

# CUT-AND-PASTE ON FOLIATED BUNDLES

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ABSTRACT. We discuss the behaviour of the signature index class of closed foliated bundles under the operation of cutting and pasting. Along the way we establish several index theoretic results: we define Atiyah-Patodi-Singer ( $\equiv$  APS) index classes for Dirac-type operators on foliated bundles with boundary; we prove a relative index theorem for the difference of two APS-index classes associated to different boundary conditions; we prove a gluing formula on closed foliated bundles that are the union of two foliated bundles with boundary; we establish a variational formula for APS-index classes of a 1-parameter family of Dirac-type operators on foliated bundles (this formula involves the noncommutative spectral flow of the boundary family). All these formulas take place in the  $K$ -theory of a suitable cross-product algebra. We then apply these results in order to find sufficient conditions ensuring the equality of the signature index classes of two cut-and-paste equivalent foliated bundles. We give applications to the question of when the Baum-Connes higher signatures of closed foliated bundles are cut-and-paste invariant.

## CONTENTS

1. Introduction	2
2. Spectral flow	6
2.1. Spectral flow through spectral sections.	6
2.2. Index and spectral flow	7
2.3. The gluing formula.	8
2.4. An analytic proof of the cut and paste invariance of the signature.	8
3. Foliated bundles, index classes and the noncommutative spectral flow	10
3.1. Preliminaries: K-Theory of $C^*$ -algebras and Fredholm operators.	10
3.2. Foliated bundles.	12
3.3. The $C^*$ -algebra $C(T) \rtimes_r \Gamma$	13
3.4. $C(T) \rtimes_r \Gamma$ -Hilbert modules.	14
3.5. Index classes on closed foliated bundles	14
3.6. Spectral sections for Dirac operators on foliated bundles.	15
3.7. Difference class associated to two spectral sections	17
3.8. The noncommutative spectral flow on foliated bundles.	19
3.9. Trivializing perturbations	20
4. Index classes on foliated bundles with boundary	21

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4.1. Preliminaries: numerical indeces on manifolds with boundary.	21
4.2. Foliated bundles with boundary.	22
4.3. Cobordism invariance	24
4.4. Dirac $b$ -index classes on foliated bundles with boundary	26
4.5. Dirac $APS$ -index classes on foliated bundles with boundary	26
4.6. $APS$ -index theory via the $b$ -calculus: an overview of the literature.	27
5. Fundamental properties of $b$ -index classes	28
5.1. The relative index theorem.	28
5.2. The gluing formula.	32
5.3. The variational formula.	32
6. On the cut-and-paste invariance of the signature index class	33
6.1. Cut-and-paste on foliated bundles.	33
6.2. The defect formula.	33
6.3. Vanishing spectral flow.	34
7. Geometric applications	35
7.1. Baum-Connes higher signatures.	35
7.2. Cut-and-paste invariance: the cyclic cohomology approach.	36
7.3. Isometric actions.	37
7.4. The Godbillon-Vey signature $\sigma_{GV}$ .	38
7.5. On the cut-and-paste invariance of $\sigma_{GV}$ .	38
7.6. Cut-and-paste invariance: the Baum-Connes approach.	39
References	41

## 1. INTRODUCTION

We recall that two oriented manifolds  $M_1, M_2$  are cut-and-paste equivalent if

$$(1.1) \quad M_1 = M_+ \cup_{(F, \phi_1)} M_-, \quad M_2 = M_+ \cup_{(F, \phi_2)} M_-$$

with  $\partial M_+ = F = -\partial M_-$  and  $\phi_j \in \text{Diffeo}^+(F)$ . In other words,  $M_1$  and  $M_2$  are obtained by gluing two manifolds with boundary but the gluing diffeomorphisms are different.

The signature of a manifold is a cut-and-paste invariant:  $\sigma(M_1) = \sigma(M_2)$  for  $M_1, M_2$  as above. An *analytic* proof of this fact is given in the book [8] of Booss-Bavnbek and Wojciechowski. The argument given there is a consequence of a more general formula concerning the numerical indeces of two Dirac-type operators obtained one from the other by a cut-and-paste construction; the formula expresses the difference of the numerical indeces in terms of the spectral flow of a suitable 1-parameter family of operators on  $F$ . For the particular case of the signature operator this spectral flow turns out to be zero, as it is simply the spectral flow of a 1-parameter family  $\{D_F(\theta)\}_{\theta \in S^1}$  of odd signature operators on  $F$  parametrized by a path of metrics. In this vanishing result the cohomological significance of the zero-eigenvalue has been used. Since the signature of a manifold is equal to the index of the signature-operator, we obtain finally

$$(1.2) \quad \sigma(M_2) - \sigma(M_1) = \text{ind}D_2 - \text{ind}D_1 = \text{sf}(\{D_F(\theta)\}_{\theta \in S^1}) = 0$$

which is what we claimed.

In this paper we shall investigate to what extent the cut-and-paste invariance of the numerical index of the signature operator can be generalized to the signature index class of *foliated bundles*, i.e.  $\Gamma$ -equivariant fibrations  $\widehat{M} \rightarrow T$  with  $T$  a manifold on which  $\Gamma$  acts, and  $\widehat{M}$  a manifold on which  $\Gamma$ -acts freely properly and cocompactly; thus the quotient  $\widehat{M}/\Gamma := M$  is a smooth manifold and  $\Gamma \rightarrow \widehat{M} \rightarrow M$  is a  $\Gamma$ -Galois covering.

We shall be therefore interested in index theory on foliated bundles, both in the closed case and in the case where a boundary is present. The relevant index classes, for Dirac type operators, will live in  $K_0(C(T) \rtimes_r \Gamma)$  if the fibers of the  $\Gamma$ -equivariant fibration  $\widehat{M} \rightarrow T$  are even-dimensional and in  $K_1(C(T) \rtimes_r \Gamma)$  if the fibers are odd-dimensional;  $C(T) \rtimes \Gamma$  denotes the reduced cross-product algebra. Notice that the manifold  $\widehat{M}/\Gamma$  is foliated by the images of the fibers of the fibration under the projection map  $\widehat{M} \rightarrow \widehat{M}/\Gamma$ . These foliations can be quite interesting; in fact it is well known that one can get any type (I, II, III) of foliation for suitable choices of  $\Gamma$ -equivariant fibrations. Index theory on foliated bundles is a particular but important case of the general foliation-index-theory developed by Connes, see [10], and Connes-Skandalis [11].

Notice that if  $T = \text{point}$  and  $\Gamma = \{1\}$  then we simply have a compact manifold  $M$ . Moreover  $C(T) \rtimes_r \Gamma = \mathbb{C}$  and (in the even dimensional case) the index class is nothing but the numeric index of the operator under the isomorphism  $K_0(\mathbb{C}) = \mathbb{Z}$ . If  $\Gamma = \{1\}$  we simply have a fibration,  $C(T) \rtimes_r \Gamma = C(T)$  and the index class reduces to the Atiyah-Singer family index in  $K_*(C(T)) = K^*(T)$ . Finally, if  $T = \text{point}$  then we have a Galois covering,  $C(T) \rtimes_r \Gamma = C_r^*\Gamma$ , the reduced group  $C^*$ -algebra associated to  $\Gamma$ , and the index class, which now lives in  $K_*(C_r^*\Gamma)$ , is nothing but the Mishchenko-Fomenko index class associated to the Dirac operator twisted by the canonical flat line bundle of the covering.

In previous work of ours, with collaborators, we investigated the cut-and-paste invariance of the signature index class in the case of Galois  $\Gamma$ -coverings  $\Gamma \rightarrow \widetilde{M} \rightarrow M$ , thus solving (at least partially) a problem raised by Lott and also by Weinberger [39]. See Leichtnam-Lott-Piazza [28] for the first positive results in this direction and then Leichtnam-Lueck-Kreck [29] and Leichtnam-Piazza [34]. It was explained in [34] that for Galois  $\Gamma$ -coverings the signature index class, in  $K_*(C_r^*\Gamma)$ , is not cut-and-paste invariant: one shows that the difference of signature index classes for two cut-and-paste equivalent coverings

$$(1.3) \quad \text{Ind}(\mathcal{D}_{M_2}^{\text{sign}}) - \text{Ind}(\mathcal{D}_{M_1}^{\text{sign}})$$

is equal to a *higher* spectral flow, in  $K_*(C_r^*\Gamma)$ , for an  $S^1$ -family of  $C_r^*\Gamma$ -linear signature operators on the cutting hypersurface  $F$ . This formula is the consequence of

- a gluing formula for the index class of a closed Galois covering which is the union of two Galois coverings with boundary;
- a variational formula for the index classes associated to a path of Dirac operators on a Galois covering with boundary

The definition of higher spectral flow was given by Dai and Zhang [13] for a path of families of Dirac operators parametrized by a compact space  $T$ , i.e. for a path of  $C(T)$ -linear operators; this definition is based on the notion of spectral section, given by Melrose and Piazza in [45]. The papers [31] and [34] extend the results of Melrose-Piazza and Dai-Zhang from the family-case, i.e.  $C(T)$ -linear operators, to the Galois-coverings case, i.e.

$C_r^*(\Gamma)$ -linear operators. This step should be thought of as the passage from a commutative to a non-commutative context.

In contrast with the numeric case explained above, the higher spectral flow appearing in formula (1.3) will *not* be equal to zero, in general. It is however possible to give sufficient conditions on the cutting hypersurface  $F$  ensuring the vanishing of this higher spectral flow and therefore the equality of the two signature index classes. This hypothesis comes from Lott's paper [38]. If, in addition, the group  $\Gamma$  satisfies the Strong Novikov Conjecture (i.e. the rational injectivity of the assembly map), then the equality of the index classes implies the equality of the Novikov higher signatures<sup>1</sup>. These ideas are now explained in the survey of Leichtnam-Piazza [36].

Summarizing:

- suitable conditions on the cutting hypersurface  $F$  ensures that the signature index class on Galois  $\Gamma$ -coverings is a cut-and-paste invariant;
- further conditions on the group  $\Gamma$  allow to deduce the cut-and-paste invariance of all Novikov higher signatures from the cut-and-paste invariance of the signature index classes

In the present paper we wish to follow the above line of reasoning for the more general case of foliated bundles. The specific problems we wish to solve are the following:

- give sufficient conditions ensuring that the signature index class, in the group  $K_*(C(T) \rtimes_r \Gamma)$ , is a cut-and-paste invariant;
- find additional conditions on  $\Gamma$  and its action on  $T$  ensuring that the Baum-Connes higher signatures (a generalization to foliated bundles of the Novikov higher signatures) are cut-and-paste invariant.

In order to solve the first problem we will need to develop a general Atiyah-Patodi-Singer index theory on foliated bundles. Some of our arguments will be easy extensions of the Galois coverings case and we will be quite brief in such cases; other arguments will be more involved and we shall explain them in detail. We shall use the  $\Gamma$ -equivariant  $b$ -pseudodifferential calculus on foliated bundles with boundary developed in [35] where a Atiyah-Patodi-Singer index theory was developed under an invertibility assumption on the boundary operator.

**The paper is structured as follows.** In **Section 2** we recall the arguments leading to the cut-and-paste invariance of the *numeric* index of the signature operator: conceptually this is the model case that will be extended to our more general situation. In **Section 3** we begin by recalling the definition of index class associated to a  $\Gamma$ -equivariant family of Dirac-type operators  $(D(\theta))_{\theta \in T}$  on a foliated bundle  $\widehat{M} \rightarrow T$ ; we denote by  $\mathcal{D}$  the  $C(T) \rtimes_r \Gamma$ -linear operator defined by the family  $(D(\theta))_{\theta \in T}$ . We then introduce the notion of spectral section associated to  $\mathcal{D}$  and prove the fundamental existence theorem: *a spectral section  $\mathcal{P}$  for  $\mathcal{D}$  exists if and only if the index class  $\text{Ind}(\mathcal{D})$  in  $K_*(C(T) \rtimes_r \Gamma)$  vanishes.* We also introduce the notion of difference class  $[\mathcal{P}] - [\mathcal{Q}]$ , in  $K_{*+1}(C(T) \rtimes_r \Gamma)$ , associated to two spectral sections  $\mathcal{P}, \mathcal{Q}$ ; following Dai and Zhang we then introduce the notion of higher spectral flow for a

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<sup>1</sup>We recall that for a Galois covering  $\Gamma \rightarrow \widetilde{M} \rightarrow M$  the Novikov higher signatures are the numbers:

$$\int_M L(M) \cup r^*[c], \quad [c] \in H^*(B\Gamma, \mathbb{C}) = H^*(\Gamma, \mathbb{C})$$

with  $r : M \rightarrow B\Gamma$  the classifying map of the covering.

path  $(\mathcal{D}_u)_{u \in [0,1]}$  of  $C(T) \rtimes_r \Gamma$ -linear operators. In **Section 4** we develop index theory on foliated bundles *with boundary*, using the  $b$ -pseudodifferential calculus on foliated bundles developed by Leichtnam and Piazza in [35]; thus we start in Subsection 4.1 by reviewing the numeric case, explaining the equality between the *generalized* APS-index on a manifold with boundary and a certain *perturbed*  $L^2$ -index on the associated manifold with cylindrical ends. The latter can also be described in the framework of Melrose'  $b$ -geometry [43]; this will be in fact the point of view that we shall adopt. In Subsection 4.2 we describe in detail the geometric set-up for foliated bundles with boundary; we also recall the index theory developed in [35] for  $\Gamma$ -equivariant families of Dirac-type operators  $(D(\theta))_{\theta \in T}$  with invertible boundary family. We explain how spectral sections for the boundary family can be used in order to remove the invertibility assumption; we then prove in Subsection 4.3 the cobordism invariance of the index class in this general case; by the existence theorem we infer that a boundary family always admits a spectral section. In Subsections 4.4 and 4.5 we define  $b$  and APS-index classes in  $K_*(C(T) \rtimes_r \Gamma)$  associated to a  $\Gamma$ -equivariant family  $(D(\theta))_{\theta \in T}$  and a choice of spectral section for the boundary family. We also prove the equality of these two index classes. In **Section 5** we establish 3 fundamental properties of these index classes: *the gluing formula* for index classes on closed foliated bundles that are union of two foliated bundles with boundary; *the relative index theorem*, equating the difference of  $b$ -index classes associated to two different choices of spectral sections  $\mathcal{P}, \mathcal{Q}$ , to the class  $[\mathcal{Q}] - [\mathcal{P}]$ ; *the variational formula* computing the variation of the  $b$ -index of a path of  $\Gamma$ -equivariant families in terms of the higher spectral flow associated to the boundary family. Section 4 and 5 are modeled on the work of Melrose-Piazza [45] [46] and the subsequent work of Leichtnam-Piazza [31] [34]. In **Section 6** we finally tackle the cut-and-paste problem: we define two cut-and-paste equivalent foliated bundles in Subsection 6.1. We then compute the difference of index classes for the signature family of two cut-and-paste equivalent manifold in terms of the higher spectral flow of a path of operators on the cutting hypersurface (Subsection 6.2); we refer to such a formula as a *defect formula*. In Subsection 6.3 we employ a gap condition on forms of middle degree (see Assumption 6.2) and spectral sections with a certain symmetry property in order to give conditions ensuring the vanishing of this defect; this will be a solution to the problem we had posed. Finally, in **Section 7** we give additional conditions on  $\Gamma$  and its action on  $T$  in order to deduce the cut-and-paste invariance of certain geometric numerical invariants generalizing the Novikov higher signatures. One of our main geometric results in Section 7 is the following:

*Assume that the rational Baum-Connes map*

$$\mu_{\mathbb{Q}} : K_{0,\tau}((E\Gamma \times T)/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(C(T) \rtimes_r \Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

*is injective. Let  $\widehat{X}_\phi \rightarrow T$  and  $\widehat{X}_\psi \rightarrow T$  be two  $\Gamma$ -equivariant fibrations that are cut-and-paste equivalent and satisfy Assumption 6.2 below. Assume moreover that the vertical tangent bundles both admit a  $\Gamma$ -invariant spin structure. Then for any  $c \in H^*((E\Gamma \times T)/\Gamma; \mathbb{Q})$  the Baum-Connes higher signatures are equal:*

$$\int_{\widehat{X}_\phi/\Gamma} L(\widehat{X}_\phi/\Gamma) \wedge r_\phi^*(c) = \int_{\widehat{X}_\psi/\Gamma} L(\widehat{X}_\psi/\Gamma) \wedge s_\psi^*(c).$$

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## 2. SPECTRAL FLOW

**2.1. Spectral flow through spectral sections.** Let  $(D_t)_{t \in [0,1]}$  be a continuous family of formally self-adjoint elliptic differential operators. For simplicity, we shall assume that  $D_0$  and  $D_1$  are invertible. The spectral flow of the family  $(D_t)_{t \in [0,1]}$ ,

$$\text{sf}((D_t)_{t \in [0,1]}) \in \mathbb{Z},$$

is by definition the net number of eigenvalues changing sign as  $t$  runs from 0 to 1.

Following Dai and Zhang [13] we shall now recall how to express the spectral flow of the family  $(D_t)_{t \in [0,1]}$  in a way which can be generalized to situations where the spectrum is not discrete. To this end we recall the notion of spectral section associated to the family  $(D_t)_{t \in [0,1]}$  and, in fact, to any family of formally self-adjoint elliptic differential operators parametrized by a compact space  $B$ . Thus, let  $\mathcal{D} = (D_z)_{z \in B}$  be a smooth family of formally self-adjoint elliptic differential operators parametrized by  $B$ . We shall eventually take  $B = [0, 1]$  but let us proceed in full generality for the time being. Each operator  $D_z$  acts on the sections of a hermitian vector bundle  $F_z$  over a closed riemannian manifold  $N_z$ . A spectral section  $\mathcal{P}$  for  $\mathcal{D}$  is a **smooth** family  $\mathcal{P} = (P_z)_{z \in B}$  of self-adjoint projections with  $P_z \in \Psi^0(N_z; F_z)$  and satisfying the following property:

$$(2.1) \quad \exists R \in \mathbb{R} \mid D_z u = \lambda u \Rightarrow P u = u \text{ if } \lambda > R, \quad P u = 0 \text{ if } \lambda < -R$$

This means that each operator  $P_z$  is equal to the identity on the eigenfunctions of  $D_z$  corresponding to large positive eigenvalues and equal to 0 on those corresponding to large negative eigenvalues. Each  $P_z$  is a finite rank perturbation of the spectral projection

$$\Pi_{\geq}(z) := \chi_{[0, \infty)}(D_z)$$

corresponding to the non-negative eigenvalues of  $D_z$ . The family  $\mathcal{D}$  is a family of self-adjoint Fredholm operators parametrized by  $B$ ; it therefore defines an index class  $\text{Ind}(\mathcal{D}) \in K^1(B)$ . One of the main results of Melrose-Piazza [45] asserts that a spectral section for  $\mathcal{D}$  exists if and only if  $\text{Ind}(\mathcal{D}) = 0$  in  $K^1(B)$ . One can prove that if  $\text{Ind}(\mathcal{D}) = 0$  then the set of spectral sections associated to  $\mathcal{D}$  is infinite.

Let us go back to our 1-parameter family of formally self-adjoint elliptic differential operators  $\mathcal{D} = (D_t)_{t \in [0,1]}$  on an odd dimensional manifold  $N$ ; since  $B = [0, 1]$  is contractible, it is certainly the case that  $\text{Ind}(\mathcal{D}) = 0$ . Thus there exists a spectral section  $\mathcal{P} = (P_t)_{t \in [0,1]}$  for the 1-parameter family  $\mathcal{D} = (D_t)_{t \in [0,1]}$ . Consider now the spectral projection  $\Pi_{\geq}(0)$  associated to the nonnegative eigenvalues of  $D_0$ ; consider the spectral projection  $\Pi_{\geq}(1)$  associated to the nonnegative eigenvalues of  $D_1$ ; consider the projection  $P_0$  and the projection  $P_1$ , also associated, respectively, to  $D_0$  and  $D_1$ . Since  $P_0$  is a finite rank perturbation of  $\Pi_{\geq}(0)$  one can define a relative index  $i(\Pi_{\geq}(0), P_0) \in \mathbb{Z}$ ; this is simply the index of the Fredholm operator  $\Pi_{\geq}(0) \circ P_0 : \text{Im } P_0 \rightarrow \text{Im } \Pi_{\geq}(0)$ . Similarly, there is a relative index  $i(\Pi_{\geq}(1), P_1) \in \mathbb{Z}$ ; we shall denote these relative indices as  $[\Pi_{\geq}(0) - P_0]$ ,  $[\Pi_{\geq}(1) - P_1]$  respectively

**Proposition 2.1.** (*Dai-Zhang [13]*)

*If  $\mathcal{D} = (D_t)_{t \in [0,1]}$  is a smooth 1-parameter family of formally self-adjoint elliptic differential operators, with  $D_0$  and  $D_1$  invertible, then*

$$(2.2) \quad \text{sf}((D_t)_{t \in [0,1]}) = [\Pi_{\geq}(1) - P_1] - [\Pi_{\geq}(0) - P_0] \in \mathbb{Z}.$$

We can see the spectral flow as an element in  $K^0(\text{point}) = K_0(\mathbb{C}) = \mathbb{Z}$  (the K-theory of an algebra will be defined in Subsection 3.1 below).

This result leads in a natural way to a small generalization of the notion of spectral flow: fix spectral sections  $Q_i$  for  $D_i$ ,  $i = 0, 1$ . If  $\mathcal{P} = (P_t)_{t \in [0,1]}$  is a total spectral section for the family  $(D_t)_{t \in [0,1]}$ , then the spectral flow  $\text{sf}((D_t)_{t \in [0,1]}; Q_0, Q_1)$  from  $(D_0, Q_0)$  to  $(D_1, Q_1)$  through  $(D_t)_{t \in [0,1]}$  is the element of  $K_0(\mathbb{C})$  given by the difference class

$$(2.3) \quad \text{sf}((D_t)_{t \in [0,1]}; Q_0, Q_1) := [Q_1 - P_1] - [Q_0 - P_0] \in K_0(\mathbb{C}) = \mathbb{Z};$$

one can prove that this class is well defined, independent of the total spectral section  $\mathcal{P} = (P_t)_{t \in [0,1]}$  chosen. The classic case explained above is obtained by making the particular choice  $Q_0 = \Pi_{\geq}(0)$ ,  $Q_1 = \Pi_{\geq}(1)$ .

**2.2. Index and spectral flow.** Let  $M$  be an even-dimensional riemannian manifold with boundary, endowed with a product metric near the boundary. In contrast with the closed case, the Atiyah-Patodi-Singer ( $\equiv$  APS) index of a Dirac-type operator, see [1], is not stable under perturbations. In fact, assume that  $(D_t)_{t \in [0,1]}$  is a smoothly varying family of Dirac operators on  $M$ ; as an important example we could consider a family of metrics  $(g(t))_{t \in [0,1]}$  on  $M$  and the associated family of signature operators  $(D^{\text{sign}}(t))_{t \in [0,1]}$ . We could also consider, on a spin manifold, a family of Dirac operators  $(\not{D}(t))_{t \in [0,1]}$  parametrized by a path of metrics  $(g(t))_{t \in [0,1]}$ . Going back to the general case, consider the family of operators induced on the boundary  $(D_{\partial M}(t))_{t \in [0,1]}$ ; let  $\Pi_{\geq}(t)$  the corresponding spectral projection associated to the non-negative eigenvalues. For simplicity, let us assume that the boundary operator is invertible at  $t = 1$  and at  $t = 0$ ; then the following variational formula for the APS-indeces holds:

$$(2.4) \quad \text{ind}(D_1^+, \Pi_{\geq}(1)) - \text{ind}(D_0^+, \Pi_{\geq}(0)) = \text{sf}((D_{\partial M}(t))_{t \in [0,1]}).$$

Formula (2.4) follows from the APS-index formula. It can also be proved analytically, without making use of the APS-index formula. See for example [13] where much more general projections are allowed. In fact, in that paper Dai and Zhang establish a more general variational formula; since such a generalization will be important to us we briefly explain it.

First of all, if  $D$  is an odd Dirac-type operator on  $M$ , acting on the sections of a  $\mathbb{Z}_2$ -graded Clifford module  $E = E^+ \oplus E^-$ , and if  $Q$  is a spectral section for  $D_{\partial M}$ , then there is a well defined *generalized* APS-boundary value problem, with a well-defined index  $\text{ind}(D^+, Q)$  (see for example [8]); the boundary problem is simply defined by taking the operator  $D^+$  with domain

$$\{u \in C^\infty(M, E^+) \mid u|_{\partial M} \in \text{Ker}Q\}.$$

This generalized boundary value problems has interesting properties. First of all, if  $Q'$  is a different spectral section for  $D_{\partial M}$ , then the following *relative index formula* holds:

$$(2.5) \quad \text{ind}(D^+, Q') - \text{ind}(D^+, Q) = [Q - Q'] \in K_0(\mathbb{C}) = \mathbb{Z}.$$

Second, let  $(D_t)_{t \in [0,1]}$  be a smoothly varying family of odd Dirac operators on  $M$ ; choose a spectral section  $Q_0$  for  $D_0$  and a spectral section  $Q_1$  for  $D_1$ ; then the following *variational formula for generalized APS-indeces* holds:

$$(2.6) \quad \text{ind}(D_1^+, Q_1) - \text{ind}(D_0^+, Q_0) = \text{sf}((D_{\partial M}(t))_{t \in [0,1]}; Q_1, Q_0).$$

**2.2.1. Remark.** If  $N$  is odd dimensional and  $(D_N^{\text{sign}}(t))_{t \in [0,1]}$  is a one-parameter family of odd signature operators parametrized by a path of metrics  $g_N(t)_{t \in [0,1]}$ , then

$$(2.7) \quad \text{sf}((D_N^{\text{sign}}(t))_{t \in [0,1]}; \Pi_{\geq}(0), \Pi_{\geq}(1)) = 0,$$

$$(2.8) \quad \text{sf}((D_N^{\text{sign}}(t))_{t \in [0,1]}; \Pi_{>}(0), \Pi_{>}(1)) = 0.$$

In fact, the kernel of the odd signature operator is equal to the space of harmonic forms on  $N$ ; from the Hodge theorem we know that such a vector space is independent of the metric we choose; thus there are not eigenvalues changing sign and we can choose as total spectral section  $\mathcal{P} = (\Pi_{\geq}(t))_{t \in [0,1]}$  for the first equation and  $\mathcal{P} = (\Pi_{>}(t))_{t \in [0,1]}$  for the second equation. In particular, if  $(D_M^{\text{sign}}(t))_{t \in [0,1]}$  is a one-parameter family of signature operators (parametrized by a path of metrics) on a  $4k$ -dimensional manifold with boundary, then we have

$$(2.9) \quad \text{ind}(D_M^{\text{sign},+}(1), \Pi_{\geq}(1)) = \text{ind}(D_M^{\text{sign},+}(0), \Pi_{\geq}(0))$$

$$(2.10) \quad \text{ind}(D_M^{\text{sign},+}(1), \Pi_{>}(1)) = \text{ind}(D_M^{\text{sign},+}(0), \Pi_{>}(0)).$$

**2.3. The gluing formula.** We consider  $X$ , a *closed* oriented compact manifold which is the union of two manifolds with boundary. Thus there exists an embedded hypersurface  $F$  which separates  $M$  into two connected components and such that

$$X = M_+ \cup_F M_-, \quad \text{with} \quad \partial M_+ = F = -\partial M_-.$$

We assume that the metric  $g$  is of product type near the hypersurface  $F$ , i.e. near the boundaries of  $M_+$  and  $M_-$ . Let  $D_X$  be a Dirac-type operator on  $X$ ; then we obtain in a natural way two Dirac operators on  $M_+$  and  $M_-$ . The following gluing formula holds:

$$(2.11) \quad \text{ind}(D_X) = \text{ind}(D_{M_+}, \Pi_{\geq}) + \text{ind}(D_{M_-}, 1 - \Pi_{\geq}).$$

The discrepancy in the spectral projections comes from the orientation of the normals to the two boundaries (if one is inward pointing, then the other is outward pointing).<sup>2</sup> In particular, for the signature operators we have the fundamental formula

$$(2.12) \quad \text{ind}(D_X^{\text{sign}}) = \text{ind}(D_{M_+}^{\text{sign}}, \Pi_{\geq}) + \text{ind}(D_{M_-}^{\text{sign}}, 1 - \Pi_{\geq}).$$

The two formulae (2.11) (2.12) can be proved directly, in a purely analytical fashion, see Bunke [9], Leichtnam-Piazza [34]. Of course they are also a consequence of the APS-index theorem.

**2.4. An analytic proof of the cut and paste invariance of the signature.** The gluing formula (2.12) for the signature operator can be generalized to a more complicated situation, where  $X_\phi$  is a closed manifold obtained by *gluing* two manifolds with boundary through a diffeomorphism  $\phi$  between their boundary. We shall concentrate on the signature operator. Thus let  $M$  and  $N$  be two oriented manifolds with boundary and let  $\phi : \partial M \rightarrow \partial N$  be an oriented diffeomorphism. We consider the manifold with boundary  $X_\phi := M \cup_\phi N^-$ , with  $N^-$  equal to  $N$  with the opposite orientation. We shall follow the notation of the previous subsection; thus we set  $M_+ := M$ ,  $M_- := N^-$  and  $X_\phi = M_+ \cup_\phi M_-$ .

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<sup>2</sup>Notice that  $1 - \Pi_{\geq}$  is not exactly the APS-projection associated to the non-negative eigenvalues of  $D_{\partial M_-}$ ; to be precise  $1 - \Pi_{\geq} = \Pi_{>}^{\partial M_-}$ , the projection onto the *positive* eigenvalues of  $D_{\partial M_-}$ .

We fix a metric  $g_\phi$  on  $X_\phi$ . Notice that giving  $g_\phi$  is equivalent to give  $g(+)$  on  $M_+$  and  $g(-)$  on  $M_-$  such that  $\phi^*(g(-)|_{\partial M_-}) = g(+)|_{\partial M_+}$ . We shall assume that these metrics are of product type near the boundary. The pull-back  $\phi^*$  defines an isometry between  $L^2(\partial M_-, \Lambda^*(\partial M_-))$  defined by  $g(-)|_{\partial M_-}$  and  $L^2(\partial M_+, \Lambda^*(\partial M_+))$  defined by  $g(+)|_{\partial M_+}$ . Let  $D_{X_\phi}^{\text{sign}}$  be the signature operator on  $X_\phi$  associated to  $g_\phi$ . We also have the signature operators on  $M_\pm$  with boundary operators  $D_{\partial M_+}^{\text{sign}}$  and  $D_{\partial M_-}^{\text{sign}}$  and it is easy to check that

$$(2.13) \quad D_{\partial M_+}^{\text{sign}} = -\phi^*(D_{\partial M_-}^{\text{sign}})(\phi^*)^{-1}.$$

Let  $\Pi_{\geq}^{\partial M_+}$  be the APS spectral projection for  $D_{\partial M_+}^{\text{sign}}$  and consider the projection

$$\Pi_{\geq}^\phi := (\phi^*)^{-1}\Pi_{\geq}^{\partial M_+}\phi^*;$$

from (2.13) we infer that  $\text{Id} - \Pi_{\geq}^\phi$  is equal to the spectral projection  $\Pi_{>}^{\partial M_-}$  onto the non-negative eigenvalues of  $D_{\partial M_-}^{\text{sign}}$ . Here the fact that we are dealing with the signature operator has been used. One can prove, analytically, the following additivity formula:

$$\begin{aligned} \text{ind}(D_{X_\phi}^{\text{sign}}) &= \text{ind}(D_{M_+}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) + \text{ind}(D_{M_-}^{\text{sign}}, 1 - \Pi_{\geq}^\phi) \\ &= \text{ind}(D_{M_+}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) + \text{ind}(D_{M_-}^{\text{sign}}, \Pi_{>}^{\partial M_-}). \end{aligned}$$

Summarizing, also in this more general case we have

$$(2.14) \quad \text{ind}(D_{X_\phi}^{\text{sign}}) = \text{ind}(D_{M_+}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) + \text{ind}(D_{M_-}^{\text{sign}}, \Pi_{>}^{\partial M_-}),$$

where it is important to notice that the operators appearing in this formula are associated to the metrics  $g_\phi$  on the left hand side and to metrics  $g_\phi(+)$ ,  $g_\psi(-)$  on the right hand side; thus we should write more precisely

$$(2.15) \quad \text{ind}(D_{(X_\phi, g_\psi)}^{\text{sign}}) = \text{ind}(D_{(M_+, g_\psi(+))}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) + \text{ind}(D_{(M_-, g_\psi(-))}^{\text{sign}}, \Pi_{>}^{\partial M_-}).$$

Let now

$$X_\phi = M_+ \cup_\phi M_-, \quad X_\psi = M_+ \cup_\psi M_-$$

with  $\phi, \psi : \partial M_+ \rightarrow \partial M_-$  diffeomorphisms, be two such manifolds. One says in this case that  $X_\phi$  and  $X_\psi$  are *cut-and-paste equivalent*. Let us fix metrics  $g_\phi$  and  $g_\psi$  on  $X_\phi$  and  $X_\psi$  respectively. We obtain metrics  $g_\phi(\pm)$ ,  $g_\psi(\pm)$  on  $M_\pm$ . We can assume these metrics to be product like near  $\partial M_\pm$ . Let  $D_{X_\phi}^{\text{sign}}$  and  $D_{X_\psi}^{\text{sign}}$  be the signature operators associated to  $g_\phi$  and  $g_\psi$ . We wish to give a proof of the equality

$$\text{ind}(D_{X_\phi}^{\text{sign},+}) = \text{ind}(D_{X_\psi}^{\text{sign},+}).$$

It will be important to keep track of the metrics involved, thus we write as above  $D_{(X_\phi, g_\phi)}^{\text{sign}}$  for the signature operator on  $X_\phi$  associated to the metric  $g_\phi$  and  $D_{(M_\pm, g_\phi(\pm))}^{\text{sign}}$  for the induced signature operators on the manifold with boundary  $M_\pm$ . Similarly we proceed for

$$D_{(X_\psi, g_\psi)}^{\text{sign}} \quad \text{and} \quad D_{(M_\pm, g_\psi(\pm))}^{\text{sign}}$$

We begin by applying the additivity formula: we obtain

$$\begin{aligned} \text{ind}(D_{(X_\phi, g_\phi)}^{\text{sign}}) &= \text{ind}(D_{(M_+, g_\phi(+))}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) + \text{ind}(D_{(M_-, g_\phi(-))}^{\text{sign}}, \Pi_{>}^{\partial M_-}) \\ \text{ind}(D_{(X_\psi, g_\psi)}^{\text{sign}}) &= \text{ind}(D_{(M_+, g_\psi(+))}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) + \text{ind}(D_{(M_-, g_\psi(-))}^{\text{sign}}, \Pi_{>}^{\partial M_-}). \end{aligned}$$

On the left hand side of these formulae we have indices of operators on manifolds which are, in general, non-diffeomorphic. On the right hand side, on the other hand, we can compare, as we have the same 2 manifolds with boundary. From the above formula we infer that

$$\begin{aligned} & \text{ind}(D_{(X_\psi, g_\psi)}^{\text{sign},+}) - \text{ind}(D_{(X_\phi, g_\phi)}^{\text{sign},+}) \\ &= \left( \text{ind}(D_{(M_+, g_\psi(+))}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) - \text{ind}(D_{(M_+, g_\phi(+))}^{\text{sign}}, \Pi_{\geq}^{\partial M_+}) \right) \\ &+ \left( \text{ind}(D_{(M_-, g_\psi(-))}^{\text{sign}}, \Pi_{>}^{\partial M_-}) - \text{ind}(D_{(M_-, g_\phi(-))}^{\text{sign}}, \Pi_{>}^{\partial M_-}) \right). \end{aligned}$$

Let  $(g_t(\pm))_{t \in [0,1]}$  be a path of metrics on  $M_\pm$  joining  $g_\phi(\pm)$  and  $g_\psi(\pm)$ . Applying the variational formula (2.4) to the two summands on the right hand side we obtain:

$$(2.16) \quad \text{ind}(D_{(X_\psi, g_\psi)}^{\text{sign},+}) - \text{ind}(D_{(X_\phi, g_\phi)}^{\text{sign},+})$$

$$(2.17) \quad = \text{sf} \left( (D_{(\partial M_+, g_t(+))}^{\text{sign}})_{t \in [0,1]}; \Pi_{\geq}^{\partial M_+}(0), \Pi_{\geq}^{\partial M_+}(1) \right)$$

$$(2.18) \quad + \text{sf} \left( (D_{(\partial M_-, g_t(-))}^{\text{sign}})_{t \in [0,1]}; \Pi_{>}^{\partial M_-}(0), \Pi_{>}^{\partial M_-}(1) \right)$$

and taking into account Remark 2.2.1, we immediately obtain

$$\text{ind}(D_{(X_\psi, g_\psi)}^{\text{sign},+}) = \text{ind}(D_{(X_\phi, g_\phi)}^{\text{sign},+});$$

this implies the equality of the signatures as required. In fact a small argument involving our various identifications shows that our conclusion can be reached through the following two equalities

$$(2.19) \quad \text{Ind}(D_{(X_\psi, g_\psi)}^{\text{sign},+}) - \text{Ind}(D_{(X_\phi, g_\phi)}^{\text{sign},+}) = \text{sf}(\{D_{\text{odd}}^{\text{sign}}(\theta)\}_{\theta \in S^1}) = 0.$$

The spectral flow appearing in this formula is associated to a  $S^1$ -family of odd signature operators acting on the fibers of the mapping torus  $M(F, \phi^{-1} \circ \psi) \rightarrow S^1$  and parametrized by a family of metrics. As already remarked this spectral flow is zero because of the cohomological significance of the zero eigenvalue for the signature operator.

**Remark.** It should be remarked that in this analytic proof of the cut-and-paste invariance of the signature, we have not used the APS-index formula; only the analytic properties of the APS boundary value problem were employed. This will be important later, when we shall extend the argument above to the foliated case.

### 3. FOLIATED BUNDLES, INDEX CLASSES AND THE NONCOMMUTATIVE SPECTRAL FLOW

**3.1. Preliminaries: K-Theory of  $C^*$ -algebras and Fredholm operators.** Let  $A$  be a unital  $C^*$ -algebra. We recall that  $K_0(A)$  is defined as the Grothendieck group associated to the semigroup of isomorphism classes of finitely generated projective left  $A$ -modules.  $K_0(A)$  is an additive group. A class in  $K_0(A)$  is represented by a formal difference  $[E] - [F]$  of isomorphism classes of finitely generated projective left  $A$ -module. Notice that a finitely generated projective left  $A$ -module is the range of a projection  $p$  in the matrix algebra  $M_n(A)$ , for a suitable  $n$ . In fact,  $K_0(A)$  can be also described in terms of such projections: one considers the inductive limit  $M_\infty(A)$ , the cartesian product  $M_\infty(A)^2$  and identifies two pairs of projections  $(p, q) \in M_n(A)^2$  and  $(p', q') \in M_{n'}(A)^2$  if for suitable  $k, k' \in \mathbb{N}$ ,

$$p \oplus q' \oplus \text{Id}_k \oplus 0_{k'} \text{ is conjugate to } p' \oplus q \oplus \text{Id}_k \oplus 0_{k'} \text{ in } M_{n+n'+k+k'}(A).$$

The quotient under this equivalence relation has a natural structure of abelian group and is naturally isomorphic to  $K_0(A)$ ; one denotes by  $[p] - [q]$  ( $= [p'] - [q']$ ) the class of  $(p, q)$ . Recall that if  $(p_1, q_1) \in M_{n_1}(A)^2$  then one has:  $([p] - [q]) + ([p_1] - [q_1]) = [p \oplus p_1] - [q \oplus q_1]$  where  $([p \oplus p_1], [q \oplus q_1]) \in M_{n+n_1}(A)^2$ . When  $A$  is a non unital  $C^*$ -algebra one introduces the unital  $C^*$ -algebra  $\tilde{A} = A \oplus \mathbb{C}$  obtained by adding the unit element  $0 \oplus 1$  to  $A$ ; one considers the morphism  $\epsilon : \tilde{A} \rightarrow \mathbb{C}$  defined by  $\epsilon(a \oplus \lambda) = \lambda$ . One then defines  $K_0(A)$  to be equal to the kernel of the map  $\epsilon_* : K_0(\tilde{A}) \rightarrow K_0(\mathbb{C})$  induced by  $\epsilon$ . Observe that  $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = \mathbb{Z}$ . We define  $K_1(A)$  to be equal to  $K_0(A \otimes C_0(\mathbb{R}))$  where  $A \otimes C_0(\mathbb{R})$  is the *suspension* of  $A$ . For instance  $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = 0$ . Alternatively,  $K_1(A)$  can be identified with the set of connected components of  $GL_\infty(A)$ . Fundamental properties of the  $K_*$ -functor are the Bott isomorphism

$$K_0(A) \simeq K_0(C_0(\mathbb{R}^2) \otimes A)$$

and the six-terms long exact sequence associated to the short exact sequence

$$0 \longrightarrow J \xrightarrow{i} A \xrightarrow{\pi} A/J \longrightarrow 0$$

with  $J$  an ideal in  $A$ :

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/J) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(A/J) & \xleftarrow{\pi_*} & K_1(A) & \xleftarrow{i_*} & K_1(J) \end{array}$$

Elements in  $K_*(A)$  arise naturally as index classes of generalized Fredholm operators between Hilbert  $A$ -modules. We recall that a Hilbert  $A$ -module  $\mathcal{E}$  is a left  $A$ -module endowed with an  $A$ -valued form  $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow A$  satisfying the following axioms:

- (i)  $\langle \eta, \xi_1 + \xi_2 \rangle = \langle \eta, \xi_1 \rangle + \langle \eta, \xi_2 \rangle$ ;
- (ii)  $\langle \eta, \xi a \rangle = \langle \eta, \xi \rangle a$ ;
- (iii)  $\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle$ ;
- (iv)  $\langle \eta, \eta \rangle \geq 0$ ;
- (v)  $\langle \eta, \eta \rangle = 0 \Leftrightarrow \eta = 0$
- (vi)  $\mathcal{E}$  is complete with respect to the norm  $\|\eta\| := \|\langle \eta, \eta \rangle\|_A$

Hilbert  $A$ -modules share *some* of the usual properties of Hilbert spaces; there are however structural differences and more care is needed in all arguments involving bounded operators, adjoints, orthogonal complements etc... See for example [56] for a clear account of these structural differences. Here we shall content ourselves by simply stating that there is indeed a natural notion of bounded  $A$ -linear adjointable operator between Hilbert  $A$ -modules <sup>3</sup>. One defines in this way the space of bounded adjointable operators between two Hilbert  $A$ -modules  $\mathcal{E}^\pm$ , denoted  $\mathcal{B}_A(\mathcal{E}^+, \mathcal{E}^-)$ . There is also a natural notion of finite rank operator and the closed subspace of compact operators,  $\mathcal{K}_A(\mathcal{E}^+, \mathcal{E}^-) \subset \mathcal{B}_A(\mathcal{E}^+, \mathcal{E}^-)$ , is defined as the norm closure in  $\mathcal{B}_A(\mathcal{E}^+, \mathcal{E}^-)$  of the space of finite rank operators.

A  $A$ -Fredholm operator  $\mathcal{L}^+ : \mathcal{E}^+ \rightarrow \mathcal{E}^-$  is a bounded adjointable operator which is invertible modulo compacts. One can prove that *up to a compact perturbation* the kernel of  $\mathcal{L}^+$  and of its adjoint are finitely generated projective  $A$ -modules; we get in this way an index

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<sup>3</sup>Notice that a bounded  $A$ -linear operator might not admit an adjoint.

class  $\text{Ind}(\mathcal{L}^+) \in K_0(A)$  by simply taking the formal difference of these finitely generated projective  $A$ -modules. If  $\mathcal{L} : \mathcal{E} \rightarrow \mathcal{E}$  is self-adjoint, then there is an index class in  $K_1(A)$ . Let us describe it. First we need to recall another fundamental property of the  $K_*$ -functor, namely its *stability*: if  $\mathcal{E}$  is a Hilbert  $A$ -module, then  $K_*(\mathcal{K}_A(\mathcal{E})) \simeq K_*(A)$ . Consider now the Calkin algebra

$$\mathcal{C}_A(\mathcal{E}) := \frac{\mathcal{B}_A(\mathcal{E})}{\mathcal{K}_A(\mathcal{E})}$$

and the canonical projection map  $\pi : \mathcal{B}_A(\mathcal{E}) \rightarrow \mathcal{C}_A(\mathcal{E})$ ; the long exact sequence in K-theory associated to

$$0 \rightarrow \mathcal{K}_A(\mathcal{E}) \rightarrow \mathcal{B}_A(\mathcal{E}) \rightarrow \mathcal{C}_A(\mathcal{E}) \rightarrow 0$$

gives the homomorphism

$$\delta : K_0(\mathcal{C}_A(\mathcal{E})) \rightarrow K_1(A),$$

since, by stability,  $K_1(\mathcal{K}_A(\mathcal{E})) \simeq K_1(A)$ . We need to assume that  $\mathcal{L}^2 = \text{Id} + \mathcal{R}$  where  $\mathcal{R} \in \mathcal{K}_A$ ; then  $\pi(\frac{1}{2}(\mathcal{L} + \text{Id}))$  is a projection in the Calkin algebra and the index class  $\text{Ind}(\mathcal{L})$  is defined by

$$\text{Ind}(\mathcal{L}) := \delta[\pi(\frac{1}{2}(\mathcal{L} + \text{Id}))] \in K_1(A).$$

**3.2. Foliated bundles.** Let  $\Gamma$  be a finitely generated discrete group. Let  $T$  be a smooth closed compact connected manifold on which  $\Gamma$  acts on the right. Let  $\widehat{M}$  be a closed manifold on which  $\Gamma$  acts freely, properly and cocompactly on the right: the quotient space  $M = \widehat{M}/\Gamma$  is thus a smooth closed compact manifold. We assume that  $\widehat{M}$  fibers over  $T$  and that the resulting fibration

$$\pi : \widehat{M} \rightarrow T$$

is a  $\Gamma$ -equivariant fibration with fibers  $\pi^{-1}(\theta)$ ,  $\theta \in T$ ,

**Remark.** We observe incidentally that what we have described is an example of a proper cocompact  $G$ -manifold  $P$  with  $G$  an étale groupoid, see Connes ([10] page 137) for the definition. In our case

$$G = T \rtimes \Gamma, \quad G^{(0)} = T, \quad (\alpha : P \rightarrow G^{(0)}) \equiv (\pi : \widehat{M} \rightarrow T).$$

The groupoid  $G = T \rtimes \Gamma$  has  $G^{(1)} = T \times \Gamma$  as set of morphisms and  $G^{(0)} = T$  as base. The range and source maps are respectively given by:

$$\forall (\theta, g) \in T \times \Gamma, \quad r(\theta, g) = \theta, \quad s(\theta, g) = \theta \cdot g.$$

The composition is defined as follows:

$$(\theta, g) \cdot (\theta', g') = (\theta, gg') \text{ if } \theta' = \theta g.$$

The inverse of  $(\theta, g)$  is  $(\theta g, g^{-1})$ .

With a small abuse of terminology we shall call  $\Gamma$ -equivariant fibrations  $\widehat{M} \rightarrow T$  with smooth compact quotient  $M = \widehat{M}/\Gamma$  a proper  $T \rtimes \Gamma$ -manifold. It is important to notice that the compact manifold  $M$  inherits a foliation  $\mathcal{F}$ , with leaves equal to the images of the fibers of  $\pi : \widehat{M} \rightarrow T$  under the quotient map  $\widehat{M} \rightarrow M = \widehat{M}/\Gamma$ . The foliated manifold  $(M, \mathcal{F})$  is usually referred to as a *foliated  $T$ -bundle* or simply as a *foliated bundle*.

*In this paper we shall refer to a  $\Gamma$ -equivariant fibrations  $\widehat{M} \rightarrow T$  with smooth compact quotient  $M = \widehat{M}/\Gamma$  either as a proper  $T \rtimes \Gamma$ -manifold or as a foliated bundle.*

**Example.** Let  $X$  be a compact closed manifold and let  $\Gamma \rightarrow \tilde{X} \rightarrow X$  be a Galois cover of  $X$ . Let  $T$  be a smooth compact manifold on which  $\Gamma$  acts by diffeomorphisms. We consider  $\widehat{M} = \tilde{X} \times T$ ,  $\pi =$  projection onto the second factor,  $M = \tilde{X} \times_{\Gamma} T := (\tilde{X} \times T)/\Gamma$  where we let  $\Gamma$  act on  $\tilde{X} \times T$  diagonally. The leaves of the foliation  $\mathcal{F}$  are the images of the manifolds  $\tilde{X} \times \{\theta\}$ ,  $\theta \in T$ .

As a *particular example* of this construction consider  $T = S^1$ ,  $\Gamma = \mathbb{Z}$ ,  $\tilde{X} = \mathbb{R}$ , so that  $\widehat{M} = \mathbb{R} \times S^1$ . We let  $n \in \mathbb{Z}$  act on  $(r, e^{i\theta}) \in \mathbb{R} \times S^1$  by  $n \cdot (r, e^{i\theta}) := (r + n, e^{i(\theta+n\alpha)})$ , for some fixed  $\alpha \in \mathbb{R}$ . Then  $M = T^2$  and if  $\alpha/2\pi$  is irrational we get as a foliation of  $T^2$  the well-known Kronecker foliation. This is a type II foliation.

As a different example, consider a smooth closed riemann surface  $\Sigma$  of genus  $g > 1$  and let  $\Gamma = \pi_1(\Sigma)$ , a discrete subgroup of  $PSL(2, \mathbb{R})$ . Then we can consider  $X := \Sigma$ ,  $\Gamma \rightarrow \tilde{X} \rightarrow X$  equal to the universal cover  $\Gamma \rightarrow \mathbb{H}^2 \rightarrow \Sigma$ ,  $T = S^1$ , with  $\Gamma \trianglelefteq PSL(2, \mathbb{R})$  acting on  $S^1 = \mathbb{R}P^1$  by fractional linear transformations. The resulting foliation is of type III.

Let us go back to the general situation; we shall also denote the typical fiber of  $\pi : \widehat{M} \rightarrow T$  by  $Z$ . We now assume that  $Z$  is of dimension  $2k - 1$ . We choose a  $\Gamma$ -invariant metric on the vertical tangent bundle  $TZ$ . Finally, we assume the existence of a  $\Gamma$ -equivariant spin structure on  $TZ$  that is fixed once and for all. We denote by  $S^Z \rightarrow \widehat{M}$  the associated spinor bundle. We consider also a  $\Gamma$ -equivariant complex hermitian vector bundle  $\widehat{V} \rightarrow \widehat{M}$  endowed with a  $\Gamma$ -invariant hermitian connection. We then set  $\widehat{E} = S^Z \otimes \widehat{V}$  which defines a smooth  $\Gamma$ -invariant family of hermitian Clifford modules on the fibers  $\pi^{-1}(\theta)$ ,  $\theta \in T$ . We thus get a  $\Gamma$ -equivariant family  $(D(\theta))_{\theta \in T}$  of Dirac type operators acting fiberwise on  $C_c^\infty(\widehat{M}, \widehat{E})$ . If  $Z$  is even-dimensional, then the spinor bundle  $S^Z$  is  $\mathbb{Z}_2$ -graded; thus  $\widehat{E}$  is also  $\mathbb{Z}_2$ -graded,  $\widehat{E} = \widehat{E}^+ \oplus \widehat{E}^-$ , and the family  $(D(\theta))_{\theta \in T}$  is now odd with respect to this grading:

$$D(\theta) = \begin{pmatrix} 0 & D^-(\theta) \\ D^+(\theta) & 0 \end{pmatrix}, \theta \in T.$$

Notice that we could assume that  $\widehat{E}$  is defined more generally by a smooth  $\Gamma$ -invariant family of  $\mathbb{Z}_2$ -graded hermitian Clifford modules on the fibers  $\pi^{-1}(\theta)$ ,  $\theta \in T$ .

*We shall now explain how such a  $\Gamma$ -equivariant family defines a class in the  $K$ -theory of the cross-product algebra  $C(T) \rtimes_r \Gamma$ ; thus we shall define suitable Hilbert  $C(T) \rtimes_r \Gamma$ -modules and see how the family  $(D(\theta))_{\theta \in T}$  defines a  $C(T) \rtimes_r \Gamma$ -Fredholm operator  $\mathcal{D}$  on them.*

**3.3. The  $C^*$ -algebra  $C(T) \rtimes_r \Gamma$ .** The *algebraic* cross-product  $C_c^\infty(T) \rtimes \Gamma$  is, by definition, the set of functions  $\sum_{g \in \Gamma} t_g(\theta)g$  such that only a finite number of the  $t_g(\cdot) \in C_c^\infty(T)$  do not vanish identically. We shall identify any function  $f$  having compact support

$$\begin{aligned} f &: T \rtimes \Gamma \rightarrow \mathbb{C} \\ (\theta, g) &\rightarrow f(\theta, g) \end{aligned}$$

with  $\sum_{g \in \Gamma} f(\theta, g)g$ . Then one has:

$$\sum_{g' \in \Gamma} f'(\theta, g')g' \cdot \sum_{g \in \Gamma} f(\theta, g)g = \sum_{h \in \Gamma} \left( \sum_{g \in \Gamma} f'(\theta, g') f(\theta \cdot g', (g')^{-1}h) \right) h$$

where we recall that

$$g' \cdot (f(\theta, g)g) = f(\theta \cdot g', g)g'g.$$

The algebraic cross-product  $C_c^\infty(T) \rtimes \Gamma$  will also be denoted by  $C_c^\infty(T \rtimes \Gamma)$ . One can introduce the reduced  $C^*$ -algebra  $C_r^*(T \rtimes \Gamma)$  associated to the groupoid  $T \rtimes \Gamma$  as a suitable completion of the algebraic cross product  $C_c^\infty(T) \rtimes \Gamma$ . See [10].

It is well known, and easy to check, that there is a natural isomorphism between the reduced  $C^*$ -algebra  $C_r^*(T \rtimes \Gamma)$  of the groupoid  $T \rtimes \Gamma$  and the cross-product algebra  $C(T) \rtimes_r \Gamma$  (see, for example, Moore-Schochet [50]); we shall henceforth identify these two  $C^*$ -algebras. Observe,  $T$  being compact, that the reduced  $C^*$ -algebra  $C(T) \rtimes_r \Gamma$  is *unital*.

**3.4.  $C(T) \rtimes_r \Gamma$ -Hilbert modules.** Recall that the action of  $\Gamma$  on  $\widehat{E}$  induces for each  $g \in \Gamma$  an operator  $R_g^*$  acting on  $C_c^\infty(\widehat{M}, \widehat{E})$ . We endow  $C_c^\infty(\widehat{M}, \widehat{E})$  with the structure of left  $C_c^\infty(T) \rtimes \Gamma$ -module by setting for any  $s \in C_c^\infty(\widehat{M}, \widehat{E})$  and  $\sum_{g \in \Gamma} f(\cdot, g)g \in C_c^\infty(T) \rtimes \Gamma$

$$(3.1) \quad \forall p \in \widehat{M}, \quad \left( \sum_{g \in \Gamma} f(\cdot, g)g \cdot s \right) (p) := \sum_{g \in \Gamma} f(\pi(p), g)(R_g^*s)(p).$$

We define a  $C_c^\infty(T) \rtimes \Gamma$ -valued hermitian product of two sections  $s$  and  $s'$  of  $C_c^\infty(\widehat{M}, \widehat{E})$  by setting:

$$(3.2) \quad \langle s; s' \rangle = \sum_{g \in \Gamma} \langle s; s' \rangle(\theta, g)g \in C_c^\infty(T) \rtimes \Gamma \subset C(T) \rtimes_r \Gamma$$

where  $\forall(\theta, g) \in T \times \Gamma$ :

$$\langle s; s' \rangle(\theta, g) = \int_{\pi^{-1}(\theta, g)} \langle R_{g^{-1}}^*(s)(y); s'(y) \rangle_{\widehat{E}} d\text{Vol}_{\pi^{-1}(\theta, g)}(y)$$

with  $d\text{Vol}_{\pi^{-1}(\theta, g)}(y)$  denoting the riemannian density in the fiber.

*Summarizing:* we have defined on  $C_c^\infty(\widehat{M}, \widehat{E})$  a structure of  $C_c^\infty(T) \rtimes \Gamma$ -module; moreover we have defined a form

$$\langle \cdot, \cdot \rangle : C_c^\infty(\widehat{M}, \widehat{E}) \times C_c^\infty(\widehat{M}, \widehat{E}) \rightarrow C_c^\infty(T) \rtimes \Gamma \subset C(T) \rtimes_r \Gamma.$$

One denotes by  $L_{C(T) \rtimes_r \Gamma}^2(\widehat{M}, \widehat{E})$  the completion of  $C_c^\infty(\widehat{M}, \widehat{E})$  with respect to the norm induced by  $\langle \cdot; \cdot \rangle$ . One can prove that  $L_{C(T) \rtimes_r \Gamma}^2(\widehat{M}, \widehat{E})$  is indeed a Hilbert  $C(T) \rtimes_r \Gamma$ -module. In a similar way, one can introduce Sobolev-modules  $H_{C(T) \rtimes_r \Gamma}^m(\widehat{M}, \widehat{E})$ ,  $m \in \mathbb{N}$ .

**3.5. Index classes on closed foliated bundles.** Let us go back to our  $\Gamma$ -equivariant family of Dirac-type operators  $(D(\theta))_{\theta \in T}$ .

**Lemma 3.1.** *The family of Dirac operators  $(D(\theta))_{\theta \in T}$  acting fiberwise on  $C_c^\infty(\widehat{M}, \widehat{E})$  defines a left  $C_c^\infty(T \rtimes \Gamma)$ -linear endomorphism  $\mathcal{D}$  of  $C_c^\infty(\widehat{M}, \widehat{E})$ .*

*Proof.* Using the above notations we have:

$$\mathcal{D}\left(\sum_{g \in \Gamma} f(\cdot, g)g \cdot s\right)(p) = \sum_{g \in \Gamma} f(\pi(p), g) D(\pi(p))(R_g^*s)(p)$$

where we have used the fact that  $(D(\theta))_{\theta \in T}$  is a family of operators, i.e. commutes with the natural action of  $C^\infty(T)$ . Since the family  $(D(\theta))_{\theta \in T}$  is  $\Gamma$ -equivariant, the right hand is by definition equal to

$$\left( \sum_{g \in \Gamma} f(\cdot, g) g \cdot \mathcal{D}(s) \right) (p)$$

which proves the lemma.  $\square$

One can prove that  $\mathcal{D}$  extends to a bounded operator

$$\mathcal{D} : H_{C(T) \rtimes_r \Gamma}^m(\widehat{M}, \widehat{E}) \rightarrow H_{C(T) \rtimes_r \Gamma}^{m-1}(\widehat{M}, \widehat{E})$$

for each  $m \in \mathbb{N}$ . Moreover, using an appropriate pseudodifferential calculus, one can show that each extension is  $C(T) \rtimes_r \Gamma$ -Fredholm. See Subsection 3.9 below for more on this point. If the fibers are even-dimensional, then the family  $(D(\theta))_{\theta \in T}$  is  $\mathbb{Z}_2$ -graded odd and we get in this way an index class  $\text{Ind}(\mathcal{D}^+) \in K_0(C(T) \rtimes_r \Gamma)$  which is independent of  $m$  by elliptic regularity. Alternatively, let  $\mathcal{E} := L_{C(T) \rtimes_r \Gamma}^2(\widehat{M}, \widehat{E})$ ; we can consider the bounded operator

$$\mathcal{L} = \frac{\mathcal{D}}{(\text{Id} + \mathcal{D}^2)^{\frac{1}{2}}};$$

the fact that  $\mathcal{L}$  is well defined in  $\mathcal{B}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$ , the algebra of bounded operators on the Hilbert  $C(T) \rtimes_r \Gamma$ -module  $\mathcal{E}$ , requires proof and is based on the continuous functional calculus for *regular* unbounded operators on Hilbert modules. The reader interested in the details may read Proposition 1 of [35] and Baaaj-Julg's work explained in [10] page 433. If the fibers are even dimensional, then  $\text{Ind}(\mathcal{D}^+) \equiv \text{Ind}(\mathcal{L}^+)$ .

If the fibers are odd dimensional we define the index class in  $K_1(C(T) \rtimes_r \Gamma)$  proceeding as in Subsection 3.1. Thus, by definition,

$$\text{Ind}(\mathcal{D}) = \delta[\pi(\frac{1}{2}(\mathcal{L} + \text{Id}))] \in K_1(\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})) \simeq K_1(C(T) \rtimes_r \Gamma)$$

with  $\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E}) \subset \mathcal{B}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$  denoting the sub-algebra of  $C(T) \rtimes_r \Gamma$ -compact operators and

$$\pi : \mathcal{B}_{C(T) \rtimes_r \Gamma}(\mathcal{E}) \rightarrow \mathcal{C}_{C(T) \rtimes_r \Gamma}(\mathcal{E}) = \frac{\mathcal{B}_{C(T) \rtimes_r \Gamma}(\mathcal{E})}{\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})}$$

denoting the projection onto the Calkin algebra.

**3.6. Spectral sections for Dirac operators on foliated bundles.** Both the notion of spectral flow and the APS-boundary value problem are based on the possibility of “dividing in two parts” the spectrum of a self-adjoint Dirac operator on a closed manifold. This is done via the self-adjoint projection  $\chi_{[0, \infty)}(D)$  associated to the non-negative spectrum of the relevant operator  $D$ . In the noncommutative context things are more complicated. Thus let  $\mathcal{D}$  be a  $C(T) \rtimes_r \Gamma$ -linear Dirac operator, associated to a  $\Gamma$ -equivariant family. As already remarked there is a well defined *continuous* functional calculus associated to  $\mathcal{D}$ ; however, as we are working in a  $C^*$ -algebraic context, there is *not* a measurable calculus; thus it does not make sense to consider the operator  $\Pi_{\geq}(\mathcal{D}) := \chi_{[0, \infty)}(\mathcal{D})$  as an element

in  $\mathcal{B}_{C(T) \rtimes_r \Gamma}(L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}))$ .<sup>4</sup> This is the reason why we need a more general notion of spectral projection for “dividing in two parts” the spectrum of  $\mathcal{D}$ ; this is provided by the definition of *spectral section*. We have already encountered this notion for families of Dirac operator parametrized by a compact manifold  $T$ , see 2.1; such a family defines a  $C(T)$ -linear operator and our task now is to pass from the commutative case to the noncommutative case where a group  $\Gamma$  is present and the relevant linearity is with respect to the noncommutative  $C^*$ -algebra  $C(T) \rtimes_r \Gamma$ .

In what follows we shall briefly denote the relevant algebras of  $C(T) \rtimes_r \Gamma$ -linear operators by

$$\mathcal{B}_{C(T) \rtimes_r \Gamma}, \quad \mathcal{K}_{C(T) \rtimes_r \Gamma}, \quad \mathcal{C}_{C(T) \rtimes_r \Gamma}.$$

**Definition 3.2.** A spectral cut  $\chi$  is by definition a function  $\chi \in C^\infty(\mathbb{R}, [0, 1])$  such that  $\chi(x) = 0$  for  $x \ll 0$  and  $\chi(x) = 1$  for  $x \gg 1$ .

Observe that  $\chi(\mathcal{D})$  induces a projection in the Calkin algebra  $\mathcal{C}_{C(T) \rtimes_r \Gamma}$  which does not depend on the choice of the spectral cut  $\chi$ . In fact, always in  $\mathcal{C}_{C(T) \rtimes_r \Gamma}$ , we have the equality

$$\pi\left(\frac{1}{2}(\mathcal{L} + \text{Id})\right) = \pi(\chi(\mathcal{D})), \text{ with } \mathcal{L} = \frac{\mathcal{D}}{(\text{Id} + \mathcal{D}^2)^{\frac{1}{2}}}.$$

Thus the index class  $\text{Ind } \mathcal{D} \in K_1(C(T) \rtimes_r \Gamma)$  is also defined by  $\text{Ind } \mathcal{D} = \delta[\chi(\mathcal{D})]$  for any spectral cut  $\chi$ .

**Definition 3.3.** A spectral section  $\mathcal{P}$  for  $\mathcal{D}$  is a self-adjoint projection  $\mathcal{P} \in \mathcal{B}_{C(T) \rtimes_r \Gamma}$  such that there exist two spectral cuts  $\chi_1, \chi_2$  such that  $\chi_2 \equiv 1$  on a neighborhood of the support of  $\chi_1$  and  $\text{Im } \chi_1(\mathcal{D}) \subset \text{Im } \mathcal{P} \subset \text{Im } \chi_2(\mathcal{D})$ .

**Theorem 3.4.**

- 1) If  $\mathcal{D}$  admits a spectral section then  $\text{Ind } \mathcal{D} = 0$  in  $K_1(C(T) \rtimes_r \Gamma)$ .
- 2) Assume that  $\text{Ind } \mathcal{D} = 0$  in  $K_1(C(T) \rtimes_r \Gamma)$ . Then  $\mathcal{D}$  admits a spectral section  $\mathcal{P}$ .

*Proof.* 1) The proof of [34] page 363 extends immediately to this context.

2) The proof of Theorem 3 of [34] (partially based on the unpublished work of Wu [58]) shows that we just have to prove that  $\text{Id} - \chi(\mathcal{D})$  and  $\chi(\mathcal{D})$  define, for any spectral cut  $\chi$ , two very full projections of  $\mathcal{C}_{C(T) \rtimes_r \Gamma}$ . Then the proof of Lemma 4 of [34] (page 360) shows that we just have to prove that for a given spectral cut  $\chi_1$  there exists  $u \in L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E})$  such that  $\langle \chi_1(\mathcal{D})(u); \chi_1(\mathcal{D})(u) \rangle$  is invertible in  $C(T) \rtimes_r \Gamma$ .

There exists a positive integer  $N$  and open connected subsets  $U_i \subset U'_i$  of  $T$  ( $1 \leq i \leq N$ ) with the following properties. Each  $U_i$  is relatively compact in  $U'_i$ ,  $\cup_{1 \leq i \leq N} U_i = T$ , and for each  $i \in \{1, \dots, N\}$  the restriction of the fibration  $\pi$  to  $\pi^{-1}(U'_i)$  is trivial:  $\pi^{-1}(U'_i) \simeq U'_i \times Z$ . Denote by  $\mu$  a given  $\Gamma$ -invariant riemannian measure on  $\widehat{M}$  and  $v_f$  the volume of a fundamental domain for the action of  $\Gamma$  on  $\widehat{M}$ . Then using an induction argument on  $i \in$

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<sup>4</sup>Needless to say, if  $\mathcal{D}$  is invertible, from  $H^1_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E})$  onto  $L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E})$ , then (see Proposition 1 of [35]) we can indeed define  $\Pi_{\geq}(\mathcal{D})$ , either by taking a smooth approximation of the characteristic function  $\chi_{[0, \infty)}$  or by setting

$$\Pi_{\geq}(\mathcal{D}) := \frac{1}{2} \left( \frac{\mathcal{D}}{|\mathcal{D}|} + \text{Id} \right).$$

$\{1, \dots, N\}$ , we may find an open connected subset  $W \subset Z$  and open subsets  $V'_i \subset \pi^{-1}(U'_i)$  ( $1 \leq i \leq N$ ) with the following three properties:

(a)  $V'_i \simeq U'_i \times W$  for each  $i \in \{1, \dots, N\}$  and  $\mu(\cup_{1 \leq i \leq N} V'_i) \leq \frac{1}{2}v_f$

(b)  $\forall \gamma \in \Gamma \setminus \{e\}, \forall i \in \{1, \dots, N\}, \overline{V'_i} \cdot \gamma \cap \overline{V'_i} = \emptyset$ .

(c)  $\forall \gamma \in \Gamma, \forall i, j \in \{1, \dots, N\}$ , with  $i \neq j$ ,  $\overline{V'_i} \cdot \gamma \cap \overline{V'_j} = \emptyset$ . Now, if the  $U'_i$  are small enough then the proof of Lemma 4 of [34] shows that for any  $\epsilon > 0$  and  $i \in \{1, \dots, N\}$  one can find  $u'_i \in C^0(\overline{V'_i}, \widehat{E})$  such that  $\forall \theta \in U'_i$  one has:

$$(d) \quad |u_i(\theta, \cdot)|_{L^2} = 1, \quad |\chi_1(D)(u_i(\theta, \cdot))|_{L^2} > \frac{2}{3}, \quad |u_i(\theta, \cdot)|_{H^{-1}} < \epsilon.$$

Now consider for each  $i \in \{1, \dots, N\}$   $\phi_i \in C_c^\infty(U'_i)$  such that  $\phi_i \equiv 1$  on  $U_i$ . Set

$$u = \frac{\sum_{i=1}^N \phi_i u_i}{\sqrt{\sum_{i=1}^N \phi_i^2}}.$$

Then properties (a), (b), (c), (d) and the proof of Lemma 4 of [34] show that if  $\epsilon > 0$  is small enough then  $\langle \chi_1(\mathcal{D})(u); \chi_1(\mathcal{D})(u) \rangle$  is invertible in  $C(T) \rtimes_r \Gamma$  which proves the result.  $\square$

### 3.7. Difference class associated to two spectral sections.

**Proposition 3.5.** (Wu) *Let  $\mathcal{P}_1, \mathcal{P}_2$  be two spectral sections for  $\mathcal{D}$ . Then there exists spectral section  $\mathcal{Q}, \mathcal{R}$  for  $\mathcal{D}$  such that for  $j \in \{1, 2\}$ :  $\mathcal{P}_j \mathcal{R} = \mathcal{P}_j = \mathcal{R} \mathcal{P}_j$  and  $\mathcal{P}_j \mathcal{Q} = \mathcal{Q} = \mathcal{Q} \mathcal{P}_j$ .*

*Proof.* We follow closely the unpublished proof of Wu ([58]).

**Lemma 3.6.** *We set  $\mathcal{E} = L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E})$ . Assume the existence of a spectral section for  $\mathcal{D}$ .*

1) *For any spectral cut  $\chi_1$ , there is a smooth spectral cut  $\chi_2$  with  $\chi_1(t)\chi_2(t) = \chi_1(t)$ , and a spectral section  $\mathcal{R}$  satisfying:*

$$\chi_1(\mathcal{D})(\mathcal{E}) \subset \mathcal{R}(\mathcal{E}) \subset \chi_2(\mathcal{D})(\mathcal{E}).$$

2) *Similarly, for any smooth spectral cut  $\chi_2$ , there a spectral cut  $\chi_1$  with  $\chi_1(t)\chi_2(t) = \chi_1(t)$  and a spectral section  $\mathcal{Q}$  satisfying*

$$\chi_1(\mathcal{D})(\mathcal{E}) \subset \mathcal{Q}(\mathcal{E}) \subset \chi_2(\mathcal{D})(\mathcal{E}).$$

*Proof.* 1) Let  $\mathcal{P}$  be a spectral section for  $\mathcal{D}$  satisfying

$$g_1(\mathcal{D})(\mathcal{E}) \subset \mathcal{P}(\mathcal{E}) \subset g_2(\mathcal{D})(\mathcal{E})$$

where  $g_1, g_2$  are smooth spectral cuts. Let  $\chi_1$  be a smooth spectral cut. Choose a smooth spectral cut  $\chi$  satisfying:

$$\chi(t)\chi_1(t) = \chi_1(t) \text{ and } \chi(t)g_2(t) = g_2(t).$$

We have  $\chi(\mathcal{D})\mathcal{P} = \mathcal{P}\chi(\mathcal{D}) = \mathcal{P}$ . Thus:

$$\begin{aligned} (\text{Id} - \chi(\mathcal{D}))(\text{Id} - \mathcal{P}) &= (\text{Id} - \mathcal{P})(\text{Id} - \chi(\mathcal{D})) = \text{Id} - \chi(\mathcal{D}) \\ &= (\text{Id} - \mathcal{P})(\text{Id} - \chi(\mathcal{D}))(\text{Id} - \mathcal{P}). \end{aligned}$$

Working in the  $C^*$ -algebras  $\mathcal{B}_{C(T) \rtimes_r \Gamma}((\text{Id} - \mathcal{P})(\mathcal{E}))$  and

$$\mathcal{K}_{C(T) \rtimes_r \Gamma}((\text{Id} - \mathcal{P})(\mathcal{E})) = (\text{Id} - \mathcal{P})\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})(\text{Id} - \mathcal{P}),$$

let  $\{P_n\} \subset (\text{Id} - \mathcal{P})\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})(\text{Id} - \mathcal{P})$  be an approximate unit,  $P_n \leq P_{n+1}$  and  $P_n$  are projections in  $\mathcal{K}_{C(T) \rtimes_r \Gamma}((\text{Id} - \mathcal{P})(\mathcal{E}))$ . Then we have

$$((\text{Id} - P) - (\text{Id} - \chi(\mathcal{D}))P_n \rightarrow (\text{Id} - \mathcal{P}) - (\text{Id} - \chi(\mathcal{D}))$$

in norm in  $(\text{Id} - \mathcal{P})\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})(\text{Id} - \mathcal{P})$ . Let  $N_0$  be such that

$$\|((\text{Id} - \mathcal{P}) - P_{N_0}) - (\text{Id} - \chi(\mathcal{D}))((\text{Id} - \mathcal{P}) - P_{N_0})\| < \frac{1}{2}.$$

As  $(\text{Id} - \mathcal{P}) - P_{N_0}$  is a projection in  $\mathcal{B}_{C(T) \rtimes_r \Gamma}((\text{Id} - \mathcal{P})(\mathcal{E}))$ , Lemma 5 of [34] (due to Wu) implies that there is a projection  $\mathcal{R}_0$  in  $\mathcal{B}_{C(T) \rtimes_r \Gamma}((\text{Id} - \mathcal{P})(\mathcal{E}))$ ,  $\mathcal{R}_0 - (\text{Id} - \mathcal{P}) \in \mathcal{K}_{C(T) \rtimes_r \Gamma}((\text{Id} - \mathcal{P})(\mathcal{E}))$ , such that

$$\begin{aligned} (\text{Id} - \mathcal{R}_0)((\text{Id} - \mathcal{P})(\mathcal{E})) &= (\text{Id} - \chi(\mathcal{D}))((\text{Id} - \mathcal{P}) - P_{N_0})(\text{Id} - \mathcal{P})(\mathcal{E}) \\ &\subset (\text{Id} - \chi(\mathcal{D}))(\mathcal{E}). \end{aligned}$$

Let  $\mathcal{R}_1 \in \mathcal{B}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$  be the projection which is  $\mathcal{R}_0$  on  $(\text{Id} - \mathcal{P})(\mathcal{E})$  and  $\text{Id}$  on  $\mathcal{P}(\mathcal{E})$ . Then  $\mathcal{R}_1 - \mathcal{P} \in \mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$  and

$$\mathcal{R}_1 \chi_1(\mathcal{D}) = \chi_1(\mathcal{D}) \mathcal{R}_1 = \chi_1(\mathcal{D}) \text{ and } \mathcal{R}_1 g_2(\mathcal{D}) = g_2(\mathcal{D}) \mathcal{R}_1 = g_2(\mathcal{D}).$$

To modify  $\mathcal{R}_1$  to the desired spectral section, let  $\psi_N(t) := \psi(t/N)$  where  $\psi$  is a smooth spectral cut such that  $\psi(t) \equiv 1$  on  $[-1, +\infty[$ . Then since  $\mathcal{D}$  is a regular operator, one checks easily that as  $N \rightarrow +\infty$

$$\psi_N(\mathcal{D})(\chi_1(\mathcal{D}) - \mathcal{R}_1) \rightarrow \chi_1(\mathcal{D}) - \mathcal{R}_1, \quad \psi_N(\mathcal{D})(g_2(\mathcal{D}) - \mathcal{R}_1) \rightarrow g_2(\mathcal{D}) - \mathcal{R}_1$$

in norm in  $\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$ . Let  $N_0$  be chosen such that  $\psi_{N_0} \chi_1 = \chi_1$  and  $\psi_{N_0} g_2 = g_2$  and

$$\|\mathcal{R}_1 - \chi_1(\mathcal{D}) + \psi_{N_0}(\mathcal{D})(\chi_1(\mathcal{D}) - \mathcal{R}_1)\| = \|\mathcal{R}_1 - \psi_{N_0}(\mathcal{D})\mathcal{R}_1\| < \frac{1}{2}.$$

Applying Lemma 5 of [34], we get a projection  $\mathcal{R} = \mathcal{R}^2 = \mathcal{R}^* \in \mathcal{B}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$ ,  $\mathcal{R} - \mathcal{R}_1 \in \mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})$ , such that

$$\mathcal{R}(\mathcal{E}) = \psi_{N_0} \mathcal{R}_1(\mathcal{E}) \subset \psi_{N_0}(\mathcal{D})(\mathcal{E}).$$

Let  $\chi_2(t) := \psi_{N_0}(t)$ . Then  $\chi_2$  is a smooth spectral cut and we have  $\mathcal{R}(\chi) \subset \chi_2(\mathcal{D})(\mathcal{E})$ . On the other hand, since

$$\chi_1(\mathcal{D}) = \psi_{N_0}(\mathcal{D})\chi_1(\mathcal{D}) = \psi_{N_0} \mathcal{R}_1 \chi_1(\mathcal{D}),$$

we get  $\chi_1(\mathcal{D})(\mathcal{E}) \subset \psi_{N_0}(\mathcal{D})\mathcal{R}_1(\mathcal{E}) = \mathcal{R}(\mathcal{E})$ . Similarly, we also have  $g_2(\mathcal{D})(\mathcal{E}) \subset \mathcal{R}(\mathcal{E})$ . Therefore,  $\mathcal{R}$  is a spectral section with the desired property:

$$\chi_1(\mathcal{D})(\mathcal{E}) \subset \mathcal{R}(\mathcal{E}) \subset \chi_2(\mathcal{D})(\mathcal{E}).$$

2) is proved similarly □

We go back to the proof of Proposition 3.5. There are smooth spectral cuts  $g_1, g_2$  such that:

$$g_1(\mathcal{D})(\mathcal{E}) \subset \mathcal{P}_j(\mathcal{E}) \subset g_2(\mathcal{D})(\mathcal{E}), \quad j = 1, 2.$$

Applying Lemma 3.6, we find smooth spectral cuts  $\chi_1, \chi_2$  with  $\chi_1 \cdot g_1 = \chi_1$ ,  $\chi_2 \cdot g_2 = g_2$  and spectral sections  $\mathcal{R}, \mathcal{Q}$  with

$$\chi_1(\mathcal{D})(\mathcal{E}) \subset \mathcal{Q}(\mathcal{E}) \subset \mathcal{P}_j(\mathcal{E}) \subset g_2(\mathcal{D})(\mathcal{E}) \mathcal{R}(\mathcal{E}) \subset \chi_2(\mathcal{D})(\mathcal{E}).$$

Now one checks easily that  $\mathcal{Q}, \mathcal{R}$  satisfy the desired property:

$$\mathcal{P}_j \mathcal{R} = \mathcal{P}_j = \mathcal{R} \mathcal{P}_j, \mathcal{P}_j \mathcal{Q} = \mathcal{Q} = \mathcal{Q} \mathcal{P}_j.$$

□

Recall the following stability result:

$$K_0(\mathcal{K}_{C(T) \rtimes_r \Gamma}(\mathcal{E})) \simeq K_0(C(T) \rtimes_r \Gamma).$$

**Definition 3.7.** Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  spectral sections for  $\mathcal{D}$ . Then there exists a difference class  $[\mathcal{P}_1 - \mathcal{P}_2] \in K_0(C(T) \rtimes_r \Gamma)$  defined as follows. Choose a spectral section  $\mathcal{P}'$  of  $\mathcal{D}$  such that  $\mathcal{P}_i \mathcal{P}' = \mathcal{P}' = \mathcal{P}' \mathcal{P}_i$  for  $i \in \{1, 2\}$ . Then  $\mathcal{P}_1 - \mathcal{P}'$  and  $\mathcal{P}_2 - \mathcal{P}'$  induce projections in  $\mathcal{K}_{C(T) \rtimes_r \Gamma}$  and  $[\mathcal{P}_1 - \mathcal{P}_2] = [\mathcal{P}_1 - \mathcal{P}'] - [\mathcal{P}_2 - \mathcal{P}']$  is well defined as an element of  $K_0(C(T) \rtimes_r \Gamma)$ .

**3.8. The noncommutative spectral flow on foliated bundles.** Now we consider a continuous family  $(g_u)_{u \in [0,1]}$  of  $\Gamma$ -equivariant vertical metrics of the fibration  $\widehat{M} \rightarrow T$ . We consider also a continuous family  $(h_u)_{u \in [0,1]}$  of  $\Gamma$ -equivariant hermitian metrics on the Clifford module  $\widehat{E} \rightarrow \widehat{M}$  and a continuous family of  $\Gamma$ -equivariant hermitian connections  $(\nabla_u)_{u \in [0,1]}$  on  $\widehat{E}$ . For each  $u \in [0, 1]$  one then gets a  $C(T) \rtimes_r \Gamma$ -Hilbert module  $L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}, u)$  which depends on  $u$  via  $g_u$  and  $h_u$ . These Hilbert modules are all isomorphic, being each one isomorphic to the standard Hilbert  $C(T) \rtimes_r \Gamma$ -module  $\mathcal{H}_{C(T) \rtimes_r \Gamma}$ . Let  $(\mathcal{D}_u)_{u \in [0,1]}$  be the associated family of  $C(T) \rtimes_r \Gamma$ -linear Dirac type operators where each  $\mathcal{D}_u$  acts on  $L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}, u)$ . This family is continuous in the following sense.

Let  $\{U_j, 1 \leq j \leq l\}$  be a finite set of open subsets of  $\widehat{M}$  satisfying the three following properties:

- (i) Each  $U_j$  is diffeomorphic to the open ball  $B(0, 1)$  of  $\mathbb{R}^n$ .
- (ii)  $\cup_{1 \leq j \leq l} U_j \cdot \Gamma = \widehat{M}$
- (iii) For each  $j \in \{1, \dots, l\}$ , the restriction to  $U_j$  of the following bundles is trivial:  $TZ \rightarrow \widehat{M}, \widehat{E} \rightarrow \widehat{M}, T\widehat{M} \rightarrow \widehat{M}$ .

Then the restriction of  $\mathcal{D}_u$  to each  $U_j$  induces a differential operator of order one acting on  $C^\infty(B(0, 1); \mathbb{C}^N)$  (for a suitable  $N$ ):  $D_{j,u} = \sum_{k=1}^n a_k(z, u) \partial_{z_k} + b(z, u)$  ( $z \in B(0, 1)$ ).

Then we observe that the coefficients of  $D_{j,u}$ ,  $u \rightarrow a_k(z, u)$ ,  $u \rightarrow b(z, u)$  belong to  $C^0([0, 1]; C^\infty(B(0, 1); \mathbb{C}^N))$ . We then say that the family  $(\mathcal{D}_u)_{u \in [0,1]}$  is continuous in this sense (the definition does not depend on the choice of the  $U_j$  and of the trivializations).

We assume that the index class defined by one (and thus any) of the  $\mathcal{D}_u$  is trivial. Recall that the family  $(\mathcal{D}_u)_{u \in [0,1]}$  defines in a standard way a  $(C^0[0, 1] \otimes C(T) \rtimes_r \Gamma)$ -linear operator acting on the  $(C^0[0, 1] \otimes C(T) \rtimes_r \Gamma)$ -Hilbert module defined by the bundle  $\mathcal{E}$  over  $[0, 1]$  with fiber  $\mathcal{E}_u = L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}, u)$ ,  $u \in [0, 1]$ . This operator is of Dirac-type and therefore  $(C^0[0, 1] \otimes C(T) \rtimes_r \Gamma)$ -Fredholm by the Mishchenko-Fomenko calculus. Recall also that we have a natural isomorphism

$$K_1(C^0[0, 1] \otimes C(T) \rtimes_r \Gamma) \simeq K_1(C(T) \rtimes_r \Gamma)$$

which is implemented by the evaluation map  $f(\cdot) \otimes \lambda \rightarrow f(0)\lambda$ . This implies that the total index class of  $(\mathcal{D}_u)_{u \in [0,1]}$ , in  $K_1(C^0[0, 1] \otimes C(T) \rtimes_r \Gamma)$ , is also zero. By the existence theorem this implies that the family  $(\mathcal{D}_u)_{u \in [0,1]}$  admits a (total) spectral section  $(\mathcal{P}_u)_{u \in [0,1]}$ . The following definition is inspired by the work of Dai-Zhang [13]:

**Definition 3.8.** If  $\mathcal{Q}_0$  (resp.  $\mathcal{Q}_1$ ) is a spectral section associated with  $\mathcal{D}_0$  (resp.  $\mathcal{D}_1$ ) then the noncommutative spectral flow from  $(\mathcal{D}_0, \mathcal{Q}_0)$  to  $(\mathcal{D}_1, \mathcal{Q}_1)$  through  $(\mathcal{D}_u)_{u \in [0,1]}$ , is the  $K_0(C(T) \rtimes_r \Gamma)$ -class:

$$\text{sf}((\mathcal{D}_u)_{u \in [0,1]}; \mathcal{Q}_0, \mathcal{Q}_1) = [\mathcal{Q}_1 - \mathcal{P}_1] - [\mathcal{Q}_0 - \mathcal{P}_0] \in K_0(C(T) \rtimes_r \Gamma).$$

Proceeding as in the work of Dai-Zhang one can prove that the definition does not depend on the choice of total spectral section  $(\mathcal{P}_u)_{u \in [0,1]}$ .

**3.9. Trivializing perturbations.** Let  $\text{Ind}(\mathcal{D})$  be equal to zero. The operator  $\mathcal{D}$  will not be, in general, invertible. Let  $\mathcal{P}$  be a spectral section for  $\mathcal{D}$ ; then  $\mathcal{P}$  fixes a specific trivialization of  $\text{Ind}(\mathcal{D})$ ; this is achieved by defining a perturbation  $\mathcal{A}_{\mathcal{P}}^0$  of  $\mathcal{D}$  such that  $\mathcal{D} + \mathcal{A}_{\mathcal{P}}^0$  be invertible. This subsection is devoted to make this statement precise. First we introduce the relevant space of pseudodifferential operators.

**Definition 3.9.**

1) Denote by  $\Psi_{C(T) \rtimes_r \Gamma}^{-\infty}(\widehat{M}, \widehat{E})$  the set of operators  $R \in \mathcal{B}_{C(T) \rtimes_r \Gamma}$  such that for any  $N \in \mathbb{N}$ ,  $R$  extends as a continuous operator

$$H_{C(T) \rtimes_r \Gamma}^{-N}(\widehat{M}, \widehat{E}) \longrightarrow H_{C(T) \rtimes_r \Gamma}^N(\widehat{M}, \widehat{E}).$$

2) Let  $k \in \mathbb{Z}$ . Denote by  $\Psi_{C(T) \rtimes_r \Gamma}^k(\widehat{M}, \widehat{E})$  the set of bounded operators  $A : H_{C(T) \rtimes_r \Gamma}^k(\widehat{M}, \widehat{E}) \rightarrow L_{C(T) \rtimes_r \Gamma}^2(\widehat{M}, \widehat{E})$  satisfying the following property. There exists  $\epsilon > 0$  and one can write  $A = B + R$  where  $R \in \Psi_{C(T) \rtimes_r \Gamma}^{-\infty}(\widehat{M}, \widehat{E})$  and  $B = (B_{\theta})_{\theta \in T}$  is a smooth  $\Gamma$ -equivariant family of fiberwise pseudo-differential operators of order  $k$  such that the Schwartz kernels of each  $B_{\theta}$  vanish outside an  $\epsilon$ -neighborhood of the diagonal.

**Remarks.**

1) The operator  $B$  appearing in definition 3.9 is an element in  $\Psi_{C(T) \rtimes_r \Gamma, c}^k(\widehat{M}, \widehat{E})$ , the space of  $\Gamma$ -equivariant families of order  $k$  pseudodifferential operators with Schwartz kernel of compact  $\Gamma$ -support, i.e. the support of the Schwartz kernel, viewed as an element in  $\widehat{M} \times_{\pi} \widehat{M}$  ( $\pi$  denotes the projection  $\widehat{M} \rightarrow T$ ), defines a compact set in  $\widehat{M} \times_{\pi} \widehat{M}/\Gamma$ .

2) One can prove, see for example Morioshi and Natsume [49, Section 3], that a  $\Gamma$ -equivariant family of Dirac operators  $(D(\theta))_{\theta \in T}$  admits a parametrix  $(Q(\theta))_{\theta \in T}$  in  $\Psi_{C(T) \rtimes_r \Gamma, c}^{-1}(\widehat{M}, \widehat{E})$  with rests in  $\Psi_{C(T) \rtimes_r \Gamma, c}^{-\infty}(\widehat{M}, \widehat{E})$ . Moreover, an element  $S = (S(\theta))_{\theta \in T} \in \Psi_{C(T) \rtimes_r \Gamma, c}^k(\widehat{M}, \widehat{E})$  defines a bounded  $C(T) \rtimes_r \Gamma$ -linear operator from  $H_{C(T) \rtimes_r \Gamma}^m(\widehat{M}, \widehat{E})$  to  $H_{C(T) \rtimes_r \Gamma}^{m-k}(\widehat{M}, \widehat{E})$ . It is precisely this result that is used in order to define the index class associated to  $(D(\theta))_{\theta \in T}$ .

Proceeding as in the proof of Propositions 2.5 and 2.10 of Leichtnam-Piazza [31] (and thus, ultimately, as in Lemma 8 of [45]) one can prove the following

**Proposition 3.10.** *Assume that  $\text{Ind } \mathcal{D} = 0$  in  $K_1(C(T) \rtimes_r \Gamma)$  and let  $\mathcal{P}$  be a spectral section for  $\mathcal{D}$ . Then  $\mathcal{P} \in \Psi_{C(T) \rtimes_r \Gamma}^0(\widehat{M}, \widehat{E})$  and there exists a self-adjoint operator  $\mathcal{A}_{\mathcal{P}}^0 \in \Psi_{C(T) \rtimes_r \Gamma}^{-\infty}(\widehat{M}, \widehat{E})$  with the following three properties.*

(1) *We can find a real  $R > 0$  such that  $\varphi(\mathcal{D}) \circ \mathcal{A}_{\mathcal{P}}^0 \equiv 0$  for any function  $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{C})$  vanishing on  $[-R, R]$ .*

(2)  *$\mathcal{D} + \mathcal{A}_{\mathcal{P}}^0$  is invertible from  $H_{C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E})$  onto  $L_{C(T) \rtimes_r \Gamma}^2(\widehat{M}, \widehat{E})$ .*

(3)  $\mathcal{P}$  is equal to the projection onto the positive part of  $\mathcal{D} + \mathcal{A}_{\mathcal{P}}^0$ ,

$$\mathcal{P} = \frac{1}{2} \left( \frac{\mathcal{D} + \mathcal{A}_{\mathcal{P}}^0}{|\mathcal{D} + \mathcal{A}_{\mathcal{P}}^0|} + \text{Id} \right).$$

We shall not reproduce the proof of this proposition here. We simply add that the construction of  $\mathcal{A}_{\mathcal{P}}^0$  does depend on choices; however, if  $\mathcal{B}_{\mathcal{P}}^0$  is a different trivializing perturbation associated to  $\mathcal{P}$ , then

$$(3.3) \quad \text{the operator } \mathcal{D} + r\mathcal{A}_{\mathcal{P}}^0 + (1-r)\mathcal{B}_{\mathcal{P}}^0 \text{ is invertible } \forall r \in [0, 1].$$

#### 4. INDEX CLASSES ON FOLIATED BUNDLES WITH BOUNDARY

##### 4.1. Preliminaries: numerical indices on manifolds with boundary.

**The invertible case.** In order to orient the reader we review in some detail the various indices that can be attached to a Dirac operator on an even dimensional manifold with boundary.

We thus consider a smooth connected compact manifold with boundary  $M$ . We fix a boundary defining function  $x \in C^\infty(M)$ . Let  $g_M$  be a riemannian metric on  $M$ ; we assume this metric to be of product type near the boundary. We consider on  $M$  a unitary Clifford module  $E$  endowed with a unitary connection  $\nabla^E$  which is Clifford with respect to the Levi-Civita connection associated to the metric  $g_M$ . We obtain in this way a Dirac-type operator  $D$ . Suppose now that  $M$  is *even* dimensional so that  $E$  is  $\mathbb{Z}_2$ -graded:  $E = E^+ \oplus E^-$ . The Clifford bundle associated to  $T^*(\partial M)$  and to the boundary metric acts in a natural way on  $E|_{\partial M}$ :

$$\forall e \in E|_{\partial M}, \quad \forall \eta \in T^*\partial M, \quad \text{cl}_\partial(\eta)(e|_{\partial M}) := \text{cl}(dx) \text{cl}(\eta)(e|_{\partial M})$$

We define  $E_0$  to be  $E^+|_{\partial M}$ . It is a unitary Clifford bundle with respect to  $\text{cl}_\partial(\cdot)$ . It is endowed with the induced Clifford connection. We denote by  $D_0$  the associated Dirac operator and we call it *the boundary operator of  $D$* . Finally, we identify  $E^-|_{\partial M}$  with  $E_0$  through Clifford multiplication by  $\text{cl}(idx)$ , denoted in the sequel by  $\sigma$ . With these identifications the operator  $D^+$  can be written near the boundary as  $\sigma(\partial_x + D_0)$ .

As already remarked the APS-boundary value problem is obtained by considering the operator  $D^+$  with domain

$$\{u \in C^\infty(M, E^+) \mid u|_{\partial M} \in \text{Ker} \Pi_{\geq}\}$$

with  $\Pi_{\geq} = \chi_{[0, \infty)}(D_0)$ . Let  $\text{ind}(D^+, \Pi_{\geq})$  be the APS-index and assume for the time being that  $D_0$  is *invertible*. Then we can describe this index in a different way: we can attach an infinite cylinder  $(-\infty, 0] \times \partial M$  to  $M$  along its boundary  $\partial M$ , thus obtaining a manifold with cylindrical ends  $M_{\text{cyl}}$  and with product metric  $dx^2 + g_{\partial M}$  along the cylinder. The operators  $D$  extends in a natural way to an operator  $D_{\text{cyl}}$  on the manifold  $M_{\text{cyl}}$ , acting on the sections of the bundle  $E_{\text{cyl}}$  obtained by extending in an obvious way  $E$ . It turns out that this operator is Fredholm, as a bounded linear map from  $H^1(M_{\text{cyl}}, E_{\text{cyl}})$  to  $L^2(M_{\text{cyl}}, E_{\text{cyl}})$ , and that its index is equal to the APS-index:

$$(4.1) \quad \text{ind}(D^+, \Pi_{\geq}) = \text{ind}(D_{\text{cyl}}^+).$$

This equality is explained in the original paper of Atiyah-Patodi-Singer where it is proved, more precisely, that the kernel and cokernel of the two operators are naturally isomorphic.

Connected with the cylindrical picture is Melrose'  $b$ -picture [43]: the change of coordinates  $x = \log y$  compactifies  $M_{\text{cyl}}$  to a compact manifold with boundary  ${}^bM$  but with a degenerate metric  ${}^bg$  which can be written as  $dy^2/y^2 + g_{\partial M}$  near the boundary. The operator on  $M_{\text{cyl}}$  then defines in a natural way a differential operator  ${}^bD$  on the compactified manifold  ${}^bM$ ; up to a bundle isomorphism the operators  ${}^bD^\pm$  can be written near the boundary as  $\pm y\partial_y + D_0$ ; this means that  ${}^bD$  is generated by the vector fields on  ${}^bM$  which are tangent to the boundary;  ${}^bD$  is therefore, by definition, a  $b$ -differential operator. Melrose has developed on  ${}^bM$  a pseudodifferential calculus, which extends the algebra of  $b$ -differential operators; this is known as the  $b$ -calculus and it can be used, among other things, in order to show that  ${}^bD$  is Fredholm on naturally defined  $b$ -Sobolev spaces  $H_b^m({}^bM, {}^bE)$ , with index equal to the APS-index. It should certainly be remarked that the cylindrical picture and the  $b$ -picture are two different descriptions of the same mathematical object.

*Summarizing:* if the boundary operator  $D_0$  is invertible

$$(4.2) \quad \text{ind}(D^+, \Pi_{\geq}) = \text{ind}({}^bD^+) = \text{ind}(D_{\text{cyl},+}).$$

**The general case.** In the  $b$ -picture ( $\equiv$  cylindrical picture) it is fundamental to assume that  $D_0$  is invertible; if the kernel of  $D_0$  is non-trivial, then  ${}^bD$  will *not* define a Fredholm operator. Still, the APS-index is indeed equal to a  $b$ -index but for a *perturbed*  $b$ -operator; we shall now describe this fundamental point in full generality, thus considering the APS-index  $\text{ind}(D^+, P)$  associated to an arbitrary spectral section  $P$  for  $D_0$ .

As already remarked in Subsection 3.9 one can prove that there is a smoothing operator  $A_P^0$  on the boundary  $\partial M$  such that  $D_0 + A_P^0$  is invertible. The perturbation  $A_P^0$  can be extended from the boundary to the interior, thus defining a smoothing  $b$ -operator  $A_P^+$ . The construction of this operator will be recalled below, in the general case of foliated bundles. The operator  ${}^bD_P^+ := {}^bD^+ + A_P^+$  is now Fredholm on  $b$ -Sobolev spaces  $H_b^m$ , with index independent of  $m$  and equal to the APS-index  $\text{ind}(D^+, P)$ . See [45] for proofs and details. As an example, consider  $P = \Pi_{\geq}$  but assume that  $\text{Ker} D_0 \neq 0$ ; then  $A_P^0$  is nothing but the  $L^2$ -orthogonal projection onto  $\text{Ker} D_0$ .

Notice that we could extend  $A_P^0$  to an operator on the cylindrical manifold  $M_{\text{cyl}}$  by simply employing a cut-off function  $\phi$  equal to 1 on the attached half-cylinder and equal to zero on the complement of the collar neighbourhood of the boundary of  $M$ . The resulting operator  $D_{\text{cyl},+} + \sigma A_P^0 \phi$  can be either viewed as an operator on the manifold  $M_{\text{cyl}}$  or as an operator on  ${}^bM$ ; as such it does not define a  $b$ -pseudodifferential operator; however, it is still possible to prove that it is Fredholm as a map  $H^1(M_{\text{cyl}}, E_{\text{cyl}}) \rightarrow L^2(M_{\text{cyl}}, E_{\text{cyl}})$  or, equivalently, as a map  $H_b^1({}^bM, {}^bE) \rightarrow L_b^2({}^bM, {}^bE)$  and with index equal to  $\text{ind}(D^+, P)$ .

*Summarizing:*

if  $P$  is a spectral section for the boundary operator  $D_0$  then

$$(4.3) \quad \text{ind}(D^+, P) = \text{ind}({}^bD^+ + A_P^+) = \text{ind}(D_{\text{cyl},+} + \sigma A_P^0 \phi).$$

We remark that in order to establish an *index formula* for one of these 3 indices, it is very useful to consider the  $b$ -perturbation  $A_P^+$ ; such a formula is obtained in [45].

**4.2. Foliated bundles with boundary.** Let  $\Gamma$  be a finitely generated discrete group. Let  $T$  be a smooth closed compact connected manifold on which  $\Gamma$  acts on the right. Let  $\widehat{M}$  be a manifold *with boundary* on which  $\Gamma$  acts freely, properly and cocompactly on the right: the

quotient space  $M = \widehat{M}/\Gamma$  is thus a smooth compact manifold with boundary. We assume that  $\widehat{M}$  fibers over  $T$  and that the resulting fibration

$$\pi : \widehat{M} \rightarrow T$$

is a  $\Gamma$ -equivariant fibration with fibers  $\pi^{-1}(\theta), \theta \in T$ , that are transverse to  $\partial\widehat{M}$  and of dimension  $2k$  (on a manifold with boundary we are always assuming that the fibers are even-dimensional). Notice that each fiber is a smooth manifold with boundary; we shall also denote the typical fiber of  $\pi : \widehat{M} \rightarrow T$  by  $Z$ . We choose a  $\Gamma$ -invariant product-like metric on the vertical tangent bundle  $TZ$ . Finally, we assume the existence of a  $\Gamma$ -equivariant spin structure on  $TZ$  that is fixed once and for all. We denote by  $S^Z \rightarrow \widehat{M}$  the associated spinor bundle.

The compact manifold with boundary  $M$  inherits a foliation  $\mathcal{F}$ , with leaves equal to the image of the fibres of  $\pi : \widehat{M} \rightarrow T$  under the quotient map  $\widehat{M} \rightarrow M = \widehat{M}/\Gamma$ . Notice that the foliation  $\mathcal{F}$  is transverse to the boundary of  $M$ .

**Example.** Let  $X$  be a compact manifold with boundary and let  $\Gamma \rightarrow \widetilde{X} \rightarrow X$  be a Galois cover of  $X$ . Let  $T$  be a smooth compact manifold on which  $\Gamma$  acts by diffeomorphisms. We consider  $\widehat{M} = \widetilde{X} \times T$ ,  $\pi =$  projection onto the second factor,  $M = \widetilde{X} \times_{\Gamma} T := (\widetilde{X} \times T)/\Gamma$  where we let  $\Gamma$  act on  $\widetilde{X} \times T$  diagonally. As a particular example of this construction consider a smooth closed riemann surface  $\Sigma$  of genus  $g > 1$  and let  $\Gamma = \pi_1(\Sigma)$ , a discrete subgroup of  $PSL(2, \mathbb{R})$ . Let  $\{p_1, \dots, p_k\}$  points in  $\Sigma$  and let  $D_j \subset \Sigma$  be a small open disc around  $p_j$ . Let  $D = \cup_{j=1}^k D_j$ . Then we can consider  $X := \Sigma \setminus D$ ,  $\Gamma \rightarrow \widetilde{X} \rightarrow X$  the Galois cover induced by the universal cover  $\mathbb{H}^2 \rightarrow \Sigma$ ,  $T = S^1$ , with  $\Gamma$  acting on  $S^1$  by fractional linear transformations.

Next we consider a  $\Gamma$ -invariant boundary defining function  $x$  of  $\partial\widehat{M}$  and a  $\Gamma$ -equivariant complex hermitian vector bundle  $\widehat{V} \rightarrow \widehat{M}$  endowed with a  $\Gamma$ -invariant hermitian connection  $\widehat{\nabla}$ . We then set  $\widehat{E} = S^Z \otimes \widehat{V} = \widehat{E}^+ \oplus \widehat{E}^-$  which defines a smooth  $\Gamma$ -equivariant family of  $\mathbb{Z}_2$ -graded hermitian Clifford modules on the fibers  $\pi^{-1}(\theta), \theta \in T$ . We then get a smooth family of  $\Gamma$ -equivariant  $\mathbb{Z}_2$ -graded Dirac type operators

$$D(\theta) = \begin{pmatrix} 0 & D^-(\theta) \\ D^+(\theta) & 0 \end{pmatrix}, \theta \in T$$

acting fiberwise on  $C_c^\infty(\widehat{M}, \widehat{E})$ . Moreover in a collar neighborhood ( $\simeq [0, 1] \times \partial\pi^{-1}(\theta) = \{(x, y)\}$ ) of  $\partial\pi^{-1}(\theta)$  we may write:

$$D^+(\theta) = \sigma(\partial_x + D_0(\theta))$$

where  $D_0(\theta)$  is the induced boundary Dirac type operator acting on

$$C^\infty(\partial\pi^{-1}(\theta), \widehat{E}_{|\partial\pi^{-1}(\theta)}^+).$$

Observe that our family can also be thought as a longitudinal operator on  $(M, \mathcal{F})$  acting on the sections of  $E := \widehat{E}/\Gamma$ .

The family  $(D(\theta))_{\theta \in T}$  defines a  $C(T) \rtimes_r \Gamma$ -linear  $\mathbb{Z}_2$ -graded Dirac-type operator  $\mathcal{D}$  acting on  $C_c^\infty(\widehat{M}, \widehat{E})$ . Similarly, the family  $(D_0(\theta))_{\theta \in T}$  defines a  $C(T) \rtimes_r \Gamma$ -linear Dirac type operator  $\mathcal{D}_0$  acting on the space  $C_c^\infty(\partial\widehat{M}, \widehat{E}_{|\partial\widehat{M}}^+)$ .

Corresponding to the above metrically-incomplete picture, there is a  $b$ -picture, obtained by attaching an infinite cylinder  $(-\infty, 0] \times \partial\widehat{M}$  to  $\widehat{M}$  and compactifying it as we did in the previous subsection. We shall keep the same notation for the resulting manifold; we shall denote by  ${}^b\mathcal{D}$  the  $C(T) \rtimes_r \Gamma$ -linear operator defined by the  $\Gamma$ -equivariant family of  $b$ -Dirac operators

$${}^bD(\theta) = \sigma(y\partial_y + D_0(\theta))$$

associated to the  $b$ -data.

In our recent paper [35] we develop a  $b$ -calculus  $\Psi_{b,C(T)\rtimes_r\Gamma}^*$  on foliated bundles with boundary; we then employ such a calculus in order to establish the following

**Theorem 4.1.** *Assume that there exists a real  $\epsilon > 0$  such that for any  $\theta \in T$ , the  $L^2$ -spectrum of  $D_0(\theta)$  acting on  $L^2(\partial\pi^{-1}(\theta), \widehat{E}_{|\partial\pi^{-1}(\theta)}^+)$  does not meet  $] -\epsilon, \epsilon[$ . Then  $({}^bD^+(\theta))$  defines  $\forall m \in \mathbb{N}^*$  a  $C(T) \rtimes_r \Gamma$ -linear bounded operator*

$${}^b\mathcal{D} : H_{b,C^0(T)\rtimes_r\Gamma}^m(\widehat{M}, \widehat{E}^+) \longrightarrow H_{b,C^0(T)\rtimes_r\Gamma}^{m-1}(\widehat{M}, \widehat{E}^-)$$

which is invertible modulo  $C^0(T) \rtimes_r \Gamma$ -compacts. There is a well defined  $b$ -index class  $\text{Ind}_b({}^bD^+)$  in  $K_0(C^0(T) \rtimes_r \Gamma)$ , independent of  $m$ .

Let us consider the operator  $\mathcal{D}_0$  defined by the boundary family  $(D_0(\theta))_{\theta \in T}$ ; it defines a (regular) unbounded operator on the  $C(T) \rtimes_r \Gamma$ -Hilbert module  $L_{C(T)\rtimes_r\Gamma}^2$ . One can prove (see [35, Proposition 1]) that if the hypothesis of the above theorem holds then  $\mathcal{D}_0$  is  $L_{C(T)\rtimes_r\Gamma}^2$ -invertible with domain  $H_{C(T)\rtimes_r\Gamma}^1$  and that its inverse is induced by the  $\Gamma$ -equivariant family of operators  $\{D_0(\theta)^{-1}\}_{\theta \in T}$ . We can thus consider the  $C(T) \rtimes_r \Gamma$ -linear bounded operator

$$(4.4) \quad \Pi_{\geq}(\mathcal{D}_0) := \frac{1}{2} \left( \frac{\mathcal{D}_0}{|\mathcal{D}_0|} + \text{Id} \right).$$

This is in fact a self-adjoint projection and can be used in order to define an APS-index class  $\text{Ind}^{APS}(\mathcal{D}^+, \Pi_{\geq}(\mathcal{D}_0)) \in K_0(C(T) \rtimes_r \Gamma)$ ; we shall see the details below. We shall also see that

$$\text{Ind}_b({}^bD^+) = \text{Ind}^{APS}(\mathcal{D}^+, \Pi_{\geq}(\mathcal{D}_0)) \in K_0(C(T) \rtimes_r \Gamma).$$

Thus in the *invertible case* we can extend to the present noncommutative context the basic results recalled for the numerical indices in Subsection 4.1. In the non-invertible case the operator  ${}^b\mathcal{D} : H_{b,C^0(T)\rtimes_r\Gamma}^m(\widehat{M}, \widehat{E}^+) \longrightarrow H_{b,C^0(T)\rtimes_r\Gamma}^{m-1}(\widehat{M}, \widehat{E}^-)$  will *not* be  $C(T) \rtimes_r \Gamma$ -Fredholm; similarly, as already remarked, in the non-invertible case the operator (4.4) does not make sense as a bounded  $C(T) \rtimes_r \Gamma$ -linear operator and we cannot define a APS-index class. The way out in the general case is therefore to consider spectral sections  $\mathcal{P}$  for the boundary operator  $\mathcal{D}_0$ ; the existence of these spectral sections follow from our basic result, Theorem 3.4, and the cobordism invariance of the index class associated to  $\mathcal{D}_0$  in  $K_1(C(T) \rtimes_r \Gamma)$ , a result that will be established in the next subsection.

### 4.3. Cobordism invariance.

**Theorem 4.2.** *Let  $\mathcal{D}_0$  be the  $C(T) \rtimes_r \Gamma$ -linear operator defined by the boundary family  $(D_0(\theta))_{\theta \in T}$ . One has  $\text{Ind } \mathcal{D}_0 = 0$  in  $K_1(C(T) \rtimes_r \Gamma)$ .*

*Proof.* The proof employs equivariant  $KK$ -theory. It is easy to see that  $\mathcal{D}_0$  defines a class  $[\mathcal{D}_0]$  in the  $\Gamma$ -equivariant Kasparov group  $KK_\Gamma^1(C_0(\partial\widehat{M}), C(T))$ . Recall that, since  $C_0(\partial\widehat{M}) \rtimes \Gamma$  is Morita equivalent to  $C(\partial M)$ , one has a natural map  $\Theta : KK_\Gamma^1(C_0(\partial\widehat{M}), C(T)) \rightarrow KK^1(C(\partial M), C(T) \rtimes_r \Gamma)$ . If  $\pi^{\partial M} : \partial M \rightarrow \text{pt}$  denotes the mapping of  $\partial M$  to a point, then, under the natural isomorphism  $KK^1(\mathbb{C}, C(T) \rtimes_r \Gamma) \simeq K_1(C(T) \rtimes_r \Gamma)$ , we have  $\text{Ind } \mathcal{D}_0 = \pi_*^{\partial M} \circ \Theta([\mathcal{D}_0])$ .

Let  $C_{0, \partial\widehat{M}}(\widehat{M}) \subset C_0(\widehat{M})$  denote the ideal of continuous functions on  $\widehat{M}$  vanishing on the boundary, let  $i$  be the natural inclusion of  $\partial\widehat{M}$  into  $\widehat{M}$  and consider the long exact sequence, in  $KK_\Gamma(\cdot, C(T))$ , associated to the semisplit short exact sequence:

$$(4.5) \quad 0 \rightarrow C_{0, \partial\widehat{M}}(\widehat{M}) \xrightarrow{j} C_0(\widehat{M}) \xrightarrow{a} C_0(\partial\widehat{M}) \rightarrow 0$$

(see Blackadar [7] page 197 and Chapter 20). We have in particular the exactness of

$$\begin{aligned} KK_\Gamma^0(C_{0, \partial\widehat{M}}(\widehat{M}), C(T)) &\xrightarrow{\delta_\Gamma} KK_\Gamma^1(C_0(\partial\widehat{M}), C(T)) \\ &\xrightarrow{i_*} KK_\Gamma^1(C_0(\widehat{M}), C(T)) \end{aligned}$$

and thus  $i_* \circ \delta_\Gamma = 0$ .

**Lemma 4.3.** *We have  $[\mathcal{D}_0] = \delta_\Gamma[\mathcal{D}]$  where  $[\mathcal{D}] \in KK_\Gamma^0(C_{0, \partial\widehat{M}}(\widehat{M}), C(T))$  is the class defined by  $\mathcal{D}$ .*

*Proof.* We are using both a  $\Gamma$ -equivariant and bivariant generalization of the proof of Theorem 5.1 of Higson [18]. We can replace  $\widehat{M}$  by a  $\Gamma$ -equivariant collar neighborhood  $\widehat{W}$  ( $\simeq [0, 1] \times \partial\widehat{M}$ ) of  $\partial\widehat{M}$  such that the restriction of  $\pi$  to  $\widehat{W}$  induces a  $\Gamma$ -equivariant fibration over  $T$ . Consider the differential operator  $d$

$$d = \begin{pmatrix} 0 & -i \frac{d}{dx} \\ -i \frac{d}{dx} & 0 \end{pmatrix}$$

acting on  $[0, 1]$ . It defines a class in  $KK_\Gamma^1(C_0(0, 1), \mathbb{C})$ . Recall that the Kasparov product  $[d] \otimes \cdot$  induces an isomorphism:

$$[d] \otimes : KK_\Gamma^1(C_0(\partial\widehat{M}), C(T)) \xrightarrow{\sim} KK_\Gamma^0(C_{0, \partial\widehat{M}}(\widehat{W}), C(T))$$

As in [18], the connecting map  $\delta_\Gamma :$

$$KK_\Gamma^0(C_{0, \partial\widehat{M}}(\widehat{W}), C(T)) \xrightarrow{\delta_\Gamma} KK_\Gamma^1(C_0(\partial\widehat{M}), C(T))$$

is given by the inverse of  $[d] \otimes \cdot$ . Denote by  $\mathcal{D}_{\widehat{W}}$  the restriction of  $\mathcal{D}$  to  $\widehat{W}$ , then one checks (as in Theorem 4.7 of [18]) that  $\mathcal{D}_{\widehat{W}} = [d] \otimes [\mathcal{D}_0]$ . One then gets  $\delta_\Gamma[\mathcal{D}_{\widehat{W}}] = [\mathcal{D}_0]$  which proves the result.  $\square$

We also have a natural map, still denoted  $\Theta :$

$$\Theta : KK_\Gamma^1(C_0(\widehat{M}), C(T)) \rightarrow KK^1(C(M), C(T) \rtimes \Gamma).$$

Denote by  $\pi^M$  the mapping of  $M$  to a point, then by functoriality we have  $\pi_*^{\partial M} \circ \Theta = \pi_*^M \circ \Theta \circ \iota_*$  as maps acting on  $KK_\Gamma^1(C_0(\partial\widehat{M}), C(T))$ . Since  $\iota_* \circ \delta_\Gamma = 0$ , the previous lemma implies that

$$\text{ind } \mathcal{D}_0 = \pi_*^{\partial M} \circ \Theta[\mathcal{D}_0] = \pi_*^M \circ \Theta \circ \delta_\Gamma[\mathcal{D}] = \pi_*^M \circ \Theta \circ \iota_* \circ \delta_\Gamma[\mathcal{D}] = 0.$$

The theorem is proved.  $\square$

**4.4. Dirac  $b$ -index classes on foliated bundles with boundary.** Now let  $\mathcal{P}$  be a spectral section for  $\mathcal{D}_0$  and consider an associated trivializing operator  $\mathcal{A}_{\mathcal{P}}^0$  as in Proposition 3.10. Let  $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$  be a nonnegative even smooth test function such that  $\int_{\mathbb{R}} \rho(x) dx = 1$ . We set  $\rho_\epsilon(x) = \frac{1}{\epsilon} \rho(\frac{x}{\epsilon})$  and then consider the Fourier transform of  $\rho_\epsilon$ :

$$\widehat{\rho}_\epsilon(z) = \int_{\mathbb{R}} e^{-itz} \rho\left(\frac{t}{\epsilon}\right) \epsilon^{-1} dt.$$

Then there exists a self-adjoint operator  $\mathcal{A}_{\mathcal{P}} \in \Psi_{b, C(T) \rtimes_r \Gamma}^{-\infty}$  (as already remarked this space is defined in [35]) such that the indicial family of  ${}^b\mathcal{D}^+ + \mathcal{A}_{\mathcal{P}}$  is given by

$$\forall z \in \mathbb{R}, I({}^b\mathcal{D}^+ + \mathcal{A}_{\mathcal{P}}, z) = \mathcal{D}_0 + iz + \widehat{\rho}_\epsilon(z) \mathcal{A}_{\mathcal{P}}^0$$

and is invertible from  $H_{C(T) \rtimes_r \Gamma}^1(\partial \widehat{M}, \widehat{E}_{|\partial \widehat{M}}^+)$  onto  $L_{C(T) \rtimes_r \Gamma}^2(\partial \widehat{M}, \widehat{E}_{|\partial \widehat{M}}^+)$  for any  $z \in \mathbb{R}$ . Recall that  $\mathcal{A}_{\mathcal{P}}$  is constructed in the following way (see formula (8.7) of [45]). Choose a  $\Gamma$ -invariant fiberwise product decomposition near the boundaries of the fibers. Let  $\mathcal{A}'_{\mathcal{P}}$  be the unique  $\mathbb{R}^+$ -invariant operator (Melrose [43] page 126) such that:

$$\forall z \in \mathbb{R}, I(\mathcal{A}'_{\mathcal{P}}, z) = \widehat{\rho}_\epsilon(z) \mathcal{A}_{\mathcal{P}}^0.$$

Then set  $\mathcal{A}_{\mathcal{P}} = \phi(x) \mathcal{A}'_{\mathcal{P}} \phi(x')$  where  $\phi \in C^\infty([0, 1], \mathbb{R})$  is such that  $\phi(x) = 1$  for  $x \in [0, \frac{1}{2}]$  and  $\phi(x) = 0$  for  $x \geq \frac{3}{4}$ .

The following theorem is proved exactly as in Section 3.4 of [35].

**Theorem 4.4.** *Let  $m \in \mathbb{N}^*$ . The operator  ${}^b\mathcal{D}^+ + \mathcal{A}_{\mathcal{P}}$  defines a  $C(T) \rtimes_r \Gamma$ -Fredholm operator*

$$H_{b, C(T) \rtimes_r \Gamma}^m(\widehat{M}, \widehat{E}^+) \longrightarrow H_{b, C(T) \rtimes_r \Gamma}^{m-1}(\widehat{M}, \widehat{E}^-).$$

*Its associated index class does not depend on  $m$  and we denote it by  $\text{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}) \in K_0(C(T) \rtimes_r \Gamma)$ .*

**4.5. Dirac APS-index classes on foliated bundles with boundary.** We keep the same geometric data as in Subsection 4.1 but we replace the  $\Gamma$ -invariant vertical  $b$ -metric by a vertical metric having a product structure near the boundary. We then get a  $C(T) \rtimes_r \Gamma$ -linear  $\mathbb{Z}_2$ -graded Dirac type operator  $\mathcal{D}$  acting on  $C_c^\infty(\widehat{M}, \widehat{E})$ :

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}.$$

One has  $\mathcal{D}^+ = \sigma(\frac{\partial}{\partial x} + \mathcal{D}_0)$  and  $\mathcal{D}^- = \sigma^{-1}(\frac{\partial}{\partial x} + \sigma \mathcal{D}_0 \sigma^{-1})$ .

Consider a spectral section  $\mathcal{P}$  for  $\mathcal{D}_0$  and define an odd operator acting on  $L_{C(T) \rtimes_r \Gamma}^2(\partial \widehat{M}, \widehat{E}_{\partial \widehat{M}})$  by

$$B_{\mathcal{P}} = \begin{pmatrix} 0 & (\text{Id} - \mathcal{P}) \sigma^{-1} \\ \sigma^{-1} \mathcal{P} & 0 \end{pmatrix}$$

Next we introduce the domain  $\text{dom } \mathcal{D}_{\mathcal{P}}$  of  $\mathcal{D}$  associated with the global APS boundary condition defined by  $\mathcal{P}$ :

$$\text{dom } (\mathcal{D}_{\mathcal{P}}) = \{\xi \in H_{C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E}) / B_{\mathcal{P}}(\xi_{|\partial \widehat{M}}) = 0\}$$

and will denote by  $\mathcal{D}_{\mathcal{P}}$  the restriction of  $\mathcal{D}$  to  $\text{dom}(\mathcal{D}_{\mathcal{P}})$ . In a similar and obvious way one defines  $\mathcal{D}_{\mathcal{P}}^{\pm}$ .

**Theorem 4.5.**

1) The operator  $\mathcal{D}_{\mathcal{P}}^+$  defines a  $C(T) \rtimes_r \Gamma$ -Fredholm operator from  $\text{dom}(\mathcal{D}_{\mathcal{P}}^+)$  to  $L^2_{C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E})$ .

We denote by  $\text{Ind}^{APS}(\mathcal{D}^+, \mathcal{P}) \in K_0(C(T) \rtimes_r \Gamma)$  the associated index class.

2) One has  $\text{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}) = \text{Ind}^{APS}(\mathcal{D}^+, \mathcal{P}) \in K_0(C(T) \rtimes_r \Gamma)$ .

*Proof.* 1) The arguments of Wu [57] page 374 can be immediately extended to our setting and allow to get easily the result.

2) One proceeds exactly as in the proof of Theorem 5 of [34].  $\square$

**4.6. APS-index theory via the  $b$ -calculus: an overview of the literature.** The use of the  $b$ -calculus on manifolds with boundary has generated a great number of interesting articles: in this subsection we shall review *only* those papers that use such a pseudodifferential calculus and are *directly* connected to an index theorem on manifolds with boundary.

The basic reference for the  $b$ -calculus on compact manifolds with boundary is of course the book by Melrose [43]. For a short introduction to the  $b$ -calculus and its use in establishing the APS-index formula the reader can also refer to the surveys of Mazzeo and Piazza [42] and Grieser [14]. The contribution of Loya in these proceedings [40] is also an excellent introduction. For pseudodifferential extensions of the Atiyah-Patodi-Singer formula in the context of the  $b$ -calculus, one can consult the work of Piazza [52] and Melrose-Nistor [44]. The work of Nistor-Weinstein-Xu [51] and Monthubert [48] in the context of groupoids should also be mentioned. Further generalizations of the index formula via the  $b$ -calculus have been given by Hassel-Mazzeo-Melrose [16] [17] to manifolds with corners. For more in this direction, see also the recent survey article of Loya [41].

The notion of spectral section and its use in establishing a general APS-*family* index theorem for Dirac operators appears for the first time in the work of Melrose and Piazza [45] [46]. Their theorem extends a result of Bismut-Cheeger [6] from the case where the boundary family is invertible to the general case. For a quick introduction to the results proved in these articles the reader can refer to the survey of Piazza [53]. A pseudodifferential extension of the Melrose-Piazza family-index theorem has been recently established by Melrose and Rochon in [47]. The family APS-index theory developed in [45] [46] was extended by Leichtnam and Piazza to the specific noncommutative context of *Galois  $\Gamma$ -coverings with boundary*, see [30] [31] [34], following a conjecture of Lott [38]. In these papers not only suitable index classes are defined in  $K_*(C_r^*\Gamma)$  but explicit formulae are also obtained for the pairing of these index classes with suitable cyclic cocycles. *Geometric applications* of these results on Galois  $\Gamma$ -coverings have been given to the problem of defining higher signatures on manifolds with boundary (see Lott [38] [39]) and proving their homotopy invariance [28], to uniqueness problems in positive scalar curvature metrics, see Leichtnam-Piazza [33], to the problem of cut-and-paste invariance of Novikov higher signatures on closed manifolds [28], [34] (see also [29], Hilsum [20]), to the homotopy invariance of the Atiyah-Patodi-Singer and Cheeger-Gromov rho-invariants for closed compact manifolds having a torsion-free fundamental group  $\Gamma$  satisfying the bijectivity of the Baum-Connes map for  $C_{\max}^*\Gamma$  (see Piazza-Schick [54]), a result due originally to Keswani [26]. The geometric applications to higher signatures, as well as the index theorems underlying them, are now also treated in the survey by Leichtnam-Piazza [36].

As already explained, in Leichtnam-Piazza [35] we define an index class on foliated bundles with boundary under an invertibility assumption on the boundary operator; we also establish an index formula for the higher indices obtained by pairing this index class with suitable cyclic cocycles. Finally, in the present paper we have just defined index classes associated to an arbitrary Dirac operator on a foliated bundle with boundary *and* the choice of a spectral section for its boundary operator. We shall now see what are the fundamental properties of this index class and how they can be employed in order to investigate the cut-and-paste invariance of the Baum-Connes higher signatures on closed foliated manifolds.

## 5. FUNDAMENTAL PROPERTIES OF $b$ -INDEX CLASSES

**5.1. The relative index theorem.** The following theorem extends the special fibration case treated by Melrose and Piazza in [45], as well as the covering case in [30].

**Theorem 5.1.** *Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two spectral sections for  $\mathcal{D}_0$ . Then one has:*

$$\text{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}_2) - \text{Ind}_b({}^b\mathcal{D}^+, \mathcal{P}_1) = [\mathcal{P}_1 - \mathcal{P}_2] \in K_0(C(T) \rtimes_r \Gamma).$$

*Proof.* We shall make precise and at the same time extend the proof sketched in [32] for  $T = \text{point}$ .

Using Lemma 3.6 and the proofs of Lemma 8 and Proposition 17 of [45] one checks easily the following five facts:

- (a) One can assume that  $\mathcal{P}_1 = \text{Id}$  on the range of  $\mathcal{P}_2$ .
- (b) There exist two spectral sections  $\mathcal{Q}, \mathcal{R}$  for  $\mathcal{D}_0$  such that for any  $j \in \{1, 2\}$ :

$$\mathcal{P}_j \mathcal{Q} = \mathcal{Q} \mathcal{P}_j = \mathcal{Q}, \quad \mathcal{P}_j \mathcal{R} = \mathcal{R} \mathcal{P}_j = \mathcal{P}_j, \quad \mathcal{Q} \mathcal{R} = \mathcal{R} \mathcal{Q} = \mathcal{Q}.$$

- (c) The four following self-adjoint projections

$$\mathcal{Q}, \quad \text{Id} - \mathcal{R}, \quad \mathcal{P}_j \mathcal{R} (\text{Id} - \mathcal{Q}), \quad (\text{Id} - \mathcal{P}_j) \mathcal{R} (\text{Id} - \mathcal{Q}), \quad 1 \leq j \leq 2$$

commute with each other and the sum of their (four) ranges provide an orthogonal decomposition of  $L^2_{C(T) \rtimes_r \Gamma}(\partial \widehat{M}; \widehat{E}^+_{|\partial \widehat{M}})$ .

- (d) One has  $(\mathcal{P}_1 - \mathcal{P}_2) \mathcal{R} (\text{Id} - \mathcal{Q}) = (\mathcal{P}_1 - \mathcal{P}_2)$ .

- (e) There exists  $s > 0$  such that for each  $j \in \{1, 2\}$  the operator  $\mathcal{D}_0^j :=$

$$\mathcal{Q} \mathcal{D}_0 \mathcal{Q} + s \mathcal{P}_j \mathcal{R} (\text{Id} - \mathcal{Q}) + (\text{Id} - \mathcal{R}) \mathcal{D}_0 (\text{Id} - \mathcal{R}) - s (\text{Id} - \mathcal{P}_j) \mathcal{R} (\text{Id} - \mathcal{Q}) = \mathcal{D}_0 + \mathcal{A}_{\mathcal{P}_j}^0$$

is invertible.

Now we set for  $r \in [-1, 1]$ :

$$\mathcal{D}_0(r) = \frac{1}{2}(1+r)\mathcal{D}_0^1 + \frac{1}{2}(1-r)\mathcal{D}_0^2.$$

### Lemma 5.2.

- 1) For any  $r \in [-1, 1] \setminus \{0\}$ ,  $\mathcal{D}_0(r)$  is invertible.
- 2) There exists  $\epsilon_1 > 0$  such that the  $L^2_{C(T) \rtimes_r \Gamma}$ -spectrum of  $\mathcal{D}_0(0)$  does not meet  $]-2\epsilon_1, 2\epsilon_1[ \setminus \{0\}$ ,  $\ker \mathcal{D}_0(0) = [\mathcal{P}_1 - \mathcal{P}_2]$  is a  $C(T) \rtimes_r \Gamma$ -finitely generated projective module,

$$L^2_{C(T) \rtimes_r \Gamma}(\partial \widehat{M}; \widehat{E}^+_{|\partial \widehat{M}}) = \ker \mathcal{D}_0(0) \oplus (\ker \mathcal{D}_0(0))^\perp$$

and  $\mathcal{D}_0(0)$  defines an  $L^2_{C(T) \rtimes_r \Gamma}$ -invertible operator from  $(\ker \mathcal{D}_0(0))^\perp$  onto

$$L^2_{C(T) \rtimes_r \Gamma}(\partial \widehat{M}; \widehat{E}^+_{|\partial \widehat{M}}).$$

*Proof.* 1) An easy computation shows that

$$(5.1) \quad \begin{aligned} \mathcal{D}_0(r) &= \mathcal{Q}\mathcal{D}_0\mathcal{Q} + s\mathcal{P}_2\mathcal{R}(\text{Id} - \mathcal{Q}) + (\text{Id} - \mathcal{R})\mathcal{D}_0(\text{Id} - \mathcal{R}) \\ &\quad - s(\text{Id} - \mathcal{P}_2 - (1+r)(\mathcal{P}_1 - \mathcal{P}_2))\mathcal{R}(\text{Id} - \mathcal{Q}). \end{aligned}$$

Using properties (b) and (c) one gets the result of 1).

2) The previous identity with  $r = 0$  and properties (a), (b), (c) show that  $\ker \mathcal{D}_0(0)$  coincides with the range of  $(\mathcal{P}_1 - \mathcal{P}_2)\mathcal{R}(\text{Id} - \mathcal{Q}) = (\mathcal{P}_1 - \mathcal{P}_2)$  and that  $\mathcal{D}_0(0)$  defines an invertible operator from  $(\ker \mathcal{D}_0(0))^\perp$  onto  $L^2_{C(T) \rtimes_r \Gamma}(\partial\widehat{M}; \widehat{E}^+_{|\partial\widehat{M}})$ . One then gets immediately part 2).  $\square$

For each  $r \in [-1, 1]$  set  $\mathcal{D}_0(r) = \mathcal{D}_0 + \mathcal{A}^0(r)$ . Just as before Theorem 4.5, we consider a  $C(T) \rtimes_r \Gamma$ -linear operator  ${}^b\mathcal{D}(r)$  such that:

$$\forall z \in \mathbb{R}, \quad I({}^b\mathcal{D}(r)^+, z) = \mathcal{D}_0 + iz \text{Id} + \widehat{\rho}_\epsilon(z)\mathcal{A}^0(r).$$

Using Lemma 5.2. 2) and the extension of the Melrose  $b$ -calculus to foliated bundles, as explained in [35], one checks easily that for  $t \in ]0, \epsilon_1[$ ,  $x^{\mp t} {}^b\mathcal{D}(0)^+ x^{\pm t}$  induces a  $C(T) \rtimes_r \Gamma$ -Fredholm operator from  $H^1_{b, C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}^+)$  into  $L^2_{b, C(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}^-)$ . Denote by  $\text{Ind}_{\pm t}^b \mathcal{D}(0)^+$  the corresponding index class in  $K_0(C(T) \rtimes_r \Gamma)$ .

**Lemma 5.3.**

1) For any  $t \in ]0, \epsilon_1[$  one has:

$$\text{Ind}_{-t}^b \mathcal{D}(0)^+ = \text{Ind}^b \mathcal{D}(-1)^+.$$

2) For any  $t \in ]0, \epsilon_1[$  one has:

$$\text{Ind}_t^b \mathcal{D}(0)^+ = \text{Ind}^b \mathcal{D}(1)^+.$$

*Proof.* 1) Fix  $t \in ]0, \epsilon_1[$ . For any  $r \in [-1, 0]$  one has:

$$\forall z \in \mathbb{R}, \quad I(x^t {}^b\mathcal{D}(r)^+ x^{-t}, z) = \mathcal{D}_0 + (iz - t) \text{Id} + \widehat{\rho}_\epsilon(z)\mathcal{A}^0(r).$$

Observe that for any  $r \in [-1, 0]$ ,  $-t - s + s(1+r) \neq 0$ . Using Lemma 5.2 and inspecting expression (5.1) (especially its last term), one checks immediately that for any  $r \in [-1, 0]$ ,  $I(x^t {}^b\mathcal{D}(r)^+ x^{-t}, 0)$  is  $L^2_{C(T) \rtimes_r \Gamma}$ -invertible. Recall that the  $\widehat{\rho}_\epsilon(z)$  take real values, then since  $\mathcal{D}_0$  and  $\mathcal{A}^0(r)$  are self-adjoint it is clear that for any  $(r, z) \in [-1, 0] \times \mathbb{R}^*$ ,  $I(x^t {}^b\mathcal{D}(r)^+ x^{-t}, z)$  is  $L^2_{C(T) \rtimes_r \Gamma}$ -invertible. Therefore, for any  $t \in ]0, \epsilon_1[$  the family  $\{x^t {}^b\mathcal{D}(r)^+ x^{-t}, r \in [-1, 0]\}$  defines a continuous family of  $C(T) \rtimes_r \Gamma$ -Fredholm operators. By the homotopy invariance of the  $C(T) \rtimes_r \Gamma$ -index in  $K_0(C(T) \rtimes_r \Gamma)$  one has:

$$\forall r \in [-1, 0], \quad \text{Ind}^b x^t {}^b\mathcal{D}(r)^+ x^{-t} = \text{Ind}^b x^t {}^b\mathcal{D}(0)^+ x^{-t}.$$

Next, the family  $\{x^{t'} {}^b\mathcal{D}(-1)^+ x^{-t'}, t' \in [0, t]\}$  defines a continuous family of  $C(T) \rtimes_r \Gamma$ -Fredholm operators so that one gets:

$$\text{Ind}^b x^t {}^b\mathcal{D}(-1)^+ x^{-t} = \text{Ind}^b {}^b\mathcal{D}(-1)^+.$$

From the last two equations one gets immediately part 1). Part 2) of the lemma is proved in a similar way.  $\square$

Since one has:

$$\text{Ind}({}^b\mathcal{D}^+, \mathcal{P}_2) = \text{Ind}({}^b\mathcal{D}(-1)^+), \quad \text{Ind}({}^b\mathcal{D}^+, \mathcal{P}_1) = \text{Ind}({}^b\mathcal{D}(1)^+)$$

Lemma 5.3 shows that Theorem 5.1 is a consequence of the following proposition

**Proposition 5.4.** *For any  $t \in ]0, \epsilon_1[$ , one has:*

$$\text{Ind}_{-t}({}^b\mathcal{D}(0)^+) - \text{Ind}_t({}^b\mathcal{D}(0)^+) = [\ker \mathcal{D}_0(0)] = [\mathcal{P}_1 - \mathcal{P}_2].$$

*Proof.* Denote by  $\mathcal{D}_{cyl}^+$  the  $\Gamma$ -equivariant family of fiberwise elliptic operators which in a collar neighborhood ( $\simeq [0, 1] \times \pi^{-1}(\theta)$ ) of the boundaries is given by

$$\mathcal{D}_{cyl}^+ = \sigma(x\partial_x + \mathcal{D}_0(0))$$

and which coincides with  ${}^b\mathcal{D}^+$  outside this collar neighborhood. Observe that unlike  ${}^b\mathcal{D}^+$ ,  $\mathcal{D}_{cyl}^+$  is not a  $b$ -operator. Proceeding as in Section 10 of [28], one proves that  $\mathcal{D}_{cyl}^+$  acting from  $x^{\pm t}H_{b,C(T)\rtimes_r\Gamma}^1(\widehat{M}, \widehat{E}^+)$  into  $x^{\pm t}L_{b,C(T)\rtimes_r\Gamma}^2(\widehat{M}, \widehat{E}^-)$  defines a  $C(T) \rtimes_r \Gamma$ -Fredholm operator whose index class  $\text{Ind}_{\pm t} \mathcal{D}_{cyl}^+$  satisfies:  $\text{Ind}_{\pm t} \mathcal{D}_{cyl}^+ = \text{Ind}_t({}^b\mathcal{D}(0)^+)$ .

Now, proceeding as in the proof of Proposition 4 of [45] one checks that there exists a positive number  $N$  and two continuous  $C(T) \rtimes_r \Gamma$ -linear maps  $\mathcal{R}_{\pm t} : (C(T) \rtimes_r \Gamma)^N \rightarrow x^{2t}H_{b,C(T)\rtimes_r\Gamma}^{2n+3}(\widehat{M}, \widehat{E}^-)$  such that the following two maps are surjective:

$$\begin{aligned} \mathcal{D}^t &: x^t H_{b,C(T)\rtimes_r\Gamma}^1(\widehat{M}, \widehat{E}^+) \oplus (C(T) \rtimes_r \Gamma)^N \rightarrow x^t L_{b,C(T)\rtimes_r\Gamma}^2(\widehat{M}, \widehat{E}^-) \\ \mathcal{D}^{-t} &: x^{-t} H_{b,C(T)\rtimes_r\Gamma}^1(\widehat{M}, \widehat{E}^+) \oplus (C(T) \rtimes_r \Gamma)^N \rightarrow x^{-t} L_{b,C(T)\rtimes_r\Gamma}^2(\widehat{M}, \widehat{E}^-). \end{aligned}$$

with

$$\mathcal{D}^t := \mathcal{D}_{cyl}^+ + \mathcal{R}_t, \quad \mathcal{D}^{-t} = \mathcal{D}_{cyl}^+ + \mathcal{R}_{-t}.$$

Then one has:

$$\text{Ind}_t \mathcal{D}_{cyl}^+ = [\ker \mathcal{D}^t] - [(C(T) \rtimes_r \Gamma)^N], \quad \text{Ind}_{-t} \mathcal{D}_{cyl}^+ = [\ker \mathcal{D}^{-t}] - [(C(T) \rtimes_r \Gamma)^N].$$

Thus we just have to prove that:

$$[\ker \mathcal{D}^{-t}] - [\ker \mathcal{D}^t] = [\mathcal{P}_1 - \mathcal{P}_2] = \ker \mathcal{D}_0(0) \in K_0(C(T) \rtimes_r \Gamma).$$

We are going now to show the existence of a short exact sequence

$$0 \rightarrow \ker \mathcal{D}^t \rightarrow \ker \mathcal{D}^{-t} \rightarrow \ker \mathcal{D}_0(0) \rightarrow 0.$$

Since  $\ker \mathcal{D}_0(0)$  is projective we shall obtain that  $\ker \mathcal{D}^{-t} = \ker \mathcal{D}^t \oplus \ker \mathcal{D}_0(0)$  which will imply the proposition. Our arguments are very much inspired by those used by Melrose in his proof of the relative index formula in [43].

Consider  $(u \oplus a) \in x^{-t}H_{b,C(T)\rtimes_r\Gamma}^1(\widehat{M}, \widehat{E}^+) \oplus (C(T) \rtimes_r \Gamma)^N$  such that  $\mathcal{D}^{-t}(u \oplus a) = 0$ . Let  $\phi \in C^\infty([0, 1], \mathbb{R})$  be such that  $\phi(x) = 1$  for  $x \leq \frac{1}{2}$  and  $\phi(x) = 0$  for  $x \geq \frac{3}{4}$ . Then  $\mathcal{D}^{-t}(\phi \cdot u) \in x^{2t}L_{b,C(T)\rtimes_r\Gamma}^2(\widehat{M}, \widehat{E}^-)$ . Denote by  $U_M(z, y)$  the Mellin transform of  $\phi \cdot u$ :

$$\forall y \in \partial \widehat{M}, \quad U_M(z, y) = \int_{\mathbb{R}} x^{-iz} (\phi \cdot u)(x, y) \frac{dx}{x}.$$

We observe that  $z \rightarrow U_M(z, y)$  is holomorphic on the half plane  $\{\text{Im } z > t\}$ . Moreover, since

$$\mathcal{D}^{-t}(\phi \cdot u) = \sigma(x \frac{\partial}{\partial x} + \mathcal{D}_0(0))(\phi \cdot u) \in x^{2t}L_{b,C(T)\rtimes_r\Gamma}^2(\widehat{M}, \widehat{E}^-)$$

one checks easily that  $z \rightarrow (iz + \mathcal{D}_0(0))U_M(z, y)$  is holomorphic on the half plane  $\{\text{Im } z > -2t\}$ . We recall the orthogonal decomposition of Lemma 5.2 2), and write for each  $z \in \{z', \text{Im } z' > -2t\}$ :

$$(iz + \mathcal{D}_0(0))U_M(z, y) = W_0(z, y) \oplus W_1(z, y) \in \ker \mathcal{D}_0(0) \oplus (\ker \mathcal{D}_0(0))^\perp.$$

Then for any  $z \in \{z', \text{Im } z' > t\}$ :

$$U_M(z, y) = \frac{1}{iz}W_0(z, y) \oplus (iz + \mathcal{D}_0(0))^{-1}W_1(z, y).$$

Considering the inverse Mellin transform one checks easily that

$$\frac{\phi(x)}{2\pi} \int_{\text{Im } z = -\frac{3t}{2}} x^{iz} \left( \frac{W_0(z, y) - e^{-z^2}W_0(0, y)}{iz} + (iz + \mathcal{D}_0(0))^{-1}W_1(z, y) \right) dz$$

belongs to  $x^t H_{b, C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E}^+)$ . Similarly, one checks that

$$\frac{\phi(x)}{2\pi} \int_{\text{Im } z = -\frac{3t}{2}} x^{iz} \left( \frac{W_0(0, y)}{iz} \right) dz \in x^t H_{b, C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E}^+)$$

if and only if  $W_0(0, y) \equiv 0$ . Then set:

$$\Pi_0(u) = W_0(0, y).$$

From our previous computations one gets the following lemma.

**Lemma 5.5.** *With the previous notations:  $u \in x^t H_{b, C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E}^+)$  if and only if  $\Pi_0(u) = W_0(0, y)$  is the null element of  $\ker \mathcal{D}_0(0)$ . Moreover, the following sequence*

$$0 \rightarrow \ker \mathcal{D}^t \rightarrow \ker \mathcal{D}^{-t} \xrightarrow{\Pi_0} \ker \mathcal{D}_0(0)$$

is exact.

Now we are going to show that the map  $\ker \mathcal{D}^{-t} \xrightarrow{\Pi_0} \ker \mathcal{D}_0(0)$  is surjective.

Consider an element  $V_0(y)$  of  $\ker \mathcal{D}_0(0)$  and set:

$$v_1(x, y) = \frac{\phi(x)}{2\pi} \int_{\text{Im } z = t} x^{iz} \frac{e^{-z^2}}{iz} V_0(y) dz.$$

It is clear that

$$\sigma^+(x\partial_x + \mathcal{D}_0(0))v_1(x, y) \in x^{2t} L_{b, C(T) \rtimes_r \Gamma}^2(\widehat{M}, \widehat{E}^-),$$

then since  $\mathcal{D}^t$  is surjective, there exists  $u_2 \oplus a_2 \in x^t H_{b, C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E}^+) \oplus (C(T) \rtimes_r \Gamma)^N$  such that  $\mathcal{D}^t(u_2 \oplus a_2) = -\mathcal{D}^{-t}(v_1 \oplus 0)$ . Then  $(v_1 - u_2) \oplus (-a_2)$  belongs to  $\ker \mathcal{D}^{-t}$  and  $\Pi_0(v_1 - u_2) = V_0(y)$ . Then, using Lemma 5.5 one obtains (as previously announced) the following short exact sequence of  $C(T) \rtimes_r \Gamma$ -finitely generated projective modules:

$$0 \rightarrow \ker \mathcal{D}^t \rightarrow \ker \mathcal{D}^{-t} \xrightarrow{\Pi_0} \ker \mathcal{D}_0(0) \rightarrow 0.$$

Since the modules are projective one obtains an isomorphism:  $\ker \mathcal{D}^{-t} \simeq \ker \mathcal{D}^t \oplus \ker \mathcal{D}_0(0)$  from which the proposition follows.  $\square$

Theorem 5.1 is thus proved.  $\square$

**5.2. The gluing formula.** Let  $T$  be a smooth closed compact connected manifold on which  $\Gamma$  acts on the right. Let  $\widehat{M}$  be a closed manifold on which  $\Gamma$  acts freely, properly and cocompactly on the right: the quotient space  $M = \widehat{M}/\Gamma$  is thus a smooth compact manifold. We assume that  $\widehat{M}$  fibers over  $T$  and that the resulting fibration

$$\pi : \widehat{M} \rightarrow T$$

is a  $\Gamma$ -equivariant fibration with fibers  $\pi^{-1}(\theta), \theta \in T$ , of dimension  $2k$ . Notice that each fiber is smooth; we shall also denote the typical fiber of  $\pi : \widehat{M} \rightarrow T$  by  $Z$ . We choose a  $\Gamma$ -invariant metric on the vertical tangent bundle  $TZ$ . Finally, we assume the existence of a  $\Gamma$ -equivariant spin structure on  $TZ$  that is fixed once and for all. We denote by  $S^Z \rightarrow \widehat{M}$  the associated spinor bundle.

We consider also a  $\Gamma$ -equivariant complex hermitian vector bundle  $\widehat{V} \rightarrow \widehat{M}$  endowed with a  $\Gamma$ -invariant hermitian connection  $\widehat{\nabla}$ . We then set  $\widehat{E} = S^Z \otimes \widehat{V} = \widehat{E}^+ \oplus \widehat{E}^-$  which defines a smooth  $\Gamma$ -invariant family of  $\mathbb{Z}_2$ -graded hermitian Clifford modules on the fibers  $\pi^{-1}(\theta), \theta \in T$ . We then get a smooth family of  $\Gamma$ -invariant  $\mathbb{Z}_2$ -graded Dirac type operators

$$D(\theta) = \begin{pmatrix} 0 & D^-(\theta) \\ D^+(\theta) & 0 \end{pmatrix}, \theta \in T$$

acting fiberwise on  $C_c^\infty(\widehat{M}, \widehat{E})$ .

The family  $(D(\theta))_{\theta \in T}$  defines a  $C(T) \rtimes_r \Gamma$ -linear  $\mathbb{Z}_2$ -graded Dirac type operator  $\mathcal{D}$  acting on  $C_c^\infty(\widehat{M}, \widehat{E})$ . This operator has a well defined index class in  $K_0(C(T) \rtimes_r \Gamma)$ . Now let  $F$  be a closed cutting  $\Gamma$ -invariant hypersurface of  $M$  such that  $\widehat{M} = \widehat{M}_+ \cup \widehat{M}_-$  where  $\widehat{M}_\pm$  are two manifolds whose common boundary is  $F$  and which both fiber over  $T$ :  $\widehat{M}_\pm \rightarrow T$ . We assume that all these data have a product structure near  $F$ . Let  $\mathcal{P}$  and  $\mathcal{Q}$  be two spectral sections for the boundary operator of the operator induced by the restriction  $\mathcal{D}|_{\widehat{M}_+}$  of  $\mathcal{D}$  to  $\widehat{M}_+$ . Observe that  $\text{Id} - \mathcal{Q}$  is a spectral section for the boundary operator of  $\mathcal{D}|_{\widehat{M}_-}$ . The following gluing formula is proved exactly as page 380 in [34] using an idea of U. Bunke [9]. It generalizes the result of Dai-Zhang [12] in the fibration case as well as the result on covering space in [30].

**Theorem 5.6.**

$$\text{Ind } \mathcal{D}^+ = \text{Ind}^{APS}(\mathcal{D}|_{\widehat{M}_+}^+, \mathcal{P}) + \text{Ind}^{APS}(\mathcal{D}|_{\widehat{M}_-}^+, \text{Id} - \mathcal{Q}) + [\mathcal{P} - \mathcal{Q}].$$

**5.3. The variational formula.** Let  $\pi : \widehat{M} \rightarrow T$  be  $\Gamma$ -equivariant fibration (whose fibers are manifold with boundary) exactly as in Section 4.1. We assume that there exists a smooth 1-parameter family of fiberwise vertical  $\Gamma$ -invariant riemannian metrics  $(g_u)_{u \in [1,2]}$  on  $\widehat{M}$  which all have a product structure near the boundary. We assume that the  $\mathbb{Z}_2$ -graded Clifford module  $\widehat{E}$  is endowed with continuous 1-parameter families of  $\Gamma$ -equivariant hermitian metrics  $(h^u)_{u \in [1,2]}$  and  $\Gamma$ -equivariant hermitian connection  $(\nabla^u)_{u \in [1,2]}$ . We denote by  $(\mathcal{D}_u)_{u \in [1,2]}$  the associated family of  $C(T) \rtimes_r \Gamma$ -linear Dirac type operator acting on  $H_{C(T) \rtimes_r \Gamma}^1(\widehat{M}, \widehat{E})$ . We can then state the following variational formula which is (as in [34] page 383) an easy consequence of the relative index theorem Theorem 5.1. It generalizes

the result of Dai-Zhang [12] in the fibration case as well as the result on covering space in [30].

**Proposition 5.7.** *Let us denote by  $\{(\mathcal{D}_u)_0, u \in [1, 2]\}$  the family of boundary operators associated to  $(\mathcal{D}_u)_{u \in [1, 2]}$ . We fix noncommutative spectral sections  $\mathcal{P}_1, \mathcal{P}_2$  for  $(\mathcal{D}_1)_0$  and  $(\mathcal{D}_2)_0$  respectively. Then:*

$$\text{Ind}^{APS}(\mathcal{D}_2^+, \mathcal{P}_2) - \text{Ind}^{APS}(\mathcal{D}_1^+, \mathcal{P}_1) = \text{sf}(\{(\mathcal{D}_u)_0\}; \mathcal{P}_2, \mathcal{P}_1) \text{ in } K_0(C(T) \rtimes_r \Gamma).$$

## 6. ON THE CUT-AND-PASTE INVARIANCE OF THE SIGNATURE INDEX CLASS

**6.1. Cut-and-paste on foliated bundles.** We first consider a  $\Gamma$ -equivariant fibrations  $\pi_{\widehat{X}} : \widehat{X} \rightarrow T$  with oriented fibers and such that the quotient  $X = \widehat{X}/\Gamma$  is a smooth compact manifolds. Let  $r : X = \widehat{X}/\Gamma \rightarrow (E\Gamma \times T)/\Gamma$  be the classifying map of the action of the groupoid  $T \rtimes \Gamma$  on  $\widehat{X}$  (Connes, [10, Chapter III], [15]). This map is the defined as follows: the  $\Gamma$ -covering  $\rho : \widehat{X} \rightarrow X = \widehat{X}/\Gamma$  is classified by a  $\Gamma$ -equivariant map  $\tilde{\rho} : \widehat{X} \rightarrow E\Gamma$ ; let  $\widehat{r} : \widehat{X} \rightarrow E\Gamma \times T$  be the map  $(\tilde{\rho}, \pi_{\widehat{X}})$ ; then  $r : X = \widehat{X}/\Gamma \rightarrow (E\Gamma \times T)/\Gamma$  is the  $\Gamma$ -quotient of  $\widehat{r}$ .

We now consider two  $\Gamma$ -equivariant fibrations  $\pi_{\widehat{M}} : \widehat{M} \rightarrow T$  and  $\pi_{\widehat{N}} : \widehat{N} \rightarrow T$  where in both cases the fibers are even  $2m$ -dimensional oriented manifolds with boundary and such that the quotient  $M = \widehat{M}/\Gamma$  and  $N = \widehat{N}/\Gamma$  are two smooth compact manifolds with boundary.

We assume the existence of two  $\Gamma$ -equivariant diffeomorphisms  $\phi, \psi : \partial\widehat{M} \rightarrow \partial\widehat{N}$  such that  $\pi_{\partial\widehat{N}} \circ \phi = \pi_{\partial\widehat{M}}$ ,  $\pi_{\partial\widehat{N}} \circ \psi = \pi_{\partial\widehat{M}}$  and  $\phi, \psi$  both preserve the orientations of the fibers. We set:

$$\widehat{X}_\phi = \widehat{M} \cup_\phi \widehat{N}^-, \quad \widehat{X}_\psi = \widehat{M} \cup_\psi \widehat{N}^-$$

where  $\widehat{N}^-$  means that the fibers of  $\widehat{N}$  are endowed with the reverse orientation.

One then consider the two  $\Gamma$ -equivariant fibrations  $\pi_\phi : \widehat{X}_\phi \rightarrow T$  and  $\pi_\psi : \widehat{X}_\psi \rightarrow T$  and the two  $\Gamma$ -covering maps:

$$\rho_\phi : \widehat{X}_\phi \rightarrow \widehat{X}_\phi/\Gamma, \quad \rho_\psi : \widehat{X}_\psi \rightarrow \widehat{X}_\psi/\Gamma.$$

Denote by  $\tilde{\rho}_\phi : \widehat{X}_\phi \rightarrow E\Gamma$  (resp.  $\tilde{\rho}_\psi : \widehat{X}_\psi \rightarrow E\Gamma$ ) the corresponding classifying map of  $\rho_\phi$  (resp.  $\rho_\psi$ ). Then, as explained above, the pairs  $(\tilde{\rho}_\phi, \pi_\phi)$  and  $(\tilde{\rho}_\psi, \pi_\psi)$  induce two classifying maps:

$$r_\phi : \widehat{X}_\phi/\Gamma \rightarrow (E\Gamma \times T)/\Gamma, \quad s_\psi : \widehat{X}_\psi/\Gamma \rightarrow (E\Gamma \times T)/\Gamma.$$

We shall briefly say that two  $T \rtimes \Gamma$ -proper manifolds obtained above are *cut-and-paste equivalent*.

**6.2. The defect formula.** We consider a fiberwise vertical metric  $g_1$  (resp.  $g_2$ ) on  $\widehat{X}_\phi$  (resp.  $\widehat{X}_\psi$ ) which is product like near  $\partial\widehat{M}$ . Consider  $(g_1)_{|\widehat{M}}$  and  $(g_2)_{|\widehat{M}}$  and let  $g_{+,u}$ , with  $u \in [1, 2]$ , be a path of vertical fiberwise riemannian metrics on  $\widehat{M}$  connecting them and having a product structure near the boundary. Similarly we choose a path  $(g_{-,u})_{u \in [1, 2]}$  of vertical fiberwise riemannian metrics on  $\widehat{N}$  connecting  $(g_1)_{|\widehat{N}}$  and  $(g_2)_{|\widehat{N}}$ . One thus gets two family of boundary  $C(T) \rtimes_r \Gamma$ -linear signature operators (as in Section 4.2)  $\{\mathcal{D}_{\partial\widehat{M}}^{\text{sign},u}\}_{u \in [1, 2]}$  and  $\{\mathcal{D}_{\partial\widehat{N}}^{\text{sign},u}\}_{u \in [1, 2]}$ . Observe that  $\mathcal{D}_{\partial\widehat{M}}^{\text{sign},1}$  is conjugated through  $\phi^*$  to  $-\mathcal{D}_{\partial\widehat{N}}^{\text{sign},2}$  and that  $\mathcal{D}_{\partial\widehat{M}}^{\text{sign},2}$

is conjugated through  $\psi^*$  to  $-\mathcal{D}_{\partial\widehat{N}}^{\text{sign},2}$ . We can thus (as in Section 6.1 of [34]) put together the family  $\{\mathcal{D}_{\partial\widehat{M}}^{\text{sign},u}\}_{u\in[1,2]}$  and the family  $\{\mathcal{D}_{\partial\widehat{N}}^{\text{sign},u}\}_{u\in[1,2]}$  and obtain a family

$$\{\mathcal{D}_{\partial\widehat{M}}^{\text{sign}}(\theta)\}_{\theta\in S^1} = \{\mathcal{D}_{\partial\widehat{M}}^{\text{sign},u}\}_{u\in[1,2]} \cup \{\mathcal{D}_{\partial\widehat{N}}^{\text{sign},u}\}_{u\in[2,1]}$$

which is an  $S^1$ -family acting on the fibers of the mapping torus defined by  $\phi^{-1} \circ \psi$ . Then one has the following formula whose proof is an easy extension of the one of Theorem 11 of [34] which is in turn modeled on the arguments given in Section 2.

**Theorem 6.1.** *Denote by  $\mathcal{D}_{\widehat{X}_\phi}^{\text{sign}}$  (resp.  $\mathcal{D}_{\widehat{X}_\psi}^{\text{sign}}$ ) the  $C(T) \rtimes_r \Gamma$ -linear signature operator of  $\widehat{X}_\phi$  (resp.  $\widehat{X}_\psi$ ) defined as in Section 4.4. The following formula holds*

$$\text{Ind } \mathcal{D}_{\widehat{X}_\phi}^{\text{sign}} - \text{Ind } \mathcal{D}_{\widehat{X}_\psi}^{\text{sign}} = \text{sf}(\{\mathcal{D}_{\partial\widehat{M}}^{\text{sign}}(\theta)\}_{\theta\in S^1}) \quad \text{in } K_0(C(T) \rtimes_r \Gamma).$$

**6.3. Vanishing spectral flow.** Let the fibers of  $Z \rightarrow \widehat{M} \rightarrow T$  have dimension  $2m$ , so that the fibers of the boundary fibration  $\partial\widehat{M} \rightarrow T$  have dimension  $2m - 1$ . We endow the boundaries of the fibers of  $\pi_{\widehat{M}} : \widehat{M} \rightarrow T$  with a  $\Gamma$ -invariant metric and make the following "middle-degree" assumption on the boundary:

*Assumption 6.2.* There exists  $\epsilon \in ]0, 1[$  such that for each  $\theta \in T$ , the  $L^2$ -spectrum of the fiberwise differential-form laplacian acting on

$$L^2(\partial\pi_{\widehat{M}}^{-1}(\theta); \wedge^{m-1}T^*\partial\pi_{\widehat{M}}^{-1}(\theta))$$

does not meet  $] - \epsilon, \epsilon[$ .

We give an example (inspired by [28] page 563) where this assumption is satisfied. Let  $\widetilde{N} \rightarrow N$  a Galois  $\Gamma$ -covering of a smooth orientable compact  $2m$ -dimensional manifold with boundary such that  $\partial N$  has a cellular decomposition without any cells of dimension  $m$ . Set  $\widehat{M} = \widetilde{N} \times T$  consider the trivial fibration

$$\pi : \widetilde{N} \times T \rightarrow T, \quad \pi(z, \theta) = \theta.$$

Then the above assumption is satisfied in this case.

**Remarks.**

- 1) In the Galois-covering case ( $T = \text{point}$ ), this assumption comes from the work of John Lott, see [38].
- 2) Under the Assumption 6.2 one can prove easily that the index of the boundary  $C(T) \rtimes_r \Gamma$ -linear signature operator vanishes in  $K_1(C(T) \rtimes_r \Gamma)$ .
- 3) Proposition 1 of [34] shows that the associated  $C(T) \rtimes_r \Gamma$ -linear signature-laplacian  $(\mathcal{D}_{\partial\widehat{M}}^{\text{sign}})^2$  induces an invertible operator from the Hilbert  $C(T) \rtimes_r \Gamma$ -module  $H_{C(T) \rtimes_r \Gamma}^2(\partial\widehat{M}, \wedge^{m-1}T^*\partial\widehat{M})$  onto  $L_{C(T) \rtimes_r \Gamma}^2(\partial\widehat{M}, \wedge^{m-1}T^*\partial\widehat{M})$ .

**Proposition 6.3.** *Let  $\widehat{X}_\phi$  and  $\widehat{X}_\psi$  be as in Subsection 6.1. Denote by  $\mathcal{D}_{\widehat{X}_\phi}^{\text{sign}}$  (resp.  $\mathcal{D}_{\widehat{X}_\psi}^{\text{sign}}$ ) the corresponding  $C(T) \rtimes_r \Gamma$ -linear signature operator of  $\widehat{X}_\phi$  (resp.  $\widehat{X}_\psi$ ). Assume that Assumption 6.2 is satisfied for  $\partial\widehat{X}_\phi$  instead of  $\partial\widehat{M}$ . Then one has:*

$$\text{Ind } \mathcal{D}_{\widehat{X}_\phi}^{\text{sign}} = \text{Ind } \mathcal{D}_{\widehat{X}_\psi}^{\text{sign}} \in K_0(C(T) \rtimes_r \Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

*Proof.* We follow pages 391-392 of [34]. We denote by  $Z$  the typical fiber of the fibration  $\pi : \widehat{\partial M} \rightarrow T$  and set  $\Omega^* = \bigcap_{j \geq 0} H_{C(T) \rtimes_r \Gamma}^j(\widehat{\partial M}; \wedge^* T^* Z)$ . We then set:

$$V = d^* \Omega^m + d \Omega^{m-1}, \quad W = \Omega^< \oplus \Omega^>, \quad \text{where}$$

$$\Omega^< = \Omega^0 \oplus \dots \oplus \Omega^{m-2} \oplus (d^* \Omega^m)^\perp, \quad \Omega^> = (d \Omega^{m-1})^\perp \oplus \Omega^{m+1} \oplus \dots \oplus \Omega^{2m-1}.$$

It is clear that the  $C(T) \rtimes_r \Gamma$ -linear signature operator  $\mathcal{D}_{\widehat{\partial M}}^{\text{sign}}$  of  $\widehat{\partial M}$  sends  $V$  (resp.  $W$ ) into itself. Using Assumption 6.2 one checks easily that  $\mathcal{D}_{\widehat{\partial M}}^{\text{sign}}$  induces an invertible operator on the  $L_{C(T) \rtimes_r \Gamma}^2$ -completion of  $V$  with domain  $H^1$  and we denote by  $\Pi_{>}$  the projection onto the positive part. Then, proceeding as in page 392 of [34], one checks that  $\mathcal{D}_{\widehat{\partial M}}^{\text{sign}}$  admits a symmetric spectral section  $\mathcal{P}$  in the sense that  $\mathcal{P}$  is diagonal with respect to the decomposition  $\sum_{j=0}^{2m-1} \Omega^j = V \oplus W$  and

$$\mathcal{P}|_V = \Pi_{>}, \quad \alpha \circ \mathcal{P}|_W + \mathcal{P}|_W \circ \alpha = \alpha$$

where  $\alpha$  is the involution of  $W$  equal to the identity of  $\Omega^<$  and to minus identity on  $\Omega^>$ . Moreover, proceeding as in the proof of Proposition 17 of [32], one checks that if  $\mathcal{Q}$  is another symmetric spectral section for  $\mathcal{D}_{\widehat{\partial M}}^{\text{sign}}$  then one has  $[\mathcal{P} - \mathcal{Q}] = 0$  in  $K_0(C(T) \rtimes_r \Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Lastly we observe that in the definition of the spectral flow  $\text{sf}(\{\mathcal{D}_{\widehat{\partial M}}^{\text{sign}}(\theta)\}_{\theta \in S^1})$  we may assume that all the spectral sections involved are symmetric. Then the result follows from Theorem 6.1.  $\square$

## 7. GEOMETRIC APPLICATIONS

In all this section we shall assume, for simplicity, that  $T$  is orientable and that  $\Gamma$  preserves the orientation of  $T$ . We shall first define the Baum-Connes higher signatures of a  $T \rtimes \Gamma$ -proper manifold; these are numeric invariants. Then we shall ask ourselves when these higher signatures are cut-and-paste invariant. The strategy here is to use the basic assumption 6.2 and the equality of the signature index classes established in Proposition 6.3 in order to deduce the equality of the higher signatures (or, at least, the equality of some of these higher signatures). When  $T = \text{point}$  there are two techniques allowing to use the equality of index classes in order to deduce the equality of (all) higher signatures: the first one employs cyclic cohomology and the second one employs the assembly map from topological  $K$ -homology to the  $K$ -Theory of the reduced  $C^*$ -algebra. We shall try to generalize these two approaches.

**7.1. Baum-Connes higher signatures.** We consider a  $(T \rtimes \Gamma)$ -proper manifold, i.e. a  $\Gamma$ -equivariant fibration  $\pi_{\widehat{X}} : \widehat{X} \rightarrow T$  with oriented fibers and such that the quotient  $X = \widehat{X}/\Gamma$  is a smooth compact manifolds. Let  $r : X = \widehat{X}/\Gamma \rightarrow (E\Gamma \times T)/\Gamma$  be the associate classifying map .

For each cohomology class  $c \in H^*((E\Gamma \times T)/\Gamma, \mathbb{C})$  the number

$$(7.1) \quad \int_X L(TX) \wedge r^*(c)$$

is called the Baum-Connes higher signature of the  $T \rtimes \Gamma$ -proper manifold  $\widehat{X}$  (see Baum-Connes [2]). We are interested in the set

$$\left\{ \int_X L(TX) \wedge r^*(c), \quad c \in H^*((E\Gamma \times T)/\Gamma, \mathbb{C}) \right\}.$$

As already explained, projecting the fibers  $\pi_{\widehat{X}}^{-1}(\theta)$  ( $\theta \in T$ ) onto the quotient  $X := \widehat{X}/\Gamma$  one gets a foliation  $\mathcal{F}$  of the compact manifold  $X$ . Then the sets of higher signatures

$$\left\{ \int_X L(TX) \wedge r^*(c), \quad c \in H^*((E\Gamma \times T)/\Gamma, \mathbb{C}) \right\}$$

can equally be described as the set

$$\left\{ \int_X L(T\mathcal{F}) \wedge r^*(c), \quad [c] \in H^*((E\Gamma \times T)/\Gamma, \mathbb{C}) \right\},$$

(here we use the fact that the  $L$ -class of the normal bundle to the foliation is the pull-back of a class in  $H^*((E\Gamma \times T)/\Gamma, \mathbb{C})$ ).

Next, with the notations of Subsection 6.1, let us mention that the higher signatures of two cut-and-paste equivalent  $T \rtimes \Gamma$ -proper manifolds  $\widehat{X}_\phi$  and  $\widehat{X}_\psi$ :

$$\int_{\widehat{X}_\phi/\Gamma} L(\widehat{X}_\phi/\Gamma) \wedge r_\phi^*(c), \quad \int_{\widehat{X}_\psi/\Gamma} L(\widehat{X}_\psi/\Gamma) \wedge s_\psi^*(c)$$

do *not* coincide in general. See Karras-Kreck-Neumann-Ossa [24] and [29] for examples when  $T$  is reduced to a point.

Our goal is to find sufficient conditions on the group  $\Gamma$  and on its action, ensuring that the Baum-Connes higher signatures are cut-and-paste invariant.

**7.2. Cut-and-paste invariance: the cyclic cohomology approach.** We begin by recalling several results from Connes [10] and Gorokhovsky-Lott [15]. Set

$$\mathcal{B}^\omega = \left\{ \sum_{\gamma \in \Gamma} c_\gamma \gamma : |c_\gamma| \text{ decays faster than any exponential in } \|\gamma\| \right\}.$$

In Section 3 of [15] is defined a certain algebra of noncommutative differential forms  $\Omega^*(T, \mathcal{B}^\omega)$ . Consider a closed graded  $N$ -trace  $\eta$  on  $\Omega^*(T, \mathcal{B}^\omega)$  which is concentrated at the identity conjugacy class of  $\Gamma$ ; then to  $\eta$  one associates a cohomology class  $\Phi_\eta \in H^{N+\dim T+2\mathbb{Z}}((E\Gamma \times T)/\Gamma; \mathbb{C})$  where  $2\mathbb{Z}$  denotes an even-odd grading. In fact, one of the main results of [15] is the proof of an isomorphism between the homology of closed graded  $N$ -traces on  $\Omega^*(T, \mathbb{C}\Gamma)$ , concentrated at the identity conjugacy class, and the cohomology space  $H^{N+\dim T+2\mathbb{Z}}((E\Gamma \times T)/\Gamma; \mathbb{C})$ . The above class is obtained by restricting our  $\eta$  from  $\Omega^*(T, \mathcal{B}^\omega)$  to  $\Omega^*(T, \mathbb{C}\Gamma)$  and applying the isomorphism.

*From now on all our graded traces will be concentrated at the identity conjugacy class.*

Let now  $\widehat{X} \rightarrow T$  a  $(T \rtimes \Gamma)$ -proper manifold. Under the hypothesis and notations of Subsection 6.1, Gorokhovsky and Lott give a heat-equation proof of Connes' index theorem for  $(T \rtimes \Gamma)$ -proper manifolds (see [10] Theorem 12, Ch III, Section 7.7 for the original statement). In particular Gorokhovsky and Lott prove the following formula:

$$\langle \text{ch Ind}_\omega \mathcal{D}^{\text{sign}}, \eta \rangle = \int_{\widehat{X}/\Gamma} L(T\mathcal{F}) \wedge r^*(\Phi_\eta).$$

The index class in this formula is not the index class defined in Subsection 3.5 but rather a refinement in the  $K$ -theory of the algebra

$$C^\infty(T, \mathcal{B}^\omega) = \mathcal{B}^\omega \otimes_{\mathbb{C}\Gamma} (C_c^\infty(T) \rtimes \Gamma),$$

a subalgebra of  $C(T) \rtimes_r \Gamma$  containing  $C_c^\infty(T) \rtimes \Gamma$ . In general, the  $K$ -theory groups of  $C^\infty(T, \mathcal{B}^\omega)$  and  $C(T) \rtimes_r \Gamma$  are different and the index class defined in Subsection 3.5 is the image of  $\text{Ind}_\omega \mathcal{D}^{\text{sign}}$  under the  $K$ -theory homomorphism  $K_0(C^\infty(T, \mathcal{B}^\omega)) \rightarrow K_0(C(T) \rtimes_r \Gamma)$  induced by the inclusion  $C^\infty(T, \mathcal{B}^\omega) \hookrightarrow C(T) \rtimes_r \Gamma$ . On the other hand, all our formulas have been established for index classes in  $K_*(C(T) \rtimes_r \Gamma)$ ; this means that we need further hypothesis in order to combine the Gorokhovsky-Lott index formula and our results in the previous sections. The result we need is stated in Corollary 3 of [15]: assume that  $\mathcal{A}$  is a dense holomorphically closed subalgebra of  $C(T) \rtimes_r \Gamma$  containing  $C^\infty(T, \mathcal{B}^\omega)$ ; then we know that  $K_*(\mathcal{A}) \simeq K_*(C(T) \rtimes_r \Gamma)$ . Let  $\eta$  be a closed graded trace on  $\Omega^*(T, \mathbb{C}\Gamma)$ , then  $\eta$  defines a cyclic cocycle on  $C^\infty(T) \rtimes \Gamma$ : we *assume* that this cyclic cocycle *extends* to a cyclic cocycle  $\eta_{\mathcal{A}}$  on  $\mathcal{A}$ , then

$$\langle \text{ch Ind } \mathcal{D}^{\text{sign}}, \eta_{\mathcal{A}} \rangle = \int_{\widehat{X}/\Gamma} L(T\mathcal{F}) \wedge r^*(\Phi_\eta).$$

where the index class is the one we defined in  $K_*(C(T) \rtimes_r \Gamma)$  and where  $\Phi_\eta$  is the cohomology class in  $H^*((E\Gamma \times T)/\Gamma, \mathbb{C})$  associated to  $\eta$  under the Gorokhovsky-Lott isomorphism.

**Definition 7.1.** We shall say that a closed graded trace on  $\Omega^*(T, \mathbb{C}\Gamma)$ , is *holomorphically extendable* if there exists a dense holomorphically closed subalgebra  $\mathcal{A} \subset C(T) \rtimes_r \Gamma$  with

$$C_c^\infty(T) \rtimes \Gamma \subset C^\infty(T, \mathcal{B}^\omega) \subset \mathcal{A}$$

and with the property that the cyclic cocycle defined by  $\eta$  is cohomologous to a cocycle that extends from  $C_c^\infty(T) \rtimes \Gamma$  to  $\mathcal{A}$ .

Making use of Proposition 6.3 in the previous section we thus obtain the following general

**Theorem 7.2.** *Let  $\widehat{X}_\phi$  and  $\widehat{X}_\psi$  two cut-and-paste equivalent  $T \rtimes \Gamma$ -proper manifolds satisfying Assumption 6.2. Let  $[c] \in H^*((E\Gamma \times T)/\Gamma; \mathbb{C})$  be equal to  $\Phi_\eta$ , with  $\eta$  a closed graded trace on  $\Omega^*(T, \mathbb{C}\Gamma)$  concentrated at the identity element. If  $\eta$  is holomorphically extendable, then*

$$\int_{\widehat{X}_\phi/\Gamma} L(T\mathcal{F}_\phi) \wedge r_\phi^*([c]) = \int_{\widehat{X}_\psi/\Gamma} L(T\mathcal{F}_\psi) \wedge s_\psi^*([c]).$$

We shall now give examples where the assumptions of the theorem are satisfied.

**7.3. Isometric actions.** Assume that  $\Gamma$  is Gromov Hyperbolic and preserves a Riemann metric of  $T$ . Let  $\omega \in \Omega^{\dim T - k}(T)$  be a  $\Gamma$ -invariant differential form, then one defines a cyclic cocycle  $\tau_\omega$  on  $C^\infty(T)$  by the formula:

$$\tau_\omega(f_0, f_1, \dots, f_k) = \int_T f_0 df_1 \wedge \dots \wedge df_k \wedge \omega.$$

Let  $\rho' \in H^l(\mathbb{C}\Gamma, \mathbb{C})$  be a group cocycle and denote by  $\tau_{\rho'} \in HC^l(\mathbb{C}\Gamma)$  the associated cyclic cocycle constructed by Connes ([10]). Consider then the cyclic cocycle  $\tau'$  of the algebraic tensor product  $C^\infty(T) \rtimes \Gamma = C_c^\infty(T \rtimes \Gamma)$  defined by:

$$\tau'(f_0\gamma_0, f_1\gamma_1, \dots, f_{k+l}\gamma_{k+l}) = \tau_{\rho'} \# \tau_\omega(f_0 \otimes \gamma_0, \dots, f_{k+l} \otimes \gamma_{k+l})$$

where the  $f_j \in C^\infty(T)$ , the  $\gamma_j \in \Gamma$  and the  $\#$  is defined in [10] page 191. Then Jiang has proven the following (see Section 3 of [21]): there exists a group cocycle  $\rho$  cohomologous to  $\rho'$  such that  $\rho$  has polynomial growth and  $\tau = \tau_\rho \# \tau_\omega$  extends as a cyclic cocycle to a dense holomorphically closed subalgebra  $\mathcal{A}$  of  $C(T) \rtimes_r \Gamma$ . We conclude that  $\tau$  is holomorphically extendable. Let  $\Phi_\rho \in H^*((E\Gamma \times T)/\Gamma; \mathbb{C})$  denote the associated cohomological class (see [15]). Then the Baum-Connes higher signature

$$\int_{\widehat{X}/\Gamma} L(T\mathcal{F}) \wedge r^*(\Phi_\rho)$$

is a cut-and-paste invariant under our basic assumption 6.2. In fact, one can prove that  $r^*(\Phi_\rho) = f^*(\rho) \wedge [\pi^*(\omega)]$  where  $f : \widehat{X}/\Gamma \rightarrow B\Gamma$  denotes the classifying map of the  $\Gamma$ -covering  $\widehat{X} \rightarrow \widehat{X}/\Gamma$  and  $[\pi^*(\omega)]$  denotes the differential form on  $\widehat{X}/\Gamma$  induced by the  $\Gamma$ -invariant differential form  $\pi^*(\omega)$ ; thus, equivalently, we have proved that  $\int_{\widehat{X}/\Gamma} L(T\mathcal{F}) \wedge f^*(\rho) \wedge [\pi^*(\omega)]$  is a cut-and-paste invariant if Assumption 6.2 is satisfied.

We should mention here that this example is automatically covered by the results of subsection 7.6 (assuming, as we do there, that the vertical tangent bundle admits a  $\Gamma$ -equivariant spin structure). The next example, on the other hand, is somewhat universal and it is *not* covered by the Baum-Connes approach explained in subsection 7.6.

**7.4. The Godbillon-Vey signature  $\sigma_{GV}$ .** Assume  $T = S^1$  and consider a  $\Gamma$ -equivariant fibration  $\pi_{\widehat{X}} : \widehat{X} \rightarrow S^1$  as in Subsection 6.1.

Let  $X := \widehat{X}/\Gamma$  and let  $\mathcal{F}$  the induced foliation. There exists a well-defined Godbillon-Vey class  $GV \in H^3(X, \mathbb{R})$ . The Godbillon-Vey signature is, by definition

$$\sigma_{GV}(X, \mathcal{F}) := \int_X L(T\mathcal{F}) \wedge GV$$

By results of Connes [10] (see also the work of Moriyoshi-Nastume [49]), we know that there exists a dense and holomorphically closed subalgebra  $\mathcal{A}$  of  $C(S^1) \rtimes_r \Gamma$  and a closed graded trace  $\eta$  on  $\Omega^*(S^1, \mathbb{C}\Gamma)$  with the following properties:

- the cyclic cocycle defined by  $\eta$  *extends* to a cyclic cocycle  $\eta_{\mathcal{A}}$  on  $\mathcal{A}$
- if  $\Phi_\eta \in H^*((E\Gamma \times S^1)/\Gamma, \mathbb{R})$  is the class corresponding to  $\eta$  under the Gorokhovskiy-Lott isomorphism between the homology of closed graded traces on  $\Omega^*(S^1, \mathbb{C}\Gamma)$  (concentrated at the identity) and the cohomology of  $(E\Gamma \times S^1)/\Gamma$ , then  $GV = r^*\Phi_\eta$ .

In other words, the so called Godbillon-Vey cyclic cocycle is holomorphically extendable and has the property that  $r^*\Phi_\eta$  is equal to the Godbillon-Vey class in  $X$ .

By applying the index theorem of Connes, we get

$$\langle \text{Ch}(\text{Ind}(\mathcal{D}^{\text{sign}})), \eta_{\mathcal{A}} \rangle = \int_X L(T\mathcal{F}) \wedge GV := \sigma_{GV}(X, \mathcal{F}).$$

It is important to remark that this formula holds with no assumption on the group  $\Gamma$  and its action on  $T$ .

**7.5. On the cut-and-paste invariance of  $\sigma_{GV}$ .** Suppose now that  $\widehat{X}_\phi$  and  $\widehat{X}_\psi$  are two cut-and-paste equivalent  $T \rtimes \Gamma$ -proper manifolds as in Subsection 6.1. By applying the above formula and Proposition 6.3, we discover that the Godbillon-Vey signature is a cut-and-paste invariant if the middle-degree assumption 6.2 is satisfied. More precisely:

**Theorem 7.3.** *Let  $\widehat{X}_\phi$  and  $\widehat{X}_\psi$  two cut-and-paste equivalent  $T \rtimes \Gamma$ -proper manifolds satisfying Assumption 6.2. Let  $(X_\phi, \mathcal{F}_\phi)$  and  $(X_\psi, \mathcal{F}_\psi)$  be the associated foliated manifolds. Then*

$$\sigma_{GV}(X_\phi, \mathcal{F}_\phi) = \sigma_{GV}(X_\psi, \mathcal{F}_\psi).$$

**7.6. Cut-and-paste invariance: the Baum-Connes approach.** Now consider (see Connes [10], page 114) the Baum-Connes rational assembly map:

$$\mu_{\mathbb{Q}} : K_{*,\tau}((E\Gamma \times T)/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C(T) \rtimes_r \Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here  $\tau := (E\Gamma \times T(T))/\Gamma$ , with  $T(T)$  denoting the tangent bundle to  $T$ , and  $K_{*,\tau}((E\Gamma \times T)/\Gamma) := K_0(B\tau, S\tau)$ , with  $B\tau$  and  $S\tau$  denoting the ball and sphere bundles of  $\tau$ . In the general foliated case ([10], Ch 2, Section 8.γ) there is a similar map

$$\mu_{\mathbb{Q}} : K_{*,\tau}(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(C_r^*(M, \mathcal{F})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with  $BG$  the classifying space of the holonomy groupoid. It is stated in work of Baum and Connes ([2] page 12) that the rational injectivity of the latter assembly map implies the leafwise homotopy invariance of the Baum-Connes higher signatures of a foliation. The line of reasoning is the following: if two foliations are leafwise homotopy equivalent, then their leafwise signature index classes are equal in  $K_0(C_r^*(M, \mathcal{F}))$  (see [2]); if moreover the Baum-Connes map is rationally injective, then from the equality of the index classes one deduces the equality of all the higher signatures. The slogan here is the following: if the Baum-Connes map is rationally injective, then *the equality of the index classes implies the equality of all higher signatures*.

In the next theorem we shall prove the analogue of this fact in our groupoid  $T \rtimes \Gamma$ -setting. In order to simplify our treatment we shall make the additional assumption that the vertical tangent bundle  $TZ$  to the fibration  $\widehat{X} \rightarrow T$ , coming into the definition of  $(T \rtimes \Gamma)$ -proper manifold, admits a  $\Gamma$ -invariant spin structure.

**Theorem 7.4.** *Assume that the rational Baum-Connes map*

$$\mu_{\mathbb{Q}} : K_{0,\tau}((E\Gamma \times T)/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(C(T) \rtimes_r \Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*is injective. Let  $\widehat{X}_\phi$  and  $\widehat{X}_\psi$  be two cut-and-paste equivalent  $T \rtimes \Gamma$ -proper manifolds satisfying Assumption 6.2 and such that the vertical tangent bundles both admit a  $\Gamma$ -invariant spin structure. Then for any  $c \in H^*((E\Gamma \times T)/\Gamma; \mathbb{Q})$  one has:*

$$\int_{\widehat{X}_\phi/\Gamma} L(\widehat{X}_\phi/\Gamma) \wedge r_\phi^*(c) = \int_{\widehat{X}_\psi/\Gamma} L(\widehat{X}_\psi/\Gamma) \wedge s_\psi^*(c).$$

*Proof.* We are going to show that for any  $c \in H^*((E\Gamma \times T)/\Gamma; \mathbb{C})$  one has:

$$\int_{\widehat{X}_\phi/\Gamma} L(T\mathcal{F}_\phi) \wedge r_\phi^*(c) = \int_{\widehat{X}_\psi/\Gamma} L(T\mathcal{F}_\psi) \wedge s_\psi^*(c)$$

which will prove the result. Let

$$r_\phi : X_\phi \rightarrow (E\Gamma \times T)/\Gamma \quad ; \quad s_\psi : X_\psi \rightarrow (E\Gamma \times T)/\Gamma$$

be the two classifying maps induced by  $(\widetilde{\rho}_\phi, \pi_\phi) : \widehat{X}_\phi \rightarrow E\Gamma \times T$  and  $(\widetilde{\rho}_\psi, \pi_\psi) : \widehat{X}_\psi \rightarrow E\Gamma \times T$  respectively (see subsection 6.1). Let  $S_{\mathcal{F},\phi}$ ,  $S_{\mathcal{F},\psi}$  be the spinor bundles induced on the

quotients  $X_\phi, X_\psi$  by the vertical spinor bundles. Since the vertical tangent bundles carry a  $\Gamma$ -invariant Spin structure, we see that the bundles

$$TX_\phi \oplus r_\phi^* \tau \quad \text{and} \quad TX_\psi \oplus s_\psi^* \tau$$

carry a  $\text{Spin}^c$  structure. Then

$$[X_\phi, S_{\mathcal{F},\phi}^*, r_\phi : X_\phi \rightarrow (E\Gamma \times T)/\Gamma]; \quad [X_\psi, S_{\mathcal{F},\psi}^*, s_\psi : X_\psi \rightarrow (E\Gamma \times T)/\Gamma]$$

define two geometric cycles in  $K_{0,\tau}((E\Gamma \times T)/\Gamma)$ , see [10] page 115. Notice, incidentally, that the Todd class is well defined for a  $\text{Spin}^c$  bundle ([10], page 115). It follows from the very definition of the Baum-Connes map that the image of  $[X_\phi, S_{\mathcal{F},\phi}^*, r_\phi]$  under  $\mu$  is precisely equal to the index class  $\text{Ind } \mathcal{D}_\phi^{\text{sign}} \in K_0(C(T) \rtimes_r \Gamma)$  and similarly for  $[X_\psi, S_{\mathcal{F},\psi}^*, s_\psi]$ . From our Assumption 6.2 we know that the two index classes are equal; thus, by the assumed rational injectivity of the Baum-Connes map, we have the equality

$$[X_\phi, S_{\mathcal{F},\phi}^*, r_\phi] = [X_\psi, S_{\mathcal{F},\psi}^*, s_\psi] \quad \text{in} \quad K_{0,\tau}((E\Gamma \times T)/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

On the other hand, there is an isomorphism

$$K_{0,\tau}((E\Gamma \times T)/\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H_{\text{even}}((E\Gamma \times T)/\Gamma, \mathbb{Q})$$

which is given by the composition of the Chern character  $\text{Ch}$  in  $K$ -homology and the Thom isomorphism  $\Phi$  in homology. Using Proposition 7 page 116 in [10] we discover that  $\Phi \circ \text{Ch}[X_\phi, S_{\mathcal{F},\phi}^*, r_\phi]$  equals

$$(r_\phi)_* ((\text{Ch}(S_{\mathcal{F},\phi}^*) \cdot \text{Td}(TX_\phi \oplus r_\phi^* \tau)) \cap [X_\phi]) \in H_{\text{even}}((E\Gamma \times T)/\Gamma, \mathbb{Q}).$$

Using Lemma 4.4 of [4] (pp. 148-150) we see that

$$(r_\phi)_* ((\text{Ch } S_{\mathcal{F},\phi}^* \cdot \text{Td}(TX_\phi \oplus r_\phi^* \tau)) \cap [X_\phi])$$

equals

$$C \cdot (r_\phi)_* ((L(T\mathcal{F}_\phi) \cdot \text{Td}_{\mathbb{C}}(r_\phi^* \tau)) \cap [X_\phi])$$

where  $C \in \mathbb{Q}^*$  depends on the rank of  $S_{\mathcal{F},\phi}^*$ . Summarizing, it follows from our assumptions that

$$\int_{X_\phi} L(T\mathcal{F}_\phi) \wedge \text{Td}_{\mathbb{C}}(r_\phi^* \tau) \wedge r_\phi^*(c) = \int_{X_\psi} L(T\mathcal{F}_\psi) \wedge \text{Td}_{\mathbb{C}}(s_\psi^* \tau) \wedge s_\psi^*(c).$$

Applying this equality to  $c := (\text{Td}_{\mathbb{C}}^{-1} \tau) \wedge b$  we get the equality

$$\int_{X_\phi} L(T\mathcal{F}_\phi) \wedge r_\phi^*(b) = \int_{X_\psi} L(T\mathcal{F}_\psi) \wedge s_\psi^*(b).$$

for each  $b \in H^*((E\Gamma \times T)/\Gamma, \mathbb{Q})$  which proves the theorem.  $\square$

**Remark.** The Baum-Connes map  $\mu_{\mathbb{Q}}$  is injective many case, we just mention three of them:

- (a)  $\Gamma$  has the Haagerup property (ie a-T-amenable) (see Higson-Kasparov [19]).
- (b)  $\Gamma$  is Gromov hyperbolic or more generally  $\Gamma$  is any discrete group acting properly by isometries on a weakly bolic, weakly geodesic metric space of bounded coarse geometry (see Kasparov-Skandalis [25]).
- (c)  $\Gamma$  is a lattice in a semi-simple Lie group  $G$  and  $T = G/P$  where  $P$  is a minimal parabolic subgroup of  $G$  (for more results in this direction see Skandalis-Tu-Yu [55]).

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