The Hecke algebra of a reductive p-adic group: a view from noncommutative geometry

A.-M. Aubert P. Baum R.J. Plymen

Abstract

Let $\mathcal{H}(G)$ be the Hecke algebra of a reductive p-adic group G. We formulate a conjecture for the ideals in the Bernstein decomposition of $\mathcal{H}(G)$. The conjecture says that each ideal is *geometrically equivalent* to an algebraic variety. Our conjecture is closely related to Lusztig's conjecture on the asymptotic Hecke algebra. We prove part (1) of our conjecture for the group GL(n), and for the Iwahori ideals of the groups PGL(n) and SL(2). We also give some relevant calculations for the Iwahori ideal of the group SO(5).

1 Introduction

The reciprocity laws in number theory have a long development, starting from conjectures of Euler, and including contributions of Legendre, Gauss, Dirichlet, Jacobi, Eisenstein, Takagi and Artin. For the details of this development, see [32]. The local reciprocity law for a local field F, which concerns the finite Galois extensions E/F such that Gal(E/F) is *commutative*, is stated and proved in [45, p. 320]. This local reciprocity law was dramatically generalized by Langlands, see [6]. The local Langlands correspondence for GL(n) is a noncommutative generalization of the reciprocity law of local class field theory [49]. The local Langlands conjectures, and the global Langlands conjectures, all involve, inter alia, the representations of reductive p-adic groups [6].

To each reductive p-adic group G there is associated the Hecke algebra $\mathcal{H}(G)$, which we now define. Let K be a compact open subgroup of G, and define $\mathcal{H}(G//K)$ as the convolution algebra of all complex-valued, compactly-supported functions on G such that $f(k_1xk_2) = f(x)$ for all $k_1, k_2 \in K$. The Hecke algebra $\mathcal{H}(G)$ is then defined as

$$\mathcal{H}(G) := \bigcup_K \mathcal{H}(G//K).$$

The smooth representations of G on a complex vector space V correspond bijectively to the nondegenerate representations of $\mathcal{H}(G)$ on V, see [5, p.2].

In this article, we consider $\mathcal{H}(G)$ from the point of view of noncommutative geometry.

We recall that the coordinate rings of affine algebraic varieties are precisely the commutative, unital, finitely generated, reduced $\mathbb{C}$-algebras, see [19, II.1.1].

The Hecke algebra $\mathcal{H}(G)$ is a non-commutative, non-unital, non-finitely-generated, non-reduced $\mathbb{C}$-algebra, and so cannot be the coordinate ring of an affine algebraic variety.

The Hecke algebra $\mathcal{H}(G)$ is non-unital, but it admits *local units*, see [5, p.2].

The algebra $\mathcal{H}(G)$ admits a canonical decomposition into ideals, the Bernstein decomposition:

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}^{\mathfrak{s}}(G).$$

Each ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is a non-commutative, non-unital, non-finitely-generated, non-reduced $\mathbb{C}$-algebra, and so cannot be the coordinate ring of an affine algebraic variety.

In section 2, we define the *extended centre* $\widetilde{\mathfrak{Z}}(G)$ of G. At a crucial point in the construction of the centre $\mathfrak{Z}(G)$ of the category of smooth representations of G, certain quotients are made: we replace each ordinary quotient by the *extended quotient* to create the *extended centre.*

In section 3 we prove that the ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is Morita equivalent to a unital k-algebra of finite type, where k is the coordinate ring of a complex affine algebraic variety. We think of the ideal $\mathcal{H}^{\mathfrak{s}}(G)$ as a noncommutative algebraic variety, and $\mathcal{H}(G)$ as a noncommutative scheme.

In section 4 we formulate a more precise conjecture. We conjecture that each ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is *geometrically* equivalent (in a sense which we make precise) to the coordinate ring of a complex affine algebraic variety $X^{\mathfrak{s}}$:

$$\mathcal{H}^{\mathfrak{s}}(G) \asymp \mathcal{O}(X^{\mathfrak{s}}) = \widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G).$$

The ring $\widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G)$ is the $\mathfrak{s}$-factor in the *extended centre* of G. The ideals $\mathcal{H}^{\mathfrak{s}}(G)$ therefore qualify as noncommutative algebraic varieties.

We have stripped away the homology and cohomology which play such a dominant role in [3], [11], leaving behind three crucial moves: *Morita equivalence, morphisms which are spectrum-preserving with respect to filtrations, and deformation of central character.*

In section 5 we discuss the first step in the proof of the conjecture.

In section 6 we review the asymptotic Hecke algebra of Lusztig.

The asymptotic Hecke algebra J plays a vital role in our conjecture, as we now proceed to explain. One of the Bernstein ideals in $\mathcal{H}(G)$ corresponds to the point $\mathfrak{i} \in \mathfrak{B}(G)$, where $\mathfrak{i}$ is the quotient variety $\Psi(T)/W_f$. Here, T is a maximal torus in G, $\Psi(T)$ is the complex torus of unramified quasicharacters of T, and W_f is the finite Weyl group of G. Let I denote an Iwahori subgroup of G, and define e as follows:

$$e(x) = \begin{cases} \mathrm{vol}(I)^{-1} & \text{if } x \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *Iwahori ideal* is the two-sided ideal generated by e:

$$\mathcal{H}^{\mathfrak{i}}(G) := \mathcal{H}(G)e\mathcal{H}(G).$$

There is a Morita equivalence $\mathcal{H}(G)e\mathcal{H}(G) \sim e\mathcal{H}(G)e$ and in fact we have

$$\mathcal{H}^{\mathfrak{i}}(G) := \mathcal{H}(G)e\mathcal{H}(G) \asymp e\mathcal{H}(G)e \cong \mathcal{H}(G//I) \cong \mathcal{H}(W, q_F) \asymp J$$

where $\mathcal{H}(W, q_F)$ is an (extended) affine Hecke algebra based on the (extended) Coxeter group W, and J is the asymptotic Hecke algebra (*c.f.,* section 6). Now the J admits a decomposition into finitely many two-sided ideals

$$J = \oplus J_{\mathbf{c}}$$

labelled by the two-sided cells $\mathbf{c}$ in W. We therefore have

$$\mathcal{H}^{\mathfrak{i}}(G) \asymp \oplus J_{\mathbf{c}}.$$

This canonical decomposition of J is well-adapted to our conjecture.

In section 7 we prove that

$$J_{\mathbf{c}_0} \asymp \mathfrak{Z}^{\mathfrak{i}}(G)$$

where $\mathbf{c}_0$ is the lowest two-sided cell, for any connected F-split adjoint simple p-adic group G. We note that $\mathfrak{Z}^{\mathfrak{i}}(G)$ is the ring of regular functions on the *ordinary quotient* $\Psi(T)/W_f$.

In section 8 we prove the conjecture for $\mathrm{GL}(n)$. We establish that for each point $\mathfrak{s} \in \mathfrak{B}(G)$, we have

$$\mathcal{H}^{\mathfrak{s}}(\mathrm{GL}(n)) \asymp \widetilde{\mathfrak{Z}}^{\mathfrak{s}}(\mathrm{GL}(n)).$$

In section 9 we prove part (1) of the conjecture for the Iwahori ideal in the Hecke algebra of SL(2).

In section 10 we prove part (1) of the conjecture for the Iwahori ideal in $\mathcal{H}(\mathrm{PGL}(n))$.

We conclude, in section 11, with some relevant calculations for the Iwahori ideal in $\mathcal{H}(\mathrm{SO}(5))$. Our proofs depend crucially on Xi's affirmation, in certain special cases, of Lusztig's conjecture on the asymptotic Hecke algebra J (see [40, §10]).

We would like to thank the referee for his/her many detailed and constructive comments, which forced us thoroughly to revise the article.

2 The extended centre

Let G be the set of rational points of a reductive group defined over a local nonarchimedean field F, and let $\mathfrak{R}(G)$ denote the category of smooth G-modules. Let (L, σ) denote a cuspidal pair. The group $\Psi(L)$ of unramified quasicharacters of L has the structure of a complex torus.

We write $[L, \sigma]_G$ for the equivalence class of (L, σ) and $\mathfrak{B}(G)$ for the set of equivalence classes, where the equivalence relation is defined by $(L, \sigma) \sim (L', \sigma')$ if $gLg^{-1} = L'$ and ${}^g\sigma \simeq \nu'\sigma'$, for some $g \in G$ and some $\nu' \in \Psi(L')$. For $\mathfrak{s} = [L, \sigma]_G$, let $\mathfrak{R}^{\mathfrak{s}}(G)$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations Π such that each irreducible subquotient of Π is a subquotient of a parabolically induced representation $\iota_P^G(\nu\sigma)$ where P is a parabolic subgroup of G with Levi subgroup L and $\nu \in \Psi(L)$. The action (by conjugation) of $\mathrm{N}_G(L)$ on L induces an action of $W(L) = \mathrm{N}_G(L)/L$ on $\mathfrak{B}(L)$. Let $W_{\mathfrak{t}}$ denote the stabilizer of $\mathfrak{t} = [L, \sigma]_L$ in $W(L)$. Thus $W_{\mathfrak{t}} = N_{\mathfrak{t}}/L$ where

$$N_{\mathfrak{t}} = \{n \in \mathrm{N}_G(L) \,:\, {}^n\sigma \simeq \nu\sigma, \text{ for some } \nu \in \Psi(L)\}\,.$$

It acts (via conjugation) on $\mathrm{Irr}^{\mathfrak{t}} L$, the set of isomorphism classes of irreducible objects in $\mathfrak{R}^{\mathfrak{t}}(L)$.

Let $\Omega(G)$ denote the set of G-conjugacy classes of pairs (L, σ) where L is a Levi subgroup and σ an irreducible supercuspidal representation of L. The group $\Psi(L)$ creates orbits in $\Omega(G)$. Each orbit is of the form $D_\sigma/W_{\mathfrak{t}}$ where $D_\sigma = \mathrm{Irr}^{\mathfrak{t}} L$ is a complex torus.

We have

$$\Omega(G) = \bigsqcup D_\sigma/W_{\mathfrak{t}}.$$

Let $\mathfrak{Z}(G)$ denote the *centre* of the category $\mathfrak{R}(G)$. The centre of an abelian category (with a small skeleton) is the endomorphism ring of the identity functor. The centre assigns to each object A in $\mathfrak{R}(G)$ a morphism $z(A)$ such that

$$f \cdot z(A) = z(B) \cdot f$$

for each morphism $f \in \mathrm{Hom}(A, B)$.

According to Bernstein's theorem [48, p. 33] we have the explicit block decomposition of $\mathfrak{R}(G)$:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

We also have

$$\mathfrak{Z}(G) \cong \prod \mathfrak{Z}^{\mathfrak{s}}$$

where

$$\mathfrak{Z}^{\mathfrak{s}}(G) = \mathcal{O}(D_\sigma / W_{\mathfrak{t}})$$

is the centre of the category $\mathfrak{R}^{\mathfrak{s}}(G)$.

Let the finite group Γ act on the space X. We define, as in [2],

$$\widetilde{X} := \{(\gamma, x) : \gamma x = x\} \subset \Gamma \times X$$

and define the Γ-action on $\widetilde{X}$ as follows:

$$\gamma_1(\gamma, x) := (\gamma_1 \gamma \gamma_1^{-1}, \gamma_1 x).$$

The *extended quotient* of X by Γ is defined to be the ordinary quotient $\widetilde{X}/\Gamma$. If Γ acts freely, then we have

$$\widetilde{X} = \{(1, x) : x \in X\} \cong X$$

and, in this case, $\widetilde{X}/\Gamma = X/\Gamma$.

We will write

$$\widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G) := \mathcal{O}(\widetilde{D_\sigma}/W_{\mathfrak{t}}).$$

We now form the *extended centre*

$$\widetilde{\mathfrak{Z}}(G) := \prod \mathcal{O}(\widetilde{D_\sigma}/W_{\mathfrak{t}}).$$

We will write

$$k := \mathcal{O}(D_\sigma / W_{\mathfrak{t}}) = \mathfrak{Z}^{\mathfrak{s}}(G).$$

3 The ideal $\mathcal{H}^{\mathfrak{s}}(G)$

Let $\mathcal{H} = \mathcal{H}(G)$ denote the Hecke algebra of G. Note that $\mathcal{H}(G)$ admits a set E of local units : for each $x \in \mathcal{H}(G)$ there exists $e \in E$ with $ex = xe = x$. As local units we may take the e_K with K a compact open subgroup of G. We have the Bernstein decomposition

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}^{\mathfrak{s}}(G)$$

of the Hecke algebra $\mathcal{H}(G)$ into two-sided ideals.

We will write

$$\mathcal{H}'(G) := \bigoplus_{\mathfrak{t}\neq\mathfrak{s}} \mathcal{H}^{\mathfrak{t}}(G)$$

so that we have

$$\mathcal{H}(G) = \mathcal{H}^{\mathfrak{s}} \oplus \mathcal{H}'(G).$$

If $h \in \mathcal{H}(G)$ then we have uniquely $h = h^{\mathfrak{s}} + h'$ with $h^{\mathfrak{s}} \in \mathcal{H}^{\mathfrak{s}}(G), h' \in \mathcal{H}'(G)$. We will write $h = (h^{\mathfrak{s}}, h')$. In particular, we will write the local unit $e = (e^{\mathfrak{s}}, e')$. Define

$$E^{\mathfrak{s}} := \{e^{\mathfrak{s}} : e \in E\}.$$

Then $E^{\mathfrak{s}}$ is a set of local units for $\mathcal{H}^{\mathfrak{s}}(G)$. For R and S rings, each with a set of local units, we call R and S *Morita equivalent* if the categories $\mathfrak{mod} - R$ and $\mathfrak{mod} - S$ are equivalent, see Abrams [1, p.816].

Theorem 1. *Let $\mathfrak{s} \in \mathfrak{B}(G)$. The ideal $\mathcal{H}^{\mathfrak{s}}(G)$ is a k-algebra Morita equivalent to a unital k-algebra $\mathcal{A}_{\mathfrak{s}}$ of finite type.*

Proof. In [14, 3.13], a *special* idempotent $e \in \mathcal{H}$ is constructed. By [14, 3.1, 3.4 − 3.6], we have $\mathcal{H}^{\mathfrak{s}}(G) \cong \mathcal{H}e\mathcal{H}$. We note that this follows from the general considerations in [14, §3], and does *not* use the existence or construction of types.

Thanks, at this point, to Gene Abrams for supplying us with reference [20]. By Proposition 2.6 in Garcia-Simon [20] we have immediately

$$\mathcal{H}e\mathcal{H} \sim_{morita} e\mathcal{H}e.$$

The category $\mathfrak{R}^{\mathfrak{s}}(G)$ is equivalent to $\mathfrak{mod} - e\mathcal{H}e$, by [14, 3.13].

Roche [46], following Bernstein, constructs a progenerator $\iota_P^G \Sigma$ in $\mathfrak{R}^{\mathfrak{s}}(G)$. We now define the unital ring

$$\mathcal{A}_{\mathfrak{s}} = \mathrm{End}_G(\iota_P^G \Sigma). \tag{1}$$

The category $\mathfrak{R}^{\mathfrak{s}}(G)$ is equivalent to $\mathfrak{mod} - \mathcal{A}_{\mathfrak{s}}$ by [46, Corollary, p.122].

Therefore $e\mathcal{H}e$ is Morita equivalent to $\mathcal{A}_{\mathfrak{s}}$. In summary, we have

$$\mathcal{H}^{\mathfrak{s}}(G) = \mathcal{H}e\mathcal{H} \sim_{morita} e\mathcal{H}e \sim_{morita} \mathcal{A}_{\mathfrak{s}}.$$

Now $\mathcal{A}_{\mathfrak{s}}$ is a free of finite rank $\mathfrak{Z}^{\mathfrak{t}}$-module, see [48, p.43]. But we have [48, 7.4.1]

$$\mathfrak{Z}^{\mathfrak{s}} = (\mathfrak{Z}^{\mathfrak{t}})^{W_{\mathfrak{t}}}.$$

Now $\mathfrak{Z}^{\mathfrak{t}}$ is finitely generated over its invariant subring $\mathfrak{Z}^{\mathfrak{s}}$. Therefore $\mathcal{A}_{\mathfrak{s}}$ is finitely generated over its centre $\mathfrak{Z}^{\mathfrak{s}}$. The ring $\mathfrak{Z}^{\mathfrak{s}}$ can be viewed as the ring of regular functions on the quotient variety $\mathrm{Irr}^{\mathfrak{t}} L/W_{\mathfrak{t}}$.

□

4 The conjecture

Let k be the coordinate ring of a complex affine algebraic variety X, $k = \mathcal{O}(X)$. Let A be an associative $\mathbb{C}$-algebra which is also a k-algebra. We work with the class of k-algebras A such that A is Morita equivalent, as a k-algebra, to a unital finite type k-algebra. We will assume that A admits a set E of local units, as in [1]. The algebras A and B, each with a set of local units, are *Morita equivalent*

$$A \sim_{morita} B$$

if the categories $\mathfrak{mod} - A$ and $\mathfrak{mod} - B$ are equivalent, see [1, p.816].

We will define an equivalence relation, called *geometric equivalence*, on the set of such algebras A.

(1) Morita equivalence. We will allow all Morita equivalences between *unital* k-algebras. If the unital k-algebras A, B are Morita equivalent, we have, with $j = 0, 1$,

$$HP_j(A) \cong HP_j(B)$$

by Morita invariance of periodic cyclic homology for unital algebras, see [33, 2.2.9]. For periodic cyclic homology, see [33] and [18].

Given $x \in A$, there exists a local unit f such that $x = xf$, therefore $A \subset A^2$. Since $A^2 \subset A$, we have $A = A^2$. Therefore A is an *idempotent* algebra in the sense of [20]. Let e be an idempotent in A. By [20, Proposition 2.6] we have

$$AeA \sim_{morita} eAe.$$

We will allow Morita equivalences of this kind, between the non-unital algebra AeA and the unital algebra eAe.

We also have $(Ae)(eA) = AeA$ and $(eA)(Ae) = eAe$. By the theorem of Cuntz [17], [18, 2.46, 2.47], we have

$$HP_*(AeA) \cong HP_*(eAe).$$

(2) Spectrum preserving morphisms with respect to filtrations of k-algebras of finite type, as in [3].

A morphism $\phi\colon A \to B$ of k-algebras of finite type is called

- *spectrum preserving* if, for each primitive ideal $\mathfrak{q}$ of B, there exists a unique primitive ideal $\mathfrak{p}$ of A containing $\phi^{-1}(\mathfrak{q})$, and the resulting map $\mathfrak{q} \mapsto \mathfrak{p}$ is a bijection from $\mathrm{Prim}(B)$ onto $\mathrm{Prim}(A)$;
- *spectrum preserving with respect to filtrations* if there exist increasing filtrations

$$(0) = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_{r-1} \subset I_r \subset \cdots \subset A$$

$$(0) = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_{r-1} \subset J_r \subset \cdots \subset B,$$

such that, for all r, we have $\phi(I_r) \subset J_r$ and the induced morphism

$$\phi_* : I_r/I_{r-1} \to J_r/J_{r-1}$$

is spectrum preserving.

(3) Character deformation of k-algebras. Let A be a unital algebra over the complex numbers. Form the algebra $A[t, t^{-1}]$ of Laurent polynomials with coefficients in A. If q is a non-zero complex number, then we have the evaluation-at-q map of algebras

$$A[t, t^{-1}] \longrightarrow A$$

which sends a Laurent polynomial $P(t)$ to $P(q)$. Suppose now that $A[t, t^{-1}]$ has been given the structure of a k-algebra i.e. we are given a unital map of algebras over the complex numbers from k to the centre of $A[t, t^{-1}]$. We assume that for any non-zero complex number q the composed map

$$k \longrightarrow A[t, t^{-1}] \longrightarrow A$$

where the second arrow is the above evaluation-at-q map makes A into a k-algebra of finite type. For q a non-zero complex number, denote the finite type k-algebra so obtained by $A(q)$. Then we decree that if q_1 and q_2 are any two non-zero complex numbers, $A(q_1)$ is equivalent to $A(q_2)$.

We fix k. The first two moves preserve the central character. This third move allows us to algebraically deform the central character.

Let $\asymp$ be the equivalence relation generated by (1), (2), (3); we say that A and B are *geometrically equivalent* if $A \asymp B$.

Since each move induces an isomorphism in periodic cyclic homology [3], we have

$$A \asymp B \Longrightarrow HP_*(A) \cong HP_*(B).$$

In order to formulate our conjecture, we need to review certain results and definitions.

The primitive ideal space of $\widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G)$ is the set of $\mathbb{C}$-points of the variety $\widetilde{D_\sigma}/W_{\mathfrak{t}}$ in the Zariski topology.

We have an isomorphism

$$HP_*(\mathcal{O}(\widetilde{D_\sigma}/W_{\mathfrak{t}})) \cong H^*(\widetilde{D_\sigma}/W_{\mathfrak{t}}; \mathbb{C}).$$

This is a special case of the Feigin-Tsygan theorem; for a proof of this theorem which proceeds by reduction to the case of smooth varieties, see [29].

Let E_σ be the maximal compact subgroup of the complex torus D_σ, so that E_σ is a compact torus.

Let $\mathrm{Prim}^t\,\mathcal{H}^{\mathfrak{s}}(G)$ denote the set of primitive ideals attached to tempered, simple $\mathcal{H}^{\mathfrak{s}}(G)$-modules.

Let $\mathcal{S}(G) = \bigoplus \mathcal{S}^{\mathfrak{s}}(G)$ be the Bernstein decomposition of the Harish-Chandra Schwartz algebra. The algebra $\mathcal{S}(G)$ is a topological algebra equipped with a *separately* continuous convolution product. The periodic cyclic homology of the topological algebra $\mathcal{S}(G)$, and of its Bernstein components $\mathcal{S}^{\mathfrak{s}}(G)$, will be constructed via the *inductive tensor product* as in [10]. The inductive tensor product respects inductive limits of locally convex spaces, by [Proposition 14.I, p.76] in Grothendieck's memoir [24]. The inductive tensor product is therefore well-adapted to the Harish-Chandra Schwartz algebra $\mathcal{S}(G)$, which is itself the inductive limit of the nuclear, Fréchet algebras $\mathcal{S}(G//K)$. The periodic cyclic homology of $\mathcal{S}^{\mathfrak{s}}(G)$ will be denoted $HP_*(\mathcal{S}^{\mathfrak{s}}(G))$.

Conjecture 1. *Let* $\mathfrak{s} \in \mathfrak{B}(G)$. *Then*

(1) $\mathcal{H}^{\mathfrak{s}}(G)$ *is geometrically equivalent to the commutative algebra* $\widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G)$*:*

$$\mathcal{H}^{\mathfrak{s}}(G) \asymp \widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G).$$

(2) *The resulting bijection of primitive ideal spaces*

$$\mathrm{Prim}\,\mathcal{H}^{\mathfrak{s}}(G) \longleftrightarrow \widetilde{D_\sigma}/W_{\mathfrak{t}}$$

restricts to give a bijection

$$\mathrm{Prim}^t\,\mathcal{H}^{\mathfrak{s}}(G) \longleftrightarrow \widetilde{E_\sigma}/W_{\mathfrak{t}}.$$

(3) *The inclusions* $E_\sigma \hookrightarrow D_\sigma, \mathcal{H}^{\mathfrak{s}}(G) \hookrightarrow \mathcal{S}^{\mathfrak{s}}(G)$ *induce the commutative diagram:*

$$\begin{array}{ccc} HP_*(\mathcal{H}^{\mathfrak{s}}(G)) & \longrightarrow & HP_*(\mathcal{S}^{\mathfrak{s}}(G)) \\ \downarrow & & \downarrow \\ H^*(\widetilde{D_\sigma}/W_{\mathfrak{t}};\mathbb{C}) & \longrightarrow & H^*(\widetilde{E_\sigma}/W_{\mathfrak{t}};\mathbb{C}) \end{array}$$

in which the vertical maps are induced by the geometric equivalence in part (1) of the conjecture, and all maps are isomorphisms.

Remark 1. If $W_{\mathfrak{t}}$ acts freely then $\widetilde{D_\sigma}/W_{\mathfrak{t}} \cong D_\sigma/W_{\mathfrak{t}}$ and the conjecture in this case predicts irreducibility of the induced representations $\iota_P^G(\sigma \otimes \chi)$. This situation is discussed in [46], with any maximal Levi subgroup of G, and also with the following (non maximal) Levi subgroup

$$L = (\mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(4)) \cap \mathrm{SL}(8)$$

of SL(8) and σ defined as in [46, p.127].

Remark 2. Part (1) of the conjecture is true for supercuspidal representations: Let $\mathfrak{s} := [G, \sigma]_G$. Then we have

$$\mathcal{H}^{\mathfrak{s}}(G) \asymp \mathfrak{Z}^{\mathfrak{s}}(G) = \widetilde{\mathfrak{Z}}^{\mathfrak{s}}(G).$$

This follows from the Paley-Wiener theorem [4, p. 81].

Remark 3. The referee has declared that he would like to see a reformulation of the conjecture in terms of the category $\mathfrak{R}(G)$ of smooth G-modules. It seems likely to us that a filtration of $\mathfrak{R}(G)$, similar to the filtrations constructed in [29] and [50], will enter into such a reformulation, which may take the following form: The extended centre $\widetilde{\mathfrak{Z}}(G)$ is the centre of an 'extended category' $\widetilde{\mathfrak{R}}(G)$ made from smooth G-modules.

5 General features of the proofs of the conjecture in the various examples

The proof of Theorem 1 shows that $\mathcal{H}^{\mathfrak{s}}(G)$ is Morita equivalent to the unital k-algebra $\mathcal{A}_{\mathfrak{s}}$ defined in (1). The next step in proving the conjecture will be to relate this algebra $\mathcal{A}_{\mathfrak{s}}$ to a *generalised Iwahori-Hecke algebra*, as defined below.

Let W' be a Coxeter group with generators $(s)_{s\in S}$ and relations

$$(ss')^{m_{s,s'}} = 1, \quad \text{for any } s, s' \in S \text{ such that } m_{s,s'} < +\infty,$$

and let L be a weight function on W', that is, a map $L\colon W' \to \mathbb{Z}$ such that $L(ww') = L(w) + L(w')$ for any w, w' in W' such that $\ell(ww') = \ell(w) + \ell(w')$, where ℓ is the usual length function on W'. Clearly, the function ℓ is itself a weight function.

Let Ω be a group acting on the Coxeter system (W', S). The group $W := W' \rtimes \Omega$ will be called an *extended Coxeter group*. We extend L to W by setting $L(w\omega) := L(w)$, for $w \in W'$, $\omega \in \Omega$.

Let $A := \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate. We set $u := v^2$ and $v_s := v^{L(s)}$ for any $s \in S$. Let $\bar{}\colon A \to A$ be the ring involution which takes v^n to v^{-n} for any $n \in \mathbb{Z}$.

Let $\mathcal{H}(W, u) = \mathcal{H}(W, L, u)$ denote the A-algebra defined by the generators $(T_s)_{s\in S}$ and the relations

$$(T_s - v_s)(T_s + v_s^{-1}) = 0 \quad \text{for } s \in S,$$

$$\underbrace{T_s T_{s'} T_s \cdots}_{m_{s,s'} \text{ factors}} = \underbrace{T_{s'} T_s T_{s'} \cdots}_{m_{s,s'} \text{ factors}}, \qquad \text{for any } s \neq s' \text{ in } S \text{ such that } m_{s,s'} < +\infty.$$

For $w \in W$, we define $T_w \in \mathcal{H}(W,u)$ by $T_w = T_{s_1}T_{s_2}\cdots T_{s_m}$, where $w = s_1s_2\cdots s_m$ is a reduced expression in W. We have $T_1 = 1$, the unit element of $\mathcal{H}(W,u)$, and $(T_w)_{w\in W}$ is an A-basis of $\mathcal{H}(W,u)$. The v_s are called the parameters of $\mathcal{H}(W,u)$.

Let $\mathcal{H}(W',u)$ be the A-subspace of $\mathcal{H}(W,u)$ spanned by all T_w with $w \in W'$. For each $q \in \mathbb{C}^\times$, we set $\mathcal{H}(W,q) := \mathcal{H}(W,u) \otimes_A \mathbb{C}$, where $\mathbb{C}$ is regarded as an A-algebra with u acting as scalar multiplication by q. The algebras of the form $\mathcal{H}(W,q)$ where W is an extended Coxeter group and $q \in \mathbb{C}$ will be called *extended Iwahori-Hecke algebras*. In the case when the Coxeter group W' is an affine Weyl group, we will say that $\mathcal{H}(W,q)$ is an *extended affine Iwahori-Hecke algebra*.

We now observe that $W_{\mathfrak{t}}$ is a (finite) extended Coxeter group. Indeed, there exists a root system $\Phi_{\mathfrak{t}}$ with associate Weyl group denoted $W'_{\mathfrak{t}}$ and a subset $\Phi_{\mathfrak{t}}^+$ of positive roots in $\Phi_{\mathfrak{t}}$, such that, setting

$$C_{\mathfrak{t}} := \left\{w \in W_{\mathfrak{t}} \,:\, w(\Phi_{\mathfrak{t}}^+) \subset \Phi_{\mathfrak{t}}^+\right\},$$

we have

$$W_{\mathfrak{t}} = W'_{\mathfrak{t}} \rtimes C_{\mathfrak{t}}.$$

This follows from [25, Prop. 4.2] and [27, Lem. 2].

It is expected, and proved, using the theory of types of [14], for level-zero representations in [43], [44], for principal series representations of split groups in [47], for the group $\mathrm{GL}(n,F)$ in [13], [15], for the group $\mathrm{SL}(n,F)$ in [23], for the group $\mathrm{Sp}(4)$ in [7], and for a large class of representations of classical groups in [30], [31], that there exists always an extended affine Iwahori-Hecke algebra $\mathcal{H}'_{\mathfrak{s}}$ such that the following holds:

1. there exists a (finite) Iwahori-Hecke algebra $H'_{\mathfrak{s}}$ with corresponding Coxeter group $W'_{\mathfrak{t}}$ and a Laurent polynomial algebra $\mathcal{B}_{\mathfrak{t}}$ satisfying $\mathcal{H}'_{\mathfrak{s}} = H'_{\mathfrak{s}} \otimes_{\mathbb{C}} \mathcal{B}_{\mathfrak{t}}$;
2. there exists a two-cocycle $\mu\colon C_{\mathfrak{t}} \times C_{\mathfrak{t}} \to \mathbb{C}^\times$ and an injective homomorphism of groups $\iota\colon C_{\mathfrak{t}} \to \mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}\mathcal{H}'_{\mathfrak{s}}$ such that $\mathcal{A}_{\mathfrak{s}}$ is Morita equivalent to the twisted tensor product algebra $\mathcal{H}'_{\mathfrak{s}}\tilde{\otimes}_\iota\mathbb{C}[C_{\mathfrak{t}}]_\mu$.

In the case of $\mathrm{GL}(n,F)$ (see [13]), and in the case of principal series representations of split groups with connected centre (see [47]), we always have $C_{\mathfrak{t}} = \{1\}$. The references quoted above give examples in which $C_{\mathfrak{t}} \neq \{1\}$. The results in [23] also show that the algebra $\mathcal{H}'_{\mathfrak{s}}\tilde{\otimes}_\iota\mathbb{C}[C_{\mathfrak{t}}]_\mu$ is not always isomorphic to an extended Iwahori-Hecke algebra.

There are no known example in which the cocycle μ is non-trivial. In the case of unipotent level zero representations [36], [37], of principal series representations [47], and of the group Sp_4 [7], it has been proved that μ is trivial.

¿From now on we will assume that $C_{\mathfrak{t}} = \{1\}$. Hence it is expected that $\mathcal{A}_{\mathfrak{s}}$ is Morita equivalent to a generalized affine Iwahori-Hecke algebra $\mathcal{H}'_{\mathfrak{s}}$.

Let G be a connected reductive F-split p-adic group. In the case when the Levi subgroup L is a torus and $\mathfrak{s} = [T, 1]_G$ the algebra $\mathcal{A}_{\mathfrak{s}}$ is isomorphic to the commuting algebra $\mathcal{H}(G//I)$ in G of the induced representation from the trivial representation of an Iwahori subgroup I. We have

$$\mathcal{H}(G//I) \simeq \mathcal{H}(W, q_F),$$

where q_F is the order of the residue field of F and W is defined as follows (see for instance [16, §3.2 and 3.5]). Here the weight function L is taken to be equal to the length function. In particular, we are in the equal parameters case.

Let T be a maximal split torus in G, and let $X^*(T)$, $X_*(T)$ denote its groups of characters and cocharacters, respectively. Let $\Phi(G,T) \subset X^*(T)$, $\Phi^\vee(G,T) \subset X_*(T)$ be the corresponding root and coroot systems, and W_f the associated (finite) Weyl group. Then

$$W = X_*(T) \rtimes W_f,$$

Now let $X'_*(T)$ denote the subgroup of $X_*(T)$ generated by $\Phi^\vee(G,T)$. Then $W' := X'_*(T) \rtimes W_f$ is a Coxeter group (an affine Weyl group) and $W = W' \rtimes \Omega$, where Ω is the group of elements in W of length zero.

6 The asymptotic Hecke algebra

There is a unique algebra involution $h \mapsto h^\dagger$ of $\mathcal{H}(W', u)$ such that $T_s^\dagger = -T_s^{-1}$ for any $s \in S$, and a unique endomorphism $h \mapsto \bar{h}$ of $\mathcal{H}(W', u)$ which is A-semilinear with respect to $^-$: $A \to A$ and satisfies $\bar{T}_s = T_s^{-1}$ for any $s \in S$. Let

$$A_{\leq 0} := \bigoplus_{m \leq 0} \mathbb{Z}\, v^m = \mathbb{Z}[v^{-1}], \quad A_{<0} := \bigoplus_{m<0} \mathbb{Z}\, v^m,$$

$$\mathcal{H}(W', u)_{\leq 0} := \bigoplus_{w \in W'} A_{\leq 0}\, T_w, \;\; \mathcal{H}(W', u)_{<0} := \bigoplus_{w \in W'} A_{<0}\, T_w.$$

Let $z \in W'$. There is a unique $c_z \in \mathcal{H}(W', u)_{\leq 0}$ such that $\bar{c}_z = c_z$ and $c_z = T_z \mod \mathcal{H}(W', u)_{<0}$, [42, Theorem 5.2 (a)]. We write $c_z = \sum_{y \in W'} p_{y,z}\, T_y$, where $p_{y,z} \in A_{\leq 0}$. For $y \in W'$, $\omega, \omega' \in \Omega$, we define $p_{y\omega, z\omega'}$ as $p_{y,z}$ if $\omega = \omega'$ and as 0 otherwise. For $w \in W$, we set

$c_w := \sum_{y \in W} p_{y,w} T_y$. Then it follows from [42, Theorem 5.2 (b)] that $(c_w)_{w \in W}$ is an A-basis of $\mathcal{H}(W, u)$.

For x, y, z in W, we define $f_{x,y,z} \in A$ by

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z.$$

From now on, we assume that W' is a bounded weighted Coxeter group, that is, that there exists an integer $N \in \mathbb{N}$ such that $v^{-N} f_{x,y,z} \in A_{\leq 0}$ for all x, y, z in W'.

For x, y, z in W, we define $h_{x,y,z} \in A$ by

$$c_x \cdot c_y = \sum_{z \in W} h_{x,y,z} c_z.$$

It follows from [42, §13.6] that, for any $z \in W$, there exists an integer $\mathbf{a}(z) \in [0, N]$ such that

$$h_{x,y,z} \in v^{\mathbf{a}(z)} \mathbb{Z}[v^{-1}] \quad \text{for all } x, y \in W,$$

$$h_{x,y,z} \notin v^{\mathbf{a}(z)-1} \mathbb{Z}[v^{-1}] \quad \text{for some } x, y \in W,$$

and we then have for any x, y, z in W:

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} \mod v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}],$$

where $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ is well-defined.

Let $\Delta(z) \geq 0$ be the integer defined by

$$p_{1,z} = n_z v^{-\Delta(z)} + \text{strictly smaller powers of } v, \quad n_z \in \mathbb{Z} - \{0\},$$

and let $\mathcal{D}$ denote the following (finite) subset of W:

$$\mathcal{D} := \left\{ z \in W' \,:\, \mathbf{a}(z) = \Delta(z) \right\}.$$

In [42, chap. 14.2] Lusztig stated a list of 15 conjectures $P_1, \ldots, P_{15}$ and proved them in several cases [42, chap. 15, 16, 17]. Assuming the validity of the conjectures, Lusztig was able to define in [42] partitions of W into left cells, right cells and two-sided cells, which extend the theory of Kazhdan-Lusztig from the case of equal parameters (that is, $v_s = v_{s'}$ for any $(s, s') \in S^2$) to the general case. In the case of equal parameters the conjectures mentioned above are known to be true.

¿From now on we shall assume the validity of these conjectures. Let us recall some of them below:

P1. For any $z \in W$ we have $\mathbf{a}(z) \leq \Delta(z)$.

P2. If $d \in \mathcal{D}$ and $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.

P3. If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y,y^{-1},d} \neq 0$.

P4. If $z' \leq_{\mathcal{LR}} z$ then $\mathbf{a}(z') \geq \mathbf{a}(z)$. Hence, if $z' \sim_{\mathcal{LR}} z$, the $\mathbf{a}(z') = \mathbf{a}(z)$.

P5. If $d \in \mathcal{D}$, $y \in W$, $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.

P6. If $d \in \mathcal{D}$, then $d^2 = 1$.

P7. For any x, y, z in W we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.

P8. Let x, y, z in W be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$.

For any $z \in W$, we set $\hat{n}_z := n_d$ where d is the unique element of $\mathcal{D}$ such that $d \sim_{\mathcal{L}} z^{-1}$.

Let J denote the free Abelian group with basis $(t_w)_{w \in W}$. We set

$$t_x \cdot t_y := \sum_{z \in W} \gamma_{x,y,z^{-1}} \, t_z.$$

(This is a finite sum.) This defines an associative ring structure on J. The ring J is called the *based ring* of W. It has a unit element $\sum_{d \in \mathcal{D}} t_d$ (see [42, §18.3]).

The $\mathbb{C}$-algebra $J(W) := J \otimes_{\mathbb{Z}} \mathbb{C}$ is called the *asymptotic Hecke algebra* of W.

According to property (P8), for each two-sided cell $\mathbf{c}$ in W, the subspace $J_{\mathbf{c}}$ spanned by the t_w, $w \in \mathbf{c}$, is a two-sided ideal of J. The ideal $J_{\mathbf{c}}$ is in fact an associative ring with unit $\sum_{d \in \mathcal{D} \cap \mathbf{c}} T_d$, which is called the based ring of the two-sided cell $\mathbf{c}$, [42, §18.3].

Let $J(W,u) := A \otimes_{\mathbb{Z}} J$. We recall from [42, Theorem 18.9] (which extends [38, 2.4]) that the A-linear map $\phi \colon \mathcal{H}(W,u) \to J(W,u)$ given by

$$\phi(c_w^{\dagger}) := \sum_{\substack{z \in W,\, d \in \mathcal{D} \\ \mathbf{a}(d)=\mathbf{a}(z)}} h_{x,d,z} \, \hat{n}_z \, t_z \quad (x \in W)$$

is a homomorphism of A-algebra with unit (note that the conjecture (P_{15}) is used here).

Let

$$\phi_q \colon \mathcal{H}(W,q) \to J \otimes_{\mathbb{Z}} \mathbb{C} \tag{2}$$

be the $\mathbb{C}$-algebra homorphism induced by ϕ.

Let $\mathcal{H}(W,q)^{\geq i}$ be the $\mathbb{C}$-subspace of $\mathcal{H}(W,q)$ spanned by all the $c_w^{\dagger}$ with $w \in W$ and $\mathbf{a}(w) \geq i$. This a two-sided ideal of $\mathcal{H}(W,q)$, because of [42, §13.1] and (P7). Let

$$\mathcal{H}(W,q)^i := \mathcal{H}(W,q)^{\geq i} / \mathcal{H}(W,q)^{\geq i+1};$$

this is an $\mathcal{H}(W,q)$-bimodule. It has as $\mathbb{C}$-basis the images $[c_w^\dagger]$ of the $c_w^\dagger \in \mathcal{H}(W,q)^{\geq i}$ such that $a(w) = i$.

We may regard $\mathcal{H}(W,q)^i$ as a J-bimodule with multiplication defined by the rule:

$$t_x * [c_w^\dagger] = \sum_{\substack{z \in W \\ a(z)=i}} \gamma_{x,w,z^{-1}} \, \hat{n}_w \, \hat{n}_z \, c_z^\dagger,$$

$$[c_w^\dagger] * t_x = \sum_{\substack{z \in W \\ a(z)=i}} \gamma_{w,x,z^{-1}} \, \hat{n}_w \, \hat{n}_z \, c_z^\dagger, \quad (w, x \in W, a(w) = i). \tag{3}$$

We have (see [42, 18.10]):

$$hf = \phi_q(h) * f, \text{ for all } f \in \mathcal{H}(W,q)^i, \, h \in \mathcal{H}(W,q). \tag{4}$$

On the other side:

$$(j * f)h = j * (fh), \text{ for all } f \in \mathcal{H}(W,q)^i, \, h \in \mathcal{H}(W,q). \tag{5}$$

Let

$$f_i := \sum_{\substack{d \in \mathcal{D} \\ a(d)=i}} [c_d^\dagger] \in \mathcal{H}(W,q)^i.$$

Lemma 1. *We have*

$$t_x * f_i = f_i * t_x = \hat{n}_x \, [c_x^\dagger].$$

Proof. By definition of $*$, we have

$$t_x * f_i = \sum_{\substack{d \in \mathcal{D} \\ a(d)=i}} \sum_{\substack{z \in W \\ a(z)=i}} \gamma_{x,d,z^{-1}} \, \hat{n}_d \, \hat{n}_z \, c_z^\dagger.$$

Since (because of P7) $\gamma_{x,d,z^{-1}} = \gamma_{d,z^{-1},x} = \gamma_{z^{-1},x,d}$, it follows from (P2) that if $\gamma_{x,d,z^{-1}} \neq 0$ then $z = x$. Then, using (P3) and (P5), we obtain $t_x * f_i = \hat{n}_x \, c_x^\dagger$. □

Let k denote the centre of $\mathcal{H}(W,q)$. Lusztig proved the following result in [39, Proposition 1.6 (i)] in the case of equal parameters. Our proof will follow the same lines.

Proposition 1. *The centre of $J \otimes_{\mathbb{Z}} \mathbb{C}$ contains $\phi_q(k)$.*

Proof. It is enough to show that $\phi_q(z) \cdot t_x = t_x \cdot \phi_q(z)$ for any $z \in k$, $x \in W$. Assume that $\mathbf{a}(x) = i$. Let $z \in k$. Using Lemma 1, we obtain

$$(\phi_q(z) t_x) * f_i = \phi_q(z) * t_x * f_i = \hat{n}_x \phi_q(z) * [c_x^\dagger] = \hat{n}_x z [c_x^\dagger].$$

On the other side, using equation (4), we get

$$(t_x\phi_q(z)) * f_i = t_x * (\phi_q(z) * f_i) = t_x * (zf_i).$$

Since $zf_i = f_i z$, it gives

$$(t_x\phi_q(z)) * f_i = t_x * (f_i z),$$

and then, using equation (5) and again Lemma 1, we obtain

$$(t_x\phi_q(z)) * f_i = (t_x * f_i)z = \hat{n}_x[c_x^\dagger]z.$$

Now, since $z \in k$, we have $z[c_x^\dagger] = [c_x^\dagger]z$. Hence

$$(\phi_q(z)t_x) * f_i = (t_x\phi_q(z)) * f_i. \tag{6}$$

It follows from the combination of (P4) and (P8) that $\gamma_{x,y,z} \neq 0$ implies $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$. Hence we have

$$\phi_q(z)t_x = \sum_{\substack{x'\in W \\ \mathbf{a}(x')=i}} \alpha_{x'} t_{x'},$$

$$t_x\phi_q(z) = \sum_{\substack{x'\in W \\ \mathbf{a}(x')=i}} \beta_{x'} t_{x'},$$

with $\alpha_{x'}$, $\beta_{x'}$ in $\mathbb{C}$. Then (6) implies that

$$\sum_{\substack{x'\in W \\ \mathbf{a}(x')=i}} \alpha_{x'}[c_{x'}^\dagger] = \sum_{\substack{x'\in W \\ \mathbf{a}(x')=i}} \beta_{x'}[c_{x'}^\dagger].$$

Hence $\alpha_{x'} = \beta_{x'}$ for all $x' \in W$ such that $\mathbf{a}(x') = i$. It gives $\phi_q(z)t_x = t_x\phi_q(z)$, as required. □

Remark 4. The above proposition provides $J \otimes_{\mathbb{Z}} \mathbb{C}$ (and also each $J_{\mathbf{c}}$) with a structure of k-algebra. This k-algebra structure is not canonical: it depends on q. Our move (3) precisely allows us to pass from one k-algebra structure, depending on q_1, to another k-algebra structure, depending on q_2.

¿From now on we will assume that the weight function L is equal to the length function ℓ and that q is either 1 or is not a root of 1.

Let E be a simple $\mathcal{H}(W, q)$-module (resp. $J \otimes_{\mathbb{Z}} \mathbb{C}$-module). We attach to E an integer $\mathbf{a}_E$ by the following two requirements:

1. $c_w E = 0$ (resp. $t_w E = 0$) for any w such $\mathbf{a}(w) > \mathbf{a}_E$;

2. $c_w E \neq 0$ (resp. $t_w E \neq 0$) for some w such $\mathbf{a}(w) = \mathbf{a}_E$.

Then Lusztig proved in [39, Cor. 3.6] (see also [41, Th. 8.1]) that there is a unique bijection $E \mapsto E'$ between the set of isomorphism classes of simple $\mathcal{H}(W,q)$-modules and the set of isomorphism classes of simple $J \otimes_{\mathbb{Z}} \mathbb{C}$-modules such that $\mathbf{a}_{E'} = \mathbf{a}_E$ and such that the restriction of E' to $\mathcal{H}(W,q)$ via ϕ_q is an $\mathcal{H}(W,q)$-module with exactly one composition factor isomorphic to E and all other composition factors of the form $\bar{E}$ with $\mathbf{a}_{\bar{E}} < \mathbf{a}_E$.

As shown in [3, Th. 9], it follows that ϕ_q is spectrum preserving with respect to filtrations. Hence

$$\mathcal{H}(W,q) \asymp J \otimes_{\mathbb{Z}} \mathbb{C}. \tag{7}$$

Let G be a connected F-split adjoint simple p-adic group. Let ${}^L G^0$ be the Langlands dual group, ${}^L T^0$ its maximal torus. By Langlands duality we have

$$W := X_*(T) \rtimes W_f = X^*({}^L T^0(\mathbb{C})) \rtimes W_f. \tag{8}$$

Lusztig proved in [40, Theorem 4.8] that the unipotent conjugacy classes in ${}^L G^0$ are in bijection with the two-sided cells in W.

Let J be the based ring attached to W. We fix a two-sided cell $\mathbf{c}$ in W. Let $\mathcal{O}_{\mathbf{c}}$ be the unipotent conjugacy class in ${}^L G^0$ corresponding to $\mathbf{c}$. Let $\varphi\colon \mathrm{SL}(2)(\mathbb{C}) \to {}^L G^0$ be a homomorphism of algebraic groups such that $u = \varphi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathcal{O}_{\mathbf{c}}$ and let $F_{\mathbf{c}}$ be a maximal reductive algebraic subgroup of the centralizer $\mathrm{C}_{{}^L G^0}(u)$. The reductive group $F_{\mathbf{c}}$ may be disconnected: the identity component of $F_{\mathbf{c}}$ will be denoted $F_{\mathbf{c}}^0$.

Let Y be a finite $F_{\mathbf{c}}$-set (that is, a set with an algebraic action of $F_{\mathbf{c}}$; thus, $F_{\mathbf{c}}^0$ acts trivially). An $F_{\mathbf{c}}$-vector bundle on Y is a collection of finite dimensional $\mathbb{C}$-vector spaces V_y ($y \in Y$) with a given algebraic representation of $F_{\mathbf{c}}$ on $\bigoplus_{y \in Y} V_y$ such that $G \cdot V_y = V_{gy}$ for all $g \in F_{\mathbf{c}}$, $y \in Y$. We now consider the finite $F_{\mathbf{c}}$-set $Y \times Y$ with diagonal action of F_c and denote by $K_{F_{\mathbf{c}}}(Y \times Y)$ the Grothendieck group of the category of $F_{\mathbf{c}}$-vector bundles on $Y \times Y$. One can define an associative ring structure on $K_{F_{\mathbf{c}}}(Y \times Y)$ (see [40, §10.2]).

Then the conjecture of Lusztig in [40, §10.5] states in particular that there should exist a finite $F_{\mathbf{c}}$-set Y and a bijection π from $\mathbf{c}$ onto the set of irreducible $F_{\mathbf{c}}$-vector bundles on $Y \times Y$ (up to isomorphism) such the $\mathbb{C}$-linear map $J_{\mathbf{c}} \to K_{F_{\mathbf{c}}}(Y \times Y) \otimes \mathbb{C}$ sending t_w to $\pi(w)$ is an algebra isomorphism (preserving the unit element).

Let $|Y|$ denote the cardinality of Y. This number is expected to be the number of left cells contained in $\mathbf{c}$. When $F_{\mathbf{c}}$ is connected,

$K_{F_{\mathbf{c}}}(Y\times Y)$ is isomorphic to the $|Y|\times|Y|$ matrix algebra $\mathrm{M}_{|Y|}(R_{\mathbb{C}}(F_{\mathbf{c}}))$ over the (complexified) rational representation ring $R_{\mathbb{C}}(F_{\mathbf{c}})$ of $F_{\mathbf{c}}$. It is important to note: when $F_{\mathbf{c}}$ is connected, the Lusztig conjecture asserts that $J_{\mathbf{c}}$ is Morita equivalent to a *commutative* algebra.

The Lusztig conjecture has been proved by Xi for any two-sided cell $\mathbf{c}$ when G is one of the following groups $\mathrm{GL}(n)$, $\mathrm{PGL}(n)$, $\mathrm{SL}(2)$, $\mathrm{SO}(5)$ and G_2, and for the lowest two-sided cell $\mathbf{c}_0$ (see next section) when G is any connected F-split adjoint simple p-adic group.

7 The ideal $J_{\mathbf{c}_0}$ in J

As above we assume that G is a connected F-split adjoint simple p-adic group. Let J be the based ring attached to W, with W as in 8. The maximal reductive subgroup of the centralizer of 1 is of course ${}^LG^0$. Under the bijection cited above, the unipotent class 1 corresponds to the *lowest two-sided cell* $\mathbf{c}_0$, that is the subset of all the elements w in W such that $\mathbf{a}(w)$ equals the number of positive roots in the root system of W_f.

Xi proved the Lusztig conjecture for this ideal $J_{\mathbf{c}_0}$ in [53, Theorem 1.10]. According to his result, we have a ring isomorphism

$$J_{\mathbf{c}_0} \cong \mathrm{M}_{|W_f|}(R_{\mathbb{C}}({}^LG^0)).$$

The character map Ch creates an isomorphism

$$R_{\mathbb{C}}({}^LG^0) \cong (R_{\mathbb{C}}({}^LT^0))^{W_f}.$$

The W_f-invariant subring of the (complexified) representation ring of ${}^LT^0$ is precisely the coordinate ring of the quotient torus ${}^LT^0/W_f$.

Since

$$\Psi(T) = {}^LT^0$$

we have a Morita equivalence

$$J_{\mathbf{c}_0} \sim \mathfrak{Z}^{\mathfrak{i}}(G)$$

where $\mathfrak{i}$ is the quotient variety $\Psi(T)/W_f$. Therefore, we obtain the following result.

Theorem 2. *Let G be a connected F-split adjoint simple p-adic group. There is a Morita equivalence between $J_{\mathbf{c}_0}$ and the coordinate ring of the Bernstein variety $\Psi(T)/W_f$.*

According to our conjecture, *the other ideals $J_{\mathbf{c}}$ account (up to Morita equivalence) for the rest of the extended quotient of $\Psi(T)$ by W_f.*

The classical Satake isomorphism is an isomorphism between the spherical Hecke algebra $\mathcal{H}(G//K)$ and the ring $R_{\mathbb{C}}({}^L G^0)$. Further, a theorem of Bernstein (see e.g. [34, Proposition 8.6]) asserts that the centre $Z(\mathcal{H}(G//I))$ of the Iwahori-Hecke algebra $\mathcal{H}(G//I)$ is also isomorphic to $R_{\mathbb{C}}({}^L G^0)$.

At this point, we need the map $\phi_{q,\mathbf{c}_0}$ defined in section 1.7 of Xi's paper [53]. This map is a the composition of ϕ_q and the projection of J onto $J_{\mathbf{c}_0}$.

Xi has proved in [53, Theorem 3.6] that the image $\phi_{q,\mathbf{c}_0}(Z(H(G//I))$ is the centre $Z(J_{\mathbf{c}_0})$ of the algebra $J_{\mathbf{c}_0}$. This creates the following diagram:

$$\begin{array}{ccc} \mathcal{H}(G//K) & \longrightarrow & R_{\mathbb{C}}({}^L G^0) \\ \downarrow & & \downarrow \\ \mathcal{H}(G//I) & \underset{\phi_{q,\mathbf{c}_0}}{\longrightarrow} & J_{\mathbf{c}_0} \end{array}$$

in which the top horizontal map is the Satake isomorphism, the left vertical map is induced by the inclusion $K \subset I$, the right vertical map sends $R_{\mathbb{C}}({}^L G^0)$ onto the centre of $J_{\mathbf{c}_0}$ and the bottom horizontal map is Xi's map $\phi_{q,\mathbf{c}_0}$. The vertical maps are injective. We expect that this diagram is commutative.

8 The Hecke algebra of $\mathcal{H}(\mathrm{GL}(n))$

Theorem 3. *The conjecture is true for* $\mathrm{GL}(n)$*.*

Proof. In this proof, we follow [11] rather closely; we have refined the proof at certain points. The occurrence of an extended quotient in the smooth dual of $\mathrm{GL}(n)$ was first recorded in [26], in the context of Deligne-Langlands parameters.

Let $G := \mathrm{GL}(n)$, $\mathfrak{s} = [L,\sigma]_G \in \mathfrak{B}(G)$ and $\mathfrak{t} = [L,\sigma]_L \in \mathfrak{B}(L)$. We can think of $\mathfrak{t}$ as a vector of irreducible supercuspidal representations of smaller general linear groups. If the vector is

$$(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_t, \ldots, \sigma_t)$$

with σ_i repeated e_i times, $1 \le i \le t$, and $\sigma_1, \ldots, \sigma_t$ pairwise distinct (after unramified twist) then we say that $\mathfrak{t}$ has *exponents* $e_1, \ldots, e_t$.

Each representation σ_i of $G_i := \mathrm{GL}(m_i)$ has a *torsion number*: the order of the cyclic group of all those unramified characters η for which $\sigma_i \otimes \eta \cong \sigma_i$. The torsion number of σ_i will be denoted r_i.

Hence

$$L \simeq \prod_{i=1}^{t} G_i^{e_i} \quad \text{and} \quad \sigma \simeq \bigotimes_{i=1}^{t} \sigma_i^{\otimes e_i},$$

Each σ_i contains a maximal simple type (K_i, λ_i) in G_i [13]. Let

$$K_L := \prod_{i=1}^{t} K_i^{e_i} \quad \text{and} \quad \tau_L := \bigotimes_{i=1}^{t} \lambda_i^{\otimes e_i}.$$

Then (K_L, τ_L) is a $\mathfrak{t}$-type in L. We have

$$W_{\mathfrak{t}} \simeq \prod_{i=1}^{t} S_{e_i}.$$

Let E_i be an extension of F of degree $[E_i : F] := m_i$ and residue index $f(E_i|F) := r_i$. Hence the order q_{E_i} of the residue field of E_i equals q^{r_i}. Let T_i be a standard maximal E_i-torus of $\mathrm{GL}(e_i)$ and let $W_{e_i} := X_*(T_i) \rtimes S_{e_i}$.

Let (K, τ) be a semisimple $\mathfrak{s}$-type, see [13, 14, 15]. Let e_τ be the idempotent attached to the type (K, τ) as in [14, Definition 2.9]:

$$e_\tau(x) = \begin{cases} (\mathrm{vol}\, K)^{-1} (\dim\, \tau)\, tr(\rho(x^{-1})) & \text{if } x \in K, \\ 0 & \text{if } x \in G, x \notin K. \end{cases}$$

The idempotent e_τ is then a *special* idempotent in the Hecke algebra $\mathcal{H}(G)$ according to [14, Definition 3.11]. Let $\mathcal{H} = \mathcal{H}(G)$. It follows from [14, §3] that

$$\mathcal{H}^{\mathfrak{s}}(G) = \mathcal{H} * e_\tau * \mathcal{H}.$$

We then have an allowed Morita equivalence

$$\mathcal{H} * e_\tau * \mathcal{H} \sim_{morita} e_\tau * \mathcal{H} * e_\tau.$$

Now let $\mathcal{H}(K, \tau)$ be the endomorphism-valued Hecke algebra attached to the semisimple type (K, τ). By [14, 2.12] we have a canonical isomorphism of unital $\mathbb{C}$-algebras :

$$\mathcal{H}(G, \tau) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}} W \cong e_\tau * \mathcal{H}(G) * e_\tau$$

so that $e_\tau * \mathcal{H}(G) * e_\tau$ is Morita equivalent to $\mathcal{H}(G, \tau)$. Now we quote the main theorem for semisimple types in $GL(n)$ [15, 1.5]: there is an isomorphism of unital $\mathbb{C}$-algebras

$$\mathcal{H}(G, \tau) \cong \bigotimes_{i=1}^{t} \mathcal{H}(W_{e_i}, q^{r_i}).$$

The factors $\mathcal{H}(W_{e_i}, q^{r_i})$ are (extended) affine Hecke algebras whose structure is given explicitly in [13, 5.6.6]. This structure is in terms of generators and relations [13, 5.4.6].

We conclude that

$$\mathcal{H}^{\mathfrak{s}}(G) \asymp \bigotimes_{i=1}^{t} \mathcal{H}(W_{e_i}, q^{r_i}).$$

On the other hand, from (7), we have

$$\bigotimes_{i=1}^{t} \mathcal{H}(W_{e_i}, q^{r_i}) \asymp \bigotimes_{i=1}^{t} J(W_{e_i}).$$

Finally we will prove that that

$$\bigotimes_{i=1}^{t} J(W_{e_i}) \asymp \widetilde{\mathfrak{Z}}^{\mathfrak{s}}.$$

Let ${}^LT^0$ be the maximal standard torus of ${}^LG^0 = \mathrm{GL}(n, \mathbb{C})$ and let W be the extended affine Weyl group associated to $\mathrm{GL}(n, \mathbb{C})$. We have $W := X^*({}^LT^0) \rtimes S_n = W_n$. For each two-sided cell $\mathbf{c}$ of W we have a corresponding partition λ of n. Let μ be the dual partition of λ. Let u be a unipotent element in $\mathrm{GL}(n, \mathbb{C})$ whose Jordan blocks are determined by the partition μ. Let the distinct parts of the dual partition μ be $\mu_1, \ldots, \mu_p$ with μ_r repeated n_r times, $1 \leq r \leq p$.

Let $C_G(u)$ be the centralizer of u in $G = \mathrm{GL}(n, \mathbb{C})$. Then the maximal reductive subgroup $F_{\mathbf{c}}$ of $C_G(u)$ is isomorphic to $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_p, \mathbb{C})$. For the non-trivial combinatorics which underlies this statement, see [21, §2.6].

Let J be the based ring of W. For each two-sided cell in W, let $|Y|$ be the number of left cells contained in $\mathbf{c}$. The Lusztig conjecture says that there is a ring isomorphism

$$J_{\mathbf{c}} \simeq \mathrm{M}_{|Y|}(R_{F_{\mathbf{c}}}), \quad t_w \mapsto \pi(w)$$

where $R_{F_{\mathbf{c}}}$ is the rational representation ring of $F_{\mathbf{c}}$. This conjecture for $\mathrm{GL}(n, \mathbb{C})$ has been proved by Xi [51, 1.5, 4.1, 8.2].

Since $F_{\mathbf{c}}$ is isomorphic to a direct product of the general linear groups $\mathrm{GL}(n_i, \mathbb{C})$ $(1 \leq i \leq p)$ we see that $R_{F_{\mathbf{c}}}$ is isomorphic to the tensor product over $\mathbb{Z}$ of the representation rings $R_{\mathrm{GL}(n_i, \mathbb{C})}$, $1 \leq i \leq p$. For the ring $R(\mathrm{GL}(n, \mathbb{C}))$ we have

$$R(\mathrm{GL}(n, \mathbb{C})) = \mathbb{Z}[X^*(T(\mathbb{C}))]^{S_n}$$

where $T(\mathbb{C})$ is the standard maximal torus in $\mathrm{GL}(n, \mathbb{C})$, and $X^*(T(\mathbb{C}))$ is the set of rational characters of $T(\mathbb{C})$, by [9, Chapter VIII]. Therefore we have

$$R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[\mathrm{Sym}^{n_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{n_p}\mathbb{C}^\times].$$

Let $\gamma \in S_n$ have cycle type μ, let $X = (\mathbb{C}^\times)^n$. Then

$$\begin{array}{rcl} X^\gamma & \simeq & (\mathbb{C}^\times)^{n_1} \times \cdots \times (\mathbb{C}^\times)^{n_p} \\ Z(\gamma) & \simeq & (\mathbb{Z}/\mu_1\mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/\mu_p\mathbb{Z}) \wr S_{n_p} \\ X^\gamma/Z(\gamma) & \simeq & \mathrm{Sym}^{n_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{n_p}\mathbb{C}^\times \end{array}$$

and so

$$R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[X^\gamma/Z(\gamma)].$$

Then

$$J \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{\mathbf{c}}(J_{\mathbf{c}} \otimes_{\mathbb{Z}} \mathbb{C}) \sim \oplus_{\mathbf{c}}(R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C}) \simeq \mathbb{C}[\widetilde{X}/S_n].$$

The algebra $J \otimes_{\mathbb{Z}} \mathbb{C}$ is Morita equivalent to a reduced, finitely generated, commutative unital $\mathbb{C}$-algebra, namely the coordinate ring of the extended quotient $\widetilde{X}/S_n$. This finishes the proof of part (1) of the conjecture.

Concerning parts (2) and (3) of the conjecture: a proof of parts (2) and (3) can be based on the results in [10], [11] and [12]. □

9 The Iwahori ideal in $\mathcal{H}(\mathrm{SL}(2))$

Even for $\mathrm{SL}(2) = \mathrm{SL}(2, F)$, there is a non-trivial story to tell. For let W be the Coxeter group with 2 generators:

$$W = \langle s_1, s_2 \rangle = \mathbb{Z} \rtimes W_f$$

where $W_f = \mathbb{Z}/2\mathbb{Z}$. Then W is the infinite dihedral group. It has the property that

$$\mathcal{H}(W, q_F) = \mathcal{H}(\mathrm{SL}(2)//I).$$

There are 2 two-sided cells in W. They are $\{e\}$ and $W - \{e\}$.

We now select the simply connected complex simple Lie group G for which

$$W = X^*(T(\mathbb{C})) \rtimes W_f$$

where $T(\mathbb{C})$ is the maximal torus of G. This means that $G = \mathrm{SL}(2, \mathbb{C})$.

There are 2 unipotent classes in $\mathrm{SL}(2, \mathbb{C})$. Let $u_1 = 1$. Then the maximal reductive subgroup F_1 of the centralizer $C_G(u_1)$ is $\mathrm{SL}(2, \mathbb{C})$. For the complexified representation ring we have

$$R_{\mathbb{C}}(\mathrm{SL}(2, \mathbb{C}) \cong \mathbb{C}[t, t^{-1}]^{W_f}.$$

We now make the substitution $t \mapsto t^2$: we obtain

$$R_{\mathbb{C}}(\mathrm{SL}(2,\mathbb{C})) \cong \mathbb{C}[t^2, t^{-2}]^{W_f} \cong \mathbb{C}[{}^LT^0]^{W_f}.$$

Let now u_2 be, in Jordan canonical form, a representative for the other unipotent class. Then the maximal reductive subgroup F_2 of the centralizer $C_G(u_2)$ is the center $Z = \{\pm 1\}$ of $\mathrm{SL}(2,\mathbb{C})$. We have

$$R_{\mathbb{C}}(Z) = \mathbb{C} \oplus \mathbb{C}.$$

Thanks to Xi's solution of the Lusztig conjecture for $\mathrm{SL}(2,\mathbb{C})$, we have the ring-isomorphism

$$J \cong \mathrm{M}_2(R_{\mathbb{C}}(\mathrm{SL}(2,\mathbb{C})) \oplus \mathbb{C} \oplus \mathbb{C}.$$

This creates the Morita equivalence:

$$J \sim \mathbb{C}[{}^LT^0]^{W_f} \oplus \mathbb{C} \oplus \mathbb{C}.$$

This is precisely the coordinate ring of the extended quotient:

$$\mathbb{C}[\widetilde{{}^LT^0}].$$

10 The Iwahori ideal in $\mathcal{H}(\mathrm{PGL}(n))$

Let $G = \mathrm{PGL}(n)$, let T be its standard maximal torus. Let $W := X_*(T) \rtimes W_f$. Then ${}^LG^0 = \mathrm{SL}(n,\mathbb{C})$ is the Langlands dual group. Its maximal torus will be denoted ${}^LT^0$.

The discrete group W is an *extended Coxeter group*:

$$W = \langle s_1, s_2, \ldots, s_n \rangle \rtimes \mathbb{Z}/n\mathbb{Z}$$

where $\mathbb{Z}/n\mathbb{Z}$ permutes cyclically the generators $s_1, \ldots, s_n$. We have

$$\mathcal{H}(W, q_F) = \mathcal{H}(G//I).$$

By Langlands duality, we have

$$W = X_*(T) \rtimes W_f = X^*({}^LT^0) \rtimes W_f.$$

The isomorphism

$${}^LT^0 \cong \Psi(T), \quad t \mapsto \chi_t$$

is fixed by the relation

$$\chi_t(\phi(\varpi_F)) = \phi(t)$$

for $t \in {}^LT^0, \phi \in X_*(T) = X^*({}^LT^0)$, and ϖ_F a uniformizer in F. This isomorphism commutes with the W_f-action, see [22, Section I.2.3].

The symmetric group $W_f = S_n$ acts on ${}^LT^0$ by permuting coordinates, and we form the quotient variety ${}^LT^0/S_n$.

Let $\mathfrak{i} \in \mathfrak{B}(G)$ be determined by the cuspidal pair $(T, 1)$. We have

$$\mathfrak{Z}^{\mathfrak{i}} = \mathbb{C}[{}^LT^0/S_n]$$

$$\widetilde{\mathfrak{Z}}^{\mathfrak{i}} = \mathbb{C}[\widetilde{{}^LT^0}/S_n]$$

Theorem 4. *Let $\mathcal{H}^{\mathfrak{i}}(G)$ denote the Iwahori ideal in $\mathcal{H}(G)$. Then $\mathcal{H}^{\mathfrak{i}}(G)$ is geometrically equivalent to the extended quotient of ${}^LT^0$ by the symmetric group S_n:*

$$\mathcal{H}^{\mathfrak{i}}(G) \asymp \mathbb{C}[\widetilde{{}^LT^0}/S_n].$$

Proof. The non-unital algebra $\mathcal{H}^{\mathfrak{i}}(G)$ is Morita equivalent to the unital affine Hecke algebra $\mathcal{H}(W, q_F)$:

$$\mathcal{H}^{\mathfrak{i}}(G) = \mathcal{H}e\mathcal{H} \sim_{morita} e\mathcal{H}e \cong \mathcal{H}(W, q_F).$$

Lusztig proved that the morphism $\phi_{q_F} : \mathcal{H}(W, q_F) \longrightarrow J$ defined in (2) is spectrum-preserving with respect to filtrations.

For each two-sided cell $\mathbf{c}$ of W we have a corresponding partition λ of n. Let μ be the dual partition of λ. Let u be a unipotent element in $\mathrm{SL}(n, \mathbb{C})$ whose Jordan blocks are determined by the partition μ. Let the distinct parts of the dual partition μ be $\mu_1 < \cdots < \mu_p$ with μ_r repeated n_r times, $1 \le r \le p$.

Let $C_G(u)$ be the centralizer of u in $G = \mathrm{SL}(n, \mathbb{C})$. Then the maximal reductive subgroup F'_λ of $C_G(u)$ is isomorphic to $\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_p, \mathbb{C}) \cap \mathrm{SL}(n, \mathbb{C})$. For details of the injective map

$$\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_p, \mathbb{C}) \longrightarrow \mathrm{SL}(n, \mathbb{C})$$

see [21].

As a special case, let the two-sided cell $\mathbf{c}$ correspond to the partition $\lambda = (1, 1, 1, \ldots, 1)$ of n. Then the dual partition $\mu = (n)$. The unipotent matrix u has one Jordan block, and its centralizer $C_G(u) = Z$ the centre of $\mathrm{SL}(n, \mathbb{C})$. The maximal reductive subgroup F'_λ of $C_G(u)$ is the finite group Z. This is the case $p = 1, \mu_1 = n, n_1 = 1$.

By the theorem of Xi [51, 8.4] we have

$$J_{\mathbf{c}} \otimes_{\mathbb{Z}} \mathbb{C} \sim_{morita} R_{\mathbb{C}}(F'_\lambda) = R_{\mathbb{C}}(Z) = \mathbb{C}^n.$$

Let γ have cycle type (n). Then the fixed set $({}^LT^0)^\gamma$ comprises the n fixed points

$$\mathrm{diag}(\omega^j, \ldots, \omega^j) \in^L T^0$$

where $\omega = \exp(2\pi i/n)$ and $0 \leq j \leq n-1$. These n fixed points correspond to the n generators in the commutative ring $\mathbb{C}$.

We expect that the corresponding points in $\mathrm{Irr}(\mathrm{PGL}(N))$ arise as follows. The unramified unitary twist

$$z^{\mathrm{val} \circ \det} \otimes \mathrm{St}(n)$$

of the Steinberg representation of $\mathrm{GL}(n)$ has trivial central character if and only if z is an nth root of unity. For these values $1, \omega, \omega^2, \ldots, \omega^{n-1}$ of z, we obtain n irreducible smooth representations of $\mathrm{PGL}(n)$.

¿From now on, we will assume that $\lambda \neq (1, 1, 1, \ldots, 1)$. Then F'_λ is a *connected* Lie group.

We will write $T_\lambda(\mathbb{C})$ for the standard maximal torus of F'_λ. The Weyl group is then

$$W(\lambda) = S_{n_1} \times \cdots \times S_{n_p}.$$

According to Bourbaki [9, Chapter VIII], the map Ch, sending each (virtual) representation to its (virtual) character, creates an isomorphism:

$$Ch : R(F'_\lambda) \cong \mathbb{Z}[X^*(T_\lambda(\mathbb{C})]^{W(\lambda)}.$$

Note that a complex linear combination of rational characters of $T_\lambda(\mathbb{C})$ is precisely a regular function on $T_\lambda(\mathbb{C})$.

For each two-sided cell $\mathbf{c}$ of W the $\mathbb{Z}$-submodule $J_\mathbf{c}$ of J, spanned by all t_w, $w \in \mathbf{c}$, is a two-sided ideal of J. The ring $J_\mathbf{c}$ is the based ring of the two-sided cell $\mathbf{c}$. Now apply the theorem of Xi [51, 8.4]. We get

$$J_\mathbf{c} \otimes_\mathbb{Z} \mathbb{C} \sim_{morita} R_\mathbb{C}(F'_\lambda) \cong \mathbb{C}[T_\lambda(\mathbb{C})]^{W(\lambda)} \cong \mathbb{C}[T_\lambda(\mathbb{C})/W(\lambda)].$$

Let $\gamma \in S_n$ have cycle type μ. Then the γ-centralizer is a direct product of wreath products:

$$Z(\gamma) \simeq (\mathbb{Z}/\mu_1\mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/\mu_p\mathbb{Z}) \wr S_{n_p}.$$

The image of $T_\lambda(\mathbb{C})$ in the inclusion $T_\lambda(\mathbb{C}) \to^L T^0$ is precisely the subtorus of ${}^LT^0$ fixed by $\mathbb{Z}/\mu_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\mu_p\mathbb{Z}$. We therefore have

$$({}^LT^0)^\gamma/Z(\gamma) \simeq T_\lambda(\mathbb{C})/W(\lambda).$$

We conclude that

$$R(F'_\lambda) \otimes_\mathbb{Z} \mathbb{C} \simeq \mathbb{C}[({}^LT^0)^\gamma/Z(\gamma)]$$

Then

$$J \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{\mathbf{c}}(J_{\mathbf{c}} \otimes_{\mathbb{Z}} \mathbb{C}) \sim \oplus_{\mathbf{c}}(R(F'_\lambda) \otimes_{\mathbb{Z}} \mathbb{C}) \simeq \mathbb{C}[\widetilde{{}^LT^0}/S_n]$$

The algebra $J \otimes_{\mathbb{Z}} \mathbb{C}$ is Morita equivalent to a reduced, finitely generated, commutative unital $\mathbb{C}$-algebra, namely the coordinate ring of the extended quotient $\widetilde{{}^LT^0}/S_n$. □

11 The Iwahori ideal in $\mathcal{H}(\mathrm{SO}(5))$

Let G denote the special orthogonal group $\mathrm{SO}(5, F)$. We view it as the group of elements of determinant 1 which stabilise the symmetric bilinear form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let T be the group of diagonal matrices

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{-1} \end{pmatrix}, \quad \lambda_1, \lambda_2 \in F^\times.$$

The extended affine Weyl group $W = X_*(T) \ltimes W_f$ is of type $\tilde{B}_2$, with $W_f \simeq S_2 \ltimes (\mathbb{Z}/2)^2$ a finite Weyl group of type B_2.

Here ${}^LG^0 = \mathrm{Sp}(4, \mathbb{C})$, and ${}^LT^0$ is the group of diagonal matrices

$$\mathrm{d}(t_1, t_2) := \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}, \quad t_1, t_2 \in \mathbb{C}^\times.$$

We have $\mathrm{N}_{{}^LG^0}({}^LT^0)/{}^LT^0 \simeq W_f$. As noticed in [8, §2.2], we have $\mathrm{Sp}(4) = \mathrm{Spin}(5)$. In particular, the group ${}^LG^0$ is simply connected. In [52, §11.1], Xi has proved the Lusztig's conjecture for the group W.

Let $\mathfrak{i} \in \mathfrak{B}(G)$ be determined by the cuspidal pair $(T, 1)$. We have

$$\mathfrak{Z}^{\mathfrak{i}} = \mathbb{C}[{}^LT^0/W_f], \quad \widetilde{\mathfrak{Z}}^{\mathfrak{i}} = \mathbb{C}[\widetilde{{}^LT^0}/W_f].$$

Let $X :=^L T^0$.

The 8 elements γ_1, ..., γ_8 of W_f can be described as follows:

$$\begin{array}{ll} \gamma_1(\mathrm{d}(t_1,t_2)) = (\mathrm{d}(t_1,t_2)) & \gamma_2(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_2,t_1) \\ \gamma_3(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_1^{-1},t_2) & \gamma_4(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_1,t_2^{-1}) \\ \gamma_5(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_2,t_1^{-1}) & \gamma_6(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_1^{-1},t_2^{-1}) \\ \gamma_7(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_2^{-1},t_1) & \gamma_8(\mathrm{d}(t_1,t_2)) = \mathrm{d}(t_2^{-1},t_1^{-1}) \end{array}.$$

We have $\gamma_5 = \gamma_2\gamma_3 = \gamma_4\gamma_2$, $\gamma_6 = \gamma_5^2 = \gamma_3\gamma_4 = \gamma_4\gamma_3$, $\gamma_7 = \gamma_5^3 = \gamma_2\gamma_4 = \gamma_3\gamma_2$, $\gamma_8 = \gamma_4\gamma_2\gamma_4 = \gamma_2\gamma_4\gamma_3$. The elements γ_2, γ_3, γ_4, γ_6 and γ_8 are of order 2, the elements γ_5 and γ_7 are of order 4. We obtain

$$X^{\gamma_1} = X, \quad X^{\gamma_2} = \left\{\mathrm{d}(t,t) \,:\, t \in \mathbb{C}^\times\right\},$$

$$X^{\gamma_3} = X^{\gamma_4} = \left\{\mathrm{d}(1,t_2), \mathrm{d}(-1,t_2) \,:\, t_2 \in \mathbb{C}^\times\right\},$$

$$X^{\gamma_5} = X^{\gamma_7} = \{\mathrm{d}(1,1), \mathrm{d}(-1,-1)\},$$

$$X^{\gamma_6} = \{\mathrm{d}(1,1), \mathrm{d}(1,-1), \mathrm{d}(-1,1), \mathrm{d}(-1,-1)\}.$$

The elements γ_1, γ_6 are central, and we have $Z(\gamma_2) = \{\gamma_1, \gamma_2, \gamma_6, \gamma_8\}$,

$$Z(\gamma_3) = \{\gamma_1, \gamma_3, \gamma_4, \gamma_6\}, \quad Z(\gamma_7) = \{\gamma_1, \gamma_5, \gamma_6, \gamma_7\}.$$

There are five W_f-conjugacy classes:

$$\{\gamma_1\}, \{\gamma_6\}, \{\gamma_2, \gamma_8\}, \{\gamma_3, \gamma_4\}, \{\gamma_5, \gamma_7\}.$$

As representatives, we will take $\gamma_1, \gamma_2, \gamma_3, \gamma_5, \gamma_6$.

- We have

  $$\mathbb{C}[X^{\gamma_6}/Z(\gamma_6)] = \mathbb{C}[X^{\gamma_6}/W_f] = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$$

  since there are three W_f-orbits in X^{γ_6}, namely

  $$\{\mathrm{d}(1,1)\}, \{\mathrm{d}(1,-1), \mathrm{d}(-1,1)\}, \{\mathrm{d}(-1,-1)\}$$

- We have

  $$\mathbb{C}[X^{\gamma_5}/Z(\gamma_5)] = \mathbb{C}[X^{\gamma_5}] = \mathbb{C} \oplus \mathbb{C} = R_{\mathbb{C}}(Z) = \mathbb{C}[\Omega] = J_{\mathbf{c}_e}$$

  since $Z(\gamma_5)$ acts trivially on X^{γ_5}.

- We have

  $$X^{\gamma_3}/Z(\gamma_3) = \{\mathrm{d}(1,t); t \in \mathbb{C}^\times\} \sqcup \{\mathrm{d}(-1,t) : t \in \mathbb{C}^\times\}$$

  The $Z(\gamma_3)$-orbit of $\mathrm{d}(1,t)$ is the unordered pair $\{\mathrm{d}(1,t), \mathrm{d}(1,t^{-1})\}$ and the $Z(\gamma_3)$-orbit of $\mathrm{d}(-1,t)$ is the unordered pair $\{\mathrm{d}(-1,t), \mathrm{d}(-1,t^{-1})\}$. Therefore we have

  $$\mathbb{C}[X^{\gamma_3}/Z(\gamma_3)] = \mathbb{L} \oplus \mathbb{L}.$$

- We have $X^{\gamma_2} = \{\mathrm{d}(t,t) : t \in \mathbb{C}^\times\} \cong \{t : t \in \mathbb{C}^\times\}$. The $Z(\gamma_2)$-orbit of the point t is the unordered pair $\{t, t^{-1}\}$. So we have

$$X^{\gamma_2}/Z(\gamma_2) \cong \mathbb{C}^\times/\mathbb{Z}/2$$

  and $\mathbb{C}[X^{\gamma_2}/Z(\gamma_2)] = \mathbb{L}$.
- $\mathbb{C}[X^{\gamma_1}/Z(\gamma_1)] \asymp J_{\mathbf{c}_0} \asymp \mathbb{L}$ by Theorem 3.

So the extended quotient of ${}^LT^0$ by W_f is

$$\mathbb{C}^2 \oplus (\mathbb{C}^3 \oplus \mathbb{L}) \oplus (\mathbb{L} \oplus \mathbb{L}) \oplus \mathbb{L}.$$

The extended Coxeter group $W = W' \rtimes \Omega$ has four two-sided cells c_e, c_1, c_2 and c_0 (see [52, §11.1]):

$$c_e = \{w \in W \,:\, a(w) = 0\} = \{e, \omega\} = \Omega,$$

$$c_1 = \{w \in W \,:\, a(w) = 1\},$$

$$c_2 = \{w \in W \,:\, a(w) = 2\},$$

$$c_0 = \{w \in W \,:\, a(w) = 4\} \quad \text{(the lowest two-sided cell)}.$$

We have

$$J = J_{\mathbf{c}_e} \oplus J_{\mathbf{c}_1} \oplus J_{\mathbf{c}_2} \oplus J_{\mathbf{c}_0}.$$

The reductive group $F_{\mathbf{c}_e}$ is the center of ${}^LG^0$, $F_{\mathbf{c}_1} = (\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{C}^\times$ where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C}^\times$ by $z \mapsto z^{-1}$, and $F_{\mathbf{c}_2} = (\mathbb{Z}/2\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{C})$ and there is a ring isomorphism $J_{\mathbf{c}_2} \simeq \mathrm{M}_4(R_{F_{\mathbf{c}_2}})$, where $R_{F_{\mathbf{c}_2}}$ is the rational representation ring of $F_{\mathbf{c}_2}$, [52, Theorem 11.2]. Hence we also have

$$J_{\mathbf{c}_e} = \mathbb{C}[\Omega] = \mathbb{C}^2$$

Let $\mathbf{c} = \mathbf{c}_1, F = F_{\mathbf{c}}$. Then we have $F = \langle\alpha\rangle \ltimes \mathbb{C}^\times$ where α generates $\mathbb{Z}/2\mathbb{Z}$. Note the crucial relation

$$\alpha z = z^{-1}\alpha$$

with $z \in \mathbb{C}^\times$. There are 4 semisimple conjugacy classes in F, namely

$$\{1\}, \quad \{-1\}, \quad \{\{z, z^{-1}\} : z \in \mathbb{C}^\times, z^2 \neq 1\}, \quad \alpha \cdot \mathbb{C}^\times$$

We have to construct the simple $J_{\mathbf{c}}$-modules explicitly, following Xi [52, p. 51, 107]. We will use Xi's explicit proof of the Lusztig conjecture for B_2. Let $Y = \{1,2,3,4\}$ be the F-set such that as F-sets we have $\{1\} \cong \{2\} \cong F/F$ and $\{3,4\} \cong F/F^0$. The simple $J_{\mathbf{c}}$-modules are given by

$$E_{s,\rho} = \mathrm{Hom}_{A(s)}(\rho, H_*(Y^s))$$

where $A(s) := C_G(s)/C_G(s)^0$ and ρ is a simple $A(s)$-module which appears in the homology group $H_*(Y^s)$. The pair (s, ρ) is chosen up to F-conjugacy.

(A). $s = 1, \rho = 1$. Then $Y^s = \{1, 2, 3, 4\}$. Also $H_*(Y^s)$ is the free $\mathbb{C}$-vector space $V := \mathbb{C}^4$ on $\{1, 2, 3, 4\}$: we will denote its basis by $\{e_1, e_2, e_3, e_4\}$. $A(s) = \mathbb{Z}/2\mathbb{Z}$. The generator of $A(s)$ permutes e_3, e_4. Let V_1 denote the span of $e_1, e_2, e_3 + e_4$, let V_2 denote the span of $e_3 - e_4$. Then

$$E_{1,1} := \mathrm{Hom}_{A(s)}(\rho, V) = V_1 = \mathbb{C}^3.$$

(B). $s = 1, \rho = \epsilon$ where ϵ is the sign representation of $\mathbb{Z}/2\mathbb{Z}$. We have

$$E_{1,\epsilon} := \mathrm{Hom}_{A(s)}(\rho, V) = V_2 = \mathbb{C}.$$

Note that we have

$$E_{1,1} \oplus E_{1,\epsilon} = V = \mathbb{C}^4$$

as $A(s)$-modules.

(C). $s = -1, \rho = 1$. We have $Y^s = Y, A(s) = \mathbb{Z}/2\mathbb{Z}$ and

$$E_{-1,1} := \mathrm{Hom}_{A(s)}(\rho, V) = V_1 = \mathbb{C}^3$$

(D). $s = -1, \rho = \epsilon$. We have $Y^s = Y, A(s) = \mathbb{Z}/2\mathbb{Z}$ and

$$E_{-1,\epsilon} := \mathrm{Hom}_{A(s)}(\rho, V) = V_2 = \mathbb{C}$$

Note that we have

$$E_{-1,1} \oplus E_{-1,\epsilon} = V = \mathbb{C}^4$$

as $A(s)$-modules.

(E) $s = z, \rho = 1$ where $z \in \mathbb{C}^\times, z^2 \neq 1$. We have $Y^s = Y, A(s) = 1$ and

$$E_{z,1} := \mathrm{Hom}(\mathbb{C}, V) = V = \mathbb{C}^4$$

(F). $s = \alpha, \rho = 1$. We have $Y^s = \{1, 2\}, H_*(Y^s) = \mathbb{C}^2, A(s) = \{1, -1, \alpha, -\alpha\} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have

$$E_{\alpha,1} := \mathrm{Hom}_{A(s)}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^2$$

We will sometimes write the simple module $E_{s,\rho}$ as the pair $(E_{s,\rho}, \pi_{s,\rho})$ with

$$\pi_{s,\rho} : J_{\mathbf{c}} \to \mathrm{End}(E_{s,\rho}).$$

We define

$$\pi_z := \pi_{z,1} \qquad z^2 \neq 1$$

$$\pi_1 := \pi_{1,1} \oplus \pi_{1,\epsilon}$$

$$\pi_{-1} := \pi_{-1,1} \oplus \pi_{-1,\epsilon}$$

We select an algebraic family of intertwining operators $\{\mathfrak{a}(z) : z \in \mathbb{C}^\times\}$ which, for all $z \in \mathbb{C}^\times$, intertwines (V, π_z) and $(V, \pi_{1/z})$. Next , we define an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{O}(\mathbb{C}^\times, \operatorname{End} V)$ as follows:

$$\alpha(f)(z) = \mathfrak{a}(z) \cdot f(z) \cdot \mathfrak{a}(z)^{-1}$$

We now consider the *Fourier Transform*

$$\mathcal{F} : J_{\mathbf{c}_1} \longrightarrow M_2(\mathbb{C}) \oplus \mathcal{O}(\mathbb{C}^\times, \operatorname{End} V)^{\mathbb{Z}/2\mathbb{Z}}$$

as follows:

$$\mathcal{F}(x) := \pi_{\alpha,1}(x) \oplus f_x$$

where

$$f_x(z) = \pi_z(x)$$

for all $z \in \mathbb{C}^\times$. The map $\mathcal{F}$ is *spectrum preserving*. For the spectrum of $J_{\mathbf{c}}$ is

$$\{E_{s(\alpha)}, E_{s(1),\epsilon}, E_{s(-1),\epsilon}, E_{s(z)} : z \in \mathbb{C}^\times\}$$

in the notation of Xi [p.111][52], wherein $s(z) = \{z, z^{-1}\}$. This is, by our construction, precisely the spectrum of

$$M_2(\mathbb{C}) \oplus \mathcal{O}(\mathbb{C}^\times, M_4(\mathbb{C}))^{\mathbb{Z}/2\mathbb{Z}}.$$

We now have to consider the noncommutative algebra

$$\mathfrak{I}_1 := \mathcal{O}(\mathbb{C}^\times, \operatorname{End} V)^{\mathbb{Z}/2\mathbb{Z}}.$$

The crossed product $\mathcal{O}(\mathbb{C}^\times) \rtimes \mathbb{Z}/2\mathbb{Z}$ is, by definition,

$$\mathfrak{I}_2 := \mathcal{O}(\mathbb{C}^\times) \rtimes \mathbb{Z}/2\mathbb{Z} := (\mathcal{O}(\mathbb{C}^\times) \otimes \operatorname{End} \mathbb{C}[\mathbb{Z}/2\mathbb{Z}])^{\mathbb{Z}/2\mathbb{Z}}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$ via the regular representation.

We will assume without proof that

$$\mathfrak{I}_1 \asymp \mathfrak{I}_2.$$

We then have

$$\mathfrak{I}_2 \cong \mathbb{C}[\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}] \cong \mathbb{C}[W_{\mathrm{SL}(2)}] \cong \mathcal{H}(2, q_F) \cong J_{\mathrm{SL}(2)}$$

with $W_{\mathrm{SL}(2)}$ the affine Weyl group of SL(2) and $J_{\mathrm{SL}(2)}$ is as in §7. We conclude that

$$J_{\mathbf{c}_1} \asymp \mathbb{C}^3 \oplus \mathbb{L}.$$

- $J_{\mathbf{c}_2} \asymp R_{\mathbb{C}}(F_{\mathbf{c}_2}) = \mathbb{L} \oplus \mathbb{L}$ since
  $$R_{\mathbb{C}}(F_{\mathbf{c}_2}) = R_{\mathbb{C}}(\mathbb{Z}/2 \times \mathrm{SL}(2,\mathbb{C})) = R_{\mathbb{C}}(\mathbb{Z}/2) \otimes R_{\mathbb{C}}(\mathrm{SL}(2,\mathbb{C})) = \mathbb{L} \oplus \mathbb{L}.$$
- $J_{\mathbf{c}_0} = R_{\mathbb{C}}({}^L G^0) = \mathbb{L}$ by Theorem 3.

We conclude that

$$\mathcal{H}^{\mathfrak{i}}(\mathrm{SO}(5)) \asymp J = J_{\mathbf{c}_e} \oplus J_{\mathbf{c}_1} \oplus J_{\mathbf{c}_2} \oplus J_{\mathbf{c}_0} \asymp \mathbb{C}[\widetilde{X}/W_f] = \widetilde{\mathfrak{Z}}^{\mathfrak{i}}(\mathrm{SO}(5)).$$

References

[1] G.D. Abrams, Morita equivalence for rings with local units, Comm. Algebra 11 (1983), 801–837.

[2] P. Baum, A. Connes, The Chern character for discrete groups, A Fête of topology, Academic Press, New York, 1988, 163–232.

[3] P. Baum, V. Nistor, Periodic cyclic homology of Iwahori-Hecke algebras, K-Theory 27 (2002), 329–357.

[4] J. Bernstein, Representations of p-adic groups, Notes by K.E. Rumelhart, Harvard University 1992.

[5] J. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vigneras, Représentations des groupes réductifs sur un corps local, Travaux en cours, Hermann, 1984.

[6] J. Bernstein and S. Gelbart, An introduction to the Langlands program. Birkhauser, Basel, 2003.

[7] L. Blasco, C. Blondel, Algèbres de Hecke et séries principales généralisées de $\mathrm{Sp}_4(F)$, Proc. London Math. Soc. (3) 85 (2002), 659–685.

[8] A. Borel, Automorphic L-functions, Proc. Symp. Pure Math. 33 (1979), part 2, 27–61, in Automorphic forms, representations, and L-functions, 1979, http://www.ams.org/online_bks/pspum331.

[9] N. Bourbaki, Lie groups and Lie algebras, Chapters VI, VIII and IX, Masson, 1981.

[10] J. Brodzki, R.J. Plymen, Periodic cyclic homology of certain nuclear algebras, C.R. Acad. Sci. Paris 329 (1999) 671–676.

[11] J. Brodzki, R.J. Plymen, Complex structure on the smooth dual of $\mathrm{GL}(n)$, Documenta Math. 7 (2002), 91–112.

[12] J. Brodzki, R.J. Plymen, Chern character for the Schwartz algebra of p-adic $\mathrm{GL}(n)$, Bull. London Math. Soc. 34 (2002), 219–228.

[13] C.J. Bushnell, P.C. Kutzko, The admissible dual of $\mathrm{GL}(n)$ via compact open subgroups, Ann. Math. Study 129, Princeton Univ. Press 1993.

[14] C.J. Bushnell, P.C. Kutzko, Smooth representations of reductive p-adic groups: Structure theory via types, Proc. London Math. Soc. 77 (1998), 582–634.

[15] C.J. Bushnell, P.C. Kutzko, Semisimple types in GL_n, Comp. Math. 119 (1999), 53–97.

[16] P. Cartier, Representations of p-adic groups: a survey, Proc. Symp. Pure Math. 33 (1979), part 1, 111–155.

[17] J. Cuntz, Morita invariance in cyclic homology for nonunital algebras, K-Theory 15 (1998), 301 - 305.

[18] J. Cuntz, G. Skandalis, B. Tsygan, Cyclic homology in noncommutative geometry, EMS 121, 2004.

[19] D. Eisenbud, J. Harris, The geometry of schemes, Springer Graduate Text 197, 2001.

[20] J.L. Garcia and J.J. Simon, Morita equivalence for idempotent rings, J. Pure Applied Algebra 76 (1991), 39–56.

[21] M. Geck, An introduction to algebraic geometry and algebraic groups, Oxford Graduate Texts in Mathematics 10, Oxford University Press, 2003.

[22] S. Gelbart, F. Shahidi, Analytic properties of automorphic L-functions, Perspectives in Math. 6 (1988).

[23] D. Goldberg, A. Roche, Hecke algebras and SL_n-types, Proc. London Math. Soc. (3) 90 (2005), no. 1, 87–131.

[24] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, AMS Memoir 16 (1955).

[25] V. Heiermann, Décomposition spectrale et représentations spéciales d'un groupe réductif p–adique, Journal de l'Institut Math. Jussieu, 3 (2004), 327–395.

[26] J.E. Hodgins and R.J. Plymen, The representation theory of p-adic $\mathrm{GL}(n)$ and Deligne-Langlands parameters, TRIM 10 (1996), 54–72.

[27] R. B. Howlett, Normalizers of parabolic subgroups of reflection groups, J. London Math. Soc. (2), 21 (1980), 62–80.

[28] J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in advanced mathematics 29 (1992).

[29] D. Kazhdan, V. Nistor, P. Schneider, Hochschild and cyclic homology of finite type algebras, Sel. Math., New ser. 4 (1998), 321–359.

[30] J.-L. Kim, Hecke algebras of classical groups over p-adic fields and supercuspidal representations, Amer. J. Math. 121 (1999), 967–1029.

[31] J.-L. Kim, Hecke algebras of classical groups over p-adic fields II, Compositio Math. 127 (2001), 117–167.

[32] F. Lemmermeyer, Reciprocity laws from Euler to Eisenstein, Springer Monograph in Mathematics, Berlin, 2000.

[33] J.-L. Loday, Cyclic homology, Springer-Verlag, Berlin, 1992.

[34] G. Lusztig, Singularities, character formulas and a q-analog of weight multiplicities, Astérisque 101 –102 (1983), 208–229.

[35] G. Lusztig, Some examples of square-integrable representations of semisimple p-adic groups, Trans. Amer. Math. Soc. 277 (1983), 623–653.

[36] G. Lusztig, Classification of unipotent representations of simple p-adic groups, Internat. Math. Res. Notices 11 (1995), 517–589.

[37] G. Lusztig, Classification of unipotent representations of simple p-adic groups, II, Represent. Theory 6 (2002), 243–289.

[38] G. Lusztig, Cells in affine Weyl groups, II, J. Algebra 109 (1987), 536–548.

[39] G. Lusztig, Cells in affine Weyl groups, III, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math, 34 (1987), 223–243.

[40] G. Lusztig, Cells in affine Weyl groups, IV, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math, 36 (1989), 297–328.

[41] G. Lusztig, Representations of affine Hecke algebras, Astérisque 171-172 (1989), 73-84.

[42] G. Lusztig, Hecke algebras with unequal parameters, CRM Monograph Series, 18, Amer. Math. Soc. 2003.

[43] L. Morris, Tamely ramified intertwining algebras, Invent. Math. 114 (1993), 1–54.

[44] L. Morris, Level zero G-types, Compositio Math. 118 (1999), 135–157.

[45] J. Neukirch, Algebraic number theory, Springer-Verlag, 1999.

[46] A. Roche, Parabolic induction and the Bernstein decomposition, Compositio Math. 134 (2002), 113–133.

[47] A. Roche, Types and Hecke algebras for principal series representations of split reductive p-adic groups, Ann. scient. Éc. Norm. Sup. **31** (1998), 361–413.

[48] A. Roche, Notes on the Bernstein decomposition (2004), http://padic.mathstat.uottawa.ca/˜workshop/aroche.dvi.

[49] J. Rogawski, The nonabelian reciprocity law for local fields, Notices of the American Math. Society, January (2000), 35–41.

[50] P. Schneider, E.-W. Zink, K-types for the tempered components of a p-adic general linear group, J. reine angew. Math. 517 (1999) 161 –208.

[51] N. Xi, The based ring of two-sided cells of affine Weyl groups of type $\widetilde{A}_{n-1}$, Amer. Math. Soc. Memoir 749, 2002.

[52] N. Xi, Representations of affine Hecke algebras, Lecture Notes in Math. 1587, Springer, 1994.

[53] N. Xi, The based ring of the lowest two-sided cell of an affine Weyl group, J. of Algebra 134 (1990), 356–368.

Anne-Marie Aubert, Institut de Mathématiques de Jussieu, U.M.R. 7586 du C.N.R.S., Paris, France
Email: aubert@math.jussieu.fr

Paul Baum, Pennsylvania State University, Mathematics Department, University Park, PA 16802, USA
Email: baum@math.psu.edu

Roger Plymen, School of Mathematics, University of Manchester, Manchester M13 9PL, England
Email: plymen@manchester.ac.uk