ON IHARA'S LEMMA FOR HILBERT MODULAR VARIETIES

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Abstract. Let $\overline{\rho}$ be a modulo p representation of the absolute Galois group of a totally real number field. Under the assumptions that $\overline{\rho}$ has *large* image and admits a *minimal* modular deformation, we show that every low weight crystalline deformation of $\overline{\rho}$ unramified outside a finite set of primes is again modular. We use the approach of Wiles [27] and Fujiwara [12]. The main new ingredient is an Ihara type lemma for the local component at $\overline{\rho}$ of the middle degree cohomology of the Hilbert modular variety. As an application we relate the algebraic p-part of the value at 1 of the adjoint L-function associated to a Hilbert modular newform, to the cardinality of the corresponding Selmer group.

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Introduction

Let F be a totally real number field of degree d, ring of integers $\mathfrak{o}$ and different $\mathfrak{d}$. Denote by J_F the set of all embeddings of F into $\overline{\mathbb{Q}}$. The absolute Galois group of a field L is denoted by $\mathcal{G}_L$. Let $p \geq 5$ be a prime unramified in F. We fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Let f be a Hilbert modular newform over F of level $\mathfrak{n}$ (an ideal of $\mathfrak{o}$), weight $k = \sum_{\tau \in J_F} k_\tau \tau$ ($k_\tau \geq 2$ of the same parity), and denote by $c(f, \mathfrak{a})$ its eigenvalue for the standard Hecke operator $T_\mathfrak{a}$. By [23, 24] and [1] one can associate to f a p-adic representation :

$$\rho_{f,p} : \mathcal{G}_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p),$$

which is irreducible, totally odd, unramified outside $\mathfrak{n}p$ and characterized by the property that for each prime v not dividing $\mathfrak{n}p$ we have $\mathrm{tr}(\rho_{f,p}(\mathrm{Frob}_v)) = \iota_p(c(f, v))$ (Frob_v denotes the geometric Frobenius at v). At the primes of F dividing p, $\rho_{f,p}$ is known to be de Rham for all p, and crystalline when p is prime to $\mathfrak{n}$ and $p > k_0 = \max\{k_\tau | \tau \in J_F\}$ (*cf* [1], [2]).

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Such a $\rho_{f,p}$ is defined over the ring of integers $\mathcal{O}$ of a finite extension E of $\mathbb{Q}_p$. Let $\overline{\rho}_{f,p}$ be the semi-simplification of the reduction of $\rho_{f,p}$ modulo a uniformizer ϖ of $\mathcal{O}$. We say that a two-dimensional irreducible p-adic (resp. modulo p) representation of $\mathcal{G}_F$ is *modular* if it can be obtained by the above construction. The following conjecture is a well known extension to an arbitrary totally real field F of a conjecture of Fontaine and Mazur [11] :

Conjecture. *An irreducible, totally odd, two-dimensional p-adic representation ρ of $\mathcal{G}_F$ unramified outside a finite set of primes and de Rham at the primes dividing p is modular, up to a twist by an integer power of the p-adic cyclotomic character χ_p.*

We provide an evidence for this conjecture by proving a result of the form :

$$\overline{\rho} \text{ minimally modular } \Rightarrow \rho \text{ modular .}$$

Such results have been obtained by Fujiwara, Taylor and Skinner-Wiles (*cf* Remark 0.1). Assuming Fujiwara's results in the minimal case [12], we show the following

Theorem A. *Let $\overline{\rho} : \mathcal{G}_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume that :*

($\mathbf{Mod}_{\overline{\rho}}$) *there exists a Hilbert modular newform f_0 of prime to p level $\mathfrak{n}_0$ and weight k, $p-1 > \sum_{\tau\in J_F}(k_\tau - 1)$, such that $\rho_{f_0,p}$ is a minimal deformation of $\overline{\rho}$ (cf Def.1.3(iii)), and*

($\mathbf{LI}_{\mathrm{Ind}\overline{\rho}}$) *there exist a power q of p, a finite extension F' of F, a partition $J_F = \coprod_{i\in I} J_F^i$ and for each $\tau \in J_F$ an element $\sigma_\tau \in \mathrm{Gal}(\mathbb{F}_q / \mathbb{F}_p)$, such that :*
– for all $i \in I$, the elements $(\sigma_\tau)_{\tau\in J_F^i}$ are two by two distinct, and
– $\mathrm{Ind}_F^{\mathbb{Q}} \overline{\rho}_{|\mathcal{G}_{F'}}$ factors as a surjection $\mathcal{G}_{F'} \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_q)^I$ followed by $(M_i)_{i\in I} \mapsto (M_i^{\sigma_\tau})_{i\in I,\tau\in J_F^i}$.

Then all crystalline deformations of $\overline{\rho}$ of weights between 0 and $p-2$ and unramified outside a finite set of primes are modular.

Remark 0.1. (i) We have greatly benefited from the work [12] of Fujiwara, though we use a different approach (*cf* §1.2 for a more detailed discussion). Let us just mention the following :

Denote by P the set of primes v of F such that $\overline{\rho}$ is of type $\mathbf{V}$ at v (*cf* §1.1) and $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(v) \equiv -1 \pmod p$. If $P = \varnothing$, then Thm.A is independent of the results of [12].

(ii) Skinner-Wiles [20, 21, 22] and Taylor [25] obtain similar results. Their assumptions on the image of $\overline{\rho}$ are much weaker, but on the other hand [20, 21, 22] assume parallel weight and ordinarity at p, while [25] assumes parallel weight and even d.

Remark 0.2. When $F = \mathbb{Q}$, it is known by the work of Ribet [19] *et al.* that :

$$\overline{\rho} \text{ modular} \Rightarrow \overline{\rho} \text{ minimally modular.}$$

For $F \neq \mathbb{Q}$, minimal modularity is established away from p by the work of Jarvis [14, 15], Fujiwara [13] and Rajaei [17]. Therefore, we can omit the word "minimal" in ($\mathbf{Mod}_{\overline{\rho}}$).

Combining Thm.A with the results of [10] and using Remark 0.2, we obtain :

Theorem B. *Let f be a Hilbert modular newform over F of level $\mathfrak{n}$ and weight k. Let p be a prime not dividing $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n}\,\mathfrak{d})$ and such that $p-1 > \sum_{\tau\in J_F}(k_\tau - 1)$. Assume that $\overline{\rho} = \overline{\rho}_{f,p}$ satisfies $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$. Then*

$$\left(\iota_p\left(\frac{\Gamma(\mathrm{Ad}^0(\rho_{f,p}),1)\,\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}),1)}{\Omega_f^+\Omega_f^-}\right)\right)_{\mathcal{O}} = \mathrm{Fitt}_{\mathcal{O}}\left(\mathrm{H}^1_{\mathrm{f}}(F,\mathrm{Ad}^0(\rho_{f,p})\otimes\mathbb{Q}_p\,/\,\mathbb{Z}_p)\right),$$

where $\mathrm{Ad}^0(\rho_{f,p})$ denotes the representation of $\mathcal{G}_F$ on the 2×2 trace zero matrices, $\Omega_f^\pm \in \mathbb{C}^\times/\,\mathcal{O}^\times$ are any two complementary Eichler-Shimura-Harder periods (cf [10, §4.2]*) and $\mathrm{H}^1_{\mathrm{f}}$ is the Selmer group defined by Bloch and Kato (cf* [8, §2.1]*).*

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1. Preliminaries.

We always assume $\mathcal{O}$ to be sufficiently large and denote by κ its residue field.

1.1. **Minimal deformations.** Let v be a place of F not dividing p and $\overline{\rho}_v : \mathcal{G}_{F_v} \to \mathrm{GL}_2(\kappa)$ be a continuous representation. Denote by I_v the inertia subgroup.

According to Diamond's classification (*cf* [6]) we distinguish four types of behavior (principal, special, vexing and harmless) :

P : $\overline{\rho}_v$ is reducible and $\overline{\rho}_v|_{I_v}$ is decomposable.

S : $\overline{\rho}_v$ is reducible and $\overline{\rho}_v|_{I_v}$ is indecomposable.

V : $\overline{\rho}_v$ is irreducible and $\overline{\rho}_v|_{I_v}$ is reducible.

H : $\overline{\rho}_v$ is irreducible and $\overline{\rho}_v|_{I_v}$ is irreducible.

Note that this classification do not see twists by characters. Let A be a local complete noetherian $\mathcal{O}$-algebra of residue field κ, and $\rho_v : \mathcal{G}_{F_v} \to \mathrm{GL}_2(A)$ be a lifting of $\overline{\rho}_v$. For a character $\overline{\mu}$ with values in $\kappa^\times$, we denote by $\widetilde{\mu}$ its Teichmüller lift.

Definition 1.1. We say that ρ_v is a minimally ramified (or finite):

– if $\overline{\rho}_v$ is of type **P** or **V**, $\overline{\rho}_v|_{I_v} \cong \begin{pmatrix}\overline{\mu} & 0\\ 0 & \overline{\mu}'\end{pmatrix}$, and $\rho_v|_{I_v} \cong \begin{pmatrix}\widetilde{\mu} & 0\\ 0 & \widetilde{\mu}'\end{pmatrix}$, or

– if $\overline{\rho}_v$ is of type **S**, $\overline{\rho}_v|_{I_v} \cong \begin{pmatrix}\overline{\mu} & *\\ 0 & \overline{\mu}\end{pmatrix}$, and $\rho_v|_{I_v} \cong \begin{pmatrix}\widetilde{\mu} & *\\ 0 & \widetilde{\mu}\end{pmatrix}$, or

– if $\overline{\rho}_v$ is of type **H**, and $\det\rho_v|_{I_v}$ is the Teichmüller lift of $\det\overline{\rho}_v|_{I_v}$.

Remark 1.2. (i) If ρ_v is a minimally ramified lifting of $\overline{\rho}_v$, then $\rho_v\otimes\widetilde{\mu}$ is a minimally ramified lifting of $\overline{\rho}_v\otimes\overline{\mu}$, for all characters $\overline{\mu} : \mathcal{G}_{F_v} \to \kappa^\times$.

(ii) If ρ_v is a minimally ramified lifting of $\overline{\rho}_v$, then the Artin conductors of ρ_v and $\overline{\rho}_v$ coincide and $\det\rho_v|_{I_v}$ is the Teichmüller lift of $\det\overline{\rho}_v|_{I_v}$. The converse holds if $\overline{\rho}_v$ has minimal conductor among its twists and $v \notin P$ (*cf* Remark 0.1(i)).

1.2. **The strategy of the proof.** The method we use originates in the work of Wiles [27] and Taylor-Wiles [26], later developed by Diamond [7] and Fujiwara [12].

Let $\overline{\rho}:\mathcal{G}_F\to \mathrm{GL}_2(\kappa)$ be a continuous, totally odd, absolutely irreducible representation. We assume that $\overline{\rho}$ is crystalline at primes dividing p with Fontaine-Laffaille weights $(\frac{k_0-k_\tau}{2},\frac{k_0+k_\tau}{2}+1)_{\tau\in J_F}$, where $k=\sum_{\tau\in J_F}k_\tau\tau$ is as in the introduction, and

$$p-1>\sum_{\tau\in J_F}(k_\tau-1).$$

Definition 1.3. Let Σ be finite set of primes of F not dividing p. Let A be a local complete noetherian $\mathcal{O}$-algebra with residue field κ. We say that a deformation $\rho:\mathcal{G}_F\to\mathrm{GL}_2(A)$ of $\overline{\rho}$ to A is Σ-ramified, if the following three conditions hold :

– ρ is minimally ramified at all primes $v\notin\Sigma$, $v\nmid p$ (*cf* Def.1.1),

– ρ is crystalline at all primes v dividing p with the same weights as $\overline{\rho}$,

– $\chi_p^{1-k_0}\cdot\det\rho$ is equal to the Teichmüller lift of $\overline{\chi}_p^{1-k_0}\cdot\det\overline{\rho}$.

A $\varnothing$-ramified deformation is called minimally ramified.

Consider the functor $\mathcal{F}_{\overline{\rho},\Sigma}$ assigning to a local complete noetherian $\mathcal{O}$-algebra A with residue field κ, the set of all Σ-ramified deformations of $\overline{\rho}$ to A. By Ramakrishna [18] and Mazur [16] $\mathcal{F}_{\overline{\rho},\Sigma}$ is representable by a $\mathcal{O}$-algebra $\mathcal{R}_\Sigma$, called the universal deformation ring.

On the other hand, let $\mathcal{T}_\Sigma$ be the $\mathcal{O}$-subalgebra of $\prod_f\mathcal{O}$ generated by the elements $(\iota_p(c(f,v)))_{v\notin\Sigma,v\nmid\mathfrak{n}p}$, where f runs over the Hilbert modular newforms such that $\rho_{f,p}$ is a Σ-ramified deformation of $\overline{\rho}$. The algebra $\mathcal{T}_\Sigma$ is local complete noetherian and reduced. Wiles' theory of pseudo-representations gives a Σ-ramified deformation of $\overline{\rho}$ to $\mathcal{T}_\Sigma$, hence a local $\mathcal{O}$-algebra homomorphism $\pi_\Sigma:\mathcal{R}_\Sigma\to\mathcal{T}_\Sigma$, surjective by Cebotarev Density Theorem.

We follow Wiles' method consisting in showing first that $\pi_\varnothing$ is an isomorphism (the minimal case) and then in proving, by induction on the cardinality of Σ, that π_Σ is an isomorphism (raising the level). In order to prove that $\mathcal{R}_\Sigma$ is "not too big" one uses Galois cohomology (*cf* (2)). In order to prove that $\mathcal{T}_\Sigma$ is "not too small" one has to realize it *geometrically* as a local component of the Hecke algebra acting on the middle degree cohomology of some Shimura variety and then to study congruences between Hilbert modular newforms. It is on this last point that our approach differs from Fujiwara's. Whereas Fujiwara's uses some quaternionic Shimura varieties over F of dimension ≤ 1, we use the d-dimensional Hilbert modular variety. The torsion freeness of some local components of the middle degree cohomology of the Hilbert modular variety, under the assumption $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$, is the main result of [10] and will be recalled in the next paragraph.

In the minimal case our result is strictly included in Fujiwara's since we only treat the case $P=\varnothing$ (*cf* Remark 0.1(i)) and furthermore we do not consider the ordinary noncrystalline case. On the other hand our level raising results are new, thanks to an Ihara type lemma for the Hilbert modular variety (*cf* Prop.3.4). Moreover, using the cohomology of the Hilbert modular variety allows us to interpret the cardinality of adjoint Selmer groups as special values of L-functions (*cf* Thm.B).

1.3. **The geometric input.** In this paragraph we make a summary of the results of [10]. Let $Y = Y_1(\mathfrak{n})$ be the Hilbert modular variety of level $K = K_1(\mathfrak{n})$ over $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n}\mathfrak{d})}]$ (*cf* [10, §1.4]). We consider the p-adic étale (or Betti) cohomology groups $\mathrm{H}^\bullet(Y_{\overline{\mathbb{Q}}}, \mathbb{V}_n)$, where $\mathbb{V}_n$ denotes the local system of weight $n = \sum_{\tau\in J_F}(k_\tau - 2)\tau \in \mathbb{N}[J_F]$ (*cf* [10, §2.1]). Let $\mathbb{T} = \mathcal{O}[T_\mathfrak{a}, \mathfrak{a} \subset \mathfrak{o}]$ be the Hecke algebra acting on $\mathrm{H}^d(Y, \mathbb{V}_n(\mathcal{O}))$ and $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}$ corresponding to f and ι_p. Let $\mathbb{T}' \subset \mathbb{T}$ be the subalgebra generated by the Hecke operators outside a finite set of places containing those dividing $\mathfrak{n}\,p$. The maximal ideal $\mathfrak{m}' = \mathfrak{m} \cap \mathbb{T}'$ of $\mathbb{T}'$ depends only on $\overline{\rho}_{f,p}$.

Theorem 1.4. [10, Thms.4.4, 6.6, 6.7] *Assume that* $p - 1 > \sum_{\tau\in J_F}(k_\tau - 1)$, *$p$ does not divide* $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n}\mathfrak{d})$, *and that* $\overline{\rho} = \overline{\rho}_{f,p}$ *satisfies* $(\mathbf{LI}_{\mathrm{Ind}\,\overline{\rho}})$. *Then*

(i) $\mathrm{H}^\bullet(Y, \mathbb{V}_n(\mathcal{O})_{\mathfrak{m}'} = \mathrm{H}^d(Y, \mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}$ *is a free* $\mathcal{O}$*-module of finite rank and the Poincaré pairing induces a perfect duality on it.*

(ii) $\mathrm{H}^\bullet(Y, \mathbb{V}_n(E/\mathcal{O})_{\mathfrak{m}'} = \mathrm{H}^d(Y, \mathbb{V}_n(E/\mathcal{O}))_{\mathfrak{m}'}$ *is a divisible* $\mathcal{O}$*-module of finite corank and* $\mathrm{H}^d(Y, \mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'} \times \mathrm{H}^d(Y, \mathbb{V}_n(E/\mathcal{O}))_{\mathfrak{m}'} \to E/\mathcal{O}$ *is a perfect Pontryagin pairing.*

(iii) $\mathrm{H}^d(Y, \mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}}$ *is a free* $\mathbb{T}_{\mathfrak{m}}$*-module of rank* 2^d, *and* $\mathbb{T}_{\mathfrak{m}}$ *is Gorenstein.*

The condition $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$ is used in the proof of the above theorem via the following lemma.

Lemma 1.5. [10, Lemma 6.5] *Denote by* $\widetilde{F}$ *the Galois closure of* F *in* $\overline{\mathbb{Q}}$. *Let* ρ_0 *be a continuous representation of* $\mathcal{G}_{\widetilde{F}}$ *on a finite dimensional* $\overline{\mathbb{F}}_p$*-vector space* W. *Assume* $(\mathbf{LI}_{\mathrm{Ind}\,\overline{\rho}})$ *and assume that, for every* $g \in \mathcal{G}_{\widetilde{F}}$, *the characteristic polynomial of* $\otimes\,\mathrm{Ind}_F^{\mathbb{Q}}\,\overline{\rho}(g)$ *annihilates* $\rho_0(g)$. *Then each* $\mathcal{G}_{\widetilde{F}}$*-irreducible subquotient of* ρ_0 *is isomorphic to* $\otimes\,\mathrm{Ind}_F^{\mathbb{Q}}\,\overline{\rho}$.

The next proposition shows that the condition $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$ is generically satisfied for Galois representations associated to Hilbert modular newforms.

Proposition 1.6. [10, Prop.3.17] *Assume that a Hilbert modular newform* f *is not a twist by a character of any of its* d *internal conjugates and is not a theta series. Then for all but finitely many primes* p, $\overline{\rho} = \overline{\rho}_{f,p}$ *satisfies* $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$.

2. Modularity of the minimal deformations.

The aim of this section is to prove the following :

Theorem 2.1. *Let* $\overline{\rho} : \mathcal{G}_F \to \mathrm{GL}_2(\kappa)$ *be a continuous representation. Assume* $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$, $(\mathbf{Mod}_{\overline{\rho}})$ *and that* $P = \varnothing$. *Then all minimally ramified deformations of* $\overline{\rho}$ *are modular.*

In the notations of §1.2, the above theorem amounts to prove that $\pi : \mathcal{R} \twoheadrightarrow \mathcal{T}$ is an isomorphism (since $\Sigma = \varnothing$ in the entire section, we shall omit the subscripts).

To show this isomorphism we use a method invented by Wiles [27] and Taylor-Wiles [26], as axiomatized by Fujiwara under the name of a Taylor-Wiles system (similar formalism has been developed independently by Diamond [7]).

The construction of a Taylor-Wiles system will occupy the entire section. It includes namely, a geometric realization of $\mathcal{T}$ as the Hecke algebra acting on the local component $\mathcal{M}$ at $\overline{\rho}$ of the degree d cohomology of the Hilbert modular variety. The torsion freeness of $\mathcal{M}$ is a crucial ingredient (*cf* Thm.1.4(i)).

2.1. **The formalism of Taylor-Wiles systems, following Fujiwara.**

Definition 2.2. Let $\mathcal{Q}$ be a family of finite sets of primes $\mathfrak{q}$ of F such that $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p}$. A Taylor-Wiles system for $\mathcal{Q}$ is a family $\{\mathcal{R}, (\mathcal{R}_Q, \mathcal{M}_Q)_{Q\in\mathcal{Q}}\}$ such that

(TW1) $\mathcal{R}_Q$ is a local complete $\mathcal{O}[\Delta_Q]$-algebra, where $\Delta_Q = \prod_{\mathfrak{q}\in Q}\Delta_{\mathfrak{q}}$ and $\Delta_{\mathfrak{q}}$ is the p-Sylow of $(\mathfrak{o}\,/\,\mathfrak{q})^\times$.

(TW2) $\mathcal{R}$ is a local complete $\mathcal{O}$-algebra and there is an isomorphism of local complete $\mathcal{O}$-algebras $\mathcal{R}_Q\,/I_Q\,\mathcal{R}_Q \cong \mathcal{R}$, where I_Q denotes the augmentation ideal of $\mathcal{O}[\Delta_Q]$.

(TW3) $\mathcal{M}_Q$ is a $\mathcal{R}_Q$-module, free over $\mathcal{O}[\Delta_Q]$, and $\mathcal{M}_Q\,/I_Q\,\mathcal{M}_Q$ is isomorphic to the same $\mathcal{R}$-module $\mathcal{M}$. We denote by $\mathcal{T}$ the image of $\mathcal{R}\to\mathrm{End}_{\mathcal{O}}(\mathcal{M})$.

When $\mathcal{Q} = \{Q_m | m\in\mathbb{N}\}$, we will write $\mathcal{R}_m, \mathcal{M}_m, \dots$ instead of $\mathcal{R}_{Q_m}, \mathcal{M}_{Q_m}, \dots$.

Theorem 2.3. *Let $\{\mathcal{R}, (\mathcal{R}_m, \mathcal{M}_m)_{m\in\mathbb{N}}\}$ be a Taylor-Wiles system. Assume that for all m:*

(i) *for all $\mathfrak{q}\in Q_m$, $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p^m}$,*

(ii) *$\mathcal{R}_m$ can be generated by $\# Q_m = r$ elements as a local complete $\mathcal{O}$-algebra.*

Then, the natural surjection $\mathcal{R}\twoheadrightarrow\mathcal{T}$ is an isomorphism. Moreover, these algebras are flat and complete intersection of relative dimension zero over $\mathcal{O}$, and $\mathcal{M}$ is free over $\mathcal{T}$.

2.2. **The rings $\mathcal{R}_Q$.**

Let Q be a finite set of auxiliary primes $\mathfrak{q}$ of F satisfying :

– $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p}$,

– $\overline{\rho}$ is unramified at $\mathfrak{q}$ and $\overline{\rho}(\mathrm{Frob}_{\mathfrak{q}})$ has two distinct eigenvalues $\overline{\alpha}_{\mathfrak{q}}$ and $\overline{\beta}_{\mathfrak{q}}$ in κ.

For such a Q we can associate by §1.2 an universal deformation ring $\mathcal{R}_Q$, endowed with a canonical surjection $\mathcal{R}_Q \twoheadrightarrow \mathcal{R}_\varnothing =: \mathcal{R}$. By a result of Faltings (*cf* [26, Appendix]) $\mathcal{R}_Q$ is a $\mathcal{O}[\Delta_Q]$-algebra and $\mathcal{R}_Q\,/I_Q\,\mathcal{R}_Q \cong \mathcal{R}$. Thus (TW1) and (TW2) hold.

2.3. **The modules $\mathcal{M}_Q$.**

Put $K = K_1(\mathfrak{n}_0)$, $K_{0,Q} = K\cap K_0(Q)$ and

$$K_Q = \left\{\gamma\in K_{0,Q} | \forall\mathfrak{q}\in Q\ ,\ \exists u\in(\mathfrak{o}/\mathfrak{q})^\times/\Delta_{\mathfrak{q}}\ ,\ \gamma\equiv\begin{pmatrix} au^{-1} & * \\ 0 & au\end{pmatrix} \pmod{\mathfrak{q}}\right\}.$$

Denote by Y (resp. $Y_{0,Q}$ et Y_Q) the integral models of the Hilbert modular varieties of level K (resp. $K_{0,Q}$ et K_Q). Then Y_Q is an étale covering of $Y_{0,Q}$ of Galois group Δ_Q.

Remark 2.4. The group $K_{\{\mathfrak{q}\}}$ does not contain the usual congruence subgroup $K_{1,\{\mathfrak{q}\}} = K\cap K_1(\mathfrak{q})$. The reason to define $K_{\{\mathfrak{q}\}}$ in this unusual way is that the Galois group of the étale covering $Y_{1,\{\mathfrak{q}\}}\to Y_{0,\{\mathfrak{q}\}}$ is equal to the cokernel of the homomorphism $\mathfrak{o}^\times\to(\mathfrak{o}/\mathfrak{q})^\times$! One could use instead the covering of fine moduli schemes $Y^1_{1,Q}\to Y^1_{0,Q}$(*cf* [10, §1.1, 1.4]).

Let ψ be the central character of f_0. Denote by $[\psi]$ the ψ-isotypic part for the action of the Hecke operators S_v, for primes $v\nmid\mathfrak{n}_0$. By Thm.1.4(i) the $\mathcal{T}$-module $\mathcal{M} := \mathrm{H}^d(Y, \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'}$ is free over $\mathcal{O}$.

Remark 2.5. The central character $\chi_p^{1-k_0}\det\rho_{g,p}$ of a Hilbert modular newform g occurring in $\mathrm{H}^d(Y,\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}$ differs from ψ by a character of a finite p-group. By twisting g by a square root of this character (p is odd!) we obtain a newform of central character ψ. Therefore this p-group acts freely on $\mathrm{H}^d(Y,\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}$.

Consider the ideal $\mathfrak{m}'_{0,Q} := (\varpi, T_v - c(f_0,v), U_{\mathfrak{q}} - \overline{\alpha}_{\mathfrak{q}}; v\nmid \mathfrak{n}_0 Q, \mathfrak{q}\in Q)$ of the Hecke algebra $\mathbb{T}'_{0,Q} = \mathcal{O}[T_v, U_{\mathfrak{q}}; v\nmid \mathfrak{n}_0 Q, \mathfrak{q}\in Q] \subset \mathrm{End}_{\mathcal{O}}(\mathrm{H}^d(Y_{0,Q},\mathbb{V}_n(E))[\psi])$ and put $\mathcal{T}_{0,Q} := (\mathbb{T}'_{0,Q})_{\mathfrak{m}'_{0,Q}}$.

By Thm.1.4(i) the $\mathcal{T}_{0,Q}$-module $\mathcal{M}_{0,Q} := \mathrm{H}^d(Y_{0,Q},\mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'_{0,Q}}$ is free over $\mathcal{O}$.

By Hensel's lemma, for all $\mathfrak{q}\in Q$, the polynomial $X^2 - T_{\mathfrak{q}}X + S_{\mathfrak{q}}\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}) \in \mathcal{T}[X]$ has an unique root $\widetilde{\alpha}_{\mathfrak{q}}$ (resp. $\widetilde{\beta}_{\mathfrak{q}}$) above $\overline{\alpha}_{\mathfrak{q}}$ (resp. above $\overline{\beta}_{\mathfrak{q}}$).

Lemma 2.6. *There exists an unique isomorphism of local complete $\mathcal{O}$-algebras $\mathcal{T}_{0,Q} \xrightarrow{\sim} \mathcal{T}$ sending T_v on T_v, for $v\nmid \mathfrak{n}_0 Qp$, and $U_{\mathfrak{q}}$ on $\widetilde{\alpha}_{\mathfrak{q}}$, for $\mathfrak{q}\in Q$. Moreover there is an isomorphism $\mathcal{M}\cong\mathcal{M}_{0,Q}$ of $\mathcal{T}_{0,Q}\cong\mathcal{T}$-modules.*

Proof : By induction, we may assume that $Q=\{\mathfrak{q}\}$. Consider the homomorphism of $\mathcal{O}$-modules $\phi:\mathcal{M}\to\mathcal{M}_{0,\mathfrak{q}}$ defined as $\phi(g) = g|(U_{\mathfrak{q}} - \widetilde{\beta}_{\mathfrak{q}})$.

First, we show that $\phi\otimes\mathrm{id}_{\mathbb{C}}$ is an isomorphism. By the Eichler-Shimura-Harder isomorphism we have to show that $S_k(K)_{\mathfrak{m}'}\to S_k(K_{0,\mathfrak{q}})_{\mathfrak{m}'_{0,\mathfrak{q}}}$ is an isomorphism. An eigenform in $S_k(K_{0,\mathfrak{q}})_{\mathfrak{m}'_{0,\mathfrak{q}}}$ is necessarily $\mathfrak{q}$-old because :

– it cannot be supercuspidal at $\mathfrak{q}$, because $\mathfrak{q}^2$ does not divide the level, and
– it cannot be a ramified principal series at $\mathfrak{q}$, having a trivial $\mathfrak{q}$-nebentypus, and
– it cannot be special at $\mathfrak{q}$, because $\overline{\alpha}_{\mathfrak{q}}\neq\overline{\beta}_{\mathfrak{q}}$ and $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})\equiv 1 \pmod p$.

Hence the inclusion $S_k(K)^{\oplus 2}\subset S_k(K_{0,\mathfrak{q}})$, $(g_1,g_2)\mapsto g_1+g_2|_{\mathfrak{q}}$ induces an isomorphism $(S_k(K)^{\oplus 2})_{\mathfrak{m}'_{0,\mathfrak{q}}} \xrightarrow{\sim} S_k(K_{0,\mathfrak{q}})_{\mathfrak{m}'_{0,\mathfrak{q}}}$. Moreover, the two eigenvalues $\widetilde{\alpha}_{\mathfrak{q}}$, $\widetilde{\beta}_{\mathfrak{q}}$ of $U_{\mathfrak{q}}$ acting on the $\mathcal{T}$-module $S_k(K)^{\oplus 2}_{\mathfrak{m}'}$ belong to $\mathcal{T}$ and are distinct modulo ϖ. Therefore $S_k(K)^{\oplus 2}_{\mathfrak{m}'}$ splits under the action of $U_{\mathfrak{q}}$ as a direct sum of two $\mathcal{T}$-modules, its $\widetilde{\alpha}_{\mathfrak{q}}$-part and its $\widetilde{\beta}_{\mathfrak{q}}$-part, and the former one is given by $(S_k(K)^{\oplus 2})_{\mathfrak{m}'_{0,\mathfrak{q}}}$. This gives an isomorphism

$$S_k(K)_{\mathfrak{m}'} \xrightarrow{\sim} (S_k(K)^{\oplus 2})_{\mathfrak{m}'_{0,\mathfrak{q}}} \ , \ \ g\mapsto (g|\widetilde{\alpha}_{\mathfrak{q}}, -g|S_{\mathfrak{q}}\mathrm{N}(\mathfrak{q})).$$

Putting all together, we find that $\phi\otimes\mathrm{id}_{\mathbb{C}} : S_k(K)_{\mathfrak{m}'}\to S_k(K_{0,\mathfrak{q}})_{\mathfrak{m}'_{0,\mathfrak{q}}}$ is an isomorphism, commuting with the action of T_v, $v\nmid\mathfrak{n}_0 Qp$, and under which the action of $U_{\mathfrak{q}}$ on the right hand side corresponds to the action of $\widetilde{\alpha}_{\mathfrak{q}}\in\mathcal{T}$ on the left hand side.

As $\mathcal{M}$ (resp. $\mathcal{M}_{0,\mathfrak{q}}$) is free over $\mathcal{O}$, the algebra $\mathcal{T}$ (resp. $\mathcal{T}_{0,\mathfrak{q}}$) acts faithfully on $S_k(K)_{\mathfrak{m}'}$ (resp. $S_k(K_{0,\mathfrak{q}})_{\mathfrak{m}'_{0,\mathfrak{q}}}$). Therefore the map $\mathcal{T}_{0,Q}\to\mathcal{T}$ defined in the lemma is an isomorphism.

Finally we show that ϕ is an isomorphism, by proving that $\phi\otimes\mathrm{id}_\kappa$ is injective.

Consider the following commutative diagram :

$$\begin{array}{ccccc}
 & & \xrightarrow{\phi\otimes \mathrm{id}_\kappa} & & \\
\mathcal{M}\otimes\kappa & \longrightarrow & ((\mathcal{M}\otimes\kappa)^{\oplus 2})_{\mathfrak{m}'_{0,\mathfrak{q}}} & \longrightarrow & \mathcal{M}_{0,\mathfrak{q}}\otimes\kappa \\
\| & & \downarrow & & \downarrow \\
\mathrm{H}^d(Y,\mathbb{V}_n(\kappa))_{\mathfrak{m}'} & \overset{j}{\hookrightarrow} & \mathrm{H}^d(Y,\mathbb{V}_n(\kappa))^{\oplus 2}_{\mathfrak{m}'} & \xrightarrow{\xi} & \mathrm{H}^d(Y_{0,\mathfrak{q}},\mathbb{V}_n(\kappa))_{\mathfrak{m}'}
\end{array}$$

where $j(g)=(-g|\widetilde{\beta}_{\mathfrak{q}},g)$ and $\xi(g_1,g_2)=g_1+g_2|_{\mathfrak{q}}$.

It remains to show that ξ is injective. Denote by $\widehat{\xi}$ the dual of ξ with respect to the twisted Poincaré pairing (that is non-degenerate by Thm.1.4(i)).

The matrix of $\widehat{\xi}\circ\xi : (\mathcal{M}\otimes\kappa)^{\oplus 2}\to(\mathcal{M}\otimes\kappa)^{\oplus 2}$ is given by $\begin{pmatrix} 1+\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}) & S_{\mathfrak{q}}^{-1}T_{\mathfrak{q}} \\ T_{\mathfrak{q}} & 1+\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})\end{pmatrix}$. It is invertible by our assumptions on $\mathfrak{q}$. Hence ξ is injective. □

Let $\mathfrak{m}'_Q := (\varpi, T_v - c(f_0,v), U_{\mathfrak{q}}-\overline{\alpha}_{\mathfrak{q}}, I_Q; v\nmid \mathfrak{n}_0 Q, \mathfrak{q}\in Q)$ be a maximal ideal of the Hecke algebra $\mathbb{T}'_Q=\mathcal{O}[T_v,U_{\mathfrak{q}},\Delta_Q; v\nmid\mathfrak{n}_0 Q,\mathfrak{q}\in Q]\subset \mathrm{End}_{\mathcal{O}}(\mathrm{H}^d(Y_Q,\mathbb{V}_n(E))[\psi])$ and $\mathcal{T}_Q=(\mathbb{T}'_Q)_{\mathfrak{m}'_Q}$.

By Thm.1.4(i) the $\mathcal{T}_Q$-module $\mathcal{M}_Q:=\mathrm{H}^d(Y_Q,\mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'_Q}$ is free over $\mathcal{O}$.

Lemma 2.7. (i) *$\mathcal{T}_Q$ is a free $\mathcal{O}[\Delta_Q]$-algebra and $\mathcal{T}_Q/I_Q\mathcal{T}_Q\cong\mathcal{T}_{0,Q}$.*
(ii) *$\mathcal{M}_Q$ is a free $\mathcal{O}[\Delta_Q]$-module and $\mathcal{M}_Q/I_Q\mathcal{M}_Q\cong\mathcal{M}_{0,Q}$ as $\mathcal{T}_{0,Q}$-modules.*

Proof : (i) As $\mathcal{M}_{0,Q}$ and $\mathcal{M}_Q$ are free over $\mathcal{O}$, the injection $\mathcal{M}_{0,Q}\otimes\mathbb{C}\subset\mathcal{M}_Q\otimes\mathbb{C}$ gives a surjective homomorphism $\mathcal{T}_Q\to\mathcal{T}_{0,Q}$, sending I_Q on 0. Therefore we have a surjection $\mathcal{T}_Q/I_Q\mathcal{T}_Q\twoheadrightarrow\mathcal{T}_{0,Q}$. The injectivity will follow from (ii).

(ii) We may assume that $\mathcal{O}$ contains the values of the characters of Δ_Q. By Nakayama's lemma, it is enough to see that :
– $\dim(\mathcal{M}_Q\otimes\mathbb{C})=\#\Delta_Q\cdot\dim(\mathcal{M}_{0,Q}\otimes\mathbb{C})$, and
– $\mathcal{M}_Q\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O}\cong\mathcal{M}_Q/I_Q\mathcal{M}_Q$ is a free $\mathcal{O}$-module.

The first point follows from the fact that the covering $Y_Q\to Y_{0,Q}$ is étale of group Δ_Q (*cf* [5, §4.3] and [12, 5.4.10]).

For the second point, we have to show that the Pontryagin dual $\mathrm{H}^d(Y_Q,\mathbb{V}_n(E/\mathcal{O}))[\psi]^{\Delta_Q}_{\mathfrak{m}_Q}$ of $\mathcal{M}_Q/I_Q\mathcal{M}_Q$ is divisible. Consider the Hochschild-Serre spectral sequence:

$$\mathrm{E}_2^{i,j}=\mathrm{H}^i(\Delta_Q,\mathrm{H}^j(Y_Q,\mathbb{V}_n(E/\mathcal{O})))\Rightarrow\mathrm{H}^{i+j}(Y_{0,Q},\mathbb{V}_n(E/\mathcal{O})).$$

Its localization at $\mathfrak{m}'_{0,Q}$ is concentrated at the line $j=d$, so degenerates at E_2.
In particular

$$\mathrm{H}^d(Y_Q,\mathbb{V}_n(E/\mathcal{O}))^{\Delta_Q}_{\mathfrak{m}_Q}=\mathrm{H}^d(Y_{0,Q},\mathbb{V}_n(E/\mathcal{O}))_{\mathfrak{m}'_{0,Q}}=\mathrm{H}^d(Y,\mathbb{V}_n(E/\mathcal{O}))_{\mathfrak{m}'},$$

and last $\mathcal{O}$-module is divisible by Thm.1.4(ii). □

Remark 2.8. In general when $F\neq\mathbb{Q}$ there does not exist a twist of $\overline{\rho}$ of smallest conductor. Instead we use a trick due to Diamond. By the ($\mathbf{LI}_{\mathrm{Ind}\,\overline{\rho}}$) assumption, there exist a prime y of F not dividing 6, such that $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(y)\not\equiv\pm1 \pmod p$ and $c(f_0,y)^2\not\equiv\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(y)^{k_0-2}(\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(y)+1)^2 \pmod\varpi$. Adding level at such a prime does not change the local component at $\overline{\rho}$ of the cohomology of the Hilbert modular variety. Therefore we may keep the same notations. Nevertheless the advantage is twofold :

– first, it allows us to assume that K is torsion free,

– second and more important, it allows us to assume that $\overline{\rho}_v$ has minimal conductor among its twists at all primes $v \neq y$.

Proposition 2.9. *There is a canonical surjection $\mathcal{R}_Q \to \mathcal{T}_Q$ of $\mathcal{O}[\Delta_Q]$-algebras.*

Proof : For the existence of a $\mathcal{O}$-algebra homomorphism we have to show that if g is a Hilbert modular newform occurring in $\mathcal{T}_Q$, then $\rho_{g,p}$ is a Q-deformation of $\overline{\rho}$, in the sense of Def.1.3. This follows from the Remark 1.2(ii), since the Artin conductors of $\overline{\rho}$ and $\rho_{g,p}$ coincide (outside y and Q) and since the central character of g is equal to ψ.

The $\Delta_{\mathfrak{q}}$-linearity follows from Carayol's Theorem [3] on the compatibility between the Local and the Global Langlands Correspondences for GL_2. In fact, if g occurs in $\mathcal{T}_Q$ then $\overline{\rho}_{g,p}|_{I_{\mathfrak{q}}}$ is of the form $\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}$, where μ is a character factorizing through $I_{\mathfrak{q}} \twoheadrightarrow \Delta_{\mathfrak{q}}$. Therefore the geometric $\Delta_{\mathfrak{q}}$-action on $\mathcal{T}_Q$ corresponds to the inertia $\Delta_{\mathfrak{q}}$-action on $\mathcal{R}_Q$. □

So far we have constructed a Taylor-Wiles system for the family of sets Q containing a finite number of primes $\mathfrak{q}$ as in §2.2. The aim of the next paragraph is to find a subfamily $\mathcal{Q} = \{Q_m | m \in \mathbb{N}\}$ satisfying the assumptions (i) and (ii) of Thm.2.3.

2.4. **Selmer groups.** We will use Galois cohomology techniques in order to control the number of generators of $\mathcal{R}_Q$. For $r \geq 1$ we put $\rho_r := \rho_{f,p} \mod \varpi^r$. So we have $\rho_1 = \overline{\rho}$.

Definition 2.10. For $v\,|\,p$ the subgroup $\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0 \rho_r) \subset \mathrm{H}^1(F_v, \mathrm{Ad}^0 \rho_r)$ consists of classes corresponding to crystalline extensions of ρ_r by itself.

For $v \nmid p$ the subgroup of unramified classes $\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0 \rho_r) \subset \mathrm{H}^1(F_v, \mathrm{Ad}^0 \rho_r)$ is defined as $\mathrm{H}^1(\mathcal{G}_{F_v}/I_v, (\mathrm{Ad}^0 \rho_r)^{I_v})$.

Definition 2.11. The Selmer groups associated to a finite set of primes Σ are defined as

$$\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0 \rho_r) = \ker\left(\mathrm{H}^1(F, \mathrm{Ad}^0 \rho_r) \to \bigoplus_{v \notin \Sigma} \mathrm{H}^1(F_v, \mathrm{Ad}^0 \rho_r)/\,\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0 \rho_r)\right),$$

$$\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0 \rho_{f,p} \otimes \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim \mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0 \rho_r).$$

The dual of $\mathrm{Ad}^0 \overline{\rho}$ is canonically isomorphic to its Tate twist, denoted $\mathrm{Ad}^0 \overline{\rho}(1)$. The corresponding dual Selmer group $\mathrm{H}^1_{\Sigma^*}(F, \mathrm{Ad}^0 \overline{\rho}(1))$ is defined as the kernel of the map

$$\mathrm{H}^1(F, \mathrm{Ad}^0 \overline{\rho}(1)) \to \bigoplus_{v \in \Sigma} \mathrm{H}^1(F_v, \mathrm{Ad}^0 \overline{\rho}(1)) \bigoplus_{v \notin \Sigma} \mathrm{H}^1(F_v, \mathrm{Ad}^0 \overline{\rho}(1))/\,\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0 \overline{\rho}(1)).$$

By the ($\mathbf{LI}_{\mathrm{Ind}\,\overline{\rho}}$) assumption, we have $\mathrm{H}^0(F, \mathrm{Ad}^0\,\overline{\rho}) = \mathrm{H}^0(F, \mathrm{Ad}^0\,\overline{\rho}(1)) = 0$. Then, the Poitou-Tate exact sequence yields the following formula (*cf* [27, Prop.1.6]) :

$$\frac{\#\,\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0\,\overline{\rho})}{\#\,\mathrm{H}^1_{\Sigma^*}(F, \mathrm{Ad}^0\,\overline{\rho}(1))} = \prod_{v \in \Sigma \cup \{\infty\}} \frac{\#\,\mathrm{H}^1(F_v, \mathrm{Ad}^0\,\overline{\rho}_v)}{\#\,\mathrm{H}^0(F_v, \mathrm{Ad}^0\,\overline{\rho}_v)} \prod_{v \mid p} \frac{\#\,\mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\overline{\rho}_v)}{\#\,\mathrm{H}^0(F_v, \mathrm{Ad}^0\,\overline{\rho}_v)}.$$

Since $\overline{\rho}$ is totally odd, for all $v \mid \infty$ we have $\dim \mathrm{H}^0(F_v, \mathrm{Ad}^0\,\overline{\rho}_v) = 2$. Since $\overline{\rho}$ is crystalline at $v \mid p$ we have $\dim \mathrm{H}^1_{\mathrm{f}}(F_v, \mathrm{Ad}^0\,\overline{\rho}_v) - \dim \mathrm{H}^0(F_v, \mathrm{Ad}^0\,\overline{\rho}_v) \leq [F_v : \mathbb{Q}_p]$ (*cf* [12, Thm.2.7.14] or [8, Cor.2.3]). Finally, $\dim \mathrm{H}^0(F_{\mathfrak{q}}, \mathrm{Ad}^0\,\overline{\rho}_{\mathfrak{q}}(1)) = 1$ for all $\mathfrak{q} \in Q$. Putting all these together we get

$$\dim \mathrm{H}^1_Q(F, \mathrm{Ad}^0\,\overline{\rho}) \leq \mathrm{H}^1_{Q^*}(F, \mathrm{Ad}^0\,\overline{\rho}(1)) + \#Q. \tag{1}$$

Lemma 2.12. [27, §3] *Let $m \geq 1$ be an integer. Then for each non-zero element x in $\mathrm{H}^0_{\varnothing^*}(F, \mathrm{Ad}^0\,\overline{\rho}(1))$, there exists a prime $\mathfrak{q}$ such that :*
– $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}) \equiv 1 \pmod{p^m}$,
– *$\overline{\rho}$ is unramified at $\mathfrak{q}$ and $\overline{\rho}(\mathrm{Frob}_{\mathfrak{q}})$ has two distinct eigenvalues in κ, and*
– *the image by the restriction map of x in $\mathrm{H}^1_{\mathrm{f}}(F_{\mathfrak{q}}, \mathrm{Ad}^0\,\overline{\rho}(1))$ is non-trivial.*

Put $r := \dim \mathrm{H}^0_{\varnothing^*}(F, \mathrm{Ad}^0\,\overline{\rho}(1))$. For each $m \geq 1$, let Q_m be the set of primes $\mathfrak{q}$ corresponding by the above lemma to the elements of a basis of $\mathrm{H}^0_{\varnothing^*}(F, \mathrm{Ad}^0\,\overline{\rho}(1))$. Then $\mathrm{H}^0_{Q_m^*}(F, \mathrm{Ad}^0\,\overline{\rho}(1)) = 0$ and so by (1) we obtain $\dim \mathrm{H}^0_{Q_m}(F, \mathrm{Ad}^0\,\overline{\rho}) \leq \#Q_m$. Therefore $\mathcal{R}_m$ is generated by at most $\#Q_m = r$ elements. This completes the proof of Thm.2.1.

3. Raising the level.

3.1. Numerical invariants.

Definition 3.1. For a local complete noetherian $\mathcal{O}$-algebra A endowed with a surjective homomorphism $\theta_A : A \to \mathcal{O}$, we define the following two invariants :
– the congruence ideal $\eta_A := \theta_A(\mathrm{Ann}_A(\ker \theta_A)) \subset \mathcal{O}$, and
– $\Phi_A := \ker \theta_A / (\ker \theta_A)^2 = \Omega^1_{A/\mathcal{O}}$.

Here we state Wiles' numerical criterion :

Theorem 3.2. [5, Thm.3.40] *Let $\pi : \mathcal{R} \twoheadrightarrow \mathcal{T}$ be a surjective homomorphism such that $\theta_{\mathcal{R}} = \pi \circ \theta_{\mathcal{T}}$. Assume that $\mathcal{T}$ is finite and flat over $\mathcal{O}$ and $\eta_{\mathcal{T}} \neq (0)$. Then the following three conditions are equivalent :*
(i) $\#\Phi_{\mathcal{R}} \leq \#(\mathcal{O}/\eta_{\mathcal{T}})$,
(ii) $\#\Phi_{\mathcal{R}} = \#(\mathcal{O}/\eta_{\mathcal{T}})$, *and*
(iii) *$\mathcal{R}$ and $\mathcal{T}$ are complete intersections over $\mathcal{O}$ and π is an isomorphism.*

We consider couples $(\mathcal{T}, \mathcal{M})$ consisting of a finite and flat $\mathcal{O}$-algebra $\mathcal{T}$, and a $\mathcal{T}$-module $\mathcal{M}$ which is finitely generated and free over $\mathcal{O}$, endowed with a perfect $\mathcal{T}$-linear pairing $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{O}$, and such that $\mathcal{M} \otimes E$ is free over $\mathcal{T} \otimes E$. The pairing induces an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}, \mathcal{O})$ of $\mathcal{T}$-modules.

From [5, Lemma 4.17] and [7, Thm.2.4] we deduce the following :

Proposition 3.3. *Let $(\mathcal{T},\mathcal{M})$ and $(\mathcal{T}',\mathcal{M}')$ be two couples as above. Assume that we have a surjective homomorphism $\mathcal{T}'\twoheadrightarrow\mathcal{T}$ and a $\mathcal{T}'$-linear injective homomorphism $\xi:\mathcal{M}\hookrightarrow\mathcal{M}'$ inducing via $\langle\cdot,\cdot\rangle$ a surjective homomorphism $\widehat{\xi}:\mathcal{M}'\twoheadrightarrow\mathcal{M}$.*

If $\mathcal{M}$ is free over $\mathcal{T}$ and if $\widehat{\xi}\circ\xi(\mathcal{M})=\Delta\cdot\mathcal{M}$ for some $\Delta\in\mathcal{T}$, then

$$\#(\mathcal{O}/\eta_{\mathcal{T}})\#(\mathcal{O}/\theta_{\mathcal{T}}(\Delta))\leq\#(\mathcal{O}/\eta_{\mathcal{T}'}).$$

Assume moreover that $\mathcal{T}'$ is Gorenstein. Then equality holds, if and only if $\mathcal{M}'$ is free over $\mathcal{T}'$.

3.2. **Ihara's lemma for Hilbert modular varieties.** Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$ divisible by the conductor of $\overline{\rho}$ and let $\mathfrak{q}$ be a prime ideal.

Let $\mathbb{T}'\subset\mathrm{End}_{\mathcal{O}}(\mathrm{H}^d(Y_1(\mathfrak{n}),\mathbb{V}_n(\mathcal{O})))$ be the subalgebra generated by the Hecke operators outside a finite set of primes containing those dividing $\mathfrak{q}\,\mathfrak{n}\,p$. Denote by $\mathfrak{m}'$ the maximal ideal of $\mathbb{T}'$ associated to $\overline{\rho}$. We consider the degeneracy maps :

$$\mathrm{pr}_1,\mathrm{pr}_2:Y_1(\mathfrak{q}\,\mathfrak{n})\rightarrow Y_1(\mathfrak{n})\ \text{and}\ \ \mathrm{pr}_3,\mathrm{pr}_4:Y_1(\mathfrak{n})\rightarrow Y_1(\mathfrak{q}^{-1}\,\mathfrak{n}).$$

Proposition 3.4. *Assume that p does not divide $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n}\,\mathfrak{d}\,\mathfrak{q})$, $p-1>\sum_{\tau\in J_F}(k_\tau-1)$ and that $(\mathbf{LI}_{\mathrm{Ind}\,\overline{\rho}})$ holds.*

(i) *If $\mathfrak{q}$ does not divide $\mathfrak{n}$, then the homomorphism*

$$\mathrm{pr}_1^*+\mathrm{pr}_2^*:\mathrm{H}^d(Y_1(\mathfrak{n}),\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}^{\oplus 2}\rightarrow\mathrm{H}^d(Y_1(\mathfrak{n}\,\mathfrak{q}),\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}$$

is injective with flat cokernel.

(ii) *If $\mathfrak{q}$ divides $\mathfrak{n}$, then we have an exact sequence*

$$0\rightarrow\mathrm{H}^d(Y_1(\mathfrak{q}^{-1}\,\mathfrak{n}),\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}\xrightarrow{(\mathrm{pr}_3^*,-\mathrm{pr}_4^*)}\mathrm{H}^d(Y_1(\mathfrak{n}),\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}^{\oplus 2}\xrightarrow{\mathrm{pr}_1^*+\mathrm{pr}_2^*}\mathrm{H}^d(Y_1(\mathfrak{n}\,\mathfrak{q}),\mathbb{V}_n(\mathcal{O}))_{\mathfrak{m}'}$$

whose last arrow has a flat cokernel.

Proof : Put $W(\mathfrak{a})=\mathrm{H}^d(Y(\mathfrak{a}),\mathbb{V}_n(\kappa))_{\mathfrak{m}'}$, for $\mathfrak{a}=\mathfrak{q}^{-1}\,\mathfrak{n}$, $\mathfrak{n}$ or $\mathfrak{q}\,\mathfrak{n}$ (and $W(\mathfrak{q}^{-1}\,\mathfrak{n})=0$ if $\mathfrak{q}$ does not divide $\mathfrak{n}$). Since $\mathcal{T}/\varpi\mathcal{T}$ is Artinian, it is sufficient to prove that the exactness of the sequence :

$$0\rightarrow W(\mathfrak{q}^{-1}\,\mathfrak{n})[\mathfrak{m}']\rightarrow W(\mathfrak{n})[\mathfrak{m}']^{\oplus 2}\rightarrow W(\mathfrak{q}\,\mathfrak{n})[\mathfrak{m}'].$$

The sequence is $\mathcal{G}_{\mathbb{Q}}$-equivariant. Moreover, by lemma 1.5 every $\mathcal{G}_{\widetilde{F}}$-irreducible subquotient of the modules in this sequence is isomorphic to $\otimes\,\mathrm{Ind}_F^{\mathbb{Q}}\,\overline{\rho}$. Thus we can check the exactness by checking it on the last graded piece of the Fontaine-Laffaille filtration. By [10, Thm.5.13] this is equivalent to the following lemma :

Lemma 3.5. (i) *If $\mathfrak{q}$ does not divide $\mathfrak{n}$, then the homomorphism*

$$\mathrm{H}^0(Y(\mathfrak{n})_{/\kappa},\underline{\omega}^k)^{\oplus 2}\rightarrow\mathrm{H}^0(Y(\mathfrak{n}\,\mathfrak{q})_{/\kappa},\underline{\omega}^k)\ ,\ (g_1,g_2)\mapsto g_1+g_2|_{\mathfrak{q}}$$

is injective.

(ii) *If $\mathfrak{q}$ divides $\mathfrak{n}$, then the kernel of the homomorphism*

$$\mathrm{H}^0(Y(\mathfrak{n})_{/\kappa},\underline{\omega}^k)^{\oplus 2}\rightarrow\mathrm{H}^0(Y(\mathfrak{n}\,\mathfrak{q})_{/\kappa},\underline{\omega}^k)\ ,\ (g_1,g_2)\mapsto g_1+g_2|_{\mathfrak{q}}$$

is isomorphic to $\{(-g|_{\mathfrak{q}},g)\ \mid\ g\in\mathrm{H}^0(Y(\mathfrak{q}^{-1}\,\mathfrak{n})_{/\kappa},\underline{\omega}^k\}$.

Proof : (i) Since $\mathfrak{q}$ does not divide $\mathfrak{n}$, the homomorphism is $U_{\mathfrak{q}}$-equivariant, for the $U_{\mathfrak{q}}$-action on the left hand side given by the matrix $\begin{pmatrix} T_{\mathfrak{q}} & \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \\ -S_{\mathfrak{q}} & 0 \end{pmatrix}$. Let (g_1, g_2) be an element of the kernel : $g_1 = -g_2|_{\mathfrak{q}}$. We may assume that (g_1, g_2) is an eigenvector for $U_{\mathfrak{q}}$. Then g_1 is a multiple of g_2, hence $g_2|_{\mathfrak{q}}$ is a multiple of g_2, and therefore $g_2 = 0$, by the q-Expansion Principle.

(ii) In this case the homomorphism is $U_{\mathfrak{q}}$-equivariant, for the $U_{\mathfrak{q}}$-action on the left hand side given by $\begin{pmatrix} U_{\mathfrak{q}} & \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) \\ 0 & 0 \end{pmatrix}$. Assume that $g_1 = -g_2|_{\mathfrak{q}}$. Then we have $g_1|U_{\mathfrak{q}} = -\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q}) g_2$ and therefore $g_2 \in \mathrm{H}^0(Y(\mathfrak{q}^{-1}\,\mathfrak{n})_{/\kappa}, \underline{\omega}^k\}$. □

3.3. **The main theorem.** We start by giving a geometric definition to $\mathcal{T}_\Sigma$: let $\mathfrak{n}_0$ be the Artin conductor of $\overline{\rho}$; denote by Y_Σ be the Hilbert modular variety of level $\mathfrak{n}_\Sigma = \mathfrak{n}_0 \prod_v v^{d_v}$, where $d_v = \dim \overline{\rho}^{I_v}$; let $\mathbb{T}'_\Sigma$ be the $\mathcal{O}$-algebra generated by the Hecke operators T_v, $v \notin \Sigma \cup \{\mathfrak{n}_0\, p\}$ and $U_{\mathfrak{q}}$, $\mathfrak{q} \in \Sigma$ acting on $\mathrm{H}^d(Y_\Sigma, \mathbb{V}_n(\mathcal{O}))[\psi]$; let $\mathfrak{m}'_\Sigma$ be the maximal ideal of $\mathbb{T}'_\Sigma$ associated to $\overline{\rho}$ and containing the $U_{\mathfrak{q}}$ for all $\mathfrak{q} \in \Sigma$; then $\mathcal{T}_\Sigma = (\mathbb{T}'_\Sigma)_{\mathfrak{m}'_\Sigma}$.

By Thm.1.4 the $\mathcal{T}_\Sigma$-module $\mathcal{M}_\Sigma = \mathrm{H}^d(Y_\Sigma, \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'_\Sigma}$ is free of finite rank over $\mathcal{O}$, endowed with a perfect $\mathcal{T}$-linear pairing $\langle\cdot,\cdot\rangle : \mathcal{M}_\Sigma \times \mathcal{M}_\Sigma \to \mathcal{O}$, and such that $\mathcal{M}_\Sigma \otimes E$ is free over $\mathcal{T}_\Sigma \otimes E$.

For every Hilbert modular newform f such that $\rho_{f,p}$ is a Σ-ramified deformations of $\overline{\rho}$, there exists a surjection $\theta_f^\Sigma : \mathcal{T}_\Sigma \to \mathcal{O}$ (the projection on the f-component). The corresponding congruence ideal is denoted by η_f^Σ.

By §1.2 we have a surjection $\pi_\Sigma : \mathcal{R}_\Sigma \to \mathcal{T}_\Sigma$. Therefore we may endow $\mathcal{R}_\Sigma$ with a surjective homomorphism $\theta_f^\Sigma \circ \pi_\Sigma : \mathcal{R}_\Sigma \to \mathcal{O}$ and we denote Φ_f^Σ the corresponding numerical invariant. Then by [27, Prop.1.2] we have

(2) $$\mathrm{Hom}_{\mathcal{O}}(\Phi_f^\Sigma, E/\mathcal{O}) = \mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Theorem 3.6. (Theorem A) *Let $\overline{\rho} : \mathcal{G}_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation satisfying ($\mathbf{LI}_{\mathrm{Ind}\overline{\rho}}$) and ($\mathbf{Mod}_{\overline{\rho}}$). Let Σ be a finite set of primes containing P. Then $\pi_\Sigma : \mathcal{R}_\Sigma \to \mathcal{T}_\Sigma$ is an isomorphism of complete intersections over $\mathcal{O}$ and $\mathcal{M}_\Sigma$ is free of finite rank over $\mathcal{T}_\Sigma$. In particular, all Σ-ramified deformations of $\overline{\rho}$ are modular.*

Moreover, if f is a Hilbert modular newform such that $\rho_{f,p}$ is a Σ-ramified deformations of $\overline{\rho}$, then we have :

$$\#\,\mathrm{H}^1_\Sigma(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = \#(\mathcal{O}/\eta_f^\Sigma) < \infty.$$

Proof : We proceed by induction $\#\Sigma$. Assume first that $\Sigma = P$. We already know that $\pi_P : \mathcal{R}_P \to \mathcal{T}_P$ is an isomorphism of complete intersections over $\mathcal{O}$ (*cf* Thm.2.1 if $P = \varnothing$ and Fujiwara [12] in general). Next we show that $\mathcal{M}_P := \mathcal{M}$ is free over $\mathcal{T}_P$. By Thm.1.4(iii), we have to show that the natural surjection $\mathcal{T}_P \to \mathbb{T}_{\mathfrak{m}}$ is an isomorphism. In other terms, we have to show that if g is a Hilbert modular form occurring in $\mathcal{T}_P$, then $c(f_0, v) \equiv c(g, v) \pmod{\varpi}$ for *all* primes v. It is clear for v not dividing $\mathfrak{n}_0$. For $v \mid \mathfrak{n}_0$, it follows from the local Langlands correspondence: in cases **V** and **H**, $c(f_0, v) = c(g, v) = 0$; in cases **P** and **S**, $c(f_0, v)$ and $c(g, v)$ are congruent to the eigenvalue of Frob_v acting on the one dimensional space $\overline{\rho}^{I_v}$ (here we use that $\overline{\rho}|_{I_v}$ has minimal conductor among its twists for $v \neq y$). Therefore the theorem holds for $\Sigma = P$.

Assume now that the theorem holds for some $\Sigma \supset P$, that is to say $\pi_\Sigma : \mathcal{R}_\Sigma \to \mathcal{T}_\Sigma$ is an isomorphism of complete intersections over $\mathcal{O}$ and that $\mathcal{M}_\Sigma$ is free over $\mathcal{T}_\Sigma$. In particular, we have $\#\Phi_{f_0}^\Sigma = \#(\mathcal{O}/\eta_{f_0}^\Sigma)$.

Let $\mathfrak{q}$ be a prime outside Σ not dividing p. Put $\Sigma' = \Sigma \cup \{\mathfrak{q}\}$.

It follows immediately from (2) and Def.2.11 that :

$$\#\Phi_{f_0}^{\Sigma'} \leq \#\Phi_{f_0}^{\Sigma} \cdot \#\,\mathrm{H}^0(F_\mathfrak{q}, (\mathrm{Ad}^0(\rho_{f_0,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)(1)).$$

By Thm.3.2 and Prop.3.3, the theorem will hold for Σ', if we construct a surjective homomorphism $\mathcal{T}_{\Sigma'} \twoheadrightarrow \mathcal{T}_\Sigma$ and a $\mathcal{T}_{\Sigma'}$-linear injective homomorphism $\xi : \mathcal{M}_\Sigma \hookrightarrow \mathcal{M}_{\Sigma'}$ with free cokernel over $\mathcal{O}$, such that $\widehat{\xi} \circ \xi(\mathcal{M}_\Sigma) = \Delta \cdot \mathcal{M}_\Sigma$ for some $\Delta \in \mathcal{T}_\Sigma$ and such that

$$\#(\mathcal{O}/\theta_{f_0}^\Sigma(\Delta)) = \#\,\mathrm{H}^0(F_\mathfrak{q}, (\mathrm{Ad}^0(\rho_{f_0,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)(1)). \tag{3}$$

This is done case by case, depending on the local behavior of $\overline{\rho}$ at $\mathfrak{q}$ (*cf* §1.1). The cases **V** and **H** are clear, because $\mathfrak{q} \notin P$. So we will distiguish two cases :

1) Assume that $\overline{\rho}_\mathfrak{q}$ is unramified.

By Prop.3.4(i) applied to $\mathfrak{n} = \mathfrak{n}_\Sigma$ and Prop.3.4(ii) applied to $\mathfrak{n} = \mathfrak{q}\,\mathfrak{n}_\Sigma$, we have that

$$\mathrm{pr}_1^*\,\mathrm{pr}_1^* + \mathrm{pr}_1^*\,\mathrm{pr}_2^* + \mathrm{pr}_2^*\,\mathrm{pr}_2^* : \mathcal{M}_\Sigma^{\oplus 3} = \mathrm{H}^d(Y_\Sigma, \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'_\Sigma}^{\oplus 3} \to \mathrm{H}^d(Y_{\Sigma'}, \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'_\Sigma}$$

is injective with flat cokernel (the $\mathfrak{m}'_\Sigma$-localization is a direct factor of the $\mathfrak{m}'$-localization).

The characteristic polynomial of $U_\mathfrak{q}$ acting on $\mathcal{M}_\Sigma^{\oplus 3}$ is $X(X^2 - T_\mathfrak{q} X + S_\mathfrak{q}\,\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}))$ and $X = 0$ is simple root modulo ϖ of this polynomial. Hence the localization of the above injection at $U_\mathfrak{q}$ yields another injection with flat cokernel :

$$\xi : \mathcal{M}_\Sigma \xrightarrow{\sim} (\mathcal{M}_\Sigma^{\oplus 3})_{U_\mathfrak{q}} \hookrightarrow \mathcal{M}_{\Sigma'}\,.$$

This gives a surjective homomorphism $\mathcal{T}_{\Sigma'} \twoheadrightarrow (\mathcal{T}_\Sigma^3)_{U_\mathfrak{q}} \cong \mathcal{T}_\Sigma$. Computations performed by Wiles [27, §2] and Fujiwara [12] show that $\widehat{\xi} \circ \xi(\mathcal{M}_\Sigma) = \Delta \cdot \mathcal{M}_\Sigma$ with

$$\Delta = (\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}) - 1)(T_\mathfrak{q}^2 - S_\mathfrak{q}(\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}) + 1)^2).$$

Then (3) follows by a straightforward computation.

2) Assume that $\overline{\rho}_\mathfrak{q}$ is ramified of type **P** or **S**.

By Prop.3.4(ii) applied to $\mathfrak{n} = \mathfrak{n}_\Sigma$ we have an exact sequence

$$0 \to \mathrm{H}^d(Y_1(\mathfrak{q}^{-1}\,\mathfrak{n}_\Sigma), \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'} \xrightarrow{(\mathrm{pr}_3^*, -\mathrm{pr}_4^*)} \mathrm{H}^d(Y_\Sigma, \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'}^{\oplus 2} \xrightarrow{\mathrm{pr}_1^* + \mathrm{pr}_2^*} \mathrm{H}^d(Y_{\Sigma'}, \mathbb{V}_n(\mathcal{O}))[\psi]_{\mathfrak{m}'}$$

whose last arrow has a flat cokernel.

The characteristic polynomial of $U_\mathfrak{q}$ acting on $(\mathrm{pr}_1^* + \mathrm{pr}_2^*)(\mathcal{M}_\Sigma^{\oplus 2})$ is $X(X - U_\mathfrak{q})$ and $X = 0$ is simple root modulo ϖ of this polynomial. Hence the localization of the map $\mathrm{pr}_1^* + \mathrm{pr}_2^*$ at $\mathfrak{m}_{\Sigma'} = (\mathfrak{m}_\Sigma, U_\mathfrak{q})$ yields an injection with flat cokernel :

$$\xi : \mathcal{M}_\Sigma \xrightarrow{\sim} (\mathcal{M}_\Sigma^{\oplus 2})_{U_\mathfrak{q}} \hookrightarrow \mathcal{M}_{\Sigma'}\,.$$

This gives a surjective homomorphism $\mathcal{T}_{\Sigma'} \twoheadrightarrow (\mathcal{T}^2_{\Sigma})_{U_{\mathfrak{q}}} \cong \mathcal{T}_{\Sigma}$. Computations performed by Wiles [27, §2] and Fujiwara [12] show that $\widehat{\xi} \circ \xi(\mathcal{M}_{\Sigma}) = \Delta \cdot \mathcal{M}_{\Sigma}$ with

$$\Delta = \begin{cases} \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})^2 - 1 \text{ if } \overline{\rho}_{\mathfrak{q}} \text{ is of type } \mathbf{P}, \text{ and} \\ \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q}) - 1 \text{ if } \overline{\rho}_{\mathfrak{q}} \text{ is of type } \mathbf{S}. \end{cases}$$

As above, (3) is obtained by a straightforward computation. □

4. Applications.

4.1. **Cardinality of the adjoint Selmer Group.** We define the imprimitive adjoint L-function $\mathrm{L}^*(\mathrm{Ad}^0(\rho_{f,p}), s)$ by removing from the standard joint L-function $\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}), s)$ the local factors at the primes v such that $\rho_{f,p}^{I_v} = \{0\}$.

The corresponding Euler factor is defined as :

$$\Gamma(\mathrm{Ad}^0(\rho_{f,p}), s) = \prod_{\tau \in J_F} \pi^{-(s+1)/2}\Gamma((s+1)/2)(2\pi)^{-(s+k_\tau-1)}\Gamma(s+k_\tau-1).$$

Denote by η_f the congruence ideal associated to the $\mathcal{O}$-algebra homomorphism $\mathbb{T}_{\mathfrak{m}} \to \mathcal{O}$, $T_{\mathfrak{a}} \mapsto c(f, \mathfrak{a})$ (*cf* Def.3.1). Using a formula of Shimura, Thm.1.4(iii) yields (*cf* [10, §4.4]) :

$$\left(\iota_p\left(\frac{\Gamma(\mathrm{Ad}^0(\rho_{f,p}),1)\,\mathrm{L}^*(\mathrm{Ad}^0(\rho_{f,p}),1)}{\Omega_f^+\Omega_f^-}\right)\right)_{\mathcal{O}} = \eta_f. \tag{4}$$

As a corollary, for a prime p as in Thm.1.4, $\iota_p\left(\frac{\Gamma(\mathrm{Ad}^0(\rho_{f,p}),1)\,\mathrm{L}^*(\mathrm{Ad}^0(\rho_{f,p}),1)}{\Omega_f^+\Omega_f^-}\right)$ has positive valuation, if and only if, there exists another normalized parabolic eigenform g of same weight, level and central character as f and such that $f \equiv g \pmod{\varpi}$, in the sense that $c(f, \mathfrak{a}) \equiv c(g, \mathfrak{a}) \pmod{\varpi}$ for each ideal $\mathfrak{a} \subset \mathfrak{o}$.

The following theorem is a first step towards the generalization to an arbitrary totally real field of the work [8] of Diamond, Flach and Guo on the Tamagawa number conjecture for $\mathrm{Ad}^0(\rho_{f,p})$.

Theorem 4.1. (Theorem B) *Let f be a Hilbert modular newform over F, of level $\mathfrak{n}$ and weight k, and p be a prime not dividing $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n}\,\mathfrak{d})$ and such that $p - 1 > \sum_{\tau \in J_F}(k_\tau - 1)$. Assume that $\overline{\rho} = \overline{\rho}_{f,p}$ satisfies $(\mathbf{LI}_{\mathrm{Ind}\overline{\rho}})$. Then, keeping the notations of Thm.B, we have :*

$$\left(\iota_p\left(\frac{\Gamma(\mathrm{Ad}^0(\rho_{f,p}),1)\,\mathrm{L}(\mathrm{Ad}^0(\rho_{f,p}),1)}{\Omega_f^+\Omega_f^-}\right)\right)_{\mathcal{O}} = \mathrm{Fitt}_{\mathcal{O}}\left(\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p / \mathbb{Z}_p)\right).$$

Proof : Let Σ be the smallest set of primes containing P such that $\rho_{f,p}$ is a Σ-ramified deformation of $\overline{\rho}_{f,p}$ (*cf* Def.1.3). By Thm.3.6 we have

$$\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p / \mathbb{Z}_p) = \mathcal{O} / \eta_f^{\Sigma}.$$

By [8, §2.1] :

$$\#\,\mathrm{H}^1_{\mathrm{f}}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p / \mathbb{Z}_p) = \#\,\mathrm{H}^1_{\Sigma}(F, \mathrm{Ad}^0(\rho_{f,p}) \otimes \mathbb{Q}_p / \mathbb{Z}_p) \prod_{v \in \Sigma} \mathrm{L}_v(\mathrm{Ad}^0(\rho_{f,p}), 1).$$

The primes diving $\mathfrak{n}_\Sigma\,\mathfrak{n}^{-1}$ are exactly the primes in Σ whose corresponding local factor in $\mathrm{L}^*(\mathrm{Ad}^0(\rho_{f,p}),1)$ is non trivial. By Prop.3.3 and the last part of the proof of Thm.3.6 applied to f we obtain :

$$\eta_f = \eta_f^P = \eta_f^\Sigma \prod_{v\in\Sigma\backslash P} \mathrm{L}_v(\mathrm{Ad}^0(\rho_{f,p}),1) = \eta_f^\Sigma \prod_{v\in\Sigma} \mathrm{L}_v^*(\mathrm{Ad}^0(\rho_{f,p}),1).$$

Now the theorem follows from (4). □

4.2. **Towards the modularity of a quintic threefold.** Consani and Scholten [4] consider the middle degree cohomology of a quintic threefold $\widetilde{X}$ (a proper and smooth $\mathbb{Z}[\frac{1}{30}]$-scheme with Hodge numbers $h^{3,0}=h^{2,1}=1$, $h^{2,0}=h^{1,0}=0$ and $h^{1,1}=141$). They show that the $\mathcal{G}_\mathbb{Q}$-representation $\mathrm{H}^3(\widetilde{X}_{\overline{\mathbb{Q}}},\mathbb{Q}_p)$ is induced from a two dimensional representation ρ of $\mathcal{G}_{\mathbb{Q}(\sqrt{5})}$, and conjecture the modularity of ρ. As explained in [9], Thm.3.6 implies the following proposition

Proposition 4.2. (Dieulefait-D.) *Assume $p\geq 7$ and that the reduction modulo p of ρ comes from a Hilbert modular form on $\mathbb{Q}(\sqrt{5})$ of weight $(2,4)$ and some prime to p level. Then ρ is modular, and in particular the L-function associated to $\mathrm{H}^3(\widetilde{X}_{\overline{\mathbb{Q}}},\mathbb{Q}_p)$ has an analytic continuation to the whole complex plane and a functional equation.*

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