\beta_2,\dots,\bar\beta_p)}.$$

Note that we can write $\mathbf{Q}_{E,W_i} = \sum_{j=1}^d v^j \otimes w^j$ where the v^j are uniformly smooth vector fields on W_i. Hence,

$$\sum_{\tau=1}^d (\mathbf{Q}_{E,W_i})_{\bar\beta_1\tau}\cdot \Phi^{\varepsilon_i}_{i,\tau,(\bar\beta_2,\dots,\bar\beta_p)} = \sum_{j=1}^d v^j_{\bar\beta_1}\,\widetilde{\Phi}^{\varepsilon_i}_{i,(\bar\beta_2,\dots,\bar\beta_p)}.$$

On the other hand,

$$\sum_{\bar\beta\in\{1,\dots,d\}^p}\int_{W_i}\sum_{\tau=1}^{d}(\mathbf{Q}_{E,W_i})_{\bar\beta_1\tau}\,\Phi^{\varepsilon_i}_{i,\tau,(\bar\beta_2,\dots,\bar\beta_p)}\partial^\beta\widetilde h$$
$$=\sum_{j=1}^{d}\sum_{\bar\beta\in\{1,\dots,d\}^p}\int_{W_i}v^j_{\bar\beta_1}\,\widetilde\Phi^{\varepsilon_i}_{i,(\bar\beta_2,\dots,\bar\beta_p)}\partial^\beta\widetilde h$$
$$=\sum_{j=1}^{d}\sum_{\bar\beta\in\{1,\dots,d\}^{p-1}}\int_{W_i}\left[D(\partial^{\bar\beta_1}\dots\partial^{\bar\beta_{p-1}}\widetilde h)(v^j)\right]\cdot\widetilde\Phi^{\varepsilon_i}_{i,(\bar\beta_1,\dots,\bar\beta_{p-1})}$$
$$=\sum_{j=1}^{d}\sum_{\bar\beta\in\{1,\dots,d\}^{p-1}}\int_{W_i}\operatorname{div}(\widetilde\Phi^{\varepsilon_i}_{i,\bar\beta}\cdot v^j)\partial^\beta\widetilde h.$$

We have thus again an expression involving only $p-1$ derivatives of h against a test function in $\mathcal{C}^q$. Iterating the above procedure p times we finally have

$$\sum_{\bar\beta\in\{1,\dots,d\}^p}\int_{W_i}\Phi^{\varepsilon_i}_{i,\bar\beta}\partial^\beta\widetilde h=\sum_{\bar\beta\in\{1,\dots,d\}^p}\int_{W_i}\Phi^{\varepsilon_i,p}_{i,\bar\beta}\partial^\beta\widetilde h+\sum_{k=0}^{p-1}\sum_{\bar\beta\in\{1,\dots,d\}^k}\int_{W_i}\widetilde\Psi^k_{i,\bar\beta}\partial^\beta\widetilde h,\tag{6.11}$$

where $\widetilde\Psi^k_{i,\bar\beta}\in\mathcal{C}^q_0$ and

$$\Phi^{\varepsilon_i,p}_{i,\bar\beta}=\prod_{j=1}^{p}(\mathbf{P}_{E,W_i}[DT^n]^{-1})_{\bar\beta_j,\bar\alpha_j}\cdot|\det DT^n|^{-1}\cdot J_WT^n\cdot(\mathbb{A}_{\varepsilon_i}\varphi)\circ T^n\cdot\rho_i.\tag{6.12}$$

Equations (6.4), (6.8), (6.9), (6.11), (6.12) and (6.10) together with the same distortion arguments as in the proof of (2.2) imply immediately

$$\|\mathcal{L}^n h\|^-_{p,q}\le\bar A_{p,q}\max\{\lambda^{-pn},\nu^{(\zeta-\bar\zeta)pn}\}\|h\|_{p,q}+D_{p,q,n}\|h\|_{p-1,q}.\tag{6.13}$$

Finally, choose n_0 such that $\bar A_{p,q}^{\frac1p}\max\{\lambda^{-n_0},\nu^{n_0(\zeta-\bar\zeta)}\}<\sigma^{n_0}$. Iterating (6.13), for $p=1$ and remembering (2.2) yields

$$\|\mathcal{L}^n h\|_{1,q}\le C\sigma^n\|h\|_{1,q}+C\|h\|_{0,q}$$

which, in particular, implies $\|\mathcal{L}^n h\|_{1,q}\le C\|h\|_{1,q}$ for each $n\in\mathbb{N}$. The result follows then by induction over p.

7. General perturbation results

It is obvious from the previous discussion that all the results discussed so far – and in particular Lemmas 2.2 and 3.2 – hold not only for the map T, but also for any map in a $\mathcal{C}^{r+1}$ open neighborhood U of T, or for any composition of such maps. We will consider perturbations of T as described in Section 2, given by a probability measure μ on a space Ω and functions $g(\omega,\cdot)\in\mathcal{C}^q(X,\mathbb{R}_+)$, and we will assume that all the random diffeomorphisms T_ω we consider belong to the above set U. In this section, we will prove Theorem 2.6.

Lemma 7.1. *For any map $\widetilde{T} \in U$ holds*

$$\|\mathcal{L}_T h - \mathcal{L}_{\widetilde{T}} h\|_{p-1,q} \leq C d_{\mathcal{C}^{p+q}}(T, \widetilde{T}) \|h\|_{p,q}.$$

Proof. Let $\widetilde{W}$ be a full admissible leaf, given by an admissible graph $\chi \in \Xi_i$ defined on a ball $\overline{B}(x, A\delta)$. We will use Lemma 3.2 with $\gamma = \gamma_0 > 1$: there exists a finite number of γ-admissible graphs $\chi_1, \dots, \chi_\ell$, such that χ_j is defined on a ball $\overline{B}(x_j, A\gamma\delta) \subset (-\frac{2r_{i(j)}}{3\gamma}, \frac{2r_{i(j)}}{3\gamma})^{d_s}$ for some index $i(j)$, and such that the corresponding γ-admissible leaves cover $T^{-1}(W)$. Write ρ_j for the corresponding partition of unity.

Take $\widetilde{T} \in U$. The projection on the first d_s coordinates of

$$\psi_{i(j)}^{-1} \circ \widetilde{T}^{-1} \circ T \circ \psi_{i(j)} \circ (\mathbf{Id}, \chi_j)(\overline{B}(x_j, \gamma A\delta))$$

contains the ball $\overline{B}(x_j, A\delta)$ if U is small enough. Hence, it is possible to define a graph $\widetilde{\chi}_j$ on $\overline{B}(x_j, A\delta)$ whose image is contained in $\psi_{i(j)}^{-1}(\widetilde{T}^{-1}(\widetilde{W}))$. Moreover, $\widetilde{T}^{-1}(W)$ is covered by the restrictions of these graphs to the balls $\overline{B}(x_j, \delta)$ if U is small enough. Finally, $\|\chi_j - \widetilde{\chi}_j\|_{\mathcal{C}^{p+q}(\overline{B}(x_j, A\delta))} \leq C d_{\mathcal{C}^{p+q}}(T, \widetilde{T})$.

Let $\varphi \in \mathcal{C}_0^q(B(x,\delta), \mathbb{R})$, $|\alpha| \leq p-1$ and set $\widetilde{h}_j := h \circ \psi_{i(j)}$. Then

$$\int_{B(x,\delta)} \partial^\alpha((\mathcal{L}_T h) \circ \psi_i)(\mathbf{Id}, \chi) \cdot \varphi = \sum_{|\beta|<p} \sum_{j=1}^{\ell} \int_{B(x_j,\delta)} \partial^\beta \widetilde{h}_j(\mathbf{Id}, \chi_j) F_{\beta,T,j} \rho_j \tag{7.1}$$

for some functions $F_{\beta,T,j}$. The same equation holds for $\mathcal{L}_{\widetilde{T}} h$, with χ_j replaced by $\widetilde{\chi}_j$ and $F_{\beta,T,j}$ replaced by a function $F_{\beta,\widetilde{T},j}$ satisfying $|F_{\beta,T,j} - F_{\beta,\widetilde{T},j}|_{\mathcal{C}^q} \leq C d_{\mathcal{C}^{p+q}}(T, \widetilde{T})$.

For $1 \leq j \leq \ell$ and $|\beta| \leq p-1$, we have

$$\begin{aligned} \left| \int_{B(x_j,\delta)} \partial^\beta \widetilde{h}_j(\mathbf{Id}, \chi_j)(F_{\beta,T,j} - F_{\beta,\widetilde{T},j})\rho_j \right| &\leq C \|h\|_{p-1,q} |F_{\beta,T,j} - F_{\beta,\widetilde{T},j}|_{\mathcal{C}^q} \\ &\leq C \|h\|_{p,q} d_{\mathcal{C}^{p+q}}(T, \widetilde{T}) \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} &\left| \int_{B(x_j,\delta)} \partial^\beta \widetilde{h}_j(\mathbf{Id}, \chi_j) F_{\beta,\widetilde{T},j}\rho_j - \int_{B(x_j,\delta)} \partial^\beta \widetilde{h}_j(\mathbf{Id}, \widetilde{\chi}_j) F_{\beta,\widetilde{T},j}\rho_j \right| \\ &\quad = \left| \int_{t=0}^1 \int_{B(x_j,\delta)} D(\partial^\beta \widetilde{h}_j)(\mathbf{Id}, \widetilde{\chi}_j + t(\chi_j - \widetilde{\chi}_j)) \cdot (0, \chi_j - \widetilde{\chi}_j) F_{\beta,\widetilde{T},j}\rho_j \right|. \end{aligned} \tag{7.3}$$

When t is fixed, each integral is an integral along an admissible graph, whence it is at most

$$C\|h\|_{p,q} |\chi_j - \widetilde{\chi}_j|_{\mathcal{C}^q} \leq C\|h\|_{p,q} d_{\mathcal{C}^{p+q}}(T, \widetilde{T}).$$

Integrating over t, we get (7.3) $\leq C\|h\|_{p,q} d_{\mathcal{C}^{p+q}}(T, \widetilde{T})$. Combining this inequality with Equations (7.2) and (7.1) yields the conclusion of the lemma. □

This lemma readily implies that, for any operator $\mathcal{L}_{\mu,g}$ satisfying the previous assumptions,

$$\|\mathcal{L}_{\mu,g} h - \mathcal{L}_T h\|_{p-1,q} \leq C\Delta(\mu, g)\|h\|_{p,q}, \tag{7.4}$$

where $\Delta(\mu, g)$ is defined in (2.4).

When $g(\omega, x) = 1$, Lemma 2.2 applied to compositions of operators of the form $\mathcal{L}_{T_\omega}$ immediately implies that

$$\|\mathcal{L}^n_{\mu,g} h\|_{p,q} \leq C\sigma^{pn}\|h\|_{p,q} + C\|h\|_{p-1,q}, \tag{7.5}$$

which is sufficient to obtain spectral stability, by [10]. In particular this suffices to prove Theorem 2.6 for deterministic perturbations.

However, in the general case, further arguments are required to obtain a uniform Lasota-Yorke type inequality:

Lemma 7.2. *For any $M > 1$ and any perturbation (μ, g) of T as above, there exists a constant $C = C\big(M, \int |g(\omega,\cdot)|_{\mathcal{C}^q(X,\mathbb{R})}\,\mathrm{d}\mu(\omega)\big)$ such that, for any $n \in \mathbb{N}$,*

$$\|\mathcal{L}^n_{\mu,g} h\|_{p,q} \leq C\sigma^{pn}\|h\|_{p,q} + CM^n\|h\|_{p-1,q}. \tag{7.6}$$

Proof. We will prove that

$$\|\mathcal{L}^n_{\mu,g} h\|_{0,q} \leq CM^n\|h\|_{0,q},$$

by adapting the proof of equation (2.2). The proof of (7.6) in the general case is similar, using the same ideas to extend the proof of (2.3). The only problem comes from the functions $g(\omega_i, x)$, and a distortion argument will show that their contribution is small. Let $c = \int |g(\omega,\cdot)|_{\mathcal{C}^q}\,\mathrm{d}\mu(\omega)$. Fix parameters $\overline{\omega}_n := (\omega_1, \dots, \omega_n) \in \Omega^n$. Fix also $\varepsilon > 0$. Write $\widetilde{g}_i(x) = g(\omega_i, x) + \varepsilon\frac{|g(\omega_i,\cdot)|_{\mathcal{C}^q}}{c}$.

We will write $T_{\overline{\omega}_i} = T_{\omega_i} \circ \dots \circ T_{\omega_1}$. Let W be an admissible leaf, $W_1, \dots, W_\ell$ a covering of $T^{-1}_{\overline{\omega}_n} W$ by admissible leaves and $\rho_1, \dots, \rho_\ell$ a corresponding partition of unity, as in the proof of (2.2). Let also φ be a test function. Then

$$\begin{aligned}
&\int_W \left(\prod_{i=1}^n g(\omega_i, T^{-1}_{\omega_i} \circ \dots \circ T^{-1}_{\omega_n} x)\right) \mathcal{L}_{T_{\overline{\omega}_n}} h(x)\varphi(x) \\
&= \sum_{k=1}^\ell \int_{W_k} |\det DT_{\overline{\omega}_n}|^{-1} h(x) \left(\prod_{i=1}^n g(\omega_i, T_{\overline{\omega}_{i-1}} x)\right) \varphi \circ T_{\overline{\omega}_n}(x) J_W T_{\overline{\omega}_n}(x)\rho_k(x).
\end{aligned}$$

Since W_k is admissible, the last integral can be estimated using the $\mathcal{C}^q$ norm of $\prod_{i=1}^n g(\omega_i, T_{\overline{\omega}_{i-1}} x)$. Since $|g(\omega_i,\cdot)|_{\mathcal{C}^q} \leq \frac{c}{\varepsilon}\widetilde{g}_i(x)$ by definition of $\widetilde{g}_i$, Lemma 6.1 shows that this norm is bounded by $C\exp(C\frac{c}{\varepsilon})\prod_{i=1}^n \widetilde{g}_i(T_{\overline{\omega}_{i-1}} x)$ for any $x \in W_k$. Combining this estimate with the distortion arguments of the proof of (2.2), we obtain

$$\begin{aligned}
&\left|\int_W \left(\prod_{i=1}^n g(\omega_i, T^{-1}_{\omega_i} \cdots T^{-1}_{\omega_n} x)\right) \mathcal{L}_{T_{\omega_n}} \cdots \mathcal{L}_{T_{\omega_1}} h(x)\varphi(x)\right| \\
&\qquad \leq C\exp\left(C\frac{c}{\varepsilon}\right)\|h\|_{0,q} \int_{\widetilde{W}} |\det DT_{\overline{\omega}_n}|^{-1} \left(\prod_{i=1}^n \widetilde{g}_i(T^{-1}_{\omega_i} \cdots T^{-1}_{\omega_n} x)\right).
\end{aligned}$$

To estimate this last integral, consider the thickening $Z = \bigcup_{x\in\widetilde{W}} W^u_\rho(x)$, where $W^u_\rho(x)$ is the local unstable manifold of $T_{\overline{\omega}_n}$ through x. Along this manifold, the function $\prod_{i=1}^n \widetilde{g}_i(T^{-1}_{\omega_i} \cdots T^{-1}_{\omega_n} x)$ changes of a multiplicative factor at most $C\exp\left(C\frac{c}{\varepsilon}\right)$, again by Lemma 6.1. Hence,

$$\begin{aligned}
\int_{\widetilde{W}} |\det DT_{\overline{\omega}_n}|^{-1} &\left(\prod_{i=1}^n \widetilde{g}_i(T_{\omega_i}^{-1}\cdots T_{\omega_n}^{-1}x)\right) \\
&\leq C\exp\left(C\frac{c}{\varepsilon}\right)\rho^{-d_u}\int_Z |\det DT_{\overline{\omega}_n}|^{-1}\left(\prod_{i=1}^n \widetilde{g}_i(T_{\omega_i}^{-1}\cdots T_{\omega_n}^{-1}x)\right) \\
&= C\exp\left(C\frac{c}{\varepsilon}\right)\int_{T^{-n}(Z)}\prod_{i=1}^n \widetilde{g}_i(T_{\omega_{i-1}}\cdots T_{\omega_1}x) \\
&\leq C\exp\left(C\frac{c}{\varepsilon}\right)\int_X\prod_{i=1}^n \widetilde{g}_i(T_{\omega_{i-1}}\cdots T_{\omega_1}x).
\end{aligned}$$

Integrating over all possible values of ω, we finally obtain

$$\begin{aligned}
\|\mathcal{L}_{\mu,g}^n h\|_{0,q} &\leq C\|h\|_{0,q}\exp\left(C\frac{c}{\varepsilon}\right) \\
&\cdot\int_X\int_{\Omega^n}\prod_{i=1}^n\left(g(\omega_i, T_{\omega_{i-1}}\cdots T_{\omega_1}x) + \varepsilon\frac{|g(\omega_i,\cdot)|_{\mathcal{C}^q}}{c}\right)\mathrm{d}\mu(\omega_1)\ldots\mathrm{d}\mu(\omega_n).
\end{aligned}$$

Integrating over ω_n gives a factor $1+\varepsilon$, since $\int g(\omega_n, y)\,\mathrm{d}\mu(\omega_n) = 1$ for any y. We can then proceed to integrate over $\omega_{n-1}, \omega_{n-2}, \ldots$, and get

$$\|\mathcal{L}_{\mu,g}^n h\|_{0,q} \leq C\|h\|_{0,q}\exp\left(C\frac{c}{\varepsilon}\right)(1+\varepsilon)^n. \qquad \square$$

The inequalities (7.4) and (7.6) suffice to apply [10], which implies Theorem 2.6.

8. An abstract perturbation theorem

Let $\mathcal{B}^0 \supset \cdots \supset \mathcal{B}^s$ be Banach spaces, $0 \in I \subset \mathbb{R}$ a fixed open interval, and $\{\mathcal{L}_t\}_{t\in I}$ a family of operators acting on each of the above Banach spaces. Moreover, assume that

$$\exists M > 0, \forall t \in I, \quad \|\mathcal{L}_t^n f\|_{\mathcal{B}^0} \leq CM^n \|f\|_{\mathcal{B}^0} \tag{8.1}$$

and

$$\exists\, \alpha < M, \ \ \forall t \in I, \quad \|\mathcal{L}_t^n f\|_{\mathcal{B}^1} \leq C\alpha^n \|f\|_{\mathcal{B}^1} + CM^n \|f\|_{\mathcal{B}^0}\,. \tag{8.2}$$

Assume also that there exist operators $Q_1, \ldots, Q_{s-1}$ satisfying the following properties:

$$\forall j = 1, \ldots, s-1, \ \ \forall i \in [j, s], \quad \|Q_j\|_{\mathcal{B}^i \to \mathcal{B}^{i-j}} \leq C \tag{8.3}$$

and, setting $\Delta_0(t) := \mathcal{L}_t$ and $\Delta_j(t) := \mathcal{L}_t - \mathcal{L}_0 - \sum_{k=1}^{j-1} t^k Q_k$ for $j \geq 1$,

$$\forall t \in I, \ \ \forall j = 0, \ldots s, \ \ \forall i \in [j, s], \quad \|\Delta_j(t)\|_{\mathcal{B}^i \to \mathcal{B}^{i-j}} \leq Ct^j. \text{[11]} \tag{8.4}$$

These assumptions mean that $\mathcal{L}_t$ is a continuous, and even a $\mathcal{C}^s$ perturbation of $\mathcal{L}_0$, but the differentials take their values in weaker spaces. This setting can be applied to the case of smooth expanding maps (see [16] for the argument limited to the case $s = 2$) and to the transfer operator associated to a perturbation of a smooth Anosov map as we will see in section 9.

[11] In fact, this property is used in the proof only for $i = s$, and for $(i, j) = (1, 1)$.

For $\varrho > \alpha$ and $\delta > 0$, denote by $V_{\delta,\varrho}$ the set of complex numbers z such that $|z| \geq \varrho$ and, for all $1 \leq k \leq s$, the distance from z to the spectrum of $\mathcal{L}_0$ acting on $\mathcal{B}^k$ is $\geq \delta$.

Theorem 8.1. *Given a family of operators $\{\mathcal{L}_t\}_{t\in I}$ satisfying conditions* (8.1), (8.2)*,* (8.3) *and* (8.4) *and setting*

$$R_s(t) := \sum_{k=0}^{s-1} t^k \sum_{\ell_1+\cdots+\ell_j=k} (z-\mathcal{L}_0)^{-1}Q_{\ell_1}(z-\mathcal{L}_0)^{-1}\dots(z-\mathcal{L}_0)^{-1}Q_{\ell_j}(z-\mathcal{L}_0)^{-1},$$

for all $z \in V_{\delta,\varrho}$ and t small enough holds true

$$\left\|(z-\mathcal{L}_t)^{-1} - R_s(t)\right\|_{\mathcal{B}^s\to\mathcal{B}^0} \leq C|t|^{s-1+\eta}$$

where $\eta = \frac{\log(\varrho/\alpha)}{\log(M/\alpha)}$.

Hence, the resolvent $(z-\mathcal{L}_t)^{-1}$ depends on t in a $\mathcal{C}^{s-1+\eta}$ way when viewed as an operator from $\mathcal{B}^s$ to $\mathcal{B}^0$.

Notice that one of the results of [10] in the present setting reads

$$\left\|(z-\mathcal{L}_t)^{-1} - (z-\mathcal{L}_0)^{-1}\right\|_{\mathcal{B}^1\to\mathcal{B}^0} \leq C|t|^{\eta}. \tag{8.5}$$

Accordingly, one has Theorem 8.1 in the case $s = 1$ where no assumption is made on the existence of the operators Q_j.

Proof of Theorem 8.1. Iterating the equation

$$(z-\mathcal{L}_t)^{-1} = (z-\mathcal{L}_0)^{-1} + (z-\mathcal{L}_t)^{-1}(\mathcal{L}_t-\mathcal{L}_0)(z-\mathcal{L}_0)^{-1},$$

and setting $A(z,t) := (\mathcal{L}_t-\mathcal{L}_0)(z-\mathcal{L}_0)^{-1}$, it follows

$$\begin{aligned}(z-\mathcal{L}_t)^{-1} &= \sum_{j=0}^{s-2}(z-\mathcal{L}_0)^{-1}A(z,t)^j + (z-\mathcal{L}_t)^{-1}A(z,t)^{s-1}\\ &= \sum_{j=0}^{s-1}(z-\mathcal{L}_0)^{-1}A(z,t)^j + \left[(z-\mathcal{L}_t)^{-1}-(z-\mathcal{L}_0)^{-1}\right]A(z,t)^{s-1}.\end{aligned} \tag{8.6}$$

Next, for each $j \in \mathbb{N}$ and $a \leq s$, using (8.4), we can write

$$A(z,t)^j = \Delta_a(t)(z-\mathcal{L}_0)^{-1}A(z,t)^{j-1} + \sum_{\ell=1}^{a-1} t^\ell Q_\ell(z-\mathcal{L}_0)^{-1}A(z,t)^{j-1}. \tag{8.7}$$

For $\epsilon = 0$ or 1, we can then prove by induction the formula, for all $1 \leq m \leq j$

$$\begin{aligned}A(z,t)^j = \sum_{k=1}^{m} &\sum_{\substack{\ell_1+\cdots+\ell_{k-1}<s-\epsilon\\ \ell_i>0}} t^{\ell_1+\cdots+\ell_{k-1}}Q_{\ell_1}(z-\mathcal{L}_0)^{-1}\cdots\\ &\cdots Q_{\ell_{k-1}}(z-\mathcal{L}_0)^{-1}\Delta_{s-\epsilon-\ell_1-\cdots-\ell_{k-1}}(t)(z-\mathcal{L}_0)^{-1}A(z,t)^{j-k}\\ +&\sum_{\substack{\ell_1+\cdots+\ell_m<s-\epsilon\\ \ell_i>0}} t^{\ell_1+\cdots+\ell_m}Q_{\ell_1}(z-\mathcal{L}_0)^{-1}\cdots Q_{\ell_m}(z-\mathcal{L}_0)^{-1}A(z,t)^{j-m}\end{aligned} \tag{8.8}$$

In fact, for $m=1$ the above formula is just (8.7) for $a=s-\epsilon$. Next, suppose (8.8) true for some m, then by (8.7) it follows

$$Q_{\ell_1}(z-\mathcal{L}_0)^{-1}\cdots Q_{\ell_m}(z-\mathcal{L}_0)^{-1}A(z,t)^{j-m}=Q_{\ell_1}(z-\mathcal{L}_0)^{-1}\cdots Q_{\ell_m}(z-\mathcal{L}_0)^{-1}$$
$$\times\Bigg[\Delta_{s-\epsilon-\sum_{i=1}^m \ell_i}(t)(z-\mathcal{L}_0)^{-1}A(z,t)^{j-m-1}$$
$$+\sum_{\ell_{m+1}=1}^{s-\epsilon-\sum_{i=1}^m \ell_i-1} t^{\ell_{m+1}}Q_{\ell_{m+1}}(z-\mathcal{L}_0)^{-1}A(z,t)^{j-m-1}\Bigg].$$

Substituting the above formula in (8.8) we have the formula for $m+1$.

We can now easily estimate the terms in which a Δ_i appears. In fact, $\|A(z,t)\|_{\mathcal{B}^s}\le C$, and (8.3) and (8.4) readily imply that

$$\|Q_{\ell_1}(z-\mathcal{L}_0)^{-1}\cdots Q_{\ell_k}(z-\mathcal{L}_0)^{-1}\Delta_{s-\epsilon-\sum_{j=1}^k \ell_j}(t)(z-\mathcal{L}_0)^{-1}\|_{\mathcal{B}^s\to\mathcal{B}^\epsilon}\le C|t|^{s-\epsilon-\sum_{j=1}^k \ell_j}.$$

The theorem follows then from (8.6), using (8.8) with $\epsilon=0$ and $m=j$ to estimate the terms $(z-\mathcal{L}_0)^{-1}A(z,t)^j$, and (8.8) with $\epsilon=1$ and $m=s-1$ together with (8.5) to show that $\left\|\left[(z-\mathcal{L}_t)^{-1}-(z-\mathcal{L}_0)^{-1}\right]A(z,t)^{s-1}\right\|_{\mathcal{B}^s\to\mathcal{B}^0}\le C|t|^{s-1+\eta}$. □

9. Differentiability results

To simplify the exposition, in this section we will abuse notations and systematically ignore the coordinate charts of the manifold X. As we have carefully discussed in the previous sections, this does not create any problem.

To start with, let us assume that δ_* is so small that $\{T_t : t\in[-\delta_*,\delta_*]\}$ is contained in the neighborhood U of T_0 in which the estimates of the Lasota-Yorke inequality hold uniformly.

By Taylor formula we have, for each $f\in\mathcal{C}^r$ and $s\le r$,

$$\mathcal{L}_{T_t}f=\sum_{k=0}^{s-1}\frac{1}{k!}\frac{d^k}{dt^k}\mathcal{L}_{T_t}f\big|_{t=0}+\int_0^t dt_1\cdots\int_0^{t_{s-1}}dt_s\left(\frac{d^s}{dt^s}\mathcal{L}_t f\right)(t_s). \tag{9.1}$$

Next, for $1\le k\le s-1$,

$$\frac{d^k}{dt^k}\mathcal{L}_{T_t}f(x)\big|_{t=0}=\sum_{\ell=1}^{k}\sum_{|\alpha|=\ell}J_\alpha(k,t,x)(\mathcal{L}_{T_t}\partial^\alpha f)(x)\big|_{t=0}=:k!Q_kf(x) \tag{9.2}$$

for appropriate functions $J_\alpha(k,t,\cdot)\in\mathcal{C}^r(X,\mathbb{R})$.

We are now ready to check the applicability of Theorem 8.1. First of all let us define $\mathcal{B}^i:=\mathcal{B}^{p_0+i-1,q}$. Conditions (8.1) and (8.2) hold with $\alpha=\sigma^{p_0}$ by our choice of δ_*, and $M=1$ by (7.5). From the definition of the norms it follows straightforwardly that, for each multi-index α with $|\alpha|=j$, ∂^α is a bounded operator from $\mathcal{B}^{p,q}$ to $\mathcal{B}^{p-j,q}$. From this condition (8.3) readily follows. By (9.1) and (9.2), it follows that Δ_k is given by the last term in (9.1). By the previous arguments

$$\left\|\frac{d^k}{dt^k}\mathcal{L}_{T_t}(f)\right\|_{p,q}\le C\|f\|_{p+k,q},$$

which obviously implies condition (8.4).

References

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature.* Proceedings of the Steklov Institue of Mathematics **90** (1967) 1–209.

[2] D.V. Anosov, Ya.G. Sinai, *Certain smooth ergodic systems.* (Russian) Uspehi Mat. Nauk 22 1967 no. 5 (137), 107–172; English transl., Russian mathematical Surveys **22** (1967), no. 5, 103–167.

[3] V. Baladi, *Positive transfer operators and decay of correlations*, Advanced Series in Nonlinear Dynamics, **16**, World Scientific (2000).

[4] M. Blank, G. Keller, C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity, **15** (2001), no. 6, 1905–1973.

[5] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes in Mathematics, Vol. **470**. Springer-Verlag, Berlin-New York (1975).

[6] D. Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems*, Invent. Math., **155** (2004), no. 2, 389–449.

[7] D. Fried, *Meromorphic zeta functions for analytic flows.* Comm. Math. Phys. **174** (1995), no. 1, 161–190.

[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren der Mathematischen Wissenschaften, **256**, Springer (1990).

[9] G. Keller, *On the rate of convergence to equilibrium in one-dimensional systems.* Comm. Math. Phys. **96** (1984), no. 2, 181–193.

[10] G. Keller, C. Liverani, *Stability of the spectrum for transfer operators*, Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, (4) **XXVIII** (1999), 141-152.

[11] A.Yu. Kitaev, *Fredholm determinants for hyperbolic diffeomorphisms of finite smoothness*, Nonlinearity, **12** (1999), no. 1, 141–179.

[12] A. Lasota and J.A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1963), 481–488.

[13] C. Liverani, *Decay of Correlations*, Annals of Mathematics **142** (1995), 239–301.

[14] C. Liverani, *Invariant measures and their properties. A functional analytic point of view*, Dynamical Systems. Part II: Topological Geometrical and Ergodic Properties of Dynamics. Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa. Centro di Ricerca Matematica "Ennio De Giorgi" : Proceedings. Published by the Scuola Normale Superiore in Pisa (2004).

[15] C. Liverani, *Rigorous numerical investigation of the statistical properties of piecewise expanding maps–A feasibility study*, Nonlinearity **14** (2001), 463–490.

[16] C. Liverani, *On Contact Anosov flows*, to appear in Annals of Mathematics.

[17] J.N. Mather, *Characterization of Anosov diffeomorphisms*, Indag. Math **30** (1968), 479–483.

[18] M. Pollicott, *Stability of mixing rates for Axiom A attractors.* Nonlinearity **16** (2003), 567–578.

[19] D. Ruelle, *A measure associated with axiom-A attractors.* Amer. J. Math. **98** (1976), 619–654.

[20] D. Ruelle, *Differentiation of SRB states*, Comm. Math. Phys. **187** (1997), 227–241.

[21] D. Ruelle, *Correction and complements: "Differentiation of SRB states" [Comm. Math. Phys. 187 (1997), no. 1, 227–241].* Comm. Math. Phys. **234** (2003), no. 1, 185–190.

[22] H.H. Rugh, *The correlation spectrum for hyperbolic analytic maps.* Nonlinearity **5** (1992), no. 6, 1237–1263 .

[23] H.H. Rugh, *Fredholm determinants for real-analytic hyperbolic diffeomorphisms of surfaces.* XIth International Congress of Mathematical Physics (Paris, 1994), 297–303, Internat. Press, Cambridge, MA, 1995.

[24] H.H. Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems.* Ergodic Theory Dynam. Systems **16** (1996), no. 4, 805–819.

[25] Ya.G. Sinai, *Gibbs measures in ergodic theory.* (Russian) Uspehi Mat. Nauk 27 (1972), no. 4 (166), 21–64; English transl. Russian mathematical Surveys **27** (1972), no.4 , 21–70.

Sébastien Gouëzel, Département de mathématiques et applications, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris, France.

E-mail address: sebastien.gouezel@ens.fr

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: liverani@mat.uniroma2.it