QUASI–COXETER ALGEBRAS, DYNKIN DIAGRAM COHOMOLOGY AND QUANTUM WEYL GROUPS

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Abstract. The author [TL1, TL2], and independently De Concini (unpublished), conjectured that the monodromy of the Casimir connection ∇_C introduced in [MTL] is described by Lusztig's quantum Weyl group operators. This conjecture was proved in [TL1] for all representations of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ and in [TL2] for a number of pairs $(\mathfrak{g}, V)$ including vector and spin representations of classical Lie algebras and the adjoint representation of all complex, simple Lie algebras. The aim of this paper, and of its sequel [TL4] is to prove this conjecture for all $\mathfrak{g}$. Our strategy is inspired by Drinfeld's proof of the equivalence of the monodromy of the Knizhnik–Zamolodchikov equations for $\mathfrak{g}$ and the R–matrix representations coming from the quantum group $U_\hbar\mathfrak{g}$. It relies on the use of *quasi–Coxeter algebras*, which are to the generalised braid group of type $\mathfrak{g}$ what Drinfeld's quasitriangular quasibialgebras are to Artin's braid groups B_n. Using this notion, and the associated deformation cohomology, which we call *Dynkin diagram cohomology*, we reduce the conjecture in this paper to a (non–cohomological) statement about the classical enveloping algebra $U\mathfrak{g}$, namely the existence of a quasi–Coxeter, quasitriangular quasibialgebra structure on it interpolating between the quasi–Coxeter algebra structure underlying the monodromy of the connection ∇_C and the quasitriangular quasibialgebra structure underlying that of the KZ equations. The existence of such a structure will be proved in [TL4].

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Introduction

Let $\mathfrak{g}$ be a complex, simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $\Phi \subset \mathfrak{h}^*$ the corresponding root system. For each $\alpha \in \Phi$, let $\mathfrak{sl}_2^\alpha = \langle e_\alpha, f_\alpha, h_\alpha \rangle \subset \mathfrak{g}$ be the corresponding three–dimensional subalgebra and denote by

$$C_\alpha = \frac{(\alpha,\alpha)}{2}\left(e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2}h_\alpha^2\right)$$

its Casimir operator with respect to the restriction to $\mathfrak{sl}_2^\alpha$ of a fixed non–degenerate, ad–invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. Note that C_α is independent of the choice of the root vectors e_α, f_α and satisfies $C_{-\alpha} = C_\alpha$. Let

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha\in\Phi} \text{Ker}(\alpha)$$

be the set of regular elements in $\mathfrak{h}$, V a finite–dimensional $\mathfrak{g}$–module, and consider the following holomorphic connection on the holomorphically trivial vector bundle $\mathbf{V}$ over $\mathfrak{h}_{\text{reg}}$ with fibre V

$$\nabla_C = d - \frac{h}{2}\sum_{\alpha\in\Phi}\frac{d\alpha}{\alpha}\cdot C_\alpha$$

The following result is due to the author and J. Millson [MTL] and was discovered independently by De Concini around 1995 (unpublished)

Theorem 0.1. *The connection ∇_C is flat for any $h \in \mathbb{C}$.*

Let $W \subset GL(\mathfrak{h})$ be the Weyl group of $\mathfrak{g}$. It is well–known that W does not act on V in general but that the triple exponentials

$$\exp(e_{\alpha_i})\exp(-f_{\alpha_i})\exp(e_{\alpha_i}) \in GL(V)$$

corresponding to a choice $\alpha_1,\ldots,\alpha_n$ of simple roots of $\mathfrak{g}$ and of $\mathfrak{sl}_2$–triples $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} \in \mathfrak{sl}_2^{\alpha_i}$ give rise to an action of an extension $\widetilde{W}$ of W by the sign group $\mathbb{Z}^n$ [Ti]. This action may be used to twist $(\mathbf{V},\nabla_C)$ into a W–equivariant, flat vector bundle on $\mathfrak{h}_{\text{reg}}$ [MTL, §2]. One therefore obtains an analytic, one–parameter family of monodromy representations

$$\mu_V^h : B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W) \longrightarrow GL(V)$$

of the generalised braid group B_W which, for $h=0$, factors through the action of $\widetilde{W}$ on V. Considering the deformation parameter h as formal and setting $\hbar = 2\pi i h$, we regard this family as a single representation

$$\mu_V : B_W \to GL(V[\![\hbar]\!])$$

Let now $U_\hbar\mathfrak{g}$ be the Drinfeld–Jimbo quantum group corresponding to $\mathfrak{g}$ and the inner product $(\cdot,\cdot)$. We regard $U_\hbar\mathfrak{g}$ as a topological Hopf algebra over the ring of formal power series $\mathbb{C}[\![\hbar]\!]$. By a finite–dimensional representation of $U_\hbar\mathfrak{g}$ we shall mean a $U_\hbar\mathfrak{g}$–module $\mathcal{U}$ which is free and finitely–generated as $\mathbb{C}[\![\hbar]\!]$–module. The isomorphism class of such a representation is uniquely determined by that of the $\mathfrak{g}$–module $\mathcal{U}/\hbar\mathcal{U}$. Lusztig [Lu], and independently Kirillov–Reshetikhin and Soibelman [KR, So], constructed operators $S_1^\hbar,\ldots,S_n^\hbar$ labelled by the simple reflections $s_i = s_{\alpha_i}$ in W, acting on any finite–dimensional representation of $U_\hbar\mathfrak{g}$. These operators satisfy the braid relations

$$S_i^\hbar S_j^\hbar \cdots = S_j^\hbar S_i^\hbar \cdots$$

for each $i \neq j$, where the number of terms on each side is equal to the order of $s_i s_j$ in W. As a consequence, each finite–dimensional representation $\mathcal{U}$ of $U_\hbar\mathfrak{g}$ carries an action of B_W called the *quantum Weyl group* action. Its reduction mod $\hbar$ factors trough the action of the Tits extension $\widetilde{W}$ on $\mathcal{U}/\hbar\mathcal{U}$.

Let V be a finite–dimensional $\mathfrak{g}$–module and let $\mathcal{V}$ be a quantum deformation of V, that is a finite–dimensional $U_\hbar\mathfrak{g}$–module such that $\mathcal{V}/\hbar\mathcal{V} \cong V$ as $\mathfrak{g}$–modules. The following conjecture was formulated in [TL1, TL2] and independently by De Concini around 1995 (unpublished).

Conjecture 0.2. *The monodromy action μ_V of B_W on $V[\![\hbar]\!]$ is equivalent to its quantum Weyl group action on $\mathcal{V}$.*

We note in passing the following interesting, and immediate consequence of the above conjecture and of the fact that the finite–dimensional $U_\hbar\mathfrak{g}$–modules and operators $S_i^\hbar$ are defined over $\mathbb{Q}[\![\hbar]\!]$

Corollary 0.3. *The monodromy representation* μ_V *is defined over* $\mathbb{Q}[\![\hbar]\!]$.

Conjecture 0.2 is proved in [TL1] for all representations of $\mathfrak{g} = \mathfrak{sl}_n$ and in [TL2] for a number of pairs $(\mathfrak{g}, V)$ including vector and spin representations of classical Lie algebras and the adjoint representation of all complex, simple Lie algebras. Its semi–classical analogue is proved for all $\mathfrak{g}$ in [Bo1, Bo2].

The aim of this paper, and of its sequel [TL4], is to give a proof of this conjecture for any complex, simple Lie algebra $\mathfrak{g}$. The strategy we follow is very much inspired by Drinfeld's proof of the equivalence of the monodromy of the Knizhnik–Zamolodchikov (KZ) equations and the R–matrix representations coming from $U_\hbar\mathfrak{g}$ [Dr4]. It proceeds along the following lines.

(i) Define a suitable category of algebras carrying representations of the generalised braid group B_W on their finite–dimensional modules. We call these algebras *quasi–Coxeter algebras*. They are the analogues for B_W of what Drinfeld's quasitriangular quasibialgebras are for the Artin braid groups B_n.
(ii) Show that the monodromy representations μ_V arise from a quasi–Coxeter algebra structure on the classical enveloping algebra $U\mathfrak{g}[\![\hbar]\!]$.
(iii) Show that the quantum Weyl group representations of B_W arise from a quasi–Coxeter algebra structure on the quantum group $U_\hbar\mathfrak{g}$.
(iv) Show that this quasi–Coxeter algebra structure on $U_\hbar\mathfrak{g}$ is (cohomologically) equivalent to one on $U\mathfrak{g}[\![\hbar]\!]$.
(v) Construct a complex which controls the deformations of quasi–Coxeter algebra structures. We call the corresponding cohomology *Dynkin diagram cohomology* HD^*. The infinitesimal of a quasi–Coxeter algebra structure on A defines a canonical class in $HD^2(A)$.
(vi) Show, using the Dynkin complex of $U\mathfrak{g}$, that there is, up to isomorphism at most one quasi–Coxeter algebra structure on $U\mathfrak{g}[\![\hbar]\!]$ having prescribed local monodromies.

Steps (i)–(v) are carried out in Part I of this paper. Step (vi) however hopelessly fails since it turns out that, for $\mathfrak{g}$ of rank greater or equal to 2, $HD^2(U\mathfrak{g})$ is infinite–dimensional. To remedy this, one must rigidify matters by taking into account the bialgebra structures on $U_\hbar\mathfrak{g}$ and $U\mathfrak{g}$. This may perhaps be guessed from the fact that both the quantum Weyl group operators $S_i^\hbar$ and the local monodromies S_i^C of the Casimir connection satisfy strikingly similar coproduct identities, namely

$$\Delta(S_i^\hbar) = R_i^{-1} \cdot S_i^\hbar \otimes S_i^\hbar$$
$$\Delta(S_i^C) = \exp(\hbar t_i/2)^{-1} \cdot S_i^C \otimes S_i^C$$

where R_i is the universal R–matrix of $U_\hbar\mathfrak{sl}_2^{\alpha_i} \subset U_\hbar\mathfrak{g}$ and

$$t_i = (\Delta(C_{\alpha_i}) - C_{\alpha_i} \otimes 1 - 1 \otimes C_{\alpha_i})/2$$

Since the failure of $S_i^\hbar$ and S_i^C to be grouplike elements involves quantum and KZ R–matrices respectively, a proper study of quasi–Coxeter bialgebras must in fact be concerned with quasi–Coxeter quasitriangular quasibialgebras, that is bialgebras which carry representations of both Artin's braid groups and the group B_W on the tensor products of their finite–dimensional modules. We carry out the analogues of steps (i) and (v) for these algebras in Part II of this paper and of steps (iii) and (iv) in Part III. Using this, we show in Part III that, up to isomorphism, there exists a unique quasi–Coxeter quasitriangular quasibialgebra structure on $U\mathfrak{g}[\![\hbar]\!]$ having prescribed local monodromies as that coming from $U_\hbar\mathfrak{g}$ (step (vi)). The final step needed to complete the proof of conjecture 0.2, namely the fact that the monodromy of the connection ∇_C and of the KZ connection for $\mathfrak{g}$ fit within a quasi–Coxeter quasitriangular quasibialgebra structure on $U\mathfrak{g}[\![\hbar]\!]$ will be proved in [TL4].

We turn now to an outline of the contents of the paper, referring the reader to the introductory paragraphs of each section for more details. We begin in section 1 by reviewing the De Concini–Procesi theory of asymptotic zones for connections of KZ–type which provides a concise, combinatorial description of their monodromy. This description forms, together with Drinfeld's theory of quasitriangular quasibialgebras, revisited through the author's duality between the connection ∇_C for $\mathfrak{sl}_n$ and the KZ connection for $\mathfrak{sl}_k$ (see [TL1] and §4.3), the basis underpinning the definition of a quasi–Coxeter algebra given in section 3. Such algebras have a type determined by a connected graph D with labelled edges. For the examples relevant to us, D is the Dynkin diagram of the Lie algebra $\mathfrak{g}$ but in most of the paper we merely assume that D is a connected graph and work in this greater generality. Just as MacLane's coherence theorem for monoidal categories is best proved using Stasheff's associahedra K_n [St1], the most compact definition of a quasi–Coxeter algebra has its relations labelled by the two–dimensional faces of a regular CW–complex $\mathcal{A}_D$, which is defined and studied in section 2. We call $\mathcal{A}_D$ the *De Concini–Procesi associahedron* since, when D is the Dynkin diagram of $\mathfrak{g}$, $\mathcal{A}_D$ is naturally realised inside their wonderful model of $\mathfrak{h}_{\text{reg}}$ [DCP2] and, when $\mathfrak{g} = \mathfrak{sl}_{n+1}$, it coincides with K_n[1]. Section 4 describes several examples of quasi–Coxeter algebras. In section 5, we define the Dynkin complex of a quasi–Coxeter algebra and show that it controls its deformation theory. In section 6 we define quasi–Coxeter quasitriangular quasibialgebras and show in section 7 that their deformation theory is controlled by a suitable bicomplex, which we call the *Dynkin–Hochschild* bicomplex. In section 8, we show that Drinfeld's R–matrix and the quantum Weyl groups operators endow the quantum group $U_\hbar\mathfrak{g}$ with the structure of a quasi–Coxeter quasitriangular quasibialgebra and that this structure may be cohomologically transferred to one on $U\mathfrak{g}[\![\hbar]\!]$. Finally, section 9 is devoted to the proof of the rigidity of quasi–Coxeter quasitriangular quasibialgebra structures on the enveloping algebra $U\mathfrak{g}$. This result relies on Drinfeld's rigidity of quasitriangular quasibialgebra deformations of $U\mathfrak{g}[\![\hbar]\!]$ [Dr3] and on the essential uniqueness of solutions of a certain relative twist equation which is studied in the companion paper [TL3].

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[1]while this paper was being written, Carr and Devadoss posted the preprint [CD] where the same CW–complex is introduced under the name *graph–associahedron* and proved to be, as in our §2.2, a convex polytope.

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Part I. Quasi–Coxeter algebras

1. Asymptotic zones for hyperplane arrangements

This section is an exposition of the paper [DCP2] and parts of [DCP1]. All results stated herein are due to De Concini and Procesi, with the minor exception of the terminology of §1.11–§1.12 and §1.14–§1.16 and of the results of §1.15–§1.16, which are however implicit in or immediate consequences of [DCP2]. The presentation in §1.10–§1.16 is a little more general than in [DCP2] since we deal with real arrangements endowed with a simplicial chamber rather than Coxeter ones, but the proofs in [DCP2] carry over verbatim to this context and are therefore mostly omitted.

Let $\mathcal{A}$ be a hyperplane arrangement in a complex, finite–dimensional vector space V. The aim of this section is to describe good solutions Ψ of the *holonomy equations* $\nabla\Psi = 0$ on $V_{\mathcal{A}} = V \setminus \mathcal{A}$. These are a generalisation of the Knizhnik–Zamolodchikov (KZ) equations, which are defined for the Coxeter arrangements of type A_n, to general hyperplane arrangements. The solutions in question are universal, *i.e.,* take values in the *holonomy algebra* A of the arrangement (see §1.1–§1.2 for definitions) and have prescribed asymptotic behaviour, that is are such that the monodromy of suitable commuting elements of $\pi_1(V_{\mathcal{A}})$ has a particularly simple form. They generalise the ones constructed by Drinfeld for the KZ equations [Dr3] and by Cherednik for his generalisation of the KZ equations to Coxeter arrangements [Ch1, Ch2]. They are described in §1.8–§1.9 in terms of local coordinates on a *wonderful model* of V that is, a smooth variety Y_X, proper over V and in which the preimage of $\mathcal{A}$ is a divisor with normal crossings. The construction and coordinatization of Y_X are reviewed in §1.4–§1.7. In §1.10–§1.11, we focus on real arrangements. This allows one to consistently define multiplicative constants, specifically elements of the holonomy algebra A, which relate different solutions. These are a generalisation of Drinfeld's associator, which arises for the Coxeter arrangement of type A_2, and allow one to concisely describe the monodromy of ∇. The main properties of these De Concini–Procesi associators, most notably their inductive structure, are described in §1.12–§1.16. Finally, in §1.17, we specialise further to the case of Coxeter arrangements and obtain that the associators satisfy, together with the local monodromies of ∇, the braid relations which give Brieskorn's presentation of the corresponding generalised braid group.

1.1. **The holonomy algebra of a hyperplane arrangement.** Let V be a finite–dimensional, complex vector space and $\mathcal{A}$ a finite collection of linear hyperplanes in V. Choose an equation $x \in H^\perp \setminus \{0\}$ for each $H \in \mathcal{A}$ and let $X = \{x\} \subset V^*$ be the corresponding collection of linear forms. Consider a connection on $V_\mathcal{A} = V \setminus \mathcal{A}$ of the form

$$\nabla = d - \sum_{x\in X} \frac{dx}{x} \cdot t_x$$

Following Chen [Chn1], we do not regard the t_x as acting on a vector space but rather as formal variables, it being understood that any finite–dimensional representation $\rho : F \to \text{End}(U)$ of the free, associative algebra $F = \mathbb{C}\langle t_x\rangle_{x\in X}$ generated by the t_x gives rise to a holomorphic connection

$$\nabla_\rho = d - \sum_{x\in X} \frac{dx}{x} \cdot \rho(t_x) \tag{1.1}$$

on the holomorphically trivial vector bundle over $V_\mathcal{A}$ with fibre U.

By [Ko2], ∇ is flat if, and only if the following relations hold for any two–dimensional subspace $B \subset V^*$ spanned by elements of X and $x \in X \cap B$,

$$[t_x, \sum_{y\in X\cap B} t_y] = 0 \tag{1.2}$$

Let $I \subset F$ be the two–sided ideal generated by these relations. I is homogeneous with respect to the grading on F for which all t_x have degree 1. The quotient F/I is the universal enveloping algebra of the graded Lie algebra with generators t_x, $x \in X$ defined by (1.2). The completion A of F/I with respect to its grading is a topological Hopf algebra called the *holonomy algebra* of the arrangement $\mathcal{A}$. The *holonomy equations* of $\mathcal{A}$ are the equations

$$d\Psi = \sum_{x\in X} \frac{dx}{x} t_x \Psi \tag{1.3}$$

for a locally defined function $\Psi : V_\mathcal{A} \to A$. Such a Ψ is necessarily *holomorphic*, *i.e.,* such that each component Ψ_m, $m \in \mathbb{N}$ with respect to the $\mathbb{N}$–grading on A is a holomorphic function with values in the finite–dimensional vector space A_m. Moreover, Ψ takes values in the group $A^\times$ of invertible elements of A if, and only if its degree zero term $\Psi_0 : V_\mathcal{A} \to A_0 \cong \mathbb{C}$ does not vanish. Since (1.3) implies that $d\Psi_0 = 0$, this is the case if, and only if $\Psi_0(v) \neq 0$ for some $v \in V_\mathcal{A}$. Finally, since the t_x are primitive elements of A, Ψ takes values in the group like elements of A if, and only if $\Delta(\Psi(v)) = \Psi(v) \otimes \Psi(v)$ for some $v \in V_\mathcal{A}$. We denote by N the group of group like elements of A with degree zero term equal to 1 and call a solution Ψ of the holonomy equations *unipotent* if it takes values in N.

Clearly, the connection (1.1) determined by a finite–dimensional representation $\rho : F \to \mathrm{End}(U)$ is flat if, and only if ρ factors through F/I. To avoid convergence issues, let h be a formal variable and note that, due to the homogeneity of the relations (1.2), the representation

$$\rho_h : F \to \mathrm{End}(U[h]), \qquad \rho_h(t_x) = h\rho(t_x) \tag{1.4}$$

factors through F/I if, and only if, ρ does. When that is the case, ρ_h extends to a representation of A on $U[\![h]\!]$ which we will denote by the same symbol. Moreover, any invertible solution Ψ of (1.3) gives rise to a fundamental solution $\Psi_{\rho_h} = \rho_h(\Psi)$ of $\nabla_{\rho_h}\Psi_{\rho_h} = 0$ with values in $\mathrm{End}(U)[\![h]\!]$.

1.2. **Monodromy of the holonomy equations.** Throughout this paper, we adopt the convention that, for a topological space X, the composition $\zeta\gamma$ of $\zeta, \gamma \in \pi_1(X, x_0)$ is given by γ *followed* by ζ, so that the holonomy of a flat vector bundle $(\mathcal{V}, \nabla)$ on X at x_0 is a homomorphism $\pi_1(X, x_0) \to GL(\mathcal{V}_{x_0})$.

Fix a basepoint $v_0 \in V_\mathcal{A}$ and let $(\widetilde{V_\mathcal{A}}, \widetilde{v_0}) \xrightarrow{p} (V_\mathcal{A}, v_0)$ be the pointed universal cover of $(V_\mathcal{A}, v_0)$. $\widetilde{V_\mathcal{A}}$ is endowed with a canonical right action of the fundamental group $\pi_1(V_\mathcal{A}, v_0)$ by deck transformation. Any solution Ψ of the holonomy equations (1.3) defined in a neighborhood of v_0 lifts uniquely to, and will be regarded as, a global, A–valued solution of $p^*\nabla\Psi = 0$ on $\widetilde{V_\mathcal{A}}$. If $\gamma \in \pi_1(V_\mathcal{A}, v_0)$, then $\gamma \bullet \Psi(\widetilde{v}) = \Psi(\widetilde{v}\gamma)$ is another such solution. Thus, if Ψ is invertible, then

$$\mu_\Psi(\gamma) = \Psi^{-1} \cdot \gamma \bullet \Psi \tag{1.5}$$

is a locally constant, and therefore constant function on $\widetilde{V_\mathcal{A}}$.

Proposition 1.1.

(i) *The map $\gamma \to \mu_\Psi(\gamma)$ is a homomorphism $\pi_1(V_\mathcal{A}, v_0) \to A$.*
(ii) *If Ψ is unipotent, μ_Ψ takes values in N.*

(iii) *If* $\Psi' = \Psi \cdot K$, $K \in A^\times$ *is another invertible solution, then*

$$\mu_{\Psi'} = \mathrm{Ad}(K^{-1}) \circ \mu_\Psi$$

Note that if $\rho : F/I \to \mathrm{End}(U)$ is a finite–dimensional representation and $\rho_h : A \to \mathrm{End}(U[\![h]\!])$ is given by (1.4), the composition $\rho_h \circ \mu_\Psi : \pi_1(V_\mathcal{A}, v_0) \to GL(U[\![h]\!])$ is the monodromy representation of the connection ∇_{ρ_h} expressed in the fundamental solution $\Psi_{\rho_h} = \rho_h(\Psi)$.

1.3. **Chen–Kohno isomorphisms.** Set $\pi = \pi_1(V_\mathcal{A}, v_0)$, let $\mathbb{C}[\pi]$ be the group algebra of π and J its augmentation ideal, that is the kernel of the counit $\varepsilon : \mathbb{C}[\pi] \to \mathbb{C}$ given by $\varepsilon(\gamma) = 1$ for any $\gamma \in \pi$. Let $\widehat{\mathbb{C}[\pi]}$ be the prounipotent completion of $\mathbb{C}[\pi]$, *i.e.,* the inverse limit

$$\widehat{\mathbb{C}[\pi]} = \lim_{\substack{\longleftarrow \\ k\to\infty}} \mathbb{C}[\pi]/J^k$$

Note that π embeds in $\widehat{\mathbb{C}[\pi]}$ if, and only if $\mathbb{C}[\pi]$ does and that this is the case if, and only if π is residually torsion free nilpotent [Chn2, prop. 2.2.1].

Let $\Psi : \widetilde{V_\mathcal{A}} \to A$ be an invertible solution of the holonomy equations (1.3). Since their reduction by the ideal $A_+ \subset A$ consisting of elements of positive degree is the differential equation $d\Psi = 0$, the diagram

$$\begin{array}{ccc} \mathbb{C}[\pi] & \xrightarrow{\mu_\Psi} & A \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow \\ \mathbb{C} & = & A/A_+ \end{array}$$

is commutative. Thus, μ_Ψ maps J to A_+ and therefore factors through an algebra homomorphism

$$\widehat{\mu_\Psi} : \widehat{\mathbb{C}[\pi]} \longrightarrow A$$

which is a homomorphism of topological Hopf algebras if Ψ is unipotent.

Theorem 1.2 (Chen–Kohno)**.** $\widehat{\mu_\Psi}$ *is an isomorphism.*

A simple proof of theorem 1.2 is given at the end of this section in §1.18.

1.4. **The wonderful model** Y_X**.** Let $\mathcal{L}$ be the (atomic) lattice of subspaces of V^* spanned by subsets of X and set $\mathcal{L}^* = \mathcal{L} \setminus \{0\}$.

Definition 1.3. *A decomposition of* $U \in \mathcal{L}^*$ *is a family* $U_1, \ldots, U_k$ *of subspaces of* U *lying in* $\mathcal{L}^*$ *such that, for every subspace* $W \subseteq U$ *with* $W \in \mathcal{L}^*$ *one has* $W \cap U_i \in \mathcal{L}$ *for every* $i = 1, \ldots, k$ *and*

$$W = (W \cap U_1) \oplus \cdots \oplus (W \cap U_k)$$

An element $U \in \mathcal{L}^*$ *is called reducible if it possesses such a decomposition with* $k \geq 2$ *summands and irreducible otherwise.*

Any $U \in \mathcal{L}^*$ possesses a unique decomposition into irreducible summands. Let $\mathcal{I} \subset \mathcal{L}^*$ be the collection of irreducible subspaces. For any $B \in \mathcal{I}$, let

$$B^\perp = \bigcap_{x \in X \cap B} x^\perp \subset V$$

be the subspace orthogonal to B and note that the rational map $\pi_B : V \to V/B^\perp \to \mathbb{P}(V/B^\perp)$ is regular outside $B^\perp$.

Definition 1.4. Y_X *is the closure of the image of* $V_\mathcal{A}$ *under the embedding*[1]

$$V_\mathcal{A} \longrightarrow V \times \prod_{B \in \mathcal{I}} \mathbb{P}(V/B^\perp)$$

Theorem 1.5.

(i) Y_X *is an irreducible, smooth algebraic variety.*
(ii) *The natural projection* $\pi : Y_X \to V$ *is proper, surjective and restricts to an isomorphism* $Y_X \setminus \pi^{-1}(\mathcal{A}) \to V_\mathcal{A}$.
(iii) $\pi^{-1}(\mathcal{A}) \subset Y_X$ *is a divisor with normal crossings.*

Remark 1.6. The points of Y_X may be parametrised by adapting Fulton and MacPherson's notion of screens for configurations of points [FM, §1] as follows. Let $v \in V$ such that $x(v) = 0$ for at least one $x \in X$, let B be the maximal element of $\mathcal{L}^*$ such that $v \in B^\perp$ and let $B = B_1 \oplus \cdots \oplus B_k$ be its irreducible decomposition. Define recursively a *sequence of screens* for v to be given by a non–zero vector v_i in each $V/B_i^\perp$, defined up to multiplication by a scalar, and, whenever $x(v_i) = 0$ for at least an $x \in X \cap B_i$, a sequence of screens for v_i relative to the hyperplane arrangement on $V/B_i^\perp$ defined by $X \cap B_i$. Then, a point $y \in Y_X$ is readily seen to determine a sequence of screens for $v = \pi(y)$ and to be uniquely determined by it. Moreover, any sequence of screens for a vector $v \in V$ arises in this way.

[1]the notation Y_X follows [DCP2]. It should be manifest however that Y_X and all constructions to follow only depend upon the arrangement $\mathcal{A}$ and not on the actual choice of the set X of linear forms describing it.

1.5. **Geometry and combinatorics of the divisor $\mathcal{D}$.** The following notion is crucial in describing the combinatorics of the divisor $\mathcal{D} = \pi^{-1}(\mathcal{A})$.

Definition 1.7. *A family $\{U_i\}_{i\in I}$ of irreducible elements of $\mathcal{L}^*$ is nested if, for any subfamily $\{U_j\}_{j\in J}$ of pairwise non comparable elements, the subspaces U_j are in direct sum and are the irreducible summands of $\bigoplus_{j\in J} U_j$.*

Theorem 1.8.

(i) *The irreducible components of $\mathcal{D}$ are smooth and labelled by the irreducible elements of $\mathcal{L}^*$, with $\mathcal{D}_B$ the unique component of $\mathcal{D}$ such that $\pi(\mathcal{D}_B) = B^\perp$.*

(ii) *A family $\{\mathcal{D}_B\}_{B\in\mathcal{S}}$ of such components has a non–zero intersection if, and only if, $\mathcal{S}$ is nested. In that case, the intersection is transversal and irreducible.*

Remark 1.9. It is easy to see that the collection $\mathcal{S}_y$ of subspaces appearing in the sequence of screens attached to a point $y \in Y_X$ is nested. If $\mathcal{S} \subset \mathcal{I}$ is nested, the intersection $\bigcap_{B\in\mathcal{S}} \mathcal{D}_B$ consists of those $y \in Y_X$ such that $\mathcal{S}_y$ contains $\mathcal{S}$, its open locus being the set of such $y \in Y$ for which $\mathcal{S}_y = \mathcal{S}$.

1.6. **Some properties of nested sets.** Let $\mathcal{S}$ be a nested set, $B \in \mathcal{S}$ and let $C_1, \ldots, C_m$ be the maximal elements of $\mathcal{S}$ properly contained in B. By nestedness, the C_i are in direct sum and

$$i_\mathcal{S}(B) = C_1 \oplus \cdots \oplus C_m \tag{1.6}$$

is properly contained in B since B is irreducible. Set

$$n(B; \mathcal{S}) = \dim(B/i_\mathcal{S}(B))$$

so that $n(B; \mathcal{S}) \geq 1$ and

$$n(\mathcal{S}) = \sum_{B\in\mathcal{S}} (n(B; \mathcal{S}) - 1)$$

Induction on $|\mathcal{S}|$ readily shows the following

Proposition 1.10. *Let $B_1, \ldots, B_k$ be the maximal elements of $\mathcal{S}$. Then,*

$$n(\mathcal{S}) = \sum_{i=1}^{k} \dim B_i - |\mathcal{S}|$$

Proposition 1.11. *Let $\mathcal{S}$ be a maximal nested set. Then,*

(i) *$n(B; \mathcal{S}) = 1$ for any $B \in \mathcal{S}$.*

(ii) *The maximal elements of $\mathcal{S}$ are the irreducible components of the subspace $\mathcal{X} \subseteq V^*$ spanned by X.*
(iii) $|\mathcal{S}| = \dim \mathcal{X}$.

1.7. **Coordinate charts on Y_X.**

1.7.1. To coordinatize Y_X, it is convenient to assume that the arrangement $\mathcal{A}$ is *essential*, that is satisfies

$$\bigcap_{H \in \mathcal{A}} H = 0 \tag{1.7}$$

so that X spans V^*. This assumption, which we henceforth make, is not truly restrictive since any linear isomorphism $V \cong V/X^\perp \oplus X^\perp$ induces an isomorphism $Y_X \cong Y_{\overline{X}} \times X^\perp$, where $\overline{X} \subset (V/X^\perp)^*$ is the collection of linear forms induced by X. Note that, by proposition 1.11, (1.7) implies that

$$|\mathcal{S}| = \dim V^*$$

for any maximal nested set $\mathcal{S}$. By theorem 1.8, the corresponding intersection $\bigcap_{B \in \mathcal{S}} \mathcal{D}_B$ therefore consists of a single point. Such a point at infinity will be denoted $y_{\mathcal{S}}$.

1.7.2.

Definition 1.12. *A basis b of V^* is adapted to a nested set $\mathcal{S}$ if, for any $B \in \mathcal{S}$, $b \cap B$ is a basis of B*[1].

If $\mathcal{S}$ is a maximal nested set and b an adapted basis, proposition 1.11 (i) implies that, for any $B \in \mathcal{S}$, there exists a unique $x \in b$ such that

$$x \in B \setminus \sum_{\substack{C \in \mathcal{S}, \\ C \subsetneq B}} C$$

Such an element will be denoted x_B. Clearly, $x_B = x_C$ implies that $B = C$ and any $x \in b$ is of this form since $|\mathcal{S}| = \dim V^*$.

1.7.3. Let $U = \mathbb{C}^{\mathcal{S}}$ with coordinates $\{u_B\}_{B \in \mathcal{S}}$ and let $\rho : U \to V$ be the map defined in the coordinates $\{x_B\}_{B \in \mathcal{S}}$ on V by

$$x_B = \prod_{C \supseteq B} u_C$$

[1]departing a little from [DCP2, §1.1], but consistently with [DCP1, §1.3], we do not assume that the elements of an adapted basis lie in X, but denote them nonetheless by the letter x.

ρ is birational, with inverse

$$u_B = \begin{cases} x_B & \text{if } B \text{ is maximal in } \mathcal{S} \\ \frac{x_B}{x_{c(B)}} & \text{otherwise} \end{cases} \tag{1.8}$$

where $c(B)$ is the smallest element of $\mathcal{S}$ properly containing B. It restricts to an isomorphism between the open set of U where all the coordinates u_B are different from zero and the open set of V where all the coordinates x_B are different from zero. Moreover, ρ maps the coordinate hyperplane defined by $u_B = 0$ into the subspace $B^\perp \subset V$.

1.7.4. For any subset Z of V^* containing a non–zero vector, the set of elements of $\mathcal{S}$ containing Z is linearly ordered. Denote by $p_{\mathcal{S}}(x)$ its infimum if it is non–empty and set $p_{\mathcal{S}}(Z) = V^*$ otherwise.

Lemma 1.13.

(i) *If $x \in X$, then $p_{\mathcal{S}}(x) \in \mathcal{S}$.*

(ii) *More generally, if $B \in \mathcal{L}^*$ is irreducible, then $p_{\mathcal{S}}(B) \in \mathcal{S}$ and there is an $x \in B \cap X$ such that $p_{\mathcal{S}}(B) = p_{\mathcal{S}}(x)$.*

Proof. (i) x lies in one of the irreducible components of the span of X. Since these are contained in $\mathcal{S}$ by proposition 1.11, the set of elements of $\mathcal{S}$ containing x is not empty. (ii) Let $C_1, \ldots, C_m \in \mathcal{S}$ be maximal among the $p_{\mathcal{S}}(x)$, $x \in X \cap B$. By nestedness, the C_i are in direct sum. Since $B \subset C_1 \oplus \cdots \oplus C_m$, the irreducibility of B implies that $B \subset C_i$ for some i ■

1.7.5.

Lemma 1.14. *Let $x \in V^* \setminus \{0\}$ be such that $B = p_{\mathcal{S}}(x)$ lies in $\mathcal{S}$. Then, $x = x_B \cdot P_x$ where P_x is a polynomial in the variables u_C, $C \subsetneq B$ such that $P_x(0) \neq 0$.*

Proof. Since x_C, $C \subseteq B$ is a basis of $B \ni x$, we have

$$\begin{aligned} x &= \sum_{C \subseteq B} \alpha_C x_C = x_B(\alpha_B + \sum_{C \subsetneq B} \alpha_C \frac{x_C}{x_B}) \\ &= x_B(\alpha_B + \sum_{C \subsetneq B} \alpha_C \prod_{C \subseteq D \subsetneq B} u_D) = x_B P_x \end{aligned}$$

where $P_x(0) = \alpha_B \neq 0$ or else $p_{\mathcal{S}}(x) \subsetneq B$ ■

The following result explains the relevance of the polynomials P_x.

Proposition 1.15. *Let $B \in \mathcal{L}^*$ be irreducible and $x \in B \cap X$ be such that $p_{\mathcal{S}}(x) = p_{\mathcal{S}}(B)$ as in lemma 1.13. Then, the rational map*

$$U \xrightarrow{\rho} V \longrightarrow \mathbb{P}(V/B^\perp)$$

restricts to a regular morphism on $U \setminus \{P_x = 0\}$.

PROOF. Let $A = p_{\mathcal{S}}(B)$. For any $y \in X \cap B$,

$$y = x_{p_{\mathcal{S}}(y)} P_y = x_A \prod_{p_{\mathcal{S}}(y) \subseteq C \subsetneq A} u_C \cdot P_y =: x_A P_y^B$$

Complete x to a basis $x = x_1, x_2, \ldots, x_n$ of B whose elements lie in X. Then, in the corresponding homogeneous coordinates on $\mathbb{P}(V/B^\perp)$, the composition above is given by

$$u \longrightarrow [x_A P_x(u), x_A P_{x_2}^B(u), \ldots, x_A P_{x_n}^B(u)] = [P_x(u), P_{x_2}^B(u), \ldots, P_{x_n}^B(u)]$$

and is therefore regular on $\{P_x(u) \neq 0\}$ ■

1.7.6.

Definition 1.16. *Let $\mathcal{U}_{\mathcal{S}}^b$ be the complement in U of the zeros of the polynomials P_x as x varies in X.*

By proposition 1.15, the rational map $U \to V \times \prod_{B \in \mathcal{S}} \mathbb{P}(V/B^\perp)$ restricts to a regular map $j_{\mathcal{S}}^b$ on $\mathcal{U}_{\mathcal{S}}^b$. Since, for any $x \in X$

$$x = x_{p_{\mathcal{S}}(x)} P_x = \prod_{B \supseteq p_{\mathcal{S}}^b(x)} u_B \cdot P_x$$

ρ maps the complement of the coordinate hyperplanes in $\mathcal{U}_{\mathcal{S}}^b$ to $V_{\mathcal{A}}$ so that $j_{\mathcal{S}}^b(\mathcal{U}_{\mathcal{S}}^b) \subset Y_X$.

Proposition 1.17. *$j_{\mathcal{S}}^b$ is an open embedding $\mathcal{U}_{\mathcal{S}}^b \hookrightarrow Y_X$.*

We shall henceforth identify $\mathcal{U}_{\mathcal{S}}^b$ with its image in Y_X under the embedding $j_{\mathcal{S}}^b$ and regard the functions u_B, $B \in \mathcal{S}$ as local coordinates on Y_X.

1.7.7. For every maximal nested set $\mathcal{S}$ and $B \in \mathcal{S}$, choose a basis b_B of B consisting of elements not lying in any $C \in \mathcal{S}$, $C \subsetneq B$. Choosing an element x from each b_B, as B varies in $\mathcal{S}$, yields a basis of V^* adapted to $\mathcal{S}$. Varying the choice of such x then gives rise to a finite set $\mathcal{B}_{\mathcal{S}}$ of such bases.

Theorem 1.18.

(i) *Y_X is covered by the open sets $\mathcal{U}_{\mathcal{S}}^b$ as $\mathcal{S}$ varies amongst the maximal nested sets and b varies in $\mathcal{B}_{\mathcal{S}}$.*

(ii) *The intersection $\mathcal{D}_B \cap \mathcal{U}_{\mathcal{S}}^b$ is non–zero if, and only if $B \in \mathcal{S}$. When that is the case, it is given by the equation $u_B = 0$.*

Let $\mathcal{S}$ be a maximal nested set and $y_{\mathcal{S}} = \bigcap_{B\in\mathcal{S}} \mathcal{D}_B$ the corresponding point at infinity in Y_X. By theorem 1.18, $y_{\mathcal{S}}$ lies in $\mathcal{U}^b_{\mathcal{S}'}$ if, and only if $\mathcal{S}' = \mathcal{S}$ and, when that is the case is the point with coordinates $u_B = 0$, $B \in \mathcal{S}$.

Remark 1.19. Although the open set $\mathcal{U}^b_{\mathcal{S}}$ depends upon the choice of the adapted basis b, the union $\mathcal{U}_{\mathcal{S}} = \bigcup_{b\in\mathcal{B}_{\mathcal{S}}} \mathcal{U}^b_{\mathcal{S}}$ has an intrinsic description as the set of those points $y \in Y_X$ such that the subspaces involved in the sequence of screens corresponding to y are all contained in $\mathcal{S}$[1].

1.8. **Some properties of residues.** For any $B \in \mathcal{L}^*$, set

$$t_B = \sum_{x\in X\cap B} t_x = \sum_{i=1}^m t_{B_i}$$

where $B_1, \ldots, B_m$ are the irreducible components of B.

Lemma 1.20. *Let B, C be irreducible and nested, then $[t_B, t_C] = 0$.*

Proof. Assume first that B and C are not comparable. By nestedness, $B \cap C = 0$ and B, C are the irreducible summands of $B \oplus C$. It follows from this that if $x \in B \cap X$ and $y \in C \cap X$, the subspace $\mathbb{C}x \oplus \mathbb{C}y \in \mathcal{L}^*$ is reducible and therefore cannot contain any other elements of X other than x and y. The relations (1.2) therefore imply that t_x and t_y commute so that

$$[t_B, t_C] = \sum_{\substack{x\in B\cap X\\ y\in C\cap X}} [t_x, t_y] = 0$$

Assume now that $B \subset C$. Let $x \in B \cap X$ and define an equivalence relation on $C \cap X \setminus x$ by $y_1 \sim y_2$ if y_1 and y_2 span the same line in $C/\mathbb{C}x$. Let Ξ be the set of equivalence classes, so that

$$t_C = t_x + \sum_{[y]\in\Xi} \sum_{y\in[y]} t_y$$

For a given equivalence class $[y] \in \Xi$, the span of x and y, with $y \in [y]$, is a two–dimensional subspace $C_{[y]} \subset C$ such that $C_{[y]} \cap X = \{x\} \cup [y]$. Thus, $[t_x, \sum_{y\in[y]} t_y] = 0$ by (1.2) and $[t_B, t_C] = 0$ ■

[1]one can show nonetheless that, if $b \subset X$, the open set $\mathcal{U}^b_{\mathcal{S}}$ is independent of b and equal to $\mathcal{U}_{\mathcal{S}}$ [DCP2, §1.1]. In order to use adapted families however (see §1.14) we shall need adapted bases whose elements do not necessarily lie in X.

1.9. **Fundamental solutions of the holonomy equations.** For any nested set $\mathcal{S}$ and $B \in \mathcal{S}$, set

$$R_B^{\mathcal{S}} = \sum_{\substack{x \in X \cap B, \\ p_{\mathcal{S}}(x) = B}} t_x = t_B - t_{i_{\mathcal{S}}(B)} \tag{1.9}$$

where $i_{\mathcal{S}}(B)$ is defined by (1.6).

Theorem 1.21.

(i) *The pull–back to Y_X of ∇ is a flat connection with logarithmic singularities on the divisor $\mathcal{D}$.*

(ii) *Its residue of the irreducible component $\mathcal{D}_B$ of $\mathcal{D}$ is equal to t_B.*

(iii) *Let $\mathcal{S}$ be a maximal nested set and b an adapted basis of V^*. Then, for any simply–connected open set $\mathcal{V} \subset \mathcal{U}_{\mathcal{S}}^b$ containing $y_{\mathcal{S}}$, there exists a unique holomorphic function $H_{\mathcal{S}}^b : \mathcal{V} \to A$ such that $H_{\mathcal{S}}^b(y_{\mathcal{S}}) = 1$ and, for every determination of $\log(x_B)$, $B \in \mathcal{S}$, the multivalued function*

$$\Psi_{\mathcal{S}}^b = H_{\mathcal{S}}^b \cdot \prod_{B \in \mathcal{S}} u_B^{t_B} = H_{\mathcal{S}}^b \cdot \prod_{B \in \mathcal{S}} x_B^{R_B^{\mathcal{S}}}$$

is a solution of the holonomy equation $\nabla\, \Psi_{\mathcal{S}}^b = 0$.

(iv) *$\Psi_{\mathcal{S}}^b$ is unipotent.*

Remark 1.22. Let $b' = \{x'_B\}_{B \in \mathcal{S}}$ be another basis of V^* adapted to $\mathcal{S}$. By lemma 1.14, $x'_B = x_B \cdot P_B$ where P_B is a polynomial in the variables u_C, $C \subsetneq B$, such that $P_B(0) \neq 0$. Thus,

$$\begin{aligned}
\Psi_{\mathcal{S}}^{b'} &= H_{\mathcal{S}}^{b'} \cdot \prod_{B \in \mathcal{S}} (x'_B)^{R_B^{\mathcal{S}}} \\
&= H_{\mathcal{S}}^{b'} \cdot \prod_{B \in \mathcal{S}} \left(\frac{P_B}{P_B(0)} \right)^{R_B^{\mathcal{S}}} \cdot \prod_{B \in \mathcal{S}} x_B^{R_B^{\mathcal{S}}} \cdot \prod_{B \in \mathcal{S}} P_B(0)^{R_B^{\mathcal{S}}}
\end{aligned}$$

Since the function $H_{\mathcal{S}}^{b'} \cdot \prod_{B \in \mathcal{S}} \left(\frac{P_B}{P_B(0)} \right)^{R_B^{\mathcal{S}}}$ is holomorphic in a neighborhood of $y_{\mathcal{S}}$ and equal to 1 at $y_{\mathcal{S}}$, it follows by the uniqueness statement of theorem 1.21 that

$$\Psi_{\mathcal{S}}^{b'} = \Psi_{\mathcal{S}}^b \cdot \prod_{B \in \mathcal{S}} P_B(0)^{R_B^{\mathcal{S}}} \tag{1.10}$$

Remark 1.23. The above solutions generalise those constructed by Drinfeld [Dr3] for the KZ equations, that is the holonomy equations for the Coxeter arrangement of type A_n, and those constructed by Cherednik for a general Coxeter arrangement $\mathcal{A}_W$, see in particular

[Ch1, §2] and [Ch2, §2] where the blowup coordinates (1.8) for $\mathcal{A}_W$ are introduced.

1.10. **Real arrangements and chambers.**

1.10.1. Assume henceforth that $\mathcal{A}$ is the complexification of a hyperplane arrangement $\mathcal{A}_{\mathbb{R}}$ in a real vector space $V_{\mathbb{R}} \subset V$ and choose the linear forms $x \in X$ so that they lie in $V_{\mathbb{R}}^* \subset V^*$. Fix a chamber $\mathcal{C}$ of $\mathcal{A}_{\mathbb{R}}$, that is a connected component of $V_{\mathbb{R}} \setminus \mathcal{A}_{\mathbb{R}}$. Up to replacing some $x \in X$ by their opposites, we may assume that $x_{|\mathcal{C}} > 0$ for any $x \in X$. Let $\Delta = \Delta(\mathcal{C}) \subset X$ be minimal for the property that any $x \in X$ is a linear combination with non–negative coefficients of elements of Δ. Δ is readily seen to consist of those $x \in X$ which are not linear combinations with positive coefficients of two or more elements of X. It is therefore unique and canonically determined by $\mathcal{C}$. An element x of V^* will be termed *real* if $x \in V_{\mathbb{R}}^*$ and *positive* if it is positive on $\mathcal{C}$. Note that $x \in V^*$ is positive and real if, and only if it is a linear combination with non–negative coefficients of the elements of Δ.

We shall assume for simplicity that Δ is a basis of V^*, so that $\mathcal{C}$ is an open simplicial cone. The contents of §1.10–§1.14 extends however, with suitable modifications, to the case of a general polyhedral chamber.

1.10.2. Let $\mathcal{I}_\Delta \subset I$ be the set of irreducible elements B which are spanned by $\Delta_B = \Delta \cap B$.

Definition 1.24. *A nested set $\mathcal{S}$ is fundamental with respect to $\mathcal{C}$ if $\mathcal{S} \subset \mathcal{I}_\Delta$.*

Let $\mathcal{S}$ be a maximal nested set, $y_{\mathcal{S}} \in Y_X$ the corresponding point at infinity and identify the chamber $\mathcal{C}$ with its preimage in Y_X. Our aim in this subsection is to prove the following

Proposition 1.25. *$y_{\mathcal{S}}$ lies in the closure of $\mathcal{C}$ in Y_X if, and only if $\mathcal{S}$ is fundamental with respect to $\mathcal{C}$.*

1.10.3. We shall need two preliminary results.

Lemma 1.26.

(i) *If $y_{\mathcal{S}}$ lies in the closure of $\mathcal{C}$ then, for any positive, real basis b of V^* adapted to $\mathcal{S}$, the polynomials $P_x \in \mathbb{R}[u_B]_{B\in\mathcal{S}}$, $x \in X$ defined by lemma 1.14 satisfy $P_x(0) > 0$.*

(ii) *Conversely, if b is a positive, real basis of V^* adapted to $\mathcal{S}$ and $P_x(0) > 0$ for any $x \in X$, then $y_{\mathcal{S}}$ lies in the closure of $\mathcal{C}$.*

PROOF. (i) follows from the identity $x = x_{p_\mathcal{S}(x)} \cdot P_x$. (ii) The sequence of points $y_n \in \mathcal{U}^b_\mathcal{S}$ with coordinates $u_B = 1/n$, $B \in \mathcal{S}$ converges to $y_\mathcal{S}$ and lies in $\mathcal{C}$ for n large enough since for any $x \in X$,

$$x = x_{p_\mathcal{S}(x)} \cdot P_x = \prod_{B \supseteq p_\mathcal{S}(x)} u_B \cdot P_x$$

■

Lemma 1.27. *If b is a basis of V^* adapted to $\mathcal{S}$, then $P_x(0) > 0$ for any $x \in X$ if, and only if, $P_x(0) > 0$ for any $x \in \Delta$.*

PROOF. Let $x \in X$ and write $x = \sum_{x' \in \Delta} k_{x'} x'$, where $k_{x'} \geq 0$. Let $B_1, \ldots, B_m \in \mathcal{S}$ be the maximal elements among the $p_\mathcal{S}(x')$ with $k_{x'} > 0$ and rewrite the above as

$$x = \sum_{i=1}^m \sum_{x' \in \Delta_{B_i}} k_{x'} x' \in B_1 \oplus \cdots \oplus B_m$$

By nestedness of $\mathcal{S}$, x is contained in one of the B_i whence $m = 1$. Thus, modulo $i_\mathcal{S}(B_1)$,

$$x = \sum_{\substack{x' \in \Delta, \\ P_\mathcal{S}(x') = B_1}} k'_x x' = \sum_{\substack{x' \in \Delta, \\ P_\mathcal{S}(x') = B_1}} k'_x P_{x'}(0) x_{B_1}$$

whence $P_x(0) = \sum_{x'} k'_x P_{x'}(0)$ ■

1.10.4. PROOF OF PROPOSITION 1.25. If $\mathcal{S}$ is fundamental, then Δ is a positive, real basis of V^* adapted to $\mathcal{S}$ and such that $P_x \equiv 1$ for any $x \in \Delta$. By lemmas 1.27 and 1.26, $y_\mathcal{S}$ therefore lies in the closure of $\mathcal{C}$. Assume now that $\mathcal{S}$ is not fundamental, let $B \in \mathcal{S}$ be minimal for the property that B is not spanned by Δ_B and set $C = i_\mathcal{S}(B)$. Let $x \in X \cap (B \setminus C)$ and write $x = \sum_{x' \in \Delta^x} k_{x'} x'$ where $\Delta^x \subseteq \Delta$ and $k_{x'} > 0$ for any $x' \in \Delta^x$. Decompose the sum as

$$x = \sum_{x' \in \Delta^x \cap B} k_{x'} x' + \sum_{x' \in \Delta^x \setminus B} k_{x'} x' = x_1 + x_2 \tag{1.11}$$

We claim that $x_2 \neq 0$. Indeed, by minimality of B, C is spanned by Δ_C so that if $x_2 = 0$, then $B = \mathbb{C}x \oplus C$ would be spanned by Δ_B, a contradiction. We claim next that there exists an $x' \in \Delta^x$ such that $p_\mathcal{S}(x') \supsetneq B$. Indeed, if $x' \in \Delta^x \setminus B$, the nestedness of $\mathcal{S}$ implies that either $B \subsetneq p_\mathcal{S}(x')$ or that the two are in direct sum. If the latter were the case for any $x' \in \Delta^x \setminus B$, (1.11) would yield $x_2 = 0$, a contradiction. Let therefore $x' \in \Delta^x \setminus B$ be such that $A = p_\mathcal{S}(x') \supsetneq B$, and set $x_A = x'$,

$x_B = x$ and complete this into a positive, real basis of V^* adapted to $\mathcal{S}$. Then, on $\mathcal{U}^b_{\mathcal{S}} \cap \mathcal{C}$,

$$k_{x'} < \frac{x_B}{x_A} = \prod_{B \subseteq D \subsetneq A} u_D$$

Thus, the product of the coordinates u_D, $B \subseteq D \subsetneq A$ is real and bounded below on $\mathcal{C} \cap \mathcal{U}^b_{\mathcal{S}}$ so that $y_{\mathcal{S}}$ does not lie in the closure of $\mathcal{C}$ by lemma 1.26 ■

1.11. **De Concini–Procesi associators.** Let $\mathcal{F}$ be a fundamental maximal nested set and $b = \{x_B\}_{B\in\mathcal{F}}$ a positive, real basis of V^* adapted to $\mathcal{F}$. Let $\mathcal{V}^b_{\mathcal{F}} \subset \mathcal{U}^b_{\mathcal{F}}$ be the complement of the real codimension one semialgebraic subvarieties $\{x_B \leq 0\}$, $B \in \mathcal{F}$. Note that the chamber $\mathcal{C}$ lies in $\mathcal{V}^b_{\mathcal{F}}$ since $x_B > 0$ on $\mathcal{C}$ for any $B \in \mathcal{F}$. We shall henceforth only consider the standard determination of the function $\log(z)$ obtained by performing a cut along the negative real axis, so that the functions $\log(x_B)$, $B \in \mathcal{F}$ are well–defined and single–valued on $\mathcal{V}^b_{\mathcal{F}}$. Theorem 1.21 then yields a single–valued fundamental solution $\Psi^b_{\mathcal{F}}$ of the holonomy equations (1.3) on the intersection of a neighborhood of $y_{\mathcal{F}}$ in $\mathcal{U}^b_{\mathcal{F}}$ with $\mathcal{V}^b_{\mathcal{F}}$. Since $y_{\mathcal{F}}$ lies in the closure of $\mathcal{C}$ by proposition 1.25, $\Psi^b_{\mathcal{F}}$ may be continued to a single–valued solution on $\mathcal{C}$ which we shall denote by the same symbol.

Let now $\mathcal{F}, \mathcal{G}$ be two fundamental maximal nested sets and b, b' two positive, real bases of V^* adapted to $\mathcal{F}$ and $\mathcal{G}$ respectively.

Definition 1.28. *The De Concini–Procesi associator $\Phi^{b'b}_{\mathcal{G}\mathcal{F}}$ is the element of A defined by*

$$\Phi^{b'b}_{\mathcal{G}\mathcal{F}} = (\Psi^{b'}_{\mathcal{G}}(y))^{-1} \cdot \Psi^b_{\mathcal{F}}(y)$$

for any $y \in \mathcal{C}$.

Note that $\Phi^{b'b}_{\mathcal{G}\mathcal{F}}$ is well–defined since the right–hand side of the above expression is a locally constant function on $\mathcal{C}$. The following properties are immediate

- **orientation** : for any pair $\mathcal{F}, \mathcal{G}$ and adapted bases b, b'

$$\Phi^{bb'}_{\mathcal{F}\mathcal{G}} = (\Phi^{b'b}_{\mathcal{G}\mathcal{F}})^{-1} \tag{1.12}$$

- **transitivity** : for any triple $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and adapted bases b, b', b''

$$\Phi^{b''b}_{\mathcal{H}\mathcal{F}} = \Phi^{b''b'}_{\mathcal{H}\mathcal{G}} \cdot \Phi^{b'b}_{\mathcal{G}\mathcal{F}} \tag{1.13}$$

1.12. **Elementary associators.** The De Concini–Procesi associators possess a number of other important properties which will be given in §1.15–1.16 and are easier to formulate in terms of elementary associators.

Definition 1.29. *A pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets (resp. an associator $\Phi^{b'b}_{\mathcal{G}\mathcal{F}}$) is called elementary if $\mathcal{G}$ and $\mathcal{F}$ differ by one element.*

Elementary associators are sufficient to reconstruct general associators in view of the transitivity relation (1.13) and the following result

Proposition 1.30. *For any pair $\mathcal{F},\mathcal{G}$ of (fundamental) maximal nested sets, there exists a sequence*

$$\mathcal{F}=\mathcal{H}_1,\ldots,\mathcal{H}_m=\mathcal{G}$$

of (fundamental) maximal nested sets such that $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ differ by an element.

The transitivity of associators implies that the elementary ones satisfy the following property :

- **coherence (I)** : if $\mathcal{H}_1,\ldots,\mathcal{H}_\ell$ and $\mathcal{K}_1,\ldots,\mathcal{K}_m$ are two sequences of fundamental maximal nested sets such that $|\mathcal{H}_{i+1}\setminus\mathcal{H}_i|=1$ for any $1\leq i\leq \ell-1$, $|\mathcal{K}_{j+1}\setminus\mathcal{K}_j|=1$ for any $1\leq j\leq m-1$,

  $$\mathcal{H}_1=\mathcal{K}_1 \qquad\text{and}\qquad \mathcal{H}_\ell=\mathcal{K}_m$$

  then,

  $$\Phi^{b_\ell b_{\ell-1}}_{\mathcal{H}_\ell\mathcal{H}_{\ell-1}}\cdots\Phi^{b_2b_1}_{\mathcal{H}_2\mathcal{H}_1}=\Phi^{b'_m b'_{m-1}}_{\mathcal{K}_m\mathcal{K}_{m-1}}\cdots\Phi^{b'_2b'_1}_{\mathcal{K}_2\mathcal{K}_1}$$

  provided $b_1=b'_1$ and $b_\ell=b'_m$.

1.13. **2–dimensional reduction.** Let $(\mathcal{G},\mathcal{F})$ be an elementary pair of fundamental maximal nested sets and $b'=\{x'_B\}$, $b=\{x_B\}$ two positive, real bases of V^* adapted to $\mathcal{G}$ and $\mathcal{F}$ respectively. Assume that

$$x_B=x'_B \qquad\text{for any}\quad B\in\mathcal{F}\cap\mathcal{G} \tag{1.14}$$

Then, as explained below, the associator $\Phi^{b'b}_{\mathcal{G}\mathcal{F}}$ coincides with one obtained from a line arrangement in a two–dimensional vector space determined by $\mathcal{F}$ and $\mathcal{G}$. This inductive structure of associators is a consequence of the factorisation property of the solutions $\Psi^b_{\mathcal{F}}$ which was pointed out for the KZ equations by Tsuchiya–Kanie [TK] and extended to the holonomy equations for Coxeter arrangements by Cherednik (Thm. 1 in [Ch1, Ch2]).

1.13.1. Note first that, by propositions 1.10 and 1.11, there is a unique $B \in \mathcal{F} \cap \mathcal{G}$ such that

$$n(B; \mathcal{F} \cap \mathcal{G}) = 2 \qquad \text{while} \qquad n(B'; \mathcal{F} \cap \mathcal{G}) = 1$$

for any other $B' \in \mathcal{F} \cap \mathcal{G}$. Set $C = i_{\mathcal{F} \cap \mathcal{G}}(B)$, so that $\dim(B/C) = 2$ and let $X_{B,C}$ be the quotient of $X \cap (B \setminus C)$ by the equivalence relation $x \sim x'$ if x is proportional to x' mod C. Then, $X_{B,C}$ defines an essential line arrangement $\mathcal{A}_{B,C}$ in the 2–dimensional vector space $C^\perp/B^\perp$.

1.13.2. Let $C = C_1 \oplus \cdots \oplus C_m$ be the irreducible decomposition of C.

Lemma 1.31. *Let $x \in X \cap (B \setminus C)$ and consider the irreducible decomposition*

$$\mathbb{C}x \oplus C = C_x \oplus C_{x,1} \oplus \cdots \oplus C_{x,p}$$

where C_x is the summand containing x. Then,

(i) *The $C_{x,j}$ are exactly the irreducible summands C_i of C such that $\mathbb{C}x \oplus C_i$ is a decomposition in $\mathcal{L}^*$.*
(ii) *$C_x = \mathbb{C}x \bigoplus_i C_i$ where C_i ranges over the irreducible summands of C such that $\mathbb{C}x \oplus C_i$ is not a decomposition in $\mathcal{L}^*$.*
(iii) *C_x only depends upon the equivalence class of x in $X_{B,C}$.*

Proof. (i) and (ii) follow easily by comparing the decompositions of C and $C \oplus \mathbb{C}_x$. (iii) If $x' \sim x$, then

$$x' \in \mathbb{C}x' \oplus C = \mathbb{C}x \oplus C = C_x \oplus C_{x,1} \oplus \cdots \oplus C_{x,p}$$

so that $x' \in C_x$ or, for some j, $x' \in C_{x,j} \subset C$ by (i). The latter however is ruled out by the fact that $x' \notin C$ ■

1.13.3. For any equivalence class $\overline{x} \in X_{B,C}$, set

$$t_{\overline{x}} = \sum_{x \in \overline{x}} t_x$$

and consider the connection on $C^\perp/B^\perp$ with logarithmic singularities on $\mathcal{A}_{B,C}$ defined by

$$\nabla_{B,C} = d - \sum_{\overline{x} \in X_{B,C}} \frac{d\overline{x}}{\overline{x}} \cdot t_{\overline{x}} \tag{1.15}$$

For any $\overline{x} \in X_{B,C}$, set $C_{\overline{x}} = C_x$ where $x \in \overline{x}$.

Proposition 1.32.

(i) *For any $\overline{x} \in X_{B,C}$ one has*

$$t_{\overline{x}} = t_{C_{\overline{x}}} - \sum_{i: C_i \subset C_{\overline{x}}} t_{C_i}$$

(ii) $t_{\overline{x}}$ *commutes with any* $t_{x'}$, $x' \in X \cap C$.
(iii) *The connection* $\nabla_{B,C}$ *is flat.*

Proof. (i) We have

$$t_{C_{\overline{x}}} = \sum_{y \in X \cap C_{\overline{x}}} t_y = \sum_{y \in X \cap C_{\overline{x}} \setminus C} t_y + \sum_{y \in X \cap C_{\overline{x}} \cap C} t_y$$

The first summand is equal to $t_{\overline{x}}$. By lemma 1.31, $y \in X \cap C$ lies in $C_{\overline{x}}$ if, and only if the irreducible component C_i of C containing y is contained in $C_{\overline{x}}$. The second sum is therefore equal to $\sum_{i:C \subset C_{\overline{x}}} t_{C_i}$ as claimed. (ii) follows from (i) and lemma 1.20. (iii) By [Ko2], the flatness of $\nabla_{B,C}$ equivalent to the fact that each $t_{\overline{x}}$ commutes with

$$\sum_{\overline{x} \in X_{B,C}} t_{\overline{x}} = t_B - t_C$$

which follows from (ii) and lemma 1.20 ■

1.13.4. We claim next that the lattice $\mathcal{L}_{B,C}$ of subspaces of B/C spanned by elements of $X_{B,C}$ contains two distinguished one–dimensional elements determined by $\mathcal{F}$ and $\mathcal{G}$ respectively. Indeed, let B_1, B_2 be the unique elements in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G} \setminus \mathcal{F}$. Note that B_1 (resp. B_2) is one of the maximal elements of $\mathcal{F}$ (resp. $\mathcal{G}$) properly contained in B since $n(B;\mathcal{F}) = 1$ (resp. $n(B;\mathcal{G}) = 1$) while $n(B;\mathcal{F} \cap \mathcal{G}) = 2$. Set, for $i = 1, 2$

$$\overline{B}_i = B_i / B_i \cap C \subseteq B/C$$

By nestedness of $\mathcal{F}$ (resp. $\mathcal{G}$), B_1 (resp. B_2) either contains, or is in direct sum with, each C_j. In particular, the maximal elements of $\mathcal{F}$ (resp. $\mathcal{G}$) properly contained in B_1 (resp. B_2) are exactly the C_j contained in B_1 (resp. B_2) so that, by proposition 1.11,

$$\dim(\overline{B}_1) = 1 = \dim(\overline{B}_2)$$

Note that if $\mathcal{A}_{B,C}$ contains at least three lines, so that $B/C = (C^\perp / B^\perp)^*$ is irreducible, then

$$\overline{\mathcal{F}} = \{\overline{B}_1, B/C\} \qquad \text{and} \qquad \overline{\mathcal{G}} = \{\overline{B}_2, B/C\}$$

are maximal nested sets of irreducible elements of $\mathcal{L}^*_{B,C}$.

1.13.5. Note next that $\mathcal{A}_{B,C}$ is the complexification of a line arrangement in $(C^\perp/B^\perp)_\mathbb{R}$ endowed with a distinguished chamber $\mathcal{C}_{B,C}$, namely the interior of the image of $\overline{\mathcal{C}} \cap C^\perp$ in $C^\perp/B^\perp$. Moreover, if x_i is the unique element in $\Delta_{B_i} \setminus \Delta_C$, $i = 1, 2$, then any element of $X_{B,C}$ is a linear combination with non–negative coefficients of $\overline{x}_1, \overline{x}_2$ so that the real lines $\overline{x}_1^\perp$ and $\overline{x}_2^\perp$ are two contiguous walls of $\mathcal{C}_{B,C}$. Since $\overline{B}_i = \mathbb{C}\overline{x}_i$,

$\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are fundamental maximal nested sets with respect to the chamber $\mathcal{C}_{B,C}$ whenever $|X_{B,C}| \geq 3$.

1.13.6. Let now $b = \{x_A\}$ and $b' = \{x'_A\}$ be positive, real bases of V^* adapted to $\mathcal{F}$ and $\mathcal{G}$ respectively and such that (1.14) holds. b and b' induce bases $\overline{b} = \{\overline{x}_B, \overline{x}_{B_1}\}$ and $\overline{b}' = \{\overline{x}_B, \overline{x'}_{B_2}\}$ of $C^\perp/B^\perp$ adapted to $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ respectively which are positive and real with respect to the chamber $\mathcal{C}_{B,C}$.

Theorem 1.33.

(i) *If the arrangement $\mathcal{A}_{B,C}$ contains at least three lines, then*

$$\Phi^{b'b}_{\mathcal{G}\mathcal{F}} = \Phi^{\overline{b}'\overline{b}}_{\overline{\mathcal{G}}\overline{\mathcal{F}}}$$

where the right–hand side is the De Concini–Procesi associator for the connection $\nabla_{B,C}$ relative to the solutions $\Psi^{\overline{b}'}_{\overline{\mathcal{G}}}$ and $\Psi^{\overline{b}}_{\overline{\mathcal{F}}}$.

(ii) *Otherwise,*

$$\Phi^{b'b}_{\mathcal{G}\mathcal{F}} = a_2^{t_{\overline{x}_{B_2}}} \cdot (a_1^{t_{\overline{x}_{B_1}}})^{-1}$$

where $a_1, a_2 \in \mathbb{R}^*_+$ are such that $x_B = a_1 x_{B_1} + a_2 x'_{B_2} \bmod C$.

1.14. **Adapted families.** Recall that $\mathcal{I}_\Delta$ is the set of irreducible elements $B \in \mathcal{L}^*$ which are spanned by $\Delta_B = \Delta \cap B$. If $\mathcal{F} \subset \mathcal{I}_\Delta$ is a fundamental nested set and $B \in \mathcal{F}$, we set

$$\underline{\alpha}^B_{\mathcal{F}} = \Delta_B \setminus \Delta_{i_{\mathcal{F}}(B)}$$

so that $|\underline{\alpha}^B_{\mathcal{F}}| = n(B;\mathcal{F})$. If $\mathcal{F}$ is maximal, we denote by $\alpha^B_{\mathcal{F}}$ the unique element in $\underline{\alpha}^B_{\mathcal{F}}$. The following notion is useful to obtain adapted bases which satisfy (1.14).

Definition 1.34. *An adapted family is a collection $\beta = \{x_B\}_{B \in \mathcal{I}_\Delta}$ such that, for any $B \in \mathcal{I}_\Delta$,*

$$x_B \in B \setminus \bigcup_{C \in \mathcal{I}_\Delta, C \subsetneq B} C$$

Clearly, if $\beta = \{x_B\}_{B \in \mathcal{I}_\Delta}$ is an adapted family then, for any fundamental maximal nested set $\mathcal{F}$, $\beta_{\mathcal{F}} = \{x_B\}_{B \in \mathcal{F}}$ is a basis of V^* adapted to $\mathcal{F}$ and all collections of adapted bases $b_{\mathcal{F}} = \{x^{\mathcal{F}}_A\}_{A \in \mathcal{F}}$ labelled by fundamental maximal nested sets and satisfying (1.14) are obtained in this way. If β is a positive, real adapted family and $\mathcal{F}, \mathcal{G}$ are fundamental maximal nested sets, set

$$\Phi^\beta_{\mathcal{G}\mathcal{F}} = \Phi^{\beta_{\mathcal{G}}\beta_{\mathcal{F}}}_{\mathcal{G}\mathcal{F}}$$

We shall henceforth only use the associators $\Phi^\beta_{\mathcal{G}\mathcal{F}}$ corresponding to a fixed such β. These clearly satisfy the orientation property $\Phi^\beta_{\mathcal{F}\mathcal{G}} =$

$(\Phi^\beta_{\mathcal{G}\mathcal{F}})^{-1}$ of §1.12 and the following simpler version of the coherence property

- **coherence (II)** : if $\mathcal{H}_1,\ldots,\mathcal{H}_\ell$ and $\mathcal{K}_1,\ldots,\mathcal{K}_m$ are two sequences of fundamental maximal nested sets such that $|\mathcal{H}_{i+1}\setminus\mathcal{H}_i| = 1$ for any $1\leq i\leq \ell-1$, $|\mathcal{K}_{j+1}\setminus\mathcal{K}_j| = 1$ for any $1\leq j\leq m-1$,
$$\mathcal{H}_1 = \mathcal{K}_1 \qquad\text{and}\qquad \mathcal{H}_\ell = \mathcal{K}_m$$
then,
$$\Phi^\beta_{\mathcal{H}_\ell\mathcal{H}_{\ell-1}}\cdots\Phi^\beta_{\mathcal{H}_2\mathcal{H}_1} = \Phi^\beta_{\mathcal{K}_m\mathcal{K}_{m-1}}\cdots\Phi^\beta_{\mathcal{K}_2\mathcal{K}_1} \tag{1.16}$$

Remark 1.35. Let $\beta' = \{x'_B\}$ be another positive, real adapted family. For any $B\in\mathcal{I}_\Delta$ and $\alpha\in\Delta_B$, let $c_{(B;\alpha)}\in\mathbb{R}^*_+$ be such that $x'_B = c_{(B;\alpha)}x_B$ modulo the span of $\Delta_B\setminus\alpha$. Then, by (1.10), the associators $\Phi^\beta_{\mathcal{G}\mathcal{F}}$ and $\Phi^{\beta'}_{\mathcal{G}\mathcal{F}}$ are related by
$$\Phi^{\beta'}_{\mathcal{G}\mathcal{F}} = a_\mathcal{G}\cdot\Phi^\beta_{\mathcal{G}\mathcal{F}}\cdot a_\mathcal{F}^{-1} \tag{1.17}$$
where, for any fundamental maximal nested set $\mathcal{F}$,
$$a_\mathcal{F} = \prod_{B\in\mathcal{F}} {c_{(B;\alpha^B_\mathcal{F})}}^{-R^\mathcal{F}_B}$$

1.15. **Support properties of elementary associators.** Let $(\mathcal{F},\mathcal{G})$ be an elementary pair of fundamental maximal nested sets.

Definition 1.36. *The support of $(\mathcal{F},\mathcal{G})$ is the unique element $B = \operatorname{supp}(\mathcal{F},\mathcal{G})$ of $\mathcal{F}\cap\mathcal{G}$ such that $n(B;\mathcal{F}\cap\mathcal{G}) = 2$. The central support of $(\mathcal{F},\mathcal{G})$ is the subspace $\mathfrak{z}\operatorname{supp}(\mathcal{F},\mathcal{G}) = i_{\mathcal{F}\cap\mathcal{G}}(\operatorname{supp}(\mathcal{F},\mathcal{G}))$.*

For any $D\in\mathcal{L}^*$, let
$$A_D\subseteq A$$
be the subalgebra topologically generated by the elements t_x, $x\in D\cap X$. Let β be a positive, real adapted family. It follows from theorem 1.33 and (ii) of proposition 1.32 that the associator $\Phi^\beta_{\mathcal{G}\mathcal{F}}$ satisfies the following property

- **support** : $\Phi^\beta_{\mathcal{G}\mathcal{F}}$ lies in $A_{\operatorname{supp}(\mathcal{F},\mathcal{G})}$ and commutes with $A_{\mathfrak{z}\operatorname{supp}(\mathcal{F},\mathcal{G})}$.

1.16. **Forgetfulness properties of elementary associators.**

Definition 1.37. *Two elementary pairs $(\mathcal{F},\mathcal{G})$, $(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ of fundamental maximal nested sets are equivalent if*
$$\operatorname{supp}(\mathcal{F},\mathcal{G}) = \operatorname{supp}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}),$$
$$\alpha_\mathcal{F}^{\operatorname{supp}(\mathcal{F},\mathcal{G})} = \alpha_{\widetilde{\mathcal{F}}}^{\operatorname{supp}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})} \qquad \textit{and} \qquad \alpha_\mathcal{G}^{\operatorname{supp}(\mathcal{F},\mathcal{G})} = \alpha_{\widetilde{\mathcal{G}}}^{\operatorname{supp}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})}$$

The following result guarantees that the equivalence of $(\mathcal{F},\mathcal{G})$ and $(\mathcal{F}',\mathcal{G}')$ implies the equality of the reduction data used in §1.13.

Proposition 1.38. *Let $(\mathcal{F},\mathcal{G})$ be an elementary pair of fundamental, maximal nested sets. Set $B=\operatorname{supp}(\mathcal{F},\mathcal{G})$ and let B_1,B_2 the unique elements in $\mathcal{F}\setminus\mathcal{G}$ and $\mathcal{G}\setminus\mathcal{F}$ respectively. Then,*

(i) *$\alpha^B_{\mathcal{F}}$ and $\alpha^B_{\mathcal{G}}$ are distinct and $\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}}=\{\alpha^B_{\mathcal{F}},\alpha^B_{\mathcal{G}}\}$.*

(ii) *$\mathfrak{z}\operatorname{supp}(\mathcal{F},\mathcal{G})$ is the span of $\Delta_{\operatorname{supp}(\mathcal{F},\mathcal{G})}\setminus\{\alpha^B_{\mathcal{F}},\alpha^B_{\mathcal{G}}\}$.*

(iii) *B_1 is the irreducible component of $\langle\Delta_B\setminus\alpha^B_{\mathcal{F}}\rangle$ containing $\alpha^B_{\mathcal{G}}$. Moreover, $\alpha^{B_1}_{\mathcal{F}}=\alpha^B_{\mathcal{G}}$.*

(iv) *B_2 is the irreducible component of $\langle\Delta_B\setminus\alpha^B_{\mathcal{G}}\rangle$ containing $\alpha^B_{\mathcal{F}}$. Moreover, $\alpha^{B_2}_{\mathcal{G}}=\alpha^B_{\mathcal{F}}$.*

Proof. (i) Clearly, $\alpha^B_{\mathcal{F}},\alpha^B_{\mathcal{G}}\in\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}}$. Since the maximal elements of $\mathcal{F}$ (resp. $\mathcal{G}$) properly contained in B are the irreducible components of the span of $\Delta_B\setminus\alpha^B_{\mathcal{F}}$ (resp. $\Delta_B\setminus\alpha^B_{\mathcal{G}}$), the equality $\alpha^B_{\mathcal{F}}=\alpha^B_{\mathcal{G}}$ would imply that these components also lie in $\mathcal{G}$ (resp. $\mathcal{F}$) and therefore that B is saturated as an element of $\mathcal{F}\cap\mathcal{G}$, a contradiction. Since $n(B;\mathcal{F}\cap\mathcal{G})=2$, this implies that $\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}}=\{\alpha^B_{\mathcal{F}},\alpha^B_{\mathcal{G}}\}$.
(ii) follows from (i) and the fact that $\mathfrak{z}\operatorname{supp}(\mathcal{F},\mathcal{G})$ is spanned by $\Delta_B\setminus\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}}$.
(iii)–(iv) Since $\alpha^B_{\mathcal{G}}\neq\alpha^B_{\mathcal{F}}$, there exists an irreducible component $B_1'\in\mathcal{F}$ of $\langle\Delta_B\setminus\alpha^B_{\mathcal{F}}\rangle$ such that $\alpha^B_{\mathcal{G}}\in B_1'$. However since $\alpha^B_{\mathcal{G}}\notin C$ for any $C\in\mathcal{G}$, $C\subsetneq B$, $B_1'\in\mathcal{F}\setminus\mathcal{G}=\{B_1\}$. Since B_1 is one of the proper maximal elements of $\mathcal{F}$ contained in B, we have $\alpha^{B_1}_{\mathcal{F}}\in\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}}=\{\alpha^B_{\mathcal{F}},\alpha^B_{\mathcal{G}}\}$ whence $\alpha^{B_1}_{\mathcal{F}}=\alpha^B_{\mathcal{G}}$ since $\alpha^B_{\mathcal{F}}\notin B_1$. Similarly, B_2 is the unique irreducible component of $\langle\Delta_B\setminus\alpha^B_{\mathcal{G}}\rangle$ containing $\alpha^B_{\mathcal{F}}$ and $\alpha^{B_2}_{\mathcal{G}}=\alpha^B_{\mathcal{F}}$ ■

Let β be a positive, real adapted family. Proposition 1.38 and theorem 1.33 imply that the associators $\Phi^\beta_{\mathcal{G}\mathcal{F}}$ satisfy the following additional property

- **forgetfulness** : if $(\mathcal{G},\mathcal{F})$ and $(\widetilde{\mathcal{G}},\widetilde{\mathcal{F}})$ are equivalent, $\Phi^\beta_{\mathcal{G}\mathcal{F}}=\Phi^\beta_{\widetilde{\mathcal{G}}\widetilde{\mathcal{F}}}$.

1.17. **Coxeter arrangements.**

1.17.1. Assume now that $\mathcal{A}_{\mathbb{R}}$ is the arrangement of reflecting hyperplanes of a finite (real) reflection group $W\subset GL(V_{\mathbb{R}})$. The set X of defining equations of $\mathcal{A}$ may then be chosen so that $\Phi=X\sqcup(-X)$ is invariant under W. Thus, Φ is a (reduced) root system with respect to any W–invariant Euclidean inner product $(\cdot,\cdot)$ on $V^*_{\mathbb{R}}$, that is a finite collection of non–zero vectors in $V^*_{\mathbb{R}}$ satisfying, for any $\alpha\in\Phi$

(R1) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$
(R2) $s_\alpha \Phi = \Phi$

where $s_\alpha \in W^t$ is the orthogonal reflection determined by α and $W \cong W^t \subset O(V_{\mathbb{R}}^*)$ is the group contragredient to W [1]. In accordance with (1.7) we assume in addition that Φ spans $V_{\mathbb{R}}^*$ so that no $v \in V \setminus \{0\}$ is fixed by W.

1.17.2. We shall need some mostly standard terminology. A root subsystem of Φ is a subset $\overline{\Phi} \subseteq \Phi$ satisfying (R1)–(R2) above and such that the intersection of its linear span $\langle\overline{\Phi}\rangle \subset V_{\mathbb{R}}^*$ with Φ is $\overline{\Phi}$[2]. A root subsystem $\overline{\Phi}$ is reducible if it possesses a non–trivial partition $\overline{\Phi} = \overline{\Phi}_1 \sqcup \overline{\Phi}_2$ into mutually orthogonal subsets, which are then necessarily root subsystems of $\overline{\Phi}$, and irreducible otherwise. Two root subsystems $\overline{\Phi}_1, \overline{\Phi}_2 \subseteq \Phi$ are said to be *completely orthogonal* if no element $\alpha \in \Phi$ is of the form $\alpha = a_1\alpha_1 + a_2\alpha_2$ with $\alpha_i \in \overline{\Phi}_i$ and $a_i \in \mathbb{R}^*$[3]. This implies in particular that $\overline{\Phi}_1 \perp \overline{\Phi}_2$.

Let now $\mathcal{C}$ be a chamber of $\mathcal{A}_{\mathbb{R}}$, $\Phi = \Phi_+ \sqcup \Phi_-$ the corresponding partition into positive and negative roots and $\Delta = \{\alpha_i\}_{i\in\mathbf{I}} \subset \Phi_+$ the basis of V^* consisting of indecomposable elements of Φ_+. We shall say that a root subsystem $\overline{\Phi} \subseteq \Phi$ is *fundamental* if $\langle\overline{\Phi}\rangle$ is spanned by $\Delta \cap \overline{\Phi}$. It is easy to see that two fundamental root subsystems $\overline{\Phi}_1, \overline{\Phi}_2 \subseteq \Phi$ are completely orthogonal if, and only if they are orthogonal.

1.17.3. In accordance with §1.10.1, we assume that $X = \Phi_+$. Let $\mathcal{L}$ be the lattice of subspaces of V^* spanned by the elements of X, as in §1.4, and $\mathcal{R}$ the lattice of root subsystems of Φ. The following result provides a dictionary between the terminology of §1.4–§1.5 and that of §1.17.2.

Proposition 1.39.

(i) *The map $\overline{\Phi} \to \langle\overline{\Phi}\rangle$ is a bijection between $\mathcal{R}$ and $\mathcal{L}$, with inverse given by $B \to \Phi_B = B \cap \Phi$.*

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[1]we follow here the terminology of [Hu]. Thus, Φ need not be crystallographic, *i.e.,* such that $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$ for any $\alpha,\beta \in \Phi$.

[2]Thus, we do not regard the long roots in the root system of type G_2 as a root subsystem.

[3]When Φ is a crystallographic root system and $\overline{\Phi}_i = \{\pm\beta_i\}$, this notion is more stringent than the strong orthogonality of β_1 and β_2 *i.e.,* the requirement that $\beta_1 \pm \beta_2 \notin \Phi$. For example, if Φ is the root system of type G_2 and α_1, α_2 are the short and long simple roots respectively, then $\beta_1 = 2\alpha_1 + \alpha_2$ and $\beta_2 = \alpha_2$ are strongly orthogonal but $\{\pm\beta_1\}$ and $\{\pm\beta_2\}$ are not completely orthogonal.

(ii) *A root subsystem* $\overline{\Phi}\subset\Phi$ *admits a partition* $\overline{\Phi}=\overline{\Phi}_1\sqcup\overline{\Phi}_2$ *into two orthogonal subsets if, and only if,* $\langle\overline{\Phi}\rangle=\langle\overline{\Phi}_1\rangle\oplus\langle\overline{\Phi}_2\rangle$ *is a decomposition in* $\mathcal{L}$*. In particular,* $\overline{\Phi}$ *is irreducible if, and only if* $\langle\overline{\Phi}\rangle$ *is an irreducible element of* $\mathcal{L}^*$*.*

(iii) *A collection* $\mathcal{S}$ *of irreducible elements of* $\mathcal{L}^*$ *is nested if, and only if the root subsystems* $\{\Phi_B\}_{B\in\mathcal{S}}$ *are pairwise completely orthogonal when non–comparable. In particular, if each* Φ_B *is fundamental, then* $\mathcal{S}$ *is nested if, and only if, the* Φ_B *are pairwise orthogonal when non–comparable.*

1.17.4. Recall that the Coxeter graph D of Φ is the graph with vertex set Δ and an edge between α_i and α_j if, and only if $\alpha_i\not\perp\alpha_j$. It follows from proposition 1.39 that the map $B\rightarrow B\cap\Delta$ induces a bijection between fundamental nested sets of irreducible elements of $\mathcal{L}^*$ and sets of connected subgraphs of D which are pairwise *compatible* that is such that $D'\subseteq D''$, $D''\subseteq D'$ or $D'\perp D''$, where the last statement means that no vertex of D' is connected to a vertex of D'' by an edge of D.

Remark 1.40. Such collections admit the following well–known alternative description when D is the Dynkin diagram of type A_{n-1}. Identify for this purpose D with the interval $[1,n-1]$ and its connected subdiagrams with the subintervals $[i,j]$, with $1\leq i\leq j\leq n-1$. This induces a bijection between the sets of pairwise compatible connected subdiagrams of D and consistent bracketings on the non–associative monomial $x_1\cdots x_n$ by attaching to $B=[i,j]$ the bracket $x_1\cdots x_{i-1}(x_i\cdots x_{j+1})x_{j+2}\cdots x_n$.

1.17.5. Fix a basepoint $v_0\in V_{\mathcal{A}}$, let $[v_0]$ be its image in $V_{\mathcal{A}}/W$ and let

$$P_W=\pi_1(V_{\mathcal{A}},v_0)\qquad\text{and}\qquad B_W=\pi_1(V_{\mathcal{A}}/W,[v_0])$$

be the generalised (topological) pure and full braid groups of type W respectively[1]. Since W acts freely on $V_{\mathcal{A}}$ [Ste], the quotient map $V_{\mathcal{A}}\xrightarrow{\pi}V_{\mathcal{A}}/W$ is a covering and gives rise to an exact sequence

$$1\longrightarrow P_W\longrightarrow B_W\longrightarrow W\longrightarrow 1 \tag{1.18}$$

where the rightmost arrow is obtained by associating to $\gamma\in B_W$ the unique $w\in W$ such that $w^{-1}v_0=\widetilde{\gamma}(1)$, with $\widetilde{\gamma}$ the unique lift of γ to a path in $V_{\mathcal{A}}$ such that $\widetilde{\gamma}(0)=v_0$.

[1]to distinguish $\pi_1(V_{\mathcal{A}}/W,[v_0])$ from the isomorphic abstract group introduced in §1.17.9, we denote them by B_W and B_D and refer to them as the topological and algebraic braid groups of W respectively.

1.17.6. Let $(\widetilde{V_{\mathcal{A}}}, \widetilde{v_0}) \xrightarrow{p} (V_{\mathcal{A}}, v_0)$ be the universal covering space of $V_{\mathcal{A}}$. Then, $(\widetilde{V_{\mathcal{A}}}, \widetilde{v_0})$ is also the universal covering space of $(V_{\mathcal{A}}/W, [v_0])$ via $\pi \circ p$ and we get a canonical right action of B_W on $\widetilde{V_{\mathcal{A}}}$ by deck transformations extending that of P_W. The group W, and therefore B_W, act on A by

$$w(t_\alpha) = t_{|w\alpha|} \tag{1.19}$$

where $|w\alpha| = \pm w\alpha$ depending on whether $w\alpha \in \Phi_\pm$. If $b \in B_W$ and $\Psi : \widetilde{V_{\mathcal{A}}} \to A$ is a solution of the holonomy equations $p^*\nabla\Psi = 0$, one readily checks that $b \bullet \Psi(\widetilde{v}) = b(\Psi(\widetilde{v}b))$ is another solution. Thus, if Ψ is invertible, then $\mu_\Psi = \Psi^{-1} \cdot b \bullet \Psi$ is a constant element of A. When $b \in P_W$, $\mu_\Psi(b)$ coincides with the element defined by (1.5) since P_W acts trivially on A. Set

$$\nu_\Psi(b) = \mu_\Psi(b) \cdot b \in A \rtimes B_W$$

Proposition 1.41.

(i) *The map $b \to \nu_\Psi(b)$ is a homomorphism $B_W \to A \rtimes B_W$.*
(ii) *If Ψ is unipotent, μ_Ψ takes values in $N \rtimes B_W$.*
(iii) *If Ψ' is another invertible solution, then $\nu_{\Psi'} = \mathrm{Ad}(K^{-1}) \circ \nu_\Psi$, where $K = \Psi^{-1} \cdot \Psi' \in A^\times$.*

1.17.7. The relevance of the map ν_Ψ is the following. To any finite–dimensional representation $\widetilde{\rho} : F/I \rtimes B_W \to \mathrm{End}(U)$, we may associate a flat holomorphic vector bundle $(\mathcal{U}_{\widetilde{\rho}}, \nabla_{\widetilde{\rho}})$ with fibre U over $V_{\mathcal{A}}/W$ as follows. Let ρ be the restriction of $\widetilde{\rho}$ to F/I,

$$\nabla_\rho = d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \cdot \rho(t_\alpha)$$

the corresponding flat connection on $V_{\mathcal{A}} \times U$ defined by (1.1) and $p^*\nabla_\rho$ its pull–back to $\widetilde{V_{\mathcal{A}}} \times U$. Then, $(\mathcal{U}_{\widetilde{\rho}}, \nabla_{\widetilde{\rho}})$ is the quotient

$$(U_{\widetilde{\rho}}, \nabla_{\widetilde{\rho}}) = (\widetilde{V_{\mathcal{A}}} \times U, p^*\nabla_\rho)/B_W$$

where B_W acts on U via $\widetilde{\rho}$. Note that if $\widetilde{\rho}$ factors through $F/I \rtimes W$, then $(\mathcal{U}_{\widetilde{\rho}}, \nabla_{\widetilde{\rho}})$ is simply the quotient of $(V_{\mathcal{A}} \times U, \nabla_\rho)$ by W. As in §1.1, we associate to $\widetilde{\rho}$ a representation $\widetilde{\rho}_h : A \rtimes B_W \to \mathrm{End}(U[\![h]\!])$ by setting, for $b \in B_W$ and $\alpha \in \Phi_+$,

$$\widetilde{\rho}_h(b) = \widetilde{\rho}(b) \qquad \text{and} \qquad \widetilde{\rho}_h(t_\alpha) = h\widetilde{\rho}(t_\alpha) \tag{1.20}$$

Then, for any invertible solution $\Psi : \widetilde{V_{\mathcal{A}}} \to A$ of $p^*\nabla\Psi = 0$, $\Psi_{\widetilde{\rho}_h} = \widetilde{\rho}_h(\Psi)$ is a fundamental solution of $\nabla_{\widetilde{\rho}_h} \Psi_{\widetilde{\rho}_h} = 0$ and, for any $b \in B_W$, $\widetilde{\rho}_h(\nu_\Psi(b)) \in GL(U[\![h]\!])$ is the monodromy of b expressed in that solution.

1.17.8. *Extended Chen–Kohno isomorphisms.* Let $I_W \subset \mathbb{C}[B_W]$ be the kernel of the epimorphism $\mathbb{C}[B_W] \to \mathbb{C}[W]$. I_W is readily seen to be the ideal generated by the augmentation ideal J of $\mathbb{C}[P_W]$ inside $\mathbb{C}[B_W]$. It follows from this that $I_W^m \cap \mathbb{C}[P_W] = J^m$ for any $m \geq 0$ and therefore that

$$0 \longrightarrow \widehat{\mathbb{C}[P_W]} \longrightarrow \widehat{\mathbb{C}[B_W]} \longrightarrow \mathbb{C}[W] \longrightarrow 0 \qquad (1.21)$$

is exact, where

$$\widehat{\mathbb{C}[P_W]} = \varprojlim_{m\to\infty} \mathbb{C}[P_W]/J^m \qquad \text{and} \qquad \widehat{\mathbb{C}[B_W]} = \varprojlim_{m\to\infty} \mathbb{C}[B_W]/I_W^m$$

are the prounipotent completion of $\mathbb{C}[P_W]$ and the completion of $\mathbb{C}[B_W]$ relative to the homomorphism $\mathbb{C}[B_W] \to \mathbb{C}[W]$ respectively [Ha].

Let $\Psi : \widetilde{V_A} \to A$ be an invertible solution of the holonomy equations (1.3) and let Θ_Ψ be the composition of the monodromy map ν_Ψ with the projection to the quotient $A \rtimes W$ of $A \rtimes B_W$. One readily checks that Θ_Ψ maps I_W into the ideal $A_+ \rtimes W$ of positive elements with respect to the grading given by $\deg(t_\alpha) = 1$ for $\alpha \in \Phi_+$ and $\deg(w) = 0$ for $w \in W$. Θ_Ψ therefore factors through a map $\widehat{\Theta}_\Psi : \mathbb{C}[B_W] \to A \rtimes W$ which fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathbb{C}[P_W]} & \longrightarrow & \widehat{\mathbb{C}[B_W]} & \longrightarrow & \mathbb{C}[W] & \longrightarrow & 0 \\
 & & \Big\downarrow \widehat{\mu_\Psi} & & \Big\downarrow \widehat{\Theta}_\Psi & & \Big\downarrow \mathrm{id} & & \\
0 & \longrightarrow & A & \longrightarrow & A \rtimes W & \longrightarrow & \mathbb{C}[W] & \longrightarrow & 0
\end{array}$$

where $\widehat{\mu_\Psi}$ is the monodromy map of §1.3. The exactness of the rows and theorem 1.2 readily yield the following

Theorem 1.42. *The monodromy map $\widehat{\Theta}_\Psi$ is an isomorphism. In particular, the exact sequence* (1.21) *is (non–canonically) split.*

1.17.9. Recall that W possesses a presentation on the reflections $s_i = s_{\alpha_i}$, $i \in \mathbf{I}$ corresponding to the walls of the chamber $\mathcal{C}$ with relations $s_i^2 = 1$ and, for any $i \neq j \in \mathbf{I}$,

$$\underbrace{s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i \cdots}_{m_{ij}}$$

where the number m_{ij} of factors on each side is the order of $s_i s_j$ in W. Assume henceforth that the base point v_0 lies in $\mathcal{C}$. Then, by Brieskorn's theorem [Br], B_W is canonically isomorphic to the algebraic braid group of W, that is the group B_D presented on generators S_i, $i \in \mathbf{I}$ with relations

$$\underbrace{S_i S_j \cdots}_{m_{ij}} = \underbrace{S_j S_i \cdots}_{m_{ij}} \tag{1.22}$$

for any $i \neq j \in \mathbf{I}$. The image of S_i in B_W is a generator of monodromy around the image of the hyperplane $\mathrm{Ker}(\alpha_i)$ in $V_{\mathcal{A}}/W$. The isomorphism is compatible with the diagrams

$$\begin{array}{ccc} B_W & \longrightarrow & B_D \\ & \searrow \quad \swarrow_{q} & \\ & W & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \pi_1(V_{\mathcal{A}}/W,[v_0]) & \xrightarrow{\imath} & \pi_1(V_{\mathcal{A}}/W,[v_0']) \\ & \searrow \quad \swarrow & \\ & B_D & \end{array}$$

where q maps S_i to s_i, $v_0' \in \mathcal{C}$ is another basepoint and $\imath$ is the canonical identification induced by the contractibility of $\mathcal{C}$.

1.17.10. Let $\mathcal{F}$ be a fundamental maximal nested sets and b a positive, real basis of V^* adapted to $\mathcal{F}$. Note that the solution $\Psi^b_{\mathcal{F}}$ lifts uniquely to $\widetilde{V_{\mathcal{A}}}$ since it is defined on $\mathcal{C} \ni v_0$.

Theorem 1.43. *If $\mathcal{F}$ contains $\mathbb{C}\alpha_i$, then*

$$\nu_{\Psi^b_{\mathcal{F}}}(S_i) = \exp(\pi\sqrt{-1}\cdot t_{\alpha_i})\cdot S_i$$

1.17.11. For any $i \in \mathbf{I}$, set

$$S_i^{\nabla} = \exp(\pi\sqrt{-1}\cdot t_{\alpha_i})\cdot S_i \in A \rtimes B_W$$

Theorem 1.43 and proposition 1.41 allow one to concisely describe the monodromy of the holonomy equations in terms of the elements S_i^{∇} and the associators $\Phi^{\beta}_{\mathcal{GF}}$ corresponding to a fixed positive, real adapted family β[1]. Indeed, let $\mathcal{F}$ be a fixed fundamental maximal nested set. If $\mathbb{C}\alpha_i \in \mathcal{F}$, then

$$\nu_{\Psi^{\beta}_{\mathcal{F}}}(S_i) = S_i^{\nabla}$$

Otherwise, if $\mathcal{G}$ is a fundamental maximal nested set such that $\mathbb{C}\alpha_i \in \mathcal{G}$, then

$$\nu_{\Psi^{\beta}_{\mathcal{F}}}(S_i) = \nu_{\Psi^{\beta}_{\mathcal{G}}\cdot\Phi^{\beta}_{\mathcal{GF}}}(S_i) = \Phi^{\beta}_{\mathcal{FG}}\cdot S_i^{\nabla}\cdot\Phi^{\beta}_{\mathcal{GF}}$$

[1] If Φ is a crystallographic root system, there are several preferred ways to obtain such families $\beta = \{x_B\}$ such that $\beta \subset \Phi_+$. One may for example set $x_B = \theta_B$ where the latter is the highest root of the irreducible subroot system $B \cap \Phi$ relative to the basis $\Delta_B = \Delta \cap B$, or $x_B = \sum \alpha_i$ where the sum ranges over those $i \in \mathbf{I}$ such that $\alpha_i \in \Delta_B$. These two choices coincide for root systems of type A_n. The latter is the choice adopted by Drinfeld [Dr3] for root systems of type A_n and by De Concini–Procesi for general root systems.

It follows in particular from (1.22) that the elementary associators $\Phi^{\beta}_{\mathcal{G}\mathcal{F}}$ satisfy the following additional property

- **braid relations** : if $(\mathcal{F}_i, \mathcal{F}_j)$ is a pair of fundamental maximal nested sets such that $\mathbb{C}\alpha_i \in \mathcal{F}_i$ and $\mathbb{C}\alpha_j \in \mathcal{F}_j$, then

$$\underbrace{S_i^{\nabla} \cdot \mathrm{Ad}(\Phi^{\beta}_{\mathcal{F}_i\mathcal{F}_j})(S_j^{\nabla}) \cdots}_{m_{ij}} = \underbrace{\mathrm{Ad}(\Phi^{\beta}_{\mathcal{F}_i\mathcal{F}_j})(S_j^{\nabla}) \cdot S_i^{\nabla} \cdots}_{m_{ij}} \tag{1.23}$$

1.18. **Appendix : proof of theorem 1.2.** We prove below that the map

$$\widehat{\mu_\Psi} : \widehat{\mathbb{C}[\pi]} \longrightarrow A$$

defined in §1.3 is an isomorphism. The surjectivity of $\widehat{\mu_\Psi}$ is due to Chen [Chn1, thm 3.4.1] and its injectivity to Kohno [Ko1, Ko3]. We merely repeat here Chen's simple proof and give an alternative approach to Kohno's based on the observation, used by Bar–Natan in the case of the Coxeter arrangements of type A_n [BN, prop. 3.6], that the defining relations (1.2) of the holonomy algebra A may be obtained by linearising suitable commutation relations in π.

Note first that $\widehat{\mathbb{C}[\pi]}$ and A are endowed with decreasing $\mathbb{N}$–filtrations. It therefore suffices to show that $\mathrm{gr}(\widehat{\mu_\Psi}) : \mathrm{gr}(\widehat{\mathbb{C}[\pi]}) \to \mathrm{gr}(A) = A$ is an isomorphism. Let $x \in X$ and $\gamma_x \in \pi$ a generator of monodromy around the hyperplane $x^\perp$. Picard iteration readily shows that, mod A_+^2

$$\mu_\Psi(\gamma_x - 1) = \int_{\gamma_x} \sum_{x' \in X} \frac{dx'}{x'} \cdot t_{x'} = 2\pi i t_x \tag{1.24}$$

so that $\mathrm{gr}(\widehat{\mu_\Psi})(\gamma_x - 1) = 2\pi i t_x$. In particular, $\mathrm{gr}(\widehat{\mu_\Psi})$ is surjective. To construct an inverse to $\mathrm{gr}(\widehat{\mu_\Psi})$ note first that, for any $\gamma, \zeta \in \pi$, the identity

$$(\gamma\zeta\gamma^{-1} - 1) - (\zeta - 1) = ((\gamma - 1)(\zeta - 1) - (\zeta - 1)(\gamma - 1))\,\gamma^{-1}$$

shows that the image of $\zeta - 1$ in J/J^2 only depends upon the conjugacy class of ζ in π. In particular, if $x \in X$, the class of $\gamma_x - 1$ in J/J^2 does not depend upon the choice of the generator of monodromy γ_x around $x^\perp$ since any two such choices are conjugate in π. Define now $\nu : A_1 \to J/J^2$ by

$$t_x \longrightarrow (2\pi i)^{-1} \cdot (\gamma_x - 1) + J^2$$

for any $x \in X$. We claim that ν extends to a homomorphism $A \to \mathrm{gr}(\widehat{\mathbb{C}[\pi]})$. It suffices to show that the elements $\delta_x = \gamma_x - 1$ satisfy the relations (1.2) modulo J^3. Let $B \subset V^*$ be a 2–dimensional subspace spanned by elements of X. We shall need the following result whose proof is given below

Lemma 1.44. *Let $x_1, \ldots, x_m$ be an enumeration of $B \cap X$. Then, there exist generators of monodromy $\gamma_i \in \pi$ around each $x_i^\perp$ such that the product $\gamma_1 \cdots \gamma_m$ commutes with each γ_i.*

Let now γ_i, $i = 1, \ldots, m$, be as in lemma 1.44 and set $\delta_i = \gamma_i - 1 \in J$. Replacing each γ_j by $\delta_j + 1$ in $\gamma_i \cdot \gamma_1 \cdots \gamma_m = \gamma_1 \cdots \gamma_m \cdot \gamma_i$ yields $[\delta_i, \sum_j \delta_j] = 0 \mod J^3$ as required. Thus, ν extends to a homomorphism $A \to \mathrm{gr}(\widehat{\mathbb{C}[\pi]})$ satisfying $\mu \circ \nu = \mathrm{id}$. Moreover, $\nu \circ \mu = \mathrm{id}$ since this holds on any generator of monodromy γ_x, $x \in X$ and these generate π [BMR, prop A.2] ■

Proof of lemma 1.44.[1] Let $\mathcal{A}_B = \bigcup_{x \in X \cap B} x^\perp \subset V$ be the arrangement determined by B and $D \subset V$ an open ball centered at $v \in B^\perp$ such that D does not intersect any hyperplane $y^\perp$, $y \in X \setminus B$. Since the composition

$$D \setminus \mathcal{A}_B \hookrightarrow V \setminus \mathcal{A} \hookrightarrow V \setminus \mathcal{A}_B$$

is a homotopy equivalence, $\pi_1(D \setminus \mathcal{A}_B)$ embeds in π. The elements γ_i will be chosen in $\pi_1(D \setminus \mathcal{A}_B) \cong \pi_1(V \setminus \mathcal{A}_B)$. Let $V' \subset V$ be a complementary subspace to $B^\perp$. The corresponding projection $V \to V'$ induces a homotopy equivalence $V \setminus \mathcal{A}_B \sim V' \setminus \mathcal{A}_B$ and therefore an isomorphism of $\pi_1(V \setminus \mathcal{A}_B)$ with the fundamental group of the complement in V' of the lines $L_x = V' \cap x^\perp$, $x \in X \cap B$. Consider now the Hopf fibration

$$\mathbb{C}^* \longrightarrow V' \setminus \bigcup_i L_{x_i} \longrightarrow \mathbb{P}^1 \setminus \{z_1, \ldots, z_m\}$$

where $z_i = [L_{x_i}] \in \mathbb{P}(V') \cong \mathbb{P}^1$. Since $m \geq 1$, the fibration is trivial and the image of $\pi_1(\mathbb{C}^*)$ in $\pi_1(V' \setminus \bigcup_i L_{x_i})$ is central. Thus, if $\overline{\gamma}_1, \ldots, \overline{\gamma}_m \in \pi_1(\mathbb{P}^1 \setminus \{z_1, \ldots, z_m\})$ are small loops around $z_1, \ldots, z_m$ such that $\overline{\gamma}_1 \cdots \overline{\gamma}_m = 1$ and each $\overline{\gamma}_i$ is lifted to a generator of monodromy γ_i around L_{x_i}, the product $\gamma_1 \cdots \gamma_m$ is central in $\pi_1(V' \setminus \bigcup_i L_{x_i})$ ■

Remark 1.45. By (1.24), the map $\mathrm{gr}(\mu_\Psi)$ does not depend upon the choice of Ψ. Thus, the monodromy of ∇ yields a canonical isomorphism identification $\mathrm{gr}(\widehat{\mathbb{C}[\pi]}) \cong A$.

[1] I owe the proof of this lemma to D. Bessis and J. Millson.

2. The De Concini–Procesi associahedron $\mathcal{A}_D$

Let D be a connected graph. The aim of this section is to construct a regular cell complex $\mathcal{A}_D$ whose face poset is that of nested sets of connected subgraphs of D, ordered by reverse inclusion. When D is the Dynkin diagram of type A_{n-1}, $\mathcal{A}_D$ is isomorphic to Stasheff's associahedron K_n [St1]. More generally, if D is the Coxeter graph of an irreducible, finite Coxeter group W, $\mathcal{A}_D$ is isomorphic to the cell complex constructed by De Concini–Procesi inside their wonderful model of the reflection arrangement of W [DCP2, §3.2]. For this reason, we call $\mathcal{A}_D$ the *De Concini–Procesi associahedron* corresponding to D. When D is the affine Dynkin diagram of type A_{n-1}, $\mathcal{A}_D$ is isomorphic to Bott and Taubes' cyclohedron W_n [BT].[1]

After defining the face poset of $\mathcal{A}_D$ in §2.1, we realise $\mathcal{A}_D$ as a convex polytope in §2.2, thereby settling its existence and proving its contractibility. The simple connectedness of $\mathcal{A}_D$ will be used in section 3 to prove an analogue for quasi–Coxeter algebras of Mac Lane's coherence theorem for monoidal categories. We then prove in §2.5 that the faces of $\mathcal{A}_D$ are isomorphic to products of associahedra corresponding to subquotients of D, a well–known fact for the associahedron K_n. In particular, each facet of $\mathcal{A}_D$ is a product of two associahedra, one corresponding to a proper, connected subgraph B of D and the other to the quotient graph D/B. Finally, in §2.7–§2.8 we describe the edges and 2–faces of $\mathcal{A}_D$ explicitly. Interestingly perhaps, the latter turn out to be squares, pentagons or hexagons.

2.1. **The poset $\mathcal{N}_D$ of nested sets on D.** By a *diagram* we shall mean a non–empty undirected graph D with no multiple edges or loops. We denote the set of vertices of D by $V(D)$ and set $|D| = |V(D)|$. A *subdiagram* $B \subset D$ is a full subgraph of D, that is a graph consisting of a subset $V(B)$ of vertices of D, together with all edges of D joining any two elements of $V(B)$. We will often abusively identify such a B with its set of vertices and write $\alpha \in B$ to mean $\alpha \in V(B)$. The *union* $B_1 \cup B_2$ of two subdiagrams $B_1, B_2 \subset D$ of D is the subdiagram having

[1] The reader should be cautioned that the De Concini–Procesi associahedra corresponding to Dynkin diagrams of finite type differ from the generalised associahedra defined by Fomin and Zelevinsky [FZ, CFZ], since the former do not depend upon the multiplicities of the edges of the diagram. For example, the De Concini–Procesi associahedra of type $\mathsf{A}_{n-1}, \mathsf{B}_{n-1}, \mathsf{C}_{n-1}$ are all isomorphic to the associahedron K_n while the Fomin–Zelevinsky associahedra of types $\mathsf{B}_n, \mathsf{C}_n$ are homeomorphic to the cyclohedron W_n.

$V(B_1) \cup V(B_2)$ as set of vertices.

Two subdiagrams $B_1, B_2 \subseteq D$ are *orthogonal* if no two vertices $\alpha_1 \in B_1, \alpha_2 \in B_2$ are joined by an edge in D. B_1 and B_2 are *compatible* if either one contains the other or they are orthogonal. Assume henceforth that D is connected.

Definition 2.1. *A nested set on D is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of D which contains D.*[1,2]

We denote by $\mathcal{N}_D$ the partially ordered set of nested sets on D, ordered by reverse inclusion. $\mathcal{N}_D$ has a unique maximal element $\hat{1} = \{D\}$. Its minimal elements are the maximal nested sets. When D is the Dynkin diagram of type A_{n-1}, $\mathcal{N}_D$ is the face poset of the associahedron K_n by remark 1.40.

2.2. **Convex realisation of $\mathcal{A}_D$.** Recall that a CW–complex X is *regular* if all its attaching maps are homeomorphisms [Ms, §IX.6]. In this case, an induction on the skeleton of X shows that its cellular isomorphism type is uniquely determined by its face poset. This justifies the following

Definition 2.2. *The De Concini–Procesi associahedron $\mathcal{A}_D$ is the regular CW–complex whose poset of (non–empty) faces is $\mathcal{N}_D$.*

We shall prove the existence of $\mathcal{A}_D$ by realising it as a convex polytope of dimension $|D| - 1$. Our construction follows the Shnider–Sternberg–Stasheff realisation of the associahedron K_n as a truncation of the $(n-2)$–simplex [SS], as presented in [St2, Appendix B], and coincides with Stasheff and Markl's convex realisation of the cyclohedron W_n [St2, Ma] when D is the affine Dynkin diagram of type A_{n-1}.

Let c be a function on the set of connected subdiagrams of D with values in $\mathbb{R}_+^*$ such that

$$c(B_1 \cup B_2) > c(B_1) + c(B_2) \tag{2.1}$$

[1] This is the opposite of Stasheff's convention in which the faces of the associahedron K_n are labelled by consistent bracketings of a monomial $x_1 \cdots x_n$ which do *not* contain the big bracket $(x_1 \cdots x_n)$, but is better suited to our needs.

[2] Such collections should perhaps be called *fundamental* nested sets on D since, when D is the graph of a finite Coxeter group W they correspond, via the dictionary of §1.17.3–§1.17.4, to fundamental nested sets of subspaces spanned by the roots of W. Since general nested sets of subspaces do not seem to have an analogue for diagrams however, we prefer to omit the adjective fundamental when speaking about diagrams.

whenever B_1 and B_2 are not compatible. An example of such a c is given by $c(B) = 3^{|B|}$. Let $\{t_\alpha\}_{\alpha\in D}$ be the canonical coordinates on $\mathbb{R}^{|D|}$ and consider, for any connected $B \subseteq D$, the linear hyperplane

$$\mathcal{L}^c_B = \{t \in \mathbb{R}^{|D|} | \sum_{\alpha\in B} t_\alpha = c(B)\} \subset \mathbb{R}^{|D|}$$

Consider next the convex polytope

$$P^c_D = \{t \in \mathbb{R}^{|D|} | \sum_{\alpha\in D} t_\alpha = c(D), \sum_{\alpha\in B} t_\alpha \geq c(B) \quad \text{for any connected } B \subsetneq D\}$$

Theorem 2.3.

(i) *The polytope P^c_D has nonempty interior in the hyperplane $\mathcal{L}^c_D$.*
(ii) *For any connected subdiagrams $B_1, \ldots, B_m \subsetneq D$, the intersection*

$$P^c_{D,B_1,\ldots,B_m} = P^c_D \cap \bigcap_{i=1}^m \mathcal{L}^c_{B_i}$$

is nonempty if, and only if $B_1, \ldots, B_m$ are pairwise compatible.
(iii) *If $B_1, \ldots, B_m$ are pairwise compatible and distinct, $P^c_{D,B_1,\ldots,B_m}$ is a face of P^c_D of dimension $|D| - 1 - m$.*
(iv) *All nonempty faces of P^c_D are obtained in this way.*

Proof. The proof given in [St2, appendix B] in the case when D is the Dynkin diagram of type A_{n-1} carries over easily to the general case ■

Corollary 2.4. *The map*

$$\mathcal{H} \longrightarrow P^c_{\mathcal{H}} = P^c_{D,B_1,\ldots,B_m}$$

where $\mathcal{H} = \{D, B_1, \ldots, B_m\}$ is a nested set on D is an isomorphism between $\mathcal{N}_D$ and the poset of nonempty faces of P^c_D.

Thus, for any function c satisfying (2.1), the polytope P^c_D gives a convex realisation of the associahedron $\mathcal{A}_D$. In particular

Corollary 2.5. *The De Concini–Procesi associahedron $\mathcal{A}_D$ is contractible.*

Remark 2.6. By theorem 2.3, the maximal nested sets on D, which label the vertices of $\mathcal{A}_D$, are of cardinality $|D|$ and any $\mathcal{H} \in \mathcal{N}_D$ of cardinality $|D| - 1$ is contained in exactly two maximal nested sets. Thus, the 1–skeleton of $\mathcal{A}_D$ may equivalently be described as having a 0–cell for each maximal nested set $\mathcal{F}$ on D and a 1–cell between $\mathcal{F}$ and $\mathcal{G}$ if, and only if $\mathcal{F}$ and $\mathcal{G}$ differ by an element. In particular, the connectedness of $\mathcal{A}_D$ gives another proof of proposition 1.30 for Coxeter arrangements.

Remark 2.7. When D is the graph of an irreducible, finite Coxeter system (W, S), the associahedron $\mathcal{A}_D$ may be obtained more geometrically as follows [DCP2, §3.2]. Let $\mathcal{A} \subset V$ be the complexified reflection arrangement of W and Y_X the wonderful model of $V_\mathcal{A} = V \setminus \mathcal{A}$ described in section 1. The irreducible component $\mathcal{D}_{V^*} \subset Y_X$ of the exceptional divisor corresponding to V^* is a smooth projective variety and Y_X is the total space of a line bundle over $\mathcal{D}_{V^*}$ in such a way that the corresponding action of $\mathbb{C}^*$ agrees, on $Y_X \setminus \mathcal{D} \cong V_\mathcal{A}$ with its natural action on V. Let $\mathcal{A}_\mathbb{R} \subset V_\mathbb{R}$ be the real reflection arrangement of W, $\mathcal{C} \subset V_\mathbb{R} \setminus \mathcal{A}_\mathbb{R}$ the chamber corresponding to S and $\overline{\mathcal{C}}$ the closure of $\mathcal{C}$ in Y_X. Then, the intersection $\overline{\mathcal{C}} \cap \mathcal{D}_{V^*}$ possesses a regular cellular structure with corresponding face poset given by $\mathcal{N}_D$.

We shall often identify a nested set $\mathcal{H} \in \mathcal{N}_D$ with the corresponding face of $\mathcal{A}_D$ and speak of the dimension $\dim(\mathcal{H}) = |D| - |\mathcal{H}|$ of $\mathcal{H}$ to mean the dimension of that face.

2.3. **The rank function of $\mathcal{N}_D$.** For any nested set $\mathcal{H} \in \mathcal{N}_D$ and $B \in \mathcal{H}$, set

$$i_\mathcal{H}(B) = B_1 \cup \cdots \cup B_m \tag{2.2}$$

where the B_i are the maximal elements of $\mathcal{H}$ properly contained in B.

Definition 2.8. *Set*

$$\underline{\alpha}^B_\mathcal{H} = B \setminus i_\mathcal{H}(B) \qquad \textit{and} \qquad n(B;\mathcal{H}) = |\underline{\alpha}^B_\mathcal{H}|$$

Note that $n(B;\mathcal{H}) \geq 1$. Indeed, if $m = 1$, then $n(B;\mathcal{H}) = |B \setminus B_1| \geq 1$. Otherwise, $B_1, \dots, B_m$ are necessarily pairwise orthogonal, their union is disconnected and cannot be equal to B. Note in passing that $B_1, \dots, B_m$ are the connected components of $B \setminus \underline{\alpha}^B_\mathcal{H}$. In particular, the latter lie in $\mathcal{H}$. Set now

$$n(\mathcal{H}) = \sum_{B \in \mathcal{H}} (n(B;\mathcal{H}) - 1)$$

The following is an analogue of propositions 1.10 and 1.11

Proposition 2.9.

(i) *For any nested set $\mathcal{H} \in \mathcal{N}_D$,*

$$n(\mathcal{H}) = |D| - |\mathcal{H}| = \dim(\mathcal{H})$$

(ii) *If $\mathcal{H}$ is a maximal nested set, then $n(B;\mathcal{H}) = 1$ for any $B \in \mathcal{H}$.*

(iii) *Any maximal nested set is of cardinality $|D|$.*

Proof. (i) If $|\mathcal{H}| = 1$, then $\mathcal{H} = \{D\}$ and $n(\mathcal{H}) = |D| - 1$ as required. Assume now that $|\mathcal{H}| \geq 2$ and let $D_1, \ldots, D_m \subsetneq D$ be the proper, maximal elements in $\mathcal{H}$ so that

$$\mathcal{H} = \{D\} \sqcup \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_m$$

where $\mathcal{H}_i$ is a nested set on D_i. By induction, $n(\mathcal{H}_i) = |D_i| - |\mathcal{H}_i|$ whence

$$\begin{aligned} n(\mathcal{H}) &= (n(D;\mathcal{H}) - 1) + \sum_i n(\mathcal{H}_i) \\ &= (n(D;\mathcal{H}) - 1) + \sum_i (|D_i| - |\mathcal{H}_i|) \\ &= |D| - |\mathcal{H}| \end{aligned}$$

(ii) Let $B \in \mathcal{H}$ and $\alpha \in \underline{\alpha}^B_{\mathcal{H}}$. Since the connected components of $B \setminus \alpha$ are compatible with the elements of $\mathcal{H}$, they lie in $\mathcal{H}$ by maximality whence $|\underline{\alpha}^B_{\mathcal{H}}| = 1$. (iii) follows from (ii) and (i) ■

If $\mathcal{F}$ is a maximal nested set and $B \in \mathcal{F}$, we denote the unique element of $\underline{\alpha}^B_{\mathcal{F}}$ by $\alpha^B_{\mathcal{F}}$.

2.4. **Quotienting by a subdiagram.** We define in this subsection the quotient D/B of D by a proper subdiagram B and relate nested sets on D/B with those on D. Let $B_1, \ldots, B_m$ be the connected components of B.

Definition 2.10. *The set of vertices of the diagram D/B is $V(D) \setminus V(B)$. Two vertices $\alpha \neq \beta$ of D/B are linked by an edge if, and only if the following holds in D*

$$\alpha \not\perp \beta \qquad \text{or} \qquad \alpha, \beta \not\perp B_i \quad \text{for some} \quad i = 1, \ldots, m$$

For any connected subdiagram $C \subseteq D$ not contained in B, we denote by $\overline{C} \subseteq D/B$ the connected subdiagram with vertex set $V(C) \setminus V(B)$. We shall need the following

Lemma 2.11. *Let $C_1, C_2 \not\subseteq B$ be two connected subdiagrams of D which are compatible. Then*

(i) *$\overline{C}_1, \overline{C}_2$ are compatible unless $C_1 \perp C_2$ and $C_1, C_2 \pitchfork B_i$ for some i.*
(ii) *If C_1 is compatible with every B_i, then $\overline{C}_1$ and $\overline{C}_2$ are compatible.*

In particular, if $\mathcal{F}$ is a nested set on D containing each B_i, then $\overline{\mathcal{F}} = \{\overline{C}\}$, where C runs over the elements of $\mathcal{F}$ such that $C \not\subseteq B$, is a nested set on D/B.

Proof. (i) Clearly, if $C_1 \subset C_2$ or $C_2 \subset C_1$ then $\overline{C}_1 \subset \overline{C}_2$ or $\overline{C}_2 \subset \overline{C}_1$ respectively. If, on the other hand, $C_1 \perp C_2$ then $\overline{C}_1 \perp \overline{C}_2$ if, and only if, for any connected component B_i of B at least one of C_1, C_2 is perpendicular to B_i. (ii) We may assume by (i) that $C_1 \perp C_2$. Since C_1 and B_i are compatible for any i and $C_1 \not\subseteq B_i$, either $B_i \subset C_1$, in which case $B_i \perp C_2$, or $B_i \perp C_1$. $\overline{C}_1$ and $\overline{C}_2$ are therefore compatible by (i) ■

Let now A be a connected subdiagram of D/B and denote by $\widetilde{A} \subseteq D$ the connected sudbdiagram with vertex set

$$V(\widetilde{A}) = V(A) \bigcup_{i:B_i \not\perp V(A)} V(B_i) \tag{2.3}$$

Clearly, $A_1 \subseteq A_2$ or $A_1 \perp A_2$ imply $\widetilde{A_1} \subseteq \widetilde{A_2}$ and $\widetilde{A_1} \perp \widetilde{A_2}$ respectively, so the lifting map $A \to \widetilde{A}$ preserves compatibility.

For any connected subdiagrams $A \subseteq D/B$ and $C \subseteq D$, we have

$$\overline{\widetilde{A}} = A \qquad \text{and} \qquad \widetilde{\overline{C}} = C \bigcup_{i:B_i \not\perp C} B_i \tag{2.4}$$

In particular, $\widetilde{\overline{C}} = C$ if, and only if, C is compatible with $B_1, \dots, B_m$ and not contained in B. The applications $C \to \overline{C}$ and $A \to \widetilde{A}$ therefore yield a bijection between the connected subdiagrams of D which are either orthogonal to or strictly contain each B_i and the connected subdiagrams of D/B. By lemma 2.11, this bijection preserves compatibility and therefore induces an embedding $\mathcal{N}_{D/B} \hookrightarrow \mathcal{N}_D$. This yields an embedding

$$\mathcal{N}_{B_1} \times \cdots \times \mathcal{N}_{B_m} \times \mathcal{N}_{D/B} \hookrightarrow \mathcal{N}_D \tag{2.5}$$

with image the poset of nested sets on D containing each B_i.

2.5. **Unsaturated elements and the faces of $\mathcal{A}_D$.** We show below that the faces of $\mathcal{A}_D$ are products of associahedra corresponding to subquotient diagrams of D. Let $\mathcal{H}$ be a nested set on D.

Definition 2.12. *An element $B \in \mathcal{H}$ is called unsaturated if $n(B;\mathcal{H}) \geq 2$.*

Let $\mathcal{A}_D^{\mathcal{H}}$ be the face of the associahedron $\mathcal{A}_D$ corresponding to $\mathcal{H}$. The face poset of $\mathcal{A}_D^{\mathcal{H}}$ is the poset $\mathcal{N}_D^{\mathcal{H}}$ of nested sets on D containing $\mathcal{H}$.

Proposition 2.13. *As posets,*

$$\mathcal{N}_D^{\mathcal{H}} \cong \prod_{C \in \mathcal{H}} \mathcal{N}_{C/i_{\mathcal{H}}(C)} \cong \prod_{j=1}^{p} \mathcal{N}_{D_j/i_{\mathcal{H}}(D_j)}$$

where $D_1, \ldots, D_p$ are the unsaturated elements of $\mathcal{H}$. In particular,

$$\mathcal{A}_D^{\mathcal{H}} \cong \prod_{j=1}^{p} \mathcal{A}_{D_j/i_{\mathcal{H}}(D_j)}$$

as CW–complexes.

Proof. Let $B_1, \ldots, B_m$ be the proper maximal elements of $\mathcal{H}$, so that $i_{\mathcal{H}}(D) = B_1 \cup \cdots \cup B_m$ and let $\mathcal{H}_i$ be the nested set on B_i induced by $\mathcal{H}$. The embedding (2.5) yields an isomorphism

$$\mathcal{N}_D^{\mathcal{H}} \cong \mathcal{N}_{D/i_{\mathcal{H}}(D)} \times \mathcal{N}_{B_1}^{\mathcal{H}_1} \times \cdots \times \mathcal{N}_{B_m}^{\mathcal{H}_m}$$

The first isomorphism now follows from an easy induction, the second from the fact that $\mathcal{N}_{C/i_{\mathcal{H}}(C)}$ consists of a single element if C is saturated. The corresponding description of $\mathcal{A}_D^{\mathcal{H}}$ follows from the fact that a regular CW–complex is determined by its face poset ■

Remark 2.14. The isomorphism $\prod_{j=1}^{p} \mathcal{N}_{D_j/i_{\mathcal{H}}(D_j)} \longrightarrow \mathcal{N}_D^{\mathcal{H}}$ is explicitly given by

$$\{\mathcal{K}_j\}_{j=1}^{p} \longrightarrow \mathcal{H} \bigcup_{j=1}^{p} (\widetilde{\mathcal{K}_j} \setminus \{D_j\}) \tag{2.6}$$

where, for a nested set $\mathcal{K}_j$ on $D_j/i_{\mathcal{H}}(D_j)$, $\widetilde{\mathcal{K}_j}$ is the nested set on D_j obtained by lifting the elements of $\mathcal{K}_j$ to connected subdiagrams of D_j.

By theorem 2.3, the facets of $\mathcal{A}_D$ are labelled by the nested sets on D of the form $\mathcal{H} = \{D, B\}$ where B is a proper, connected subdiagrams of D.

Corollary 2.15. *The facet of $\mathcal{A}_D$ corresponding to B is isomorphic, as cell complex, to the product $\mathcal{A}_B \times \mathcal{A}_{D/B}$.*

When D is the finite or the affine Dynkin diagram of type A_{n-1}, we recover from corollary 2.15 the familiar fact that each facet of the associahedron K_n or of the cyclohedron W_n is the product $K_r \times K_s$ or $K_r \times W_s$, with $r + s = n + 1$ of two smaller associahedra or an associahedron and a cyclohedron respectively.

Remark 2.16. The set of Dynkin diagrams of is not closed under quotienting. For example, if D is the Dynkin diagram of type D_n and α is the trivalent node of D, then D/α is the affine Dynkin diagram of type A_2 if $n = 4$ and a tadpole if $n \geq 5$. Thus, if D is a Dynkin diagram with a trivalent node other than D_4, the faces of $\mathcal{A}_D$ are products of associahedra some of which correspond to non–Dynkin diagrams.

2.6. **An alternative description of the lifting map.** We shall need an alternative description of the map (2.6). Let $\mathcal{H}$ be a nested set on D with $|\mathcal{H}| < |D|$ and let $D_1, \ldots, D_p$ be the unsaturated elements of $\mathcal{H}$. For any $1 \leq j \leq p$, let $\underline{\alpha}_j = \underline{\alpha}_{\mathcal{H}}^{D_j}$ and, for any subset $\emptyset \neq \underline{\beta}_j \subseteq \underline{\alpha}_j$, set

$$\underline{\beta}_j^c = \underline{\alpha}_j \setminus \underline{\beta}_j \qquad \text{and} \qquad D_{\underline{\beta}_j} = \mathsf{C}_{\underline{\beta}_j}^{D_j \setminus \underline{\beta}_j^c} \tag{2.7}$$

where the latter denotes the connected component of $D_j \setminus \underline{\beta}_j^c$ containing $\underline{\beta}_j$ if one such exists and the empty set otherwise.

Lemma 2.17. *Let $B_j \subset D_j/i_{\mathcal{H}}(D_j)$ be the subdiagram with vertex set $\underline{\beta}_j$.*

(i) *$D_{\underline{\beta}_j}$ is non–empty if, and only if B_j is connected.*
(ii) *When that is the case,*

$$\widetilde{B_j} = D_{\underline{\beta}_j} \qquad \textit{and} \qquad \overline{D_{\underline{\beta}_j}} = B_j$$

where $\bar{\cdot}$ and $\tilde{\cdot}$ are the quotient and lifting maps for the quotient $D_j/i_{\mathcal{H}}(D_j)$.

Proof. If $D_{\underline{\beta}_j}$ is non–empty, $\overline{D_{\underline{\beta}_j}}$ is a subdiagram of $D_j/i_{\mathcal{H}}(D_j)$ with vertex set $\underline{\beta}_j$. Thus, $\overline{D_{\underline{\beta}_j}} = B_j$ and the latter is connected since $D_{\underline{\beta}_j}$ is. Conversely, if B_j is connected, $\widetilde{B_j}$ is a connected subdiagram of $D_j \setminus \underline{\beta}_j^c$ containing $\underline{\beta}_j$. Thus, $D_{\underline{\beta}_j}$ is non–empty and, by (2.4), $D_{\underline{\beta}_j} = \widetilde{\overline{D_{\underline{\beta}_j}}} = \widetilde{B_j}$ ■

2.7. **Edges of the associahedron $\mathcal{A}_D$.** Let $\mathcal{H} \in \mathcal{N}_D$ be a nested set of dimension 1. By proposition 2.9, $\mathcal{H}$ has a unique unsaturated element B and $\underline{\alpha}_{\mathcal{H}}^B$ consists of two vertices α_1, α_2. Thus, $\overline{B} = B/i_{\mathcal{H}}(B)$ is the connected diagram with vertices $\overline{\alpha}_1, \overline{\alpha}_2$ and $\mathcal{A}_{\overline{B}}$ is the interval

$$\{\overline{B}, \overline{\alpha}_1\} \circ\!\!\xrightarrow{\{\overline{B}\}}\!\!\circ \{\overline{B}, \overline{\alpha}_2\} \tag{2.8}$$

Setting $B_1 = \mathsf{C}_{\alpha_1}^{B \setminus \alpha_2}$ and $B_2 = \mathsf{C}_{\alpha_2}^{B \setminus \alpha_1}$, we see that, by proposition 2.13 and lemma 2.17, the edge of $\mathcal{A}_D$ corresponding to $\mathcal{H}$ is of the form

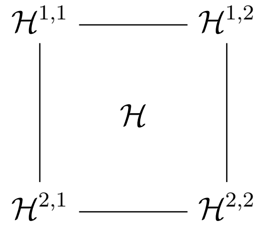
$$\mathcal{H} \cup \{B_1\} \circ\!\!\xrightarrow{\mathcal{H}}\!\!\circ \mathcal{H} \cup \{B_2\}$$

2.8. **Two–faces of the associahedron $\mathcal{A}_D$.** We work out below the two–faces of $\mathcal{A}_D$ and show that they are squares, pentagons or hexagons. Let $\mathcal{H} \in \mathcal{N}_D$ be a nested set of dimension 2. By proposition 2.9, $\mathcal{H}$ either has two unsaturated elements B_1, B_2, and $|\underline{\alpha}_{\mathcal{H}}^{B_1}| = 2 = |\underline{\alpha}_{\mathcal{H}}^{B_2}|$, or a unique unsaturated element B and $|\underline{\alpha}_{\mathcal{H}}^B| = 3$. We treat these two cases separately.

2.8.1. *Square 2–faces.* Assume first that $\mathcal{H}$ has two unsaturated elements B_1, B_2. By proposition 2.13, $\mathcal{A}_D^{\mathcal{H}} \cong \mathcal{A}_{B_1/i_{\mathcal{H}}(B_1)} \times \mathcal{A}_{B_2/i_{\mathcal{H}}(B_2)}$ is the product of two intervals of the form (2.8). Thus, setting for $i, j \in \{1, 2\}$

$$\underline{\alpha}_{\mathcal{H}}^{B_i} = \{\alpha_i^1, \alpha_i^2\} \qquad \text{and} \qquad B_i^j = \complement_{\alpha_i^j}^{B_i \setminus \alpha_i^{3-j}} \subset B_i$$

we see that $\mathcal{A}_D^{\mathcal{H}}$ is the square



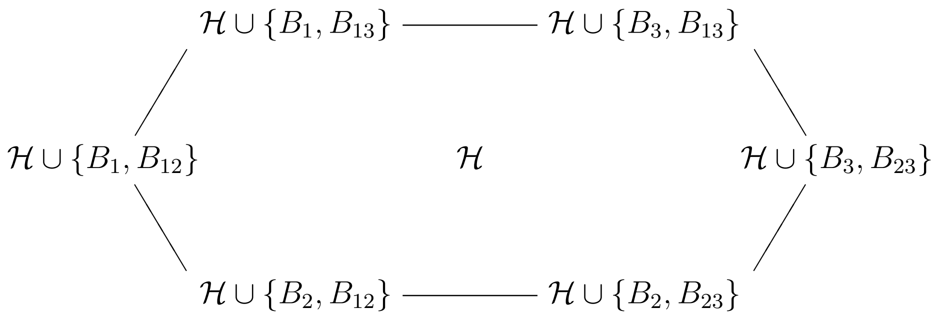
where $\mathcal{H}^{j,k} = \mathcal{H} \cup \{B_1^j, B_2^k\}$.

2.8.2. *Pentagonal and hexagonal 2–faces.* Assume now that $\mathcal{H}$ has a unique unsaturated element B and set $\underline{\alpha}_{\mathcal{H}}^B = \{\alpha_1, \alpha_2, \alpha_3\}$. Then $\overline{B} = B/i_{\mathcal{H}}(B)$ is a connected diagram with vertices $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3$ and is therefore, up to a relabelling of the α_i, one of the following diagrams

For any $1 \leq i \leq 3$ and $1 \leq j \neq k \leq 3$, set

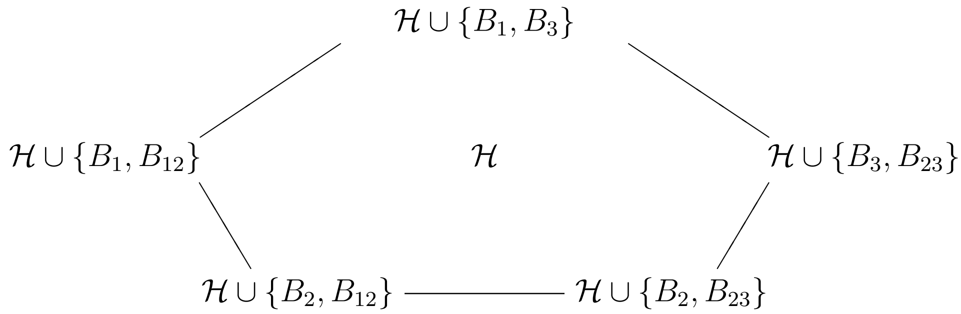
$$B_i = \complement_{\alpha_i}^{B \setminus (\underline{\alpha}_{\mathcal{H}}^B \setminus \alpha_i)} \qquad \text{and} \qquad B_{jk} = \complement_{\{\alpha_j, \alpha_k\}}^{B \setminus (\underline{\alpha}_{\mathcal{H}}^B \setminus \{\alpha_j, \alpha_k\})}$$

If $\overline{B}$ is the affine Dynkin diagram of type A_2, proposition 2.13 and lemma 2.17 imply that the vertices of $\mathcal{A}_D^{\mathcal{H}}$ are of the form $\mathcal{H} \cup \{B_i, B_{ij}\}$ with $1 \leq i \neq j \leq 3$, so that $\mathcal{A}_D^{\mathcal{H}}$ is the hexagon



If, on the other hand $\overline{B}$ is the Dynkin diagram of type A_3, the vertices of $\mathcal{A}_D^{\mathcal{H}}$ are $\mathcal{H} \cup \{B_i, B_{ij}\}$, with $1 \leq i \neq j \leq 3$ and $(i, j) \neq (1, 3)$, and

$\mathcal{H} \cup \{B_1, B_3\}$ so that $\mathcal{A}_D^{\mathcal{H}}$ is the pentagon



Remark 2.18. If B is linear, B_{ij} is empty for a unique pair (i, j) and $\mathcal{H}$ is therefore a pentagon. Thus, the associahedron K_n, and more generally the associahedra corresponding to linear Dynkin diagrams, only have squares and pentagons as 2–faces. The associahedra of type $\mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ and those corresponding to the affine Dynkin diagrams of type $\mathsf{A}_n, \mathsf{B}_n, \mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ on the other hand all have some hexagonal 2–faces.

3. D–algebras and quasi–Coxeter algebras

The aim of this section is to define the category of quasi–Coxeter algebras. We begin in §3.1–§3.4 by describing the underlying notion of D–algebras. We then give three equivalent definitions of quasi–Coxeter algebras. The first two, in §3.8 and §3.13 respectively, are more closely inspired by the De Concini–Procesi theory of asymptotic zones reviewed in section 1, as well as by Drinfeld's theory of quasi–bialgebras [Dr3]. The first definition is better suited to the study of examples, which are considered in section 4, while the second is more convenient to show that quasi–Coxeter algebras define representations of braid groups, as explained in §3.14. The third definition, given in §3.17, is the most compact one and will be used in section 5 to study the deformation theory of quasi–Coxeter algebras. The equivalence of this definition with the first two is the analogue for quasi–Coxeter algebras of Mac Lane's coherence theorem for monoidal categories and relies on the simple connectedness of the De Concini–Procesi associahedron $\mathcal{A}_D$ introduced in section 2. In §3.18, we define the twisting of a quasi–Coxeter algebra , show that it does not change its isomorphism class and in particular that it defines equivalent braid group representations.

3.1. **D–algebras.** Let k be a fixed commutative ring with unit. By an algebra we shall henceforth mean a unital, associative k–algebra. All algebra homomorphisms will be tacitly assumed to be unital. Let D be a connected diagram.

Definition 3.1. *A D–algebra is an algebra A endowed with subalgebras $A_{D'}$ labelled by the non–empty connected subdiagrams D' of D such that the following holds*

- $A_{D'} \subseteq A_{D''}$ *whenever* $D' \subseteq D''$.
- $A_{D'}$ *and* $A_{D''}$ *commute whenever* D' *and* D'' *are orthogonal.*

If A is a D–algebra and α_i is a vertex of D we denote A_{α_i} by A_i. If $D_1, D_2 \subseteq D$ are subdiagrams with D_1 connected, we denote by $A_{D_1}^{D_2}$ the centraliser in A_{D_1} of the subalgebras $A_{D_2'}$ where D_2' runs over the connected components of D_2.

3.2. **Examples.** Most, but not all examples of D–algebras arise in the following way. An algebra A is endowed with subalgebras A_i labelled by the vertices α_i of D with $[A_i, A_j] = 0$ whenever α_i and α_j are orthogonal. In this case, letting $A_{D'} \subseteq A$ be the subalgebra generated by the A_i corresponding to the vertices of D' endows A with a D–algebra structure.

3.2.1. Let W be an irreducible Coxeter group with system of generators $S = \{s_i\}_{i\in\mathbf{I}}$ and let D be the Coxeter graph of W. For any $i \in \mathbf{I}$, let $\mathbb{Z}_2 \cong W_i \subset W$ be the subgroup generated by s_i. Then, $(k[W], k[W_i])$ is a D–algebra. Similarly, let $q_i \in k$ be invertible elements in k such that $q_i = q_j$ whenever s_i and s_j are conjugate in W and let $\mathcal{H}(W)$ be the Iwahori–Hecke algebra of W, that is the algebra with generators $\{S_i\}_{i\in\mathbf{I}}$ and relations

$$(S_i - q_i)(S_i + q_i^{-1}) = 0$$
$$\underbrace{S_iS_j\cdots}_{m_{ij}} = \underbrace{S_jS_i\cdots}_{m_{ij}}$$

where m_{ij} is the order of s_is_j in W. Then, $(\mathcal{H}(W), \mathcal{H}(W_i))$ is a D–algebra.

3.2.2. Let $\mathbf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ be an irreducible, generalised Cartan matrix, $\mathfrak{g} = \mathfrak{g}(\mathbf{A})$ the corresponding Kac–Moody algebra and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ its derived subalgebra with generators e_i, f_i, h_i, $i \in \mathbf{I}$ [Ka]. Let $D = D(\mathbf{A})$ be the Dynkin diagram of $\mathfrak{g}$, that is the connected graph having $\mathbf{I}$ as its vertex set and an edge between i and j if $a_{ij} \neq 0$. For any $i \in \mathbf{I}$, let $\mathfrak{sl}_2^i \subseteq \mathfrak{g}'$ be the three–dimensional subalgebra spanned by e_i, f_i, h_i. Then $(U\mathfrak{g}', U\mathfrak{sl}_2^i)$ is a D–algebra over $k = \mathbb{C}$. Similarly, if $\mathbf{A}$ is symmetrisable and $U_\hbar\mathfrak{g}'$ is the corresponding quantum enveloping algebra (see [Lu], or §4.1.3), then $(U_\hbar\mathfrak{g}', U_\hbar\mathfrak{sl}_2^i)$ is a D–algebra over the ring $\mathbb{C}[\![\hbar]\!]$ of formal power series in $\hbar$.

3.3. **Morphism of D–algebras.** In example 3.2.1, set $k = \mathbb{C}[\![\hbar]\!]$ and $q_i = \exp(\hbar k_i)$, where $k_i \in \mathbb{C}^*$ and $k_i = k_j$ if s_i and s_j are conjugate in W. It is well–known in this case that if W is finite and $\mathfrak{g} = \mathfrak{g}'$ is a complex, simple Lie algebra, $\mathcal{H}(W)$ and $U_\hbar\mathfrak{g}$ are isomorphic to $\mathbb{C}[W][\![\hbar]\!]$ and $U\mathfrak{g}[\![\hbar]\!]$ respectively. We will need to use such isomorphisms to compare the corresponding structures of D–algebras. The following result shows that the naïve notion of isomorphism between D–algebras is too restrictive for this purpose.

Proposition 3.2. *Assume that* $|\mathbf{I}| \geq 2$. *Then,*

(i) *There exists no algebra isomorphism* $\Psi : \mathcal{H}(W) \to \mathbb{C}[W][\![\hbar]\!]$ *such that* $\Psi(\mathcal{H}[W_i]) = \mathbb{C}[W_i][\![\hbar]\!]$ *for any* $i \in \mathbf{I}$.

(ii) *There exists no algebra isomorphism* $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ *equal to the identity mod* $\hbar$ *such that* $\Psi(U_\hbar\mathfrak{sl}_2^i) = U\mathfrak{sl}_2^{\alpha_i}[\![\hbar]\!]$ *for any* $i \in \mathbf{I}$.

The proof of proposition 3.2 is given in §3.4 below. The following gives the correct notion of morphism of D–algebras.

Definition 3.3. *A morphism of D–algebras A, A' is a collection of algebra homomorphisms $\Psi_\mathcal{F} : A \to A'$ labelled by the maximal nested sets $\mathcal{F}$ on D such that for any $\mathcal{F}$ and $D' \in \mathcal{F}$, $\Psi_\mathcal{F}(A_{D'}) \subseteq A'_{D'}$.*

Remark 3.4. We will prove in theorems 4.6 and 8.3 that $\mathcal{H}(W)$ and $U_\hbar\mathfrak{g}$ are isomorphic, as D–algebras to $\mathbb{C}[W][\![\hbar]\!]$ and $U\mathfrak{g}[\![\hbar]\!]$ respectively.

3.4. **Proof of proposition 3.2.** We may assume that $|\mathbf{I}| = 2$. (i) Let $\Psi : \mathcal{H}(W) \to \mathbb{C}[W][\![\hbar]\!]$ be an isomorphism such that $\Psi(\mathcal{H}(W_i)) = \mathbb{C}[W_i][\![\hbar]\!]$ for any $i \in \mathbf{I}$. Then, $\Psi(S_i) = x_i s_i + y_i$ for some $x_i, y_i \in \mathbb{C}[\![\hbar]\!]$ with $x_i \neq 0$. Since $S_i^2 = (q_i - q_i^{-1})S_i + 1$, we get $y_i = (q_i - q_i^{-1})/2$. Equating the coefficients of $s_1 s_2 \cdots$ ($m_{12} - 2$ factors) in

$$\Psi(\underbrace{S_1 S_2 \cdots}_{m_{12}}) = (x_1 s_1 + y_1)(x_2 s_2 + y_2) \cdots$$

and in

$$\Psi(\underbrace{S_2 S_1 \cdots}_{m_{12}}) = (x_2 s_2 + y_2)(x_1 s_1 + y_1) \cdots$$

yields $2y_i y_j = y_i y_j$, a contradiction.

(ii) Recall that any algebra isomorphism $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ equal to the identity mod $\hbar$ canonically identifies the centre of $U_\hbar\mathfrak{g}$ with $Z(U\mathfrak{g})[\![\hbar]\!]$ [Dr2, p. 331]. Let $C_i = e_i f_i + f_i e_i + 1/2\, h_i^2 \in U\mathfrak{sl}_2^i$ be the Casimir operator of $\mathfrak{sl}_2^i$ and $C_i^\hbar$ the corresponding element of $Z(U_\hbar\mathfrak{sl}_2^i)$. If $\Psi(U_\hbar\mathfrak{sl}_2^i) = U\mathfrak{sl}_2^i[\![\hbar]\!]$ for any i, then $\Psi(C_i^\hbar) = C_i$. We will prove that such a Ψ does not exist by showing that $C_1C_2 = \Psi(C_1^\hbar C_2^\hbar)$ and $C_1^\hbar C_2^\hbar$ have different eigenvalues on the adjoint representation V of $\mathfrak{g}$ and its quantum deformation $\mathcal{V}$ respectively. Lusztig has given an explicit presentation of $\mathcal{V}$ [Lu2, §2.1]. Its zero weight space $\mathcal{V}[0]$ is spanned by t_1, t_2 and

$$E_i\, t_j = -[a_{ji}]_j X_i$$

with $E_i\, X_i = 0$.[1] Thus, for $i \neq j$, t_i and $t_j - [a_{ji}]_j/[2]_i\, t_i$ transform like the zero weight spaces of the quantum deformations of the simple $\mathfrak{sl}_2^i$–modules V_2^i and V_0^i of highest weights 2 and 0 respectively. Since C_i acts as multiplication by $m(m+2)/2$ on V_m^i, we get

$$C_i^\hbar t_i = 4t_i \qquad \text{and} \qquad C_i^\hbar(t_j - [a_{ji}]_j/[2]_i t_i) = 0$$

so that, on $\mathcal{V}[0]$,

$$C_1^\hbar = \begin{pmatrix} 4 & 4[a_{21}]_2/[2]_1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C_2^\hbar = \begin{pmatrix} 0 & 0 \\ 4[a_{12}]_1/[2]_2 & 4 \end{pmatrix}$$

It follows that

$$C_1^\hbar C_2^\hbar = \begin{pmatrix} 16[a_{12}]_1[a_{21}]_2/[2]_1[2]_2 & 16[a_{21}]_2/[2]_1 \\ 0 & 0 \end{pmatrix}$$

has eigenvalues 0 and $16[a_{12}]_2[a_{21}]_1/[2]_1[2]_2 \in \mathbb{C}[\![\hbar]\!] \setminus \mathbb{C}$ on $\mathcal{V}[0]$. Since the isomorphism class of a finite–dimensional $U_\hbar\mathfrak{g}$–module $\mathcal{U}$ is uniquely determined by that of the $\mathfrak{g}$–module $\mathcal{U}/\hbar\mathcal{U}$, $\mathcal{V}$ is isomorphic as $U_\hbar\mathfrak{g}$–module to $V[\![\hbar]\!]$, where $U_\hbar\mathfrak{g}$ acts on the latter via $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$. The eigenvalues of $C_1^\hbar C_2^\hbar$ on $\mathcal{V}$ cannot therefore depend upon $\hbar$ ■

3.5. **Completion with respect to finite–dimensional representations.** Let $\mathbf{Vec_k}$ be the category of finitely–generated, free k–modules and $\mathbf{Mod_{fd}}(A)$ that of finite–dimensional A–modules, that is the A–modules whose underlying k–module lies in $\mathbf{Vec_k}$. Consider the forgetful functor

$$\mathbf{F} : \mathbf{Mod_{fd}}(A) \to \mathbf{Vec_k}$$

By definition, the completion of A with respect to its finite–dimensional representations is the algebra $\widehat{A}$ of endomorphisms of $\mathbf{F}$. Thus, an element of $\widehat{A}$ is a collection $\Theta = \{\Theta_V\}$, with $\Theta_V \in \mathrm{End}_k(V)$ for any $V \in \mathbf{Mod_{fd}}(A)$, such that for any $U, V \in \mathbf{Mod_{fd}}(A)$ and $f \in \mathrm{Hom}_A(U, V)$

$$\Theta_V \circ f = f \circ \Theta_U$$

There is a natural homomorphism $A \to \widehat{A}$ mapping $a \in A$ to the element $\Theta(a)$ which acts on a finite–dimensional representation ρ :

[1] for a definition of the q–numbers $[n]_i$, see §4.1.3.

$A \to \mathrm{End}_k(V)$ as $\rho(a)$. The following is a straightforward consequence of the above definitions

Proposition 3.5.

(i) *$A \to \widehat{A}$ is a functor.*

(ii) *For any algebra A, the natural map $\widehat{A} \to \widehat{\widehat{A}}$ is an isomorphism.*

(iii) *Let $\hbar$ be a formal variable and $A[\![\hbar]\!]$ the formal power series in $\hbar$ with coefficients in A, regarded as an algebra over $k[\![\hbar]\!]$. Then, the natural homomorphism $\widehat{A}[\![\hbar]\!] \to \widehat{A[\![\hbar]\!]}$ induced by mapping $U \in \mathbf{Mod_{fd}}(A)$ to $U[\![\hbar]\!] \in \mathbf{Mod_{fd}}(A[\![\hbar]\!])$ is an isomorphism if finite–dimensional A–modules do not admit non–trivial deformations.*

(iv) *If A is a bialgebra, $\widehat{A}$ is a bialgebra with coproduct and counit*

$$\Delta(\Theta)_{U\otimes V} = \Theta_U \otimes \Theta_V \qquad \textit{and} \qquad \varepsilon(\Theta) = \Theta_k$$

where A acts on $U \otimes V$ via the coproduct $\Delta : A \to A^{\otimes 2}$ and on k via the counit $\varepsilon : A \to k$.

3.6. **Elementary pairs of maximal nested sets.** The terminology below corresponds, via the dictionary of §1.17.4, to that of §1.12 and §1.15–§1.16.

Definition 3.6. *An ordered pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D is called elementary if $\mathcal{G}$ and $\mathcal{F}$ differ by an element. A sequence $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of maximal nested sets on D is called elementary if $|\mathcal{H}_{i+1} \setminus \mathcal{H}_i| = 1$ for any $i = 1, \ldots, m-1$.*

By remark 2.6, elementary pairs correspond to oriented edges of the associahedron $\mathcal{A}_D$ and elementary sequences to edge–paths in $\mathcal{A}_D$.

Definition 3.7. *The support* $\mathrm{supp}(\mathcal{G}, \mathcal{F})$ *of an elementary pair of maximal nested sets on D is the unique unsaturated element of $\mathcal{F} \cap \mathcal{G}$. The central support* $\mathfrak{z}\mathrm{supp}(\mathcal{G}, \mathcal{F})$ *of $(\mathcal{G}, \mathcal{F})$ is the union of the maximal elements of $\mathcal{F} \cap \mathcal{G}$ properly contained in* $\mathrm{supp}(\mathcal{G}, \mathcal{F})$. *Thus,*

$$\mathfrak{z}\mathrm{supp}(\mathcal{G}, \mathcal{F}) = \mathrm{supp}(\mathcal{G}, \mathcal{F}) \setminus \underline{\alpha}_{\mathcal{G}\cap\mathcal{F}}^{\mathrm{supp}(\mathcal{G},\mathcal{F})}$$

Definition 3.8. *Two elementary pairs $(\mathcal{F}, \mathcal{G})$, $(\mathcal{F}', \mathcal{G}')$ of maximal nested sets on D are equivalent if*

$$\mathrm{supp}(\mathcal{F}, \mathcal{G}) = \mathrm{supp}(\mathcal{F}', \mathcal{G}'),$$

$$\alpha_{\mathcal{F}}^{\mathrm{supp}(\mathcal{F},\mathcal{G})} = \alpha_{\mathcal{F}'}^{\mathrm{supp}(\mathcal{F}',\mathcal{G}')} \qquad \textit{and} \qquad \alpha_{\mathcal{G}}^{\mathrm{supp}(\mathcal{F},\mathcal{G})} = \alpha_{\mathcal{G}'}^{\mathrm{supp}(\mathcal{F}',\mathcal{G}')}$$

Remark 3.9. As in proposition 1.38, one readily shows that for an elementary pair $(\mathcal{F}, \mathcal{G})$,

$$\underline{\alpha}_{\mathcal{F}\cap\mathcal{G}}^{\mathrm{supp}(\mathcal{F},\mathcal{G})} = \{\alpha_{\mathcal{F}}^{\mathrm{supp}(\mathcal{F},\mathcal{G})}, \alpha_{\mathcal{G}}^{\mathrm{supp}(\mathcal{F},\mathcal{G})}\}$$

In particular, two equivalent elementary pairs have the same central support.

3.7. **Labelled diagrams and Artin braid groups.**

Definition 3.10. *A labelling of the diagram D is the assignement of an integer $m_{ij} \in \{2, 3, \ldots, \infty\}$ to any pair α_i, α_j of distinct vertices of D such that*

$$m_{ij} = m_{ji} \qquad \text{and} \qquad m_{ij} = 2$$

if, and only if α_i and α_j are orthogonal.

Let D be a labelled diagram.

Definition 3.11 (Brieskorn–Saito [BS]). *The Artin group B_D is the group generated by elements S_i labelled by the vertices α_i of D with relations*

$$\underbrace{S_i S_j \cdots}_{m_{ij}} = \underbrace{S_j S_i \cdots}_{m_{ij}}$$

for any $\alpha_i \neq \alpha_j$.

We shall also refer to B_D as the braid group corresponding to D.

3.8. **Quasi–Coxeter algebras.** Let D be a labelled diagram.

Definition 3.12. *A quasi–Coxeter algebra of type D is a D–algebra A endowed with the following additional data*

- **Local monodromies :** *for each $\alpha_i \in D$, an invertible element*

$$S_i^A \in \widehat{A_i}$$

  where $\widehat{A_i}$ is the completion of A_i with respect to its finite–dimensional representations.
- **Elementary Associators :** *for each elementary pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D, an invertible element*

$$\Phi_{\mathcal{G}\mathcal{F}} \in A$$

satisfying the following axioms

- **Orientation :** *for any elementary pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D,*

$$\Phi_{\mathcal{F}\mathcal{G}} = \Phi_{\mathcal{G}\mathcal{F}}^{-1}$$

- **Coherence :** *for any pair of elementary sequences $\mathcal{H}_1, \ldots, \mathcal{H}_m$ and $\mathcal{K}_1, \ldots, \mathcal{K}_\ell$ of maximal nested sets on D such that $\mathcal{H}_1 = \mathcal{K}_1$ and $\mathcal{H}_m = \mathcal{K}_\ell$,*

$$\Phi_{\mathcal{H}_m \mathcal{H}_{m-1}} \cdots \Phi_{\mathcal{H}_2 \mathcal{H}_1} = \Phi_{\mathcal{K}_\ell \mathcal{K}_{\ell-1}} \cdots \Phi_{\mathcal{K}_2 \mathcal{K}_1}$$

- **Support :** *for any elementary pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D,

$$\Phi_{\mathcal{G}\mathcal{F}} \in A_{\operatorname{supp}(\mathcal{G},\mathcal{F})}^{\mathfrak{z}\operatorname{supp}(\mathcal{G},\mathcal{F})}$$

- **Forgetfulness :** *for any equivalent elementary pairs* $(\mathcal{G},\mathcal{F})$, $(\mathcal{G}',\mathcal{F}')$ *of maximal nested sets on* D,

$$\Phi_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{G}'\mathcal{F}'}$$

- **Braid relations :** *for any pair* α_i,α_j *of distinct vertices of* D *such that* $m_{ij}<\infty$, *and elementary pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D *such that* $\alpha_i\in\mathcal{F}$ *and* $\alpha_j\in\mathcal{G}$,

$$\operatorname{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^A)\cdot S_j^A\cdots = S_j^A\cdot\operatorname{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^A)\cdots$$

where the number of factors on each side is equal to m_{ij}.

Definition 3.13. *A morphism of quasi–Coxeter algebras* A, A' *of type* D *is a morphism* $\{\Psi_{\mathcal{F}}\}$ *of the underlying* D*–algebras such that*

- *for any* $\alpha_i\in D$ *and maximal nested set* $\mathcal{F}$ *on* D *with* $\{\alpha_i\}\in\mathcal{F}$,

$$\Psi_{\mathcal{F}}(S_i^A) = S_i^{A'}$$

- *for any elementary pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D,

$$\Psi_{\mathcal{G}}\circ\operatorname{Ad}(\Phi^A_{\mathcal{G}\mathcal{F}}) = \operatorname{Ad}(\Phi^{A'}_{\mathcal{G}\mathcal{F}})\circ\Psi_{\mathcal{F}}$$

3.9. **The symmetric difference $\mathcal{F}\Delta\mathcal{G}$.** Let $\mathcal{F},\mathcal{G}$ be maximal nested sets on D. We characterise below the unsaturated elements $B_1,\ldots,B_m$ of $\mathcal{F}\cap\mathcal{G}$ and the subsets of vertices $\alpha^{B_i}_{\mathcal{F}\cap\mathcal{G}}$ in terms of the symmetric difference

$$\mathcal{F}\Delta\mathcal{G} = \mathcal{F}\setminus\mathcal{G}\cup\mathcal{G}\setminus\mathcal{F}$$

These results will be used in §3.10–§3.12. For any (maximal) nested set $\mathcal{H}$ on D and $B\in\mathcal{H}$, let

$$\mathcal{H}_B = \{B'\in\mathcal{H}|\, B'\subseteq B\}$$

be the (maximal) nested set on B induced by $\mathcal{H}$.

Lemma 3.14. *Let* C *be an unsaturated element of* $\mathcal{F}\cap\mathcal{G}$. *Then,*

$$C = \bigcup_{B\in\mathcal{F}_C\Delta\mathcal{G}_C} B \tag{3.1}$$

Proof. The right–hand side is clearly contained in the left–hand side. Let $\alpha_1=\alpha^C_{\mathcal{F}}$, $\alpha_2=\alpha^C_{\mathcal{G}}$ and note that $\alpha_1\neq\alpha_2$ since C is unsaturated. Set $C_1=\complement^{C\setminus\alpha_1}_{\alpha_2}\in\mathcal{F}\setminus\mathcal{G}$ and $C_2=\complement^{C\setminus\alpha_2}_{\alpha_1}\in\mathcal{G}\setminus\mathcal{F}$. Since C_1 and C_2 are not compatible, $C_1\cup C_2$ is connected and properly contains C_1,C_2. Let $\overline{C}$ be the connected component of the right–hand side of (3.1) containing C_1,C_2. We claim that $\overline{C}$ is compatible with any $B\in\mathcal{F}$. This is clear if $B \perp C$, if $B \supseteq C$ or if $B \subset C$ and $C \notin \mathcal{G}$. If, on the other hand, $B \in \mathcal{G}$ then B is compatible with any element in $\mathcal{F}\Delta\mathcal{G}$ and therefore with $\overline{C}$. By maximality of $\mathcal{F}$, $\overline{C} \in \mathcal{F}$ whence $\overline{C} = C$ since $C_1 \subsetneq \overline{C} \subset C$ ■

Proposition 3.15. *The connected components of*

$$\bigcup_{B \in \mathcal{F}\Delta\mathcal{G}} B$$

are the maximal unsaturated elements of $\mathcal{F} \cap \mathcal{G}$.

Proof. We claim that any $B \in \mathcal{F}\Delta\mathcal{G}$ is contained in an unsaturated element of $\mathcal{F} \cap \mathcal{G}$. Let $\overline{B}$ be the minimal element in $\mathcal{F} \cap \mathcal{G}$ containing B. If $B \in \mathcal{F} \setminus \mathcal{G}$ (resp. $\mathcal{G} \setminus \mathcal{F}$), B is contained in a connected components $\overline{B}'$ of $\overline{B} \setminus \alpha_{\mathcal{F}}^{\overline{B}}$ (resp. $\overline{B} \setminus \alpha_{\mathcal{G}}^{\overline{B}}$). If $\overline{B}$ were saturated as an element of $\mathcal{F} \cap \mathcal{G}$, then $\alpha_{\mathcal{F}}^{\overline{B}} = \alpha_{\mathcal{F}\cap\mathcal{G}}^{\overline{B}} = \alpha_{\mathcal{G}}^{\overline{B}}$ and $\overline{B}' \in \mathcal{F} \cap \mathcal{G}$, in contradiction with the minimality of $\overline{B}$. It follows that

$$\bigcup_{B \in \mathcal{F}\Delta\mathcal{G}} B = \bigcup_{\substack{C \in \mathcal{F}\cap\mathcal{G}, \\ C \text{ unsaturated}}} \bigcup_{\substack{B \in \mathcal{F}\Delta\mathcal{G}, \\ B \subseteq C}} B = \bigcup_{\substack{C \in \mathcal{F}\cap\mathcal{G}, \\ C \text{ unsaturated}}} C$$

where the second equality holds by lemma 3.14 ■

Proposition 3.16. *Let C be an unsaturated element of $\mathcal{F} \cap \mathcal{G}$. Then, the connected components of*

$$\bigcup_{\substack{B \subsetneq C, \\ B \text{ compatible with any } B' \in \mathcal{F}\Delta\mathcal{G}}} B \tag{3.2}$$

are the maximal elements of $\mathcal{F} \cap \mathcal{G}$ properly contained in C.

Proof. Any $B \in \mathcal{F} \cap \mathcal{G}$ with $B \subsetneq C$ is clearly contained in (3.2). Let now $B \subsetneq C$ be connected and compatible with any $B' \in \mathcal{F}\Delta\mathcal{G}$. Assume that B is not contained in one of the maximal elements $C_1, \ldots, C_m$ of $\mathcal{F} \cap \mathcal{G}$ properly contained in C, set $\underline{C} = C_1 \sqcup \cdots \sqcup C_m$ and let $\overline{B}$ be the image of B in $\overline{C} = C/\underline{C}$. Let $\overline{\mathcal{F}}_C$ and $\overline{\mathcal{G}}_C$ be the maximal nested sets on $\overline{C}$ induced by $\mathcal{F}_C, \mathcal{G}_C$ respectively. By (ii) of lemma 2.11, $\overline{B}$ is compatible with

$$\overline{\mathcal{F}}_C \setminus \overline{\mathcal{G}}_C = \overline{\mathcal{F}}_C \setminus \{\overline{C}\} \qquad \text{and} \qquad \overline{\mathcal{G}}_C \setminus \overline{\mathcal{F}}_C = \overline{\mathcal{G}}_C \setminus \{\overline{C}\}$$

and therefore with $\overline{\mathcal{F}}_C, \overline{\mathcal{G}}_C$. By maximality of $\overline{\mathcal{F}}_C$ and $\overline{\mathcal{G}}_C$, $\overline{B}$ lies in $\overline{\mathcal{F}}_C \cap \overline{\mathcal{G}}_C = \{\overline{C}\}$. We claim that this is a contradiction. It suffices for this to prove the existence of a $B' \in \mathcal{F}_C\Delta\mathcal{G}_C$ such that $B \subseteq B'$, for then $\overline{C} = \overline{B} \subset \overline{B}' \subset \overline{C}$, whence, by (2.4), $B' = \widetilde{\overline{B}'} = C \in \mathcal{F} \cap \mathcal{G}$. Assume for this purpose that $B \not\subseteq B'$ for any $B' \in \mathcal{F}_C \Delta \mathcal{G}_C$ and set

$$C_\perp = \bigcup_{\substack{B' \in \mathcal{F}_C \Delta \mathcal{G}_C, \\ B' \perp B}} B' \qquad \text{and} \qquad C_\subset = \bigcup_{\substack{B' \in \mathcal{F}_C \Delta \mathcal{G}_C, \\ B' \subset B}} B'$$

By lemma 3.14, $C = C_\perp \cup C_\subset$. Since C is connected and $C_\perp$ and $C_\subset$ are orthogonal, one has $C_\perp = \emptyset$ or $C_\subset = \emptyset$ and therefore $C \subset B$ or $C \perp B$ respectively, both of which contradict $B \subsetneq C$ ■

The following is a direct consequence of propositions 3.15 and 3.16.

Corollary 3.17. *The unsaturated elements $B_1, \dots, B_m$ of $\mathcal{F} \cap \mathcal{G}$ and subsets of vertices $\underline{\alpha}^{B_i}_{\mathcal{F}\cap\mathcal{G}} \subset B_i$ only depend upon the symmetric difference $\mathcal{F} \Delta \mathcal{G}$.*

3.10. **Support and central support of a pair of maximal nested sets.** We extend below the notions of support and central support to a general pair $(\mathcal{F}, \mathcal{G})$ of maximal nested sets on D.

Definition 3.18. *The support of $(\mathcal{F}, \mathcal{G})$ is the union*

$$\operatorname{supp}(\mathcal{F}, \mathcal{G}) = \bigcup_{B \in \mathcal{F} \Delta \mathcal{G}} B$$

By proposition 3.15, $\operatorname{supp}(\mathcal{F}, \mathcal{G})$ is the union of the maximal unsaturated elements of $\mathcal{F} \cap \mathcal{G}$. In particular, definition 3.18 is consistent with definition 3.7 when $\mathcal{F}$ and $\mathcal{G}$ differ by an element.

For any collection $\mathcal{C}$ of connected subdiagrams of D, set now

$$\kappa(\mathcal{C}) = \{B \subseteq D | \; B \perp C \text{ or } B \subseteq C \text{ for any } C \in \mathcal{C}\}$$

One readily checks that if $B_1, B_2 \in \kappa(\mathcal{C})$ are incompatible, then $B_1 \cup B_2 \in \kappa(\mathcal{C})$. In particular, the maximal elements $B_1, \dots, B_m$ of $\kappa(\mathcal{C})$ are pairwise orthogonal and

$$\bigcup_{B \in \kappa(\mathcal{C})} B = B_1 \sqcup \cdots \sqcup B_m$$

Note that if $B \in \kappa(\mathcal{F} \Delta \mathcal{G})$, then either $B \subset \operatorname{supp}(\mathcal{F}, \mathcal{G})$ or $B \perp \operatorname{supp}(\mathcal{F}, \mathcal{G})$.

Definition 3.19. *The central support $\mathfrak{z}\operatorname{supp}(\mathcal{F}, \mathcal{G})$ is the union $\bigcup B$, where B ranges over the elements of $\kappa(\mathcal{F} \Delta \mathcal{G})$ contained in $\operatorname{supp}(\mathcal{F}, \mathcal{G})$.*

The following result shows that definition 3.19 is consistent with definition 3.7.

Proposition 3.20. *Assume that $\mathcal{G}$ and $\mathcal{F}$ differ by an element and let $B = \operatorname{supp}(\mathcal{F}, \mathcal{G})$ be the unique unsaturated element of $\mathcal{F} \cap \mathcal{G}$. Then,*

$$\bigcup_{\substack{C \in \kappa(\mathcal{F}\Delta\mathcal{G}),\\ C \subseteq B}} C = B \setminus \underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}}$$

Proof. Set $\alpha_1 = \alpha^B_{\mathcal{F}}$ and $\alpha_2 = \alpha^B_{\mathcal{G}}$, so that $\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}} = \{\alpha_1, \alpha_2\}$, $B_1 = \complement^{B\setminus\alpha_1}_{\alpha_2} \in \mathcal{F} \setminus \mathcal{G}$ and $B_2 = \complement^{B\setminus\alpha_2}_{\alpha_1} \in \mathcal{G} \setminus \mathcal{F}$. Since any $C \in \kappa(\{B_1, B_2\})$ does not contain α_1 and α_2, the left–hand side is contained in the right–hand side. The opposite inclusion is easy to check ■

Remark 3.21. Note that $\kappa(\mathcal{F}\Delta\mathcal{G})$ and $\mathcal{F}\Delta\mathcal{G}$ are disjoint. Indeed, any $B \in \kappa(\mathcal{F}\Delta\mathcal{G}) \cap \mathcal{F}\Delta\mathcal{G}$ is compatible with $\mathcal{G} = \mathcal{G} \setminus \mathcal{F} \cup (\mathcal{F} \cap \mathcal{G})$ and $\mathcal{F} = \mathcal{F} \setminus \mathcal{G} \cup (\mathcal{F} \cap \mathcal{G})$ and therefore lies in $\mathcal{F} \cap \mathcal{G}$ by maximality of $\mathcal{F}$ and $\mathcal{G}$, a contradiction. Thus, if $B \in \kappa(\mathcal{F}\Delta\mathcal{G})$ and $C \in \mathcal{F}\Delta\mathcal{G}$, then

$$B \perp C \qquad \text{or} \qquad B \subsetneq C$$

3.11. **Equivalence of pairs of maximal nested sets.** We shall need to extend the notion of equivalence to general pairs of maximal nested sets on D. We begin by giving an alternative characterisation of the equivalence of two elementary such pairs.

Proposition 3.22. *Two elementary pairs $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ of maximal nested sets on D are equivalent if, and only if*

$$\mathcal{F} \setminus \mathcal{G} = \mathcal{F}' \setminus \mathcal{G}' \qquad \textit{and} \qquad \mathcal{G} \setminus \mathcal{F} = \mathcal{G}' \setminus \mathcal{F}'$$

Proposition 3.22 is an immediate corollary of the following.

Proposition 3.23. *Let $(\mathcal{F}, \mathcal{G})$ be an elementary pair of maximal nested sets on D. Let $B = \operatorname{supp}(\mathcal{F}, \mathcal{G})$ be the unique unsaturated element of $\mathcal{F} \cap \mathcal{G}$ and B_1, B_2 the unique elements in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G} \setminus \mathcal{F}$ respectively. Then,*

(i) *$\alpha^B_{\mathcal{F}}$ and $\alpha^B_{\mathcal{G}}$ are distinct and $\underline{\alpha}^B_{\mathcal{F}\cap\mathcal{G}} = \{\alpha^B_{\mathcal{F}}, \alpha^B_{\mathcal{G}}\}$.*
(ii) *$B_1 = \complement^{B\setminus\alpha^B_{\mathcal{F}}}_{\alpha^B_{\mathcal{G}}}$ and $B_2 = \complement^{B\setminus\alpha^B_{\mathcal{G}}}_{\alpha^B_{\mathcal{F}}}$.*
(iii) *$\alpha^{B_1}_{\mathcal{F}} = \alpha^B_{\mathcal{G}}$ and $\alpha^{B_2}_{\mathcal{G}} = \alpha^B_{\mathcal{F}}$.*
(iv) *B_1, B_2 are not compatible and $B_1 \cup B_2 = B$.*
(v) *$\alpha^B_{\mathcal{F}}$, $\alpha^B_{\mathcal{G}}$ are uniquely determined by (ii).*

Proof. (i)–(iii) are proved exactly as in proposition 1.38. (iv) the incompatibility of B_1, B_2 is a direct consequence of (ii). The fact that $B = B_1 \cup B_2$ follows by lemma 3.14. (v) Let α_1 a vertex of B such that B_1 is a connected component of $B \setminus \alpha_1$. Then, $\alpha_1 \not\perp B_1$ so that, if $\alpha_1 \neq \alpha^B_{\mathcal{F}}$, α_1 and B_1 lie in the same connected component of $B \setminus \alpha^B_{\mathcal{F}}$.

By (ii), this implies that $\alpha_1 \in B_1$, a contradiction. Similarly, $\alpha_{\mathcal{G}}^B$ is the unique vertex of B such that B_2 is a connected component of $B \setminus \alpha_{\mathcal{G}}^B$ ■

Proposition 3.22 ensures that the following is consistent with definition 3.8.

Definition 3.24. *Two ordered pairs $(\mathcal{F},\mathcal{G})$, $(\mathcal{F}',\mathcal{G}')$ of maximal nested sets on D are equivalent if*

$$\mathcal{F} \setminus \mathcal{G} = \mathcal{F}' \setminus \mathcal{G}' \qquad \text{and} \qquad \mathcal{G} \setminus \mathcal{F} = \mathcal{G}' \setminus \mathcal{F}'$$

Note that the equivalence of $(\mathcal{F},\mathcal{G})$ and $(\mathcal{F}',\mathcal{G}')$ implies that

$$\operatorname{supp}(\mathcal{F},\mathcal{G}) = \operatorname{supp}(\mathcal{F}',\mathcal{G}') \qquad \text{and} \qquad \mathfrak{z}\operatorname{supp}(\mathcal{F},\mathcal{G}) = \mathfrak{z}\operatorname{supp}(\mathcal{F}',\mathcal{G}')$$

3.12. **Existence of good elementary sequences.**

Proposition 3.25.

(i) *For any pair $(\mathcal{F},\mathcal{G})$ of maximal nested sets on D, there exists an elementary sequence*

$$\mathcal{F} = \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m = \mathcal{G}$$

such that, for any $i = 1, \ldots, m-1$,

$$\mathcal{H}_i \cap \mathcal{H}_{i+1} \supseteq \mathcal{F} \cap \mathcal{G},$$
$$\operatorname{supp}(\mathcal{H}_i, \mathcal{H}_{i+1}) \subseteq \operatorname{supp}(\mathcal{F},\mathcal{G})$$

and, for any component B of $\mathfrak{z}\operatorname{supp}(\mathcal{F},\mathcal{G})$, either

$$B \perp \operatorname{supp}(\mathcal{H}_i, \mathcal{H}_{i+1}) \quad \text{or} \quad B \subseteq \mathfrak{z}\operatorname{supp}(\mathcal{H}_i, \mathcal{H}_{i+1})$$

(ii) *If $(\mathcal{G},\mathcal{F})$ and $(\mathcal{G}',\mathcal{F}')$ are equivalent pairs of maximal nested sets on D, the corresponding elementary sequences*

$$\mathcal{F} = \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m = \mathcal{G} \quad \text{and} \quad \mathcal{F}' = \mathcal{H}_1', \mathcal{H}_2', \ldots, \mathcal{H}_\ell' = \mathcal{G}'$$

may be chosen such that $\ell = m$ and such that, for any $i = 1 \ldots m-1$, $(\mathcal{H}_i, \mathcal{H}_{i+1})$ is equivalent to $(\mathcal{H}_i', \mathcal{H}_{i+1}')$.

Proof. (i) By the connectedness of the face of the associahedron $\mathcal{A}_D$ corresponding to $\mathcal{K} = \mathcal{F} \cap \mathcal{G}$, there exists an elementary sequence $\mathcal{F} = \mathcal{H}_1, \ldots, \mathcal{H}_m = \mathcal{G}$ such that $\mathcal{K} \subset \mathcal{H}_i$ for any i. Let $B_1, \ldots, B_p$ be the unsaturated elements of $\mathcal{K}$ and set $\underline{\alpha}_j = \underline{\alpha}_{\mathcal{K}}^{B_j}$. By proposition 2.13 and lemma 2.17, each $\mathcal{H}_i$ is the union of $\mathcal{K}$ and of a compatible family of diagrams of the form $D_{\underline{\beta}_j} = \complement_{\underline{\beta}_j}^{B_j \setminus (\underline{\alpha}_j \setminus \underline{\beta}_j)}$ for some $1 \leq j \leq p$ and $\emptyset \neq \underline{\beta}_j \subsetneq \underline{\alpha}_j$. For any $i = 1, \ldots, p-1$, set

$$D_{\underline{\beta}_{j_i}} = \mathcal{H}_i \setminus \mathcal{H}_{i+1} \qquad \text{and} \qquad D_{\underline{\gamma}_{k_i}} = \mathcal{H}_{i+1} \setminus \mathcal{H}_i$$

Since $D_{\underline{\beta}_{j_i}}$ and $D_{\underline{\gamma}_{k_i}}$, one has $j_i = k_i$ whence,

$$\mathrm{supp}(\mathcal{H}_i, \mathcal{H}_{i+1}) = D_{\underline{\beta}_{j_i}} \cup D_{\underline{\gamma}_{k_i}} \subseteq B_{j_i} \subsetneq \mathrm{supp}(\mathcal{F}, \mathcal{G})$$

Let now B be a component of $\mathfrak{z}\mathrm{supp}(\mathcal{F}, \mathcal{G})$. We shall need the following

Lemma 3.26. *For any $1 \leq j \leq p$, one has $B \perp B_{j_i}$ or $B \subseteq B_{j_i} \setminus \underline{\alpha}_{j_i}$.*

Proof. By lemma 3.14, $B_{j_i} = \bigcup_{B' \in \mathcal{F}_{B_{j_i}} \Delta \mathcal{G}_{B_{j_i}}} B'$. Since B is compatible with any such B' and does not contain it by remark 3.21, either $B \perp B_{j_i}$ or $B \subsetneq B_{j_i}$. In the latter case, $B \subseteq \bigcup B'$ where B' now ranges over the proper connected subdiagrams of B_{j_i} which are compatible with any element of $\mathcal{F}_{B_{j_i}} \Delta \mathcal{G}_{B_{j_i}}$, whence $B \subseteq B_{j_i} \setminus \underline{\alpha}_{j_i}$ by proposition 3.16 ■

If $B \perp B_{j_i}$, then $B \perp \mathrm{supp}(\mathcal{H}_i, \mathcal{H}_{i+1})$ as required. If, on the other hand $B \subseteq B_{j_i} \setminus \underline{\alpha}_{j_i}$, then B is compatible with $D_{\underline{\beta}_{j_i}}$ and $D_{\underline{\gamma}_{j_i}}$ and contains neither since $\underline{\alpha}_{j_i} \cap D_{\underline{\beta}_{j_i}}, \underline{\alpha}_{j_i} \cap D_{\underline{\gamma}_{j_i}} \neq \emptyset$. Thus, either $B \perp D_{\underline{\beta}_{j_i}}, D_{\underline{\gamma}_{j_i}}$, in which case $B \perp D_{\underline{\beta}_{j_i}} \cup D_{\underline{\gamma}_{j_i}} = \mathrm{supp}(\mathcal{H}_i, \mathcal{H}_{i+1})$, or $B \subseteq \bigcup B'$, where the union ranges over the connected subdiagrams of $\mathrm{supp}(\mathcal{H}_i, \mathcal{H}_{i+1})$ compatible with, but not containing either of $D_{\underline{\beta}_{j_i}}, D_{\underline{\gamma}_{j_i}}$ and therefore $B \subseteq \mathfrak{z}\mathrm{supp}(\mathcal{H}_i, \mathcal{H}_{i+1})$.

(ii) Let $\mathcal{H}_i = \mathcal{K} \cup \{D_{\underline{\beta}_j}\}_{j \in J_i}$ be the elementary sequence obtained in (i). By corollary 3.17, $\mathcal{K}' = \mathcal{F}' \cap \mathcal{G}'$ and $\mathcal{K}$ have the same unsaturated elements $B_1, \ldots, B_p$ and $\underline{\alpha}_{\mathcal{K}'}^{B_i} = \underline{\alpha}_{\mathcal{K}}^{B_i}$ for any $i = 1, \ldots, p$. It follows from this, proposition 2.13 and lemma 2.17 that $\mathcal{H}_i' = \mathcal{K}' \cup \{D_{\underline{\beta}_j}\}_{j \in J_i}$ is a maximal nested set on D ■

3.13. **General associators.** Let A be a quasi–Coxeter algebra of type D. By the connectedness of the associahedron $\mathcal{A}_D$, there exists, for any pair $\mathcal{G}, \mathcal{F}$ of maximal nested sets on D, an elementary sequence $\mathcal{H}_1, \ldots, \mathcal{H}_m$ such that $\mathcal{H}_1 = \mathcal{F}$ and $\mathcal{H}_m = \mathcal{G}$. Set

$$\Phi_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{H}_m \mathcal{H}_{m-1}} \cdots \Phi_{\mathcal{H}_2 \mathcal{H}_1}$$

The coherence axiom implies that this definition is independent of the choice of the elementary sequence and that $\Phi_{\mathcal{G}\mathcal{F}}$ is the elementary associator corresponding to $(\mathcal{G}, \mathcal{F})$ if $\mathcal{F}$ and $\mathcal{G}$ differ by an element. The following result summarises the main properties of the general associators $\Phi_{\mathcal{G}\mathcal{F}}$ and gives an equivalent characterisation of quasi–Coxeter algebras in terms of them.

Theorem 3.27. *The associators $\Phi_{\mathcal{G}\mathcal{F}}$ satisfy the following properties*

- **Orientation :** *for any pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D*,*
$$\Phi_{\mathcal{F}\mathcal{G}}=\Phi_{\mathcal{G}\mathcal{F}}^{-1}$$
- **Transitivity :** *for any triple* $\mathcal{H},\mathcal{G},\mathcal{F}$ *of maximal nested sets on* D*,*
$$\Phi_{\mathcal{H}\mathcal{F}}=\Phi_{\mathcal{H}\mathcal{G}}\cdot\Phi_{\mathcal{G}\mathcal{F}}$$
- **Forgetfulness :** *for any equivalent pairs* $(\mathcal{G},\mathcal{F})$ *and* $(\mathcal{G}',\mathcal{F}')$ *of maximal nested sets on* D*,*
$$\Phi_{\mathcal{G}\mathcal{F}}=\Phi_{\mathcal{G}'\mathcal{F}'}$$
- **Support :** *for any pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D*,*
$$\Phi_{\mathcal{G}\mathcal{F}}\in A_{\mathrm{supp}(\mathcal{F},\mathcal{G})}^{\mathfrak{z}\mathrm{supp}(\mathcal{F},\mathcal{G})}$$
- **Braid relations :** *for any pair* α_i,α_j *of distinct vertices of* D *such that* $m_{ij}<\infty$*, and pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D *such that* $\alpha_i\in\mathcal{F}$ *and* $\alpha_j\in\mathcal{G}$*,*
$$\mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^A)\cdot S_j^A\cdots=S_j^A\cdot\mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^A)\cdots$$
where the number of factors on each side is equal to m_{ij}*.*

Conversely, if A *is a* D*–algebra endowed with invertible elements* $S_i^A\in\widehat{A}_i$ *for any* $\alpha_i\in D$ *and* $\Phi_{\mathcal{G}\mathcal{F}}\in A$ *for any pair* $(\mathcal{G},\mathcal{F})$ *of maximal nested sets on* D *which satisfy the above properties, then the* S_i^A *and associators* $\Phi_{\mathcal{G}\mathcal{F}}$ *corresponding to elementary pairs give* A *the structure of a quasi–Coxeter algebra of type* D*.*

Proof. Orientation and transitivity follow at once from the orientation and coherence axioms satisfied by the elementary associators. If $(\mathcal{G},\mathcal{F})$ and $(\mathcal{G}',\mathcal{F}')$ are two equivalent pairs of maximal nested sets on D and
$$\mathcal{F}=\mathcal{H}_1,\ldots,\mathcal{H}_m=\mathcal{G}\qquad\text{and}\qquad\mathcal{F}'=\mathcal{H}'_1,\ldots,\mathcal{H}'_m=\mathcal{G}'$$
are two elementary sequences such that $(\mathcal{H}_i,\mathcal{H}_{i+1})$ and $(\mathcal{H}'_i,\mathcal{H}'_{i+1})$ are equivalent for any $i=1,\ldots,m-1$ as in proposition 3.25 (ii), then, by the forgetfulness axiom satisfied by elementary associators
$$\Phi_{\mathcal{G}\mathcal{F}}=\Phi_{\mathcal{H}_m\mathcal{H}_{m-1}}\cdots\Phi_{\mathcal{H}_2\mathcal{H}_1}=\Phi_{\mathcal{H}'_m\mathcal{H}'_{m-1}}\cdots\Phi_{\mathcal{H}'_2\mathcal{H}'_1}=\Phi_{\mathcal{G}'\mathcal{F}'}$$
Similarly, if $\mathcal{F}=\mathcal{H}_1,\ldots,\mathcal{H}_m=\mathcal{G}$ is an elementary sequence of maximal nested sets on D as in proposition 3.25 (i), then, by the support properties of elementary associators,
$$\Phi_{\mathcal{H}_{i+1}\mathcal{H}_i}\in A_{\mathrm{supp}(\mathcal{H}_{i+1},\mathcal{H}_i)}^{\mathfrak{z}\mathrm{supp}(\mathcal{H}_{i+1},\mathcal{H}_i)}\subseteq A_{\mathrm{supp}(\mathcal{G},\mathcal{F})}^{\mathfrak{z}\mathrm{supp}(\mathcal{G},\mathcal{F})}$$
so that
$$\Phi_{\mathcal{G}\mathcal{F}}=\Phi_{\mathcal{H}_m\mathcal{H}_{m_1}}\cdots\Phi_{\mathcal{H}_2\mathcal{H}_1}\in A_{\mathrm{supp}(\mathcal{G},\mathcal{F})}^{\mathfrak{z}\mathrm{supp}(\mathcal{G},\mathcal{F})}$$

Let now $\alpha_i \neq \alpha_j$ be such that $m_{ij} < \infty$, $\mathcal{G}, \mathcal{F}$ such that $\alpha_i \in \mathcal{F}$, $\alpha_j \in \mathcal{G}$ and choose an elementary pair $(\mathcal{F}', \mathcal{G}')$ such that $\alpha_i \in \mathcal{F}'$ and $\alpha_j \in \mathcal{G}'$. Since $\alpha_i \in \mathcal{F} \cap \mathcal{F}'$ and α_i does not contain any elements of $\mathcal{F}_i \Delta \mathcal{F}'_i$, α_i is either contained in $\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{F}')$ or perpendicular to $\text{supp}(\mathcal{F}, \mathcal{F}')$. In either case,

$$\text{Ad}(\Phi_{\mathcal{F}'\mathcal{F}})\,(S_i^A) = S_i^A \quad \text{and, similarly,} \quad \text{Ad}(\Phi_{\mathcal{G}'\mathcal{G}})\,(S_j^A) = S_j^A$$

Thus

$$\begin{aligned}\text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^A) \cdot S_j^A \cdots &= \text{Ad}(\Phi_{\mathcal{G}\mathcal{G}'} \cdot \Phi_{\mathcal{G}'\mathcal{F}'} \cdot \Phi_{\mathcal{F}'\mathcal{F}})(S_i^A) \cdot S_j^A \cdots \\ &= \text{Ad}(\Phi_{\mathcal{G}\mathcal{G}'})\left(\text{Ad}(\Phi_{\mathcal{G}'\mathcal{F}'})(S_i^A) \cdot S_j^A \cdots\right)\end{aligned}$$

and

$$S_j^A \cdot \text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S_i^A) \cdots = \text{Ad}(\Phi_{\mathcal{G}\mathcal{G}'})\left(S_j^A \cdot \text{Ad}(\Phi_{\mathcal{G}'\mathcal{F}'})(S_i^A) \cdots\right)$$

so the two are equal because of the braid relations satisfied by the elementary associators. The converse implication is clear ■

We record for later use the following consequence of proposition 3.25 and of the definition of general associators

Proposition 3.28. *If A is a quasi–Coxeter algebra of type D then, for any pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D and $B \in \mathcal{G} \cap \mathcal{F}$,*

$$\text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(A_B) \subseteq A_B$$

Proof. Assume first that $\mathcal{G}$ and $\mathcal{F}$ differ by an element and let $C = \text{supp}(\mathcal{F}, \mathcal{G})$ be the unique unsaturated element of $\mathcal{G} \cap \mathcal{F}$, so that $\Phi_{\mathcal{G}\mathcal{F}} \in A_C^{C \setminus \underline{\alpha}^C_{\mathcal{G}\cap\mathcal{F}}}$. If $B \perp C$ or $B \subseteq C$, in which case $B \subseteq C \setminus \underline{\alpha}^C_{\mathcal{G}\cap\mathcal{F}}$, then $[\Phi_{\mathcal{G}\mathcal{F}}, A_B] = 0$ and the result follows. If, on the other hand $B \supseteq C$, then $\Phi_{\mathcal{G}\mathcal{F}} \in A_C \subseteq A_B$ and the result follows again. If $(\mathcal{G}, \mathcal{F})$ is a general pair of maximal nested sets on D, proposition 3.25 implies the existence of an elementary sequence $\mathcal{F} = \mathcal{H}_0, \cdots, \mathcal{H}_m = \mathcal{G}$ such that, for any $i = 1, \ldots, m-1$, $B \in \mathcal{H}_i \cap \mathcal{H}_{i+1}$. The result now follows from our previous analysis ■

3.14. **Braid group representations from quasi–Coxeter algebras.** By mimicking the monodromy computations of §1 (in particular §1.17.11), we show below that a quasi–Coxeter algebra A of type D defines representations of the braid group B_D on any finite–dimensional A–module, with isomorphic quasi–Coxeter algebras defining equivalent representations of B_D. It is important to keep in mind that, just as the action of Artin's braid group B_n on the n–fold tensor product $V^{\otimes n}$ of an object in a braided tensor category depends upon the choice of a complete bracketing on $V^{\otimes n}$, the procedure described below yields not one, but a *family* of canonically equivalent representations

$$\pi_{\mathcal{F}} : B_D \longrightarrow \widehat{A}$$

labelled by the maximal nested sets $\mathcal{F}$ on D.

Let $\mathcal{F}$ be a maximal nested set on D. For any $\alpha_i \in D$, choose a maximal nested set $\mathcal{G}_i$ such that $\alpha_i \in \mathcal{G}_i$ and set

$$\pi_{\mathcal{F}}(S_i) = \Phi_{\mathcal{F}\mathcal{G}_i} \cdot S_i^A \cdot \Phi_{\mathcal{G}_i\mathcal{F}}$$

Theorem 3.29.

(i) *The above assignement is independent of the choice of $\mathcal{G}_i$ and extends to a homomorphism $\pi_{\mathcal{F}} : B_D \to \widehat{A}$.*

(ii) *If $\alpha_i \in \mathcal{F}$, then*

$$\pi_{\mathcal{F}}(S_i) = S_i^A$$

(iii) *For any $D' \in \mathcal{F}$,*

$$\pi_{\mathcal{F}}(B_{D'}) \subset \widehat{A_{D'}}$$

(iv) *If $\mathcal{G}$ is another maximal nested set on D then, for any $b \in B_D$,*

$$\pi_{\mathcal{G}}(b) = \Phi_{\mathcal{G}\mathcal{F}} \cdot \pi_{\mathcal{F}}(b) \cdot \Phi_{\mathcal{F}\mathcal{G}}$$

so that $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ are canonically equivalent.

(v) *If $\mathcal{G}$ is another maximal nested set and $D' \in \mathcal{G} \cap \mathcal{F}$ is such that the induced maximal nested sets $\mathcal{F}_{D'}, \mathcal{G}_{D'}$ on D' coincide, the restrictions of $\pi_{\mathcal{F}}, \pi_{\mathcal{G}}$ to $B_{D'}$ are equal.*

(vi) *If $\{\Psi_{\mathcal{F}}\}_{\mathcal{F}} : A \to A'$ is a morphism of quasi–Coxeter algebras, then for any maximal nested set $\mathcal{F}$ and $b \in B_D$,*

$$\Psi_{\mathcal{F}}(\pi_{\mathcal{F}}^A(b)) = \pi_{\mathcal{F}}^{A'}(b)$$

In particular, isomorphic quasi–Coxeter algebras yield equivalent representations of B_D.

Proof. (i) If $\mathcal{G}_i'$ is such that $\alpha_i \in \mathcal{G}_i'$, then either $\alpha_i \perp \text{supp}(\mathcal{G}_i, \mathcal{G}_i')$ or $\alpha_i \subseteq \mathfrak{z}\text{supp}(\mathcal{G}_i, \mathcal{G}_i')$ so that $\Phi_{\mathcal{G}_i'\mathcal{G}_i}$ centralises A_i by theorem 3.27. Thus,

$$\Phi_{\mathcal{F}\mathcal{G}_i} \cdot S_i^A \cdot \Phi_{\mathcal{G}_i\mathcal{F}} = \Phi_{\mathcal{F}\mathcal{G}_i'} \cdot \Phi_{\mathcal{G}_i'\mathcal{G}_i} \cdot S_i^A \cdot \Phi_{\mathcal{G}_i\mathcal{G}_i'} \cdot \Phi_{\mathcal{G}_i'\mathcal{F}} = \Phi_{\mathcal{F}\mathcal{G}_i'} \cdot S_i^A \cdot \Phi_{\mathcal{G}_i'\mathcal{F}}$$

Let now $\alpha_i \neq \alpha_j$ be such that $m_{ij} < \infty$. Then,

$$\begin{aligned}\pi_{\mathcal{F}}(S_i)\pi_{\mathcal{F}}(S_j)\cdots &= \Phi_{\mathcal{F}\mathcal{G}_i} \cdot S_i^A \cdot \Phi_{\mathcal{G}_i\mathcal{F}} \cdot \Phi_{\mathcal{F}\mathcal{G}_j} \cdot S_j^A \cdot \Phi_{\mathcal{G}_j\mathcal{F}} \cdots \\ &= \Phi_{\mathcal{F}\mathcal{G}_i} \cdot \left(S_i^A \cdot \Phi_{\mathcal{G}_i\mathcal{G}_j} \cdot S_j^A \cdot \Phi_{\mathcal{G}_j\mathcal{G}_i} \cdots\right) \cdot \Phi_{\mathcal{G}_i\mathcal{F}}\end{aligned}$$

and

$$\begin{aligned}\pi_{\mathcal{F}}(S_j)\pi_{\mathcal{F}}(S_i)\cdots &= \Phi_{\mathcal{F}\mathcal{G}_j}\cdot S_j^A\cdot\Phi_{\mathcal{G}_j\mathcal{F}}\cdot\Phi_{\mathcal{F}\mathcal{G}_i}\cdot S_i^A\cdot\Phi_{\mathcal{G}_i\mathcal{F}}\cdots\\ &= \Phi_{\mathcal{F}\mathcal{G}_i}\cdot\left(\Phi_{\mathcal{G}_i\mathcal{G}_j}\cdot S_j^A\cdot\Phi_{\mathcal{G}_j\mathcal{G}_i}\cdot S_i^A\cdots\right)\cdot\Phi_{\mathcal{G}_i\mathcal{F}}\end{aligned}$$

so that the two coincide by theorem 3.27.
(ii) follows by choosing $\mathcal{G}_i=\mathcal{F}$.
(iii) For any $\alpha_i\in D'$, let $\mathcal{G}_{D'}$ be a maximal nested set on D' such that $\alpha_i\in\mathcal{G}_{D'}$ and set

$$\mathcal{G}_i=(\mathcal{F}\setminus\mathcal{F}_{D'})\cup\mathcal{G}_{D'}$$

Then, $\text{supp}(\mathcal{F},\mathcal{G}_i)\subseteq D'$ so that, by theorem 3.27

$$\pi_{\mathcal{F}}(S_i)=\Phi_{\mathcal{F}\mathcal{G}_i}\cdot S_i^A\cdot\Phi_{\mathcal{G}_i\mathcal{F}}\in\widehat{A_{D'}}$$

(iv) we have

$$\pi_{\mathcal{G}}(S_i)=\Phi_{\mathcal{G}\mathcal{G}_i}\cdot S_i^A\cdot\Phi_{\mathcal{G}_i\mathcal{G}}=\Phi_{\mathcal{G}\mathcal{F}}\cdot\Phi_{\mathcal{F}\mathcal{G}_i}\cdot S_i^A\cdot\Phi_{\mathcal{G}_i\mathcal{F}}\cdot\Phi_{\mathcal{F}\mathcal{G}}=\Phi_{\mathcal{G}\mathcal{F}}\cdot\pi_{\mathcal{F}}(S_i)\cdot\Phi_{\mathcal{F}\mathcal{G}}$$

(v) By assumption, either $D'\perp\text{supp}(\mathcal{F},\mathcal{G})$ or $D'\subseteq\mathfrak{z}\text{supp}(\mathcal{F},\mathcal{G})$. It follows that $\Phi_{\mathcal{G}\mathcal{F}}$ centralises $A_{D'}$ whence, by (iii)

$$\pi_{\mathcal{G}}(S_i)=\Phi_{\mathcal{G}\mathcal{F}}\cdot\pi_{\mathcal{F}}(S_i)\cdot\Phi_{\mathcal{F}\mathcal{G}}=\pi_{\mathcal{F}}(S_i)$$

(vi) By definition,

$$\begin{aligned}\pi_{\mathcal{F}}^{A'}(S_i)&=\Phi^{A'}_{\mathcal{F}\mathcal{G}_i}\cdot S_i^{A'}\cdot\Phi^{A'}_{\mathcal{G}_i\mathcal{F}}\\ &=\Phi^{A'}_{\mathcal{F}\mathcal{G}_i}\cdot\Psi_{\mathcal{G}_i}(S_i^A)\cdot\Phi^{A'}_{\mathcal{G}_i\mathcal{F}}\\ &=\Psi_{\mathcal{F}}(\Phi^A_{\mathcal{F}\mathcal{G}_i}\cdot S_i^A\cdot\Phi^A_{\mathcal{G}_i\mathcal{F}})\\ &=\Psi_{\mathcal{F}}(\pi^A_{\mathcal{F}}(S_i))\end{aligned}$$

■

Remark 3.30. The group algebra $A'=k[B_D]$ of B_D may be regarded as a quasi–Coxeter algebra of type D by setting

$$A'_{D'}=k[B_{D'}],\qquad S_i^{A'}=S_i\qquad\text{and}\qquad\Phi^{A'}_{\mathcal{G}\mathcal{F}}=1$$

Theorem 3.29 may then be rephrased as saying that the collection $\{\pi_{\mathcal{F}}\}$ is a morphism of quasi–Coxeter algebras $k[B_D]\to\widehat{A}$ which is functorial in A.

3.15. **Generalised pentagon relations.** The coherence relations satisfied by the elementary associators of a quasi–Coxeter algebra are convenient for most applications but somewhat redundant. In this subsection, we use the simple connectedness of the associahedron $\mathcal{A}_D$ to reduce them to a smaller number of identities labelled by the pentagonal and hexagonal faces of $\mathcal{A}_D$.

Let A be a D–algebra endowed with invertible elements $\Phi_{\mathcal{G}\mathcal{F}}$ labelled by elementary pairs of maximal nested sets on D. For any 2–face $\mathcal{H}$ of $\mathcal{A}_D$, orientation $\overline{\varepsilon}$ of $\mathcal{H}$ and maximal nested set $\mathcal{F}_0$ on the boundary of $\mathcal{H}$, let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{k-1}, \mathcal{F}_k = \mathcal{F}_0$ be the vertices of $\mathcal{H}$ listed in their order of appearance along $\partial\mathcal{H}$ when the latter is endowed with the orientation $\overline{\varepsilon}$. Set

$$\mu(\mathcal{H}; \mathcal{F}_0, \overline{\varepsilon}) = \Phi_{\mathcal{F}_0\mathcal{F}_{k-1}} \cdots \Phi_{\mathcal{F}_1\mathcal{F}_0} \in A$$

The following is immediate

Lemma 3.31.

(i) *For any* $i = 0, \ldots, k-1$,

$$\mu(\mathcal{H}; \mathcal{F}_i, \overline{\varepsilon}) = \mathrm{Ad}(\Phi_{\mathcal{F}_i\mathcal{F}_{i-1}} \cdots \Phi_{\mathcal{F}_1\mathcal{F}_0})\, \mu(\mathcal{H}; \mathcal{F}_0, \overline{\varepsilon})$$

(ii) *If* $\Phi_{\mathcal{F}\mathcal{G}} = \Phi_{\mathcal{G}\mathcal{F}}^{-1}$ *for any elementary pair* $(\mathcal{G}, \mathcal{F})$ *of maximal nested sets on* D*, then*

$$\mu(\mathcal{H}; \mathcal{F}_0, -\overline{\varepsilon}) = \mu(\mathcal{H}; \mathcal{F}_0, \overline{\varepsilon})^{-1}$$

where $-\overline{\varepsilon}$ *is the opposite orientation to* $\overline{\varepsilon}$.

By lemma 3.31, the identity $\mu(\mathcal{H}; \mathcal{F}_0, \overline{\varepsilon}) = 1$, regarded as an identity in the variables $\Phi_{\mathcal{G}\mathcal{F}}$ does not depend upon the choice of $\overline{\varepsilon}$ and $\mathcal{F}_0$ provided the $\Phi_{\mathcal{G}\mathcal{F}}$ satisfy the orientation axiom of definition 3.12. We shall henceforth denote this identity by $\mu(\mathcal{H}) = 1$.

Proposition 3.32. *Assume that the elements* $\Phi_{\mathcal{G}\mathcal{F}}$ *satisfy the orientation, forgetfulness and support axioms of definition 3.12. Then, for any square 2–face* $\mathcal{H}$ *of the associahedron* $\mathcal{A}_D$, $\mu(\mathcal{H}) = 1$.

Proof. Let D_1, D_2 be the unsaturated elements of $\mathcal{H}$ and set, for $i, j, k \in \{1, 2\}$,

$$\underline{\alpha}_i = \underline{\alpha}_{\mathcal{H}}^{D_i} = \{\alpha_i^1, \alpha_i^2\}, \quad D_{i,j} = \complement_{\alpha_i^j}^{D_i \setminus \alpha_i^{3-j}} \quad \text{and} \quad \mathcal{H}_{j,k} = \mathcal{H} \cup \{D_{1,j}, D_{2,k}\}$$

By §2.8.1, $\mathcal{H}$ is given by

$$\begin{array}{ccc} \mathcal{H}_{1,1} & \text{———} & \mathcal{H}_{1,2} \\ \Big| & \mathcal{H} & \Big| \\ \mathcal{H}_{2,1} & \text{———} & \mathcal{H}_{2,2} \end{array}$$

Set $\mathcal{F}_0 = \mathcal{H}_{1,1}$ and let $\overline{\varepsilon}$ be the clockwise orientation of $\mathcal{H}$ so that

$$\mu(\mathcal{H}; \mathcal{F}_0, \overline{\varepsilon}) = \Phi_{\mathcal{H}_{1,1}\mathcal{H}_{2,1}} \Phi_{\mathcal{H}_{2,1}\mathcal{H}_{2,2}} \Phi_{\mathcal{H}_{2,2}\mathcal{H}_{1,2}} \Phi_{\mathcal{H}_{1,2}\mathcal{H}_{1,1}}$$

For $j=1,2$, the unsaturated element of $\mathcal{H}_{j,1}\cap\mathcal{H}_{j,2}=\mathcal{H}\cup\{D_{1,j}\}$ is D_2 with

$$\underline{\alpha}_{\mathcal{H}_{j,1}}^{D_2}=\alpha_2 \qquad \text{and} \qquad \alpha_{\mathcal{H}_{j,2}}^{D_2}=\alpha_1$$

It follows from the forgetfulness and support axioms that

$$\Phi_{\mathcal{H}_{1,2}\mathcal{H}_{1,1}}=\Phi^{-1}_{\mathcal{H}_{2,1}\mathcal{H}_{2,2}}\in A_{D_2}^{D_2\backslash\underline{\alpha}_2}$$

and similarly that

$$\Phi_{\mathcal{H}_{2,2}\mathcal{H}_{1,2}}=\Phi^{-1}_{\mathcal{H}_{1,1}\mathcal{H}_{2,1}}\in A_{D_1}^{D_1\backslash\underline{\alpha}_1}$$

Since $D_1,D_2\in\mathcal{H}$ are compatible, $[A_{D_2}^{D_2\backslash\underline{\alpha}_2},A_{D_1}^{D_1\backslash\underline{\alpha}_1}]=0$ and

$$\begin{aligned}&\Phi_{\mathcal{H}_{1,1}\mathcal{H}_{2,1}}\Phi_{\mathcal{H}_{2,1}\mathcal{H}_{2,2}}\Phi_{\mathcal{H}_{2,2}\mathcal{H}_{1,2}}\Phi_{\mathcal{H}_{1,2}\mathcal{H}_{1,1}}\\ &\qquad=\Phi_{\mathcal{H}_{1,1}\mathcal{H}_{2,1}}\Phi_{\mathcal{H}_{2,2}\mathcal{H}_{1,2}}\Phi_{\mathcal{H}_{2,1}\mathcal{H}_{2,2}}\Phi_{\mathcal{H}_{1,2}\mathcal{H}_{1,1}}=1\end{aligned}$$

as claimed ■

The following result is the analogue for quasi–Coxeter algebras of Mac Lane's coherence theorem for monoidal categories.

Theorem 3.33. *Let A be a D–algebra and $\{\Phi_{\mathcal{G}\mathcal{F}}\}$ a collection of invertible elements of A labelled by elementary pairs of maximal nested sets on D. Assume that $\Phi_{\mathcal{G}\mathcal{F}}$ satisfy the orientation, forgetfulness and support axioms of definition 3.12. Then, the coherence axiom of definition 3.12 is equivalent to the identities*

$$\mu(\mathcal{H})=1$$

for any pentagonal or hexagonal 2–face $\mathcal{H}$ of the associahedron $\mathcal{A}_D$.

Proof. By the simple–connectedness of $\mathcal{A}_D$, the coherence axiom is equivalent to the identities $\mu(\mathcal{H})=1$ for any two–face $\mathcal{H}$. The conclusion now follows since, by §2.8, the two–faces of $\mathcal{A}_D$ are either squares, pentagons or hexagons and, by proposition 3.32, $\mu(\mathcal{H})=1$ for any square 2–face $\mathcal{H}$ ■

Remark 3.34. The identities $\mu(\mathcal{H})=1$ corresponding to the pentagonal and hexagonal 2–faces of $\mathcal{A}_D$ are analogous to the pentagon identity satisfied by the associator of a quasi–bialgebra. We shall refer to them as *generalised pentagon relations.*

3.16. **Diagrammatic notation for elementary pairs.**

Proposition 3.35. *The map*

$$\imath:(\mathcal{G},\mathcal{F})\longrightarrow(\mathrm{supp}(\mathcal{G},\mathcal{F});\alpha_{\mathcal{G}}^{\mathrm{supp}(\mathcal{G},\mathcal{F})},\alpha_{\mathcal{F}}^{\mathrm{supp}(\mathcal{G},\mathcal{F})})$$

induces a bijection between equivalence classes of elementary pairs of maximal nested sets on D and triples $(B;\alpha,\beta)$ consisting of a connected subdiagram $B\subseteq D$ and an ordered pair (α,β) of distinct vertices of B.

Proof. $\imath$ is injective by definition of equivalence. To show that it is surjective, let $B\subseteq D$ be connected and let $\alpha_1\neq\alpha_2$ be two vertices of B. Set $B_1=\mathsf{C}_{\alpha_2}^{B\setminus\alpha_1}$ and $B_2=\mathsf{C}_{\alpha_1}^{B\setminus\alpha_2}$. Let $B^1,\ldots,B^k$ be the connected components of $B\setminus\{\alpha_1,\alpha_2\}$ and choose a maximal nested set $\mathcal{H}^j$ on each B^j. Since B^j is either contained in, or orthogonal to, each of B_1,B_2,

$$\overline{\mathcal{F}}=\mathcal{H}^1\sqcup\cdots\sqcup\mathcal{H}^k\sqcup\{B_1,B\}\qquad\text{and}\qquad\overline{\mathcal{G}}=\mathcal{H}^1\sqcup\cdots\sqcup\mathcal{H}^k\sqcup\{B_2,B\}$$

are an elementary pair of maximal nested sets on B such that $\operatorname{supp}(\overline{\mathcal{F}},\overline{\mathcal{G}})=B$, $\alpha^B_{\overline{\mathcal{F}}}=\alpha_1$ and $\alpha^B_{\overline{\mathcal{G}}}=\alpha_2$. Choose next an increasing sequence $B=D_1\subset\cdots\subset D_m=D$ of connected subdiagrams such that $|D_{j+1}\setminus D_j|=1$ for any $j=1,\ldots,m-1$ and set

$$\mathcal{F}=\overline{\mathcal{F}}\sqcup\{D_2,\ldots,D_m\}\qquad\text{and}\qquad\mathcal{G}=\overline{\mathcal{G}}\sqcup\{D_2,\ldots,D_m\}$$

Then, $\imath(\mathcal{G},\mathcal{F})=(B;\alpha_2,\alpha_1)$ ■

3.17. **Diagrammatic notation for elementary associators.** Let A be a quasi–Coxeter algebra of type D. For any connected subdiagram $B\subseteq D$ and ordered pair (α_i,α_j) of distinct vertices of B, there exists by proposition 3.35 an elementary pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D such that $B=\operatorname{supp}(\mathcal{F},\mathcal{G})$, $\alpha^B_{\mathcal{F}}=\alpha_i$ and $\alpha^B_{\mathcal{G}}=\alpha_j$. Set

$$\Phi_{(B;\alpha_j,\alpha_i)}=\Phi_{\mathcal{G}\mathcal{F}}$$

The forgetfulness axioms implies that this definition is independent of the choice of $\mathcal{G},\mathcal{F}$.

Theorem 3.36. *The associators $\Phi_{(B;\alpha_j,\alpha_i)}$ satisfy the following properties*

- **Orientation :**

$$\Phi_{(B;\alpha_i,\alpha_j)}=\Phi^{-1}_{(B;\alpha_j,\alpha_i)}$$

- **Generalised pentagon relations :** *For any connected $B\subseteq D$ and triple $(\alpha_i,\alpha_j,\alpha_k)$ of distinct vertices of B, set*

$$B_i=\mathsf{C}_{\alpha_i}^{B\setminus\{\alpha_j,\alpha_k\}}\qquad\text{and}\qquad B_{jk}=\mathsf{C}_{\{\alpha_j,\alpha_k\}}^{B\setminus\alpha_i}$$

Then, if $D'_{jk}=\emptyset$,

$$\Phi_{(B;\alpha_k,\alpha_i)}\cdot\Phi_{(B;\alpha_i,\alpha_j)}\cdot\Phi_{(B_{ik};\alpha_i,\alpha_k)}\cdot\Phi_{(B;\alpha_j,\alpha_k)}\cdot\Phi_{(B_{ij};\alpha_j,\alpha_i)}=1$$

whereas, if $B_{ij}, B_{jk}, B_{ik} \neq \emptyset$,

$$\Phi_{(B;\alpha_k,\alpha_i)} \cdot \Phi_{(B_{jk};\alpha_k,\alpha_j)} \cdot \Phi_{(B;\alpha_i,\alpha_j)} \cdot \\ \cdot \Phi_{(B_{ik};\alpha_i,\alpha_k)} \cdot \Phi_{(B;\alpha_j,\alpha_k)} \cdot \Phi_{(B_{ij};\alpha_j,\alpha_i)} = 1$$

- **Support :**

$$\Phi_{(B;\alpha_j,\alpha_i)} \in A_B^{B\setminus\{\alpha_i,\alpha_j\}}$$

- **Braid relations :** *if* $m_{ij} < \infty$ *and* B *is the connected diagram with vertices* $\{\alpha_i, \alpha_j\}$*, then*

$$\mathrm{Ad}(\Phi_{(B;\alpha_i,\alpha_j)})(S_i^A) \cdot S_j^A \cdots = S_j^A \cdot \mathrm{Ad}(\Phi_{(B;\alpha_i,\alpha_j)})(S_i^A) \cdots$$

where the number of factors on each side is equal to m_{ij}*.*

Conversely, if A *is a* D*–algebra endowed with invertible elements* $S_i^A \in \widehat{A}_i$ *for any* $\alpha_i \in D$ *and* $\Phi_{(B;\alpha_j,\alpha_i)} \in A$ *for any connected subdiagram* $B \subseteq D$ *and ordered pair of distinct vertices* $(\alpha_i, \alpha_j) \in B$ *which satisfy the above properties, then the* S_i^A *and the associators*

$$\Phi_{\mathcal{GF}} = \Phi_{(\mathrm{supp}(\mathcal{G},\mathcal{F});\alpha_{\mathcal{G}}^{\mathrm{supp}(\mathcal{G},\mathcal{F})},\alpha_{\mathcal{F}}^{\mathrm{supp}(\mathcal{G},\mathcal{F})})} \tag{3.3}$$

corresponding to elementary pairs of maximal nested sets on D *give* A *the structure of a quasi–Coxeter algebra of type* D*.*

PROOF. Orientation and support are equivalent to the orientation and support axioms satisfied by the elementary associators $\Phi_{\mathcal{GF}}$. Let $B \subseteq D$ be connected and $(\alpha_i, \alpha_j, \alpha_k)$ a triple of distinct elements of B. Let $\mathcal{H}$ be a 2–face of $\mathcal{A}_D$ such that $\mathcal{H}$ has B as its unique unsaturated element and $\underline{\alpha}_{\mathcal{H}}^B = \{\alpha_i, \alpha_j, \alpha_k\}$. A simple exercise, using §2.8.2 shows that the identity $\mu(\mathcal{H}) = 1$ corresponding to $\mathcal{H}$ is the first or the second of the two generalised pentagonal relations above depending on whether B_{ij}, B_{jk}, B_{ki} are all non–empty or one of them, which up to a relabelling we may assume to be B_{jk}, is empty. Let now $\alpha_i \neq \alpha_j$ be such that $m_{ij} < \infty$ and let B be the connected diagram with vertices α_i and α_j. Let $(\mathcal{F}, \mathcal{G})$ be an elementary pair of maximal nested sets on D such that $\alpha_i \in \mathcal{F}$ and $\alpha_j \in \mathcal{G}$. Since α_i and α_j are not compatible, they are the unique elements in $\mathcal{F} \setminus \mathcal{G}$ and $\mathcal{G} \setminus \mathcal{F}$ respectively. Thus, $\mathrm{supp}(\mathcal{F}, \mathcal{G}) = B$ and clearly $\alpha_{\mathcal{F}}^B = \alpha_j$ and $\alpha_{\mathcal{G}}^B = \alpha_i$. It follows that $\Phi_{\mathcal{GF}} = \Phi_{(B;\alpha_i,\alpha_j)}$ and the braid relations for $\Phi_{\mathcal{GF}}$ coincide with those for $\Phi_{(B;\alpha_i,\alpha_j)}$. The converse follows from the fact that the associators $\Phi_{\mathcal{GF}}$ defined by (3.3) clearly satisfy the forgetfulness axiom and the fact that, by theorem 3.33, the coherence axiom for the associators $\Phi_{\mathcal{GF}}$ is equivalent to the relations $\mu(\mathcal{H}) = 1$ which, as pointed out, coincide with the generalised pentagonal relations above ∎

3.18. **Twisting of quasi–Coxeter algebras.** Let A be a quasi–Coxeter algebra of type D.

Definition 3.37. *A twist $a = \{a_{(B;\alpha)}\}$ is a collection of invertible elements of A labelled by pairs $(B;\alpha)$ consisting of a connected subdiagram $B \subseteq D$ and a vertex α of B such that*

$$a_{(B;\alpha)} \in A_B^{B\setminus\alpha}$$

Let a be a twist. For any connected $B \subseteq D$ and maximal nested set $\mathcal{F}$ on B, set

$$a_\mathcal{F} = \prod_{B'\in\mathcal{F}} a_{(B';\alpha_\mathcal{F}^{B'})}$$

Note that the product does not depend upon the order of the factors. Indeed, if $B' \neq B'' \in \mathcal{F}$, then either $B' \perp B''$ in which case $A_{B'} \ni a_{(B';\alpha_\mathcal{F}^{B'})}$ and $A_{B''} \ni a_{(B'';\alpha_\mathcal{F}^{B''})}$ commute, or, up to a permutation, $B' \subsetneq B''$ so that $B' \subseteq B'' \setminus \alpha_\mathcal{F}^{B''}$ and $a_{(B'';\alpha_\mathcal{F}^{B''})}$ commutes with $A_{B'} \ni a_{(B';\alpha_\mathcal{F}^{B'})}$.

We shall need the following

Lemma 3.38. *Let a be a twist, $\mathcal{F}$ a maximal nested set on D and $B \in \mathcal{F}$. Then,*

(i) *for any $x \in A_B$: $\mathrm{Ad}(a_\mathcal{F})\, x = \mathrm{Ad}(a_{\mathcal{F}_B})\, x$.*

(ii) *For any $x \in A^B$: $\mathrm{Ad}(a_\mathcal{F})\, x = \mathrm{Ad}(\prod_{\substack{B'\in\mathcal{F},\\ B'\not\subseteq B}} a_{(B';\alpha_\mathcal{F}^{B'})})\, x$.*

Proof. (i) If $B' \in \mathcal{F}$ and $B' \not\subseteq B$, then either $B' \perp B$, so that $a_{(B';\alpha_\mathcal{F}^{B'})} \in A_{B'}$ commutes with A_B, or $B' \supsetneq B$, so that $B' \setminus \alpha_\mathcal{F}^{B'} \supseteq B$ and again $a_{(B';\alpha_\mathcal{F}^{B'})} \in A^{B'\setminus\alpha_\mathcal{F}^{B'}}$ commutes with A_B. Thus, for any $x \in A_B$,

$$\mathrm{Ad}(a_\mathcal{F})\, x = \prod_{B'\subseteq B} \mathrm{Ad}(a_{(B';\alpha_\mathcal{F}^{B'})})\, x = \mathrm{Ad}(a_{\mathcal{F}_B})\, x$$

(ii) If $B' \subseteq B$, then $a_{(B';\alpha_\mathcal{F}^{B'})} \in A_{B'} \subseteq A_B$ commutes with x. The result follows ■

For any $\alpha_i \in D$ and elementary pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D, set

$$S_i^{A^a} = a_{(\alpha_i;\alpha_i)} \cdot S_i^A \cdot a_{(\alpha_i;\alpha_i)}^{-1}$$
$$\Phi_{\mathcal{G}\mathcal{F}}^a = a_\mathcal{G} \cdot \Phi_{\mathcal{G}\mathcal{F}} \cdot a_\mathcal{F}^{-1}$$

Let $B = \text{supp}(\mathcal{F},\mathcal{G})$ and set $\alpha_1 = \alpha^B_{\mathcal{F}}$, $\alpha_2 = \alpha^B_{\mathcal{G}}$, $B_1 = \complement^{B\setminus\alpha_1}_{\alpha_2}$ and $B_2 = \complement^{B\setminus\alpha_2}_{\alpha_1}$.

Lemma 3.39.

$$\Phi^a_{\mathcal{G}\mathcal{F}} = a_{(B;\alpha_2)} \cdot a_{(B_2;\alpha_1)} \cdot \Phi_{\mathcal{G}\mathcal{F}} \cdot a^{-1}_{(B_1;\alpha_2)} \cdot a^{-1}_{(B;\alpha_1)}$$

Proof. Let $B' \in \mathcal{G}\cap\mathcal{F} = \mathcal{F}\setminus\{B_1\} = \mathcal{G}\setminus\{B_2\}$ be distinct from B so that $\alpha^{B'}_{\mathcal{F}} = \alpha^{B'}_{\mathcal{F}\cap\mathcal{G}} = \alpha^{B'}_{\mathcal{G}}$. Lemma 3.38 readily implies that $a_{(B';\alpha^{B'}_{\mathcal{G}})} = \alpha_{(B';\alpha^{B'}_{\mathcal{F}})}$ commutes with $\Phi_{\mathcal{G}\mathcal{F}} \in A^{B\setminus\{\alpha_1,\alpha_2\}}_B$. The identity above now follows from this and the fact that, by proposition 3.23, $\alpha^{B_1}_{\mathcal{F}} = \alpha_2$ and $\alpha^{B_2}_{\mathcal{G}} = \alpha_1$ ■

Proposition 3.40. *$A^a = (A, A_{D'}, S^{A^a}_i, \Phi^a_{\mathcal{G}\mathcal{F}})$ is a quasi–Coxeter algebra of type D called the twist of A by a.*

Proof. The elements $\Phi^a_{\mathcal{G}\mathcal{F}}$ clearly satisfy the orientation and coherence axioms of definition 3.12. Lemma 3.39 implies that they also satisfy the support and forgetfulness axiom. Let now $\alpha_i \neq \alpha_j \in D$ be such that $m_{ij} < \infty$ and $(\mathcal{F},\mathcal{G})$ an elementary pair of maximal nested sets on D such that $\alpha_i \in \mathcal{F}$ and $\alpha_j \in \mathcal{G}$. By lemma 3.38, $\text{Ad}(a_{\mathcal{F}})(S^A_i) = \text{Ad}(a_{(\alpha_i;\alpha_i)})(S^A_i) = S^{A^a}_i$ and similarly $\text{Ad}(a_{\mathcal{G}})(S^A_j) = S^{A^a}_j$. Thus,

$$\text{Ad}(\Phi^a_{\mathcal{G}\mathcal{F}})(S^{A^a}_i) \cdot S^{A^a}_j \cdots = \text{Ad}(a_{\mathcal{G}}) \left(\text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S^A_i) \cdot S^A_j \cdots\right)$$

and

$$S^{A^a}_j \cdot \text{Ad}(\Phi^a_{\mathcal{G}\mathcal{F}})(S^{A^a}_i) \cdots = \text{Ad}(a_{\mathcal{G}}) \left(S^A_j \cdot \text{Ad}(\Phi_{\mathcal{G}\mathcal{F}})(S^A_i) \cdots\right)$$

so that $\Phi^a_{\mathcal{G}\mathcal{F}}$ satisfies the braid relations with respect to $S^{A^a}_i$, $S^{A^a}_j$ ■

The following result shows that twisting does not change the isomorphism class of a quasi–Coxeter algebra.

Proposition 3.41. *If a is a twist of A, then $\Psi_{\mathcal{F}} = \text{Ad}(a_{\mathcal{F}})$ is an isomorphism of A onto the quasi–Coxeter algebra A^a. In particular, A and A^a define equivalent representations of the Artin group B_D.*

Proof. If $\alpha_i \in \mathcal{F}$, then, by lemma 3.38, $\text{Ad}(a_{\mathcal{F}})(S^A_i) = S^{A^a}_i$. Moreover, for any elementary pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D,

$$\text{Ad}(a_{\mathcal{G}}) \circ \text{Ad}(\Phi_{\mathcal{G}\mathcal{F}}) = \text{Ad}(\Phi^a_{\mathcal{G}\mathcal{F}}) \circ \text{Ad}(a_{\mathcal{F}})$$

■

Remark 3.42. By lemma 3.39, the twist by a of the associators of a quasi–Coxeter algebra reads, in diagrammatic notation

$$\Phi^a_{(B;\alpha_j,\alpha_i)} = a_{(B;\alpha_j)} \cdot a_{(\mathsf{C}^{B\setminus\alpha_j}_{\alpha_i};\alpha_i)} \cdot \Phi_{(B;\alpha_j,\alpha_i)} \cdot a^{-1}_{(\mathsf{C}^{B\setminus\alpha_i}_{\alpha_j};\alpha_j)} \cdot a^{-1}_{(B;\alpha_i)} \tag{3.4}$$

4. Examples of quasi–Coxeter algebras

This section is devoted to the study of several examples of quasi–Coxeter algebras. In §4.1 we consider 'quantum' examples, the defining feature of which is that their associators are all trivial. We begin with the universal one given by the braid group of a Coxeter group, then consider the corresponding Hecke algebra and finally Lusztig's quantum Weyl group operators for a symmetrisable Kac–Moody algebra. In §4.2, we consider examples which underlie the monodromy representations of several flat connections, specifically the holonomy equations (1.3) of a Coxeter arrangement, Cherednik's KZ connection and the Casimir connection described in the Introduction. The study of these examples relies heavily on the De Concini–Procesi theory of asymptotic zones described in section 1. Finally, in §4.3 we show how to obtain quasi–Coxeter algebras of type A_n as commutants of quasibialgebras and of quasitriangular quasibialgebras.

Throughout this section, W denotes an irreducible Coxeter group with system of generators $S = \{s_i\}_{i\in\mathbf{I}}$. We denote by D the Coxeter graph of (W,S) and label the pair $i \neq j \in D$ with the order m_{ij} of s_is_j in W.

4.1. **Quantum examples.**

4.1.1. *Universal example.* For any connected subgraph $D' \subseteq D$ with vertex set $\mathbf{I}' \subseteq \mathbf{I}$, let $W_{D'} \subseteq W$ be the parabolic subgroup generated by s_i, $i \in \mathbf{I}'$ and $B_{D'}$ the (algebraic) braid group of $W_{D'}$, that is the group with generators S_i, $i \in \mathbf{I}'$ and relations (1.22) for any $i \neq j \in \mathbf{I}'$. Then, as noted in remark 3.30, the assignement

$$A_{D'} = k[B_{D'}], \qquad S_i^A = S_i \qquad \text{and} \qquad \Phi^A_{\mathcal{G}\mathcal{F}} = 1$$

endows $A = k[B_D]$ with the structure of a quasi–Coxeter algebra of type D and if A' is a quasi–Coxeter algebra of type D, the maps $\pi_{\mathcal{F}} : B_D \to \widehat{A'}$ given by theorem 3.29 define a morphism of quasi–Coxeter algebras $k[B_D] \to \widehat{A'}$.

4.1.2. *Hecke algebras.* Let $q_i \in k$ be invertible elements such that $q_i = q_j$ whenever s_i is conjugate to s_j in W. For any $D' \subseteq D$ with vertex set $\mathbf{I}' \subseteq \mathbf{I}$, let $\mathcal{H}(W_{D'})$ be the Iwahori–Hecke algebra of $W_{D'}$, that is the quotient of $k[B_{D'}]$ by the quadratic relations

$$(S_i - q_i)(S_i + q_i^{-1}) = 0 \tag{4.1}$$

Then, the assignement

$$A_{D'} = \mathcal{H}(W_{D'}), \qquad S_i^A = S_i \qquad \text{and} \qquad \Phi^A_{\mathcal{G}\mathcal{F}} = 1$$

endows $A = \mathcal{H}(W)$ with a structure of quasi–Coxeter algebra of type D. The corresponding maps $\pi_{\mathcal{F}} : B_D \to A$ are all equal to the quotient map $k[B_D] \to \mathcal{H}(W)$.

4.1.3. *Quantum Weyl groups.* Assume now that W is the Weyl group of a Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(\mathbf{A})$ whose generalised Cartan matrix $\mathbf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ we assume to be irreducible [Ka]. Let $(\mathfrak{h}, \Delta, \Delta^\vee)$ be the unique realisation of A. Thus, $\mathfrak{h}$ is a complex vector space of dimension $2|\mathbf{I}| - \text{rank}(A)$,

$$\Delta = \{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^* \qquad \text{and} \qquad \Delta^\vee = \{\alpha_i^\vee\}_{i\in\mathbf{I}} \subset \mathfrak{h}$$

are linearly independent sets of cardinality $|\mathbf{I}|$ and, for any $i, j \in \mathbf{I}$, $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Then, W is the subgroup of $GL(\mathfrak{h})$ generated by the reflections s_i acting on $t \in \mathfrak{h}$ by

$$s_i(t) = t - \alpha_i(t) \cdot \alpha_i^\vee$$

and $m_{ij} = 2, 3, 4, 6, \infty$ according to whether $a_{ij}a_{ji} = 0, 1, 2, 3, \geq 4$.

Assume further that $\mathbf{A}$ is symmetrisable, that is that there exists relatively prime integers $d_i \geq 1$, $i \in \mathbf{I}$ such that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in \mathbf{I}$. Then, there exists a non–degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$, unique up to a scalar, such that $a_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the derived subalgebra of $\mathfrak{g}$, $k = \mathbb{C}[\![\hbar]\!]$ the ring of formal power series in the variable $\hbar$ and $U_\hbar\mathfrak{g}'$ the Drinfeld–Jimbo quantum group corresponding to $\mathbf{A}$ and $(\cdot,\cdot)$, that is the algebra over $\mathbb{C}[\![\hbar]\!]$ topologically generated by elements E_i, F_i, H_i, $i \in \mathbf{I}$, subject to the relations[1]

$$[H_i, H_j] = 0$$

$$[H_i, E_j] = a_{ij}E_j \qquad [H_i, F_j] = -a_{ij}F_j$$

$$[E_i, F_j] = \delta_{ij}\frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_i^{-1}}$$

[1]we follow here the conventions of [Lu]

where

$$q_i = q^{(\alpha_i,\alpha_i)/2} \qquad \text{with} \qquad q = e^{\hbar}$$

and the q–Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i E_i^k E_j E_i^{1-a_{ij}-k} = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i F_i^k F_j F_i^{1-a_{ij}-k} = 0$$

where for any $k \leq n$,

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

$$[n]_i! = [n]_i[n-1]_i \cdots [1]_i \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i![n-k]_i!}$$

For any connected $D' \subseteq D$ with vertex set $\mathbf{I}' \subseteq \mathbf{I}$, let $U_\hbar\mathfrak{g}'_{D'} \subseteq U_\hbar\mathfrak{g}'$ be the subalgebra topologically generated by the elements E_i, F_i and H_i with $i \in \mathbf{I}'$. Then, $U_\hbar\mathfrak{g}_{D'}$ is the Drinfeld–Jimbo quantum group corresponding to the Cartan matrix $(a_{ij})_{i,j\in\mathbf{I}'}$ and the unique non–degenerate bilinear form on its realisation which coincides with the restriction of $(\cdot,\cdot)$ on the span of the α_i, $i \in \mathbf{I}'$. If $\mathbf{I}' = \{i\}$, we denote $U_\hbar\mathfrak{g}_{D'}$ by $U_\hbar\mathfrak{sl}_2^i$.

For any i, let $\overline{S}_i^\hbar$ be the operator acting on an integrable $U_\hbar\mathfrak{g}$–module $\mathcal{V}$ as[1]

$$\overline{S}_i^\hbar\, v = \sum_{\substack{a,b,c\in\mathbb{Z}: \\ a-b+c=\lambda(\alpha_i^\vee)}} (-1)^b q_i^{ac-b} F_i^{(a)} E_i^{(b)} F_i^{(c)} v \tag{4.2}$$

where

$$E_i^{(a)} = \frac{E_i^a}{[a]_i!} \qquad F_i^{(a)} = \frac{F_i^a}{[a]_i!}$$

and $v \in V$ if of weight $\lambda \in \mathfrak{h}^*$. Set

$$S_i^\hbar = \overline{S}_i^\hbar \cdot q_i^{-H_i^2/4} = q_i^{-H_i^2/4} \cdot \overline{S}_i^\hbar \in \widehat{U_\hbar\mathfrak{sl}_2^i} \tag{4.3}$$

We shall refer to S_i as the *quantum Weyl group operator* corresponding to i. The following result is due to Lusztig and, independently to Kirillov–Reshetikhin and Soibelman [Lu, KR, So]

[1] the element $\overline{S}_i^\hbar$ is, in the notation of [Lu, §5.2.1], the operator $T'_{i,-1}$. Although any of Lusztig's operators $T'_{i,\pm1}, T''_{i,\pm1}$ would have worked for our purposes, we choose $T'_{i,-1}$ for definiteness.

Proposition 4.1. *If the order m_{ij} of $s_i s_j$ in W is finite, then*

$$\underbrace{S_i^\hbar S_j^\hbar \cdots}_{m_{ij}} = \underbrace{S_j^\hbar S_i^\hbar \cdots}_{m_{ij}}$$

Proof. this is an immediate consequence of the braid relations satisfied by the operators $\overline{S}_i^\hbar$ [Lu, §39.4] ■

It follows from proposition 4.1 that the assignement

$$A_{D'} = U_\hbar \mathfrak{g}'_{D'}, \qquad S_i^A = S_i^\hbar \qquad \text{and} \qquad \Phi_{\mathcal{GF}}^A = 1$$

endows $A = U_\hbar \mathfrak{g}'$ with the structure of a quasi–Coxeter algebra of type D. The corresponding representations $B_D \to GL(\mathcal{V})$, with $\mathcal{V}$ a finite–dimensional $U_\hbar \mathfrak{g}'$–module are called quantum Weyl group representations.

4.2. **Differential examples.** Assume now W is finite. Let $V_\mathbb{R}$ be its reflection representation, $\mathcal{A}_\mathbb{R} \subset V_\mathbb{R}$ the corresponding arrangement of reflecting hyperplanes of W and $V, \mathcal{A}$ their complexifications. Retain the notation of section 1, particularly §1.17. Thus, $\Phi \subset V_\mathbb{R}^*$ is a root system for W, $\mathcal{C} \subset V_\mathbb{R} \setminus \mathcal{A}_\mathbb{R}$ a chamber, which we choose to be the one bound by the reflecting hyperplanes of the generators s_i of W, $\Phi = \Phi_+ \sqcup \Phi_-$ the corresponding partition into positive and negative roots, $\Delta = \{\alpha_i\}_{i \in \mathbf{I}} \subset \Phi_+$ the basis of $V_\mathbb{R}^*$ consisting of indecomposable elements of Φ_+, and we choose $X = \Phi_+$ as the set of defining equations for $\mathcal{A}$. Fix $v_0 \in \mathcal{C}$ and identify $B_W = \pi_1(V_\mathcal{A}/W, v_0)$ with B_D via the presentation (1.22).

Recall that, by proposition 1.39, there is a bijection between nested sets of connected subdiagrams of D and fundamental nested sets of irreducible subspaces V^* which contain V^*. This correspondence maps $D' \subseteq D$ with vertex set $\mathbf{I}' \subseteq \mathbf{I}$ to the subspace $\langle D' \rangle$ spanned by the roots α_i, $i \in \mathbf{I}'$ and $B \subseteq V^*$ to the connected subdiagram with vertex set $B \cap \Delta$.

4.2.1. *Universal example.* For any subdiagram $D' \subseteq D$, let $F_{D'} \subseteq F$ be the subalgebra generated by the elements t_α, $\alpha \in \langle D' \rangle$, $I_{D'} = I \cap F_{D'}$ the ideal defined by the relations (1.2) for $B \subseteq \langle D' \rangle$ and $A_{D'}$ the completion of $F_{D'}/I_{D'}$ with respect to its $\mathbb{N}$–grading. The braid group $B_{D'}$ acts on F/I by (1.19) via the homomorphism $B_{D'} \to W_{D'}$ and leaves $F_{D'}/I_{D'}$ invariant.

Let h be a formal variable and $\imath_h : A \rtimes B_D \rightarrow F/I[\![h]\!] \rtimes B_D$ the embedding given on $b \in B_D$ and $t_\alpha \in A$ by

$$\imath_h(b) = b \qquad \text{and} \qquad \imath_h(t_\alpha) = ht_\alpha \tag{4.4}$$

Note that if $\widetilde{\rho} : F/I \rtimes B_D \rightarrow \mathrm{End}(U)$ is a finite–dimensional representation, the action $\widetilde{\rho}_h$ of $A \rtimes B_D$ on $U[\![h]\!]$ defined by (1.20) is given by $\rho \circ \imath_h$.

Proposition 4.2. *Set $k = \mathbb{C}[\![h]\!]$.*

(i) *The assignement*

$$A^\nabla_{D'} = F_{D'}/I_{D'}[\![h]\!] \rtimes B_{D'}$$

endows $A^\nabla = F/I[\![h]\!] \rtimes B_D$ with the structure of a D–algebra over k.

(ii) *Let $\beta \subset V^*$ be a positive, real adapted family. Then, the elements*

$$S^\nabla_i = \exp(\pi\sqrt{-1} \cdot ht_{\alpha_i}) \cdot S_i \quad \textit{and} \quad \Phi^\nabla_{\mathcal{GF}} = \imath_h(\Phi^\beta_{\mathcal{GF}})$$

where the latter are the De Concini–Procesi associators corresponding to β, endow A^∇ with the structure of a quasi–Coxeter algebra of type D.

(iii) *If $\widetilde{\rho} : F/I \rtimes B_D \longrightarrow End(U)$ is a finite–dimensional representation, the action $\widetilde{\rho} \circ \pi_{\mathcal{F}}$ of B_D on $U[\![h]\!]$ induced by the quasi–Coxeter algebra structure on A^∇ coincides with the monodromy of the flat vector bundle $(\widetilde{V_\mathcal{A}} \times U, p^*\nabla_{\widetilde{\rho}_h})/B_D$ over $V_\mathcal{A}/W$, where*

$$\nabla_{\widetilde{\rho}_h} = d - h \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \widetilde{\rho}(t_\alpha)$$

expressed in the fundamental solution $\widetilde{\rho}_h(\Psi^{\beta_\mathcal{F}}_\mathcal{F})$.

(iv) *The quasi–Coxeter algebra structures on A^∇ given by two positive, real adapted families β, β' differ by a canonical twist.*

Proof. (i) Clearly, $A^\nabla_{D'} \subset A^\nabla_{D''}$ whenever $D' \subset D''$. If, on the other hand $D' \perp D''$, then $\{\langle D' \rangle, \langle D'' \rangle\}$ is nested by proposition 1.39 so that $[A_{D'}, A_{D''}] = 0$ by lemma 1.20. It follows that $[A^\nabla_{D'}, A^\nabla_{D''}] = 0$ since, whenever $D' \perp D''$, $B_{D'}$ fixes $A_{D''}$ and commutes with $B_{D''}$.
(ii) The orientation, transitivity, forgetfulness and braid relations axioms have been proved in §1.12, §1.14, §1.16 and §1.17.11 respectively. Let now $(\mathcal{G}, \mathcal{F})$ be an elementary pair of maximal nested sets on D, $D' = \mathrm{supp}(\mathcal{G}, \mathcal{F})$ and $D'' = \mathfrak{z}\mathrm{supp}(\mathcal{F}, \mathcal{G})$. By §1.15, the associator $\Phi^\beta_{\mathcal{GF}}$ lies in $A_{D'}$ and commutes with $A_{D''}$. We therefore need only prove that it is invariant under $W_{D''}$. It is sufficient to show that this is the case

for the coefficients $t_{\overline{x}}$ of the connection $\nabla_{\langle D'\rangle,\langle D''\rangle}$ defined by (1.15). However, if $x \in X \cap (\langle D'\rangle \setminus \langle D''\rangle)$ and $\alpha_i \in D''$, then

$$s_i x = x - 2(x,\alpha_i)/(\alpha_i,\alpha_i)\alpha_i = x \mod \langle D''\rangle$$

so that

$$s_i(t_{\overline{x}}) = s_i(\sum_{x\in\overline{x}} t_x) = \sum_{x\in\overline{x}} t_{s_i(x)} = t_{\overline{x}}$$

(iii) Follows from the discussion in §1.17.7.
(iv) By remark 1.35, $\Phi^{\beta'}_{\mathcal{G}\mathcal{F}} = a_{\mathcal{G}} \cdot \Phi^{\beta}_{\mathcal{G}\mathcal{F}} \cdot a_{\mathcal{F}}^{-1}$, where

$$a_{\mathcal{F}} = \prod_{D'\in\mathcal{F}} c_{(D';\alpha^{D'}_{\mathcal{F}})}^{\;-R^{\mathcal{F}}_{\langle D'\rangle}} \tag{4.5}$$

for some $c_{(D';\alpha)} \in \mathbb{R}^*_+$ and

$$R^{\mathcal{F}}_{\langle D'\rangle} = t_{\langle D'\rangle} - t_{i_{\mathcal{F}}\langle D'\rangle} = t_{\langle D'\rangle} - t_{\langle D'\setminus\alpha^{D'}_{\mathcal{F}}\rangle} \in A_{D'}$$

is defined by (1.9). The claim now follows since, by lemma 1.20, $R^{\mathcal{F}}_{\langle D'\rangle}$, commutes with $A_{D'\setminus\alpha^{D'}_{\mathcal{F}}}$ and $c^{-t_{\alpha_i}}_{(\alpha_i;\alpha_i)}$ is fixed by s_i so that $\mathrm{Ad}(c^{-t_{\alpha_i}}_{(\alpha_i;\alpha_i)})S_i^\nabla = S_i^\nabla$ ■

Remark 4.3. Since the elements S_i^∇ lie in A^∇, the maps $\pi_{\mathcal{F}} : B_D \to \widehat{A^\nabla}$ given by theorem 3.29 factor through A^∇. Moreover, by theorem 1.42, they induce isomorphism

$$\widehat{\pi_{\mathcal{F}}} : \widehat{\mathbb{C}[B_D][\![h]\!]} \longrightarrow F/I[\![h]\!] \rtimes W$$

where $\widehat{\mathbb{C}[B_D][\![h]\!]}$ is the completion of $\mathbb{C}[B_D][\![h]\!]$ with respect to the kernel of the epimorphism $\mathbb{C}[B_D][\![h]\!] \to \mathbb{C}[W][\![h]\!]$.

4.2.2. *Cherednik's rational KZ connection.* We recast below the monodromy of Cherednik's KZ connection for W [Ch1, Ch2] in the language of quasi–Coxeter algebras. For any $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding orthogonal reflection and consider the connection on $V_{\mathcal{A}}$ with values in $\mathbb{C}[W]$ given by[1]

$$\nabla_{\text{CKZ}} = d - \sum_{\alpha\in\Phi_+} k_\alpha \frac{d\alpha}{\alpha} s_\alpha$$

where the k_α are complex numbers such that $k_\alpha = k_{\alpha'}$ whenever s_α and $s_{\alpha'}$ are conjugate in W.

Theorem 4.4 (Cherednik). *The connection ∇_{CKZ} is flat.*

By theorem 4.4, the W–equivariant homomorphism $F \to \mathbb{C}[W]$ given by $t_\alpha \to k_\alpha s_\alpha$ factors through F/I. Composing with the embedding (4.4) yields a homomorphism $\imath_{\text{CKZ}} : A \rtimes B_D \to \mathbb{C}[\![h]\!][W]$ restricting to the canonical projection $B_D \to W$. The following result is an immediate consequence of proposition 4.2.

Proposition 4.5.

(i) *The assignement $A_{D'}^{\text{CKZ}} = \mathbb{C}[\![h]\!][W_{D'}]$ endows $A^{\text{CKZ}} = \mathbb{C}[\![h]\!][W]$ with the structure of a D–algebra over $k = \mathbb{C}[\![h]\!]$.*

(ii) *Let $\beta \subset V^*$ be a positive, real adapted family. Then, the elements*

$$S_i^{\text{CKZ}} = \exp(\pi\sqrt{-1}\cdot hk_{\alpha_i} s_i)\cdot s_i \quad \textit{and} \quad \Phi_{\mathcal{GF}}^{\text{CKZ}} = \imath_{\text{CKZ}}(\Phi_{\mathcal{GF}}^\beta)$$

give A^{CKZ} the structure of a quasi–Coxeter algebra of type D.

(iii) *If $\rho : W \to GL(U)$ is a finite–dimensional representation, the action $\rho\circ\pi_{\mathcal{F}}$ of B_D on $U[\![h]\!]$ induced by the quasi–Coxeter algebra structure on A^{CKZ} coincides with the monodromy representation of the flat connection*

$$\nabla_{\text{CKZ}}^\rho = d - h\sum_{\alpha\in\Phi_+} k_\alpha \frac{d\alpha}{\alpha}\rho(s_\alpha)$$

on the holomorphic vector bundle $\mathcal{U}_\rho = V_{\mathcal{A}} \times_W U[\![h]\!]$ over $V_{\mathcal{A}}/W$ expressed in the fundamental solution $\rho\circ\imath_{\text{CKZ}}(\Psi_{\mathcal{F}}^{\beta_{\mathcal{F}}})$.

(iv) *The quasi–Coxeter algebra structures on A^{CKZ} given by two positive, real adapted families β, β' differ by a twist.*

The following rephrases a well–known result of Cherednik [Ch1, Ch2]

[1]The letter C is used to distinguish ∇_{CKZ} from the Knizhnik–Zamolodchikov connection ∇_{KZ} which arises in Conformal Field Theory. The latter depends upon the choice of a complex, reductive Lie algebra $\mathfrak{r}$ and takes values in the n–fold tensor product $V^{\otimes n}$ of a finite–dimensional $\mathfrak{r}$–module V. It coincides with ∇_{CKZ} for the Coxeter group $W = \mathfrak{S}_n$ when $\mathfrak{r} = \mathfrak{gl}_n$ and $V = \mathbb{C}^n$ is the vector representation, via the identication $V^{\otimes n}[0] \cong \mathbb{C}[\mathfrak{S}_n]$ (see, *e.g.*, [TL2, §4]). In [TL2], the name Coxeter–KZ connection was used for ∇_{CKZ}. The name Cherednik–KZ connection seems far more appropriate however.

Theorem 4.6. *For any maximal nested set $\mathcal{F}$ on D, the map $\pi_{\mathcal{F}}$: $B_D \rightarrow A^{\text{CKZ}}$ induced by the quasi–Coxeter algebra structure on A^{CKZ} factors through the Iwahori–Hecke algebra $\mathcal{H}(W)$ of B_D, that is the quotient of $\mathbb{C}[\![h]\!][B_D]$ by the quadratic relations*

$$(S_i - q_i)(S_i + q_i^{-1}) = 0 \qquad \textit{where} \qquad q_i = \exp(\pi\sqrt{-1}\cdot hk_{\alpha_i})$$

and induces an isomorphism $\pi_{\mathcal{F}} : \mathcal{H}(W) \longrightarrow \mathbb{C}[\![h]\!][W]$ which restricts to an isomorphism

$$\mathcal{H}(W_{D'}) \longrightarrow \mathbb{C}[\![h]\!][W_{D'}] \quad \textit{for any } D' \in \mathcal{F} \tag{4.6}$$

Proof. That $\pi_{\mathcal{F}}$ factors through $\mathcal{H}(W)$ follows at once from theorem 1.43 and proposition 1.41. $\mu_{\mathcal{F}}$ is surjective by Nakayama's lemma, since its reduction mod h is the canonical projection $B_D \rightarrow W$ and it is injective since the dimension of $\mathcal{H}(W)$ is bounded above by that of $\mathbb{C}[W]$ ■

Remark 4.7. The only significant difference between theorem 4.6 and theorem 2 in [Ch1, Ch2] is that we consider a *collection* of isomorphisms $\mu_{\mathcal{F}} : \mathcal{H}(W) \rightarrow \mathbb{C}[\![h]\!][W]$. These are labelled by maximal nested sets on the Coxeter graph D of W, respect the parabolic structure on $\mathcal{H}(W)$ and $\mathbb{C}[\![h]\!][W]$ in the sense of (4.6) and are related by

$$\mu_{\mathcal{G}} = \text{Ad}(\Phi^{\text{CKZ}}_{\mathcal{GF}}) \circ \mu_{\mathcal{F}}$$

Remark 4.8. Cherednik has given an explicit computation of the monodromy of ∇_{CKZ} when W is of type $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ in terms of the classical hypergeometric function [Ch3].

4.2.3. *The Casimir connection.* Assume now that W is the Weyl group of a complex, simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, identify $V_{\mathcal{A}}$ with the set

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha\in\Phi} \text{Ker}(\alpha)$$

of regular elements in $\mathfrak{h}$ and choose $\Phi \subset \mathfrak{h}^*$ to be the root system of $\mathfrak{g}$. For any $\alpha \in \Phi$, let $\mathfrak{sl}_2^{\alpha} = \langle e_\alpha, f_\alpha, h_\alpha \rangle \subset \mathfrak{g}$ be the corresponding three–dimensional subalgebra and denote by

$$C_\alpha = \frac{(\alpha,\alpha)}{2}\left(e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2}h_\alpha^2\right)$$

its Casimir operator with respect to the restriction to $\mathfrak{sl}_2^{\alpha}$ of a a fixed non–degenerate, invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. Note that C_α is independent of the choice of the root vectors e_α, f_α and satisfies $C_{-\alpha} = C_\alpha$. Let V a finite–dimensional $\mathfrak{g}$–module and consider the following holomorphic connection on the holomorphically trivial vector bundle over $\mathfrak{h}_{\text{reg}}$ with fibre V

$$\nabla_C = d - \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha}\cdot C_\alpha \tag{4.7}$$

The following result is due to the author and J. Millson [MTL] and was discovered independently by De Concini around 1995 (unpublished)

Theorem 4.9. *The connection* ∇_C *is flat.*

It will be convenient to use the following, closely related auxiliary connection

$$\nabla_\kappa = d - \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha}\cdot \kappa_\alpha \qquad \text{where} \qquad \kappa_\alpha = \frac{(\alpha,\alpha)}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha)$$

is the truncated Casimir operator of $\mathfrak{sl}_2^\alpha$. ∇_κ is also flat [MTL] and therefore determines a homomorphism $F/I \to U\mathfrak{g}$ given by $t_\alpha \to \kappa_\alpha$. We extend it to a morphism $F/I \rtimes B_D \to \widehat{U\mathfrak{g}}$ by mapping the generator $S_i \in B_D$ to the triple exponential

$$\widetilde{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i) \tag{4.8}$$

where $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$ are a fixed choice of simple root vectors. By [Ti], the assignement $S_i \to \widetilde{s}_i$ extends to a homomorphism $\sigma : B_D \to N(H) \subset G$ where G is the connected and simply–connected complex Lie group corresponding to $\mathfrak{g}$, $H \subset G$ the maximal torus with Lie algebra $\mathfrak{h}$ and $N(H)$ its normaliser. The image of σ is an extension $\widetilde{W}$ of $W = N(H)/H$ by the sign group $\mathbb{Z}_2^{\dim(\mathfrak{h})}$ which we shall call the *Tits extension*. We shall however regard the elements $\widetilde{s}_i$ as lying in the completion $\widehat{U\mathfrak{g}}$ of $U\mathfrak{g}$ with respect to to its finite–dimensional representations rather than in $N(H)$.

Composing with the embedding (4.4) we therefore obtain a homomorphism $\imath_\kappa : A \rtimes B_D \to \widehat{U\mathfrak{g}}[\![h]\!] = \widehat{U\mathfrak{g}[\![h]\!]}$ given by

$$\imath_\kappa(t_\alpha) = h\kappa_\alpha \qquad \text{and} \qquad \imath_\kappa(S_i) = \widetilde{s}_i$$

Similarly, we have a homomorphism $\imath_C : A \rtimes B_D \to \widehat{U\mathfrak{g}}[\![h]\!]$ given by $\imath_C(t_\alpha) = hC_\alpha$ and $\imath_C(S_i) = \widetilde{s}_i$. For any subdiagram $D' \subseteq D$ with vertex set $\mathbf{I}' \subseteq \mathbf{I}$, let $\mathfrak{g}_{D'} \subseteq \mathfrak{g}_D$ be the subalgebra generated by e_i, f_i, h_i, $i \in \mathbf{I}'$.

Theorem 4.10.

(i) *The assignement* $D' \to U\mathfrak{g}_{D'}[\![h]\!]$ *defines a* D*–algebra structure on* $U\mathfrak{g}[\![h]\!]$.

(ii) *Let β be a positive, real adapted family. Then, the elements*

$$S_i^C = \widetilde{s}_i \cdot \exp(\pi\sqrt{-1} \cdot hC_{\alpha_i}) \qquad \text{and} \qquad \Phi^C_{\mathcal{G}\mathcal{F}} = \imath_\kappa(\Phi^\beta_{\mathcal{G}\mathcal{F}})$$

define a quasi–Coxeter algebra structure of type D on $U\mathfrak{g}[\![h]\!]$.

(iii) *If $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a finite–dimensional $\mathfrak{g}$–module, the corresponding action $\pi_\mathcal{F} : B_D \to GL(V[\![h]\!])$ given by the quasi–Coxeter algebra structure is equivalent to the monodromy representation of the flat vector bundle*

$$(\widetilde{\mathfrak{h}_{\mathrm{reg}}} \times V, p^*\nabla_C^h)/B_D \quad \text{where} \quad \nabla_C^h = d - h \sum_{\alpha\in\Phi_+} \frac{d\alpha}{\alpha} \cdot C_\alpha$$

expressed in the fundamental solution $\rho \circ \imath_C(\Psi^\beta_\mathcal{F})$.

(iv) *The quasi–Coxeter algebra structures corresponding to two adapted families β, β' differ by a canonical twist.*

Proof. (i) is clear. (ii) For any $i \in \mathbf{I}$, set

$$S_i^\kappa = \widetilde{s}_i \cdot \exp(\pi\sqrt{-1} \cdot h\kappa_{\alpha_i}) = \imath_\kappa(S_i \cdot \exp(\pi\sqrt{-1} \cdot t_{\alpha_i}))$$

We first prove that the S_i^κ and associators $\Phi^C_{\mathcal{G}\mathcal{F}}$ endow $U\mathfrak{g}[\![h]\!]$ with the structure of a quasi–Coxeter algebra of type D. This follows by transport of structure from proposition 4.2 except for the statement that, for an elementary pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D with $D' = \mathrm{supp}(\mathcal{F}, \mathcal{G})$ and $D'' = \mathfrak{z}\mathrm{supp}(\mathcal{G}, \mathcal{F})$ the associator $\Phi^C_{\mathcal{G}\mathcal{F}}$ commutes with $U\mathfrak{g}_{D''}$. To see this, it is sufficient to prove that the coefficients $\imath_\kappa(t_{\overline{x}})$ of the connection $\nabla_{\langle D'\rangle,\langle D''\rangle}$ defined by (1.15) are invariant under the adjoint action of $\mathfrak{g}_{D''}$[1]. Let $\alpha \in \Phi_+ \cap (\langle D'\rangle \setminus \langle D''\rangle)$ and $\langle D''\rangle_\alpha$ the irreducible component of $\langle D''\rangle \oplus \mathbb{C}\alpha$ containing α as in lemma 1.31. Set $\langle D''\rangle^\alpha = \bigoplus_i \langle D''\rangle_i$ where the latter are the irreducible components of $\langle D''\rangle$ contained in $\langle D''\rangle_\alpha$. By proposition 1.32,

$$t_{\overline{\alpha}} = t_{\langle D''\rangle_\alpha} - t_{\langle D''\rangle^\alpha}$$

For any subdiagram $B \subset D$, with corresponding subalgebra $\mathfrak{g}_B \subset \mathfrak{g}$, one has

$$\imath_\kappa(t_{\langle B\rangle}) = h \sum_{\alpha\in\Phi_+\cap\langle B\rangle} \kappa_\alpha = h\left(C_{\mathfrak{g}_B} - \sum_j t_j^2\right)$$

where $C_{\mathfrak{g}_B}$ is the Casimir operator of $\mathfrak{g}_B$ relative to the restriction of $(\cdot,\cdot)$ to it and t_j is an orthonormal basis of the Cartan subalgebra

[1]this is not true of the coefficients $\imath_C(t_{\overline{x}})$ which is why we prefer to work with the connection ∇_κ. ∇_C on the other hand has better local monodromies S_i^C since their squares are almost central in the corresponding $U_\hbar\mathfrak{sl}_2^i[\![h]\!]$.

$\mathfrak{g}_B \cap \mathfrak{h} = \langle B \rangle$ of $\mathfrak{g}_B$. Thus,

$$\imath_\kappa(t_{\overline{\alpha}}) = h \left(C_{\mathfrak{g}_{\langle D'' \rangle_\alpha}} - C_{\mathfrak{g}_{\langle D'' \rangle^\alpha}} - t^2 \right)$$

where t is a vector in $\langle D'' \rangle_\alpha$ which is orthogonal to $\langle D'' \rangle^\alpha$ and of norm one so that $\imath_\kappa(t_{\overline{\alpha}})$ is invariant under $\mathfrak{g}_{D''}$ as claimed. To prove that the S_i^C and $\Phi^C_{\mathcal{GF}}$ endow $U\mathfrak{g}[\![h]\!]$ with a quasi–Coxeter algebra structure, it suffices to show that the braid relations hold. This follows easily from the fact that $S_i^C = S_i^\kappa \cdot \exp(\pi\sqrt{-1} \cdot h^2_{\alpha_i}/2)$ and that the elements $\Phi^\kappa_{\mathcal{GF}}$ are of weight 0.

(iii) Consider the multivalued function $\Theta : \mathfrak{h}_{\text{reg}} \longrightarrow U\mathfrak{g}[\![h]\!]$ given by

$$\Theta = \prod_{\alpha \in \Phi_+} \alpha^{h \frac{\langle \alpha, \alpha \rangle}{4} h_\alpha^2}$$

One readily checks that Ψ satisfies $\nabla_\kappa \Psi = 0$ if, and only if, $\nabla_C(\Psi\Theta) = 0$. The claim follows easily from this and from the discussion in §1.17.7.

(iv) This again follows by transport of structure from proposition 4.2, except for the statement that the explicit twisting element $\imath_\kappa(c_{(D';\alpha^{D'}_{\mathcal{F}})}^{-R^{\mathcal{F}}_{<D'>}})$ of (4.5) commutes with $U\mathfrak{g}_{D' \setminus \alpha^{D'}_{\mathcal{F}}}$. This however follows from the already noted fact that

$$\imath_\kappa(R^{\mathcal{F}}_{<D'>}) = h(C_{\mathfrak{g}_{D'}} - C_{\mathfrak{g}_{D' \setminus \alpha^{D'}_{\mathcal{F}}}} - t^2)$$

where t is a unitary vector in $\langle D' \rangle$ which is orthogonal to $\langle D' \setminus \alpha^{D'}_{\mathcal{F}} \rangle$ ■

4.3. **From quasibialgebras to quasi–Coxeter algebras.** We show below that the commutant in the n–fold tensor product $A^{\otimes n}$ of a quasibialgebra A with a coassociative coproduct, or of a quasitriangular quasibialgebra A with a cocommutative and coassociative coproduct, has the structure of a quasi–Coxeter algebra of type A_{n-1}. In the latter case, the corresponding quasi–Coxeter representations of Artin's braid group B_n coincide with the R–matrix representations obtained from A. This construction abstracts the author's duality between the Knizhnik–Zamolodchikov connection for $\mathfrak{gl}_k$ and the Casimir connection ∇_C for $\mathfrak{gl}_n$ [TL1, §3] which can be used to relate the quasibialgebra structure on $U\mathfrak{gl}_k[\![\hbar]\!]$ responsible for the monodromy of the KZ connection to the quasi–Coxeter algebra structure on $U\mathfrak{gl}_n[\![\hbar]\!]$ underlying the monodromy of ∇_C.

4.3.1. Recall [Dr3, §1] that a quasibialgebra $(A,\Delta,\varepsilon,\Phi)$ is an algebra A endowed with algebra homomorphisms $\Delta: A\to A^{\otimes 2}$ and $\varepsilon: A\to k$ called the coproduct and counit, and an invertible element $\Phi\in A^{\otimes 3}$ called the associator which satisfy, for any $a\in A$

$$\mathrm{id}\otimes\Delta(\Delta(a)) = \Phi\cdot\Delta\otimes\mathrm{id}(\Delta(a))\cdot\Phi^{-1} \tag{4.9}$$

$$\mathrm{id}^{\otimes 2}\otimes\Delta(\Phi)\cdot\Delta\otimes\mathrm{id}^{\otimes 2}(\Phi) = 1\otimes\Phi\cdot\mathrm{id}\otimes\Delta\otimes\mathrm{id}(\Phi)\cdot\Phi\otimes 1 \tag{4.10}$$

$$\varepsilon\otimes\mathrm{id}\circ\Delta = \mathrm{id} \tag{4.11}$$

$$\mathrm{id}\otimes\varepsilon\circ\Delta = \mathrm{id} \tag{4.12}$$

$$\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}(\Phi) = 1 \tag{4.13}$$

Recall also that a twist of a quasibialgebra A is an invertible element $F\in A^{\otimes 2}$ satisfying

$$\varepsilon\otimes\mathrm{id}(F) = 1 = \mathrm{id}\otimes\varepsilon(F)$$

Given such an F, the twisting of A by F is the quasibialgebra $(A,\Delta_F,\varepsilon,\Phi_F)$ where the coproduct Δ_F and associator Φ_F are given by

$$\Delta_F(a) = F\cdot\Delta(a)\cdot F^{-1}$$

$$\Phi_F = 1\otimes F\cdot\mathrm{id}\otimes\Delta(F)\cdot\Phi\cdot\Delta\otimes\mathrm{id}(F^{-1})\cdot F^{-1}\otimes 1$$

A naïve, or strict morphism $\Psi: A\to A'$ of quasibialgebras is an algebra homomorphism satisfying

$$\varepsilon=\varepsilon'\circ\Psi,\qquad \Psi^{\otimes 2}\circ\Delta=\Delta'\circ\Psi\qquad\text{and}\qquad\Psi^{\otimes 3}(\Phi)=\Phi'$$

A morphism $A\to A'$ of quasibialgebras is a pair (Ψ,F') where F' is a twist of A' and Ψ is a naïve morphism of A to the twisting of A' by F'.

4.3.2. Let A be a quasibialgebra. For any $n\geq 1$, let $\mathcal{B}r_n$ be the set of complete bracketings on the non–associative monomial $x_1\cdots x_n$. If $n\geq 2$, we require that such bracketings contain the parentheses $(x_1\cdots x_n)$. Define, for any $\mathcal{B}\in\mathcal{B}r_n$, a homomorphism $\Delta_{\mathcal{B}}: A\to A^{\otimes n}$ in the following way. If $n=1$, set $\Delta_{x_1}=\mathrm{id}$. Otherwise, let

$$1\leq i_1<i_2-1<\cdots<i_k-(k-1)\leq n-k \tag{4.14}$$

be the indices i such that $\mathcal{B}$ contains the bracket $(x_i x_{i+1})$. Let $\overline{\mathcal{B}}$ be the bracketing on $\overline{x}_1\cdots\overline{x}_{n-k}$ obtained from $\mathcal{B}$ by performing the substitutions

$$x_\ell\longrightarrow\left\{\begin{array}{cl}\overline{x}_\ell & \text{if } 1\leq\ell<i_1\\ \overline{x}_{\ell-(j-1)} & \text{if } i_j+1<\ell<i_{j+1}\\ \overline{x}_{\ell-(k-1)} & \text{if } i_k+1<\ell\leq n\end{array}\right.\qquad\text{and}\qquad (x_{i_j}x_{i_j+1})\longrightarrow\overline{x}_{i_j} \tag{4.15}$$

and set

$$\Delta_{\mathcal{B}} = \mathrm{id}^{\otimes(i_1-1)} \otimes\Delta\otimes\mathrm{id}^{\otimes(i_2-i_1-2)} \otimes\cdots\otimes\mathrm{id}^{\otimes(i_k-i_{k-1}-2)} \otimes\Delta\otimes\mathrm{id}^{\otimes(n-1-i_k)} \circ\Delta_{\overline{\mathcal{B}}}$$

Let now $\mathcal{B},\mathcal{B}' \in \mathcal{B}r_n$ be two bracketings differing by the change of one pair of parentheses. To such a data one associates an element $\Phi_{\mathcal{B}'\mathcal{B}}$ of $A^{\otimes n}$ such that, for any $a \in A$,

$$\Phi_{\mathcal{B}'\mathcal{B}} \cdot \Delta_{\mathcal{B}}(a) = \Delta_{\mathcal{B}'}(a) \cdot \Phi_{\mathcal{B}'\mathcal{B}}$$

in the following way. Up to a permutation, $\mathcal{B}$ and $\mathcal{B}'$ differ by the replacement of a monomial $((\mathcal{B}_1\mathcal{B}_2)\mathcal{B}_3)$ by $(\mathcal{B}_1(\mathcal{B}_2\mathcal{B}_3)$ where $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3$ are parenthesised monomials in the variables $x_i\cdots x_{i+n_1-1}$, $x_{i+n_1}\cdots x_{i+n_1+n_2-1}$ and $x_{i+n_1+n_2}\cdots x_{i+n_1+n_2+n_3-1}$ respectively. Set

$$\Phi_{\mathcal{B}'\mathcal{B}} = 1^{\otimes(i-1)} \otimes \Delta_{\mathcal{B}_1} \otimes \Delta_{\mathcal{B}_2} \otimes \Delta_{\mathcal{B}_3}(\Phi) \otimes 1^{\otimes(n-(i+n_1+n_2+n_3)+1)} \tag{4.16}$$

For any bracketing $\mathcal{B} \in \mathcal{B}r_n$, and twist F of A, one also defines an element $F_{\mathcal{B}} \in A^{\otimes n}$ such that

$$(\Delta_F)_{\mathcal{B}} = \mathrm{Ad}(F_{\mathcal{B}}) \circ \Delta_{\mathcal{B}} \tag{4.17}$$

as follows. If $n = 1$, $F_{x_1} = 1$. Otherwise, let $1 \leq i_1 < \cdots < i_k-(k-1) \leq n-k$ and $\overline{\mathcal{B}}$ be as in (4.14) and (4.15) respectively and set

$$\begin{aligned} F_{\mathcal{B}} &= 1^{\otimes(i_1-1)} \otimes F \otimes 1^{\otimes(i_2-i_1-2)} \otimes\cdots\otimes 1^{\otimes(i_k-i_{k-1}-2)} \otimes F \otimes 1^{\otimes(n-1-i_k)} \\ &\quad\cdot \mathrm{id}^{\otimes(i_1-1)} \otimes\Delta\otimes\mathrm{id}^{\otimes(i_2-i_1-2)} \otimes\cdots\otimes\mathrm{id}^{\otimes(i_k-i_{k-1}-2)} \otimes\Delta\otimes\mathrm{id}^{\otimes(n-1-i_k)}(F_{\overline{\mathcal{B}}}) \end{aligned} \tag{4.18}$$

Then, one also has

$$(\Phi_F)_{\mathcal{B}'\mathcal{B}} = F_{\mathcal{B}'} \cdot \Phi_{\mathcal{B}'\mathcal{B}} \cdot F_{\mathcal{B}}^{-1} \tag{4.19}$$

4.3.3. Assume henceforth that the coproduct Δ of A is coassociative, *i.e.*, satisfies $\Delta\otimes\mathrm{id}\circ\Delta = \mathrm{id}\otimes\Delta\circ\Delta$, so that (A,Δ,ε) is a bialgebra. Then, $\Delta_{\mathcal{B}} = \Delta_{\mathcal{B}'}$ for any $\mathcal{B},\mathcal{B}' \in \mathcal{B}r_n$. Denote their common value by $\Delta^{(n)}$ and set

$$(A^{\otimes n})^A = \{\alpha \in A^{\otimes n} | \, [\alpha, \Delta^{(n)}(a)] = 0 \text{ for any } a \in A\}$$

Fix $n \in \mathbb{N}$, with $n \geq 2$ and let D be the Dynkin diagram of type A_{n-1}. We wish to define a quasi–Coxeter algebra structure of type D on $(A^{\otimes n})^A$. Identify for this purpose D with the interval $[1, n-1]$, its connected subdiagrams with subintervals $[i,j] \subseteq [1,n-1]$ with integral endpoints and maximal nested sets on D with elements in $\mathcal{B}r_n$ by attaching to $D' = [i,j]$ the bracket $x_1\cdots x_{i-1}(x_i\cdots x_{j+1})x_{j+2}\cdots x_n$ as in remark 1.40.

Theorem 4.11. *Let A be a quasibialgebra with coassociative coproduct and set $\mathcal{T}_n(A) = (A^{\otimes n})^A$. Then,*

(i) *The assignement*

$$[i,j] \longrightarrow \mathcal{T}_n(A)_{[i,j]} = 1^{\otimes(i-1)} \otimes (A^{\otimes(j-i+2)})^A \otimes 1^{\otimes(n-1-j)}$$

endows $\mathcal{T}_n(A)$ *with the structure of a* D*–algebra.*

(ii) *If the edges of* D *are each given an infinite multiplicity, the elements*

$$S_i^{\mathcal{T}_n(A)} = 1^{\otimes n},\ i = 1, \ldots, n-1 \qquad \textit{and} \qquad \Phi_{\mathcal{B}'\mathcal{B}}$$

give $\mathcal{T}_n(A)$ *the structure of a quasi–Coxeter algebra of type* D*.*

(iii) *If* $\Psi : A \to A'$ *is a naïve epimorphism of quasibialgebras, then* $\Psi_{\mathcal{B}} = \Psi^{\otimes n}$*,* $\mathcal{B} \in \mathcal{B}r_n$*, is a morphism of quasi–Coxeter algebras* $\mathcal{T}_n(A) \to \mathcal{T}_n(A')$*.*

(iv) *If* F *is a twist of* A *such that* Δ_F *is coassociative*[1]*, then* $\Psi_{\mathcal{B}} = \mathrm{Ad}(F_{\mathcal{B}})$*,* $\mathcal{B} \in \mathcal{B}r_n$*, is an isomorphism of quasi–Coxeter algebras* $\mathcal{T}_n(A) \to \mathcal{T}_n(A_F)$*.*

(v) *If, in addition,* F *is an invariant twist, then* $\{\mathrm{Ad}(F_{\mathcal{B}})\}_{\mathcal{B}\in\mathcal{B}r_n}$ *is the isomorphism induced by a canonical twist of* $\mathcal{T}_n(A)$*.*

The following is an immediate consequence of theorem 4.11

Corollary 4.12. *Let* $\mathcal{Q}$ *be the category of quasibialgebras with coassociative coproducts and morphisms* (Ψ, F') *with* Ψ *surjective. Then, the assignements* $A \to \mathcal{T}_n(A)$ *and* $(\Psi, F') \to \{\mathrm{Ad}(F'_{\mathcal{B}}) \circ \Psi^{\otimes n}\}_{\mathcal{B}\in\mathcal{B}r_n}$ *define a functor from* $\mathcal{Q}$ *to the category of quasi–Coxeter algebras of type* A_{n-1}*.*

Remark 4.13. Note that $\mathcal{T}_n(A)_{[i,j]}$ is not generated by the subalgebras $\mathcal{T}_n(A)_k$ corresponding to the vertices k of $D' = [i,j]$.

4.3.4. Proof of theorem 4.11. (i) The identity

$$\Delta^{(n)} = \mathrm{id}^{\otimes(i-1)} \otimes \Delta^{(j-i+2)} \otimes \mathrm{id}^{\otimes(n-1-j)} \circ \Delta^{(n-1-(j-i))}$$

shows that $\mathcal{T}_n(A)_{[i,j]}$ is a subalgebra of $\mathcal{T}_n(A)$ and that $\mathcal{T}_n(A)_{[i,j]} \subseteq \mathcal{T}_n(A)_{[i',j']}$ if $[i,j] \subseteq [i',j']$. It is moreover clear that $[\mathcal{T}_n(A)_{[i,j]}, \mathcal{T}_n(A)_{[i',j']}] = 0$ if either $j < i'-1$ or $j' < i-1$.

(ii) By definition, the associators $\Phi_{\mathcal{B}'\mathcal{B}}$ satisfy $\Phi_{\mathcal{B}\mathcal{B}'} = \Phi_{\mathcal{B}'\mathcal{B}}^{-1}$. When Δ is coassociative, (4.16) reduces to

$$\Phi_{\mathcal{B}'\mathcal{B}} = 1^{\otimes(i-1)} \otimes \Delta^{(n_1)} \otimes \Delta^{(n_2)} \otimes \Delta^{(n_3)}(\Phi) \otimes 1^{\otimes(n-(i+n_1+n_2+n_3)+1)} \quad (4.20)$$

This element lies in $1^{\otimes(i-1)} \otimes (A^{\otimes(n_1+n_2+n_3)})^A \otimes 1^{\otimes(n-(i+n_1+n_2+n_3)+1)}$, and therefore in $\mathcal{T}_n(A)$, because of the identity

$$\Delta^{(n_1+n_2+n+3)} = \Delta^{(n_1)} \otimes \Delta^{(n_2)} \otimes \Delta^{(n_3)} \circ \Delta^{(3)}$$

[1]this is equivalent to the requirement that $\mathrm{id} \otimes \Delta(F^{-1}) \cdot 1 \otimes F^{-1} \cdot F \otimes 1 \cdot \Delta \otimes \mathrm{id}(F) \in (A^{\otimes 3})^A$.

and of the fact that, when Δ is coassociative, Φ lies in $(A^{\otimes 3})^A$. (4.20) also shows that $\Phi_{\mathcal{B}'\mathcal{B}}$ satisfies the forgetfulness axiom as well as the support axiom since it commutes with

$$\begin{aligned}&\mathcal{T}_n(A)_{[i,i+n_1+n_2+n_3-1]\setminus\alpha_{\mathcal{B}\cap\mathcal{B}'}^{[i,i+n_1+n_2+n_3-1]}}\\&\quad= 1^{\otimes(i-1)}\otimes(A^{\otimes n_1})^A\otimes(A^{\otimes n_2})^A\otimes(A^{\otimes n_3})^A\otimes 1^{\otimes(n-(i+n_1+n_2+n_3)+1)}\end{aligned}$$

The coherence axiom follows from MacLane's coherence theorem and the braid relations are void in this case since $m_{ij}=\infty$ for any i,j.
(iii) The surjectivity of Ψ guarantees that $\Psi^{\otimes n}$ maps $(A^{\otimes n})^A$ to $(A'^{\otimes n})^{A'}$, the rest of the claim is clear.
(iv) When Δ is coassociative, the relation (4.17) reduces to $\Delta_F^{(n)} = \mathrm{Ad}(F_{\mathcal{B}})\circ\Delta^{(n)}$ and shows that $\mathrm{Ad}(F_{\mathcal{B}})$ maps $\mathcal{T}_n(A)$ to $\mathcal{T}_n(A_F)$. Set now, for $1\le i\le k\le j\le n-1$,

$$\begin{aligned}F_{([i,j];k)} &= 1^{\otimes(i-1)}\otimes\Delta^{(k-i+1)}\otimes\Delta^{(j-k+1)}(F)\otimes 1^{\otimes(n-1-j)}\\&\in 1^{\otimes(i-1)}\otimes A^{\otimes(j-i+2)}\otimes 1^{\otimes(n-1-j)}\end{aligned}\tag{4.21}$$

An induction based on (4.18) shows that

$$F_{\mathcal{B}} = \overrightarrow{\prod_{[i,j]\in\mathcal{B}}} F_{([i,j];\alpha_{\mathcal{B}}^{[i,j]})}\tag{4.22}$$

where the product is taken with $F_{([i,j];\alpha_{\mathcal{B}}^{[i,j]})}$ written to the left of $F_{([i',j'];\alpha_{\mathcal{B}}^{[i',j']})}$ whenever $[i,j]\subset[i',j']$[1]. Since $F_{([i,j];k)}$ commutes with $\mathcal{T}_n(A)_{[i,k-1]\cup[k+1,j]}$ we get, for any $[i,j]\in\mathcal{B}$

$$\mathrm{Ad}(F_{\mathcal{B}})\left(\mathcal{T}_n(A)_{[i,j]}\right) = \mathrm{Ad}(F_{\mathcal{B}_{[i,j]}})\left(\mathcal{T}_n(A)_{[i,j]}\right) = \mathcal{T}_n(A_F)_{[i,j]}$$

where $\mathcal{B}_{[i,j]}$ is the bracketing on $x_i\cdots x_{j+1}$ induced by $\mathcal{B}$. Thus $\{\mathrm{Ad}(F_{\mathcal{B}})\}$ is a D–algebra morphism $\mathcal{T}_n(A)\to\mathcal{T}_n(A_F)$ and therefore a morphism of quasi–Coxeter algebras by (4.19).
(iv) Assume that $F\in(A^{\otimes 2})^A$, so that $\Delta_F=\Delta$. The identity $\Delta^{\otimes(j-i+2)} = \Delta^{\otimes(k-i+1)}\otimes\Delta^{\otimes(j-k+1)}\circ\Delta$ shows that the element $F_{([i,j];k)}$ defined by (4.21) is invariant under A. In other words,

$$a_{([i,j];k)} = F_{([i,j];k)}\in\mathcal{T}_n(A)_{[i,j]}^{[i,j]\setminus k}$$

is a twist of $\mathcal{T}_n(A)$ and $\mathcal{T}_n(A_F)$ is obtained from $\mathcal{T}_n(A)$ by twisting by $\{a_{([i,j];k)}\}$ ■

[1] this does not specify the order of the factors uniquely, but any two orders satisfying this requirement are easily seen to yield the same result.

4.3.5. We give next a similar construction for quasitriangular quasibialgebras with a coassociative and cocommutative coproduct. Recall first [Dr3, §3] that a quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ is quasitriangular if it is endowed with an invertible element $R \in A^{\otimes 2}$ satisfying, for any $a \in A$,

$$\Delta^{\mathrm{op}}(a) = R \cdot \Delta(a) \cdot R^{-1} \tag{4.23}$$

$$\Delta \otimes \mathrm{id}(R) = \Phi_{312} \cdot R_{13} \cdot \Phi_{132}^{-1} \cdot R_{23} \cdot \Phi_{123} \tag{4.24}$$

$$\mathrm{id} \otimes \Delta(R) = \Phi_{231}^{-1} \cdot R_{13} \cdot \Phi_{213} \cdot R_{12} \cdot \Phi_{123}^{-1} \tag{4.25}$$

A twist F of a quasitriangular quasibialgebra A is a twist of the underyling quasibialgebra. The twisting of A by F is the quasitriangular quasibialgebra $(A, \Delta_F, \varepsilon, \Phi_F, R_F)$ where

$$R_F = F_{21} \cdot R \cdot F^{-1}$$

A morphism $(\Psi, F') : A \to A'$ of quasitriangular quasibialgebras is a morphism of the underlying quasibialgebras such that $\Phi^{\otimes 2}(R) = R'_{F'}$.

4.3.6. Let A be a quasibialgebra with cocommutative and coassociative coproduct. Then $\sigma \circ \Delta^{(n)} = \Delta^{(n)}$ for any $\sigma \in \mathfrak{S}_n$, so that $(A^{\otimes n})^A$ is invariant under $\mathfrak{S}_n$. Set $\widetilde{\mathcal{T}}_n(A) = \mathcal{T}_n(A) \rtimes \mathfrak{S}_n = (A^{\otimes n})^A \rtimes \mathfrak{S}_n$ and, for any $1 \leq i \leq j \leq n-1$, let $\mathfrak{S}_{[i,j]} \subseteq \mathfrak{S}_n$ be the subgroup generated by the transpositions $(k\, k+1)$, $k = i, \ldots, j$.

Theorem 4.14.

(i) *The assignement*

$$\widetilde{\mathcal{T}}_n(A)_{[i,j]} = 1^{\otimes(i-1)} \otimes (A^{\otimes(j-i+2)})^A \otimes 1^{\otimes(n-1-j)} \rtimes \mathfrak{S}_{[i,j]}$$

endows $\widetilde{\mathcal{T}}_n(A)$ *with the structure of a* D*–algebra.*

(ii) *If the edges of* D *are each given the multiplicity* 3*, the elements*

$$S_i^{\widetilde{\mathcal{T}}_n(A)} = (i\, i+1) \cdot 1^{\otimes(i-1)} \otimes R \otimes 1^{\otimes(n-i-1)}$$

and the associators $\Phi_{\mathcal{B}'\mathcal{B}}$ *give* $\widetilde{\mathcal{T}}_n(A)$ *the structure of a quasi–Coxeter algebra of type* D*.*

(iii) *If* $(\Psi, F') : A \to A'$ *is a surjective morphism of quasitriangular quasibialgebras,* $\Psi_{\mathcal{B}} = \mathrm{Ad}(F'_{\mathcal{B}}) \circ \Psi^{\otimes n}$*,* $\mathcal{B} \in \mathcal{B}r_n$*, is a quasi–Coxeter algebra morphism* $\widetilde{\mathcal{T}}_n(A) \to \widetilde{\mathcal{T}}_n(A')$*.*

(iv) *If* F *is an invariant twist of* A*, then* $\widetilde{T}_n(A)$ *and* $\widetilde{T}_n(A_F)$ *differ by a twist.*

Proof. (i) is clear. (ii) Note first that the cocommutativity of Δ and (4.23) imply that $R \in (A^{\otimes 2})^A$ so that $S_i^{\widetilde{\mathcal{T}}_n(A)} \in \widetilde{\mathcal{T}}_n(A)_i$ for any

$i = 1, \ldots, n-1$. By theorem 4.11, the associators $\Phi_{\mathcal{B}'\mathcal{B}}$ satisfy the orientation, forgetfulness and coherence axioms. The support axioms follows from the fact that, by (4.20), $\Phi_{\mathcal{B}'\mathcal{B}}$ commutes with $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \mathfrak{S}_{n_3}$ since Δ is cocommutive. The braid relations follow from the hexagon relations (4.24)–(4.25) in the usual way.
(iii) The claim is obvious if $F' = 1$. Assume now that $\Psi = \mathrm{id}_A$ and that F is a twist of A such that Δ_F is cocommutative and coassociative[1]. By (4.17), $\mathrm{Ad}(F_{\mathcal{B}})$ maps $\mathcal{T}_n(A)$ to $\mathcal{T}_n(A_F)$. Moreover, if $\sigma \in \mathfrak{S}_n$ and $a \in A$, then

$$\begin{aligned} F_{\mathcal{B}}\sigma F_{\mathcal{B}}^{-1}\sigma^{-1}\Delta_F^{(n)}(a) &= F_{\mathcal{B}}\sigma F_{\mathcal{B}}^{-1}\Delta_F^{(n)}(a)\sigma^{-1} = F_{\mathcal{B}}\sigma\Delta^{(n)}(a)F_{\mathcal{B}}^{-1}\sigma^{-1} \\ &= F_{\mathcal{B}}\Delta^{(n)}(a)\sigma F_{\mathcal{B}}^{-1}\sigma^{-1} = \Delta_F^{(n)}(a)F_{\mathcal{B}}\sigma F_{\mathcal{B}}^{-1}\sigma^{-1} \end{aligned}$$

where the first and third equalities follow from the cocommutativity of Δ_F and Δ respectively and the second and fourth from (4.17). It follows from this that $\mathrm{Ad}(F_{\mathcal{B}})$ maps $\widetilde{\mathcal{T}}_n(A)$ to $\widetilde{\mathcal{T}}_n(A_F)$. If $[i,j] \in \mathcal{B}$, the factorisation (4.22) together with the cocommutativity of Δ readily imply that

$$\mathrm{Ad}(F_{\mathcal{B}})\left(\widetilde{\mathcal{T}}_n(A)_{[i,j]}\right) = \mathrm{Ad}(F_{\mathcal{B}_{[i,j]}})\left(\widetilde{\mathcal{T}}_n(A)_{[i,j]}\right) \tag{4.26}$$

Thus, $\mathrm{Ad}(F_{\mathcal{B}})$ defines a D–algebra morphism $\widetilde{\mathcal{T}}_n(A) \rightarrow \widetilde{\mathcal{T}}_n(A_F)$ and therefore a morphism of quasi–Coxeter algebras by (4.19) since, by (4.26), for any $i = 1, \ldots, n-1$ such that $[i,i] \in \mathcal{B}$,

$$\begin{aligned} \mathrm{Ad}(F_{\mathcal{B}})S_i^{\widetilde{\mathcal{T}}_n(A)} &= \mathrm{Ad}(F_{([i,i];i)})S_i^{\widetilde{\mathcal{T}}_n(A)} \\ &= (i\, i+1)1^{\otimes i-1} \otimes F_{21}RF^{-1} \otimes 1^{\otimes n-1-i} \\ &= S_i^{\widetilde{\mathcal{T}}_n(A_F)} \end{aligned}$$

(iv) follows as in the proof of theorem 4.11 ■

Corollary 4.15. *Let $\widetilde{\mathcal{Q}}$ be the category of quasitriangular quasibialgebras with cocommutative and coassociative coproduct and morphisms (Ψ, F') where Ψ is surjective. Then, for any $n \geq 2$,*

- (i) *The assignement $A \rightarrow \widetilde{\mathcal{T}}_n(A)$ is a functor from $\widetilde{\mathcal{Q}}$ to the category of quasi–Coxeter algebras of type A_{n-1}.*
- (ii) *The R–matrix representation of Artin's braid group B_n corresponding to an A–module V and a bracketing $\mathcal{B} \in \mathcal{B}r_n$ coincides with the quasi–Coxeter algebra representation $\pi_{\mathcal{B}}$ of B_n on the $\widetilde{\mathcal{T}}_n(A)$–module $V^{\otimes n}$.*

[1]the cocommutativity of Δ_F is equivalent to the requirement that $F_{21}^{-1}F \in (A^{\otimes 2})^A$.

5. The Dynkin complex and deformations of quasi–Coxeter algebras

Let D be a connected diagram and A a D–algebra. We define in §5.1 the *Dynkin complex* of A and study some of its elementary properties in §5.2–§5.3. Its main property, which we establish in §5.7, is that it controls the deformation theory of quasi–Coxeter algebra structures on A. This is obtained by showing in §5.6 that, in degrees greater or equal to 2, the Dynkin complex of A embeds into the cellular cochain complex of the De Concini–Procesi associahedron $\mathcal{A}_D$. In turn, this embedding is obtained from an explicit presentation of the cellular chain complex of $\mathcal{A}_D$ in terms of the poset $\mathcal{N}_D$ of nested sets on D which is described in §5.4 and §5.5.

5.1. The Dynkin complex of A.

We begin by defining the category of coefficients of the Dynkin complex of A.

Definition 5.1. *A D–bimodule over A is an A–bimodule M, with left and right actions denoted by am and ma respectively, endowed with a family of subspaces M_{D_1} indexed by the connected subdiagrams $D_1 \subseteq D$ of D such that the following properties hold*

- *for any $D_1 \subseteq D$,*
$$A_{D_1} M_{D_1} \subseteq M_{D_1} \qquad \text{and} \qquad M_{D_1} A_{D_1} \subseteq M_{D_1}$$
- *For any pair $D_2 \subseteq D_1 \subseteq D$,*
$$M_{D_2} \subseteq M_{D_1}$$
- *For any pair of orthogonal subdiagrams D_1, D_2 of D, $a_{D_1} \in A_{D_1}$ and $m_{D_2} \in M_{D_2}$,*
$$a_{D_1} m_{D_2} = m_{D_2} a_{D_1}$$

A morphism of D–bimodules M, N over A is an A–bimodule map $T : M \to N$ such that $T(M_D) \subseteq N_D$ for all $D \subseteq D$.

Clearly, A is a D–bimodule over itself. We denote by $\text{Bimod}_D(A)$ the abelian subcategory of $\text{Bimod}(A)$ consisting of D–bimodules over A. If $M \in \text{Bimod}_D(A)$, and $D_1 \subseteq D$ is a subdiagram, we set

$$M^{D_1} = \{m \in M \,|\, am = ma \quad \text{for any } a \in A_{D_1^i}\}$$

where D_1^i are the connected components of D_1. In particular, if $D_1, D_2 \subseteq D$ are orthogonal, and D_1 is connected, then $M_{D_1} \subseteq M^{D_2}$.

Let $M \in \mathrm{Bimod}_D(A)$. For any integer $0 \leq p \leq n = |D|$, set

$$C^p(A;M) = \bigoplus_{\substack{\underline{\alpha}\subseteq D_1\subseteq D,\\ |\underline{\alpha}|=p}} M_{D_1}^{D_1\setminus\underline{\alpha}}$$

where the sum ranges over all connected subdiagrams D_1 of D and ordered subsets $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_p\}$ of cardinality p of D_1 and $M_{D_1}^{D_1\setminus\underline{\alpha}} = (M_{D_1})^{D_1\setminus\underline{\alpha}}$. We denote the component of $m \in C^p(A;M)$ along $M_{D_1}^{D_1\setminus\underline{\alpha}}$ by $m_{(D_1;\underline{\alpha})}$.

Definition 5.2. *The group of Dynkin p–cochains on A with coefficients in M is the subspace $CD^p(A;M) \subset C^p(A;M)$ of elements m such that*

$$m_{(D_1;\sigma\underline{\alpha})} = (-1)^\sigma m_{(D_1;\underline{\alpha})}$$

where, for any $\sigma \in \mathfrak{S}_p$, $\sigma\{\alpha_1, \ldots, \alpha_p\} = \{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}\}$.

Note that

$$CD^0(A;A) = \bigoplus_{D_1\subseteq D} Z(A_{D_1}) \qquad \text{and} \qquad CD^n(A;M) \cong M_D$$

For $1 \leq p \leq n-1$, define a map $d_D^p : C^p(A;M) \to C^{p+1}(A;M)$ by

$$d_D m_{(D_1;\underline{\alpha})} = \sum_{i=1}^{p+1} (-1)^{i-1} \left(m_{(D_1;\underline{\alpha}\setminus\alpha_i)} - m_{(\complement_{\underline{\alpha}\setminus\alpha_i}^{D_1\setminus\alpha_i};\underline{\alpha}\setminus\alpha_i)} \right)$$

where $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_{p+1}\}$, $\complement_{\underline{\alpha}\setminus\alpha_{i_j}}^{D_1\setminus\alpha_i}$ is the connected component of $D_1 \setminus \alpha_i$ containing $\underline{\alpha} \setminus \alpha_i$ if one such exists and the empty set otherwise, and we set $m_{(\emptyset;-)} = 0$. For $p = 0$, define $d_D^0 : C^0(A;M) \to C^1(A;M)$ by

$$d_D^0 m_{(D_1;\alpha_i)} = m_{D_1} - m_{D_1\setminus\alpha_i}$$

where $m_{D_1\setminus\alpha_i}$ is the sum of m_{D_2} with D_2 ranging over the connected components of $D_1 \setminus \alpha_i$. Finally, set $d_D^n = 0$. It is easy to see that the Dynkin differential d_D is well–defined and that it leaves $CD^*(A;M)$ invariant.

Theorem 5.3. $(CD^*(A;M), d_D)$ *is a complex. The cohomology groups*

$$HD^p(A;M) = \mathrm{Ker}(d_D^p)/\,\mathrm{Im}(d_D^{p-1})$$

for $p = 0, \ldots, |D|$ are called the Dynkin diagram cohomology groups of A with coefficients in M.

Proof. Assume first that m is a zero–cochain. Let $B \subseteq D$ be a connected diagram, α_1, α_2 two distinct elements of D and set

$$B_1 = \complement_{\alpha_1}^{B\setminus\alpha_2} \qquad \text{and} \qquad B_2 = \complement_{\alpha_2}^{B\setminus\alpha_1}$$

Then,

$$\begin{aligned} d_D^2\, m_{(B;\alpha_1,\alpha_2)} &= d_D m_{(B;\alpha_2)} - d_D m_{(B_2;\alpha_2)} - d_D m_{(B;\alpha_1)} + d_D m_{(B_1;\alpha_1)} \\ &= m_B - m_{B\setminus\alpha_2} - m_{B_2} + m_{B_2\setminus\alpha_2} - m_B + m_{B\setminus\alpha_1} + m_{B_1} - m_{B_1\setminus\alpha_1} \\ &= (m_{B\setminus\alpha_1} - m_{B_2} + m_{B_2\setminus\alpha_2}) - (m_{B\setminus\alpha_2} - m_{B_1} + m_{B_1\setminus\alpha_1}) \\ &= m_{B\setminus\{\alpha_1,\alpha_2\}} - m_{B\setminus\{\alpha_1,\alpha_2\}} \\ &= 0 \end{aligned}$$

To treat the general case, we shall need the following

Lemma 5.4. *Let $\underline{\alpha} = \{\alpha_1, \dots, \alpha_k\} \subseteq B$ be a subset of cardinality $k \geq 3$. For any $1 \leq i \neq j \leq k$, set*

$$B_{ij} = \complement_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}}^{\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i}\setminus\alpha_j}$$

Then,

(i) *if $\complement_{\underline{\alpha}\setminus\alpha_j}^{B\setminus\alpha_j} = \emptyset$, then, $B_{ij} = \complement_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}}^{B\setminus\alpha_j}$.*

(ii) *If $\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i} \neq \emptyset$, then, $B_{ij} = \complement_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}}^{B\setminus\{\alpha_i,\alpha_j\}}$.*

Proof. (i) The left–hand side is contained in the right–hand side since, when it is not empty, it is connected, contained in $B \setminus \alpha_j$ and contains $\underline{\alpha} \setminus \{\alpha_i, \alpha_j\} \neq \emptyset$. Similarly, the right–hand side is contained in $\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i}$ since, when non–empty, it is connected, does not contain α_i by assumption, and contains $\underline{\alpha} \setminus \{\alpha_i, \alpha_j\}$. Since the right–hand side does not contain α_j, it is therefore contained in the left–hand side. (ii) is proved in a similar way ■

Write $d_D = d_1 - d_2$ where, for $\underline{\alpha} = \{\alpha_1, \dots, \alpha_k\}$,

$$d_1 m_{(B;\underline{\alpha})} = \sum_{i=1}^{k} (-1)^{i-1} m_{(B;\underline{\alpha}\setminus\alpha_i)} \qquad d_2 m_{(B;\underline{\alpha})} = \sum_{i=1}^{k} (-1)^{i-1} m_{(\complement_{\underline{\alpha}\setminus\alpha_i}^{B\setminus\alpha_i};\underline{\alpha}\setminus\alpha_i)}$$

Note that

$$d_1^2\, m_{(B;\underline{\alpha})} = \sum_{1\leq i\neq j\leq k} (-1)^{i+j} \operatorname{sign}(i-j)\, m_{(B;\underline{\alpha}\setminus\{\alpha_i,\alpha_j\})} = -d_1^2\, m_{(B;\underline{\alpha})}$$

whence $d_1^2 = 0$. We next have

$$\begin{aligned} d_1 d_2\, m_{(B;\underline{\alpha})} = &\sum_{i=1}^{k} (-1)^{i-1} d_2 a_{(B;\underline{\alpha}\setminus\alpha_i)} \\ = &\sum_{1\leq i\neq j\leq k} (-1)^{i+j} \operatorname{sign}(i-j)\, m_{(\complement_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}}^{B\setminus\alpha_j};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\})} \end{aligned}$$

and

$$\begin{aligned} d_2 d_1\, m_{(B;\underline{\alpha})} &= \sum_{j=1}^{k} (-1)^{j-1} d_1 a_{(\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j};\underline{\alpha}\setminus\alpha_j)} \\ &= \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(j-i)\, m_{(\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\})} \end{aligned}$$

so that

$$\begin{aligned} &(d_1 d_2 + d_2 d_1)\, m_{(B;\underline{\alpha})} \\ &\quad= \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, \delta_{\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j}=\emptyset} \cdot m_{(\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\})} \end{aligned}$$

Finally,

$$\begin{aligned} d_2^2\, m_{(B;\underline{\alpha})} &= \sum_{i=1}^{k} (-1)^{i-1} d_2 a_{(\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i};\underline{\alpha}\setminus\alpha_i)} \\ &= \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, m_{\left(\mathsf{C}^{\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i}\setminus\alpha_j}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}\right)} \\ &= \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, \delta_{\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j}=\emptyset} \cdot m_{\left(\mathsf{C}^{\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i}\setminus\alpha_j}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}\right)} \\ &\quad+ \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, \delta_{\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j}\neq\emptyset} \cdot m_{\left(\mathsf{C}^{\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i}\setminus\alpha_j}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}\right)} \end{aligned}$$

We claim that the second summand is zero. Indeed, it is equal to

$$\begin{aligned} &\sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, \delta_{\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j}\neq\emptyset} \cdot \delta_{\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i}\neq\emptyset} \cdot m_{\left(\mathsf{C}^{\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i}\setminus\alpha_j}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}\right)} \\ &= \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, \delta_{\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j}\neq\emptyset} \cdot \delta_{\mathsf{C}^{B\setminus\alpha_i}_{\underline{\alpha}\setminus\alpha_i}\neq\emptyset} \cdot m_{(\mathsf{C}^{B\setminus\{\alpha_i,\alpha_j\}}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\})} \end{aligned}$$

where we used (ii) of lemma 5.4, and it therefore vanishes since the summand is antisymmetric in the interchange $i \leftrightarrow j$. Thus, by (i) of lemma 5.4,

$$\begin{aligned} d_2^2\, m_{(B;\underline{\alpha})} &= \sum_{1\leq i\neq j\leq k} (-1)^{i+j}\,\mathrm{sign}(i-j)\, \delta_{\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\alpha_j}=\emptyset}\, m_{(\mathsf{C}^{B\setminus\alpha_j}_{\underline{\alpha}\setminus\{\alpha_i,\alpha_j\}};\underline{\alpha}\setminus\{\alpha_i,\alpha_j\})} \\ &= (d_1 d_2 + d_2 d_1) m_{(B;\underline{\alpha})} \end{aligned}$$

so that $d_D^2 = d_1^2 - (d_1 d_2 + d_2 d_1) + d_2^2 = 0$ ■

5.2. **Elementary Properties of the Dynkin complex.**

5.2.1. *Functoriality with respect to restriction to subdiagrams.* Let $M \in \text{Bimod}_D(A)$, $B \subseteq D$ a connected subdiagram and consider M_B as an B–bimodule over A_B. Then, the map

$$p_{B,D} : CD^*(A;M) \longrightarrow CD^*(A_B;M_B)$$

given by $p_{B,D} m_{(B;\underline{\alpha})} = m_{(B;\underline{\alpha})}$ for any $\underline{\alpha} \subseteq B \subseteq B$ is a chain map satisfying

$$p_{C,B} \cdot p_{B,D} = p_{C,D}$$

for any $C \subseteq B \subseteq D$. More generally, the Dynkin complex is functorial with respect to naïve morphisms of D–algebras but not with respect to general morphisms.

5.2.2. *Low–dimensional cohomology groups.*

Proposition 5.5. *Let M be a D–bimodule over A. Then*

(i) $H^0(A;M) = 0$.

(ii) *If, for each connected $B \subseteq D$, the algebra A_B is generated by the subalgebras A_α, $\alpha \in B$, the map*

$$\bigoplus_{\alpha \in D} p_{\alpha,D} : H^1(A;M) \longrightarrow \bigoplus_{\alpha \in D} H^1(A_\alpha;M_\alpha)$$

is injective.

Proof. (i) We need to prove that the differential d_D^0 is injective. Let $m = \{m_B\}_{B \subseteq D}$ be a zero–cocycle and assume by induction that $m_B = 0$ whenever $|B| \leq k$. Let $B \subseteq D$ be a connected subdiagram of cardinality $k+1$ and let $\alpha \in B$. Then

$$0 = dm_{(B;\alpha)} = m_B - m_{B \setminus \alpha}$$

implies that $m_B = 0$ whence $m = 0$.

(ii) Note that $H^1(A_\alpha;M_\alpha) = M_\alpha / M_\alpha^\alpha$. Let $m = \{m_{(B;\alpha)}\}_{\alpha \in B \subseteq D}$ be a one–cocycle such that $m_{(\alpha;\alpha)} \in M_\alpha^\alpha$ for any $\alpha \in D$. Replacing m by $m - d_D n$, where $n \in CD^0(A;M)$ is given by

$$n_B = \begin{cases} m_{(\alpha;\alpha)} & \text{if } B = \alpha \\ 0 & \text{if } |B| \geq 2 \end{cases}$$

we may assume that $m_{(\alpha;\alpha)} = 0$ for any $\alpha \in D$. Assume therefore that, up to the addition of a 1–coboundary, $m_{(B;\alpha)} = 0$ for any $\alpha \in B \subseteq D$ such that $|B| \leq k$. For any B of cardinality $k+1$ and $\alpha \neq \beta \in B$, we have

$$0 = d_D m_{(B;\alpha,\beta)} = m_{(B;\beta)} - m_{(\mathsf{C}_\beta^{B\setminus\alpha};\beta)} - m_{(B;\alpha)} + m_{(\mathsf{C}_\alpha^{B\setminus\beta};\alpha)}$$

whence, given that $|\mathsf{C}_\beta^{B\setminus\alpha}| \leq k$,

$$m_{(B;\alpha)} = m_{(B;\beta)}$$

This implies that $m_{(B;\alpha)} \in M_B^{B\setminus\alpha} \cap M_B^{B\setminus\beta}$ which, by assumption is equal to M_B^B, and that $m_{(B;\alpha)}$ is independent of $\alpha \in B$. Thus, replacing m by $m - d_D n$ where

$$n_B = \begin{cases} m_{(B;\alpha)} & \text{if } |B| = k+1 \text{ and } \alpha \in B \\ 0 & \text{if } |B| \neq k+1 \end{cases}$$

we find that $m_{(B;\alpha)} = 0$ whenever $|B| \leq k+1$ ■

Corollary 5.6. *If* (W, S) *is a Coxeter system, then* $HD^i(k[W]; k[W]) = 0$ *for* $i = 0, 1$.

Proof. For any simple reflection $s_i \in S$, $H^1(A_{s_i}; A_{s_i}) = A_{s_i}/A_{s_i}^{s_i} = 0$ since $A_{s_i} \cong k[\mathbb{Z}_2]$ is commutative. The result now follows from proposition 5.5 ■

5.3. **Dynkin cohomology and Hochschild cohomology.** Let $n \geq 2$, D the Dynkin diagram of type A_{n-1}, A a bialgebra and $\mathcal{T}_n(A) = (A^{\otimes n})^A$ the D–algebra constructed in §4.3.3. We relate below the cobar complex of A with the Dynkin complex of $\mathcal{T}_n(A)$.

Recall first (see *e.g.*, [Ks, §XVIII.5]) that if (A, Δ, ε) is a coalgebra endowed with an element $1 \in A$ such that $\Delta(1) = 1 \otimes 1$, the cobar complex $(C^k(A), d_H)$ of A is defined by setting $C^k(A) = A^{\otimes k}$ for $k \geq 0$, $d_H = 0$ in degree 0 and, for $a \in A^{\otimes k}$, $k \geq 1$,

$$d_H a = 1 \otimes a + \sum_{i=1}^{k} (-1)^i \, \mathrm{id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(k-i)}(a) + (-1)^{k+1} 1 \otimes a$$

If A is a bialgebra with unit 1, the subspaces $(A^{\otimes k})^A$ are readily seen to form a subcomplex of $C^k(A)$. For $k = 1, \ldots, n$, define a map $\phi_k : (A^{\otimes k})^A \to CDD^{k-1}(\mathcal{T}_n(A); \mathcal{T}_n(A))$ by

$$\phi_1 a_{([i,j])} = \sum_{\ell=0}^{j-i+1} 1^{\otimes(i-1+\ell)} \otimes a \otimes 1^{\otimes(n-i-\ell)} - 1^{\otimes(i-1)} \otimes \Delta^{(j-i+2)}(a) \otimes 1^{\otimes(n-1-j)}$$

and, for $k \geq 2$,

$$\phi_k a_{([i,j];\ell_1,\ldots,\ell_{k-1})} = (-1)^k \cdot 1^{\otimes(i-1)} \otimes \Delta^{(\ell_1-i+1)} \otimes \Delta^{(\ell_2-\ell_1)} \otimes \cdots$$
$$\cdots \otimes \Delta^{(\ell_{k-1}-\ell_{k-2})} \otimes \Delta^{(j-\ell_k+1)}(a) \otimes 1^{\otimes(n-i-j)}$$

where $1 \leq i \leq \ell_1 < \cdots < \ell_{k-1} \leq j \leq n-1$.

Proposition 5.7. *For any* $k = 1, \ldots, n-2$,

$$d_{DD} \circ \phi_k = d_H \circ \phi_{k+1}$$

so that ϕ is a degree -1 chain map from the invariant, truncated cobar complex $((A^{\otimes k})^A, d_H)_{k=1}^n$ of A to the Dynkin complex of $\mathcal{T}_n(A)$.

PROOF. This follows by a straightforward computation ■

5.4. **The chain complex $C_*(\mathcal{N}_D)$.** We define in this subsection the chain complex $C_*(\mathcal{N}_D)$ of *oriented* nested sets on D. We then show in §5.5 that $C_*(\mathcal{N}_D)$ is isomorphic to the cellular chain complex of the De Concini–Procesi associahedron $\mathcal{A}_D$.

Definition 5.8. *Let $\mathcal{H} \in \mathcal{N}_D$ be a nested set with $|\mathcal{H}| < D$. An orientation ε of $\mathcal{H}$ is a choice of*

(i) *an enumeration $D_1, \ldots, D_m$ of the unsaturated elements of $\mathcal{H}$,*
(ii) *a total order on each $\underline{\alpha}_{\mathcal{H}}^{D_i}$.*

By convention, a maximal nested set has a unique orientation.

Definition 5.9. *Let $C_k(\mathcal{N}_D)$ be the free $\mathbb{Z}$–module generated by symbols $\mathcal{H}_\varepsilon$ where $\mathcal{H}$ is a nested set of dimension $k = 0, \ldots, |D|$ and ε is an orientation of $\mathcal{H}$, modulo the following relations if $k \geq 1$*

$$\mathcal{H}_{\varepsilon'} = (-1)^{(|\underline{\alpha}_{\mathcal{H}}^{D_i}|-1)\cdot(|\underline{\alpha}_{\mathcal{H}}^{D_{i+1}}|-1)} \cdot \mathcal{H}_\varepsilon \tag{5.1}$$

if ε' is obtained from ε by permuting the unsaturated elements D_i, D_{i+1} of $\mathcal{H}$ while leaving the total order on each $\underline{\alpha}_{\mathcal{H}}^{D_j}$ unchanged and

$$\mathcal{H}_{\varepsilon'} = (-1)^{\sigma_i} \cdot \mathcal{H}_\varepsilon \tag{5.2}$$

if ε' is obtained from ε by changing the order on $\underline{\alpha}_{\mathcal{H}}^{D_i}$ by a permutation σ_i.

Let $\mathcal{H}_\varepsilon$ be an oriented nested set of positive dimension. Let $D_1, \ldots, D_m$ be the unsaturated elements of $\mathcal{H}$ and set $\underline{\alpha}_i = \underline{\alpha}_{\mathcal{H}}^{D_i}$. Let $\mathcal{G}$ a boundary facet of $\mathcal{H}$, that is $\mathcal{G} \supset \mathcal{H}$ and $|\mathcal{G} \setminus \mathcal{H}| = 1$. By proposition 2.13 and lemma 2.17, $\mathcal{G} = \mathcal{H} \cup D_{\underline{\beta}_i}$ for a unique $i \in [1, m]$ and $\emptyset \neq \underline{\beta}_i \subsetneq \underline{\alpha}_i$. The unsaturated elements of $\mathcal{G}$ are D_j, with $j \neq i$, and $D_{\underline{\beta}_i}$, D_i, provided $|\underline{\beta}_i| \geq 2$ and $|\underline{\beta}_i| \leq |\underline{\alpha}_i| - 2$ respectively. The corresponding subsets of vertices are

$$\underline{\alpha}_{\mathcal{G}}^{D_j} = \underline{\alpha}_j, \qquad \underline{\alpha}_{\mathcal{G}}^{D_{\underline{\beta}_i}} = \underline{\beta}_i \qquad \text{and} \qquad \underline{\alpha}_{\mathcal{G}}^{D_i} = \underline{\alpha}_i \setminus \underline{\beta}_i$$

Definition 5.10. *The orientation $\overline{\varepsilon}$ of $\mathcal{G} = \mathcal{H} \cup D_{\underline{\beta}_i}$ induced by ε is obtained by enumerating the unsaturated elements of $\mathcal{G}$ as*

$$D_1, \ldots, D_{i-1}, D_{\underline{\beta}_i}, D_i, D_{i+1}, \ldots, D_m,$$

ordering $\underline{\alpha}_j$, $j \neq i$ as prescribed by ε and endowing $\underline{\beta}_i$ and $\underline{\alpha}_i \setminus \underline{\beta}_i$ with the restriction of the total order on $\underline{\alpha}_i$ prescribed by ε.

Define the *shuffle number* $s(\underline{\beta}_i;\underline{\alpha}_i)$ of $\underline{\beta}_i$ with respect to $\underline{\alpha}_i$ to be the number of elementary transpositions required to move all elements of $\underline{\beta}_i$ to the left of the first element of $\underline{\alpha}_i$. In other words, if

$$\underline{\alpha}_i = \{\alpha_i^1,\ldots,\alpha_i^{n_i}\} \qquad \text{and} \qquad \underline{\beta}_i = \{\alpha_i^{j_1},\ldots,\alpha_i^{j_p}\}$$

for some $1 \leq j_1 < \cdots < j_p \leq n_i$, then

$$s(\underline{\beta}_i;\underline{\alpha}_i) = (j_1-1)+(j_2-2)+\cdots+(j_p-p)$$

Proposition 5.11. *Let $\partial_k : C_k(\mathcal{N}_D) \longrightarrow C_{k-1}(\mathcal{N}_D)$ be the operator given by*

$$\partial_k \mathcal{H}_\varepsilon = \sum_{\substack{1\leq i\leq m,\\ \emptyset\neq\underline{\beta}_i\subsetneq\underline{\alpha}_i}} (-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)} \cdot (-1)^{|\underline{\beta}_i|-1} \cdot (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot (H \cup D_{\underline{\beta}_i})_{\overline{\varepsilon}}$$

if $k = 1,\ldots,|D|$ and $\partial_0 = 0$. Then, ∂_k is well–defined and $\partial_{k-1}\circ\partial_k = 0$.

Proof. By proposition 2.13 and lemma 2.17, the summands which arise in writing $\partial^2\mathcal{H}_\varepsilon$ are of the form $\mathcal{H}\cup\{D_{\underline{\beta}_i}, D_{\underline{\gamma}_j}\}$ with $D_{\underline{\beta}_i}$ and $D_{\underline{\gamma}_j}$ compatible. We must therefore prove that the sign contributions corresponding to the two sides of the diamond

$$\begin{array}{ccccc} & & \mathcal{H} & & \\ & \swarrow & & \searrow & \\ \mathcal{H}\cup\{D_{\underline{\beta}_i}\} & & & & \mathcal{H}\cup\{D_{\underline{\gamma}_j}\} \\ & \searrow & & \swarrow & \\ & & \mathcal{H}\cup\{D_{\underline{\beta}_i}, D_{\underline{\gamma}_j}\} & & \end{array}$$

are opposite to each other. We consider the various cases corresponding to the relative position of $\underline{\beta}_i$ and $\underline{\gamma}_j$.

5.4.1. *$i \neq j$.* In this case, the orientations induced on $\mathcal{H}\cup\{D_{\underline{\beta}_i}, D_{\underline{\gamma}_j}\}$ by the two sides of the diamond are the same. We may assume, up to a permutation of i and j that $i < j$. The sign contribution of the left side is then

$$\begin{aligned} &(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)} \cdot (-1)^{|\underline{\beta}_i|-1} \cdot (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \\ &\cdot(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)+(|\underline{\beta}_i|-1)+(|\underline{\alpha}_i\setminus\underline{\beta}_i|-1)+(|\underline{\alpha}_{i+1}|-1)+\cdots+(|\underline{\alpha}_{j-1}|-1)} \\ &\cdot(-1)^{|\underline{\gamma}_j|-1} \cdot (-1)^{s(\underline{\gamma}_j;\underline{\alpha}_j)} \\ =& -(-1)^{(|\underline{\alpha}_i|-1)+\cdots+(|\underline{\alpha}_{j-1}|-1)} \cdot (-1)^{|\underline{\beta}_i|+|\underline{\gamma}_j|} \cdot (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)+s(\underline{\gamma}_j;\underline{\alpha}_j)} \end{aligned}$$

while that of the right side is

$$
\begin{aligned}
&(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{j-1}|-1)}\cdot(-1)^{|\underline{\gamma}_j|-1}\cdot(-1)^{s(\underline{\gamma}_j;\underline{\alpha}_j)}\\
&\cdot(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)}\cdot(-1)^{|\underline{\beta}_i|-1}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)}\\
=&(-1)^{(|\underline{\alpha}_i|-1)+\cdots+(|\underline{\alpha}_{j-1}|-1)}\cdot(-1)^{|\underline{\beta}_i|+|\underline{\gamma}_j|}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)+s(\underline{\gamma}_j;\underline{\alpha}_j)}
\end{aligned}
$$

as required.

5.4.2. *$i=j$ and $\underline{\beta}_i\subset\underline{\gamma}_i$ or $\underline{\gamma}_i\subset\underline{\beta}_i$.* Up to a permutation, we may assume that $\underline{\beta}_i\subset\underline{\gamma}_i$. In this case again, the orientations induced on $\mathcal{H}\cup\{D_{\underline{\beta}_i},D_{\underline{\gamma}_j}\}$ by the two sides of the diamond are the same. The sign contribution from the left side is

$$
\begin{aligned}
&(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)}\cdot(-1)^{|\underline{\beta}_i|-1}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)}\\
&\cdot(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)+(|\underline{\beta}_i|-1)}\cdot(-1)^{|\underline{\gamma}_i\setminus\underline{\beta}_i|-1}\cdot(-1)^{s(\underline{\gamma}_i\setminus\underline{\beta}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}\\
=&-(-1)^{|\underline{\beta}_i|+|\underline{\gamma}_i|}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)+s(\underline{\gamma}_i\setminus\underline{\beta}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}
\end{aligned}
$$

while that of the right one is

$$
\begin{aligned}
&(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)}\cdot(-1)^{|\underline{\gamma}_i|-1}\cdot(-1)^{s(\underline{\gamma}_i;\underline{\alpha}_i)}\\
&\cdot(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)}\cdot(-1)^{|\underline{\beta}_i|-1}\cdot(-1)^{s(\underline{\beta}_i;\underline{\gamma}_i)}\\
=&(-1)^{|\underline{\beta}_i|+|\underline{\gamma}_i|}\cdot(-1)^{s(\underline{\beta}_i;\underline{\gamma}_i)+s(\underline{\gamma}_i;\underline{\alpha}_i)}
\end{aligned}
$$

These are opposite to each other in view of the following

Lemma 5.12. $(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)+s(\underline{\gamma}_i\setminus\underline{\beta}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}=(-1)^{s(\underline{\beta}_i;\underline{\gamma}_i)+s(\underline{\gamma}_i;\underline{\alpha}_i)}$

PROOF. The left–hand side is the parity of the number of elementary transpositions required to shuffle $\underline{\beta}_i$ to the left of $\underline{\alpha}_i$ and then $\underline{\gamma}_i\setminus\underline{\beta}_i$ to the left of $\underline{\alpha}_i\setminus\underline{\beta}_i$, thus arriving at the ordered configuration $\underline{\beta}_i,\underline{\gamma}_i\setminus\underline{\beta}_i,\underline{\alpha}_i\setminus\underline{\gamma}_i$. The right–hand side on the other hand is the parity of the number of transpositions needed to shuffle $\underline{\gamma}_i$ to the left of $\underline{\alpha}_i$ and then $\underline{\beta}_i$ to the left of $\underline{\gamma}_i$ resulting in the very same configuration ∎

5.4.3. *$i=j$ and $\underline{\beta}_i\cap\underline{\gamma}_i=\emptyset$.* In this last case, the sign contribution of the left side of the diamond is

$$
\begin{aligned}
&(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)}\cdot(-1)^{|\underline{\beta}_i|-1}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)}\\
&\cdot(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)+(|\underline{\beta}_i|-1)}\cdot(-1)^{|\underline{\gamma}_i|-1}\cdot(-1)^{s(\underline{\gamma}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}\\
=&(-1)^{|\underline{\gamma}_i|-1}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)+s(\underline{\gamma}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}
\end{aligned}
$$

so that, by symmetry, that of the right side is

$$(-1)^{|\underline{\beta}_i|-1}\cdot(-1)^{s(\underline{\gamma}_i;\underline{\alpha}_i)+s(\underline{\beta}_i;\underline{\alpha}_i\setminus\underline{\gamma}_i)}$$

In this case however, the orientation induced on $\mathcal{H}\cup\{D_{\underline{\beta}_i}, D_{\underline{\gamma}_i}\}$ by each side may differ. Indeed, the left side leads to enumerating the unsaturated elements of $\mathcal{H}\cup\{D_{\underline{\beta}_i}, D_{\underline{\gamma}_i}\}$ as

$$D_1,\ldots,D_{i-1},D_{\underline{\beta}_i},D_{\underline{\gamma}_i},D_i,\ldots,D_m$$

while the right branch leads to enumerating them as

$$D_1,\ldots,D_{i-1},D_{\underline{\gamma}_i},D_{\underline{\beta}_i},D_i,\ldots,D_m$$

In view of the relation (5.1) we must therefore prove that

$$\begin{aligned}(-1)^{|\underline{\gamma}_i|}\cdot(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)+s(\underline{\gamma}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}\\ = -(-1)^{(|\underline{\beta}_i|-1)(|\underline{\gamma}_i|-1)}\cdot(-1)^{|\underline{\beta}_i|}\cdot(-1)^{s(\underline{\gamma}_i;\underline{\alpha}_i)+s(\underline{\beta}_i;\underline{\alpha}_i\setminus\underline{\gamma}_i)}\end{aligned}$$

which is settled by the following

Lemma 5.13. $(-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)-s(\underline{\beta}_i;\underline{\alpha}_i-\underline{\gamma}_i)} = (-1)^{|\underline{\beta}_i||\underline{\gamma}_i|}(-1)^{s(\underline{\gamma}_i;\underline{\alpha}_i)-s(\underline{\gamma}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}$

PROOF. The left–hand side is the parity of the set $N_{\underline{\beta}_i,\underline{\gamma}_i}$ of pairs $(\beta,\gamma)\in\underline{\beta}_i\times\underline{\gamma}_i$ which are permuted when $\underline{\beta}_i$ is shuffled to the left of $\underline{\alpha}_i$. Similarly, $(-1)^{s(\underline{\gamma}_i;\underline{\alpha}_i)-s(\underline{\gamma}_i;\underline{\alpha}_i\setminus\underline{\beta}_i)}$ is the parity of the set $N_{\underline{\gamma}_i,\underline{\beta}_i}$ of pairs $(\beta,\gamma)\in\underline{\beta}_i\times\underline{\gamma}_i$ which are permuted when $\underline{\gamma}_i$ is shuffled to the left of $\underline{\alpha}_i$. Since $N_{\underline{\beta}_i,\underline{\gamma}_i}\cup N_{\underline{\gamma}_i,\underline{\beta}_i}=\underline{\beta}_i\times\underline{\gamma}_i$, the product of these parities is equal to $(-1)^{|\underline{\beta}_i||\underline{\gamma}_i|}$ ■

5.5. **The cellular chain complex of $\mathcal{A}_D$.** We construct below an isomorphism between the complex $C_*(\mathcal{N}_D)$ and the cellular chain complex of the associahedron $\mathcal{A}_D$ by using its realisation as a convex polytope P^c_D given in section 2.2. We begin by explaining how an orientation of a nested set $\mathcal{H}\in\mathcal{N}_D$ determines one of the corresponding face $P^c_{\mathcal{H}}$ of P^c_D.

For any $t\in P^c_D\subset\mathbb{R}^{|D|}$ with coordinates $\{t_\alpha\}_{\alpha\in D}$ and subset $B\subset D$, set

$$t_B=\sum_{\gamma\in B}t_\gamma$$

If $t\in P^c_{\mathcal{H}}$, then, for any $A\in\mathcal{H}$

$$t_{\underline{\alpha}^A_{\mathcal{H}}}=t_A-t_{A\setminus\underline{\alpha}^A_{\mathcal{H}}}=c(A)-c(A\setminus\underline{\alpha}^A_{\mathcal{H}}) \tag{5.3}$$

where we extend the function c to non–connected subdiagrams $B\subset D$ with connected components $B_1,\ldots,B_m$ by setting $c(B)=c(B_1)+\cdots+c(B_m)$. It follows that if $D_1,\ldots,D_m$ are the unsaturated elements of $\mathcal{H}$, a redundant system of coordinates on $P^c_{\mathcal{H}}$ is given by the components t_γ with γ ranging over $\underline{\alpha}^{D_1}_{\mathcal{H}}\cup\cdots\cup\underline{\alpha}^{D_m}_{\mathcal{H}}$. These coordinates are only subject to the constraints that equation (5.3) should hold whenever $A = D_i$ for some i.

Assume that $\mathcal{H}$ is of positive dimension. Let ε be an orientation of $\mathcal{H}$ and let

$$D_1, \dots, D_m \qquad \text{and} \qquad \underline{\alpha}_i = \underline{\alpha}_{\mathcal{H}}^{D_i} = \{\alpha_i^1, \dots, \alpha_i^{n_i}\} \subset D_i$$

be the corresponding enumeration of the unsaturated elements of $\mathcal{H}$ and ordered subsets of vertices respectively.

Definition 5.14. *The orientation of the face $P_{\mathcal{H}}^c$ induced by ε is the one determined by the volume element*

$$\omega_{\mathcal{H}_\varepsilon} = dt_{\alpha_1^1} \wedge \cdots \wedge dt_{\alpha_1^{n_1-1}} \wedge \cdots \wedge dt_{\alpha_m^1} \wedge \cdots \wedge dt_{\alpha_m^{n_m-1}}$$

Note that the assignement $\mathcal{H}_\varepsilon \longrightarrow \omega_{\mathcal{H}_\varepsilon}$ is consistent with relations (5.1)–(5.2). This is clear if one permutes D_i and D_{i+1} or α_i^j and α_i^{j+1} within $\underline{\alpha}_i$, so long as $1 \leq j \leq n_i - 2$. If $j = n_i - 1$, the new contribution of D_i to the volume form on $P_{\mathcal{H}}^c$ is

$$\begin{aligned} dt_{\alpha_i^1} \wedge \cdots \wedge dt_{\alpha_i^{n_i-2}} \wedge dt_{\alpha_i^{n_i}} &= dt_{\alpha_i^1} \wedge \cdots \wedge dt_{\alpha_i^{n_i-2}} \wedge d(c(D_i) - c(D_i \setminus \underline{\alpha}_i) - \sum_{\ell=1}^{n_i-1} t_{\alpha_i^\ell}) \\ &= -dt_{\alpha_i^1} \wedge \cdots \wedge dt_{\alpha_i^{n_i-2}} \wedge dt_{\alpha_i^{n_i-1}} \end{aligned}$$

as required. We next work out the orientation given by the volume form $\omega_{\mathcal{H}_\varepsilon}$ more explicitly. Assume first that $\mathcal{H}$ is of dimension 1 so that $m = 1$ and $\underline{\alpha}_1 = \{\alpha_1^1, \alpha_1^2\}$. By §2.7, the corresponding edge $P_{\mathcal{H}}^c$ has boundary points labelled by $\mathcal{H} \cup D_{\alpha_1^1}$ and $\mathcal{H} \cup D_{\alpha_1^2}$.

Lemma 5.15. *The orientation of $P_{\mathcal{H}}^c$ induced by ε is given by*

$$\mathcal{H} \cup D_{\alpha_1^1} \xrightarrow{\quad \mathcal{H} \quad} \mathcal{H} \cup D_{\alpha_1^2}$$

Proof. In this case $\omega_{\mathcal{H}_\varepsilon} = dt_{\alpha_1^1}$. By (5.3),

$$t_{\alpha_1^1}(P_{\mathcal{H} \cup D_{\alpha_1^1}}) = c(D_{\alpha_1^1}) - c(D_{\alpha_1^1} \setminus \alpha_1^1) \quad \text{and} \quad t_{\alpha_1^1}(P_{\mathcal{H} \cup D_{\alpha_1^2}}) = c(D_1) - c(D_1 \setminus \alpha_1^1)$$

Since the connected components of $D_1 \setminus \alpha_1^1$ not containing α_1^2 are the connected components of $D_{\alpha_1^1} \setminus \alpha_1^1$ which are orthogonal to α_1^2, we have

$$c(D_1) - (c(D_1 \setminus \alpha_1^1) - c(D_{\alpha_1^1} \setminus \alpha_1^1)) - c(D_{\alpha_1^1}) \geq c(D_1) - c(D_{\alpha_1^2}) - c(D_{\alpha_1^1}) > 0$$

where the inequality follows from (2.1) and the fact $D_{\alpha_1^1}$ and $D_{\alpha_1^2}$ are not compatible and such that $D_{\alpha_1^1} \cup D_{\alpha_1^2} = D_1$ ■

Assume now that $\mathcal{H}$ is of dimension greater or equal to two and let $\mathcal{G} = \mathcal{H} \cup \underline{\beta}_i$ be a boundary facet of $\mathcal{H}$. Note that the function $t_{\underline{\beta}_i} = \sum_{\gamma \in \underline{\beta}_i} t_\gamma$ is identically equal to $c(D_{\underline{\beta}_i}) - c(D_{\underline{\beta}_i} \setminus \underline{\beta}_i)$ on $P^c_{\mathcal{G}} \subset \partial P^c_{\mathcal{H}}$ and strictly greater than that value on the interior of $P^c_{\mathcal{H}}$ since $D_{\underline{\beta}_i} \notin \mathcal{H}$ while the connected components of $D_{\underline{\beta}_i} \setminus \underline{\beta}_i$ lie in $\mathcal{H}$. The orientation of $P^c_{\mathcal{G}}$ as a boundary component of $P^c_{\mathcal{H}}$ induced by the volume form $\omega_{\mathcal{H}_\varepsilon}$ is therefore given by the form $\partial_{\underline{\beta}_i} \omega_{\mathcal{H}_\varepsilon}$ such that

$$\omega_{\mathcal{H}_\varepsilon} = -dt_{\underline{\beta}_i} \wedge \partial_{\underline{\beta}_i} \omega_{\mathcal{H}_\varepsilon}$$

We wish to relate $\partial_{\underline{\beta}_i} \omega_{\mathcal{H}_\varepsilon}$ to the volume form $\omega_{\mathcal{G}_{\overline{\varepsilon}}}$ where $\overline{\varepsilon}$ is the orientation of $\mathcal{G}$ induced by ε.

Lemma 5.16.

$$\partial_{\underline{\beta}_i} \omega_{\mathcal{H}_\varepsilon} = -(-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)} \cdot (-1)^{|\underline{\beta}_i|-1} \cdot (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot \omega_{\mathcal{G}_{\overline{\varepsilon}}}$$

Proof. For any ordered set $\underline{\gamma} = \{\alpha_1, \ldots, \alpha_k\} \subset D$, set

$$\omega_{\underline{\gamma}} = dt_{\alpha_1} \wedge \cdots \wedge dt_{\alpha_{k-1}}$$

so that

$$\begin{aligned}
\omega_{\mathcal{H}_\varepsilon} &= \omega_{\underline{\alpha}_1} \wedge \cdots \wedge \omega_{\underline{\alpha}_m} \\
\omega_{\mathcal{G}_{\overline{\varepsilon}}} &= \omega_{\underline{\alpha}_1} \wedge \cdots \wedge \omega_{\underline{\alpha}_{i-1}} \wedge \omega_{\underline{\beta}_i} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i} \wedge \omega_{\underline{\alpha}_{i+1}} \wedge \cdots \wedge \omega_{\underline{\alpha}_m}
\end{aligned}$$

If $\underline{\beta}_i = \{\alpha_i^{j_1}, \ldots, \alpha_i^{j_p}\} \subset \{\alpha_i^1, \ldots, \alpha_i^{n_i}\} = \underline{\alpha}_i$, then assuming first $j_p < n_i$,

$$\begin{aligned}
\omega_{\underline{\alpha}_i} &= (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot dt_{\alpha_i^{j_1}} \wedge \cdots \wedge dt_{\alpha_i^{j_p}} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i} \\
&= (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot dt_{\alpha_i^{j_1}} \wedge \cdots \wedge dt_{\alpha_i^{j_{p-1}}} \wedge d(t_{\alpha_i^{j_1}} + \cdots + t_{\alpha_i^{j_p}}) \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i} \\
&= (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot (-1)^{|\underline{\beta}_i|-1} \cdot dt_{\underline{\beta}_i} \wedge \omega_{\underline{\beta}_i} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i}
\end{aligned}$$

If, on the other hand, $j_p = n_i$, then, denoting the maximal element of $\underline{\alpha}_i \setminus \underline{\beta}_i$ by α_ℓ, we get

$$\begin{aligned}
\omega_{\underline{\alpha}_i} &= (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)-(|\underline{\alpha}_i|-|\underline{\beta}_i|)} \cdot \omega_{\underline{\beta}_i} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i} \wedge dt_{\alpha_\ell} \\
&= (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)-(|\underline{\alpha}_i|-|\underline{\beta}_i|)} \\
&\quad \cdot \omega_{\underline{\beta}_i} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i} \wedge d\big(c(D_i) - c(D_i \setminus \underline{\alpha}_i) - t_{\underline{\beta}_i} - t_{\underline{\alpha}_i \setminus (\underline{\beta}_i \cup \{\alpha_\ell\})}\big) \\
&= (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot (-1)^{|\underline{\beta}_i|-1} \cdot dt_{\underline{\beta}_i} \wedge \omega_{\underline{\beta}_i} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i}
\end{aligned}$$

Thus, in either case, we find

$$\begin{aligned}
\omega_{\mathcal{H}_\varepsilon} &= (-1)^{|\underline{\beta}_i|-1} \cdot (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \\
&\quad \cdot \omega_{\underline{\alpha}_1} \wedge \cdots \wedge \omega_{\underline{\alpha}_{i-1}} \wedge dt_{\underline{\beta}_i} \wedge \omega_{\underline{\beta}_i} \wedge \omega_{\underline{\alpha}_i \setminus \underline{\beta}_i} \wedge \omega_{\underline{\alpha}_{i+1}} \wedge \cdots \wedge \omega_{\underline{\alpha}_m} \\
&= (-1)^{(|\underline{\alpha}_1|-1)+\cdots+(|\underline{\alpha}_{i-1}|-1)} \cdot (-1)^{|\underline{\beta}_i|-1} \cdot (-1)^{s(\underline{\beta}_i;\underline{\alpha}_i)} \cdot dt_{\underline{\beta}_i} \wedge \omega_{\mathcal{G}_{\overline{\varepsilon}}}
\end{aligned}$$

as required ■

Theorem 5.17. *The map $\mathcal{H}_\varepsilon \longrightarrow (P^c_{\mathcal{H}}, \omega_{\mathcal{H}_\varepsilon})$ associating to each oriented nested set the corresponding face of the polytope P^c_D with orientation given by the volume form $\omega_{\mathcal{H}_\varepsilon}$ is an isomorphism between $C_*(\mathcal{N}_D)$ and the cellular chain complex of P^c_D.*

Proof. Recall (see *e.g.*, [Ms, §IX.4]) that the cellular chain complex of a CW–complex X is defined by $C^{\text{cell}}_n(X) = H_n(X^n, X^{n-1})$, where X^n is the n–skeleton of X, with differential ∂^{cell}_n given by the composition

$$H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

Each $C^{\text{cell}}_n(X)$ is a free abelian group of rank equal to the number of n–cells in X. Identifying a given factor with $\mathbb{Z}$ when $n \geq 1$ amounts to choosing an orientation of the corresponding cell. For a regular CW–complex such as P^c_D, that is one where all attaching maps are homeomorphisms, the boundary ∂^{cell} has a very simple description [Ms, §IX.6]. Given an oriented n–cell b^n_λ,

$$\partial^{\text{cell}} b^n_\lambda = \sum_\mu [b^n_\lambda : b^{n-1}_\mu] \cdot b^{n-1}_\mu$$

where the sum ranges over the $(n-1)$–cells of X, each taken with a chosen orientation if $n \geq 2$. The incidence number $[b^n_\lambda : b^{n-1}_\mu]$ is zero if b^{n-1}_μ is not contained in the boundary of b^n_λ and ± 1 otherwise. In the latter case, the sign depends on whether the induced orientation on the boundary of b^n_λ agrees with that on b^{n-1}_μ if $n \geq 2$ and is otherwise given, for $n = 1$ by

$$\partial^{\text{cell}} b^1_\lambda = b^0_\mu - b^0_\nu$$

where, under an orientation preserving identification $b^1_\lambda \cong [0,1]$, the attaching map send 1 to b^0_μ and 0 to b^0_ν. Thus, by lemmas 5.15 and 5.16, the map $\mathcal{H}_\varepsilon \longrightarrow (P^c_{\mathcal{H}_\varepsilon}, \omega_{\mathcal{H}_\varepsilon})$ identifies the differential ∂ on $C_*(\mathcal{N}_D)$ to the opposite of the cellular boundary ■

Theorem 5.17 and the contractibility of P^c_D imply in particular the following

Corollary 5.18. *The complex $C_*(\mathcal{N}_D)$ is acyclic.*

5.6. **The Dynkin complex and the cellular cochain complex of $\mathcal{A}_D$.** Let A be a D–algebra. We relate in this subsection the Dynkin complex of A to the cellular cochain complex of the associahedron $\mathcal{A}_D$, when both are taken with coefficients in a D–bimodule M over A. We show in particular that in degrees greater or equal to two, the Dynkin differential is a geometric boundary operator, albeit in combinatorial guise. We shall need some terminology.

Definition 5.19.

(i) *A nested set $\mathcal{H} \in \mathcal{N}_D$ of positive dimension is called irreducible if it has a unique unsaturated element and reducible otherwise.*
(ii) *Two oriented nested sets $\mathcal{H}_\varepsilon$ and $\mathcal{H}'_{\varepsilon'}$ of positive dimension are equivalent if they have the same unsaturated elements $D_1, \ldots, D_m$, if $\underline{\alpha}_{\mathcal{H}}^{D_i} = \underline{\alpha}_{\mathcal{H}'}^{D_i}$ for any $i = 1 \ldots m$ and if the orientations $\varepsilon, \varepsilon'$ agree in the obvious sense.*

Note that a nested set of dimension 1 is clearly irreducible. If $\mathcal{H} \in \mathcal{N}_D$ is of dimension 2, §2.8 shows that $\mathcal{H}$ is irreducible when the corresponding face $\mathcal{A}_D^{\mathcal{H}}$ of $\mathcal{A}_D$ is a pentagon or a hexagon and reducible when $\mathcal{A}_D^{\mathcal{H}}$ is a square. More generally, by proposition 2.13, $\mathcal{H}$ is reducible precisely when $\mathcal{A}_D^{\mathcal{H}}$ is the product of $p \geq 2$ smaller associahedra.

Let $CD^*(A; M)$ be the Dynkin complex of A with coefficients in M and

$$\widetilde{C}^*(\mathcal{N}_D; M) = 0 \longrightarrow M \xrightarrow{\epsilon^*} \operatorname{Hom}_{\mathbb{Z}}(C_*(\mathcal{N}_D); M)$$

be the augmented cellular cochain complex of $\mathcal{A}_D$ with coefficients in M. We regard M as sitting in degree -1 in $\widetilde{C}^*(\mathcal{N}_D; M)$. For any $k = 0, \ldots, |D|$, define a map $g^k : CD^k(A; M) \longrightarrow \widetilde{C}^{k-1}(\mathcal{N}_D; M)$ by

$$\begin{aligned} g^0 m &= m_D \\ g^1 m(\mathcal{H}) &= \sum_{B \in \mathcal{H}} m_{(B; \alpha_{\mathcal{H}}^B)} \\ g^k m(\mathcal{H}_\varepsilon) &= \begin{cases} m_{(B; \underline{\alpha}_{\mathcal{H}}^B)} & \text{if } \mathcal{H} \text{ is irreducible with unsaturated set } B \\ 0 & \text{if } \mathcal{H} \text{ is reducible} \end{cases} \end{aligned}$$

for any $k \geq 2$.

Theorem 5.20. *g is a chain map from $CD^*(A; M)$ to $\widetilde{C}^{*-1}(\mathcal{N}_D; M)$. For $k \geq 2$, g^k is an embedding whose image consists of those cochains $c \in \widetilde{C}^{k-1}(\mathcal{N}_D; M)$ such that*

(i) *$c(\mathcal{H}_\varepsilon) = 0$ for any reducible nested set $\mathcal{H}$.*
(ii) *$c(\mathcal{H}_\varepsilon) = c(\mathcal{H}'_{\varepsilon'})$ for any equivalent oriented nested sets $\mathcal{H}_\varepsilon, \mathcal{H}'_\varepsilon$.*
(iii) *$c(\mathcal{H}_\varepsilon) \in M_B^{B \setminus \underline{\alpha}_{\mathcal{H}}^B}$ for any irreducible $\mathcal{H}$ with unsaturated element B.*

PROOF. Denote the differential on $\widetilde{C}^k(\mathcal{N}_D;M)$ by d^k, with $d^{-1}=\epsilon^*$. For $m=\{m_B\}_{B\subseteq D}\in CD^0(A;M)$ and $\mathcal{H}$ a maximal nested set,

$$g^1 d^0_D m(\mathcal{H})=\sum_{B\in\mathcal{H}}(m_B-m_{B\setminus\alpha^B_{\mathcal{H}}})=m_D=(d^{-1}g^0 m)(\mathcal{H})$$

Next, if $m=\{m_{(B;\alpha)}\}_{\alpha\in B\subseteq D}$ is a one–chain, and $\mathcal{H}_\varepsilon$ an oriented nested set of dimension one with unsaturated element B and ordered $\underline{\alpha}^B_{\mathcal{H}}=\{\alpha_1,\alpha_2\}$, then

$$d^0 g^1 m(\mathcal{H}_\varepsilon)=g^1 m(\mathcal{H}\cup B_1)-g^1 m(\mathcal{H}\cup B_2)$$

where $B_1=\complement^{B\setminus\alpha_2}_{\alpha_1}$ and $B_2=\complement^{B\setminus\alpha_1}_{\alpha_2}$. This is equal to

$$m_{(B;\alpha_2)}-m_{(B_2;\alpha_2)}-m_{(B;\alpha_1)}+m_{(B_1;\alpha_1)}=g^2 d^1_D m(\mathcal{H}_\varepsilon)$$

since $\alpha^{B_i}_{\mathcal{H}\cup B_i}=\alpha_i$ and the only element B' of $\mathcal{H}$ for which $\alpha^{B'}_{\mathcal{H}\cup B_1}\neq\alpha^{B'}_{\mathcal{H}\cup B_2}$ is B with $\alpha^B_{\mathcal{H}\cup B_1}=\alpha_2$ and $\alpha^B_{\mathcal{H}\cup B_2}=\alpha_1$.

Let now $m\in CD^k(A;M)$ with $k\geq 2$, and let $\mathcal{H}_\varepsilon$ be an oriented nested set or dimension k. Let p be the number of unsaturated elements of $\mathcal{H}$ and assume first that $p\geq 2$. Then, $\mathcal{H}$ is reducible so that

$$g^{k+1}d^k_D\, m(\mathcal{H}_\varepsilon)=0$$

If $p\geq 3$, any boundary facet $\mathcal{G}=\mathcal{H}\cup D_{\underline{\beta}_i}$ of $\mathcal{H}$ is also reducible and

$$d^{k-1}g^k\, m(\mathcal{H}_\varepsilon)=0$$

as required. If $\mathcal{H}$ only has two unsaturated elements D_1 and D_2, a boundary facet $\mathcal{G}=\mathcal{H}\cup D_{\underline{\beta}_i}$ of $\mathcal{H}$, with $\emptyset\neq\underline{\beta}_i\subsetneq\underline{\alpha}_i=\underline{\alpha}^{D_i}_{\mathcal{H}}$ and $i=1,2$, is reducible unless $|\underline{\alpha}_i|=2$. In that case, the unique unsaturated element of $\mathcal{G}$ is D_{3-i}. It follows that, when $p=2$,

$$\begin{aligned}d^{k-1}g^k\, m(\mathcal{H}_\varepsilon)&=\delta_{|\underline{\alpha}_1|=2}\cdot(m(D_2;\underline{\alpha}_2)-m(D_2;\underline{\alpha}_2))\\&\quad+(-1)^{|\underline{\alpha}_1|-1}\cdot\delta_{|\underline{\alpha}_2|=2}\cdot(m(D_1;\underline{\alpha}_1)-m(D_1;\underline{\alpha}_1))\\&=0\end{aligned}$$

There remains to consider the case when $\mathcal{H}$ is irreducible with unsaturated element B and $\underline{\alpha}=\underline{\alpha}^B_{\mathcal{H}}$ of cardinality $k+1$. If $\emptyset\neq\underline{\beta}\subsetneq\underline{\alpha}$ is of cardinality $2\leq|\underline{\beta}|\leq|\underline{\alpha}|-2$, then $H\cup D_{\underline{\beta}}$ is reducible with unsaturated elements $D_{\underline{\beta}}$ and B. It follows that the only non–trivial contributions in $d^{k-1}g^k\, m(\mathcal{H}_\varepsilon)$ arise when $\underline{\beta}=\{\alpha_j\}$ or $\underline{\beta}=\underline{\alpha}\setminus\{\alpha_j\}$ for some $j=1,\dots,k+1$, where $\underline{\alpha}=\{\alpha_1,\dots,\alpha_{k+1}\}$. The corresponding sign contributions are

$$(-1)^{|\underline{\beta}|-1}\cdot(-1)^{s(\underline{\beta};\alpha)} = (-1)^{j-1}$$
$$(-1)^{|\underline{\beta}|-1}\cdot(-1)^{s(\underline{\beta};\alpha)} = (-1)^{k-1}\cdot(-1)^{k-j+1}$$

respectively, so that

$$\begin{aligned}
d^{k-1}g^k\, m(\mathcal{H}_\varepsilon) &= \sum_{j=1}^{k+1}(-1)^{j-1}\left(g^k\, m\left((\mathcal{H}\cup \mathsf{C}_{\alpha_j}^{B\setminus(\underline{\alpha}\setminus\alpha_j)})_{\overline{\varepsilon}}\right) - g^k\, m\left((\mathcal{H}\cup \mathsf{C}_{\underline{\alpha}\setminus\alpha_j}^{B\setminus\alpha_j})_{\overline{\varepsilon}}\right)\right)\\
&= \sum_{j=1}^{k+1}(-1)^{j-1}(m_{(B;\underline{\alpha}\setminus\alpha_j)} - m_{(\mathsf{C}_{\underline{\alpha}\setminus\alpha_j}^{B\setminus\alpha_j};\underline{\alpha}\setminus\alpha_j)})\\
&= d_D^k\, m_{(B;\underline{\alpha})}\\
&= g^{k+1}d_D^k\, m(\mathcal{H}_\varepsilon)
\end{aligned}$$

Thus, g is a chain map as claimed. g^k is an embedding for $k\geq 2$ because for any connected subdiagram $B\subseteq D$ and nonempty subset $\underline{\alpha}\subseteq B$, there exists an irreducible nested set $\mathcal{H}$ with unique unsaturated element B such that $\underline{\alpha}_{\mathcal{H}}^B=\underline{\alpha}$. Such an $\mathcal{H}$ may be constructed as follows : let $B_1,\ldots,B_m$ be the connected components of $B\setminus\underline{\alpha}$ and choose a maximal nested set $\mathcal{H}_i$ on each B_i. Let moreover

$$D_1 = B\subset D_2\subset\cdots\subset D_{p-1}\subset D_p = D$$

be a sequence of encased connected subdiagrams of D such that $|D_i\setminus D_{i+1}|=1$ for $i=1\ldots p-1$. Then,

$$\mathcal{H} = \mathcal{H}_1\sqcup\cdots\sqcup\mathcal{H}_m\sqcup\{D_1,\ldots,D_p\}$$

is a suitable nested set. The image of g^k, for $k\geq 2$ is clearly characterised by conditions (i)–(iii) ■

Remark 5.21. Note that the fact that g is a chain map and that g^k is an embedding in degrees ≥ 2 gives another proof of the fact that the Dynkin differential squares to zero.

5.7. **Deformations of quasi–Coxeter algebras.** Let A be a D–algebra. Regard D as labelled by attaching an infinite multiplicity to each edge. By a *trivial* quasi–Coxeter algebra structure on A we shall mean one whose underlying D–algebra is A and for which all associators $\Phi_{\mathcal{GF}}^A$ are equal to 1. We do not assume that the local monodromies S_i^A are trivial however. When considering deformations of a trivial quasi–Coxeter algebra structure on A, the elements S_i^A will be assumed to remain undeformed.

Theorem 5.22. *The Dynkin complex $CD^*(A;A)$ controls the formal, one–parameter deformations of trivial quasi–Coxeter algebra structures on A. Specifically,*

(i) *a weak quasi–Coxeter algebra structure on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$ which is trivial mod $\hbar$ canonically determines a Dynkin 3–cocyle ξ and lifts to a quasi–Coxeter algebra structure on $A[\![\hbar]\!]/\hbar^{n+2}A[\![\hbar]\!]$ if, and only if ξ is a coboundary.*

(ii) *Two quasi–Coxeter algebra structures on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$ which agree mod $\hbar^n$ and are trivial mod $\hbar$ differ by a Dynkin 2–cocycle φ. They are related by a twist of the form $\{1+\hbar^n a_{(B;\alpha)}\}_{\alpha\in B\subseteq D}$ if, and only if, $\varphi = d_D a$ and $Ad(S_i^A)a_{(\alpha_i;\alpha_i)} = a_{(\alpha_i;\alpha_i)}$ for any $\alpha_i \in D$.*

Proof. We prove (ii) first. Let $\Phi^i_{\mathcal{GF}}$, $i = 1, 2$, be the associators of the quasi–Coxeter algebra structures and, for any elementary pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D, define $\varphi_{\mathcal{GF}} \in A$ by

$$\Phi^2_{\mathcal{GF}} = \Phi^1_{\mathcal{GF}} + \hbar^n \cdot \varphi_{\mathcal{GF}} \mod \hbar^{n+1}$$

Since $\Phi^1_{\mathcal{GF}}$ and $\Phi^2_{\mathcal{GF}}$ satisfy the coherence and orientation axioms mod $\hbar^{n+1}$, $(\mathcal{G}, \mathcal{F}) \to \varphi_{\mathcal{GF}}$ is a cellular one–cocycle on the associahedron $\mathcal{A}_D$ with values in A. Proposition 3.32, and the support and forgetfulness properties of the associators Φ^1, Φ^2 imply that φ satisfies the constraints (i)–(iii) of theorem 5.20 respectively so that φ is a Dynkin 2–cocycle. The rest of (ii) is a simple exercise. The proof of (i) is given in §5.7.1–§5.7.5.

5.7.1. Let

$$\Phi_{\mathcal{GF}} = 1 + \hbar\, \varphi^1_{\mathcal{GF}} + \cdots + \hbar^n \varphi^n_{\mathcal{GF}}$$

be the associators of the quasi–Coxeter algebra structure on A. By assumption, $\Phi^{-1}_{\mathcal{GF}} = \Phi_{\mathcal{FG}} \mod \hbar^{n+1}$ for any elementary pair of maximal nested sets $(\mathcal{G}, \mathcal{F})$ on D. We begin by modifying each $\Phi_{\mathcal{GF}}$ so that this identity holds mod $\hbar^{n+2}$. Define $\eta_{\mathcal{GF}} \in A$ by

$$\Phi_{\mathcal{GF}}\Phi_{\mathcal{FG}} = 1 + \hbar^{n+1}\eta_{\mathcal{GF}} \mod \hbar^{n+2}$$

Clearly, $\eta_{\mathcal{GF}} \in A_B^{B\setminus\{\alpha,\beta\}}$, where $B = \text{supp}(\mathcal{F}, \mathcal{G})$ is the unique unsaturated element of $\mathcal{H} = \mathcal{G} \cap \mathcal{F}$ and $\{\alpha, \beta\} = \underline{\alpha}^B_{\mathcal{H}}$, and $\eta_{\mathcal{GF}} = \eta_{\mathcal{G}'\mathcal{F}'}$ whenever $(\mathcal{G}, \mathcal{F})$ and $(\mathcal{G}', \mathcal{F}')$ are equivalent elementary pairs. Moreover, modulo $\hbar^{n+2}$,

$$\Phi_{\mathcal{GF}}\Phi_{\mathcal{FG}} = \Phi_{\mathcal{GF}}(\Phi_{\mathcal{FG}}\Phi_{\mathcal{GF}} - \hbar^{n+1}\eta_{\mathcal{FG}})\Phi_{\mathcal{FG}} = (\Phi_{\mathcal{GF}}\Phi_{\mathcal{FG}})^2 - \hbar^{n+1}\eta_{\mathcal{FG}}$$

whence $\eta_{\mathcal{FG}} = \eta_{\mathcal{GF}}$. It follows from this that the associators

$$\widetilde{\Phi}_{\mathcal{GF}} = \Phi_{\mathcal{GF}} - \frac{1}{2}\hbar^{n+1}\eta_{\mathcal{GF}}$$

satisfy $\widetilde{\Phi}_{\mathcal{G}\mathcal{F}}\widetilde{\Phi}_{\mathcal{F}\mathcal{G}} = 1 \mod \hbar^{n+2}$, as well as all the required relations to endow $A[\![\hbar]\!]/\hbar^{n+2}A[\![\hbar]\!]$ with a quasi–Coxeter algebra structure, except possibly for the coherence one.

5.7.2. We define next the obstruction ξ as a cellular 2–cochain on $\mathcal{A}_D$ with values in A. Let $\mathcal{H}_\varepsilon$ be an oriented nested set of dimension 2, fix a maximal nested set $\mathcal{F}_0$ on the boundary of $\mathcal{H}$ and let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{k-1}, \mathcal{F}_k = \mathcal{F}_0$ be the vertices of $\mathcal{H}$ listed in their order of appearence on $\partial\mathcal{H}$ when the latter is endowed with the orientation $\overline{\varepsilon}$. Define $\xi(\mathcal{H}_\varepsilon; \mathcal{F}_0) \in A$ by

$$\widetilde{\Phi}_{\mathcal{F}_k\mathcal{F}_{k-1}} \cdots \widetilde{\Phi}_{\mathcal{F}_1\mathcal{F}_0} = 1 + h^{n+1}\xi(\mathcal{H}_\varepsilon; \mathcal{F}_0) \mod \hbar^{n+2}$$

By lemma 3.31, $\xi(\mathcal{H}_\varepsilon; \mathcal{F}_0)$ does not depend upon the choice of $\mathcal{F}_0$ and will be hereafter denoted by $\xi(\mathcal{H}_\varepsilon)$. Moreover ξ satisfes $\xi(\mathcal{H}_{-\varepsilon}) = -\xi(\mathcal{H}_\varepsilon)$, where $-\varepsilon$ is the opposite orientation of $\mathcal{H}$ since $\widetilde{\Phi}_{\mathcal{G}\mathcal{F}}^{-1} = \widetilde{\Phi}_{\mathcal{F}\mathcal{G}}$ mod $\hbar^{n+2}$.

5.7.3. We show next that ξ is a Dynkin 3–cochain. In view of theorem 5.20, it is sufficient to prove the following

Lemma 5.23.

(i) $\xi(\mathcal{H}_\varepsilon) = 0$ *if* $\mathcal{H}$ *is a reducible nested set.*
(ii) $\xi(\mathcal{H}_\varepsilon) = \xi(\mathcal{H}'_{\varepsilon'})$ *if* $\mathcal{H}_\varepsilon, \mathcal{H}'_{\varepsilon'}$ *are equivalent.*
(iii) $\xi(\mathcal{H}_\varepsilon) \in A_B^{B\setminus\{\alpha,\beta\}}$ *if* $\mathcal{H}$ *is irreducible with unsaturated element* B *and* $\underline{\alpha}_{\mathcal{H}}^B = \{\alpha, \beta\}$.

Proof. (i) is a consequence of proposition 3.32. (ii) and (iii) follow from the analysis of the 2–faces of $\mathcal{A}_D$ given in §2.8 and the forgetfulness and support axioms respectively ■

5.7.4. We claim now that ξ is a Dynkin 3–cocycle. By theorem 5.20, it suffices to prove the following

Proposition 5.24. *ξ is a cellular 2–cocycle.*

Proof. We shall in fact prove that ξ is a cellular 2–coboundary. Let $\gamma = \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k = \mathcal{F}_0$ be an elementary sequence of maximal nested sets on D. Since the associators $\widetilde{\Phi}_{\mathcal{G}\mathcal{F}}$ satisfy the coherence axiom, we have

$$\Phi_{\mathcal{F}_k\mathcal{F}_{k-1}} \cdots \Phi_{\mathcal{F}_1\mathcal{F}_0} = 1 + \hbar^{n+1}\zeta(\gamma) \mod \hbar^{n+2}$$

for some $\zeta(\gamma) \in A$. Fix a reference maximal nested set $\mathcal{F}_0$ on D and, for any maximal nested set $\mathcal{F}$, use the connectedness of $\mathcal{A}_D$ to choose an edge–path $p_{\mathcal{F}}$ from $\mathcal{F}_0$ to $\mathcal{F}$. For any oriented 1–edge $e = (\mathcal{G}, \mathcal{F})$ in $\mathcal{A}_D$, set

$$\eta(e) = \zeta(p_{\mathcal{G}} \vee e \vee p_{\mathcal{F}}) \in A$$

where $\vee$ is concatenation. One readily checks that $\eta(\overline{e}) = -\eta(e)$, where $\overline{e} = (\mathcal{F}, \mathcal{G})$, so that η defines a 1–cochain on $\mathcal{A}_D$ with values in A and that $d\eta = \xi$, where d is the cellular differential ■

5.7.5. To complete the proof of (i), we must show

Proposition 5.25. *The quasi–Coxeter algebra structure on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$ given by the associators $\{\Phi_{\mathcal{G}\mathcal{F}}\}$ lifts to one on $A[\![\hbar]\!]/\hbar^{n+2}A[\![\hbar]\!]$ if, and only if ξ is a Dynkin coboundary.*

Proof. If $\xi = d_D\Theta$ for some $\Theta = \{\Theta_{\mathcal{G}\mathcal{F}}\}$, one readily checks that the associators $\widetilde{\Phi}_{\mathcal{G}\mathcal{F}} - \hbar^{n+1}\Theta_{\mathcal{G}\mathcal{F}}$ give the required lift. Conversely, if

$$\overline{\Phi}_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{G}\mathcal{F}} + \hbar^{n+1}\overline{\Theta}_{\mathcal{G}\mathcal{F}} = \widetilde{\Phi}_{\mathcal{G}\mathcal{F}} + \hbar^{n+1}\Theta_{\mathcal{G}\mathcal{F}}$$

endow $A[\![\hbar]\!]/\hbar^{n+2}A[\![\hbar]\!]$ with a weak quasi–Coxeter algebra structure, one readily checks using theorem 5.20 that Θ is a Dynkin one–cochain and that $d_D\Theta = \xi$ ■

Part II. Quasi–Coxeter quasibialgebras

6. Quasi–Coxeter quasitriangular quasibialgebras

The aim of this section is to define the category of quasi–Coxeter quasitriangular quasibialgebras. We proceed in several stages.

6.1. D–bialgebras.

Definition 6.1. *A D–bialgebra $(A, \{A_B\}, \Delta, \varepsilon)$ is a D–algebra A endowed with a bialgebra structure, with coproduct Δ and counit ε, such that each A_B is a sub–bialgebra of A, that is satisfies $\Delta(A_B) \subseteq A_B \otimes A_B$.*

Definition 6.2. *A morphism of D–bialgebras A, A' is a morphism $\{\Psi_{\mathcal{F}}\}$ of the underlying D–algebras such that each $\Psi_{\mathcal{F}}$ is a bialgebra morphism $(A, \Delta, \varepsilon) \to (A', \Delta', \varepsilon')$.*

If A is a bialgebra, we denote by $\Delta^{(n)} : A \to A^{\otimes n}$, $n \geq 0$, the iterated coproduct defined by $\Delta^{(0)} = \varepsilon$, $\Delta^{(1)} = \mathrm{id}$, and

$$\Delta^{(n+1)} = \Delta \otimes \mathrm{id}^{\otimes n-1} \circ \Delta^{(n)}$$

if $n \geq 1$. Each tensor power $A^{\otimes n}$ of A is an A–bimodule, where $a \in A$ acts by left and right multiplication by $\Delta^{(n)}(a)$ respectively. If A is a D–bialgebra, this endows each $A^{\otimes n}$ with the structure of a D–bimodule over A by setting $(A^{\otimes n})_B = A_B^{\otimes n}$. In the notation of §5.1 we then have, for any $B_1, B_2 \subseteq D$, with B_1 connected

$$(A^{\otimes n})_{B_1}^{B_2} = \{\omega \in A_{B_1}^{\otimes n} |\, [\omega, \Delta^{(n)}(a)] = 0 \text{ for all } a \in A_{B_2^i}\}$$

where B_2^i are the connected components of B_2.

6.2. **D–quasibialgebras.** Retain the definitions and notation of §4.3.1.

Definition 6.3. *A D–quasibialgebra $(A, \{A_B\}, \Delta, \varepsilon, \{\Phi_B\}, \{F_{(B;\alpha)}\})$ is a D–bialgebra A endowed with the following additional data :*

- **Associators :** *for each connected subdiagram $B \subseteq D$, an invertible element*
$$\Phi_B \in (A_B^{\otimes 3})^B$$
- **Structural twists :** *for each connected subdiagram $B \subseteq D$ and vertex $\alpha \in B$, a twist*
$$F_{(B;\alpha)} \in (A_B^{\otimes 2})^{B\setminus\alpha}$$

satisfying the following axioms :

- *for any connected $B \subseteq D$, $(A_B, \Delta, \varepsilon, \Phi_B)$ is a quasibialgebra.*
- *For any connected $B \subseteq D$ and $\alpha \in B$,*
$$(\Phi_B)_{F_{(B;\alpha)}} = \Phi_{B\setminus\alpha} \tag{6.1}$$
where $\Phi_{B\setminus\alpha} = \prod_{B'} \Phi_{B'}$, with the product ranging over the connected components of $B \setminus \alpha$ if $B \neq \alpha$, and $\Phi_\emptyset = 1^{\otimes 3}$.

The essence of the above definition is that for each maximal nested set $\mathcal{F}$ on D, one can coherently twist the family of quasibialgebras $(A_B, \Delta, \varepsilon, \Phi_B)$, with $B \in \mathcal{F}$, in the following way[1]. For any connected $B \subseteq D$ and maximal nested set $\mathcal{F}_B$ on B, set

$$F_{\mathcal{F}_B} = \overrightarrow{\prod_{C\in\mathcal{F}_B}} F_{(C;\alpha^C_{\mathcal{F}_B})} \in A_B^{\otimes 2} \tag{6.2}$$

where the product is taken with $F_{(C_1;\alpha^{C_1}_{\mathcal{F}_B})}$ written to the left of $F_{(C_2;\alpha^{C_2}_{\mathcal{F}_B})}$ whenever $C_1 \subset C_2$. This does not specify the order of the factors uniquely, but two orders satisfying this requirement are readily seen to yield the same product. The factorised form of the twist $F_{\mathcal{F}_B}$ implies the following

Lemma 6.4. *Let $\mathcal{F}$ be a maximal nested set on D and $B \in \mathcal{F}$. Then, for any $a \in A_B$,*
$$F_{\mathcal{F}} \cdot \Delta(a) \cdot F_{\mathcal{F}}^{-1} = F_{\mathcal{F}_B} \cdot \Delta(a) \cdot F_{\mathcal{F}_B}^{-1}$$
where $\mathcal{F}_B = \{C \in \mathcal{F} |\, C \subseteq B\}$ is the maximal nested set on B induced by $\mathcal{F}$.

[1]The reader may object that a D–quasibialgebra is not truly 'quasi' since the coproduct Δ is assumed to be coassociative. In particular, in view of the coassociativity axiom (4.9), the invariance of the associator Φ_B is a necessary condition for $(A_B, \Delta, \varepsilon, \Phi_B)$ to be a quasibialgebra.

Thus, if $\mathcal{F}$ is a maximal nested set on D and $B \in \mathcal{F}$, the twisted coproduct

$$\Delta_{\mathcal{F}}(a) = F_{\mathcal{F}} \cdot \Delta(a) \cdot F_{\mathcal{F}}^{-1} \tag{6.3}$$

corresponding to $\mathcal{F}$ restricts to $\Delta_{\mathcal{F}_B}$ on A_B so that $(A_B, \Delta_{\mathcal{F}_B}, \varepsilon, (\Phi_B)_{F_{\mathcal{F}_B}})$ is a sub–quasibialgebra of $(A, \Delta_{\mathcal{F}}, \varepsilon, (\Phi_D)_{F_{\mathcal{F}}})$. Turning now to the associators Φ_B, an inductive application of (6.1) readily yields the following[1]

Lemma 6.5. *For any connected $B \subseteq D$ and maximal nested set $\mathcal{F}_B$ on B,*

$$(\Phi_B)_{F_{\mathcal{F}_B}} = 1^{\otimes 3}$$

In particular, the twisted coproduct $\Delta_{F_{\mathcal{F}_B}}$ is in fact coassociative so that $(A_B, \Delta_{\mathcal{F}_B}, \varepsilon)$ is a bialgebra which is a sub–bialgebra of $(A, \Delta_{\mathcal{F}}, \varepsilon)^2$.

Remark 6.6. Lemma 6.5 implies that the associators of a D–quasibialgebra are not independent variables since, for any connected B and maximal nested set $\mathcal{F}_B$ on B,

$$\Phi_B = \mathrm{id} \otimes \Delta(F_{\mathcal{F}_B}^{-1}) \cdot 1 \otimes F_{\mathcal{F}_B}^{-1} \cdot F_{\mathcal{F}_B} \otimes 1 \cdot \Delta \otimes \mathrm{id}(F_{\mathcal{F}_B}) \tag{6.4}$$

The axioms involving Φ_B are in fact equivalent to the requirement that the right–hand side of (6.4) be invariant under A_B and independent of the choice of $\mathcal{F}_B$. It is, however, more convenient to work with the associators Φ_B.

Remark 6.7. Relation (6.1) may be rephrased as follows. For any subdiagram $B \subseteq D$, let A_B be the algebra generated by the A_{B_i}, where B_i runs over the connected components of B and set $\Phi_B = \prod_i \Phi_{B_i}$. Consider the (Drinfeld) tensor category $\mathrm{Rep}_{\Phi_B}(A_B)$ of representations of A_B where the associativity constraints are given by the action of the associator Φ_B. Then, for any $\alpha \in B \subseteq D$, the twist $F_{(B;\alpha)}$ gives rise to a tensor structure on the restriction functor $\mathrm{Rep}_{\Phi_B}(A_B) \rightarrow \mathrm{Rep}_{\Phi_{B\setminus\alpha}}(A_{B\setminus\alpha})$.

6.3. Morphism of D–quasibialgebras.

Definition 6.8. *A morphism of D–quasibialgebras A, A' is a morphism $\{\Psi_{\mathcal{F}}\}$ of the underlying D–algebras such that, for any maximal nested set $\mathcal{F}$ on D, $\Psi_{\mathcal{F}}$ is a bialgebra morphism $(A, \Delta_{\mathcal{F}}, \varepsilon) \longrightarrow (A', \Delta'_{\mathcal{F}}, \varepsilon')$.*

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[1] this consequence of (6.1) arose during a conversation with R. Nest.

[2] For any such $\mathcal{F}$ and B one may of course also consider the quasibialgebra $(A_B, \Delta_{\mathcal{F}_B}, \varepsilon, \Psi_{\mathcal{F}_B})$ where $\Psi_{\mathcal{F}_B} = (1^{\otimes 3})_{F_{\mathcal{F}_B}} = 1 \otimes F_{\mathcal{F}_B} \cdot \mathrm{id} \otimes \Delta(F_{\mathcal{F}_B}) \cdot \Delta \otimes \mathrm{id}(F_{\mathcal{F}_B}^{-1}) \cdot F_{\mathcal{F}_B}^{-1} \otimes 1$. We will not however use the associators $\Psi_{\mathcal{F}_B}$.

Remark 6.9. Note that a morphism of D–quasibialgebras is not a morphism of the underlying D–bialgebras in general.

6.4. **Twisting of D–quasibialgebras.** Let A be a D–quasibialgebra.

Definition 6.10. *A twist of A is a family $F = \{F_B\}_{B\subseteq D}$ of invertible elements labelled by the connected subdiagrams of D where $F_B \in (A_B^{\otimes 2})^B$ satisfies*

$$\varepsilon \otimes \mathrm{id}(F_B) = 1 = \mathrm{id} \otimes \varepsilon(F_B)$$

Given a twist F of A, set, for any connected $B \subseteq D$ and vertex $\alpha \in B$

$$\Phi_B^F = (\Phi_B)_{F_B} \tag{6.5}$$

$$F_{(B;\alpha)}^F = F_{B\setminus\alpha} \cdot F_{(B;\alpha)} \cdot F_B^{-1} \tag{6.6}$$

where $F_{B\setminus\alpha} = \prod_i F_{B_i}$, with the product ranging over the connected components of $B \setminus \alpha$ if $B \neq \alpha$, and $F_\emptyset = 1^{\otimes 2}$ otherwise.

Definition 6.11. *The twist of A by F is the D–quasibialgebra*

$$A^F = (A, \{A_B\}, \Delta, \varepsilon, \{\Phi_B^F\}, \{F_{(B;\alpha)}^F\})$$

Remark 6.12. In the notation (6.2), the twist by F of $F_{\mathcal{F}_B}$ is given by

$$F_{\mathcal{F}_B}^F = F_{\mathcal{F}_B} \cdot F_B^{-1}$$

Since F_B is invariant under A_B, twisting by F does not change the coproduct $\Delta_{F_{\mathcal{F}_B}}$ on A_B. Thus, A^F and A are isomorphic as D–quasibialgebras via the identity map.[1].

6.5. **D–quasitriangular quasibialgebras.** Retain the definitions of §4.3.5.

Definition 6.13. *A D–quasitriangular quasibialgebra*

$$(A, \{A_B\}, \Delta, \varepsilon, \{\Phi_B\}, \{F_{(B;\alpha)}\}, \{R_B\})$$

is a D–quasibialgebra A endowed with an invertible element $R_B \in A_B^{\otimes 2}$ for each connected subdiagram $B \subseteq D$ such that $(A_B, \Delta, \varepsilon, \Phi_B, R_B)$ is a quasitriangular quasibialgebra.

Let A be a D–quasitriangular quasibialgebra and $\mathcal{F}$ a maximal nested set on D. For any $B \in \mathcal{F}$, the twist by $F_{\mathcal{F}_B}$ of $(A_B, \Delta, \varepsilon, \Phi_B, R_B)$ yields a quasitriangular bialgebra $(A_B, \Delta_{\mathcal{F}_B}, \varepsilon, (R_B)_{F_{\mathcal{F}_B}})$.

[1]Notice the apparent difference between the definition of morphism of D–quasibialgebras and quasibialgebras. As pointed out in §4.3.1, a morphism of quasibialgebras $A \to A'$ is given by a twist F of A' followed by a naïve morphism $A \to A'_F$. Due to the restrictive nature of the twists we use, the twist of a D–quasibialgebra A' is still 'naively' isomorphic to A' in the sense of definition 6.8.

Definition 6.14. *A morphism of D–quasitriangular quasibialgebras A, A' is a morphism $\{\Psi_{\mathcal{F}}\}$ of the underlying D–quasibialgebras such that, for any maximal nested set $\mathcal{F}$ on D and $B \in \mathcal{F}$, $\Psi_{\mathcal{F}}$ satisfies*

$$\Psi_{\mathcal{F}}((R_B)_{F_{\mathcal{F}_B}}) = (R'_B)_{F'_{\mathcal{F}_B}}$$

and therefore restricts to a morphism of quasitriangular bialgebras

$$(A, \Delta_{\mathcal{F}}, \varepsilon, (R_B)_{F_{\mathcal{F}_B}}) \longrightarrow (A', \Delta'_{\mathcal{F}}, \varepsilon', (R'_B)_{F'_{\mathcal{F}_B}})$$

Definition 6.15. *A twist $F = \{F_B\}_{B \subseteq D}$ of a D–quasitriangular quasibialgebra A is a twist of the underlying D–quasibialgebra. The twisting of A by F is the D–quasitriangular quasibialgebra*

$$A^F = (A, \{A_B\}, \Delta, \varepsilon, \{\Phi_B^F\}, \{F^F_{(B;\alpha)}\}, \{R_B^F\})$$

where Φ_B^F and $F^F_{(B;\alpha)}$ are given by (6.5)*–*(6.6) *and $R_B^F = (R_B)_{F_B}$.*

Remark 6.16. Since for any maximal nested set $\mathcal{F}$ on D and $B \in \mathcal{F}$,

$$(R_B^F)_{\mathcal{F}^F_{\mathcal{F}_B}} = (R_B)_{\mathcal{F}_{\mathcal{F}_B}}$$

the twisting A^F of A by F and A are isomorphic as D–quasitriangular quasibialgebras via the identity map.

6.6. **Quasi–Coxeter quasibialgebras.** Assume henceforth that the diagram D is labelled.

Definition 6.17. *A quasi–Coxeter quasibialgebra of type D is a set*

$$(A, \{A_B\}, \{S_i\}, \{\Phi_{(B;\alpha,\beta)}\}, \Delta, \varepsilon, \{F_{(B;\alpha)}\}, \{\Phi_B\})$$

where

- $(A, \{A_B\}, \{S_i\}, \{\Phi_{(B;\alpha,\beta)}\})$ *is a quasi–Coxeter algebra of type D.*
- $(A, \{A_B\}, \Delta, \varepsilon, \{F_{(B;\alpha)}\}, \{\Phi_B\})$ *is a D–quasibialgebra*

and, for any pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D, the following holds

$$F_{\mathcal{G}} \cdot \Delta(\Phi_{\mathcal{G}\mathcal{F}}) = \Phi_{\mathcal{G}\mathcal{F}}^{\otimes 2} \cdot F_{\mathcal{F}} \tag{6.7}$$

Since $\varepsilon \otimes \mathrm{id}(F_{\mathcal{F}}) = 1 = \varepsilon \otimes \mathrm{id}(F_{\mathcal{G}})$ and $\varepsilon \otimes \mathrm{id} \circ \Delta = \mathrm{id}$, (6.7) implies that

$$\varepsilon(\Phi_{\mathcal{G}\mathcal{F}}) = 1 \tag{6.8}$$

It follows that $\mathrm{Ad}(\Phi_{\mathcal{G}\mathcal{F}})$ is an isomorphism of bialgebras $(A, \Delta_{\mathcal{F}}, \varepsilon) \longrightarrow (A, \Delta_{\mathcal{G}}, \varepsilon)$. Moreover, by proposition 3.28 this isomorphism restricts to an isomorphism $(A_B, \Delta_{\mathcal{F}_B}, \varepsilon) \longrightarrow (A_B, \Delta_{\mathcal{G}_B}, \varepsilon)$ for any element $B \in \mathcal{F} \cap \mathcal{G}$.

Remark 6.18. It was pointed out in remark 6.6 that, for a D–quasibialgebra the axiom (6.1) is equivalent to the invariance of the right–hand side of (6.4) and its independence on the choice of the maximal nested set $\mathcal{F}_B$. Since in a quasi–Coxeter quasibialgebra the twists $F_\mathcal{F}$ are related by gauge transformations, the invariance of (6.4) implies its independence on the choice of $\mathcal{F}_B$. Thus, for quasi–Coxeter quasibialgebra axiom (6.1) is equivalent to the invariance of the right–hand side of (6.4).

Let us spell out (6.7) in diagrammatic notation. By the connectedness of the associahedron $\mathcal{A}_D$, (6.7) holds for any pair $(\mathcal{G},\mathcal{F})$ if, and only if it holds for any elementary pair of maximal nested sets on D. Let $(\mathcal{G},\mathcal{F})$ be one such pair, $B=\operatorname{supp}(\mathcal{G},\mathcal{F})$ the unique unsaturated element of $\mathcal{F}\cap\mathcal{G}$ and set $\alpha_1=\alpha^B_\mathcal{F}$ and $\alpha_2=\alpha^B_\mathcal{G}$, so that $\Phi_{\mathcal{G}\mathcal{F}}=\Phi_{(B;\alpha_2,\alpha_1)}$.

Lemma 6.19. *The relation* (6.7) *is equivalent to*

$$F_{(\mathsf{C}^{B\setminus\alpha_2}_{\alpha_1};\alpha_1)}\cdot F_{(B,\alpha_2)}\cdot\Delta(\Phi_{(B;\alpha_2,\alpha_1)})=\Phi^{\otimes 2}_{(B;\alpha_2,\alpha_1)}\cdot F_{(\mathsf{C}^{B\setminus\alpha_1}_{\alpha_2};\alpha_2)}\cdot F_{(B,\alpha_1)}$$

Proof. By definition,

$$F_\mathcal{G}=\overrightarrow{\prod_{C\in\mathcal{G}}}F_{(C;\alpha^C_\mathcal{G})}\qquad\text{and}\qquad F_\mathcal{F}=\overrightarrow{\prod_{C\in\mathcal{F}}}F_{(C;\alpha^C_\mathcal{F})}$$

Let $C\in\mathcal{F}\cap\mathcal{G}$, with $C\neq B$ so that $\alpha^C_\mathcal{F}=\alpha^C_\mathcal{G}$. If $C\perp B$, then

$$F_{(C;\alpha^C_\mathcal{G})}\cdot\Delta(\Phi_{(B;\alpha_2,\alpha_1)})=\Delta(\Phi_{(B;\alpha_2,\alpha_1)})\cdot F_{(C;\alpha^C_\mathcal{F})}\tag{6.9}$$

since $F_{(C;\alpha^C_\mathcal{G})}\in A_C^{\otimes 2}$, $\Delta(\Phi_{(B;\alpha_2,\alpha_1)})\in A_B^{\otimes 2}$ and $[A_C,A_B]=0$. If $C\supsetneq B$, then $B\subseteq C\setminus\alpha^C_\mathcal{G}$ and (6.9) holds since $F_{(C;\alpha^C_\mathcal{G})}$ commutes with $\Delta(A_{C\setminus\alpha^C_\mathcal{G}})$. Finally, if $C\subsetneq B$, then $C\subseteq B\setminus\{\alpha_1,\alpha_2\}$ and

$$\Phi^{\otimes 2}_{(B;\alpha_2,\alpha_1)}\cdot F_{(C;\alpha^C_\mathcal{G})}=F_{(C;\alpha^C_\mathcal{G})}\cdot\Phi^{\otimes 2}_{(B;\alpha_2,\alpha_1)}$$

since $\Phi_{(B;\alpha_2,\alpha_1)}$ centralises $A_{C\setminus\{\alpha_1,\alpha_2\}}$. This implies the stated equivalence since, by proposition 3.23, $\mathcal{G}\setminus\mathcal{F}=\mathsf{C}^{B\setminus\alpha_2}_{\alpha_1}$ and $\mathcal{F}\setminus\mathcal{G}=\mathsf{C}^{B\setminus\alpha_2}_{\alpha_1}$ ■

6.7. **Morphism of quasi–Coxeter quasibialgebras.**

Definition 6.20. *A morphism of quasi–Coxeter quasibialgebras of type D is a morphism of the underlying D–algebras which is a morphism of the underlying quasi–Coxeter algebras and D–quasibialgebras.*

6.8. **Twisting of quasi–Coxeter quasibialgebras.** Let A be a quasi–Coxeter quasibialgebra of type D.

Definition 6.21. *A twist of A is a pair (a, F) where*

- $a = \{a_{(B;\alpha)}\}$ *is a twist of the underlying quasi–Coxeter algebra such that* $\varepsilon(a_{(B;\alpha)}) = 1$ *for any connected* $B \subseteq D$ *and* $\alpha \in B$.
- $F = \{F_B\}_B$ *is a twist of the underlying D–quasibialgebra.*

The twisting of A by (a, F) is the quasi–Coxeter quasibialgebra

$$(A, \{A_B\}, \{S_i^a\}, \{\Phi^a_{(B;\alpha,\beta)}\}, \Delta, \varepsilon, \{F^{(a,F)}_{(B;\alpha)}\}, \{\Phi_B^F\})$$

where

$$\begin{aligned} F^{(a,F)}_{(B;\alpha)} &= a^{\otimes 2}_{(B;\alpha)} \cdot F^F_{(B;\alpha)} \cdot \Delta(a_{(B;\alpha)})^{-1} \\ &= F_{B\setminus\alpha} \cdot a^{\otimes 2}_{(B;\alpha)} \cdot F_{(B;\alpha)} \cdot \Delta(a_{(B;\alpha)})^{-1} \cdot F_B^{-1} \end{aligned}$$

In the notation (6.2), the twist by (a, F) of $F_{\mathcal{F}'}$ is given by

$$F^{(a,F)}_{\mathcal{F}'} = a^{\otimes 2}_{\mathcal{F}'} \cdot F_{\mathcal{F}'} \cdot \Delta(a_{\mathcal{F}'})^{-1} \cdot F_B^{-1} = a^{\otimes 2}_{\mathcal{F}'} \cdot F_{\mathcal{F}'} \cdot F_B^{-1} \cdot \Delta(a_{\mathcal{F}'})^{-1}$$

Note that the twisting of A by (a_1, F_1) followed by a twisting by (a_2, F_2) is equal to the twisting by $(a_2 \cdot a_1, F_2 \cdot F_1)$.

Proposition 6.22. *Let (a, F) be a twist of A. Then, the assignement $\mathcal{F} \to \mathrm{Ad}(a_{\mathcal{F}})$ defines an isomorphism of the quasi–Coxeter quasibialgebras A and $A^{(a,F)}$.*

6.9. **Quasi–Coxeter, quasitriangular quasibialgebras.**

Definition 6.23. *A quasi–Coxeter quasitriangular quasibialgebra of type D is a quasi–Coxeter quasibialgebra A of type D endowed with an invertible element $R_B \in A_B^{\otimes 2}$ for any connected subdiagram $B \subseteq D$giving it the structure of a quasitriangular quasibialgebra such that, for $\alpha_i \in D$, the following holds*

$$\Delta_{F_{(\alpha_i;\alpha_i)}}(S_i) = (R_{\alpha_i})^{-1}_{F_{(\alpha_i;\alpha_i)}} \cdot S_i \otimes S_i \tag{6.10}$$

A morphism of quasi–Coxeter quasitriangular quasibialgebra is a morphism of the underlying quasi–Coxeter quasibialgebra and quasitriangular quasibialgebra structures. A twist (a, F) of A is one of the underlying quasi–Coxeter quasibialgebra. The twisting of A by (a, F) the quasi–Coxeter quasitriangular quasibialgebra

$$(A, \{A_B\}, \{S_i^a\}, \{\Phi^a_{(B;\alpha,\beta)}\}, \Delta, \varepsilon, \{F^{(a,F)}_{(B;\alpha)}\}, \{\Phi_B^F\}, \{R_B^F\})$$

Remark 6.24. Since $\varepsilon \otimes \mathrm{id}(R) = 1 = \mathrm{id} \otimes \varepsilon(R)$ in any quasitriangular quasibialgebra $(A, \Delta, \varepsilon, \Phi, R)$ [Dr3, §3], applying $\varepsilon \otimes \mathrm{id}$ to (6.10) yields

$$\varepsilon(S_i) = 1$$

By (6.8), this implies that, for any maximal nested set $\mathcal{F}$ on D, the action $\varepsilon \circ \pi_{\mathcal{F}}$ of the braid group B_D on the trivial A–module is trivial.

7. The Dynkin–Hochschild bicomplex of a D–bialgebra

Let D be a connected diagram and A a D–bialgebra. By combining the Dynkin complex of A with the cobar complexes of its subalgebras A_B, we define in this section a bicomplex which controls the deformations of quasi–Coxeter quasibialgebra structures on A.

7.1. Let A be a bialgebra and $C^*(A)$ the cobar complex of A, regarded as a coalgebra, defined in §5.3. If $C \subseteq A$ a sub–bialgebra, let

$$C^n(A)^C = \{a \in C^n(A) | \, [a, \Delta^{(n)}(c)] = 0 \text{ for any } c \in C\}$$

where $\Delta^{(n)} : C \to C^{\otimes n}$ is the nth iterated coproduct, be the submodule of C–invariants. It is easy to check that $C^*(A)^C$ is a subcomplex of $C^*(A)$.

Assume now that A is a D–bialgebra. For $p \in \mathbb{N}$ and $0 \leq q \leq |D|$, let

$$CD^q(A; A^{\otimes p}) \subset \bigoplus_{\substack{\underline{\alpha} \subseteq B \subseteq D, \\ |\underline{\alpha}|=q}} (A_B^{\otimes p})^{B \setminus \underline{\alpha}}$$

be the group of Dynkin q–cochains with values in the D–bimodule $A^{\otimes p}$ over A. The Dynkin differential d_{DD} defines a vertical differential $CD^q(A; A^{\otimes p}) \to CD^{q+1}(A; A^{\otimes p})$ while the Hochschild differential d_H defines a horizontal differential $CD^q(A; A^{\otimes p}) \to CD^q(A; A^{\otimes (p+1)})$. A straightforward computation yields the following.

Theorem 7.1. *One has $d_{DD} \circ d_H = d_H \circ d_{DD}$. The corresponding cohomology of the bicomplex $CD^q(A; A^{\otimes p})$ is called the Dynkin–Hochschild cohomology of the D–bialgebra A.*

7.2. Regard D as labelled by attaching an infinite multiplicity to each edge. By a trivial quasi–Coxeter quasibialgebra structure on A we shall mean one whose underlying D–bialgebra structure is that of A and for which

$$\Phi_{(B;\beta,\alpha)} = 1, \qquad F_{(B;\alpha)} = 1^{\otimes 2} \qquad \text{and} \qquad \Phi_B = 1^{\otimes 3}$$

We do not assume that the local monodromies S_i are trivial however. When considering deformations of a trivial quasi–Coxeter quasibialgebra structure, the local monodromies will be assumed to remain undeformed.

Theorem 7.2. *The Dynkin–Hochschild bicomplex of A controls the formal, one–parameter deformations of trivial quasi–Coxeter quasibialgebra structures on A. Specifically,*

(i) *A quasi–Coxeter quasibialgebra structure on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$ which is trivial mod $\hbar$ canonically determines a Dynkin–Hochschild 4–cocycle ξ and lifts to a quasi–Coxeter quasibialgebra structure on $A[\![\hbar]\!]/\hbar^{n+2}A[\![\hbar]\!]$ if, and only if ξ is a coboundary.*

(ii) *Two quasi–Coxeter quasibialgebra structures on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$ which are trivial mod $\hbar$ and agree mod $\hbar^n$ differ by a Dynkin–Hochschild 3–cocycle η and can be obtained from each other by a twist of the form $(1+\hbar^n a, 1+\hbar^n F)$ if, and only if, $\eta = d(a,F)$ and $\mathrm{Ad}(S_i^A)a_{(\alpha_i;\alpha_i)} = a_{(\alpha_i;\alpha_i)}$.*

Proof. (i) Let $(\Phi_{(B;\alpha,\beta)}, F_{(B;\alpha)}, \Phi_B)$ be the associators and structural twists of the quasi–Coxeter quasibialgebra structure on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$. We construct in §7.3–7.6 a cochain $(\xi,\eta,\chi,\theta) \in \bigoplus_{i+j=4} CD^j(A;A^{\otimes i})$ and check that it is a Dynkin–Hochschild cocycle.

7.3. Proceeding as in §5.7.1, we may assume that $\Phi_{\mathcal{F}\mathcal{G}} = \Phi_{\mathcal{G}\mathcal{F}}^{-1}$ mod $\hbar^{n+2}$ for any elementary pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D. Let $\mathcal{H}_\varepsilon$ be an oriented nested set of dimension 2 on D, $\mathcal{F}_0$ a maximal nested set on the boundary of $\mathcal{H}$ and $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{k-1}, \mathcal{F}_k = \mathcal{F}_0$ the vertices of $\mathcal{H}$ listed in their order of appearence on $\partial\mathcal{H}$ when the latter is endowed with the orientation $\overline{\varepsilon}$. Define $\xi(\mathcal{H}_\varepsilon) \in A$ by

$$\Phi_{\mathcal{F}_k\mathcal{F}_{k-1}} \cdots \Phi_{\mathcal{F}_1\mathcal{F}_0} = 1 + h^{n+1}\xi(\mathcal{H}_\varepsilon) \mod \hbar^{n+2}$$

It was proved in §5.7.2–5.7.4 that ξ is independent of the choice of $\mathcal{F}_0$, satisfies $\xi(\mathcal{H}_{-\varepsilon}) = -\xi(\mathcal{H}_\varepsilon)$, where $-\varepsilon$ is the opposite orientation, and ξ is a Dynkin 3–cocycle with values in A.

7.4. For any elementary pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D, define $\eta_{\mathcal{G}\mathcal{F}} \in A^{\otimes 2}$ by

$$F_{\mathcal{G}} \cdot \Delta(\Phi_{\mathcal{G}\mathcal{F}}) - \Phi_{\mathcal{G}\mathcal{F}}^{\otimes 2} \cdot F_{\mathcal{F}} = \hbar^{n+1}\eta_{\mathcal{G}\mathcal{F}} \mod \hbar^{n+2} \tag{7.1}$$

Lemma 7.3. *The following holds*

(i) *$\eta_{\mathcal{F}\mathcal{G}} = -\eta_{\mathcal{G}\mathcal{F}}$ so that η is a cellular 1–cochain on the associahedron $\mathcal{A}_D$ with values in $A^{\otimes 2}$.*

(ii) *If d is the cellular differential on $\mathcal{A}_D$, then $d\eta = -d_H\xi$.*

(iii) *η is a Dynkin 2–cochain.*

(iv) *$d_{DD}\eta = -d_H\xi$.*

Proof. (i) Multiplying (7.1) on the left by $\Phi_{\mathcal{F}\mathcal{G}}^{\otimes 2} = 1^{\otimes 2}$ mod $\hbar$ and on the right by $\Delta(\Phi_{\mathcal{F},\mathcal{G}}) = 1^{\otimes 2}$ mod $\hbar$ and using the fact that $\Phi_{\mathcal{G}\mathcal{F}}\cdot\Phi_{\mathcal{F}\mathcal{G}} = 1$

mod $\hbar^{n+2}$, we get, working mod $\hbar^{n+2}$

$$\hbar^{n+1}\eta_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{F}\mathcal{G}}^{\otimes 2} \cdot F_{\mathcal{G}} - F_{\mathcal{F}} \cdot \Delta(\Phi_{\mathcal{F}\mathcal{G}}) = -\hbar^{n+1}\eta_{\mathcal{F}\mathcal{G}}$$

(ii) Since $\Phi_{\mathcal{F}\mathcal{G}} = 1 \bmod \hbar$ and $\Phi_{\mathcal{G}\mathcal{F}} \cdot \Phi_{\mathcal{F}\mathcal{G}} = 1 \bmod \hbar^{n+2}$, (7.1) may be rewritten as

$$F_{\mathcal{G}} = \hbar^{n+1}\eta_{\mathcal{G}\mathcal{F}} + \Phi_{\mathcal{G}\mathcal{F}}^{\otimes 2} \cdot F_{\mathcal{F}} \cdot \Delta(\Phi_{\mathcal{F}\mathcal{G}}) \mod \hbar^{n+2}$$

Let $\mathcal{H}_\varepsilon$, $\mathcal{F}_0, \ldots, \mathcal{F}_{k-1}$ be as in §7.3. Then, mod $\hbar^{n+2}$,

$$\begin{aligned}
F_{\mathcal{F}_0} &= \hbar^{n+1}\eta_{\mathcal{F}_0,\mathcal{F}_{k-1}} + \Phi_{\mathcal{F}_0,\mathcal{F}_{k-1}}^{\otimes 2} \cdot F_{\mathcal{F}_{k-1}} \cdot \Delta(\Phi_{\mathcal{F}_{k-1},\mathcal{F}_0}) \\
&= \hbar^{n+1}(\eta_{\mathcal{F}_0,\mathcal{F}_{k-1}} + \cdots + \eta_{\mathcal{F}_1,\mathcal{F}_0}) \\
&+ \left(\Phi_{\mathcal{F}_0,\mathcal{F}_{k-1}} \cdots \Phi_{\mathcal{F}_1,\mathcal{F}_0}\right)^{\otimes 2} \cdot F_{\mathcal{F}_0} \cdot \Delta\left(\Phi_{\mathcal{F}_0,\mathcal{F}_1} \cdots \Phi_{\mathcal{F}_{k-1},\mathcal{F}_0}\right) \\
&= \hbar^{n+1} d\eta(\mathcal{H}_\varepsilon) + \left(1 + h^{n+1}\xi(\mathcal{H}_\varepsilon)\right)^{\otimes 2} \cdot F_{\mathcal{F}_0} \cdot \Delta\left(1 - h^{n+1}\xi(\mathcal{H}_\varepsilon)\right) \\
&= \hbar^{n+1}\left(d\eta(\mathcal{H}_\varepsilon) + 1 \otimes \xi(\mathcal{H}_\varepsilon) - \Delta(\xi(\mathcal{H}_\varepsilon)) + \xi(\mathcal{H}_\varepsilon) \otimes 1\right) + F_{\mathcal{F}_0}
\end{aligned}$$

(iii) Let $(\mathcal{G}, \mathcal{F})$ be an elementary pair of maximal nested sets on D and set $B = \operatorname{supp}(\mathcal{F}, \mathcal{G})$, $\alpha_1 = \alpha_{\mathcal{F}}^B$ and $\alpha_2 = \alpha_{\mathcal{G}}^B$ so that $\Phi_{\mathcal{G}\mathcal{F}} = \Phi_{(B;\alpha_2,\alpha_1)}$. Reasoning as in the proof of lemma 6.19, shows that, modulo $\hbar^{n+2}$,

$$\begin{aligned}
\hbar^{n+1}\eta_{\mathcal{G}\mathcal{F}} = & F_{(\mathsf{C}_{\alpha_1}^{B\setminus\alpha_2};\alpha_1)} \cdot F_{(B;\alpha_2)} \cdot \Delta(\Phi_{(B;\alpha_2,\alpha_1)}) \\
& - \Phi_{(B;\alpha_2,\alpha_1)}^{\otimes 2} \cdot F_{(\mathsf{C}_{\alpha_2}^{B\setminus\alpha_1};\alpha_2)} \cdot F_{(B;\alpha_1)}
\end{aligned} \tag{7.2}$$

from which it readily follows that $\eta_{\mathcal{G}\mathcal{F}} \in (A_B^{\otimes 2})^{B\setminus\{\alpha_1,\alpha_2\}}$ and that $\eta_{\mathcal{G}\mathcal{F}}$ only depends on the equivalence class of the elementary pair $(\mathcal{G}, \mathcal{F})$. By theorem 5.20, ξ is therefore a Dynkin 2–cochain.
(iv) is a direct consequence of (ii) and theorem 5.20 ■

7.5. For any $\alpha \in B \subseteq D$, define $\chi_{(B;\alpha)} \in A^{\otimes 3}$ by

$$\begin{aligned}
\hbar^{n+1}\chi_{(B;\alpha)} = & 1 \otimes F_{(B;\alpha)} \cdot \mathrm{id} \otimes \Delta(F_{(B;\alpha)}) \cdot \Phi_B \\
& - \Phi_{B\setminus\alpha} \cdot F_{(B;\alpha)} \otimes 1 \cdot \Delta \otimes \mathrm{id}(F_{(B;\alpha)}) \mod \hbar^{n+2}
\end{aligned} \tag{7.3}$$

Lemma 7.4. *The following holds*

(i) *χ is a Dynkin 1–cochain with values in $A^{\otimes 3}$.*
(ii) $d_{DD}\chi = d_H\eta$

PROOF. (i) We must show that $\chi_{(B;\alpha)} \in (A_B^{\otimes 2})^{B\setminus\alpha}$. This readily follows from the support properties of $\Phi_B, \Phi_{B\setminus\alpha}$ and $F_{(B;\alpha)}$.
(ii) Since $F_{(B;\alpha)} = 1 \bmod \hbar$, (7.3) may be rewritten as

$$(\Phi_B)_{F_{(B;\alpha)}} = \Phi_{B\setminus\alpha} + \hbar^{n+1}\chi_{(B;\alpha)} \mod \hbar^{n+2} \tag{7.4}$$

Let $\alpha_1 \neq \alpha_2 \in B$ and set $D_1 = \mathsf{C}_{\alpha_1}^{B\setminus\alpha_2}$, $D_2 = \mathsf{C}_{\alpha_2}^{B\setminus\alpha_1}$. Then, mod $\hbar^{n+2}$

$$\begin{aligned}(\Phi_B)_{F_{(D_2;\alpha_2)}\cdot F_{(B;\alpha_1)}} &= (\Phi_{B\setminus\alpha_1} + \hbar^{n+1}\chi_{(B;\alpha)})_{F_{(D_2;\alpha_2)}} \\ &= \prod_{D"} \Phi_{D"} \cdot (\Phi_{D_2})_{F_{(D_2;\alpha_2)}} + \hbar^{n+1}\chi_{(B;\alpha_1)} \\ &= \prod_{D"} \Phi_{D"} \cdot (\Phi_{D_2\setminus\alpha_2} + \hbar^{n+1}\chi_{(D_2;\alpha_2)}) + \hbar^{n+1}\chi_{(B;\alpha_1)} \\ &= \Phi_{B\setminus\{\alpha_1,\alpha_2\}} + \hbar^{n+1}\left(\chi_{(B;\alpha_1)} + \chi_{(D_2;\alpha_2)})\right)\end{aligned}$$

where the product in the third equality ranges over the connected components D" of $B \setminus \alpha_1$ not containing α_2. Permuting α_1 and α_2, we get mod $\hbar^{n+2}$

$$(\Phi_B)_{F_{(D_1;\alpha_1)}\cdot F_{(B;\alpha_2)}} = \Phi_{B\setminus\{\alpha_1,\alpha_2\}} + \hbar^{n+1}\left(\chi_{(B;\alpha_2)} + \chi_{(D_1;\alpha_1)})\right) \tag{7.5}$$

By (7.2) however,

$$\begin{aligned}F_{(D_1;\alpha_1)} \cdot F_{(B;\alpha_2)} = (\Phi_{(B;\alpha_2,\alpha_1)})^{\otimes 2} \cdot F_{(D_2;\alpha_2)} \cdot F_{(B;\alpha_1)} \cdot \Delta(\Phi^{-1}_{(B;\alpha_2,\alpha_1)}) \\ + \hbar^{n+1}\eta_{(B;\alpha_2,\alpha_1)} \quad \text{mod } \hbar^{n+2}\end{aligned}$$

Since for any $\Phi \in (A_B^{\otimes 3})^B$, $F \in A_B^{\otimes 2}$ and $a \in A_B$,

$$(\Phi)_{a^{\otimes 2}\cdot F\cdot\Delta(a)^{-1}} = a^{\otimes 3} \cdot (\Phi)_F \cdot (a^{\otimes 3})^{-1}$$

the left–hand side of (7.5) is also equal mod $\hbar^{n+2}$ to

$$\begin{aligned}(\Phi_B)_{F_{(D_1;\alpha_1)}\cdot F_{(B;\alpha_2)}} &= \mathrm{Ad}(\Phi^{\otimes 3}_{(B;\alpha_2,\alpha_1)})(\Phi_{B\setminus\{\alpha_1,\alpha_2\}} + \hbar^{n+1}(\chi_{(B;\alpha_1)} + \chi_{(D_2;\alpha_2)})) \\ &\quad + \hbar^{n+1}d_H\eta_{(B;\alpha_2,\alpha_1)} \\ &= \Phi_{B\setminus\{\alpha_1,\alpha_2\}} + \hbar^{n+1}(\chi_{(B;\alpha_1)} + \chi_{(D_2;\alpha_2)} + d_H\eta_{(B;\alpha_2,\alpha_1)})\end{aligned}$$

where the last equality follows from the fact that $\Phi_{(B;\alpha_2,\alpha_1)}$ centralises $A_{B\setminus\{\alpha_1,\alpha_2\}}$ and the fact that $\Phi_{B\setminus\{\alpha_1,\alpha_2\}} \in A^{\otimes 3}_{B\setminus\{\alpha_1,\alpha_2\}}$. Comparing the two expressions for $(\Phi_B)_{F_{(D_1;\alpha_1)}\cdot F_{(B;\alpha_2)}}$ yields

$$d_{DD}\chi_{(B;\alpha_2,\alpha_1)} = \chi_{(B;\alpha_1)} - \chi_{(D_1,\alpha_1)} - \chi_{(B;\alpha_2)} + \chi_{(D_2,\alpha_2)} = d_H\eta_{(B;\alpha_2,\alpha_1)}$$

as claimed ■

7.6. For any algebra homomorphism $\widetilde{\Delta} : A \to A^{\otimes 2}$ and element $\Phi \in A^{\otimes 3}$, set

$$\mathrm{Pent}_{\widetilde{\Delta}}(\Phi) = \mathrm{id}^{\otimes 2} \otimes\widetilde{\Delta}(\Phi) \cdot \widetilde{\Delta} \otimes \mathrm{id}^{\otimes 2}(\Phi) - 1 \otimes \Phi \cdot \mathrm{id} \otimes\widetilde{\Delta} \otimes \mathrm{id}(\Phi) \cdot 1 \otimes \Phi$$

Let now $B \subseteq D$ be a connected subdiagram and define $\theta_B \in A^{\otimes 3}$ by

$$\hbar^{n+1}\Theta_B = \mathrm{Pent}_\Delta(\Phi_B) \quad \text{mod } \hbar^{n+2} \tag{7.6}$$

Lemma 7.5.

(i) *θ is a Dynkin 0–cocycle with values in $A^{\otimes 4}$.*

(ii) $d_{DD}\theta = d_H\eta$.
(iii) $d_H\theta = 0$.

PROOF. (i) we must prove that $\theta_B \in (A_B^{\otimes 4})^B$ which readily follows from the fact that $\Phi_B \in (A_B^{\otimes 3})^B$.
(ii) Let $\alpha \in B \subseteq D$. One readily checks that

$$\begin{aligned}\mathrm{Pent}_{\Delta_{F_{(B;\alpha)}}}((\Phi_B)_{F_{(B;\alpha)}}) = &\, 1^{\otimes 2} \otimes F_{(B;\alpha)} \cdot 1 \otimes \mathrm{id} \otimes \Delta(F_{(B;\alpha)}) \cdot \mathrm{id} \otimes \Delta^{(3)}(F_{(B;\alpha)}) \\ &\cdot \mathrm{Pent}_\Delta(\Phi_B) \\ &\cdot \Delta^{(3)} \otimes \mathrm{id}(F^{-1}_{(B;\alpha)}) \cdot \Delta \otimes \mathrm{id}(F^{-1}_{(B;\alpha)}) \otimes 1 \cdot F^{-1}_{(B;\alpha)} \otimes 1^{\otimes 2}\end{aligned}$$

By (7.6), the right–hand side of the above equation is equal to $\hbar^{n+1}\theta_B$ mod $\hbar^{n+2}$. On the other hand, by (7.4), the left–hand side is equal to

$$\begin{aligned}\mathrm{Pent}_{\Delta_{F_{(B;\alpha)}}}(\Phi_{B\setminus\alpha} + \hbar^{n+1}\chi_{(B;\alpha)}) &= \mathrm{Pent}_{\Delta_{F_{(B;\alpha)}}}(\Phi_{B\setminus\alpha}) + \hbar^{n+1} d_H\chi_{(B;\alpha)} \\ &= \mathrm{Pent}_\Delta(\Phi_{B\setminus\alpha}) + \hbar^{n+1} d_H\chi_{(B;\alpha)} \\ &= \hbar^{n+1}(\theta_{B\setminus\alpha} + d_H\chi_{(B;\alpha)})\end{aligned}$$

where the second equality follows from the fact that $\Delta_{F_{(B;\alpha)}}$ restricts to Δ on $A_{B\setminus\alpha}$. Equating these two expressions, we therefore get

$$d_{DD}\theta_{(B;\alpha)} = \theta_B - \theta_{B\setminus\alpha} = d_H\eta_{(B;\alpha)}$$

(iii) It is known that an obstruction defined by (7.6) satisfies $d_H\theta = 0$ [Dr3, pp. 1448–9]. In the case at hand, a simpler proof can be given owing to the fact that by remark 6.6, Φ_B is a non–abelian Hochschild coboundary. Let $B \subseteq D$ and $\alpha_1, \ldots, \alpha_k$ an enumeration of the vertices of B. Then

$$\begin{aligned}\theta_B &= \sum_{i=0}^{k-1}(\theta_{B\setminus\{\alpha_1,\ldots,\alpha_i\}} - \theta_{B\setminus\{\alpha_1,\ldots,\alpha_{i+1}\}}) \\ &= \sum_{i=0}^{k-1} d_{DD}\theta_{(B\setminus\{\alpha_1,\ldots,\alpha_i\};\alpha_{i+1})} \\ &= \sum_{i=0}^{k-1} d_H\eta_{(B\setminus\{\alpha_1,\ldots,\alpha_i\};\alpha_{i+1})}\end{aligned}$$

so that θ_B is a Hochschild coboundary ■

7.7. Let now

$$\phi_{(D;\alpha,\beta)} \in A_B^{B\setminus\{\alpha,\beta\}} \qquad f_{(B;\alpha)} \in (A_B^{\otimes 2})^{B\setminus\alpha} \qquad \text{and} \qquad \psi_B \in (A_B^{\otimes 3})^B$$

with $\varepsilon\otimes\mathrm{id}(f_{(B;\alpha)})=\mathrm{id}\otimes\varepsilon(f_{(B;\alpha)})=0$. The cocycle $(\xi',\eta',\chi',\theta')$ corresponding to

$$\Phi_{(B;\alpha,\beta)}+\hbar^{n+1}\phi_{(B;\alpha,\beta)}\qquad F_{(B;\alpha)}+\hbar^{n+1}f_{(B;\alpha)}\qquad\text{and}\qquad \Phi_B+\hbar^{n+1}\psi_B$$

is given by

$$\begin{aligned}\xi'&=\xi+d_{DD}\phi\\ \eta'&=\eta+d_{DD}f-d_H\phi\\ \chi'&=\chi+d_{DD}\psi+d_Hf\\ \theta'&=\theta+d_H\psi\end{aligned}$$

so that the given quasi–Coxeter quasibialgebra structure lifts mod $\hbar^{n+2}$ if, and only if, (ξ,η,χ,θ) is a Dynkin–Hochschild coboundary. This concludes the proof of (i).

7.8. Let now $(\{S_i\},\{\Phi^j_{(B;\alpha,\beta)}\},\{F^j_{(B;\alpha)}\},\{\Phi_B\})$, with $j=1,2$, be two quasi–Coxeter quasibialgebra structures on $A[\![\hbar]\!]/\hbar^{n+1}A[\![\hbar]\!]$ which are trivial mod $\hbar$ and agree mod $\hbar^n$. Define $\phi_{(B;\alpha,\beta)},f_{(B;\alpha)}$ and ψ_B by the following equalities mod $\hbar^{n+1}$

$$\begin{aligned}\Phi^2_{(B;\alpha,\beta)}&=\Phi^1_{(B;\alpha,\beta)}+\hbar^n\phi_{(B;\alpha,\beta)}\\ F^2_{(B;\alpha)}&=F^1_{(B;\alpha)}+\hbar^nf_{(B;\alpha)}\\ \Phi^2_B&=\Phi^1_B+\hbar^n\psi_B\end{aligned}$$

Then, linearlising the defining identities of a quasi–Coxeter quasibialgebra readily yields that (ϕ,f,ψ) is a Dynkin–Hochschild 3–cocycle. It is easy to check that (ϕ,f,ψ) is a coboundary if, and only if the two structures differ by a twist equal to 1 mod $\hbar^n$ ■

Part III. Quantum Weyl groups

8. $U_\hbar\mathfrak{g}$ as a quasi–Coxeter quasitriangular quasibialgebra

Let $\mathfrak{g}$ be a complex, simple Lie algebra and let $D_\mathfrak{g}$ be its Dynkin diagram. We point out in 8.1–8.2 that the quantum group $U_\hbar\mathfrak{g}$, when endowed with the quantum Weyl group operators and the collection of universal R–matrices corresponding to all subdiagrams of $D_\mathfrak{g}$, has the structure of a quasi–Coxeter quasitriangular quasibialgebra of type $D_\mathfrak{g}$ with trivial associators and relative twists. We then prove that this structure may be transferred to one on $U\mathfrak{g}[\![\hbar]\!]$. This requires the cohomological construction of non–trivial associators and structural twists and is similar in spirit to the fact that $U_\hbar\mathfrak{g}$ is twist equivalent to a quasitriangular quasibialgebra of the form $(U\mathfrak{g}[\![\hbar]\!],\Delta_0,\exp(\hbar\Omega),\Phi)$ where Δ_0 is the cocommutative coproduct on $U\mathfrak{g}$, $\Omega\in\mathfrak{g}\otimes\mathfrak{g}$ the Casimir operator of $\mathfrak{g}$ and Φ some associator. The proof is somewhat lengthier however and occupies the rest of this section.

8.1. Retain the notation of §4.1.3 and regard the quantum group $U_\hbar\mathfrak{g}$ as a topological Hopf algebra over the ring of formal power series $\mathbb{C}[\![\hbar]\!]$ by endowing it with the coproduct given by

$$\Delta(E_i) = E_i \otimes 1 + q_i^{H_i} \otimes E_i$$
$$\Delta(F_i) = F_i \otimes q_i^{-H_i} + 1 \otimes F_i$$
$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i$$

For any subdiagram $D \subseteq D_\mathfrak{g}$, the operators E_i, F_i, H_i, with i such that $\alpha_i \in D$, topologically generate a subalgebra $U_\hbar\mathfrak{g}_D \subseteq U_\hbar\mathfrak{g}$ canonically isomorphic to the quantum group corresponding to $\mathfrak{g}_D$ and the restriction of the bilinear form $(\cdot,\cdot)$ to it. Let

$$R_{D,\hbar} \in 1^{\otimes 2} + \hbar U_\hbar\mathfrak{g}_D^{\otimes 2}$$

be the universal R–matrix of $U_\hbar\mathfrak{g}_D$ [Dr1, Dr2]. For $D = \alpha_i$, we denote $U_\hbar\mathfrak{g}_D$ by $U_\hbar\mathfrak{sl}_2^i$ and $R_{D,\hbar}$ by $R_i^\hbar$.

For any $\alpha_i \in D_\mathfrak{g}$, let $\overline{S}_i^\hbar \in \widehat{U_\hbar\mathfrak{sl}_2^i}$ be the corresponding quantum Weyl group element defined in §4.1.3. The following result is due to Lusztig and, independently to Kirillov–Reshetikhin and Soibelman [Lu, KR, So]

Proposition 8.1.

(i) *The following holds in* $\widehat{U_\hbar\mathfrak{sl}_2^i}$

$$(S_i^\hbar)^2 = \exp(i\pi H_i) \cdot q^{-C_i}$$

where $\exp(i\pi H_i)$ *and* C_i *are the sign and Casimir operators of* $U_\hbar\mathfrak{sl}_2^i$*, that is the central elements of* $\widehat{U_\hbar\mathfrak{sl}_2^i}$ *acting on the indecomposable representation* $\mathcal{V}_m$ *of dimension* $m+1$ *as multiplication by* $(-1)^m$ *and* $\frac{(\alpha_i,\alpha_i)}{2} \cdot \frac{m(m+2)}{2}$ *respectively.*

(ii) *The following holds in* $\widehat{U_\hbar\mathfrak{sl}_2^i}^{\otimes 2}$,

$$\Delta(S_i^\hbar) = R_i^{\hbar\,-1} \cdot S_i^\hbar \otimes S_i^\hbar$$

PROOF. By [Lu, §5.2.2], $(\overline{S}_i^\hbar)^2$ acts on the subspace of $\mathcal{V}_m$ of weight $\overline{\jmath} = m - 2j$, $j = 0 \dots m$, as multiplication by

$$(-1)^m q_i^{-2j(m-j)-m} = (-1)^m q_i^{-\frac{1}{2}(m-\overline{\jmath})(m+\overline{\jmath})-m} = (-1)^m q_i^{-\frac{m(m+2)}{2} - \frac{\overline{\jmath}}{2}^2}$$

so that $(\overline{S}_i^\hbar)^2 = \exp(i\pi H_i) q^{-C_i} q_i^{H_i^2/2}$. Since $\mathrm{Ad}(S_i^\hbar)(H_i) = -H_i$, (i) holds. (ii) readily follows from the fact that $\overline{S}_i^\hbar$ satisfies, by propositions

5.3.4 and 5.2.3 of [Lu],

$$\Delta(\overline{S}_i^\hbar) = \overline{R}_i^{\hbar^{-1}} \cdot \overline{S}_i^\hbar \otimes \overline{S}_i^\hbar$$

where $\overline{R}_i^\hbar = q_i^{-\frac{H_i \otimes H_i}{2}} \cdot R_i^\hbar$ ■

8.2. As a corollary of proposition 8.1 and [Dr1, §13], we get

Proposition 8.2. *For any $\alpha_i \neq \alpha_j \in D \subseteq D_\mathfrak{g}$, set*

$$\Phi_{(D;\alpha_i,\alpha_j)} = 1, \qquad F_{(D;\alpha_i)} = 1^{\otimes 2} \qquad \text{and} \qquad \Phi_D = 1^{\otimes 3}$$

Then,

$$\left(U_\hbar\mathfrak{g}, \{U_\hbar\mathfrak{g}_D\}, \{S_i^\hbar\}, \{\Phi_{(D;\alpha_i,\alpha_j)}\}, \Delta, \{R_{D,\hbar}\}, \{F_{(D;\alpha_i)}\}, \{\Phi_D\}\right)$$

is a quasi–Coxeter quasitriangular quasibialgebra of type $D_\mathfrak{g}$. The corresponding braid group representations are the quantum Weyl group representations of $B_\mathfrak{g}$ on finite–dimensional $U_\hbar\mathfrak{g}$–modules.

8.3. For any $\alpha_i \in D_\mathfrak{g}$, choose root vectors $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}$ such that $[e_i, f_i] = h_i$. Then, the assignement

$$E_i \to e_i, \quad F_i \to f_i, \quad H_i \to h_i$$

extends uniquely to an isomorphism of Hopf algebras

$$\Psi_0 : U_\hbar\mathfrak{g}/\hbar U_\hbar\mathfrak{g} \longrightarrow U\mathfrak{g}$$

We shall say that a $\mathbb{C}[\![\hbar]\!]$–linear map $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ is equal to the identity mod $\hbar$ if its reduction mod $\hbar$ is equal to Ψ_0. Since finite–dimensional $\mathfrak{g}$–modules do not possess non–trivial deformations, the canonical map $\widehat{U\mathfrak{g}}[\![\hbar]\!] \to \widehat{U\mathfrak{g}[\![\hbar]\!]}$ is an isomorphism. Any algebra isomorphism $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ therefore extends to an isomorphism

$$\widehat{U_\hbar\mathfrak{g}} \longrightarrow \widehat{U\mathfrak{g}[\![\hbar]\!]} = \widehat{U\mathfrak{g}}[\![\hbar]\!]$$

which we denote by the same symbol.

8.4. Extend the bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}$ to a non–degenerate, symmetric, bilinear, ad–invariant form on $\mathfrak{g}$. For any subdiagram $D \subseteq D_\mathfrak{g}$, let $\mathfrak{g}_D \subseteq \mathfrak{l}_D \subseteq \mathfrak{g}$ be the corresponding simple and Levi subalgebras. Denote by

$$\Omega_D = x_a \otimes x^a, \quad C_D = x_a \cdot x^a \quad \text{and} \quad r_{\mathfrak{g}_D} = \sum_{\substack{\alpha \succ 0: \\ \mathrm{supp}(\alpha) \subseteq D}} \frac{(\alpha,\alpha)}{2} \cdot e_\alpha \wedge f_\alpha$$

where $\{x_a\}_a, \{x^a\}_a$ are dual basis of $\mathfrak{g}_D$ with respect to $(\cdot,\cdot)$, the corresponding invariant tensor, Casimir operator and standard solution of the modified classical Yang–Baxter equation (MCYBE) for $\mathfrak{g}_D$ respectively. Abbreviate $\mathfrak{sl}_2^{\alpha_i}$, Ω_{α_i} and C_{α_i} to $\mathfrak{sl}_2^i$, Ω_i and C_i respectively and let $\widetilde{s}_i$ be the triple exponentials (4.8).

Label $D_\mathfrak{g}$ by attaching to each pair $\alpha_i \neq \alpha_j$ the order m_{ij} of the product $s_i s_j \in W$ of the corresponding simple reflections. The aim of this section is to prove the following

Theorem 8.3. *$U_\hbar\mathfrak{g}$ is equivalent to a quasi–Coxeter quasitriangular quasibialgebra of the form*

$$\Big(U\mathfrak{g}[\![\hbar]\!], \{U\mathfrak{g}_D[\![\hbar]\!]\}, \{S_{i,C}\}, \{\Phi_{(D;\alpha_i,\alpha_j)}\}, \Delta_0, \{\Phi_D\}, \{R^{\text{KZ}}_D\}, \{F_{(D;\alpha_i)}\}\Big)$$

where Δ_0 is the cocommutative coproduct on $U\mathfrak{g}$,

$$\begin{aligned} S_{i,C} &= \widetilde{s}_i \cdot \exp(-\hbar/2 \cdot C_i) \\ \Phi_D &= 1^{\otimes 3} \mod \hbar^2 \\ R^{\text{KZ}}_D &= \exp(\hbar \cdot \Omega_D) \\ \operatorname{Alt}_2 F_{(D;\alpha_i)} &= \hbar \cdot (r_{\mathfrak{g}_D} - r_{\mathfrak{g}_{D\setminus\{\alpha_i\}}}) \mod \hbar^2 \end{aligned}$$

and $\Phi_{(D;\alpha_i,\alpha_j)}$, $F_{(D;\alpha_i)}$ are of weight 0.

Proof. We begin by constructing in §8.5–§8.9 two families $\{\Psi_{(D;\mathcal{F})}\}$ and $\{\Phi_{(D;\mathcal{F},\mathcal{G})}\}$ labelled by connected subdiagrams $D \subseteq D_\mathfrak{g}$ and (elementary pairs of) maximal nested sets on D satisfying the following properties

(i)$_D$ For any maximal nested set $\mathcal{F}$ on D,

$$\Psi_{(D;\mathcal{F})} : U_\hbar\mathfrak{g}_D \longrightarrow U\mathfrak{g}_D[\![\hbar]\!]$$

is an algebra isomorphism equal to the identity mod $\hbar$ and restricting to the identity on $\mathfrak{h}_D$. Moreover, for any $B \in \mathcal{F}$, $\Psi_{(D;\mathcal{F})}$ restricts to $\Psi_{(B;\mathcal{F}_B)}$ on $U_\hbar\mathfrak{g}_B$. Lastly, for any $\alpha_i \in D_\mathfrak{g}$,

$$\Psi_{(\alpha_i;\alpha_i)}(S_i^\hbar) = S_{i,C}$$

(ii)$_D$ For any elementary pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D, the associator

$$\Phi_{(D;\mathcal{G},\mathcal{F})} \in 1 + \hbar U\mathfrak{g}_{D_1}[\![\hbar]\!]^{\mathfrak{l}_{D_2}}$$

where $D_1 = \operatorname{supp}(\mathcal{G},\mathcal{F}) \supset D_2 = \mathfrak{z}\operatorname{supp}(\mathcal{G},\mathcal{F})$, satisfies

$$\Psi_{(D;\mathcal{G})} = \operatorname{Ad}(\Phi_{(D;\mathcal{G},\mathcal{F})}) \circ \Psi_{(D;\mathcal{F})} \tag{8.1}$$

and $\Phi_{(D;\mathcal{F},\mathcal{G})} = \Phi^{-1}_{(D;\mathcal{G},\mathcal{F})}$. Moreover, if $\alpha^D_\mathcal{G} = \alpha^D_\mathcal{F} = \alpha_i$, then

$$\Phi_{(D;\mathcal{G},\mathcal{F})} = \Phi_{(D\setminus\alpha_i;\mathcal{G}\setminus D,\mathcal{F}\setminus D)} \tag{8.2}$$

(iii)$_D$ For any pair of maximal nested sets $\mathcal{F},\mathcal{G}$ on D and elementary sequences

$$\mathcal{F}=\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_l=\mathcal{G}\quad\text{and}\quad\mathcal{F}=\mathcal{K}_1,\mathcal{K}_2,\ldots,\mathcal{K}_m=\mathcal{G}$$

one has

$$\Phi_{(D;\mathcal{H}_1,\mathcal{H}_2)}\cdots\Phi_{(D;\mathcal{H}_{l-1},\mathcal{H}_l)}=\Phi_{(D;\mathcal{K}_1,\mathcal{K}_2)}\cdots\Phi_{(D;\mathcal{K}_{m-1},\mathcal{K}_m)}$$

(iv)$_D$ For any equivalent elementary pairs of maximal nested sets $(\mathcal{G},\mathcal{F})$ and $(\mathcal{G}',\mathcal{F}')$ on D, one has

$$\Phi_{(D;\mathcal{G},\mathcal{F})}=\Phi_{(D;\mathcal{G}',\mathcal{F}')}$$

Here and in the sequel, we follow the convention that the isomorphisms and associators corresponding to non–connected diagrams are the product of those corresponding to their connected components. Specifically, let $\alpha_i\in D$ and let $D_1,\ldots,D_k$ be the connected components of $D\setminus\alpha_i$, so that

$$U_\hbar\mathfrak{g}_{D\setminus\alpha_i}\cong U_\hbar\mathfrak{g}_{D_1}\otimes\cdots\otimes U_\hbar\mathfrak{g}_{D_k}\quad\text{and}\quad U\mathfrak{g}_{D\setminus\alpha_i}[\![\hbar]\!]\cong U\mathfrak{g}_{D_1}[\![\hbar]\!]\otimes\cdots\otimes U\mathfrak{g}_{D_k}[\![\hbar]\!]$$

If $\mathcal{F}$ is a maximal nested set on D with $\alpha^D_{\mathcal{F}}=\alpha_i$, so that

$$\mathcal{F}=\{D\}\sqcup\mathcal{F}_1\sqcup\cdots\mathcal{F}_k$$

where $\mathcal{F}_i$ is a maximal nested set on D_i, we set

$$\Psi_{(D\setminus\alpha_i;\mathcal{F}\setminus D)}=\Psi_{(D_1;\mathcal{F}_1)}\otimes\cdots\otimes\Psi_{(D_k;\mathcal{F}_k)}$$

If $\mathcal{G}$ is another maximal nested set on D with $\alpha^D_{\mathcal{G}}=\alpha_i$, so that

$$\mathcal{G}=\{D\}\sqcup\mathcal{G}_1\sqcup\cdots\sqcup\mathcal{G}_k$$

with $\mathcal{G}_i$ a maximal nested set on D_i, we set

$$\Phi_{(D\setminus\alpha_i;\mathcal{G}\setminus D;\mathcal{F}\setminus D)}=\Phi_{(D_1;\mathcal{G}_1,\mathcal{F}_1)}\otimes\cdots\otimes\Psi_{(D_k;\mathcal{G}_k,\mathcal{F}_k)}$$

Once constructed, the associators $\Phi_{(D_\mathfrak{g};\mathcal{G},\mathcal{F})}$ endow $U\mathfrak{g}[\![\hbar]\!]$ with the structure of a quasi–Coxeter algebra $\mathcal{Q}$ which is equivalent, via the isomorphisms $\Psi_{(D_\mathfrak{g};\mathcal{F})}$, to the quasi–Coxeter structure on $U_\hbar\mathfrak{g}$ determined by the quantum Weyl group operators $S_i^\hbar$. A suitable collection of associators Φ_D and structural twists $F_{(D;\alpha_i)}$ promoting $\mathcal{Q}$ to a quasi–Coxeter quasitriangular quasibialgebra structure on $U\mathfrak{g}[\![\hbar]\!]$ equivalent to that on $U_\hbar\mathfrak{g}$ will then be constructed in §8.10–§8.16.

8.5. We first construct, for any $\alpha_i \in D_{\mathfrak{g}}$, an algebra isomorphism

$$\Psi_{(\alpha_i;\alpha_i)} : U_\hbar \mathfrak{sl}_2^i \longrightarrow U\mathfrak{sl}_2^i[\![\hbar]\!]$$

equal to the identity mod $\hbar$ and mapping H_i to h_i and $S_i^\hbar$ to $S_{i,C}$.

Lemma 8.4. *Let A be a complete, topological algebra over $\mathbb{C}[\![\hbar]\!]$ and $a, b \in A$ two invertible elements such that*

$$a = b \mod \hbar \qquad \textit{and} \qquad a^2 = b^2$$

Then

$$b = gag^{-1} \qquad \textit{where} \qquad g = (ba^{-1})^{1/2} \in 1 + \hbar A$$

Proof. Let $\delta = ba^{-1} \in 1 + \hbar A$ so that $b = \delta a$. Then $b^2 = a^2$ implies that $\delta a \delta = a$ and therefore that $F(\delta) a = aF(\delta^{-1})$ for any formal power series F. In particular $\delta^{1/2} a = a\delta^{-1/2}$ so that

$$b = \delta^{1/2}\delta^{1/2} a = \delta^{1/2} a \delta^{-1/2}$$

as claimed ■

Let

$$\Psi_i : U_\hbar \mathfrak{sl}_2^i \longrightarrow U\mathfrak{sl}_2^i[\![\hbar]\!]$$

be an algebra isomorphism equal to the identity mod $\hbar$ and mapping H_i to h_i [Dr2, Prop. 4.3] and set $S_i = \Psi_i(S_i^\hbar)$. Then, $S_i = \widetilde{s}_i$ mod $\hbar$ and, by proposition 8.1,

$$S_i^2 = \Psi_i({S_i^\hbar}^2) = \Psi_i(\exp(i\pi H_i) \cdot q^{-C_i}) = \exp(i\pi h_i) \cdot q^{-C_i} = S_{i,C}^2$$

Thus, by lemma 8.4,

$$\Psi_{(\alpha_i;\alpha_i)} = \mathrm{Ad}(S_{i,C} \cdot S_i^{-1})^{1/2} \circ \Psi_i$$

maps $S_i^\hbar$ to $S_{i,C}$ and H_i to h_i since

$$\mathrm{Ad}(S_i) h_i = -h_i = \mathrm{Ad}(S_{i,C}) h_i$$

8.6. Assume that, for some $1 \le m \le |D_{\mathfrak{g}}|-1$, the isomorphisms $\Psi_{(D;\mathcal{F})}$ and associators $\Phi_{(D;\mathcal{F},\mathcal{G})}$ have been constructed for all D with $|D| \le m$ in such a way that properties (i)$_D$–(iv)$_D$ hold. We now construct isomorphisms $\Psi_{(D;\mathcal{F})}$ for all D with $|D| = m+1$ which satisfy (i)$_D$. We shall need the following.

Proposition 8.5. *Let $D \subseteq D_{\mathfrak{g}}$ be a subdiagram. Then, for any algebra isomorphism*

$$\Psi_D : U_\hbar \mathfrak{g}_D \longrightarrow U\mathfrak{g}_D[\![\hbar]\!]$$

equal to the identity mod $\hbar$, there exists an algebra isomorphism

$$\Psi : U_\hbar \mathfrak{g} \longrightarrow U\mathfrak{g}[\![\hbar]\!]$$

equal to the identity mod $\hbar$ and restricting to Ψ_D on $U_\hbar\mathfrak{g}_D$. If Ψ_D restricts to the identity on $\mathfrak{h}_D$, then Ψ may be chosen such that $\Psi|_\mathfrak{h} =$ id.

Proof. Let $\widetilde{\Psi} : U_\hbar\mathfrak{g} \longrightarrow U\mathfrak{g}[\![\hbar]\!]$ be an algebra isomorphism equal to the identity mod $\hbar$. Set

$$\tau = \widetilde{\Psi} \circ \Psi_D^{-1} : U\mathfrak{g}_D \longrightarrow U\mathfrak{g}[\![\hbar]\!]$$

so that $\tau = \mathrm{id} + \hbar\tau_1 \mod \hbar^2$ for some linear map $\tau_1 : U\mathfrak{g}_D \to U\mathfrak{g}$. Since τ is an algebra homomorphism, we readily find that, for any $x, y \in \mathfrak{g}_D$,

$$\tau_1([x,y]) = [x, \tau_1(y)] + [\tau_1(x), y] = \mathrm{ad}(x)\tau_1(y) - \mathrm{ad}(y)\tau_1(x)$$

so that the restriction of τ_1 to $\mathfrak{g}_D$ is a 1–cocycle with values in $U\mathfrak{g}$ endowed with the adjoint action of $\mathfrak{g}_D$. Since $H^1(\mathfrak{g}_D, U\mathfrak{g}) = 0$, there exists $a_1 \in U\mathfrak{g}$ such that

$$\tau_1(x) = \mathrm{ad}(x)a_1 = -[a_1, x]$$

for any $x \in \mathfrak{g}_D$. It follows that

$$\mathrm{Ad}(1 + \hbar a_1) \circ \widetilde{\Psi} \circ \Psi_D^{-1} = \mathrm{id} + \hbar^2\tau_2 \mod \hbar^3$$

for some linear map $\tau_2 : U\mathfrak{g}_D \to U\mathfrak{g}$. Continuing in this way, we find a sequence of elements $a_n \in U\mathfrak{g}$, $n \geq 2$ such that

$$\mathrm{Ad}(1 + \hbar^n a_n) \circ \cdots \circ \mathrm{Ad}(1 + \hbar a_1) \circ \widetilde{\Psi} \circ \Psi_D^{-1} = \mathrm{id} \mod \hbar^{n+1}$$

so that, setting

$$a = \lim_{n\to\infty} (1 + \hbar^n a_n) \cdots (1 + \hbar a_1) \in 1 + \hbar U\mathfrak{g}[\![\hbar]\!]$$

and

$$\Psi = \mathrm{Ad}(a) \circ \widetilde{\Psi} : U_\hbar\mathfrak{g} \longrightarrow U\mathfrak{g}[\![\hbar]\!]$$

we find that Ψ is an algebra isomorphism equal to the identity mod $\hbar$ and extending Ψ_D. If $\Psi_D|_{\mathfrak{h}_D} = \mathrm{id}$, and $\widetilde{\Psi}$ is chosen such that $\widetilde{\Psi}_{|\mathfrak{h}} = \mathrm{id}$ [Dr2, prop. 4.3], the obstructions τ_i constructed above are readily seen to be equivariant for the adjoint actions of $\mathfrak{h}$ on $U\mathfrak{g}_D$ and $U\mathfrak{g}$ so that the a_n, $n \geq 2$ may be chosen of weight 0 thus implying that $\Psi|_\mathfrak{h} = \mathrm{id}$ ■

Let now $D \subseteq D_\mathfrak{g}$ be a connected subdiagram with $|D| = m + 1$. For any $\alpha_i \in D$, choose a reference maximal nested set $\mathcal{F}_i$ on D such that $\alpha^D_{\mathcal{F}_i} = \alpha_i$ and, using proposition 8.5, an algebra isomorphism

$$\Psi_{(D;\mathcal{F}_i)} : U_\hbar\mathfrak{g}_D \longrightarrow U\mathfrak{g}_D[\![\hbar]\!]$$

such that

$$\Psi_{(D;\mathcal{F}_i)} = \mathrm{id} \mod \hbar, \qquad \Psi_{(D;\mathcal{F}_i)}\big|_{\mathfrak{h}_D} = \mathrm{id}$$

and

$$\Psi_{(D;\mathcal{F}_i)}\Big|_{U_\hbar\mathfrak{g}_{D\setminus\alpha_i}} = \Psi_{(D\setminus\alpha_i;\mathcal{F}_i\setminus D)}$$

For any maximal nested set $\mathcal{F}$ on D with $\alpha^D_\mathcal{F} = \alpha_i$, set

$$\Psi_{(D;\mathcal{F})} = \mathrm{Ad}(\Phi_{(D\setminus\alpha_i;\mathcal{F}\setminus D,\mathcal{F}_i\setminus D)}) \circ \Psi_{(D;\mathcal{F}_i)} \tag{8.3}$$

We claim that $\Psi_{(D;\mathcal{F})}$ satisfies (i)$_D$. Since $\Phi_{(D\setminus\alpha_i;\mathcal{F}\setminus D,\mathcal{F}_i\setminus D)}$ is of weight 0, this amount to showing that the restriction of $\Psi_{(D;\mathcal{F})}$ to $U_\hbar\mathfrak{g}_{D\setminus\alpha_i}$ is equal to $\Psi_{(D\setminus\alpha_i;\mathcal{F}\setminus D)}$. By construction, this restriction is equal to

$$\mathrm{Ad}(\Phi_{(D\setminus\alpha_i;\mathcal{F}\setminus D,\mathcal{F}_i\setminus D)}) \circ \Psi_{(D\setminus\alpha_i;\mathcal{F}_i\setminus D)}$$

which, by (ii)$_{D\setminus\alpha_i}$, is equal to $\Psi_{(D\setminus\alpha_i;\mathcal{F}\setminus D)}$.

8.7. We next construct associators $\Phi_{(D;\mathcal{G},\mathcal{F})}$ and prove that they satisfy property (ii)$_D$. For any $\alpha_i,\alpha_j \in D$, the automorphism $\tau_{ji} = \Psi_{(D;\mathcal{F}_j)} \circ \Psi^{-1}_{(D;\mathcal{F}_i)}$ of $U\mathfrak{g}[\![\hbar]\!]$ is equal to the identity mod $\hbar$ and fixes $\mathfrak{h}_D$. Since $H^1(\mathfrak{g}_D, U\mathfrak{g}_D) = 0$, τ_{ji} is inner and there exists an element

$$\Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)} \in 1 + \hbar U\mathfrak{g}_D[\![\hbar]\!]^{\mathfrak{h}_D}$$

such that

$$\Psi_{(D;\mathcal{F}_j)} = \mathrm{Ad}(\Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)}) \circ \Psi_{(D;\mathcal{F}_i)} \tag{8.4}$$

We choose the associators $\Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)}$ in such a way that

$$\Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)} = \Phi^{-1}_{(D;\mathcal{F}_i,\mathcal{F}_j)} \quad \text{and} \quad \Phi_{(D;\mathcal{F}_i,\mathcal{F}_i)} = 1$$

For any pair $\mathcal{F},\mathcal{G}$ of maximal nested sets on D with $\alpha^D_\mathcal{F} = \alpha_i$ and $\alpha^D_\mathcal{G} = \alpha_j$, set

$$\Phi_{(D;\mathcal{G};\mathcal{F})} = \Phi_{(D\setminus\alpha_j;\mathcal{G}\setminus D,\mathcal{F}_j\setminus D)} \cdot \Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)} \cdot \Phi_{(D\setminus\alpha_i;\mathcal{F}_i\setminus D,\mathcal{F}\setminus D)} \tag{8.5}$$

so that, by (iii)$_{D\setminus\alpha_i}$,

$$\Phi_{(D;\mathcal{G};\mathcal{F})} = \Phi_{(D\setminus\alpha_i,\mathcal{G}\setminus D,\mathcal{F}\setminus D)} \tag{8.6}$$

whenever $\alpha^D_\mathcal{G} = \alpha_i = \alpha^D_\mathcal{F}$. We claim that these associators satisfy (ii)$_D$. For any $\mathcal{F},\mathcal{G}$ with $\alpha^D_\mathcal{F} = \alpha_i$, $\alpha^D_\mathcal{G} = \alpha_j$, we have,

$$\begin{aligned}
\Psi_{(D;\mathcal{G})} &= \mathrm{Ad}(\Phi_{(D;\mathcal{G};\mathcal{F}_j)}) \circ \Psi_{(D;\mathcal{F}_j)} \\
&= \mathrm{Ad}(\Phi_{(D;\mathcal{G};\mathcal{F}_j)} \cdot \Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)}) \circ \Psi_{(D;\mathcal{F}_i)} \\
&= \mathrm{Ad}(\Phi_{(D;\mathcal{G};\mathcal{F}_j)} \cdot \Phi_{(D;\mathcal{F}_j,\mathcal{F}_i)} \cdot \Phi_{(D;\mathcal{F}_i,\mathcal{F})}) \circ \Psi_{(D;\mathcal{F})} \\
&= \mathrm{Ad}(\Phi_{(D;\mathcal{G};\mathcal{F})}) \circ \Psi_{(D;\mathcal{F})}
\end{aligned} \tag{8.7}$$

where the first and third equalities follow from (8.3), the second from (8.4) and the last one from (8.5). Set now

$$D_1 = \mathrm{supp}(\mathcal{F},\mathcal{G}) \qquad \text{and} \qquad D_2 = \mathfrak{z}\mathrm{supp}(\mathcal{F},\mathcal{G})$$

We claim that $\Phi_{(D;\mathcal{G},\mathcal{F})}$ lies in $U\mathfrak{g}_{D_1}[\![\hbar]\!]$ and is invariant under $\mathfrak{g}_{D_2}$. It suffices to show that $\Phi_{(D;\mathcal{G},\mathcal{F})} \in U\mathfrak{g}_{D_1}[\![\hbar]\!]$ since, by (8.7) and the fact that, by (i)$_D$, $\Psi_{(D,\mathcal{G})}$ and $\Psi_{(D,\mathcal{F})}$ have the same restriction on $U\mathfrak{g}_{D_2}$, $\Phi_{(D;\mathcal{G},\mathcal{F})}$ centralises $U\mathfrak{g}_{D_2}$. Now if $\alpha^D_{\mathcal{G}} = \alpha^D_{\mathcal{F}}$, then

$$\operatorname{supp}(\mathcal{G},\mathcal{F}) = \operatorname{supp}(\mathcal{G}\setminus D, \mathcal{F}\setminus D) \tag{8.8}$$

and the claim follows from the inductive assumption and (8.6). If, on the other hand $\alpha^D_{\mathcal{G}} \neq \alpha^D_{\mathcal{F}}$, then $\operatorname{supp}(\mathcal{G},\mathcal{F}) = D$ and there is nothing to prove.

8.8. We now modify the associators $\Phi_{(D;\mathcal{G},\mathcal{F})}$ so that they also satisfy (iii)$_D$. Introduce to this end some terminology. Call a fundamental nested set $\mathcal{H}$ on D *old* (resp. *new*) if

$$|D \setminus \bigcup_{B\in\mathcal{H}\setminus D} B| = 1 \quad (\text{resp. } \geq 2)$$

For example, if $(\mathcal{G},\mathcal{F})$ is an elementary pair of maximal nested sets on D, then $\mathcal{H} = \mathcal{G}\cap\mathcal{F}$ is old precisely when $\alpha^D_{\mathcal{G}} = \alpha^D_{\mathcal{F}}$ and therefore when the associator $\Phi_{(D;\mathcal{G},\mathcal{F})}$ is inductively determined by (8.2). If, on the other hand, $\mathcal{H}$ is new then $\operatorname{supp}(\mathcal{G},\mathcal{F}) = D$ and $\Phi_{(D;\mathcal{G},\mathcal{F})}$ is determined by (8.1) only up to multiplication by an element of

$$\zeta_{(D;\mathcal{G},\mathcal{F})} \in 1 + \hbar\cdot Z(U\mathfrak{g}_D)[\![\hbar]\!]$$

Our goal is to modify these new associators by suitable elements $\zeta_{(D;\mathcal{G},\mathcal{F})}$ while keeping the old ones fixed, in such a way that the generalised pentagon identities corresponding to the two–faces of the associahedron $\mathcal{A}_D$ hold. Note first the following straightforward

Lemma 8.6. *The collection of faces $\mathcal{A}^{\text{old}}_D \subseteq \mathcal{A}_D$ corresponding to old maximal nested sets is a subcomplex of $\mathcal{A}_D$. For any $\alpha_i \in D$, let $D^i_1,\ldots,D^i_{k_i}$ be the connected components of $D\setminus\alpha_i$ and set*

$$\mathcal{A}_{D\setminus\alpha_i} = \mathcal{A}_{D_1}\times\cdots\times\mathcal{A}_{D_{k_i}}$$

Then, the map $\mathcal{A}_{D\setminus\alpha_i} \longrightarrow \mathcal{A}^{\text{old}}_D$ given by

$$(\mathcal{H}_1,\ldots,\mathcal{H}_{k_i}) \longrightarrow \{D\}\sqcup\mathcal{H}_1\sqcup\cdots\sqcup\mathcal{H}_{k_i}$$

yields an isomorphism

$$\mathcal{A}^{\text{old}}_D \cong \bigsqcup_{\alpha_i\in D} \mathcal{A}_{D\setminus\alpha_i}$$

Let now Σ be an oriented two–face of $\mathcal{A}_D$ with vertices

$$\mathcal{F}_1,\ldots,\mathcal{F}_k,\mathcal{F}_{k+1} = \mathcal{F}_1$$

listed in their order of appearence along the boundary of Σ. Thus, for each $i = 1 \dots k$, $(\mathcal{F}_{i+1}, \mathcal{F}_i)$ is an elementary pair of maximal nested sets on D and we may set

$$\zeta(\Sigma) = \Phi_{(D;\mathcal{F}_{k+1},\mathcal{F}_k)} \cdots \Phi_{(D;\mathcal{F}_2,\mathcal{F}_1)}$$

Proposition 8.7.

(i) *The element* $\zeta(\Sigma)$ *lies in* $\mathcal{Z}_D = 1 + \hbar Z(U\mathfrak{g}_D)[\![\hbar]\!]$, *only depends upon the orientation of* Σ *and satisfies* $\zeta(-\Sigma) = \zeta(\Sigma)^{-1}$.

(ii) *The assignement* $\Sigma \to \zeta(\Sigma)$ *defines a 2-cocycle on* $\mathcal{A}_D$ *relative to the subcomplex* $\mathcal{A}_D^{\text{old}}$ *with coefficients in the abelian group* $\mathcal{Z}_D$.

Proof. (i) By $(\text{ii})_D$,

$$\begin{aligned}\operatorname{Ad}(\zeta(\Sigma)) &= \operatorname{Ad}(\Phi_{(D;\mathcal{F}_{k+1},\mathcal{F}_k)} \cdots \Phi_{(D;\mathcal{F}_2,\mathcal{F}_1)}) \\ &= \Psi_{(D;\mathcal{F}_{k+1})} \circ \Psi^{-1}_{(D,\mathcal{F}_k)} \circ \cdots \circ \Psi_{(D;\mathcal{F}_2)} \circ \Psi^{-1}_{(D,\mathcal{F}_1)} \\ &= \text{id}\end{aligned}$$

since $\mathcal{F}_{k+1} = \mathcal{F}_1$, so that $\zeta(\Sigma) \in \mathcal{Z}_D$ as claimed. It follows that $\zeta(\Sigma)$ only depends upon the orientation of Σ since $\Phi_{(D;\mathcal{F}_2,\mathcal{F}_1)}$ commutes with $\zeta(\Sigma)$ and we therefore have

$$\Phi_{(D;\mathcal{F}_{k+1},\mathcal{F}_k)} \cdots \Phi_{(D;\mathcal{F}_2,\mathcal{F}_1)} = \Phi_{(D;\mathcal{F}_2,\mathcal{F}_1)} \cdot \Phi_{(D;\mathcal{F}_{k+1},\mathcal{F}_k)} \cdots \Phi_{(D;\mathcal{F}_3,\mathcal{F}_2)}$$

(ii)[1] We claim that $\zeta(\Sigma)$ is of the form $\widetilde{\zeta}(\partial\Sigma)$ where $\widetilde{\zeta}$ is a homomorphism mapping one–chains in $\mathcal{A}_D$ to $\mathcal{Z}_D$, so that $d\zeta = \widetilde{\zeta} \circ \partial \circ \partial = 0$. Note first that we may attach an element $z(p) \in \mathcal{Z}_D$ to any closed edge–path in $\mathcal{A}_D$ *i.e.*, a sequence $p = f_n, \dots, f_1$ of oriented one–faces in $\mathcal{A}_D$ such that the end point of f_i is the start point of f_{i+1}, with $f_{n+1} = f_1$, by setting

$$z(p) = \Phi_{(D;f_1,f_n)} \cdots \Phi_{(D;f_2,f_1)}$$

Fix now a maximal nested set $\mathcal{F}_0$ on D and, for each maximal nested set $\mathcal{F}$ on D, an edge–path $p_\mathcal{F}$ from $\mathcal{F}_0$ to $\mathcal{F}$. For any oriented 1–face $e = (\mathcal{G}, \mathcal{F})$ of $\mathcal{A}_D$, set

$$\widetilde{\zeta}(e) = z(p_\mathcal{G}^{-1} \vee e \vee p_\mathcal{F})$$

where $p_\mathcal{G}^{-1}$ is the edge–path from $\mathcal{G}$ to $\mathcal{F}_0$ obtained by reversing the orientation of $p_\mathcal{G}$ and $\vee$ is the concatenation. It is clear that $\zeta(\Sigma) = \widetilde{\zeta}(\partial\Sigma)$ so that ζ is a two–cocycle on $\mathcal{A}_D$ which, by the inductive assumption is equal to 1 on the 2–faces of $\mathcal{A}_D^{\text{old}}$ ■

[1] I owe this proof to G. Skandalis

Since $\mathcal{A}_D$ and $\mathcal{A}_D^{\mathrm{old}}$ are contractible, $H^2(\mathcal{A}_D, \mathcal{A}_D^{\mathrm{old}}; \mathcal{Z}_D) = 1$. Thus, there exists a 1–cochain ξ on $\mathcal{A}_D$ such that

$$d\xi = \zeta \qquad \text{and} \qquad \xi(\mathcal{G}, \mathcal{F}) = 1 \tag{8.9}$$

whenever $(\mathcal{G}, \mathcal{F})$ is an elementary pair of maximal nested sets on D such that $\mathrm{supp}(\mathcal{G}, \mathcal{F}) \subsetneq D$. Replacing each $\Phi_{(D;\mathcal{G},\mathcal{F})}$ by $\Phi_{(D;\mathcal{G},\mathcal{F})} \cdot \xi(\mathcal{G}, \mathcal{F})^{-1}$ yields a collection of associators satisfying (ii)$_D$ and (iii)$_D$.

8.9. We now show that the associators $\Phi_{(D;\mathcal{G},\mathcal{F})}$ satisfy property (iv)$_D$. Let $(\mathcal{G}, \mathcal{F})$ and $(\mathcal{G}', \mathcal{F}')$ be two equivalent elementary pairs of maximal nested sets on D. If $\mathrm{supp}(\mathcal{G}, \mathcal{F}) \subsetneq D$, then

$$\alpha_{\mathcal{G}}^D = \alpha_{\mathcal{F}}^D = \alpha_i = \alpha_{\mathcal{F}'}^D = \alpha_{\mathcal{G}'}^D$$

for some $\alpha_i \in D$ and, by (ii)$_{D\setminus\alpha_i}$ and (iv)$_{D\setminus\alpha_i}$,

$$\Phi_{(D;\mathcal{G},\mathcal{F})} = \Phi_{(D\setminus\alpha_i;\mathcal{G}\setminus D,\mathcal{F}\setminus D)} = \Phi_{(D\setminus\alpha_i;\mathcal{G}'\setminus D,\mathcal{F}'\setminus D)} = \Phi_{(D;\mathcal{G}',\mathcal{F}')}$$

Assume now that $\mathrm{supp}(\mathcal{G}, \mathcal{F}) = D = \mathrm{supp}(\mathcal{G}', \mathcal{F}')$ and set

$$\alpha_i = \alpha_{\mathcal{F}}^D = \alpha_{\mathcal{F}'}^D \qquad \text{and} \qquad \alpha_j = \alpha_{\mathcal{G}}^D = \alpha_{\mathcal{G}'}^D$$

Lemma 8.8. *There exist two sequences*

$$\mathcal{F} = \mathcal{F}_1, \ldots, \mathcal{F}_m = \mathcal{F}' \qquad \text{and} \qquad \mathcal{G} = \mathcal{G}_1, \ldots, \mathcal{G}_m = \mathcal{G}'$$

of maximal nested sets on D such that, for any $i = 1 \ldots m-1$, the following holds

(i) *$(\mathcal{F}_i, \mathcal{F}_{i+1})$ and $(\mathcal{G}_i, \mathcal{G}_{i+1})$ are equivalent elementary pairs of maximal nested sets on D such that*

$$\mathrm{supp}(\mathcal{F}_i, \mathcal{F}_{i+1}) = \mathrm{supp}(\mathcal{G}_i, \mathcal{G}_{i+1}) \subseteq D \setminus \{\alpha_i, \alpha_j\}$$

(ii) *$(\mathcal{G}_i, \mathcal{F}_i)$ and $(\mathcal{G}_{i+1}, \mathcal{F}_{i+1})$ are equivalent elementary pairs of maximal nested sets on D.*

Proof. Let $D_1, \ldots, D_p$ be the connected components of $D \setminus \{\alpha_i, \alpha_j\}$, so that

$$\mathcal{F} = \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_p \sqcup \{\complement_{\alpha_j}^{D\setminus\alpha_i}\} \qquad \mathcal{F}' = \mathcal{H}'_1 \sqcup \cdots \sqcup \mathcal{H}'_p \sqcup \{\complement_{\alpha_j}^{D\setminus\alpha_i}\}$$
$$\mathcal{G} = \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_p \sqcup \{\complement_{\alpha_i}^{D\setminus\alpha_j}\} \qquad \mathcal{G}' = \mathcal{H}'_1 \sqcup \cdots \sqcup \mathcal{H}'_p \sqcup \{\complement_{\alpha_i}^{D\setminus\alpha_j}\}$$

where $\mathcal{H}_k, \mathcal{H}'_k$ are maximal nested sets on D_k. By connectedness of $\mathcal{A}_{D_k}$, there exists an elementary sequence $\mathcal{H}_k = \mathcal{H}_k^1, \ldots, \mathcal{H}_k^{m_k} = \mathcal{H}'_k$ of maximal nested sets on D_k. Setting $m = m_1 + \cdots + m_k$ and

$$\mathcal{F}_i = \mathcal{H}'_1 \sqcup \cdots \mathcal{H}'_{k-1} \sqcup \mathcal{H}_k^{i-m_1-\cdots-m_k-1} \sqcup \mathcal{H}_{k+1} \sqcup \cdots \mathcal{H}_p \sqcup \{\complement_{\alpha_j}^{D\setminus\alpha_i}\}$$
$$\mathcal{G}_i = \mathcal{H}'_1 \sqcup \cdots \mathcal{H}'_{k-1} \sqcup \mathcal{H}_k^{i-m_1-\cdots-m_k-1} \sqcup \mathcal{H}_{k+1} \sqcup \cdots \mathcal{H}_p \sqcup \{\complement_{\alpha_i}^{D\setminus\alpha_j}\}$$

for any $m_1 + \cdots + m_{k-1} + 1 \leq i \leq m_1 + \cdots + m_k$ yields the required sequences ■

By (iii)$_D$,

$$\begin{aligned}\Phi_{(D;\mathcal{G},\mathcal{F})} \cdot \Phi_{(D;\mathcal{F}_1,\mathcal{F}_2)} \cdots \Phi_{(D;\mathcal{F}_{m-1},\mathcal{F}_m)} &= \Phi_{(D;\mathcal{G},\mathcal{G}_2)} \cdots \Phi_{(\mathcal{G}_{m-1},\mathcal{G}_m)} \cdot \Phi_{(D;\mathcal{G}',\mathcal{F}')} \\ &= \Phi_{(D;\mathcal{G}',\mathcal{F}')} \cdot \Phi_{(D;\mathcal{G}_1,\mathcal{G}_2)} \cdots \Phi_{(G_{m-1},\mathcal{G}_m)} \\ &= \Phi_{(D;\mathcal{G}',\mathcal{F}')} \cdot \Phi_{(D;\mathcal{F}_1,\mathcal{F}_2)} \cdots \Phi_{(F_{m-1},\mathcal{F}_m)}\end{aligned}$$

where the second equality follows from the fact that $\Phi_{(D;\mathcal{G},\mathcal{F})}$ commutes with

$$U\mathfrak{g}_{\mathfrak{z}\mathrm{supp}(\mathcal{G},\mathcal{F})}[\![\hbar]\!] \ni \Phi_{(D;G_i,G_{i+1})}$$

and the last one from (iv)$_{D\setminus\{\alpha_i,\alpha_j\}}$ and the fact that $(\mathcal{G}_i, \mathcal{G}_{i+1})$ and $(\mathcal{F}_i, \mathcal{F}_{i+1})$ are equivalent pairs.

8.10. We next graft on to the previously constructed isomorphisms $\Psi_{(D;\mathcal{F})}$ and associators $\Phi_{(D;\mathcal{G},\mathcal{F})}$ a collection $\{F_{(D;\alpha_i)}\}$ of relative twists such that, for any connected subdiagram $D \subseteq D_\mathfrak{g}$, the following properties hold

(v)$_D$ For any $\alpha_i \in D$,

$$F_{(D;\alpha_i)} \in 1^{\otimes 2} + \hbar(U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{l}_{D\setminus\alpha_i}}$$

(vi)$_D$ For any maximal nested set $\mathcal{F}$ on D,

$$\Psi^{\otimes 2}_{(D;\mathcal{F})} \circ \Delta \circ \Psi^{-1}_{(D;\mathcal{F})} = \mathrm{Ad}(F_{(D;\mathcal{F})}) \circ \Delta_0 \tag{8.10}$$

where, as customary

$$F_{(D;\mathcal{F})} = \overrightarrow{\prod_{B\in\mathcal{F}}} F_{(B;\alpha^B_\mathcal{F})}$$

(vii)$_D$ For any pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on D

$$F_{(D;\mathcal{G})} = \Phi^{\otimes 2}_{(D;\mathcal{G},\mathcal{F})} \cdot F_{(D;\mathcal{F})} \cdot \Delta_0(\Phi_{(D;\mathcal{F},\mathcal{G})})$$

8.11. Assume that, for some $0 \leq m \leq |D_\mathfrak{g}| - 1$, the relative twists $F_{(D;\alpha_i)}$ have been constructed for all D with $|D| \leq m$ in such a way that properties (v)$_D$–(vii)$_D$ hold. Let $D \subseteq D_\mathfrak{g}$ be a connected subdiagram such that $|D| = m+1$. For any maximal nested set $\mathcal{F}$ on D, denote by

$$\Delta_{(D;\mathcal{F})} : U\mathfrak{g}_D[\![\hbar]\!] \longrightarrow U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!]$$

the algebra homomorphism defined by the left–hand side of (8.10). Note that if $\mathcal{F}, \mathcal{G}$ are maximal nested sets on D, property (ii)$_D$ of §8.4 implies that

$$\Delta_{(D;\mathcal{G})} = \mathrm{Ad}(\Phi^{\otimes 2}_{(D;\mathcal{G},\mathcal{F})}) \circ \Delta_{(D;\mathcal{F})} \circ \mathrm{Ad}(\Phi_{(D;\mathcal{F},\mathcal{G})}) \tag{8.11}$$

Fix $\alpha_i \in D$ and a maximal nested set $\mathcal{F}_i$ on D such that $\alpha^D_{\mathcal{F}_i} = \alpha_i$. Since $\Delta_{(D;\mathcal{F}_i)} = \Delta_0 \bmod \hbar$ and $H^1(\mathfrak{g}_D, U\mathfrak{g}_D^{\otimes 2}) = 0$, where $U\mathfrak{g}_D$ is regarded as a $\mathfrak{g}_D$–module under the adjoint action, there exists a twist

$$F_i \in 1^{\otimes 2} + \hbar U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!]$$

such that

$$\Delta_{(D;\mathcal{F}_i)} = \mathrm{Ad}(F_i) \circ \Delta_0 \tag{8.12}$$

This implies in particular that F_i is invariant under $\mathfrak{h}_D$ since $\Delta_{(D;\mathcal{F}_i)}$ and Δ_0 coincide on $\mathfrak{h}_D$.

For any $\alpha_i \neq \alpha_j \in D$, choose a maximal nested set $\mathcal{F}_j$ on D such that $\alpha^D_{\mathcal{F}_j} = \alpha_j$ and set

$$F_j = \Phi^{\otimes 2}_{(D;\mathcal{F}_j,\mathcal{F}_i)} \cdot F_i \cdot \Delta_0(\Phi_{(D;\mathcal{F}_i,\mathcal{F}_j)}) \in 1^{\otimes 2} + \hbar(U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{h}_D} \tag{8.13}$$

It follows from (8.12) and (8.11) that, for any $\alpha_j \in D$,

$$\Delta_{(D;\mathcal{F}_j)} = \mathrm{Ad}(F_j) \circ \Delta_0 \tag{8.14}$$

For any such α_j, set

$$F_{(D;\alpha_j)} = F^{-1}_{(D\setminus\alpha_j;\mathcal{F}_j\setminus D)} \cdot F_j \in 1^{\otimes 2} + \hbar(U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{h}_D}$$

We claim that these relative twists satisfy (v)$_D$–(vii)$_D$

Proposition 8.9.

(i) *$F_{(D;\alpha_j)}$ is invariant under $\mathfrak{l}_{D\setminus\alpha_j}$.*

(ii) *For any pair $(\mathcal{G},\mathcal{F})$ of maximal nested sets on D*

$$F_{(D;\mathcal{G})} = \Phi^{\otimes 2}_{(D;\mathcal{G},\mathcal{F})} \cdot F_{(D;\mathcal{F})} \cdot \Delta_0(\Phi_{(D;\mathcal{F},\mathcal{G})})$$

(iii) *For any maximal nested set $\mathcal{F}$ on D,*

$$\Delta_{(D;\mathcal{F})} = \mathrm{Ad}(F_{(D;\mathcal{F})}) \circ \Delta_0$$

Proof. (i) By (8.14),

$$\mathrm{Ad}(F_{(D;\alpha_j)}) \circ \Delta_0 = \mathrm{Ad}(F^{-1}_{(D\setminus\alpha_j;\mathcal{F}_j\setminus D)}) \circ \Delta_{(D;\mathcal{F}_j)}$$

By (i)$_D$ and (vi)$_{D\setminus\alpha_j}$, the right–hand side restricts to Δ_0 on $U\mathfrak{g}_{D\setminus\alpha_j}$. This implies the invariance of $F_{(D;\alpha_j)}$ under $\mathfrak{g}_{D\setminus\alpha_j}$. (ii) The stated identity certainly holds if $\mathcal{F} = \mathcal{F}_i$ and $\mathcal{G} = \mathcal{F}_j$ for some $\alpha_j \in D$ since in that case $F_{(D;\mathcal{F})} = F_i$ and $F_{(D;\mathcal{G})} = F_j$ is given by (8.13). By transitivity of the associators, it therefore suffices to check it when $\alpha^D_{\mathcal{F}} = \alpha_j = \alpha^D_{\mathcal{G}}$ for some $\alpha_j \in D$. In that case,

$$\begin{aligned}
F_{(D;\mathcal{G})} &= F_{(D\setminus\alpha_j;\mathcal{G}\setminus D)} \cdot F_{(D;\alpha_j)} \\
&= \Phi^{\otimes 2}_{(D\setminus\alpha_j;\mathcal{G}\setminus D,\mathcal{F}\setminus D)} \cdot F_{(D\setminus\alpha_j;\mathcal{F}\setminus D)} \cdot \Delta_0(\Phi_{(D\setminus\alpha_j;\mathcal{F}\setminus D,\mathcal{G}\setminus D)}) \cdot F_{(D;\alpha_j)} \\
&= \Phi^{\otimes 2}_{(D\setminus\alpha_j;\mathcal{G}\setminus D,\mathcal{F}\setminus D)} \cdot F_{(D\setminus\alpha_j;\mathcal{F}\setminus D)} \cdot F_{(D;\alpha_j)} \cdot \Delta_0(\Phi_{(D\setminus\alpha_j;\mathcal{F}\setminus D,\mathcal{G}\setminus D)}) \\
&= \Phi^{\otimes 2}_{(D;\mathcal{G},\mathcal{F})} \cdot F_{(D;\mathcal{F})} \cdot \Delta_0(\Phi_{(D;\mathcal{F},\mathcal{G})})
\end{aligned}$$

where the second equality follows by $(\mathrm{vii})_{D\setminus\alpha_j}$, the third one by the invariance of $F_{(D;\alpha_j)}$ under $\mathfrak{g}_{D\setminus\alpha_j}$ and the last one from property $(\mathrm{ii})_D$ of §8.4. (iii) Let $\alpha_j = \alpha^D_{\mathcal{F}}$. By (8.14), the stated identity holds if $\mathcal{F} = \mathcal{F}_j$ since in that case

$$F_{(D;\mathcal{F})} = F_{(D\setminus\alpha_j;\mathcal{F}_j\setminus D)} \cdot F_{(D;\alpha_j)} = F_j$$

In the general case, we have, by (8.11) and (ii)

$$\begin{aligned}
\Delta_{(D;\mathcal{F})} &= \mathrm{Ad}(\Phi^{\otimes 2}_{(D;\mathcal{F},\mathcal{F}_j)}) \circ \Delta_{(D;\mathcal{F}_j)} \circ \mathrm{Ad}(\Phi_{(D;\mathcal{F}_j,\mathcal{F})}) \\
&= \mathrm{Ad}(\Phi^{\otimes 2}_{(D;\mathcal{F},\mathcal{F}_j)}) \circ \mathrm{Ad}(F_{(D;\mathcal{F}_j)}) \circ \Delta_0 \circ \mathrm{Ad}(\Phi_{(D;\mathcal{F}_j,\mathcal{F})}) \\
&= \mathrm{Ad}(F_{(D;\mathcal{F})}) \circ \mathrm{Ad}(\Delta_0(\Phi_{(D;\mathcal{F},\mathcal{F}_j)})) \circ \Delta_0 \circ \mathrm{Ad}(\Phi_{(D;\mathcal{F}_j,\mathcal{F})}) \\
&= \mathrm{Ad}(F_{(D;\mathcal{F})}) \circ \Delta_0
\end{aligned}$$

■

8.12. We now construct associators Φ_D and R–matrices R_D such that for any connected subdiagram $D \subseteq D_{\mathfrak{g}}$, the following holds

$(\mathrm{viii})_D$ $\Phi_D \in 1^{\otimes 3} + \hbar (U\mathfrak{g}_D^{\otimes 3}[\![\hbar]\!])^{\mathfrak{g}_D}$ satisfies the pentagon equations with respect to Δ_0 and, for any maximal nested set $\mathcal{F}$ on D,

$$(\Phi_D)_{F_{(D;\mathcal{F})}} = 1^{\otimes 3}$$

$(\mathrm{ix})_D$ $R_D \in 1^{\otimes 2} + \hbar (U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{g}_D}$ satisfies the hexagon equations with respect to Δ_0 and Φ_D and, for any maximal nested set $\mathcal{F}$ on D,

$$(R_D)_{F_{(D;\mathcal{F})}} = \Psi^{\otimes 2}_{(D;\mathcal{F})}(R_{D,\hbar})$$

For any maximal nested set $\mathcal{F}$ on D, set

$$\begin{aligned}
\Phi_{(D;\mathcal{F})} &= 1 \otimes F^{-1}_{(D;\mathcal{F})} \cdot \mathrm{id} \otimes \Delta_{(D;\mathcal{F})}(F^{-1}_{(D;\mathcal{F})}) \cdot \Delta_{(D;\mathcal{F})} \otimes \mathrm{id}(F_{(D;\mathcal{F})}) \cdot F_{(D;\mathcal{F})} \otimes 1 \\
&\in 1^{\otimes 3} + \hbar U\mathfrak{g}_D^{\otimes 3}[\![\hbar]\!]
\end{aligned}$$

and

$$\begin{aligned}
R_{(D;\mathcal{F})} &= (F^{-1}_{(D;\mathcal{F})})^{21} \cdot \Psi^{\otimes 2}_{(D;\mathcal{F})}(R_{D,\hbar}) \cdot F_{(D;\mathcal{F})} \\
&\in 1^{\otimes 2} + \hbar U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!]
\end{aligned}$$

so that

$$(U\mathfrak{g}_D[\![\hbar]\!],\Delta_0,\Phi_{(D;\mathcal{F})},R_{(D;\mathcal{F})}) = (U\mathfrak{g}_D[\![\hbar]\!],\Delta_{(D;\mathcal{F})},1^{\otimes 3},\Psi^{\otimes 2}_{(D;\mathcal{F})}(R_{D,\hbar}))_{F^{-1}_{(D;\mathcal{F})}}$$
$$\cong (U_\hbar\mathfrak{g}_D,\Delta,1^{\otimes 3},R_{D,\hbar})_{\Psi^{-1}_{(D;\mathcal{F})}{}^{\otimes 2}(F^{-1}_{(D;\mathcal{F})})}$$

is a quasitriangular quasibialgebra. In particular, $\Phi_{(D;\mathcal{F})}$ and $R_{(D;\mathcal{F})}$ satisfy the pentagon and hexagon equations with respect to Δ_0 and are invariant under $\mathfrak{g}_D$ since Δ_0 is coassociative and cocommutative. We claim that $\Phi_{(D;\mathcal{F})}$ and $R_{(D;F)}$ are independent of the choice of $\mathcal{F}$, so that $(\text{viii})_D$ and $(\text{ix})_D$ hold with $\Phi_D = \Phi_{(D;\mathcal{F})}$ and $R_D = R_{(D;\mathcal{F})}$ respectively. From $(\text{vii})_D$, one readily finds that

$$\Phi_{(D;\mathcal{G})} = \Delta_0^{(3)}(\Phi_{(D;\mathcal{G},\mathcal{F})})\cdot\Phi_{(D;\mathcal{F})}\cdot\Delta_0^{(3)}(\Phi_{(D;\mathcal{F},\mathcal{G})})$$
$$R_{(D;\mathcal{G})} = \Delta_0(\Phi_{(D;\mathcal{G},\mathcal{F})})\cdot R_{(D;\mathcal{F})}\cdot\Delta_0(\Phi_{(D;\mathcal{F},\mathcal{G})})$$

where

$$\Delta_0^{(3)} = \Delta_0\otimes\mathrm{id}\circ\Delta_0 = \mathrm{id}\otimes\Delta_0\circ\Delta_0 : U\mathfrak{g}_D \longrightarrow U\mathfrak{g}_D^{\otimes 3}$$

Thus $\Phi_{(D;\mathcal{G})} = \Phi_{(D;\mathcal{F})}$ and $R_{(D;\mathcal{G})} = R_{(D;\mathcal{F})}$ since $\Phi_{(D;\mathcal{F})}$ and $R_{(D;\mathcal{F})}$ are invariant under $\mathfrak{g}_D$.

8.13. **The coproduct identity.** Note that, for any $\alpha_i\in D_\mathfrak{g}$, we have

$$\begin{aligned}\mathrm{Ad}(F_{(\alpha_i;\alpha_i)})\Delta_0(S_{i,C}) &= \Delta_{(\alpha_i;\alpha_i)}(\Psi_{(\alpha_i;\alpha_i)}(S_i^\hbar))\\ &= \Psi^{\otimes 2}_{(\alpha_i;\alpha_i)}\left(R_i^{\hbar\,-1}\cdot S_i^\hbar\otimes S_i^\hbar\right)\\ &= (R_{\alpha_i})^{-1}_{F_{(\alpha_i;\alpha_i)}}\cdot S_{i,C}\otimes S_{i,C}\end{aligned}$$

Thus, the relative twists, associators and R–matrices constructed in §8.10–§8.12 endow $U\mathfrak{g}[\![\hbar]\!]$ with a quasi–Coxeter quasitriangular quasibialgebra structure $\mathcal{Q}$ extending the quasi–Coxeter algebra structure constructed in §8.5–§8.9 and isomorphic, via the isomorphisms $\Psi_{(D;\mathcal{F})}$, to the quasi–Coxeter quasitriangular quasibialgebra structure $\mathcal{Q}_\hbar$ on $U_\hbar\mathfrak{g}$. In the next two subsections, we apply suitable F–twists to $\mathcal{Q}$ which, while clearly preserving its equivalence to $\mathcal{Q}_\hbar$, bring the R–matrices and associators to the form required by the statement of theorem 8.3.

8.14. **Symmetrising the R-matrices R_D.** By proposition 3.16. of [Dr3], there exists, for each $D\subseteq D_\mathfrak{g}$, an invariant twist

$$F_D\in 1^{\otimes 2}+\hbar(U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{g}_D}$$

such that $(R_D)_{F_D} = R_D^{\mathrm{KZ}}$. Performing an F–twist of $\mathcal{Q}$ by the collection $\{F_D\}_{D\subset D_\mathfrak{g}}$, we obtain an equivalent structure for which $R_D = R_D^{\mathrm{KZ}}$.

8.15. **Normalising the associators Φ_D.** By lemma 9.2, there exists, for each $D \subseteq D_{\mathfrak{g}}$, a symmetric invariant twist

$$F_D \in 1^{\otimes 2} + \hbar (U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{g}_D}$$

such that $(\Phi_D)_{F_D} = 1^{\otimes 3} \mod \hbar^2$. Twisting by $\{F_D\}_{D \subseteq D_{\mathfrak{g}}}$ we may therefore assume that Φ_D is equal to $1^{\otimes 3} \mod \hbar^2$. This twist does not alter

$$R_D = R_D^{\mathrm{KZ}} = \Delta_0(e^{\hbar/2C_D}) \cdot e^{-\hbar/2C_D} \otimes e^{-\hbar/2C_D}$$

since F_D is invariant and symmetric.

8.16. **Computing the 1-jet of $F_{(D;\alpha_i)}$.** To complete the proof of theorem 8.3, we need to check that the relative twists satisfy

$$\mathrm{Alt}_2(F_{(D;\alpha_i)}) = \hbar \cdot (r_{\mathfrak{g}_D} - r_{\mathfrak{g}_{D\setminus\{\alpha_i\}}}) \mod \hbar^2$$

We shall need the following well–known

Lemma 8.10. *Let $\Psi : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ be an algebra isomorphism equal to the identity mod $\hbar$. Then, the following holds mod $\hbar^2$,*

$$\Psi^{\otimes 2} \circ (\Delta - \Delta^{21}) \circ \Psi^{-1} = 2\hbar \cdot \mathrm{ad}(r_{\mathfrak{g}}) \circ \Delta_0$$

PROOF. It is sufficient to show that both sides agree on the generators e_i, f_i, h_i of $\mathfrak{g}$. Set

$$\Delta_\Psi = \Psi^{\otimes 2} \circ \Delta \circ \Psi^{-1}$$

Then, modulo $\hbar^2$,

$$\begin{aligned}\Delta_\Psi(e_i) &= \Psi^{\otimes 2}(E_i \otimes 1 + q_i^{H_i} \otimes E_i + \hbar\Delta\varepsilon_i)\\ &= e_i \otimes 1 + \hbar\varepsilon_i \otimes 1 + \hbar\alpha_i \otimes e_i + 1 \otimes e_i + \hbar 1 \otimes \varepsilon' + \hbar\Psi^{\otimes 2} \circ \Delta(\varepsilon)\end{aligned}$$

where $\Psi^{-1}(e_i) = E_i + \hbar\varepsilon_i$, $\Psi(E_i) = e_i + \hbar\varepsilon_i'$ and $q_i^{H_i} = 1 + \hbar(\alpha_i, \alpha_i)/2H_i$ mod $\hbar^2$. Antisymmetring, and using the fact that $\Delta(x) - \Delta^{21}(x) \in \hbar U_\hbar\mathfrak{g}$ for any $x \in U_\hbar\mathfrak{g}$, we find that

$$\Delta_\Psi(e_i) - \Delta_\Psi^{21}(e_i) = \hbar \cdot (\alpha_i \otimes e_i - e_i \otimes \alpha_i) = 2\hbar \cdot \alpha_i \wedge e_i \mod \hbar^2$$

A similar calculation yields

$$\Delta_\Psi(f_i) - \Delta_\Psi^{21}(f_i) = 2\hbar \cdot \alpha_i \wedge f_i \mod \hbar^2$$

and

$$\Delta_\Psi(h_i) - \Delta_\Psi^{21}(h_i) = 0$$

Let now $\mathfrak{n}_{\alpha_i}^{\pm} \subset \mathfrak{n}^{\pm}$ be the span of the root vectors e_α (resp. f_α) with $\alpha \neq \alpha_i$. $\mathfrak{n}_{\alpha_i}^{\pm}$ are invariant under the adjoint action of $\mathfrak{sl}_2^i$ and the inner product $(\cdot,\cdot)$ yields an $\mathfrak{sl}_2^i$–equivariant identification $(\mathfrak{n}_{\alpha_i}^{+})^* \cong \mathfrak{n}_{\alpha_i}^{-}$. Since

$$r_{\mathfrak{g}} = \sum_{\alpha \succ 0} \frac{(\alpha,\alpha)}{2} \cdot e_\alpha \wedge f_\alpha = r_{D\setminus\alpha_i} + r_{\alpha_i}$$

where $r_{D\setminus\alpha_i} = r_{\mathfrak{g}} - r_{\alpha_i}$ is the image in $\bigwedge^2(\mathfrak{n}^+_{\alpha_i} \oplus \mathfrak{n}^-_{\alpha_i})$ of

$$\mathrm{id}_{\mathfrak{n}^+_{\alpha_i}} \in \mathrm{End}(\mathfrak{n}^+_{\alpha_i}) \cong \mathfrak{n}^+_{\alpha_i} \otimes \mathfrak{n}^-_{\alpha_i} \subset (\mathfrak{n}^+_{\alpha_i} \oplus \mathfrak{n}^-_{\alpha_i})$$

we find

$$[r_{\mathfrak{g}}, \Delta_0(e_i)] = [r_{\alpha_i}, \Delta_0(e_i)] = -\frac{(\alpha_i, \alpha_i)}{2} \cdot \mathrm{ad}(e_i)e_i \wedge f_i = \alpha_i \wedge e_i$$

and similarly

$$[r_{\mathfrak{g}}, \Delta_0(f_i)] = \alpha_i \wedge f_i$$

Since $r_{\mathfrak{g}}$ is of weight 0, $[r_{\mathfrak{g}}, \Delta_0(h_i)] =$ and the claim is proved ■

For any connected subdiagram $D \subseteq D_{\mathfrak{g}}$ and maximal nested set $\mathcal{F}$ on D, write

$$F_{(D;\mathcal{F})} = 1^{\otimes 2} + \hbar \cdot f_{(D;\mathcal{F})} \mod \hbar^2$$

where $f_{(D;\mathcal{F})} \in U\mathfrak{g}_D^{\otimes 2}$. Taking the coefficient of $\hbar$ in

$$(\Phi_D)_{F_{(D;\mathcal{F})}} = 1^{\otimes 3}$$

and using the fact that $\Phi_D = 1^{\otimes 3} \mod \hbar^2$, we find

$$d_H f_{(D;\mathcal{F})} = 1 \otimes f_{(D;\mathcal{F})} - \Delta_0 \otimes \mathrm{id}(f_{(D;\mathcal{F})}) + \mathrm{id} \otimes \Delta_0(f_{(D;\mathcal{F})}) - f_{(D;\mathcal{F})} \otimes 1 = 0$$

where $d_H : U\mathfrak{g}_D^{\otimes 2} \to U\mathfrak{g}_D^{\otimes 3}$ is the Hochschild differential. It follows that $\mathrm{Alt}_2(f_{(D;\mathcal{F})})$ lies in $\bigwedge^2 \mathfrak{g}_D$. On the other hand, using (vi)$_D$, we find that, mod $\hbar^2$,

$$\Psi^{\otimes 2}_{(D;\mathcal{F})} \circ (\Delta - \Delta^{21}) \circ \Psi_{(D;\mathcal{F})} = 2\hbar \cdot \mathrm{ad}(\mathrm{Alt}_2(f_{(D;\mathcal{F})})) \circ \Delta_0$$

By lemma 8.10, this implies that

$$\mathrm{Alt}_2(f_{(D;\mathcal{F})}) - r_{\mathfrak{g}_D} \in (\bigwedge^2 \mathfrak{g}_D)^{\mathfrak{g}_D} = 0$$

as required ■

9. Rigidity of $U\mathfrak{g}$

9.1. Retain the notation of section 8, particularly §8.4. Contrary to §8.4, label $D_{\mathfrak{g}}$ by attaching an infinite multiplicity to each edge.[1] The aim of this section is to prove the following

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[1]A quasi–Coxeter structure on $U\mathfrak{g}$ with respect to the usual labelling of $D_{\mathfrak{g}}$ is clearly also a quasi–Coxeter structure with respect to to its present infinite labelling. Surprisingly, the proof of the rigidity theorem 9.1 does not use the braid relations (1.22). This is why the result is stated in this slightly greater generality.

Theorem 9.1. *Up to twisting, there exists a unique quasi–Coxeter quasitriangular quasibialgebra structure of type $D_{\mathfrak{g}}$ on $U\mathfrak{g}[\![\hbar]\!]$ of the form*

$$\Big(U\mathfrak{g}[\![\hbar]\!], \{U\mathfrak{g}_D[\![\hbar]\!]\}, \{S_{i,C}\}, \{\Phi_{(D;\alpha_i,\alpha_j)}\}, \Delta, \{R_D\}, \{\Phi_D\}, \{F_{(D;\alpha_i)}\}\Big)$$

where Δ is the cocommutative coproduct on $U\mathfrak{g}$,

$$S_{i,C} = \widetilde{s}_i \cdot \exp(-\hbar/2 \cdot C_i), \tag{9.1}$$

$$R_D = \exp(\hbar \cdot \Omega_D), \tag{9.2}$$

$$\mathrm{Alt}_2\, F_{(D;\alpha_i)} = \hbar \cdot (r_{\mathfrak{g}_D} - r_{\mathfrak{g}_{D\setminus\{\alpha_i\}}}) \mod \hbar^2 \tag{9.3}$$

and $\Phi_{(D;\alpha_i,\alpha_j)}$, $F_{(D;\alpha_i)}$ are of weight 0.

Proof. Let

$$\mathcal{Q}^a = \Big(U\mathfrak{g}[\![\hbar]\!], \{U\mathfrak{g}_D[\![\hbar]\!]\}, \{S_{i,C}\}, \{\Phi^a_{(D;\alpha_i,\alpha_j)}\}, \Delta, \{R_D\}, \{\Phi^a_D\}, \{F^a_{(D;\alpha_i)}\}\Big)$$

$a = 1, 2$ be two quasi–Coxeter quasitriangular quasibialgebra structures of the above form. We proceed in four steps

9.2. **Normalising the 1–jets of Φ^a_D.** We claim first that, up to a suitable twist, we may assume that $\Phi^a_D = 1^{\otimes 3} \mod \hbar^2$ for any $D \subseteq D_{\mathfrak{g}}$ and $a = 1, 2$.

Lemma 9.2. *Let $\Phi \in 1^{\otimes 3} + \hbar (U\mathfrak{g}^{\otimes 3}[\![\hbar]\!])^{\mathfrak{g}}$ be a solution of the pentagon and hexagon equations with respect to $R = e^{\hbar\Omega}$. Then, there exists a symmetric, invariant twist*

$$F \in 1^{\otimes 2} + \hbar (U\mathfrak{g}^{\otimes 2}[\![\hbar]\!])^{\mathfrak{g}}$$

such that $(\Phi)_F = 1^{\otimes 3} \mod \hbar^2$.

Proof. Write

$$\Phi = 1^{\otimes 3} + \hbar\varphi \mod \hbar^2$$

where $\varphi \in (U\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}$. The pentagon equation for Φ implies that

$$d_H\varphi = 1 \otimes \varphi - \Delta \otimes \mathrm{id}^{\otimes 2}(\varphi) + \mathrm{id} \otimes \Delta \otimes \mathrm{id}(\varphi) - \mathrm{id}^{\otimes 2} \otimes \Delta(\varphi) + \varphi \otimes 1 = 0$$

where d_H is the Hochschild differential. By [Dr2, Prop. 3.5], $(\Phi)^{-1} = (\Phi)^{321}$. Substituting this into the second of the hexagon relations

$$\Delta \otimes \mathrm{id}(R) = \Phi^{312} \cdot R^{13} \cdot (\Phi^{132})^{-1} \cdot R^{23} \cdot \Phi^{123} \tag{9.4}$$

$$\mathrm{id} \otimes \Delta(R) = (\Phi^{231})^{-1} \cdot R^{13} \cdot \Phi^{213} \cdot R^{12} \cdot (\Phi^{123})^{-1} \tag{9.5}$$

substracting them, and taking the coefficient of $\hbar$ yields that $\mathrm{Alt}_3\,\varphi = 0$. Thus, $\varphi = d_H f$ where $f \in U\mathfrak{g}^{\otimes 2}$ may be chosen invariant under $\mathfrak{g}$. Since

$$d_H f^{21} = -(d_H f)^{321} = -\varphi^{321} = \varphi = d_H f$$

we may further assume, up to replacing f by $(f+f^{21})/2$, that f is symmetric . Setting $F = 1 - \hbar f$ yields the required twist ■

For any connected $D \subseteq D_{\mathfrak{g}}$, let $F^a_D \in 1^{\otimes 2} + \hbar (U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{g}_D}$ be a symmetric invariant twist such that $(\Phi^a_D)_{F^a_D} = 1^{\otimes 3} \mod \hbar^2$. Twisting $\mathcal{Q}^a$ by $F^a = \{F^a_D\}_{D \subseteq D_{\mathfrak{g}}}$ yields the claimed result. Note that

$$(R_D)_{F^a_D} = F^{a\,21}_D R_D F^{a\,-1}_D = R_D$$

since $R_D = \Delta(\exp(\hbar/2C_D)) \cdot \exp(-\hbar/2C_D)^{\otimes 2}$ and F^a_D is symmetric and invariant under $\mathfrak{g}_D$, and that

$$\mathrm{Alt}_2(F^a_{(D;\alpha_i)})_{F^a} = F^a_{D\setminus\{\alpha_i\}} \cdot \mathrm{Alt}_2(F^a_{(D;\alpha_i)}) \cdot F^{a\,-1}_D = \hbar \cdot (r_{\mathfrak{g}_D} - r_{\mathfrak{g}_{D\setminus\{\alpha_i\}}})$$

since F^a is symmetric. Thus, twisting $\mathcal{Q}^a$ by F^a preserves the conditions (9.1)–(9.3).

9.3. **Matching the associators Φ^a_D.** We claim next that, up a twist, we may assume that $\Phi^2_D = \Phi^1_D$ for all $D \subseteq D_{\mathfrak{g}}$. Indeed, for any such D there exists, by Drinfeld's uniqueness theorem [Dr3, prop. 3.12], a symmetric, invariant twist

$$F_D \in 1^{\otimes 2} + \hbar (U\mathfrak{g}_D^{\otimes 2}[\![\hbar]\!])^{\mathfrak{g}_D}$$

such that $(\Phi^2_D)_{F_D} = \Phi^1_D$. Twisting $\mathcal{Q}^2$ by $F = \{F_D\}_{D \subseteq D_{\mathfrak{g}}}$ yields the claimed equality of associators and, as in the previous step, preserves the conditions (9.1)–(9.3).

9.4. **Matching the twists $F^a_{(D;\alpha_i)}$.** We claim now that, up to a further twist which does not alter the associators $\Phi^1_D = \Phi^2_D$, we may assume that $F^2_{(D;\alpha_i)} = F^1_{(D;\alpha_i)}$ for any $\alpha_i \in D \subseteq D_{\mathfrak{g}}$. We need two preliminary results.

Lemma 9.3. *If $\Phi = 1 + \hbar^2\varphi + \cdots \in 1^{\otimes 3} + \hbar^2 (U\mathfrak{g}^{\otimes 3}[\![\hbar]\!])^{\mathfrak{g}}$ satisfies the pentagon and hexagon equations with respect to $R = e^{\hbar\Omega}$, then*

$$\mathrm{Alt}_3\,\varphi = \frac{1}{6}[\Omega_{12}, \Omega_{23}]$$

Proof. Since R is symmetric, proposition 3.5 of [Dr2] implies that $\Phi^{-1} = \Phi^{321}$. Substituting this into the second hexagon equation (9.5), substracting it from (9.4) and taking the coefficient of $\hbar^2$ shows that

$$\begin{aligned}
6\,\mathrm{Alt}_3(\varphi) &= \frac{1}{2}\left(\Delta\otimes\mathrm{id}(\Omega^2) - \mathrm{id}\otimes\Delta(\Omega^2) + \Omega_{12}^2 - \Omega_{23}^2\right) + \Omega_{13}(\Omega_{12}-\Omega_{23})\\
&= \frac{1}{2}\left(\Omega_{13}\Omega_{23} + \Omega_{23}\Omega_{13} - \Omega_{12}\Omega_{13} - \Omega_{13}\Omega_{12}\right) + \Omega_{13}(\Omega_{12}-\Omega_{23})\\
&= \frac{1}{2}\left(\Omega_{13}(\Omega_{12}+\Omega_{23}+\Omega_{13}) - \Omega_{13}(\Omega_{12}+\Omega_{13}) + \Omega_{23}\Omega_{13} - \Omega_{13}\Omega_{12}\right.\\
&\quad \left. - (\Omega_{12}+\Omega_{23}+\Omega_{13})\Omega_{13} + (\Omega_{23}+\Omega_{13})\Omega_{13}\right) + \Omega_{13}(\Omega_{12}-\Omega_{23})\\
&= [\Omega_{23},\Omega_{13}]\\
&= -[\Omega_{23},\Omega_{12}]
\end{aligned}$$

where the fourth equality uses the fact that $\Omega_{12}+\Omega_{23}+\Omega_{13}$ commutes with Ω_{ij} ■

Lemma 9.4. *Let $F\in 1+\hbar U\mathfrak{sl}_2^{i\,\otimes 2}[\![\hbar]\!]$ be a twist of weight zero and set*

$$S_{i,C} = \widetilde{s}_i\cdot\exp(-\hbar/2\cdot C_i),\qquad R_i = \exp(\hbar\cdot\Omega^i)$$

Then, the equation

$$\Delta_F(S_{i,C}) = (R_i)_F^{-1}\cdot S_{i,C}\otimes S_{i,C} \tag{9.6}$$

is equivalent to $F^\Theta = F^{21}$ where $\Theta\in\mathrm{Aut}(\mathfrak{sl}_2^i)$ is any involution such that $\Theta(h_{\alpha_i}) = -h_{\alpha_i}$.

Proof. Since $\Delta(C_i) = C_i\otimes 1 + 1\otimes C_i + 2\Omega^i$,

$$\begin{aligned}
\Delta_F(S_{i,C}) &= F\cdot\exp(-\hbar\Omega^i)\cdot S_{i,C}\otimes S_{i,C}\cdot F^{-1}\\
&= (R_i)_F^{-1}\cdot F^{21}\cdot S_{i,C}\otimes S_{i,C}\cdot F^{-1}
\end{aligned}$$

which is equal to the right–hand side of (9.6) if, and only if

$$\mathrm{Ad}(\widetilde{s}_i^{\otimes 2})F = F^{21}$$

The claim follows since $\Theta = \mathrm{Ad}(\widetilde{s}_i)\,\mathrm{Ad}(c\cdot h_{\alpha_i})$, for some $c\in\mathbb{C}$ so that Θ and $\mathrm{Ad}(\widetilde{s}_i^{\otimes 2})$ coincide on zero weight elements of $U\mathfrak{g}^{\otimes 2}$ ■

Fix now $\alpha_i\in D\subseteq D_\mathfrak{g}$ and, for $D' = D, D\setminus\{\alpha_i\}$, denote $\Phi^2_{D'} = \Phi^1_{D'}$ by $\Phi_{D'}$. Write

$$\Phi_{D'} = 1 + \hbar^2\varphi_{D'} \mod \hbar^3$$

By lemma 9.3,

$$\mathrm{Alt}_3\,\varphi_D = [\Omega_{12}^D,\Omega_{23}^D]\qquad\text{and}\qquad \mathrm{Alt}_3\,\varphi_{D\setminus\{\alpha_i\}} = [\Omega_{12}^{D\setminus\{\alpha_i\}},\Omega_{23}^{D\setminus\{\alpha_i\}}]$$

so that Φ_D and $\Phi_{D\setminus\alpha_i}$ are non–degenerate in the sense of definition 5.1 of [TL3] and $\overline{\pi}^3(\mathrm{Alt}_3\,\varphi_D) = \mathrm{Alt}_3\,\varphi_{D\setminus\alpha_i}$ where

$$\overline{\pi}^3 : (U\mathfrak{g}_D^{\otimes 3})^{\mathfrak{g}_D} \longrightarrow (U\mathfrak{g}_{D\setminus\{\alpha_i\}}^{\otimes 3})^{\mathfrak{g}_{D\setminus\{\alpha_i\}}}$$

is the generalised Harish–Chandra homomorphism defined in §2 of [TL3]. Since $(\Phi_D)_{F^a_{(D;\alpha_i)}} = \Phi_{D\setminus\{\alpha_i\}}$ for $a = 1,2$, there exists, by [TL3, thm. 6.1(iv)] a gauge transformation

$$a_{(D;\alpha_i)} \in 1 + \hbar U\mathfrak{g}_D[\![\hbar]\!]^{\mathfrak{l}_{D\setminus\alpha_i}}$$

such that

$$F^1_{(D;\alpha_i)} = a^{\otimes 2}_{(D;\alpha_i)} \cdot F^2_{(D;\alpha_i)} \cdot \Delta(a^{-1}_{(D;\alpha_i)})$$

Moreover, by lemma 9.4 and [TL3, thm. 6.1(iii)], we may assume that

$$\mathrm{Ad}(\widetilde{s}_i)a_{(\alpha_i;\alpha_i)} = a_{(\alpha_i;\alpha_i)} \tag{9.7}$$

for any $i = 1\ldots n$. Twisting $\mathcal{Q}^2$ by $a = \{a_{(D;\alpha_i)}\}_{\alpha_i\in D\subseteq D_{\mathfrak{g}}}$ yields the required equality of twists while preserving (9.1)–(9.3) since

$$(S_{i,C})_a = a_{(\alpha_i;\alpha_i)} \cdot \widetilde{s}_i \cdot \exp(-\hbar/2C_i)a^{-1}_{(\alpha_i;\alpha_i)} = S_{i,C}$$

by (9.7) and, mod $\hbar^2$,

$$\mathrm{Alt}_2((F^2_{(D;\alpha_i)})_a) = \mathrm{Alt}_2(F^2_{(D;\alpha_i)}) + \hbar\,\mathrm{Alt}_2\, d_H(a_{(\alpha_i;\alpha_i)_1}) = \mathrm{Alt}_2(F^2_{(D;\alpha_i)})$$

where $a_{(\alpha_i;\alpha_i)} = 1 + \hbar a_{(\alpha_i;\alpha_i)_1} \mod \hbar^2$.

9.5. **Matching the associators $\Phi^a_{(D;\alpha_i,\alpha_j)}$.** We may henceforth assume that

$$\Phi^2_D = \Phi^1_D \qquad \text{and} \qquad F^2_{(D;\alpha_i)} = F^1_{(D;\alpha_i)}$$

for any $\alpha_i \in D \subseteq D_{\mathfrak{g}}$ and that $\Phi^1_{(D;\alpha_i,\alpha_j)} = \Phi^2_{(D;\alpha_i,\alpha_j)} \mod \hbar^n$ for some $n \geq 1$ and all $\alpha_i \neq \alpha_j \in D \subseteq D_{\mathfrak{g}}$. Thus,

$$\Phi^2_{(D;\alpha_i,\alpha_j)} = \Phi^1_{(D;\alpha_i,\alpha_j)} + \hbar^n\varphi_{(D;\alpha_i,\alpha_j)} \mod \hbar^{n+1}$$

for some $\varphi_{(D;\alpha_i,\alpha_j)} \in U\mathfrak{g}_D^{\mathfrak{g}_{D\setminus\{\alpha_i,\alpha_j\}}}$. Let $\alpha_i \neq \alpha_j \in D \subseteq D_{\mathfrak{g}}$ and $\mathcal{F}, \mathcal{G}$ two fundamental maximal nested set such that

$$\mathcal{F}\setminus\mathcal{G} = \complement^{D\setminus\{\alpha_i\}}_{\alpha_j} \quad \text{and} \quad \mathcal{G}\setminus\mathcal{F} = \complement^{D\setminus\{\alpha_j\}}_{\alpha_i}$$

Substracting the equations

$$F_{\mathcal{F}} \cdot \Delta(\Phi^1_{(D;\alpha_i,\alpha_j)}) = {\Phi^2_{(D;\alpha_i,\alpha_j)}}^{\otimes 2} \cdot F_{\mathcal{G}} F_{\mathcal{F}} \cdot \Delta(\Phi^2_{(D;\alpha_i,\alpha_j)}) = {\Phi^2_{(D;\alpha_i,\alpha_j)}}^{\otimes 2} \cdot F_{\mathcal{G}}$$

where, as usual

$$F_{\mathcal{F}} = \overrightarrow{\prod}_{D'\in\mathcal{F}} F_{(D',\alpha^{D'}_{\mathcal{F}})}$$

and $F_{(D',\alpha')} = F^1_{(D',\alpha')} = F^2_{(D',\alpha')}$, and equating the coefficients of $\hbar^{n+1}$, we find

$$\Delta(\varphi_{(D;\alpha_i,\alpha_j)}) - \varphi_{(D;\alpha_i,\alpha_j)} \otimes 1 - 1 \otimes \varphi_{(D;\alpha_i,\alpha_j)} = 0$$

so that $\varphi_{(D;\alpha_i,\alpha_j)}$ is a primitive element of $U\mathfrak{g}_D$ and therefore lies in $\mathfrak{g}_D$. Since $\varphi_{(D;\alpha_i,\alpha_j)}$ is also of weight 0, we find that

$$\varphi_{(D;\alpha_i,\alpha_j)} \in \mathfrak{h}_D$$

where $\mathfrak{h}_D \subset \mathfrak{g}_D$ is the span of the simple roots $\alpha_i \in D$. Since $\Phi^a_{(D;\alpha_i,\alpha_j)}$ satisfy the generalised pentagon identities corresponding to the 2–faces of the De Concini–Procesi associahedron $\mathcal{A}_\mathfrak{g}$, we also find

$$d_D\{\varphi_{(D;\alpha_i,\alpha_j)}\} = 0$$

Proposition 9.5. *Let* $\varphi = \{\varphi_{(D;\alpha_i,\alpha_j)}\}$ *be a 2–cochain in the Dynkin complex of* $\mathfrak{g}$ *satisfying*

$$\varphi_{(D;\alpha_i,\alpha_j)} \in \mathfrak{h}_D \quad \textit{and} \quad d_D\varphi = 0$$

Then, there exists a Dynkin 1–cochain $a = \{a_{(D;\alpha_i)}\}$ *satisfying*

$$a_{(D;\alpha_i)} \in \mathfrak{h}_D \quad \textit{and} \quad d_D a = \varphi$$

The element a *may be chosen such that* $a_{(\alpha_i;\alpha_i)} = 0$ *for all* α_i *and is then unique with this additional property.*

Proof. It will be convenient to fix an order $\alpha_1, \ldots, \alpha_n$ of the simple roots and identify the group of Dynkin cochains $CD^p(A;M)$ with elements $m = \{m_{(B;\underline{\alpha})}\}$ where B ranges over the connected subdiagrams of $D_\mathfrak{g}$ and $\underline{\alpha}$ over the subsets $\{\alpha_{i_1}, \ldots, \alpha_{i_p}\} \subseteq B$ such that $i_1 < \ldots < i_p$. We wish to solve the equation $\varphi = d_D a$. In components, this reads

$$\varphi_{(D;\alpha_i,\alpha_j)} = a_{(D;\alpha_j)} - a_{(\complement^{D\setminus\{\alpha_i\}}_{\alpha_j};\alpha_j)} - a_{(D;\alpha_i)} + a_{(\complement^{D\setminus\{\alpha_j\}}_{\alpha_i};\alpha_i)} \tag{9.8}$$

for any connected subdiagram $D \subseteq D_\mathfrak{g}$ and $i < j$ such that $\alpha_i, \alpha_j \in D$. The assumptions $\varphi_{(D;\alpha_i,\alpha_j)}, a_{(D;\alpha_i)} \in \mathfrak{h}_D$ and the fact that φ, a lie in the Dynkin complex of $\mathfrak{g}$ imply that

$$\varphi_{(D;\alpha_i,\alpha_j)} \in \mathbb{C}\lambda_i^\vee \oplus \mathbb{C}\lambda_j^\vee \quad \text{and} \quad a_{(D;\alpha_i)} \in \mathbb{C}\lambda_i^\vee$$

respectively where $\lambda_i^\vee$ is the fundamental coweight dual to α_i. Projecting (9.8) on $\lambda_i^\vee$ and $\lambda_j^\vee$ we therefore find that it is equivalent to

$$a_{(D;\alpha_i)} = a_{(\complement^{D\setminus\{\alpha_j\}}_{\alpha_i};\alpha_i)} - \varphi^i_{(D;\alpha_i,\alpha_j)}$$
$$a_{(D;\alpha_j)} = a_{(\complement^{D\setminus\{\alpha_i\}}_{\alpha_j};\alpha_j)} + \varphi^j_{(D;\alpha_i,\alpha_j)}$$

where

$$\varphi_{(D;\alpha_i,\alpha_j)} = \varphi^i_{(D;\alpha_i,\alpha_j)}\lambda_i^\vee + \varphi^j_{(D;\alpha_i,\alpha_j)}\lambda_j^\vee$$

and we are identifying $a_{(D;\alpha_i)}, a_{(D,\alpha_j)}$ with their components along $\lambda_i^\vee, \lambda_j^\vee$ respectively. Thus, $\varphi = d_D a$ iff

$$a_{(D;\alpha_i)} = a_{(\complement^{D\setminus\{\alpha_j\}}_{\alpha_i};\alpha_i)} - (-1)^{(i:j)}\varphi^i_{(D;\alpha_i,\alpha_j)} \tag{9.9}$$

for all D and $1 \leq i \neq j \leq n$ with $\alpha_i, \alpha_j \in D$, where $(i:j) = 0$ if $i < j$ and 1 otherwise. Induction on the cardinality of D readily shows that the above equations possess at most one solution once the values of $a_{(\alpha_i;\alpha_i)}$ are fixed. To prove that one solution exists, assume that $a_{(D;\alpha_i)}$ have been constructed for all D with at most m vertices in such a way that the equations (9.9) hold for all such D. We claim that (9.9) may be used to define $a_{(D;\alpha_i)}$ for all D with $|D| = m+1$ in a consistent way, *i.e.,* independently of $j \neq i$ such that $\alpha_j \in D$. This amounts to showing that for all such D, and distinct vertices $\alpha_i, \alpha_j, \alpha_k \in D$, one has

$$a_{(\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j\}};\alpha_i)} - (-1)^{(i:j)}\varphi^i_{(D;\alpha_i,\alpha_j)} = a_{(\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_k\}};\alpha_i)} - (-1)^{(i:k)}\varphi^i_{(D;\alpha_i,\alpha_k)} \tag{9.10}$$

To see this, consider the $(D;\alpha_i,\alpha_j,\alpha_k)$ component of $d_D\varphi$ *i.e.,* the sum

$$\begin{aligned}
&(-1)^{(i:j)+(i:k)}\left(\varphi_{(D;\alpha_j,\alpha_k)} - \varphi_{(\mathsf{C}_{\alpha_j,\alpha_k}^{D\setminus\{\alpha_i\}};\alpha_j,\alpha_k)}\right)\\
+&(-1)^{(j:i)+(j:k)}\left(\varphi_{(D;\alpha_i,\alpha_k)} - \varphi_{(\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}};\alpha_i,\alpha_k)}\right)\\
+&(-1)^{(k:i)+(k:j)}\left(\varphi_{(D;\alpha_i,\alpha_j)} - \varphi_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i,\alpha_j)}\right)
\end{aligned}$$

Since $d_D\varphi = 0$ we get, by projecting on $\lambda_i^\vee$,

$$\varphi^i_{(D;\alpha_i,\alpha_j)} = \varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i,\alpha_j)} + (-1)^{(i:j)+(i:k)}\left(\varphi^i_{(D;\alpha_i,\alpha_k)} - \varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}};\alpha_i,\alpha_k)}\right)$$

so that (9.10) holds iff

$$a_{(\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j\}};\alpha_i)} + (-1)^{(i:k)}\varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}};\alpha_i,\alpha_k)} = a_{(\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_k\}};\alpha_i)} + (-1)^{(i:j)}\varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i,\alpha_j)} \tag{9.11}$$

We consider four separate cases.

9.5.1. $\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}} = \emptyset$ *and* $\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}} = \emptyset$. This case cannot arise since the first condition implies that any path in D from α_i to α_k must pass through α_j before it reaches α_k while the second one implies that the portion of this path linking α_i to α_j must first pass through α_k.

9.5.2. $\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}} = \emptyset$ *and* $\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}} \neq \emptyset$. In this case,

$$\varphi_{(\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}};\alpha_i,\alpha_k)} = 0 \quad\text{and}\quad \mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_k\}} = \mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}$$

and (9.11) reads

$$\begin{aligned} a_{(\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j\}};\alpha_i)} &= a_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i)} + (-1)^{(i:j)}\varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i,\alpha_j)} \\ &= a_{(\mathsf{C}_{\alpha_i}^{\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}\setminus\{\alpha_j\}};\alpha_i)} \end{aligned}$$

where the last equality follows from (9.9) applied to the diagram $\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}$. This equation however holds since

$$\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j\}} = \mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j,\alpha_k\}} = \mathsf{C}_{\alpha_i}^{\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}\setminus\{\alpha_j\}}$$

where the first equality holds because $\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}} = \emptyset$.

9.5.3. $\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}} \neq \emptyset$ *and* $\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}} = \emptyset$. This case reduces to the previous one under the interchange $\alpha_j \leftrightarrow \alpha_k$.

9.5.4. $\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}} \neq \emptyset$ *and* $\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}} \neq \emptyset$. In this case

$$\mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j\}} = \mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}} \quad \text{and} \quad \mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_k\}} = \mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}$$

so that (9.10) reads

$$\begin{aligned} a_{(\mathsf{C}_{\alpha_i}^{\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}}\setminus\{\alpha_k\}};\alpha_i)} &= a_{(\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}};\alpha_i)} + (-1)^{(i:k)}\varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}};\alpha_i,\alpha_k)} \\ &= a_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i)} + (-1)^{(i:j)}\varphi^i_{(\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}};\alpha_i,\alpha_j)} \\ &= a_{(\mathsf{C}_{\alpha_i}^{\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}\setminus\{\alpha_j\}};\alpha_i)} \end{aligned}$$

where the first and last equalities follow from (9.10) for the diagrams $\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}}$ and $\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}$ respectively. This equation however holds because

$$\mathsf{C}_{\alpha_i}^{\mathsf{C}_{\alpha_i,\alpha_k}^{D\setminus\{\alpha_j\}}\setminus\{\alpha_k\}} = \mathsf{C}_{\alpha_i}^{D\setminus\{\alpha_j,\alpha_k\}} = \mathsf{C}_{\alpha_i}^{\mathsf{C}_{\alpha_i,\alpha_j}^{D\setminus\{\alpha_k\}}\setminus\{\alpha_j\}}$$

This concludes the proof of proposition 9.5 ■

We may now conclude the proof of theorem 9.1. Twist $\mathcal{Q}^2$ by

$$a = \{1 - \hbar^n \cdot a_{(D;\alpha_i)}\}_{\alpha_i \in D \subseteq D_{\mathfrak{g}}} \tag{9.12}$$

where the $a_{(D;\alpha_i)}$ are given by proposition 9.5 and $a_{(\alpha_i;\alpha_i)} = 0$ for any i. Then $\Phi^2_{(D;\alpha_i,\alpha_j)} = \Phi^1_{(D;\alpha_i,\alpha_j)} \mod \hbar^{n+1}$ for any $\alpha_i \neq \alpha_j \in D \subseteq D_{\mathfrak{g}}$ and the conditions (9.1)–(9.3) are preserved since, owing to the fact that $a_{(\alpha_i;\alpha_i)} = 0$, we have $(S_{i,C})_a = S_{i,C}$ for any $i = 1 \ldots n$ ■

References

[BN] D. Bar–Natan, *On associators and the Grothendieck–Teichmuller group. I*, Selecta Math. (N.S.) **4** (1998), 183–212.

[Bo1] P. P. Boalch, *Stokes matrices, Poisson Lie groups and Frobenius manifolds*, Invent. Math. **146** (2001), 479–506.

[Bo2] P. P. Boalch, *G-bundles, isomonodromy, and quantum Weyl groups*, Int. Math. Res. Not. **2002**, 1129–1166.

[BT] R. Bott, C. Taubes, *On the self–linking of knots*, Topology and physics. J. Math. Phys. **35** (1994), 5247–5287.

[Br] E. Brieskorn, *Die Fundamentalgruppe des Raumes der Regulären Orbits einer Endlichen Komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.

[BS] E. Brieskorn, K. Saito, *Artin–Gruppen und Coxeter–Gruppen*, Invent. Math. **17** (1972), 245–271.

[BMR] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[CD] M. Carr, S. Devadoss, *Coxeter complexes and graph–associahedra*, preprint, math.QA/0407229.

[CFZ] F. Chapoton, S. Fomin, A. Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2002), 537–566.

[Chn1] K.–T. Chen, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. **97** (1973), 217–246.

[Chn2] K.–T. Chen, *Extension of C^∞ function algebra by integrals and Malcev completion of π_1*, Advances in Math. **23** (1977), 181–210.

[Ch1] I. V. Cherednik, *Generalized braid groups and local r–matrix systems*, Soviet Math. Dokl. **40** (1990), 43–48.

[Ch2] I. V. Cherednik, *Monodromy representations for generalized Knizhnik–Zamolodchikov equations and Hecke algebras*, Publ. Res. Inst. Math. Sci. **27** (1991), 711–726.

[Ch3] I. V. Cherednik, *Calculation of the monodromy of some W–invariant local systems of type B, C and D*, Funct. Anal. Appl. **24** (1990), 78–79.

[DCP1] C. De Concini, C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), 459–494.

[DCP2] C. De Concini, C. Procesi, *Hyperplane arrangements and holonomy equations*, Selecta Math. (N.S.) **1** (1995), 495–535.

[Dr1] V. G. Drinfeld, *Quantum groups*, Proceedings of the International Congress of Mathematicians (Berkeley 1986), 798–820, AMS, 1987.

[Dr2] V. G. Drinfeld, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990), 321–342.

[Dr3] V. G. Drinfeld, *Quasi–Hopf algebras*, Leningrad Math. J. **1** (1990), 1419–1457.

[Dr4] V. G. Drinfeld, *On quasitriangular quasi–Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{Q}/Q)$, Leningrad Math. J. **2** (1991), 829–860.

[FZ] S. Fomin, A. V. Zelevinsky, *Y–systems and generalized associahedra*, Ann. of Math. **158** (2003), 977–1018.

[FM] W. Fulton, R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. **139** (1994), 183–225.

[Ha] R. Hain, *Completions of mapping class groups and the cycle* $C - C^-$, Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991), 75–105, Contemp. Math., 150, Amer. Math. Soc., Providence, 1993.

[Hu] J. E. Humphreys, *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990.

[Ka] V.G. Kac, *Infinite–dimensional Lie algebras.* Third edition. Cambridge University Press, Cambridge, 1990.

[Ks] C. Kassel, *Quantum groups.* Graduate Texts in Mathematics, 155. Springer–Verlag, New York, 1995.

[Ko1] T. Kohno, *On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces*, Nagoya Math. J. **92** (1983), 21–37.

[Ko2] T. Kohno, *Integrable connections related to Manin and Schechtman's higher braid groups*, Illinois J. Math. **34** (1990), 476–484.

[Ko3] T. Kohno, *Bar complex of the Orlik–Solomon algebra*, Arrangements in Boston: a Conference on Hyperplane Arrangements (1999), Topology Appl. **118** (2002), 147–157.

[KR] A. N. Kirillov, N. Reshetikhin, *q–Weyl Group and a Multiplicative Formula for Universal R–Matrices*, Comm. Math. Phys. **134** (1990), 421–431.

[Lu] G. Lusztig, *Introduction to quantum groups.* Progress in Mathematics, 110. Birkhäuser, Boston, 1993.

[Lu2] G. Lusztig, *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), 89–113.

[Ma] M. Markl, *Simplex, associahedron, and cyclohedron*, Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), 235–265, Contemp. Math. 227, AMS, 1999.

[Ms] W. S. Massey, *A basic course in algebraic topology.* Graduate Texts in Mathematics, 127. Springer–Verlag, New York, 1991.

[MTL] J. J. Millson, V. Toledano Laredo, *Casimir operators and monodromy representations of generalised braid groups*, to appear in Transform. Groups **10** (2005), math.QA/0305062.

[SS] S. Shnider, S. Sternberg, *Quantum groups. From coalgebras to Drinfel'd algebras. A guided tour.* Graduate Texts in Mathematical Physics, II. International Press, 1993.

[So] Y. S. Soibelman, *Algebra of Functions on a Compact Quantum Group and its Representations*, Leningrad Math. J. **2** (1991), 161–178.

[St1] J. D. Stasheff, *Homotopy associativity of H–spaces. I*, Trans. Amer. Math. Soc. **108** (1963), 275–292.

[St2] J. D. Stasheff, *From operads to "physically" inspired theories*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), 53–81, Contemp. Math. 202, AMS, 1997.

[Stl] R. P. Stanley *Enumerative combinatorics.* Vol. 1. Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, Cambridge, 1997.

[Ste] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112** (1964), 392–400.

[Ti] J. Tits, *Normalisateurs de tores. I. Groupes de Coxeter étendus*, J. Algebra **4** (1966), 96–116.

[TL1] V. Toledano Laredo, *A Kohno–Drinfeld theorem for quantum Weyl groups*, Duke Math. J. **112** (2002), 421–451.

[TL2] V. Toledano Laredo, *Flat connections and quantum groups*, The 2000 Twente Conference on Lie Groups. Acta Appl. Math. **73** (2002), 155–173.

[TL3] V. Toledano Laredo, *Cohomological construction of relative twists*, math.QA/0506527.

[TL4] V. Toledano Laredo, *Quasi–Coxeter quasitriangular quasi bialgebras and the Casimir connection*, in preparation.

[TK] A. Tsuchiya, Y. Kanie, *Vertex operators in conformal field theory on P^1 and monodromy representations of braid group*, in Conformal field theory and solvable lattice models, 297–372, Adv. Stud. Pure Math., 16. Academic Press, Boston, 1988.

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