p-ADIC L-FUNCTIONS FOR UNITARY SHIMURA VARIETIES, I: CONSTRUCTION OF THE EISENSTEIN MEASURE

To John Coates, with admiration

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Introduction

This is the first of a projected series of papers devoted to studying the relations between p-adic L-functions for $GL(n)$ (and unitary groups), congruences between stable and endoscopic automorphic forms on unitary groups, and Selmer groups for p-adic representations. The goals of these papers are outlined in the survey article [HLS]. The purpose of the present installment is set the ground for the construction of p-adic L-functions in sufficient generality for the purposes of the subsequent applications to congruences and Selmer groups.

The first general conjectures on the construction of p-adic L-functions for ordinary motives were elaborated by Coates in [Co]. The conjectured p-adic analytic functions of [Co] interpolate the quotients of normalized values of L-functions at critical points, in the sense of Deligne. The normalization proceeds in two steps. The critical values are first rendered algebraic, by dividing by their Deligne periods. Next, they are p-stabilized: the Euler factors at p and ∞ are modified according to a complicated but explicit recipe. Coates' conjecture is that the resulting values are p-adically interpolated by a p-adic analytic function of Iwasawa type, associated to a p-adic measure. In our setting, the Deligne period is generally replaced by a certain Petersson norm or an algebraic multiple thereof; the relation of this Petersson norm to the Deligne period is discussed at length in [H3]. In [Pa], Panchishkin points out that Coates' recipe can be adapted unchanged for motives

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satisfying a condition weaker than ordinarity, which he calls *admissibility* and which Perrin-Riou and Greenberg have called the *Panchishkin condition.* Although this is somewhat obscured by the automorphic normalization, we work in the generality of Panchishkin's admissibility condition. Panchishkin also conjectures the existence of more general p-adic L-functions in the absence of admissibility; we do not address this question.

We work with automorphic forms on the unitary groups of hermitian vector spaces over a CM field $\mathcal{K}$, with maximal totally real subfield E. We assume every prime of E dividing p splits in $\mathcal{K}$; we also impose a hypothesis (1.1.2) linking primes above p to signatures of the unitary group at real places of E. Unitary groups, unlike $GL(n)$, are directly related to Shimura varieties. We show that the special values of L-functions of automorphic forms on unitary groups satisfy the congruences needed for the construction of p-adic L-functions by appealing to the fact that the corresponding Shimura varieties are moduli spaces for abelian varieties of PEL type. In this our approach is directly modeled on Katz's construction [K] of p-adic L-functions for Hecke characters of CM fields; indeed, for groups of type $U(1)$ our results reduce to those of Katz.

The starting point of Katz's construction is Damarell's formula and its generalizations due to Shimura, which relate the values of arithmetic Eisenstein series at CM points to special values of L-functions of arithmetic Hecke characters. A generalization of Damerell's formula in higher dimensions is the construction of standard L-functions of unitary groups by the doubling method. This was first developed systematically in the article [PSR] of Piatetski-Shapiro and Rallis, though special cases had been discovered independently by Garrett, and a more thorough development in classical language is contained in the books [S97, S00] of Shimura. The local theory for unitary groups was ignored in [PSR] but was worked out in [L2] and [HKS].

Our p-adic L-functions are actually attached to Hida families of nearly ordinary modular forms on a unitary group $G = U(V)$. As in [K], the main step is the construction of an Eisenstein measure on a large unitary group H, attached to the sum of two copies $V \oplus (-V)$ of V. The hermitian form on $-V$ has been multiplied by -1, so that H is quasi-split and its associated Shimura variety has a point boundary component, stabilized by a maximal parabolic subgroup, the *Siegel parabolic.* The Eisenstein series attached to the Siegel parabolic are the direct generalizations of the classical Eisenstein series on $GL(2)$. The Eisenstein measure is a p-adic measure on a product T of copies of $\mathbb{Z}_p^\times$ with values in the algebra of p-adic modular forms on H interpolating such Eisenstein series. The theory of p-adic modular forms on H was developed by Hida in [Hi04, Hi05]. As in [K], these forms belong to the algebra of functions on the Igusa tower, which is a rigid analytic étale covering of the ordinary locus of the Shimura variety attached to H. The existence of the Eisenstein measure relies crucially on the irreducibility of the Igusa tower; this was established in some generality by Hida, though an easier argument due to Chai is sufficient for the case at hand.

The Eisenstein measure associates, by integrating over T with respect to this measure, p-adic modular forms to continuous functions on T. The integrals of characters of T of finite order, which determine the measure, are classical holomorphic (Siegel) Eisenstein series on H and as such are associated to explicit functions ("sections") belonging to degenerate principal series induced from characters of the Siegel parabolic. These sections factor as tensor products of local sections over the primes of E. At almost all finite primes the local sections are unramified and present no difficulty, and we simplify the theory by choosing local sections at ramified primes, other than those dividing p, that are insensitive to p-adic variation of the character of T. With our choice of data, the Fourier coefficients of the Eisenstein series at a chosen point boundary component also factor over primes. All the work in constructing the Eisenstein measure then comes down to choosing local data at primes above p such that the corresponding local coefficients satisfy the necessary Kummer congruences. Our strategy for choosing local data follows [K] in making use of a partial Fourier transform. Unlike in [K], our construction is systematically adelic and isolates the local considerations at p. The Eisenstein measure is designed to pair with Hida families – on $G \times G$, not on G itself – and thus depends on several variables, considerably complicating the calculations.

The doubling method was used by Böcherer and Schmidt in [BS] to construct standard p-adic L-functions for Siegel modular forms. They do not use p-adic modular forms; their approach is to construct the p-adic measure directly in terms of normalized special values of complex L-functions. Their approach applies to all critical values, unlike the present paper, which avoids reference to non-holomorphic differential operators (and their p-adic analogues). Presumably their techniques work for quasi-split unitary groups as well. We have not attempted to compare our results where they can be compared, namely in the local analysis at the prime p, since our group is locally isomorphic to $GL(2n)$, in principle much simpler than a symplectic group.

As predicted by Coates, the shape of the modified Euler factor at a prime v dividing p depends on the p-adic valuations of the eigenvalues of Frobenius at v. On the other hand, as in [H3], the fact that a critical value of the standard L-function is an algebraic multiple of a period of an arithmetic modular form on the doubled group $G \times G$ – in other words, the Petersson norm of an arithmetic modular form on G – can be expressed in terms of Hodge numbers. Then the Panchishkin condition, applied to the standard L-function for $GL(n)_{\mathcal{K}}$, roughly states that, for each v dividing p, the modified Euler factor at v is given by a natural partition of the Frobenius eigenvalues at v that corresponds to the signature of the unitary group at real places assigned to v by Hypothesis (1.1.2).

The form of the modified Euler factor at p is thus linked to the real form of G. This is reflected in the fact that the natural embedding of the Shimura variety attached to $G \times G$ in that attached to H in general does not define a map of Igusa towers. In order to pair p-adic modular forms on H with p-adic modular forms on $G \times G$, the natural embedding has to be replaced by a p-adic translation (cf. (2.1.11)), which is exactly what is needed to provide the expected modification of the Euler factor.

The main innovation of our construction concerns the zeta integral at p. As in [K], the use of a partial Fourier transform to define local data at p with the appropriate congruence properties to construct the Eisenstein measure is precisely what is needed to obtain the modified Euler factor at p directly as a local zeta integral, up to some volume factors. For $U(1)$, this was proved by Katz by direct computation. In general, we obtain the result as an immediate application of the local functional equation for the Godement-Jacquet integral representation of the standard L-function of $GL(n)$. These calculations are presented in Part II.

Why the present construction is not altogether satisfactory.

The first reasons have to do with somewhat arbitrary restrictions on the scope of our result. We have only constructed the p-adic L-function for holomorphic automorphic forms of scalar weight. Moreover, for any fixed scalar weight, we have only studied the p-adic interpolation of the critical values at a fixed point s_0, though we allow the inertial characters at p to vary freely. Relaxing these restrictions would require the construction of the p-adic analogues of the classical non-holomorphic weight-raising operators of Maass, as in [K]. There is no doubt that Katz's constructions can be generalized, but the paper was already quite long without this additional generality, which is not necessary for our intended applications to Selmer groups. Moreover, although Garrett has determined the special values of the archimedean zeta integrals up to rational factors in general, his method does not permit identification up to p-adic units in general.[4]

As mentioned above, our choice of Eisenstein measure is insensitive to p-adic variation at ramified primes not dividing p, and the resulting p-adic L-function is missing its local Euler factors at the corresponding primes. A construction taking ramification away from p into account would probably require at the very least a p-integral version of the Godement-Jacquet theory of local zeta integrals (at primes not dividing p), based on Vignéras' modular representation theory of $GL(n)$ over local fields. We hope to return to this question in the future. Ignoring a finite number of Euler factors at places prime to p introduces a bounded error in expected applications to Selmer groups.

There are also local restrictions at primes dividing p. Working with general r-dimensional Hida families, we expect the values of our p-adic L-functions at algebraic (classical) points to be explicitly related to normalized special values of archimedean L-functions. The normalization involves dividing by a complex period invariant, to which we return momentarily. Our main results assert this to be the case under certain restrictions: at algebraic points corresponding to r-tuples of characters lying in a certain positive cone (the *regular* case); or when $r = 1$, where the Hida family is just the family of twists by characters composed with the determinant; or finally when $r \leq 2$ but only along an "anticyclotomic" direction. This is sufficient for our intended applications but is certainly less than optimal, and we hope to be able to relax at least the anticyclotomic condition in the final version of Part II. The restrictions allow us to identify the specialization of the Hida family at an algebraic point as an explicit vector in a principal series representation, which can then be used as a test vector in a local zeta integral.

The most serious defect of our construction is global. The conjectures of [Co] and [Pa] are expressed in the language of motives, and relate the special values of the p-adic L-function to the special values of the quotient of an archimedean L-function by a complex period invariant attached to the motive. In order for this relation to make sense, one needs to know that this quotient is an algebraic number, and so the statements of the conjectures of [Co] and [Pa] require Deligne's conjecture on the critical values of motivic L-functions as a preliminary hypothesis.[5] Our archimedean L-functions are attached to automorphic forms rather than to motives,

[4] Shimura calculates the archimedean zeta integrals precisely in [S97], but only for forms of scalar weight. His scalar weights, unlike those treated here, are non-constant functions on the set of real primes; thus he is forced to work with Maass operators.

[5] The more general conjectures of Perrin-Riou concern non-critical values of motivic L-functions, and the normalizing periods are defined by Beilinson's conjectures; in general, this is far beyond the scope of the automorphic theory as it presently stands.

and the period invariants are defined, as in Shimura's work, as (suitable algebraic multiples of) Petersson norms of arithmetic holomorphic modular forms on the appropriate Shimura varieties. The conjectural relation of these Petersson norms to Deligne's motivic periods, up to rational factors, is discussed in [H3], at least when the ground field is $\mathbb{Q}$. Partial results in this direction are obtained in [H4, H5], using an elaborate inductive argument, based on the theta correspondence, for establishing period relations between automorphic forms on unitary groups of different signatures. It is not beyond the realm of imagination that such techniques can eventually provide relations between Petersson norms up to integral factors, though it may well be beyond the limits of anyone's patience. Even the relatively favorable case of Shimura curves, where no products of periods are involved, required extraordinary efforts on the part of Prasanna [Pr]. However, and this is the most important point, even assuming integral period relations for Petersson norms, we still need to compare products of Petersson norms to motivic periods. When $n = 2$ and the ground field is $\mathbb{Q}$, Hida realized long ago that the ratio of the Petersson norm to the motivic period generates the congruence ideal, and is itself the specialization of a p-adic L-function. When $n > 2$ we do not know how to use the automorphic theory to study the analogous ratios.

Contents of this paper.

To keep this first paper in the series to a reasonable length we have decided to break it into two parts. Part I, by recalling the theory of p-adic modular forms on unitary groups and constructing the Eisenstein measure, sets up the ground work for the construction of the p-adic L-functions.

More precisely, §1 recalls the theory of modular forms on unitary Shimura varieties, a theory ultimately due to Shimura but presented here in the setting of [H1]. We present the theory of p-adic modular forms on unitary Shimura varieties in §2, following Hida's generalization of the constructions of Deligne and Katz for $GL(2)$. Most of these results are at least implicitly due to Hida, but we have highlighted some special features adapted to the embedding of Igusa towers mentioned above. The calculation of the local coefficients at p of Eisenstein series occupies the greater part of §3, the rest of which is concerned with the local coefficients at the remaining places, and the relation of local to global coefficients, due essentially to Shimura. We conclude §3 with the construction of the Eisenstein measures.

Part II will develop Hida theory for p-adic modular forms on unitary groups G, carry out the related zeta-integral calculations from the doubling method, and complete the construction of p-adic L-functions. It will also establish a dictionary between the motivic and automorphic normalizations, and in particular will verify that the modified Euler factors at p are as predicted in [Co] and [Pa].

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Finally, it is a special privilege and pleasure to dedicate this article to John Coates. His insights and taste have shaped our field for a generation; his generosity, especially in supporting young researchers, is unparalleled; and his personal charm is in large part responsible for making number theory a most enviable profession.

0. Notation and Conventions

Let G be a reductive algebraic group over the number field F. If v is a place of F we let $G_v = G(\mathbb{Q}_v)$; if v is archimedean we let $\mathfrak{g}_v = Lie(G_v)_{\mathbb{C}}$. We let G_∞ denote $\prod_{v|\infty} G_v$, the product being over all archimedean places of F, and let $\mathfrak{g}_\infty = \prod_{v|\infty} \mathfrak{g}_v$. In practice we will denote by K_∞ a subgroup of G_∞ which is maximal compact modulo the center of G.

We let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Thus for any number field L we identify the set Σ_L of complex embeddings of L with the set $Hom(L, \overline{\mathbb{Q}})$. Let $\mathbb{C}_p$ denote the completion of an algebraic closure of $\mathbb{Q}_p$, with integer ring $\mathcal{O}_{\mathbb{C}_p}$. We choose once and for all an embedding $incl_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$, and let $\bar{\mathbb{Z}}_{(p)} = incl_p^{-1}(\mathcal{O}_{\mathbb{C}_p})$, the corresponding valuation ring. When necessary, we denote by $incl_\infty$ the given inclusion of $\overline{\mathbb{Q}}$ in $\mathbb{C}$. Via this pair of inclusions, any embedding $\tau : L \to \mathbb{C}$ of a number field L gives rise to an embedding $\tau_p = incl_p \circ \tau : L \to \mathbb{C}_p$.

(0.1) Unitary groups over CM fields.

Let E be a totally real number field of degree d over $\mathbb{Q}$ and let $\mathcal{K}$ be a totally imaginary quadratic extension of E, with ring of integers $\mathcal{O}$. Let $c \in Gal(\mathcal{K}/E)$ denote the non-trivial automorphism, and $\varepsilon_{\mathcal{K}}$ the character of the idele classes of E associated to the quadratic extension $\mathcal{K}$. We fix a CM type of $\mathcal{K}$, i.e. a subset $\Sigma \subset \Sigma_{\mathcal{K}}$ such that $\Sigma \coprod \Sigma c = \Sigma_{\mathcal{K}}$.

Let V be an n-dimensional $\mathcal{K}$-vector space, endowed with a non-degenerate hermitian form $< \bullet, \bullet >_V$ relative to the extension $\mathcal{K}/E$. For each $\sigma \in \Sigma_{\mathcal{K}}$, $< \bullet, \bullet >_V$ defines a hermitian form $< \bullet, \bullet >_\sigma$ on the complex space $V_\sigma = V \otimes_{\mathcal{K},\sigma} \mathbb{C}$. We let (a_σ, b_σ) denote the signature of the form $< \bullet, \bullet >_\sigma$. Note that $(a_{c\sigma}, b_{c\sigma}) = (b_\sigma, a_\sigma)$ for all $\sigma \in \Sigma_{\mathcal{K}}$.

The hermitian pairing $< \bullet, \bullet >_V$ defines an involution $\tilde{c}$ on the algebra $End(V)$ via

$$< a(v), v' >_V = < v, a^{\tilde{c}}(v') >, \tag{0.1.1}$$

and this involution extends to $End(V \otimes_{\mathbb{Q}} R)$ for any $\mathbb{Q}$-algebra R. We define $\mathbb{Q}$-algebraic groups $U(V) = U(V, < \bullet, \bullet >_V)$ and $GU(V) = GU(V, < \bullet, \bullet >_V)$ over $\mathbb{Q}$ such that, for any $\mathbb{Q}$-algebra R,

$$\begin{gathered} U(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot \tilde{c}(g) = 1\}; \\ GU(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot \tilde{c}(g) = \nu(g) \text{ for some } \nu(g) \in R^\times\}. \end{gathered} \tag{0.1.2}$$

Thus $GU(V)$ admits a homomorphism $\nu : GU(V) \to \mathbb{G}_m$ with kernel $U(V)$. There is an algebraic group $U_E(V)$ over E such that $U(V) \xrightarrow{\sim} R_{E/\mathbb{Q}} U_E(V)$, where $R_{E/\mathbb{Q}}$ denotes Weil's restriction of scalars functor. This isomorphism identifies automorphic representations of $U(V)$ and $U_E(V)$.

The groups $U(V)$ (resp. $GU(V)$) are all inner forms of the same quasi-split unitary group (resp. unitary similitude group), denoted U_0 (resp. GU_0). The group U_0 is of the form $U(D_0, \tilde{\chi}(*)_0)$ where D_0 is the matrix algebra and $\tilde{\chi}(*)_0$ is an appropriate involution. Then $U_{0,\infty} \cong U(\frac{n}{2}, \frac{n}{2})^{[E:\mathbb{Q}]}$ if n is even, $U_{0,\infty} \cong U(\frac{n-1}{2}, \frac{n+1}{2})^{[E:\mathbb{Q}]}$ if n is odd.

(0.2) Haar measures.

The bulk of this article and its companion, Part II, is devoted to calculations involving Fourier transforms, zeta integrals, and Petersson inner products of automorphic forms on the groups $U(V)$ of (0.1). The integrals are defined with respect to local and adelic Haar measures. The natural adelic Haar measure on $G = U_E(V)$ is Tamagawa measure $d^\tau g$, associated to an invariant top differential ω rational over E on G. Let $\delta(E)$ denote the discriminant of E. The adelic Tamagawa measure $d^\tau g$ factors up to normalization as a product of local measures

$$d^\tau g = |\delta(E)|^{-\frac{\dim G}{2}} L(1, \varepsilon_{\mathcal{K}})^{-1} \prod_v d^\tau g_v \tag{0.2.1}$$

where $d^\tau g_v$ is the measure defined by ω_v if v is real and by $L_v(1, \varepsilon_{\mathcal{K}})\omega_v$ if v is finite. The Tamagawa number $\tau(G)$ of G is $vol(G(\mathbb{Q})\backslash G(\mathbf{A}), d^\tau g) = 2$. For finite v the volume of any compact open set with respect to $d^\tau g_v$ is always a rational number.

An alternative measure, traditionally used in the calculation of zeta integrals, is $dg = \prod_v dg_v$ where $dg_v = d^\tau g_v$ for archimedean v but dg_v is chosen to give volume 1 to a hyperspecial maximal compact subgroup K_v at almost all finite primes. Let S_G be the set of finite places v of E where ω_v is not an $\mathcal{O}_{E,v}$ generator of the module of top differentials; in particular, the group G is unramified at $v \notin S_G$ and so $G(E_v)$ has hyperspecial maximal compacts. The relation is

$$d^\tau g_v = L_v(1, \varepsilon_{\mathcal{K}}) \cdot A_v(n) dg_v, \ A_v(n) = (q_v)^{-\dim G} \cdot |G_v(k_v)| \tag{0.2.2}$$

where G_v is the smooth reductive group scheme over $Spec(\mathcal{O}_{E,v})$ associated to the hyperspecial subgroup K_v. If for $v \in S_G$ (which includes the finite places where G has no hyperspecial maximal compact) we arbitrarily set $d^* g_v = dg_v$ for $v \in S_G$, then

(0.2.3)

$$vol(G(\mathbb{Q})\backslash G(\mathbf{A}), dg))/vol(G(\mathbb{Q})\backslash G(\mathbf{A}), d^\tau g) = |\delta(E)|^{\frac{\dim G}{2}} \cdot \prod_{v \notin S_G} A_v(n)^{-1} = \prod_{j=1}^{n} L^{S_G}(j, \varepsilon_{\mathcal{K}}^j)$$

where L^{S_G} denotes the partial L-function with the factors at S_G removed.

Given an open compact subgroup $K \subset G(\mathbf{A}_f)$, we let $d\mu_K(g)$ be the Haar measure that gives each connected component of ${}_K S(G) = G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty K$ total volume 1, for any maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. When V is totally definite, so $G(\mathbb{R}) = K_\infty$, $d\mu_K(g)$ is counting measure on the finite set ${}_K S(G)$. In general,

$$d\mu_K(g) = \frac{\mathcal{C}(G, K)}{2} d^\tau g \tag{0.2.4}$$

where the class number $\mathcal{C}(G, K) = |\pi_0({}_K S(G)|$ can be determined explicitly.

1. Automorphic forms on unitary groups

(1.1) Ordinary primes for unitary groups.

Let $(V, < \bullet, \bullet >_V)$ be a hermitian pairing as in (0.1). Let p be a rational prime which is unramified in $\mathcal{K}$ (hence in particular in the associated reflex field $E(V)$), and such that every divisor of p in E splits completely in $\mathcal{K}$. Choose an inclusion $incl_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ as above. Composition with $incl_p$ defines an identification $\Sigma_{\mathcal{K}} \xrightarrow{\sim} Hom(\mathcal{K}, \mathbb{C}_p)$, hence for every $\tau \in Hom(\mathcal{K}, \mathbb{C}_p)$ we can define a signature

$$(a_\tau, b_\tau) = (a_\sigma, b_\sigma) \text{ if } \tau = incl_p \circ \sigma. \tag{1.1.1}$$

We assume the triple $(\Sigma, incl_p, (a_\sigma, b_\sigma)_{\sigma \in \Sigma_{\mathcal{K}}})$ to be *ordinary* in the following sense:

(1.1.2) Hypothesis. *Suppose $\sigma, \sigma' \in \Sigma$ have the property that $incl_p \circ \sigma$ and $incl_p \circ \sigma'$ define the same p-adic valuations. Then $a_\sigma = a_{\sigma'}$.*

When $a_\sigma = n$ for all $\sigma \in \Sigma$ – this is the *definite case*, to be described in detail later – or more generally, when $a_\sigma = a$ for all $\sigma \in \Sigma$ is constant, this comes down to the following hypothesis, used by Katz in the case $n = 1$:

(1.1.3) Hypothesis. *For $\sigma, \sigma' \in \Sigma$, the p-adic valuations defined by $incl_p \circ \sigma$ and $incl_p \circ \sigma' \dot{c}$ are* distinct.

As Katz observes in [K], our hypotheses on p guarantee that Σ's satisfying (1.1.2) exist.

We let Σ_p denote the set $\{\sigma_p \mid \sigma \in \Sigma\}$ of $\mathbb{C}_p$-embeddings of $\mathcal{K}$. Complex conjugation c acts on the set of primes of $\mathcal{K}$ dividing p, and the set of all such primes of $\mathcal{K}$ is the disjoint union

$$Hom(\mathcal{K}, \mathbb{C}_p) = \Sigma_p \coprod \Sigma_p c. \tag{1.1.4}$$

Hypothesis (1.1.2) was suggested by Fargues, who observed that it is equivalent to the condition that the completion of the reflex field of the Shimura variety attached to G (see §1.2) at the place defined by $incl_p$ is $\mathbb{Q}_p$. This is in turn equivalent, by a criterion of Wedhorn [We], to the condition that the ordinary locus of the completion of the Shimura variety at $incl_p$ is non-empty (see (2.1.7), below). We reformulate the elementary condition (1.1.2) in equally elementary terms. We have a canonical isomorphism $V_p \xrightarrow{\sim} \oplus_{w|p} V_w$ where $V_w = V \otimes_{\mathcal{K}} \mathcal{K}_w$. Let V_{p,Σ_p} and $V_{p,\Sigma_p c}$ be, respectively, the preimages of the subspaces $\oplus_{w|p, w \in \Sigma_p} V_w$ and $\oplus_{w|p, w \in \Sigma_p c} V_w$, where the notation $w \in \Sigma_p$ designates those w such that w is the valuation determined by some $\sigma_p \in \Sigma_p$. In particular,

$$V_{p,\Sigma_p} \xrightarrow{\sim} \oplus_{w|p, w \in \Sigma_p} V_w. \tag{1.1.5}$$

The fact that all primes of E above p split in $\mathcal{K}/E$ is equivalent to the condition that the $\mathbb{Q}_p$-vector space $V_p = R_{\mathcal{K}/\mathbb{Q}} V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ decomposes $\mathbb{Q}_p$-rationally as $V_p = V_{p,\Sigma_p} \oplus V_{p,\Sigma_p c}$. The decomposition (1.1.5) is tautologically $\mathbb{Q}_p$-rational. For any w dividing p, let

$$\Sigma_w = \{\sigma \in \Sigma_{\mathcal{K}} \mid \sigma_p = w\}.$$

Equivalent to (1.1.2) is the hypothesis:

(1.1.6) Hypothesis. *There is a* $\mathbb{Q}_p$*-rational* $\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p$*-submodule* $W(sig)\subset V_{p,\Sigma_p}$ *(resp.* $F^0V_p\subset V_p$*) such that* $W(sig)=\oplus_{w|p,w\in\Sigma_p}W(sig)_w$ *(resp.* $F^0V_p=\oplus_{w|p}F^0V_w$*) with* $\dim W(sig)_w=a_\sigma$ *for any* $\sigma\in\Sigma$ *(resp.* $\dim F^0V_w=a_\sigma$ *for any* $\sigma\in\Sigma_w$*).*

In the definite case we just have $W(sig)=V_{p,\Sigma_p}$. Under hypothesis (1.1.2) we write $(a_w,b_w)=(a_\sigma,b_\sigma)$ for any $\sigma\in\Sigma_w$.

(1.2) Shimura varieties and automorphic vector bundles.

Let $(V,<\bullet,\bullet>_V)$ be an n-dimensional hermitian space over $\mathcal{K}$ as above. As in [H4], we let $-V$ denote the space V with hermitian form $<\bullet,\bullet>_{-V}=-<\bullet,\bullet>_V$ and $2V$ denote the doubled hermitian space $V\oplus(-V)$ with hermitian form the sum of $<\bullet,\bullet>_V$ and $<\bullet,\bullet>_{-V}$. We define $U(2V)$ and $GU(2V)$ as in (1.1); in particular, $GU(2V)$ denotes the *rational* similitude group.

The stabilizer in $U(2V)$ of the direct sum decomposition $2V=V\oplus(-V)$ is naturally isomorphic to the product $U(V)\times U(-V)$, embedded naturally in $U(2V)$. Similarly, the stabilizer in $GU(2V)$ is isomorphic to the subgroup $G(U(V)\times U(-V))\subset GU(V)\times GU(-V)$, defined by

$$G(U(V)\times U(-V))=\{(g,g')\in GU(V)\times GU(-V)\mid \nu(g)=\nu(g')\}. \tag{1.2.1}$$

Let $(W,<\bullet,\bullet>_W)$ be any hermitian space over $\mathcal{K}$. To the group $G=GU(W)$ one can canonically attach a Shimura datum (G,X), and hence a Shimura variety $Sh(W)=Sh(G,X)$, as follows. For each $\sigma\in\Sigma$, let (a_σ,b_σ) denote the signature of the hermitian form induced by $<\bullet,\bullet>_W$ on the complex space $W_\sigma=W\otimes_{\mathcal{K},\sigma}\mathbb{C}$. Let $GU(a_\sigma,b_\sigma)=GU(W_\sigma)$ denote the real unitary similitude group, and let X^{a_σ,b_σ} denote the $GU(a_\sigma,b_\sigma)(\mathbb{R})$-conjugacy class of homomorphisms $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}\to GU(a_\sigma,b_\sigma)$ defined in [H4, p. 143]. The product $X=X(W)=\prod_{\sigma\in\Sigma}X^{a_\sigma,b_\sigma}$ is naturally a $G(\mathbb{R})$-conjugacy class of homomorphisms $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}\to G_{\mathbb{R}}$, and the pair (G,X) satisfies the axioms of [D] defining a Shimura variety – unless W_σ is definite for all σ, in which case one can attach a zero-dimensional Shimura variety to (G,X) all the same, as in [H3]. We recall that the complex-valued points of $Sh(G,X)$ are given by

$$Sh(G,X)(\mathbb{C})=\varprojlim_K G(\mathbb{Q})\backslash X\times G(\mathbf{A}_f)/K, \tag{1.2.2}$$

where K runs over open compact subgroups of $G(\mathbf{A}_f)$. We let ${}_KSh(G,X)$ denote the associated variety whose complex points are given by $G(\mathbb{Q})\backslash X\times G(\mathbf{A}_f)/K$.

If W' is a second hermitian space, the above construction applies to groups of the form $G(U(W)\times U(W'))$, defined by analogy with (1.2.1), yielding a Shimura datum $(G(U(W)\times U(W')),X(W,W'))$. With the above conventions, it is immediate that the natural map $G(U(W)\times U(W'))\to GU(W\oplus W')$ defines a map of Shimura data $(G(U(W)\times U(W')),X(W,W'))\to G(U(W\oplus W')),X(W\oplus W'))$, hence a morphism of Shimura varieties

$$Sh(W,W')=Sh((G(U(W)\times U(W')),X(W,W')))\to Sh(W\oplus W'). \tag{1.2.3}$$

When $E=\mathbb{Q}$, this is worked out in detail in [H4]. In particular, we obtain a map

$$Sh(V,-V)\to Sh(2V). \tag{1.2.4}$$

The group $GU(2V)$ is always quasi-split; in particular, up to isomorphism, it does not depend on the choice of V of dimension n. The corresponding Shimura variety always has a canonical model over $\mathbb{Q}$. The more general Shimura varieties $Sh(W)$, $Sh(W,W')$ are defined over reflex fields $E(W)$, $E(W,W')$, respectively, of which one can only say in general that they are contained in the Galois closure of $\mathcal{K}$ over $\mathbb{Q}$. It is easy to see, however, that $E(V,-V) = E(V)$, and the general theory of canonical models implies that the map (1.2.4) is rational over $E(V)$. If $E = \mathbb{Q}$ then $\mathcal{K}$ is a quadratic imaginary field, and $E(V) = \mathcal{K}$ unless V is quasi-split, in which case $E(V) = \mathbb{Q}$. When V is a definite hermitian space, $E(V)$ is the reflex field $E(\mathcal{K},\Sigma)$ of the CM type $(\mathcal{K},\Sigma)$.

We will be working with holomorphic automorphic forms on G, when G is of the form $G = GU(W)$ or $GU(W,W')$. These are constructed as follows; for details, see [H1]. Let $K_\infty \subset G(\mathbb{R})$ be the stabilizer of a point $x \in X$ $(= X(W)$ or $X(W,W'))$; thus K_∞ contains a maximal connected compact subgroup of $G(\mathbb{R})$, as well as the real points of the center Z_G of G. In fact, K_∞ is the group of real points of an algebraic subgroup, also denoted K_∞, of G, the centralizer of the torus $x(R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}})$. Moreover, the derived subgroup of G is simply connected, hence K_∞ is connected. Hence one can speak of algebraic representations of K_∞ and their extreme weights. If $\tau : K_\infty \to GL(W_\tau)$ is an algebraic representation, then there exists a holomorphic vector bundle $[W_\tau]$ on $Sh(G,X)$; more precisely, there exists a canonical holomorphic structure on the C^∞ vector bundle

$$[W_\tau] = \varprojlim_K G(\mathbb{Q})\backslash G(\mathbb{R}) \times W_\tau \times G(\mathbf{A}_f)/K_\infty K, \tag{1.2.5}$$

where K_∞ acts on the right on $G(\mathbb{R})$ and on the left on W_τ, yielding a natural map to

$$\varprojlim_K G(\mathbb{Q})\backslash G(\mathbb{R}) \times G(\mathbf{A}_f)/K_\infty K = \varprojlim_K G(\mathbb{Q})\backslash X \times G(\mathbf{A}_f)/K = Sh(G,X)(\mathbb{C}).$$

A holomorphic automorphic form on G of type τ is a global section $f \in H^0(Sh(G,X),[W_\tau])$; when G contains a rational normal subgroup isogenous to $SL(2)_\mathbb{Q}$ one needs to add a growth condition at infinity. The representation τ is included in the notation for $[W_\tau]$, but is superfluous; $[W_\tau]$ can be defined without reference to a choice of K_∞ (or, equivalently, a choice of $p \in X$), and has a canonical model rational over a number field $E(W_\tau)$, containing the reflex field $E(G,X)$, and attached canonically to the set of extreme weights of W_τ. In particular, the space $H^0(Sh(G,X),[W_\tau])$ has a canonical rational structure over $E(W_\tau)$. However, since we have chosen K_∞, we can also realize holomorphic automorphic forms of type τ as W_τ-valued functions on the adèle group of G via (1.2.5). Let $\mathcal{A}(G)$ denote the space of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$. Then

$$H^0(Sh(G,X),[W_\tau]) \xrightarrow{\sim} \mathcal{A}_{hol,\tau}(G) := \{f \in (\mathcal{A}(G)\otimes W_\tau)^{K_\infty} \mid \mathfrak{p}^- f = 0\}, \tag{1.2.6}$$

canonically. Here

$$\mathfrak{g}_\infty = Lie(K_\infty)_\mathbb{C} \oplus \mathfrak{p}^- \oplus \mathfrak{p}^+ \tag{1.2.7}$$

is the Harish-Chandra decomposition, and the choice a base point $x \in X$, and hence K_∞ and the decomposition (1.2.7), is implicit in the notation $\mathcal{A}_{hol,\tau}(G)$. We also write the right-hand side of (1.2.6) as

$$(\mathcal{A}(G)\otimes W_\tau)^{K_\infty}[\mathfrak{p}^-],$$

the $\mathfrak{p}^-$-torsion in $(\mathcal{A}(G) \otimes W_\tau)^{K_\infty}$.

If $X = X(V, -V)$ with V a definite hermitian space, then $K_\infty = GU(V, -V)(\mathbb{R})$. If $X = X(2V)$, with V again definite, we can take K_∞ to be $GU(V, -V)(\mathbb{R}) \subset GU(2V)$. With this choice, the Harish-Chandra decomposition (1.2.7) is rational over $E(V, -V) = E(V) = E(\mathcal{K}, \Sigma)$.

Restricting forms.

Let $G = GU(V, -V)$, $X = X(V, -V)$, $G' = GU(2V)$, and $X' = X(2V)$. Pick $x \in X(V, -V)$. This determines a base point in X' and hence $K'_\infty \subseteq G'_\infty$ in addition to K_∞, with K_∞ being identified with a subgroup of K'_∞ via the canonical embedding of G into G'.

Suppose τ is a one-dimensional representation of K'_∞. This then determines a one-dimensional representation of K_∞ by restriction, and we obtain holomorphic vector bundles $[W\tau]$ and $[W'_\tau]$ on $Sh(V, -V) = Sh(G, X)$ and $Sh(2V) = Sh(G', X')$, respectively, having canonical models over the respective fields $E(W_\tau)$ and $E(W'_\tau)$. There is canonical map from the pull-back of $[W'_\tau]$ under the morphism (1.2.4) to $[W_\tau]$ and therefore a homomorphism:

$$\mathrm{res}_{V,\tau} : H^0(Sh(2V), [W'_\tau]) \to H^0(Sh(V, -V), [W_\tau]). \tag{1.2.8}$$

This is rational over $E(W_\tau)$. Over the complex numbers (1.2.8) is compatible in the obvious way with the isomorphisms in (1.2.6) and the restriction of forms in $\mathcal{A}_{hol,\tau}(G')$ to $G(\mathbf{A})$, which gives forms in $\mathcal{A}_{hol,\tau}(G)$.

Connected components.

We let $G = GU(V)$. Let C denote the algebraic group G/G^{der} over $\mathbb{Q}$. Let $G(\mathbb{R})^+$ denote the identity component of $G(\mathbb{R})$, $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. For any open compact subgroup $K \subset G(\mathbf{A}_f)$, the set $\pi_0({}_KSh(G, X)(\mathbb{C}))$ of connected components of ${}_KSh(G, X)(\mathbb{C})$ is given by $\overline{G(\mathbb{Q})^+}\backslash G(\mathbf{A}_f)/K$, where $\overline{G(\mathbb{Q})^+}$ denotes the closure of $G(\mathbb{Q})^+$ in $G(\mathbf{A}_f)$. Let $C_K \subset C(\mathbf{A}_f)$ denote the image of K under the natural map; let $C^+ \subset C(\mathbf{A}_f)$ denote the image of $G(\mathbb{Q})^+$. Now G^{der} is an inner form of the simply-connected group $SL(n)$, hence satisfies strong approximation. It follows (cf. [D, (2.1.3.1)]) that

$$\pi_0({}_KSh(G, X)(\mathbb{C})) = C(K) \stackrel{def.}{=} C(\mathbf{A}_f)/C_K \cdot C^+. \tag{1.2.9}$$

We can define a Shimura datum $(C, X(C))$ to be the quotient of (G, X) by G^{der}. The corresponding Shimura variety $Sh(C, X(C))$ also has a modular interpretation in terms of level structures on certain direct factors of rank one over $\mathcal{K}$ of certain tensor powers of the Tate modules of abelian varieties with CM by $\mathcal{K}$. The tensor power in question depends on the signatures (a_σ, b_σ). The natural map ${}_KSh(G, X)(\mathbb{C}) \to C(K) = \pi_0({}_KSh(G, X)(\mathbb{C}))$ becomes a morphism of moduli spaces. This interpretation will not be used in the sequel.

(1.3) PEL structures.

Let $G = GU(V)$. Notation is as in the previous section. Write

$$\mathcal{K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \oplus_{\sigma \in \Sigma_{\mathcal{K}}} \overline{\mathbb{Q}}\sigma,$$

and let $e_\sigma \in \mathcal{K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ be the corresponding orthogonal idempotents. We decompose $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ as a $\mathcal{K} \otimes \mathbb{Q}\overline{\mathbb{Q}}$-module as $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = V_\Sigma \oplus \mathcal{V}_{\Sigma c}$, where $\mathcal{V}_\Sigma$ is the sum of the spaces $V_\sigma = e_\sigma(V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ for $\sigma \in \Sigma$, and similarly for $V_{\Sigma c}$. Inside $V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ we consider a variable $\mathcal{K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$- submodule F^0V satisfying

(1.3.0) Property. *For any $\sigma \in \Sigma_{\mathcal{K}}$, the projection $F^0V_\sigma = e_\sigma F^0V$ of F^0V on V_σ is of dimension a_σ.*

Let T be an indeterminate and, for $x \in \mathcal{K}$, let $P_0(x,T) \in \overline{\mathbb{Q}}[T]$ denote the characteristic polynomial of x, acting on F^0V. It follows from the definition of the reflex field $E(V)$ that $P_\Sigma(x,T) \in E(V)[T]$, independently of the choice of F^0V. Indeed, Shimura defined $E(V)$ to be the field generated by traces of elements of $\mathcal{K}$ acting on F^0V.

Choose a purely imaginary element $\daleth \in \mathcal{K}$, i.e. an element such that $Tr_{\mathcal{K}/E}(\daleth) = 0$. The form $< \bullet,\bullet >_{V,\daleth} = \daleth\cdot < \bullet,\bullet >_V$ is skew-hermitian. When we fix a prime p we will always assume $\daleth$ to be a unit at p. Fix a compact open subgroup $K \subset GU(V)(\mathbf{A}_f)$. We consider the following functor from the category of schemes over $E(V)$ to the category of sets:

$$S \mapsto {}_K\mathcal{A}_V(S) = {}_K\mathcal{A}_{V,\daleth}(S) = \{(A,\lambda,\iota,\alpha)\} \tag{1.3.1}$$

where

(1.3.1.1) A is an abelian scheme over S, viewed as an abelian scheme up to isogeny;
(1.3.1.2) $\lambda : A \to \hat{A}$ is a polarization;
(1.3.1.3) $\iota : \mathcal{K} \to End_S(A) \otimes \mathbb{Q}$ is an embedding of $\mathbb{Q}$-algebras;
(1.3.1.4) $\alpha : V(\mathbf{A}_f) \xrightarrow{\sim} V^f(A)$ is an isomorphism of $\mathcal{K}$-spaces, modulo K.

Here $V^f(A) = \prod_\ell T_\ell(A) \otimes \mathbb{Q}$ is the adelic Tate module, viewed as a ind-pro-étale sheaf over S; it's $\mathcal{K}$-structure comes from (1.2.1.3). The level K structure of (1.3.1.4) is understood in the sense of Kottwitz [Ko]. These data satisfy the usual compatibility conditions:

(1.3.1.5) The Rosati involution on $End_S(A) \otimes \mathbb{Q}$ defined by λ fixes $\iota(\mathcal{K})$ and acts as complex conjugation;
(1.3.1.6) The isomorphism α identifies the Weil pairing on $V^f(A)$ with an ${\mathbf{A}_f}^\times$- multiple of the skew-symmetric pairing on $V(\mathbf{A}_f)$ defined by $tr_{\mathcal{K}/\mathbb{Q}} < \bullet,\bullet >_{V,\daleth}$.

Finally, the action induced by ι on $Lie_{A/S}$ satisfies Shimura's trace condition, which we state here in the equivalent formulation due to Kottwitz. Let $P_\iota(x,T) \in \mathcal{O}_S[T]$ denote the characteristic polynomial of x, acting on $Lie(A/S)$. We view $E(V)[T]$ as a subalgebra of $\mathcal{O}_S[T]$. The Shimura-Kottwitz condition is

$$P_\iota(x,T) = P_0(x,T) \in \mathcal{O}_S[T],\ \forall x \in \mathcal{K}. \tag{1.3.1.7}$$

Two quadruples (A,λ,ι,α) and $(A',\lambda',\iota',\alpha')$ are identified if and only if there is an isogeny $\phi : A \to A'$, commuting with ι', prime to the level K in the obvious sense and taking α to α', and identifying λ' with a positive rational multiple of λ.

(1.3.2) Theorem (Shimura). *For K sufficiently small, the functor (1.3.1) is representable by a quasi-projective scheme over $E(V)$, and this is precisely the canonical model of ${}_KSh(V)$. As K varies, the natural maps between these functors induce the natural maps between the various ${}_KSh(V)$. The action of $GU(V)(\mathbf{A}_f)$ on the tower ${}_KSh(V)$ preserves the $E(V)$-rational structure.*

For $U \subset GU(V)(\mathbf{A}_f)$ a closed compact subgroup, we write ${}_USh(V) = \varprojlim_{K\supset U} {}_KSh(V)$, as K runs over compact open subgroups of $GU(V)(\mathbf{A}_f)$. This is simply a shorthand for referring to the full tower of the ${}_KSh(V)$ for $K \subset U$, and we will not need to worry about the nature of the projective limit.

The above theory applies in particular to the Shimura varieties $Sh(2V)$ and $Sh(V) \times Sh(-V)$. The Shimura variety $Sh(V,-V)$ is defined as the subvariety of $Sh(V) \times Sh(-V)$, which parametrizes pairs of quadruples $((A,\lambda,\iota,\alpha),(A^-,\lambda^-,\iota^-,\alpha^-))$, determined by compatibility of polarizations in the obvious sense. As a subvariety of $Sh(2V)$, $Sh(V) \times Sh(-V)$ is then the set of quadruples (B,μ,ι_2,β) which decompose as a product

$$(B,\mu,\iota_2,\beta) \xrightarrow{\sim} (A \times A^-, \lambda \times \lambda^-, \iota \times \iota^-, \alpha \times \alpha^-).$$

In particular, β respects the $\mathbf{A}_{\mathcal{K},f}$-decomposition $2V(\mathbf{A}_f) = V(\mathbf{A}_f) \oplus (-V)(\mathbf{A}_f)$. The most important level structures β for our purposes do not, however, respect this decomposition. In other words, in the applications, we will not be working with the Shimura variety $Sh(V,-V)$ via its natural embedding in $Sh(2V)$, but rather with a translate of the latter, cf. (2.1.11).

For the remainder of this section, let $G = GU(V)$, $X = X(V)$, $Sh = Sh(G,X)$. We identify

$$GU(V)(\mathbb{Q}_p) = U(V)(\mathbb{Q}_p) \times {\mathbb{Q}_p}^\times \xrightarrow{\sim} U_E(V)(E \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times {\mathbb{Q}_p}^\times,$$

where the map to ${\mathbb{Q}_p}^\times$ is the similitude factor and $U_E(V)$ is as in (0.1). The ordinarity hypothesis (1.1.2) allows us to define subspaces V_{p,Σ_p} and $V_{p,c\cdot\Sigma_p}$ of V_p as in (1.1.5). The hermitian pairing

$$V_p \times V_p \to E \otimes \mathbb{Q}_p$$

determines, and is determined by, a perfect duality $V_{p,\Sigma_p} \otimes V_{p,c\cdot\Sigma_p} \to E \otimes_{\mathbb{Q}} (\mathbb{Q}_p)$ of free $E \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \prod_{w \in \Sigma_p} \mathcal{K}_w$-modules. There is thus a natural isomorphism

$$G(\mathbb{Q}_p) \xrightarrow{\sim} GL(V_{p,\Sigma_p}) \times {\mathbb{Q}_p}^\times \xrightarrow{\sim} \prod_{w,\Sigma_w \subset \Sigma_p} GL(n,\mathcal{K}_w) \times \mathbb{Q}_p^\times \quad (1.3.3)$$

The indexing by w such that $\Sigma_w \subset \Sigma_p$ is a reminder of the fact that several elements of Σ_p can correspond to the same divisor w of p. This is just a way of saying, somewhat more carefully than usual, that the unitary group at a split place is isomorphic to a general linear group. We identify $G_{\mathbb{Q}_p}$ with the product of algebraic groups $G_0 \times GL(1)$, where $G_0 = GL(V_{p,\Sigma_p})$ as algebraic groups and the map to $GL(1)$, as before, is the similitude factor ν.

Fix a compact open subgroup $K = K_p \times K^p \subset G(\mathbf{A}_f)$, with $K_p \subset G(\mathbb{Q}_p)$, $K^p \subset G(\mathbf{A}_f^p)$, and let ${}_KSh$ denote the Shimura variety at level K. Our hypotheses imply that $G_{\mathbb{Q}_p}$ is an unramified group over $\mathbb{Q}_p$, hence that $G(\mathbb{Q}_p)$ contains hyperspecial maximal compact subgroups; we assume that K_p is one such. Then K_p is the group of $\mathbb{Z}_p$-points of an extension of G to a smooth group scheme, also denoted G, over $Spec(\mathbb{Z}_p)$. The choice of K_p is equivalent to the choice of a self-dual $\mathcal{O} \otimes \mathbb{Z}_p$-lattice $M_V \subset V_p$. Let $M_{V,\Sigma_p} \subset V_{p,\Sigma_p}$ be the projection of M_V. We can extend G_0 to a group scheme over $\mathbb{Z}_p$ as $G_0 = GL(M_{V,\Sigma_p})$. Then there are isomorphisms

$$K_p = G(\mathbb{Z}_p) \xrightarrow{\sim} G_0(\mathbb{Z}_p) \times GL(1,\mathbb{Z}_p) = \prod_{w,\Sigma_w \subset \Sigma_p} GL(n,\mathcal{O}_w) \times \mathbb{Z}_p^\times \quad (1.3.4)$$

compatible with the factorization (1.3.3). We also assume K^p is sufficiently small, in a sense we will make precise later.

When $G = G(2V) \supset G(V,-V)$, we choose K_p so that $M_{2V} = M_V \oplus M_{-V}$ with $M_V \subset V \otimes \mathbb{Q}_p$ and $M_{-V} \subset (-V) \otimes \mathbb{Q}_p$ self-dual lattices; this is equivalent to the assumption that $K_p \cap G(V,-V)(\mathbb{Q}_p)$ is a hyperspecial maximal compact subgroup of $G(V,-V)(\mathbb{Q}_p)$. In (2.1) we will impose additional conditions on the choice of M_V in the general case.

(1.4) Automorphic vector bundles on unitary Shimura varieties, again.

Notation is as in the previous sections: $G = GU(2V)$, resp. $GU(V,-V)$, $X = X(2V)$, resp. $X(V,-V)$, and $[W_\tau]$ is an automorphic vector bundle on $Sh(G,X)$.

In (1.2) we have fixed the stabilizer $K_\infty \subset G(\mathbb{R})$ of a point $x \in X$. Choose a maximal torus $T_\infty \subset K_\infty$, an algebraic subgroup over $\mathbb{R}$ necessarily containing the image of x. Then T_∞ is also a maximal torus in G. A specific choice of pair (T_∞, K_∞) can be obtained as follows. Decompose $(V, <\bullet,\bullet>_V)$ as an orthogonal direct sum of one-dimensional hermitian spaces over $\mathcal{K}$:

$$(V, <\bullet,\bullet>_V) = \oplus_{i=1}^n (V_i, <\bullet,\bullet>_i). \tag{1.4.1}$$

We assume the V_i are numbered so that, for any $\sigma \in \Sigma$, $V_{i,\sigma} = V_i \otimes_{\mathcal{K},\sigma} \mathbb{C}$ has signature $(1,0)$ for $i \leq r_\sigma$ and signature $(0,1)$ for $i > r_\sigma$. Let $-V_i$ denote V_i with the hermitian form $- <\bullet,\bullet>_i$. Let $GU^{\oplus_i}(V,-V)$ denote the subgroup of the torus $\prod_i GU(V_i) \times \prod_i GU(-V_i)$ defined by equality of similitude factors. We obtain embeddings of Shimura data

(1.4.2)
$$(GU^{\oplus_i}(V,-V), \prod_i (X_i \times X_i')) \hookrightarrow (GU(V,-V), X(V,-V)) \hookrightarrow (GU(2V), X(2V))$$

where $\prod_i (X_i \times X_i')$ is an appropriate product of point symmetric spaces determined by the signatures of each V_i and $-V_i$. We write

$$Sh^{\oplus_i}(V,-V) = Sh(GU^{\oplus_i}(V,-V), \prod_i (X_i \times X_i')),$$

the superscript $^{\oplus_i}$ serving as a reminder of the choice of direct sum decomposition above. Define $(GU^{\oplus_i}(V), \prod_i X_i) \subset (GU(V), X(V))$, $GU^{\oplus_i}(-V), \prod_i X_i') \subset (GU(-V), X(-V))$ analogously. The groups $GU^{\oplus_i}(V)$, $GU^{\oplus_i}(-V)$, and $GU^{\oplus_i}(V,-V)$, defined over $\mathbb{Q}$, are maximal $\mathbb{R}$-elliptic tori in $GU(V)$, $GU(-V)$, and $GU(V,-V)$ or $GU(2V)$, and we take T_∞ to be the group of real points of one of these tori. We can of course find K_∞ containing T_∞, though K_∞ will in general not be defined over $\mathbb{Q}$. The Shimura data $(GU^{\oplus_i}(V), \prod_i X_i)$, etc., define CM points of the corresponding unitary Shimura varieties.

The group T_∞ is a maximal torus in a reductive group of type A, and we parametrize its roots in the usual way. In the case $G = GU(V)$, $G_\mathbb{C}$ is naturally isomorphic to $\prod_{\sigma\in\Sigma} GL(n,\mathbb{C}) \times GL(1,\mathbb{C})$, the last term coming from the similitude factor. Thus the group $X(T_\infty)$ of characters λ of the algebraic torus T_∞ consists of d-tuples $(a_{1,\sigma},\ldots,a_{n,\sigma})_{\sigma\in\Sigma}$ of n-tuples of integers, indexed by $\sigma \in \Sigma$, together with a single integer a_0 for the similitude factor. The $(a_{j,\sigma})$ are given by the restriction of the character λ to $T_\infty \cap U(V)$, whereas a_0 is given by the restriction of λ to the maximal $\mathbb{R}$-split torus in $T_\infty \cap Z_G$: if $tI_n \in G(\mathbb{R})$ is a real central element then $\lambda(tI_n) = t^{a_0}$. The parameters satisfy the relation

$$a_0 \equiv \sum_{j,\sigma} a_{j,\sigma} \pmod 2. \tag{1.4.3}$$

Given an ordering on the roots of the maximal torus $T_\infty \subset G$, the dominant weights are then the characters parametrized as above, with $a_{i,\sigma} \geq a_{i+1,\sigma}$, for all σ and $i = 1, \ldots, n-1$. We choose a set of positive roots containing the roots in $\mathfrak{p}^-$. The n-tuple corresponding to σ will often be written with a semi-colon $(a_{1,\sigma}, \ldots, a_{a_\sigma,\sigma}; -b_{b_\sigma,\sigma}, \ldots, -b_{1,\sigma})$ or occasionally $(a_{1,\sigma}, \ldots, a_{a_\sigma,\sigma}; -b_{b_\sigma,\sigma}, \ldots, -b_{1,\sigma}; a_0)$ when the term a_0 needs to be stressed, in such a way that it gives a dominant weight of the σ-factor of $K_\infty \cap U(V)_\infty$, $U(V)_\infty \equiv \prod_\sigma U(a_\sigma, b_\sigma)$ if and only if

$$a_{1,\sigma} \geq \cdots \geq a_{a_\sigma,\sigma},\ b_{1,\sigma} \geq \cdots \geq b_{b_\sigma,\sigma} \tag{1.4.4}$$

The parametrization in $G = GU(2V)$ is the same as above, except that n is replaced by $2n$ and $a_\sigma = b_\sigma = n$. For $G = GU(V, -V)$, we place the parameters for $GU(V)$ and $GU(-V)$ side by side.

If K is sufficiently small, ${}_KSh(V)$ carries a universal abelian scheme ${}_KA$ endowed with PEL structure of the appropriate type. Let $p_K : {}_KA \to {}_KSh(V)$ denote the structure map and put

$$\omega = \omega_V = p_{K,*}\Omega^1_{{}_KA/{}_KSh(V)}.$$

This is a locally-free sheaf on ${}_KSh(V)$ of rank $dn = [E:\mathbb{Q}] \dim_{\mathcal{K}} V$ with a natural action of $\mathcal{O}_{{}_KSh(V)} \otimes_{\mathbb{Q}} \mathcal{K}$, the $\mathcal{K}$-action coming from (1.3.1.3). If we extend the ground field to contain E, then ω breaks up as $\omega = \oplus_{\sigma\in\Sigma}\omega_\sigma$ corresponding to the canonical decomposition $E \otimes_{\mathbb{Q}} \mathcal{K} = \oplus_{\sigma\in\Sigma}\mathcal{K}$. Each ω_σ is a locally-free $\mathcal{O}_{{}_KSh(V)}$-sheaf of rank n. The sheaf $\otimes_{\sigma\in\Sigma}\omega_\sigma$ is the canonical bundle associated to $(0, ..., 0; 1, 0, ..., 0; 1)_{\sigma\in\Sigma}$.

The canonical bundles for other τ's can be constructed as follows, again assuming the ground field contains E. Let $Fl(\omega_\sigma)_{{}_KSh(V)}$ be the scheme representing the functor

$$S \mapsto (\mathcal{E}_1 = \omega_{\sigma/S} \supset \mathcal{E}_2 \supset \cdots \supset \mathcal{E}_n \supset \mathcal{E}_{n+1} = 0\ ;\ \phi_i : \mathcal{E}_i/\mathcal{E}_{i+1} \xrightarrow{\sim} \mathcal{O}_S, i = 1, ..., n).$$

There is an obvious action of $D_\sigma = \mathbb{G}^n_{m/{}_KSh(V)}$ on $FL(\omega_\sigma)$: $d = (d_1, ..., d_n)$ acts by multiplying ϕ_i by d_i. Let $\pi_\sigma : FL(\omega_\sigma) \to {}_KSh(V)$ be the structure map. For each $\tau_\sigma \in X(D_\sigma)$ we define a locally-free sheaf ρ_{τ_σ} on ${}_KSh(V)$ by $H^0(U, \rho_{\tau_\sigma}) = H^0(\pi_\sigma^{-1}(U), \mathcal{O}_{FL(\omega_\sigma)})[\tau_\sigma]$, where the $[\tau_\sigma]$ signifies the submodule on which D_σ acts through τ_σ. We identify each τ_σ with an n-tuple of integers $(m_{1,\sigma}, ..., m_{n,\sigma})$ in the usual way and say that such a τ_σ is dominant if $m_{1,\sigma} \geq \cdots \geq m_{n,\sigma}$. Given a d-tuple $\tau = (\tau_\sigma)_{\sigma\in\Sigma}$ of dominant characters, let $\rho_\tau = \otimes_{\sigma\in\Sigma}\rho_{\tau_\sigma}$. Then we can naturally identify ρ_τ with $[W_\tau]$, where the character of T_∞ associated to τ is $(m_{1,\sigma}, ..., m_{a_\sigma,\sigma}; m_{a_\sigma+1,\sigma}, ..., m_{n,\sigma})$. These identifications respect the maps in (1.2.8) in the obvious way.

(1.5) Fourier expansions of modular forms.

In this section we consider the Shimura datum $(GU(2V), X(2V))$. The symmetric domain $X(2V)$ is holomorphically isomorphic to the product of $[E:\mathbb{Q}] = |\Sigma|$ copies of the irreducible tube domain $X_{n,n}$ of dimension n^2 attached to the group $U(n,n)$. Let $P = P_\Delta \subset G$ be the maximal parabolic defined in §1.5. The group of real points of P stabilizes the 0-dimensional boundary component of this product of tube domains. Fourier expansion with respect to $U(\mathbb{R})$ defines the q-expansion of a holomorphic automorphic form on X relative to a congruence subgroup of $GU(2V,\mathbb{Q})$. By work of Fujiwara [F], extending the results of Chai and Faltings, one can also define q-expansions for sections of the automorphic vector bundles $[W_\tau]$ over ${}_K\mathbb{S}$ when K_p is hyperspecial. In [Hi04, Hi05], Hida defined q-expansions on the closed Igusa tower. We will formulate this theory in an adelic version analogous to the characteristic zero formulation in [H1, §6] and [P].

In [H1, §6] we attach a Shimura datum (G_P, X_P) to the rational parabolic subgroup $P \subset GU(2V)$. The domain X_P is a version of the point boundary component mentioned above, and G_P is a torus; specifically, G_P is contained in the center of the standard Levi component of P. Recall the definition of G_P: the standard rational representation of G on $R_{\mathcal{K}/\mathbb{Q}}(2V)$ carries a family of Hodge structures of type $(-1,0)+(0,-1)$, corresponding to the family of abelian varieties of PEL type over $Sh(2V)$. In a neighborhood of the boundary component corresponding to P, this family degenerates to a mixed Hodge structure of type $(0,0)+(-1,-1)$.

Actually, the formulation in [H1] is not quite correct: in general the boundary Shimura datum should be defined as in [Pink], where X_P is a homogeneous space for $G_P(\mathbb{R})$ finitely fibered over a $G_P(\mathbb{R})$ conjugacy class of homomorphisms $R_{\mathbb{C}/\mathbb{R}} \to G_{P,\mathbb{R}}$. In the present case, $G_P(\mathbb{R})$ has two connected components, corresponding to upper and lower hermitian half-spaces, and X_P consists of two points. The Shimura variety $Sh(G_P, X_P)$ is zero-dimensional, and one easily verifies it is of PEL type.

Indeed, it parametrizes pairs $(\alpha_\Sigma, \alpha_m)$ where α_Σ is a complete level structure on the abelian variety with complex multiplication type $(\mathcal{K}, \Sigma)$, and α_m is an isomorphism

$$\alpha_m : \prod_q \mathbb{Q}_q/\mathbb{Z}_q \xrightarrow{\sim} \prod_q \mu_{q^\infty}$$

Thus as long as one works in finite level K_P prime to p, there is no difficulty defining an integral model ${}_{K_P}\mathbb{S}(G_P)$ of ${}_{K_P}Sh(G_P, X_P)$. For general level K_P, there is a unique normal integral model, and we define this to be ${}_{K_P}\mathbb{S}(G_P)$.

We let U_P denote the unipotent radical of P, and let $U^* = Hom(U_P(\mathbb{Q}), \mathbb{Q})$. This is the vector space denoted $\mathfrak{g}^{-2}(\mathbb{Q})^*$ in [H1,]. The space $U^* \otimes \mathbb{R}$ contains a self-adjoint cone, homogeneous under $P(\mathbb{R})/U_P(\mathbb{R})$, and denoted C in [H1, 5.1]; we let $U^*(C) = U^* \cap C$. Let $[W_\tau]$ be an automorphic vector bundle over $\bar{S}$, as above. There is an automorphic vector bundle $[W_{\tau_P}]$ over $Sh(G_P, X_P)$, and a map

$$(1.5.1) \qquad F.J.^{P,0} : \Gamma(Sh(2V), [W_\tau]) \to \hat{\bigoplus_{\beta \in U^*}} \Gamma(Sh(G_P, X_P), [W_{\tau_P}])$$

defined, with slightly different notation, in [H1, (6.3.3)], and in [Pink, §12]. Here $\hat{\bigoplus}$ is understood as the subset (f_β) of the direct product over $\beta \in U^*$ such that $f_\alpha = 0$ for all but finitely many $\beta \notin U^*(C)$. If $F.J.^{P,0}(f) = (f_\beta)$ for some $f \in$

$\Gamma(Sh(2V), [W_\tau])$, then the usual Fourier expansion is written $\sum f_\beta q^\beta$. The Koecher principle asserts that, for $n > 0$, $F.J.^{P,0}$ is supported on $U^*(C)$, and even for $n = 0$ one takes care only to consider f with that property.

Since C is self-adjoint, it can also be viewed as a cone in $U_P(\mathbb{R})$. One obtains a more reassuring variant of the q-expansion in the following way. Let $N = \dim U_P$, and let

$$\Lambda = \Lambda(K^p) = U_P(\mathbb{Q}) \cap K(U, m) \subset U_P(\mathbb{Q}). \tag{1.5.2}$$

Note that Λ is a lattice in $U_P(\mathbb{Q})$ and does not depend on m. We choose a polyhedral cone $\mathfrak{c} \subset C$ generated by a basis $\{\lambda_1, \dots, \lambda_N\}$ of Λ:

$$\mathfrak{c} = \{\sum_{i=1}^{N} a_i\lambda_i \mid a_i \geq 0\}$$

and let $\mathfrak{c}^* \supset U^*(C)$ be the dual cone:

$$\mathfrak{c}^* = \{v \in U^*(C) \mid v(\lambda_i) \geq 0, i = 1, \dots N\}.$$

Let $\Lambda^* = Hom(\Lambda, \mathbb{Z})$, viewed as a subgroup of $U^*(C)$. Let R be a $\mathcal{O}_v$-algebra and $\mathcal{M}$ a free R-module. The intersection $\Lambda^* \cap \mathfrak{c}^*$ is a free monoid on N generators β_i, $i = i, \dots, N$, and the ring of formal series

$$R[[q^{\Lambda^* \cap \mathfrak{c}^*}]] = \{\sum_{\beta \in \Lambda^* \cap \mathfrak{c}^*} f_\beta q^\beta\}, \tag{1.5.3}$$

with $f_\beta \in R$, and with the usual multiplication rule $q^\beta \cdot q^{\beta'} = q^{\beta+\beta'}$, is then isomorphic to $R[[q^{\beta_1}, \dots q^{\beta_N}]]$. We define the $R[[q^{\Lambda^* \cap \mathfrak{c}^*}]]$-module

$$\mathcal{M}[[q^{\Lambda^* \cap \mathfrak{c}^*}]] = \mathcal{M} \otimes_R R[[q^{\Lambda^* \cap \mathfrak{c}^*}]] = \{\sum_{\beta \in \Lambda^* \cap \mathfrak{c}^*} f_\beta q^\beta\}$$

where now $f_\beta \in \mathcal{M}$ for all β. Taking

$$\mathcal{M}^0 = \mathcal{M}^0([W_{\tau_P}], K_P(m)) = \Gamma({}_{K_P(m)}\mathbb{S}(G_P), [W_{\tau_P}])$$

for appropriate m, $F.J.^{P,0}$ can be regarded as a map

$$F.J.^{P,0} : \Gamma({}_{K(U,m)}Sh(2V), [W_\tau]) \to \mathcal{M}^0([W_{\tau_P}], K_P(m))[[q^{\Lambda^* \cap \mathfrak{c}^*}]]. \tag{1.5.4}$$

Letting K^p run over a fundamental set of open subgroups of $G(\mathbf{A}_f^p)$ corresponds to letting Λ^* grow to a $\mathbb{Z}_{(p)}$-lattice in $U_P(\mathbb{Q})$, or equivalently to adding nth roots of the generators q^{β_i} of $R[[q^{\Lambda^* \cap \mathfrak{c}^*}]]$ for all n prime to p.

(1.5.5) *One-dimensional τ's.*

In the present article we will mainly consider W_τ of dimension one. More precisely, $[W_{\tau_P}]$ is the automorphic vector bundle associated to an algebraic character, say τ_P, of the torus G_P. Fix a base point $x \in {}_{K_P(m)}\mathbb{S}(G_P)(\mathbb{C})$; for instance, we can take x to be the image of the element $1 \in GU(2V)(\mathbf{A})$ under the isomorphism $G_P(\mathbb{Q})\backslash G_P(\mathbf{A})/K_P(m) \xrightarrow{\sim} {}_{K_P(m)}\mathbb{S}(G_P)(\mathbb{C})$. Let W_{τ_P} be the stalk at x of $[W_{\tau_P}]$. Then $H^0(\mathbb{S}(G_P), [W_{\tau_P}])$ can be canonically identified with the space $\mathcal{M}(W_{\tau_P}(\mathbb{C}), K_P(m))$ of $W_{\tau_P}(\mathbb{C})$-valued automorphic forms on G_P of infinity type τ_P^{-1}; i.e., the space of functions

$$c : G_P(\mathbb{Q})\backslash G_P(\mathbf{A})/K_P(m) \to W_{\tau_P}(\mathbb{C})$$

such that $c(g_\infty g) = \tau_P(g_\infty)^{-1}c(g)$ for all $g \in G_P(\mathbf{A})$ and all $g_\infty \in G_P(\mathbb{R})$. Choosing a basis of $W_{\tau_P}(\mathbb{C})$ identifies $\mathcal{M}(W_{\tau_P}(\mathbb{C}), K_P(m))$ with the space

(1.5.5.1)
$$X_{\tau_P}(G_P; K_P(m)) = \{c : G_P(\mathbb{Q})\backslash G_P(\mathbf{A})/K_P(m) \to \mathbb{C} \mid c(g_\infty g) = \tau_P(g_\infty)^{-1}c(g)\}$$

spanned by $\mathbb{C}$-valued Hecke characters of the indicated infinity type. This in turn identifies the Fourier expansion of a holomorphic modular form with an element of $X_{\tau_P}(G_P; K_P(m))[[q^{\Lambda^*\cap\mathfrak{c}^*}]]$. In this notation we can regard $\mathbb{C}$ as a $\mathcal{O}_v$-algebra or, more prudently, regard both $\mathbb{C}$ and $\mathcal{O}_v$ as algebras over the ring of integers of some number field.

(1.5.6) *Comparison with the transcendental theory*

Let $\psi : \mathbf{A}/\mathbb{Q} \to \mathbb{C}^\times$ be a non-trivial additive character, with local component ψ_v at the place v of $\mathbb{Q}$, such that $\psi_\infty(x) = e^{2\pi i x}$. For any $\beta \in U^*(\mathbb{Q})$ we define the character

$$(1.5.6.1) \qquad \psi_\beta : U(\mathbb{Q})\backslash U(\mathbf{A}) \to \mathbb{C}^\times \mid \psi_\beta(u) = \psi(\beta(u)), u \in U(\mathbf{A}).$$

A section $f \in \Gamma(Sh(2V), [W_\tau])$ can be identified with a $W_\tau(\mathbb{C})$-valued automorphic form on $GU(2V)(\mathbf{A})$, belonging to the space on the right-hand side of (1.2.6), which This automorphic form will again be denoted f. We assume we are given an isomorphism of $W_\tau(\mathbb{C})$ with $\mathbb{C}$, so that f is viewed as a complex-valued automorphic form. The Fourier coefficients of such an f are then defined, classically, as functions on $GU(2V, \mathbf{A})$ by

$$(1.5.6.2) \qquad f_\beta(h) = \int_{U(\mathbb{Q})\backslash U(\mathbf{A})} f(uh)\psi_{-\beta}(u)du$$

For $h = (h_\infty, h_f) \in GU(2V, \mathbf{A})$, the holomorphy of f implies a factorization $f_\beta(h) = f_{\beta,\infty}(h_\infty)f_{\beta,f}(h_f)$ where $f_{\beta,\infty}$ depends only on τ and β. Explicitly, if we write $h_\infty = p_\infty k_\infty$ with $p_\infty \in P(\mathbb{R})$ and $k_\infty \in K_\infty$, we have

$$(1.5.6.3) \qquad f_{\beta,\infty}(p_\infty k_\infty) = \tau(k_\infty)^{-1}e^{2\pi i\beta(Z(p_\infty))}$$

where $Z(p_\infty) = p_\infty(x) \in U(\mathbb{C})$, with x the fixed point of K_∞ in $X(2V)$ and $X(2V)$ is realized as the tube domain $U(\mathbb{C})$ over the self-adjoint cone C in $U(\mathbb{R})$ and the action of $P(\mathbb{R})$ on the tube domain is the standard one. For more details, see [H1, II].

We write $q^\beta(h_\infty) = f_{\beta,\infty}(h_\infty)$. The function f can be recovered from the Fourier coefficients by Fourier inversion, to which we add Koecher's principle:

$$(1.5.6.4) \qquad f(h) = \sum_{\beta\in U^*\cap C} f_\beta(h) = \sum_{\beta\in U^*\cap C} q^\beta(h_\infty)f_{\beta,f}(h_f).$$

It follows that the finite parts $f_{\beta,f}$ of f_β, as β varies, suffice to determine the form f. Suppose f is invariant under the compact open subgroup $K \subset GU(2V)(\mathbf{A}_f)$. Now the derived subgroup $GU(2V)^{der}$ is simply-connected, hence strong approximation is valid, and it follows that the coefficients $f_{\beta,f}$ are uniquely determined by their values on any subset $C' \subset GU(2V)(\mathbf{A}_f)$ which maps surjectively onto the quotient $C(K)$ defined as in (1.2). Let $L_P \subset P$ be the standard Levi component, the centralizer of G_P. Then we can take $C' = L_P(\mathbf{A}_f^p)$. It follows that

(1.5.6.5) Transcendental q-expansion principle. *A form $f \in \mathcal{A}_{hol,\tau}(GU(2V))$ is determined by the values $f_{\beta,f}(h_f)$ for $h_f \in L_P(\mathbf{A}_f^p)$.*

To simplify the comparison of the algebraic and transcendental theories, we introduce the "Shimura variety" $Sh(L_P, X_P)$ attached to L_P:

(1.5.6.6) $$Sh(L_P, X_P) = Sh(G_P, X_P) \times^{G_P(\mathbf{A}_f)} L_P(\mathbf{A}_f).$$

This can be interpreted as an inductive limit of profinite schemes over $E(G_P, X_P) = \mathbb{Q}$, with natural $L_P(\mathbf{A}_f)$-action. The normal integral model $\mathbb{S}(G_P)$ extends similarly to an $L_P(\mathbf{A}_f)$-equivariant normal integral model $\mathbb{S}_P$ of $Sh(L_P, X_P)$. The automorphic vector bundles $[W_{\tau_P}]$ on $Sh(G_P, X_P)$ extend trivially to $L_(\mathbf{A}_f)$-equivariant vector bundles on $Sh(L_P, X_P)$. As in (1.5.5), we can write

$$\mathcal{M} = \mathcal{M}([W_{\tau_P}], K_{L(P)}(m)) = \Gamma({}_{K_{L(P)}(m)}\mathbb{S}_P, [W_{\tau_P}])$$

for an appropriate compact open subgroup $K_{L(P)}(m) \subset L_P(\mathbf{A}_f)$, and identify the latter with

(1.5.6.7)
$$X_{\tau_P}(L_P; K_{L_P}(m)) = \{c : G_P(\mathbb{Q})\backslash G_P(\mathbb{R}){\cdot}L_P(\mathbf{A}_f)/K_{L(P)}(m) \to \mathbb{C} \mid c(g_\infty h) = \tau_P(g_\infty)^{-1}c(h)\}$$

where now $h \in L_P(\mathbf{A}_f)$ but $g_\infty \in G_P(\mathbb{R})$. Ignoring the level structure, the Fourier expansion (1.5.6.4), with h_f restricted to $L_P(\mathbf{A}_f^p)$, then corresponds to a map

(1.5.6.8) $$F.J.^P : \Gamma(Sh(2V), [W_\tau]) \to \hat{\bigoplus_{\beta \in U^*}} \Gamma(Sh(L_P, X_P), [W_{\tau_P}])$$

defined over $\mathbb{Q}$. By (1.5.6.5), this map is injective.

(1.5.7) *Trivializations.*

A good choice of basis of W_{τ_P} is provided by the theory of degenerating abelian varieties of type ${}_{K(U,m)}\mathcal{A}_{2V}$ (1.3.1); cf. [K, p. 212 ff.], [H1, Lemma 6.6], and [Pink,12.20]. The automorphic vector bundle W_{τ_P} is some power, say the kth, of the relative canonical sheaf (bundle of top differentials) on the universal degenerating abelian scheme over the toroidal compactification. Its natural basis is then the product

(1.5.7.1) $$(\bigwedge_{j=1}^{N} dq^{\beta_j}/q^{\beta_j})^{\otimes k} = (2\pi i)^{Nk}(\bigwedge_j dz_j)^k,$$

where the tube domain coordinate z_j on $X(2V)$ is defined by $q^{\beta_j} = e^{2\pi i z_j}$. This basis is defined over $\mathbb{Z}_{(p)}$ because the coordinates q^{β_j} are used to define the toroidal compactification over $\mathbb{Z}_{(p)}$ in [F]. Thus the trivialization (1.5.7.1) is compatible with the theory of p-adic modular forms, just as in [K], and allows us to identify

(1.5.7.2) $$F.J.^P(f)_\beta(h) = q^\beta(h_\infty)f_{\beta,f}(h_f),\ h = (h_\infty, h_f) \in L_P(\mathbf{A})$$

where the left-hand term is (1.5.6.8) and the right-hand expression is from (1.5.6.4).

(1.6) p-integral models and p-integral sections.

Let p be a rational prime, and assume hypothesis (1.1.2) is satisfied. Let L' be a finite extension of $\mathbb{Q}$ containing $E(V)$ and let $\mathcal{O}'$ be the ring of integers of L'. For simplicity we will assume that L' also contains E. Fix a sufficiently small compact open subgroup $K = K_p \times K^p \subset G(\mathbf{A}_f)$, as in §1.3. Then it is known (cf. [Ko]) that ${}_KSh(G,X)$ admits a smooth integral model ${}_K\mathbb{S} = {}_K\mathbb{S}(G,X)$ over the valuation ring $\mathcal{O}'_{(p)}$ that is a moduli space for abelian varieties with additional structure of PEL type (the moduli problem is that of (1.3.1) but with ι now an embedding $\mathcal{O}_{(p)} \to End_S(A) \otimes \mathbb{Z}_{(p)}$). Moreover, if L' also contains $E(W_\tau)$ for every τ (a finite set of τ suffices) then the automorphic vector bundles $[W_\tau]$ extend naturally to locally free sheaves over ${}_K\mathbb{S}$. In particular, the construction of the ρ_τ's from §1.4 can be carried out over ${}_K\mathbb{S}$; these then provide integral structures on the various $[W_\tau]$'s. Both ${}_K\mathbb{S}$ and the integral structures on the $[W_\tau]$'s are functorial with respect to change of the level subgroup K^p away from p. In particular, we occasionally drop the notation K in what follows.

By our hypotheses on p, and by an elementary approximation argument, the decomposition (1.4.1) can be taken integral over $\mathcal{O}_{(p)}$. We assume that K is so defined so that $K_p \cap GU^{\oplus_i}(V,-V)(\mathbb{Q}_p)$ is again a maximal compact. Then ${}_KSh^{\oplus_i}(V,-V)$ (where the subscript ${}_K$ has the obvious meaning) also has a model over $\mathcal{O}'_{(p)}$, which we denote ${}_K\mathbb{S}^{\oplus_i} = {}_K\mathbb{S}^{\oplus_i}(V,-V)$. The natural map ${}_KSh^{\oplus_i}(V,-V) \to {}_KSh(G,X)$ (which is just the inclusion of certain CM points) extend to a map ${}_K\mathbb{S}^{\oplus_i} \to {}_K\mathbb{S}$, which can be uses to detect p-integrality of sections of the $[W_\tau]$'s, as we now explain.

Let A_Σ be an abelian variety over $\overline{\mathbb{Q}}$, of dimension $2d$, with complex multiplication by $\mathcal{K}$ of type Σ, and assume $End(A_\Sigma) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathcal{O}_{(p)}$. In other words, $\mathcal{O}_{(p)}$ acts on the object "$A_\Sigma \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$" defined by A_Σ in the category of abelian varieties up to prime-to-p isogeny. One knows A_Σ extends to an abelian scheme, also denoted A_Σ, over the valuation ring $\bar{\mathbb{Z}}_{(p)}$, also with action by $\mathcal{O}_{(p)}$ up to prime-to-p isogeny. There is a decomposition

$$(1.6.1) \qquad H^1_{DR}(A_\Sigma/\bar{\mathbb{Z}}_{(p)}) \xrightarrow{\sim} \oplus_{\sigma\in\Sigma_{\mathcal{K}}} \Omega(\Sigma)_\sigma,$$

with each $\Omega(\Sigma)_\sigma$ a free $\bar{\mathbb{Z}}_{(p)}$-module of rank one. Choose $\bar{\mathbb{Z}}_{(p)}$-generators $\omega_\sigma, \sigma \in \Sigma_{\mathcal{K}}$ of $\Omega(\Sigma)_\sigma$. On the other hand, the topological (Betti) homology $H_1(A_\Sigma(\mathbb{C}), \bar{\mathbb{Z}}_{(p)})$ is a free rank one $\mathcal{O} \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}_{(p)}$-module, hence admits a decomposition

$$(1.6.2) \qquad H_1(A_\Sigma(\mathbb{C}), \bar{\mathbb{Z}}_{(p)}) \xrightarrow{\sim} \oplus_{\sigma\in\Sigma_{\mathcal{K}}} (\bar{\mathbb{Z}}_{(p)})_\sigma$$

where $(\bar{\mathbb{Z}}_{(p)})_\sigma$ is the submodule of $H_1(A_\Sigma(\mathbb{C}), \bar{\mathbb{Z}}_{(p)})$, isomorphic to $\bar{\mathbb{Z}}_{(p)}$, on which $\mathcal{O}$ acts via σ. Choose $\bar{\mathbb{Z}}_{(p)}$-generators $\gamma_{\sigma'} \in (\bar{\mathbb{Z}}_{(p)})_{\sigma'}$, for $\sigma' \in \Sigma_{\mathcal{K}}$. The natural pairing (integration)

$$Int : H^1_{DR}(A_\Sigma/\bar{\mathbb{Z}}_{(p)}) \otimes H_1(A_\Sigma(\mathbb{C}), \bar{\mathbb{Z}}_{(p)}) \to \mathbb{C}$$

defines invariants

$$(1.6.3) \qquad p_{\mathcal{K}}(\sigma, \Sigma) = Int(\omega_\sigma, \gamma), \ \ \sigma \in \Sigma_{\mathcal{K}}, \gamma \in H_1(A_\Sigma(\mathbb{C}), \bar{\mathbb{Z}}_{(p)})$$

where γ is taken to be a free $\mathcal{O} \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}_{(p)}$ generator of $H_1(A_\Sigma(\mathbb{C}), \bar{\mathbb{Z}}_{(p)})$. It is easy to see that $Int(\omega_\sigma, \gamma)$ depends only on the projection of γ on $(\bar{\mathbb{Z}}_{(p)})_{c\sigma}$, hence that the complex number $p_{\mathcal{K}}(\sigma,\Sigma)$ is well defined up to multiplication by units in $(\bar{\mathbb{Z}}_{(p)})^{\times}$. Indeed, both $H_1(A_{\Sigma}(\mathbb{C}),\bar{\mathbb{Z}}_{(p)})$ and $H^1_{DR}(A_{\Sigma}/\bar{\mathbb{Z}}_{(p)})$ are invariants of the prime-to-p isogeny class of A_{Σ}, so the invariants $p_{\mathcal{K}}(\sigma,\Sigma)$ are independent of the choice of base point in the prime-to-p isogeny class of A_{Σ}, up to $(\bar{\mathbb{Z}}_{(p)})^{\times}$-multiples. It is well-known that any two choices of A_{Σ} can be related by a prime-to-p isogeny (concretely, any idèle class of $\mathcal{K}$ mod $\mathcal{K}_{\infty}^{\times}$ can be represented by an idèle trivial at p). Thus the $p_{\mathcal{K}}(\sigma,\Sigma)$ can be considered well-defined invariants of Σ, once a base point in the isogeny class is chosen.

Now the elements of $\Sigma_{\mathcal{K}}$ generate the character group of the torus $R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_m$, hence their restrictions to the subtorus $GU(V_i)$, for any V_i as above, generate the character group of the latter. We only consider characters of $R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_m$ trivial on the Zariski closure of a sufficiently small congruence subgroup of the units in $\mathcal{K}$. These are characters of the Serre group, and can be identified with the formal linear combinations $\sum_{\sigma\in\Sigma_{\mathcal{K}}} n_{\sigma}\sigma$ with $n_{\sigma}\in\mathbb{Z}$ such that $n_{\sigma}+n_{\sigma c}$ is independent of σ. For such characters we define

$$p_{\mathcal{K}}(\sum_{\sigma} n_{\sigma}\sigma,\Sigma,V_i) = \prod_{\sigma} p_{\mathcal{K}}(\sigma e(i,\sigma),\Sigma)^{n_{\sigma}} \tag{1.6.4}$$

where $e(i,\sigma)=1$ if $i\leq a_{\sigma}$ and $e(i,\sigma)=c$ otherwise. More generally, if κ is a character of $\prod_i GU(V_i)\times\prod_i GU(-V_i)$, written as an n-tuple of pairs of formal linear combinations

$$(\sum_{\sigma\in\Sigma_{\mathcal{K}}} n_{i,\sigma}\sigma, \sum_{\sigma\in\Sigma_{\mathcal{K}}} n^{-}_{i,\sigma}\sigma)$$

we define

$$p_{\mathcal{K}}(\kappa,\Sigma,2V) = \prod_i p_{\mathcal{K}}(\sum_{\sigma} n_{i,\sigma}\sigma,\Sigma,V_i)\cdot\prod_i p_{\mathcal{K}}(\sum_{\sigma} n^{-}_{i,\sigma}\sigma,\Sigma,-V_i). \tag{1.6.5}$$

Here $p_{\mathcal{K}}(\sum_{\sigma} n_{i,\sigma}\sigma,\Sigma,-V_i)$ is defined as in (1.6.4) but with a_{σ} replaced by $n-a_{\sigma}$

The subgroup $T=GU^{\oplus_i}(V,-V)_{\infty}\subset\prod_i GU(V_i)\times\prod_i GU(-V_i)$ is a maximal torus in K_{∞} (maximal compact mod center in $G=GU(2V)$). The formalism of CM periods implies that the product on the right in (1.6.5) depends only on the restriction of the algebraic character κ to the subgroup T. Indeed, if the restriction of κ to $\prod_i U(V_i)\times\prod_i U(-V_i)$ is trivial, then in particular $n_{i,\sigma}=n_{i,\sigma c}$ for all i and all σ. Since $n_{i,\sigma}+n_{i,\sigma c}$ is independent of σ for each i, it follows that $n_i=n_{i,\sigma}$ is independent of σ for each i, and one can define n_i^{-} likewise. One then has

$$p_{\mathcal{K}}(\sum_{\sigma} n_{\sigma}\sigma,\Sigma,V_i) = p_{\mathcal{K}}(\sum_{\sigma}\sigma,\Sigma,V_i)^{n_i} = p_{\mathcal{K}}(||\bullet||,1)^{n_i} = (2\pi i)^{-dn_i}$$

as in [H2,Lemma 1.8.3]. If moreover $\kappa|_T\equiv 1$, then $\sum_i n_i+n_i^{-}=0$, and so the product of powers of $2\pi i$ is in fact algebraic. Hence the statement of the following Proposition makes sense:

(1.6.6) Proposition. *Let $G=GU(2V)$. Let κ be a character of the torus T that extends to a one-dimensional representation of K_{∞}. Let $[W_{\tau}]$ be the corresponding automorphic line bundle over ${}_K\mathbb{S}$. Let $D\subset Sh^{\oplus_i}(V,-V)(\overline{\mathbb{Q}})$ be a set of points with the following property: the $G(\mathbf{A}_f^p)$ orbit of the image of D under specialization is Zariski dense in the special fiber of ${}_K\mathbb{S}$. Then $f\in H^0({}_K\mathbb{S},[W_{\tau}])\otimes_{L'}\mathbb{C}$ belongs to*

$H^0({}_K\mathbb{S}, [W_\tau]) \otimes_{O'} \bar{\mathbb{Z}}_{(p)}$ *if and only if, for all* $g \in G(\mathbf{A}_f^p)$*, the weight* κ *component* $f^g[\kappa]$ *of the restriction of the* g *translate* f^g *of* f *to* D *satisfies*

$$p_{\mathcal{K}}(\kappa, \Sigma, 2V)^{-1} f^g[\kappa](x) \in \bar{\mathbb{Z}}_{(p)} \tag{1.6.7}$$

for all $x \in D$*. Here the section* $f^g \in H^0({}_K\mathbb{S}, [W_\tau]) \otimes_{L'} \mathbb{C}$ *is identified with a classical automorphic form on* $X(2V) \times G(\mathbf{A}_f)$ *via* (1.2.6)*. The same holds with* $\mathbb{C}$ *replaced by* $\mathbb{C}_p$ *and* $\bar{\mathbb{Z}}_{(p)}$ *replaced by* $\mathcal{O}_{\mathbb{C}_p}$*.*

Remark. There is an analogous proposition for $[W_\tau]$ of arbitrary dimension, but we will not be needing it in the present paper.

Proof. Write $H = H^0({}_K\mathbb{S}, [W_\tau])$, $\bar{H} = H \otimes_{\mathcal{O}'_{(p)}} \overline{\mathbb{Q}}$. Our hypothesis on D implies that $D \cdot G(\mathbf{A}_f^p)$ is Zariski dense in the generic fiber ${}_K Sh(G, X)$. Then (1.6.7), with $\bar{\mathbb{Z}}_{(p)}$ replaced by $\overline{\mathbb{Q}}$, is a version of Shimura's criterion for f to belong to $\bar{H}$ (cf. ([H1,§5.3], cf. [H3, III, Lemma 3.10.2] for an explicit statement when $\mathcal{K}$ is imaginary quadratic). Then there is a number field L, containing L, such that $f \in H \otimes_{L'} L$. Let $H_p = H \otimes_{\mathcal{O}'_{(p)}} \mathcal{O}_{L,(p)}$. Thus H_p is a free $\mathcal{O}_{L,(p)}$-module of finite rank, and $\bar{H} = H_p \otimes_{\mathcal{O}_{L,(p)}} \overline{\mathbb{Q}}$.

Let $\mathfrak{p}$ be a prime of $\mathcal{O}_{L,(p)}$, necessarily dividing p, and let ϖ be a uniformizer of $\mathfrak{p}$. Thus for some positive integer m we have $\varpi^m f \in H_p$. Write $F = \varpi^m f$. Condition (1.6.7) asserts that

$$p_{\mathcal{K}}(\kappa, \Sigma, 2V)^{-1} F^g[\kappa](x) \equiv 0 \pmod{\mathfrak{p}^m}, \forall x \in D \tag{1.6.8}$$

The proposition then comes down to showing that any F satisfying (1.6.8) belongs to $\mathfrak{p}^m H_p$.

Since belonging to $\mathfrak{p}^m H_p$ is a local condition on ${}_K\mathbb{S} \times \mathcal{O}_{L,p}$, we can replace the latter by an affine open subset $U = Spec(A)$ flat over $\mathcal{O}_{L,p}$, and H_p by a free A-module M_p; F is an element of M_p. By induction we reduce to the case $m = 1$. Let $\bar{U} = Spec(A/pA)$ denote the special fiber of U; for a geometric point y of $\bar{U}$ let $I_y \subset A$ denote the maximal ideal at y. Condition (1.6.8) is the condition that $F \in I_y \cdot M_p$ for y in a Zariski dense subset $\bar{D}$ of $\bar{U}$; this is essentially the obvious p-integal version of the results of [H3, (3.10)]. By definition, the intersection $\bigcap_{\bar{D}} I_y = \mathfrak{p} \cdot A$. Since M_p is free of finite rank over the noetherian ring A, the proposition is clear.

A simple continuity argument now provides the proof in the case where $\bar{\mathbb{Z}}_{(p)}$ is replaced by $\mathcal{O}_{\mathbb{C}_p}$.

2. p-ADIC AUTOMORPHIC FORMS ON UNITARY GROUPS

(2.1) The Igusa tower, I: Definitions.

Notation is as in §1. Recall the $\mathbb{Q}_p$-rational $\mathcal{K} \otimes_{\mathbb{Q}} \mathbb{Q}_p$-submodule $F^0V_p \subset V_p$, defined in (1.1.6), and the $\mathcal{K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$-submodule subspace $F^0V \subset V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ introduced at the beginning of (1.3). The flag variety $\hat{X}$ of $\mathcal{K}$-linear subspaces of V satisfying (1.3.0) has a natural $E(V)$-rational structure. Hypothesis (1.1.6) is equivalent to the condition that the completion $E(V)_{w_0}$ of $E(V)$ at the place w_0 of $E(V)$ corresponding to $incl_p$ is isomorphic to $\mathbb{Q}_p$, and the $\mathcal{K}$-linear subspace $F^0V_p \subset V_p$ is indeed a $\mathbb{Q}_p = E(V)_{w_0}$-rational point of $\hat{X}$.

The skew-hermitian pairing $tr_{\mathcal{K}/\mathbb{Q}} < \bullet, \bullet >_{V,\daleth}$ on $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ defines a perfect duality

$$V_{p,\Sigma_p} \otimes V_{p,c\Sigma_p} \to \mathbb{Q}_p. \tag{2.1.1}$$

This duality identifies

$$V_w/F^0V_w \xrightarrow{\sim} Hom(F^0V_{c\cdot w}, \mathbb{Q}_p)$$

for any w dividing p. In this way

$$\prod_{w|p} GL(F^0V_w) \xrightarrow{\sim} \prod_{w,\Sigma_w\subset\Sigma_p} GL(F^0V_w) \times GL(V_w/F^0V_w), \tag{2.1.2}$$

is naturally isomorphic to the Levi quotient L^0 of the parabolic

$$P^0 = \prod_{w,\Sigma_w\subset\Sigma_p} Stab(F^0V_w).$$

Here P^0 is a viewed as a parabolic subgroup of the unitary group G_0 rather than the unitary similitude group G. Bear in mind that the action of L^0 on V_w/F^0V_w is dual to that on $F^0V_{c\cdot w}$.

We return to the situation of (1.3.1), and let $K = K^p \times K_p$ where $K_p = G_0(\mathbb{Z}_p) \times GL(1, \mathbb{Z}_p)$ is the hyperspecial maximal compact subgroup of (1.3.4), viewed as the group of $\mathbb{Z}_p$-points of a smooth reductive group scheme $\mathbb{K}$ over $\mathbb{Z}_p$ with generic fiber $G \times_{\mathbb{Q}} \mathbb{Q}_p$. We assume the subspace F^0V_p and K_p are chosen compatibly, in the sense that P^0 is the $\mathbb{Q}_p$-points of a parabolic subgroup $\mathbb{P}^0 \subset \mathbb{K}$ (parabolic in the subgroup of $\mathbb{K}$ corresponding to $G_0(\mathbb{Z}_p)$), and we can define $\mathbb{L}^0$ to be the Levi quotient of $\mathbb{P}^0$ so that $L^) = (L)^0(\mathbb{Q}_p)$. Equivalently, V_p and F^0V_p contain compatible $\mathcal{O}_p$-stable lattices M and M^0, respectively, with K_p the stabilizer of M, and the decomposition $F^0V_p = \oplus_{w|p}F^0V_w$ of (1.1.6) is obtained by extension of scalars from a decomposition $M^0 = \oplus_{w|p}M^0_w$; $\mathbb{P}^0$ is then the stabilizer in $\mathbb{K}$ of M^0. Where necessary, we write $M = M(V)$, $M^0 = M(V)^0$, etc., to emphasize the relation with the hermitian space V defining the moduli problem.

We write

$$M^0_{\Sigma_p} = \oplus_{w,\Sigma_w\subset\Sigma_p} M^0_w,\ M^{-1}_{\Sigma_p} = \oplus_{w,\Sigma_w\subset\Sigma_p} M_w/M^0_w \tag{2.1.3}$$

As in the preceding paragraph, the skew-hermitian form $tr_{\mathcal{O}/\mathbb{Z}} < \bullet, \bullet >_{V,\daleth}$ can be normalized to define a natural skew-hermitian perfect duality.

$$M^0 \otimes M/M^0 \to \mathbb{Z}_p. \tag{2.1.4}$$

There is also a natural isomorphism

$$M^0 \xrightarrow{\sim} M^0_{\Sigma_p} \oplus Hom^c(M^{-1}_{\Sigma_p}, \mathbb{Z}_p), \tag{2.1.5}$$

where

$$Hom^c(M^{-1}_{\Sigma_p}, \mathbb{Z}_p) = \mathcal{O}_p \otimes_{\mathcal{O}_p,c} Hom(M^{-1}_{\Sigma_p}, \mathbb{Z}_p)$$

i.e. the natural action of $\mathcal{O}_p$ on $Hom(M^{-1}_{\Sigma_p}, \mathbb{Z}_p)$ is composed with complex conjugation.

Let ${}_{K^p}\mathcal{A}^p = {}_{K^p}\mathcal{A}^p_{V,\daleth}$ be the functor

$$S \mapsto \{(A, \lambda, \iota, \alpha^p)\}$$

where A is now an abelian scheme over S up to *prime-to* p-isogeny, λ is a polarization of degree prime to p, $\iota : \mathcal{O}_{(p)} \to End_S(A) \otimes \mathbb{Z}_{(p)}$ is an embedding of $\mathbb{Z}_{(p)}$-algebras, and $\alpha^p : V(\mathbf{A}^p_f) \xrightarrow{\sim} V^{f,p}(A)$ is a prime-to-p $\mathcal{O}_{(p)}$-linear level structure modulo K^p. The forgetful map ${}_{K^p}\mathcal{A}^p \to {}_K\mathcal{A}$ is obviously an isomorphism. The functor ${}_{K^p}\mathcal{A}^p$ is representable over the integer ring $\mathcal{O}_{w_0}$ of $E(V)_{w_0}$ by a scheme we will denote ${}_K\mathbb{S}$, as in (1.4).

(2.1.6) *Igusa Schemes*

The following constructions are compatible with change of the level subgroup K^p, and with passage to the limit over all K^p. Hence we drop the subscript ${}_{K^p}$ for the time being. We view $\mathcal{A}^p$ as a functor on the category of schemes over $\mathcal{O}_{w_0}$. Points of $\mathcal{A}^p(S)$ will be denoted $\underline{A}$. Define three families of functors above $\mathcal{A}^p$, indexed by non-negative integers m:

(2.1.6.1) $$Ig_{1,m}(S) = \{(\underline{A}, j^{et})\}, \quad j^{et} : A[p^m] \twoheadrightarrow (M/M^0) \otimes \mathbb{Z}/p^m\mathbb{Z}.$$

(2.1.6.2) $$Ig_{2,m}(S) = \{(\underline{A}, j^o)\}, \ \underline{A} = (A, \lambda, \iota, \alpha^p), \quad j^o : M^0 \otimes \mu_{p^m} \hookrightarrow A[p^m].$$

(2.1.6.3) $$Ig_{3,m}(S) = \{(\underline{A}, j^0, j^{(-1)})\},$$
$$j^0 : M^0_{\Sigma_p} \otimes \mu_{p^m} \hookrightarrow A[p^m]_{\Sigma_p}, \ j^{(-1)} : A[p^m]_{\Sigma_p} \twoheadrightarrow M^{-1}_{\Sigma_p} \otimes \mathbb{Z}/p^m\mathbb{Z}.$$

In each case $\underline{A}$ designates a quadruple $(A, \lambda, \iota, \alpha^p) \in \mathcal{A}^p(S)$. The maps j^0, j^{et}, j^0, and $j^{(-1)}$ are all assumed $\mathcal{O}/p^m\mathcal{O}$-linear.

(2.1.6.4) Lemma. *The functors $Ig_{i,m}$, $i = 1, 2, 3$, are all relatively representable over $\mathcal{A}^p$, and are canonically isomorphic for all m. These isomorphisms are compatible with the natural forgetful projection maps $Ig_{i,m+1} \to Ig_{i,m}$ for all i; moreover, these projection maps are étale for all m.*

Proof. Since the polarization λ is assumed of degree prime to p, we can use it to identify $\hat{A}[p^m] \xrightarrow{\sim} A[p^m]$. The isomorphism $Ig_{1,m} \xrightarrow{\sim} Ig_{2,m}$ is then obtained by combining the duality (2.1.4) with Cartier duality $A[p^m] \times \hat{A}[p^m] \to \mu_{p^m}$. The isomorphism between $Ig_{2,m}$ and $Ig_{3,m}$ is obtained in a similar way from (2.1.5). Compatibility of these isomorphisms with the forgetful projection maps is obvious. Finally, the projection $Ig_{1,m+1} \to Ig_{1,m}$ is obviously étale, since it corresponds to lifting a trivialization of the étale quotient of $A[p^m]$ to one of the étale quotient of $A[p^{m+1}]$.

Since the isomorphisms in (2.1.6) are canonical, we write Ig_m for $Ig_{i,m}$, $i = 1, 2, 3$, or $Ig(V)_m$ when we need to emphasize V. For any $m > 0$, the natural forgetful map $Ig_m \to \mathbb{S}$ obviously factors through the inclusion of the ordinary locus $\mathbb{S}^{ord} \subset \mathbb{S}$. The limit $Ig_\infty = \varprojlim_m Ig_m$ is an étale Galois covering of $\mathbb{S}^{ord}$ with covering group

$$\mathbb{L}^0(\mathbb{Z}_p) = Aut(M^0) \xrightarrow{\sim} Aut(M^0_{\Sigma_p}) \times Aut(M^{-1}_{\Sigma_p}).$$

Let $\mathbb{F}$ denote the algebraic closure of the residue field of $\mathcal{O}_{w_0}$, and let $\bar{S} = {}_K\mathbb{S} \times_{\mathcal{O}_{w_0}} \mathbb{F}$ denote the geometric special fiber of the moduli scheme ${}_K\mathbb{S}$. Let $\bar{S}^{ord} = \mathbb{S}^{ord} \times_{Spec(\mathcal{O}_{w_0})} \bar{S} \subset \bar{S}$ denote the ordinary locus of the special fiber. The following theorem is a special case of a result of Wedhorn [We]:

(2.1.7) Theorem. *The ordinary locus $\bar{S}^{ord}$ contains an open dense subscheme of every irreducible component of $\bar{S}$.*

(2.1.8) *Modular interpretation of the Igusa tower in the limit*

In the limit as m tends to infinity we can reformulate the definition of the Igusa tower in terms of abelian varieties up to isogeny. We prefer to use the models $Ig_{3,m}$. Let $T(\mathbb{G}_m) = \varprojlim_m \mu_{p^m}$ denote the Tate module of the multiplicative group, viewed as a profinite flat group scheme over $Spec(\mathbb{Z}_p)$. For any vector space W over $\mathbb{Q}_p$ we let $W(1) = W \otimes_{\mathbb{Z}_p} T(\mathbb{G}_m)$. Consider the functor on schemes over $\mathcal{O}_{w_0}$:

(2.1.8.1)
$$\begin{gathered} Ig'_{3,\infty}(S) = \{(\underline{A}, j^0, j^{-1})\}, \\ j^0 : F^0V_{\Sigma_p}(1) \hookrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A[p^\infty]_{\Sigma_p},\ j^{-1} : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A[p^\infty]_{\Sigma_p} \twoheadrightarrow V/F^0V_{\Sigma_p}. \end{gathered}$$

Here $\underline{A} = (A, \lambda, \iota, \alpha^p)$ as above, but now A is an abelian variety up to isogeny, and $A[p^\infty]$ is a p-divisible group up to isogeny, or rather quasi-isogeny (cf. [RZ], 2.8). For fixed m we define $Ig'_{3,m}$ by the same functor as $Ig'_{3,\infty}$ but with j^0 and j^{-1} defined only modulo the principal congruence subgroups modulo p^m of $GL(M^0_{\Sigma_p})$ and $GL(M^{-1}_{\Sigma_p})$, respectively. The usual argument shows that

(2.1.8.2) Lemma. *There are canonical isomorphisms $Ig'_{3,m} \xrightarrow{\sim} Ig_{3,m}$ for all m, compatible with the forgetful maps from level p^{m+1} to level p^m for all m.*

In particular, the natural action of $\mathbb{L}^0(\mathbb{Z}_p)$ on Ig_∞ extends canonically to an action of $L^0(\mathbb{Q}_p)$.

The final assertion is completely analogous to the existence of an action of $G(\mathbf{A}_f^p)$ in the inverse limit over K^p.

(2.1.9) *Irreducibility of the Igusa tower*

We reintroduce the prime-to-p level subgroups K^p, and the level subgroup $K = K^p \times K_p$. The fiber over $\mathbb{Q}_p$ of the ordinary locus ${}_K\mathbb{S}^{ord}$ coincides with ${}_K\mathbb{S} \times_{\mathbb{Z}_p} \mathbb{Q}_p = {}_KSh(V)_{\mathbb{Q}_p}$; here, as above, we identify $\mathbb{Q}_p = E(V)_{w_0}$. The generic fibers ${}_{K^p}Ig_{m,\mathbb{Q}_p}$ can be identified with Shimura varieties attached to appropriate level subgroups, as follows. Let $U \subset \mathbb{P}^0$ denote the unipotent radical. For any non-negative integer m, let $K(U,m)_p \subset K_p$ denote the inverse image of $U(\mathbb{Z}_p/p^m Zp) \times GL(1, \mathbb{Z}_p)$ under the natural map $K_p \to \mathbb{K}(\mathbb{Z}_p/p^m Zp)$. Let $K(U,m) = K(U,m)_p \times K^p$. The variety ${}_{K(U,m)}Sh$, as m tends to infinity, parametrizes quadruples $(A, \lambda, \iota, \alpha)$ where $\alpha = (\alpha_{p,m}, \alpha^p)$ with α^p as above and

$$\alpha^0_{p,m} : M/p^m M \xrightarrow{\sim} A[p^m] \pmod{K(U,m)}$$

is an $\mathcal{O}/p^m\mathcal{O}$-linear injection that identifies the given skew-symmetric pairing on M/p^mM with the Weil pairing on $A[p^m]$. This comes down to an inclusion of $(M/p^mM)^{K(U,m)} = M^0/p^mM^0$ in $A[p^m]_{\Sigma_p}$ and a Cartier dual surjection of $A[p^m]_{\Sigma_p}$ onto M^{-1}/p^mM^{-1}. It follows that there are natural isomorphisms

$$ {}_{K^p}Ig_{m,\mathbb{Q}_p} \xrightarrow{\sim} {}_{K(U,m)}Sh \tag{2.1.9.1} $$

compatible with the forgetful maps from level $m+1$ to level m.

Over $\mathbb{C}$, the connected components of ${}_{K(U,m)}Sh(G,X)$ are in bijection with the class group $C(m) = C(K(U,m))$, as at the end of (1.2). Consider the normalization $\widehat{{}_K\mathbb{S}}$ of ${}_K\mathbb{S}$ in ${}_{K(U,m)}Sh(G,X)$. This is an $\mathcal{O}_v$-model of ${}_{K(U,m)}Sh(G,X)$, though not a very good one. However the non-singular locus $(\widehat{\bar{S}}^{o}$ is étale over ${}_K\bar{S}^{ord}$, and ${}_{K^p}Ig_m$ is naturally isomorphic to an open subscheme of $(\widehat{\bar{S}})^o$. In particular, there is a map $c_m :{}_{K^p} Ig_m \to C(m)$, which can be given a modular interpretation as in (1.2). A special case of Corollary 8.17 of [Hi04], (cf. also [Hi05, §10]) is that

(2.1.10) Theorem. ([Hida]) *The fibers of c_m are geometrically irreducible for all m.*

This is proved in [loc. cit.] under a hypothesis labeled (ord), which is equivalent to our hypothesis (1.1.2). Lemma 8.10 of [loc. cit.] makes this explicit, but only for imaginary quadratic $\mathcal{K}$.

(2.1.11) *Inclusion of Igusa towers for $Sh(V,-V)$ in $Sh(2V)$*

Applying the previous discussion to the hermitian space $2V$, we identify $Ig(2V)_m = Ig(2V)_{2,m}$ with the moduli space of quintuples

$$ \{\underline{B} = (B, \mu, \iota_2, \beta^p), j^o_{2V} : M(2V)^0 \otimes \mu_{p^m} \hookrightarrow B[p^m]). $$

Now $M(2V)^0$ is a lattice in the $\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p$-submodule $F^0(2V)_p$ of $(2V)_p$, which we can choose arbitrarily as long as we respect Hypothesis (1.1.6). For example, we can choose

$$ F^0(2V)_p = F^0V_p \oplus F^0(-V)_p \tag{2.1.11.1} $$

where $F^0(-V)_p \subset (-V)_p$ is any $\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p$-submodule satisfying (1.1.6), which for $-V$ amounts to the condition that $\dim F^0(-V)_w = n - a_\sigma = a_{c\sigma}$ for any $\sigma \in \Sigma_w$. As $\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p$-module $-V$ is isomorphic to V, and it is particularly convenient to choose $F^0(-V)_p \subset (-V)_p = V_p$ to be a subspace mapping isomorphically to V_p/F^0V_p under the projection, or equivalently such that (2.1.1) restricts to a duality between $F^0(-V)_w$ and F^0V_{cw} for any w dividing p.

We define $Ig(V,-V)_m \subset Ig(V)_m \times Ig(-V)_m$ as $Sh(V,-V)$ in (1.3) as the subvariety with compatible polarizations. Then, ignoring prime-to p level structures, the reduction modulo p of the natural morphism $Sh(V,-V) \subset Sh(2V)$ defines a family of morphisms $Ig(V,-V)_m \to Ig(2V)_m$ whose image, in the version $Ig_{2,m}$, is the moduli space of quintuples as above where

$$ (B, \mu, \iota_2, \beta) \xrightarrow{\sim} (A \times A^-, \lambda \times \lambda^-, \iota \times \iota^-, \alpha \times \alpha^-) $$

as in §1.4 and where

$$ j^o_{2V} = j^o_V \times j^o_{-V} : M(V)^0 \otimes \mu_{p^m} \times M(-V)^0 \otimes \mu_{p^m} \hookrightarrow A[p^m] \times A^-[p^m]. \tag{2.1.11.2} $$

We make this more explicit. Fix w dividing p, let $(a,b) = (a_w, b_w)$, and choose bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ for V_w and $(-V)_w$, respectively, with $e_1, \ldots e_a$ a basis for $F^0 V_w$, $f_1, \ldots f_b$ a basis for $F^0(-V_w)$. We regard the natural identification of V_w with $(-V)_w$ as an isomorphism between the two halves of $2V$, in such a way that e_i is taken to f_{b+i} for $1 \leq i \leq a$ and e_{a+j} is taken to f_j for $1 \leq j \leq b$. The $2n \times 2n$-matrix $\gamma_1 = \gamma_{1;a,b}$:

$$\gamma_1 = \gamma_{1;a,b} = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & 0 & I_b \\ 0 & 0 & I_a & 0 \\ 0 & I_b & 0 & 0 \end{pmatrix}, \tag{2.1.11.3}$$

in the basis $(e_1, \ldots, e_n, f_1, \ldots f_n)$ of $2V_w$, takes the subspace $V_w \subset 2V_w$ to the subspace $F^0(2V)_w$ defined by (2.1.11.1).

(2.2) The Igusa tower, II: p-adic modular forms.

We now recall Hida's generalization of the Deligne-Katz construction of p-adic modular forms, for the Shimura varieties $Sh(G,X)$. In the present article we will only need p-adic modular forms in order to define a good notion of p-integrality for certain holomorphic Eisenstein series ramified at p, but later we will use them to construct p-adic L-functions and establish their boundedness . So for the moment we let $(G,X) = (GU(V), X(V))$ or $(GU(V,-V), X(V,-V))$.

We work with a smooth, projective, toroidal compactification ${}_K\mathbb{S}^{\sim}$ of ${}_K\mathbb{S}$. The construction of such compactifications in this setting is due to Fujiwara. The choice of ${}_K\mathbb{S}^{\sim}$ is not canonical. However, the universal abelian scheme ${}_K\mathbb{A}$ over ${}_K\mathbb{S}$ extends to a semi-abelian scheme over ${}_K\mathbb{S}^{\sim}$. Hence ω, and therefore each ρ_τ, also extends.

Let v be the prime of $\mathcal{K}$ determined by $incl_p$. We begin by choosing a lifting of ${}_K\bar{S}^{ord}$ to an $\mathcal{O}_v$-flat open subscheme of ${}_K\mathbb{S}^{\sim}$. (This is possible since under (1.1.2) $E(V)_{w_0} = \mathbb{Q}_p$ so our schemes are all defined over $\mathcal{O}_v$.) More precisely, ${}_K\bar{S}^{ord}$ is defined by the non-vanishing of the Hasse invariant H, which can be regarded as a section of a certain automorphic line bundle $[\mathcal{L}]$ over $\bar{S}$. The line bundle $\mathcal{L}$ is known to be ample, hence for some power $\kappa >> 0$ the section H^κ lifts to a section $\tilde{H} \in \Gamma({}_K\mathbb{S}, [\mathcal{L}]^\kappa)$. We let ${}_K\mathbb{S}^{ord} \subset {}_K\mathbb{S}$ be the open subscheme defined by non-vanishing of $\tilde{H}$. This is slightly abusive, since it depends on the choice of lifting $\tilde{H}$, but different choices yield isomorphic theories. For all this, see [Hi05, p. 213 ff.] or [SU].

We let W be a finite flat $\mathcal{O}_v$ algebra, $W_r = W/p^r W$, and let $S_m = {}_K\mathbb{S}^{ord} \otimes_{\mathcal{O}_v} W_r$. The S_r form a sequence of flat W_r schemes, with given isomorphisms

$$S_{r+1} \otimes_{W_{r+1}} W_r \xrightarrow{\sim} S_r.$$

For $m \geq 1$, let $P_m = \mathbb{A}[p^m]^{et} = \mathbb{A}[p^m]/\mathbb{A}[p^m]^0$ over ${}_K\bar{S}^{ord}$, the quotient of the ${}_K\bar{S}^{ord}$ group scheme of p^m-division points of $\mathbb{A}$ by its maximal connected subgroup scheme. This is a free étale sheaf in $\mathcal{O}_v/p^m\mathcal{O}_v$-modules over ${}_K\bar{S}^{ord}$, hence lifts canonically, together with its $\mathcal{O}_v$-action, to an étale sheaf over S_r for all r. Following [Hi04], we define $\mathcal{T}_{r,m}$ to be the lifting to S_r of the corresponding principal $GL(n, \mathcal{O}/p^m\mathcal{O})$-bundle (resp.., $GL(n, \mathcal{O}/p^m\mathcal{O}) \times GL(n, \mathcal{O}/p^m\mathcal{O})$-bundle) $Ig_m(V) = Ig_{1,m}$ (resp., $Ig_m(V,-V)$), defined by (2.1.6.1) (resp., as in (2.1.11)); note that our indices are not the same as Hida's. Let

$$\mathcal{V}_{r,m} = \Gamma(\mathcal{T}_{r,m}, \mathcal{O}_{\mathcal{T}_{r,m}});\ \mathcal{V}_{r,\infty} = \varinjlim_m \mathcal{V}_{r,m};\ \mathcal{V}_{\infty,\infty} = \varprojlim_r \mathcal{V}_{r,\infty}$$

Note that these carry actions of $GL(n, \mathcal{O}_p)$ or of $GL(n, \mathcal{O}_p) \times GL(n, \mathcal{O})$, depending on whether $G = GU(V)$ or $GU(V, -V)$. Let U be the upper-triangular unipotent radical of $GL(n, \mathcal{O}_p)$ or $GL(n, \mathcal{O}_p) \times GL(n, \mathcal{O}_p)$, depending. We then define our space of p-adic modular forms to be

$$\mathcal{V} := \mathcal{V}^U_{\infty,\infty}.$$

We will adopt the convention of adding a superscript V or $V, -V$ when it is necessary to distinguish the groups in question. Hence, $\mathcal{V}_V$ is the ring of p-adic modular forms for $GU(V)$.

It is clear that the construction of the spaces of p-adic modular forms for $GU(V, -V)$ and $GU(2V)$ can be done compatibly, at least when the various prime-to-p level structures are compatible (i.e., there are morphisms ${}_KSh(V, -V) \to {}_{K'}Sh(2V)$). This gives rise to a restriction map

$$r_V : \mathcal{V}_{2V} \to \mathcal{V}_{(V,-V)}.$$

The primary goal of this section is to explain why this is a good definition and how it naturally contains all p-adic sections of $[W_\tau]$ for all τ, and, in the case $G = GU(2V)$, is contained in the power series ring $R[[q^{\Lambda^* \cap \mathfrak{c}^*}]]$ of (1.5.3) for an appropriate R. For $n > 1$, the sections of $[W_\tau]$ are vector-valued functions. To compare them for different τ, we follow Hida and trivialize the $[W_\tau]$, using the modular definition of $\mathcal{T}_{r,m}$, and then apply the theorem of the highest weight in integral form. The discussion below follows [Hi04,8.1], to which we refer for missing details.

Let $\omega_{r,m}$ denote the pullback of ω to $\mathcal{T}_{r,m}$. By Cartier duality, the universal surjection (2.1.6.1), with $S = \mathcal{T}_{1,m}$, is equivalent to an isomorphism of group schemes

$$\mathfrak{d}^{-1} \otimes (\mu_{p^m})^n \xrightarrow{\sim} \hat{\mathbb{A}}[p^m]^0. \tag{2.2.1}$$

Here $\mathfrak{d}^{-1}$ is the different of $\mathcal{K}$ over $\mathbb{Q}$, μ_{p^m} is the kernel of multiplication by p^m in the multiplicative group scheme, $\hat{\mathbb{A}}$ is the abelian scheme dual to $\mathbb{A}$, and the superscript ^{0} denotes the maximal connected subgroup scheme. Since (2.2.1) is Cartier dual to an isomorphism of étale group schemes induced by (2.1.6.1), it lifts canonically to $\mathcal{T}_{r,m}$. Since there are canonical isomorphisms

$$\omega_{r,m} \xrightarrow{\sim} Lie(\hat{\mathbb{A}}) \otimes W_r \xrightarrow{\sim} Lie(\hat{\mathbb{A}}[p^m]^0) \otimes W_r$$

we can identify

$$\omega_{r,m} \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes Lie(\mu_{p^m})^n \otimes W_r \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mathcal{O}^n_{\mathcal{T}_{r,m}}. \tag{2.2.2}$$

as $\mathcal{O}_p \otimes_{\mathbb{Z}_p} W_r$ modules.

Since $\mathcal{K}$ is unramified at p, $\mathfrak{d}^{-1}$ is prime to p, and (2.2.2) reduces to a family of $\mathcal{O}_p \otimes_{\mathbb{Z}_p} W_r$ isomorphisms

$$\omega_{r,m} \xrightarrow{\sim} \mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}^n_{\mathcal{T}_{r,m}}, \tag{2.2.3}$$

compatible as m and r vary. Note that in (2.2.1), (2.2.2), and (2.2.3) the n should be replaced by a $2n$ if $G = GU(V, -V)$.

Suppose that $G = GU(V)$. Now we apply the highest weight formalism as in [Hi05]. Let $G_1 = Res_{\mathcal{O}_p/\mathbb{Z}_p} GL(n)$, let B_1 be the upper-triangular Borel of G_1, U_1 its unipotent radical, and T_1 the torus of diagonal elements. Let $\mathcal{H} = G_1/U_1$. Then (2.2.3) yields a family of isomorphisms

$$G_{1/\mathcal{T}_{r,m}} \xrightarrow{\sim} GL_{\mathcal{O}}(\omega_{r,m}) \tag{2.2.4}$$

and

$$\mathcal{H}_{\mathcal{T}_{r,m}} \xrightarrow{\sim} Y_{r,m} \stackrel{def}{=} GL_{\mathcal{O}}(\omega_{r,m})/U_{can} \tag{2.2.5}$$

where U_{can} is the $\mathcal{T}_{r,m}$-unipotent group scheme corresponding to U_1 under (2.2.4). The isomorphisms (2.2.5) are compatible with the natural $G_1 \times T_1$ actions on the two sides (G_1 acting on the left and T_1 on the right) and patch together as r and m vary. Not that for any character κ of T_1, taking κ-equivariant sections (indicated by $[\kappa]$) of $\mathcal{O}_{Y_{r,m}}$ makes sense.

Continuing as in [Hi05, §7], and writing $Y = Y_{r,m}$, $p_Y : Y \to T_{r,m}$ the natural map, note that $p_{Y,*}(\mathcal{O}_Y[\kappa])$ inherits an action of $G_1(\mathbb{Z}_p)$, covering the trivial action on $T_{r,m}$, because p_Y is a fibration in $G_1(\mathbb{Z}/p^m\mathbb{Z})$-homogeneous spaces. On the other hand, $T_{r,m}$ is a $G_1(\mathbb{Z}/p^m\mathbb{Z})$-torsor over S_r. We let δ_m denote the diagonal action of $G_1(\mathbb{Z}/p^m\mathbb{Z})$ on $p_{Y,*}(\mathcal{O}_Y[\kappa])$ over S_r. Over S_r

$$\rho_\kappa = p_{Y,*}(\mathcal{O}_Y[\kappa])/\delta_m(G_1(\mathbb{Z}/p^m\mathbb{Z})), \tag{2.2.6}$$

From the isomorphism (2.2.5) one obtains an isomorphism

$$\begin{aligned}\phi_m : H^0(S_m, \rho_\kappa) \\ \xrightarrow{\sim} \{f \in Mor_{\mathcal{V}_{m,m}}(G_{1/\mathcal{V}_{m,m}}, \mathbf{G}_{a/\mathcal{V}_{m,m}}) \mid f(hgut) = \kappa(t)h \cdot f(g), \\ h \in G_1(\mathbb{Z}_p), u \in U_1, t \in T_1\}.\end{aligned}$$

These isomorphisms are clearly compatible with varying m. Composing with the evaluation at the identity map yields a map

$$\beta_\kappa : H^0(S_m, \rho_\kappa) \to \mathcal{V}_{m,m}^{U_1}.$$

Because of the compatibilities as m varies, this also makes sense for $m = \infty$, in which case we have an injection

$$\beta_\kappa : H^0(S_\infty, \rho_\kappa) \to \mathcal{V} = \mathcal{V}_{\infty,\infty}^{U_1}. \tag{2.2.7}$$

The image of β_κ is naturally contained in $\mathcal{V}[\kappa]$.

From (2.2.7) we obtain an injection

(2.2.8)

$$Ig : H^0({}_{K(U,\infty)}Sh(V), [W_\tau]) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p = H^0({}_{K(U,\infty)}Sh(V), \rho_\tau) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p \to \mathcal{V} \otimes_{\mathcal{O}_v} \mathbb{C}_p.$$

This is defined by restricting a section of $H^0({}_{K(U,\infty)}Sh(V), \rho_\tau)$ to a formal neighborhood of the Igusa tower in the special fibre of the normalization $\widehat{{}_K\mathbb{S}}$ of ${}_K\mathbb{S}$ in ${}_K Sh(V)$.

When $G = GU(V,-V)$ the same arguments apply, but in the definition of G_1, $GL(n)$ is replaced by $GL(n) \times GL(n)$, and in (2.2.4) $GL_{\mathcal{O}}(\omega_{r,m})$ is replaced by the subgroup preserving the splitting of $Lie(\hat{\mathbb{A}})$ coming from the splitting of $\mathbb{A}$. In particular, when the prime-to-p levels are compatible, there is a commutative diagram

(2.2.9)

$$\begin{array}{ccc} H^0({}_{K(U,\infty)}Sh(2V),[W_\tau]) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p & \xrightarrow{res'} & H^0({}_{K(U,\infty)}Sh(V,-V),[W_\tau]) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p \\ {\scriptstyle Ig_{2V}}\downarrow & & {\scriptstyle Ig_{V,-V}}\downarrow \\ \mathcal{V}_{2V} \otimes_{\mathcal{O}_v} \mathbb{C}_p & \xrightarrow{r_V} & \mathcal{V}_{V,-V} \otimes_{\mathcal{O}_v} \mathbb{C}_p \end{array}$$

where res' is the map coming from the inclusion of Igusa towers as in (2.1.11).

(2.3) p-adic modular forms and the q-expansion principle.

Now we return to the situation of (1.5), with the Shimura datum $(GU(2V), X(2V))$. We write $Sh(L_P)$ instead of $Sh(L_P, X_P)$. For simplicity, we again restrict attention to one-dimensional $[W_\tau]$. Then the Fourier expansion of (1.5.6.8), applied to

$$H^0({}_{K(U,\infty)}Sh(2V),[W_\tau]) := \varinjlim_m H^0({}_{K(U,m)}Sh(2V),[W_\tau]),$$

takes values in

$$\hat{\bigoplus_{\beta\in U^*}} H^0({}_{K_P(\infty)}Sh(L_P),[W_{\tau_P}]) := \hat{\bigoplus_{\beta\in U^*}} \varinjlim_m H^0({}_{K_{L_P}(m)}Sh(L_P),[W_{\tau_P}]).$$

These can be translated into locally constant functions on $L_P(\mathbf{A}_f)$ as in the discussion following (1.5.6.5), and as indicated there, it suffices to consider values on $L_P(\mathbf{A}_f^p)$. In [Hi04, 8.3.2], Hida explains how to fill in the lower horizontal arrow in the following commutative diagram:

(2.3.1)

$$\begin{array}{ccc} H^0({}_{K(U,\infty)}Sh(2V),[W_\tau]) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p & \xrightarrow{F.J.^P} & \hat{\bigoplus}_{\beta\in U^*} H^0({}_{K_P(\infty)}Sh(L_P),[W_{\tau_P}]) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p \\ {\scriptstyle Ig}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{V} \otimes_{\mathcal{O}_v} \mathbb{C}_p & \xrightarrow{(F.J.^P)_{\mathbb{C}_p}} & \hat{\bigoplus}_{\beta\in U^*} H^0({}_{K_P(\infty)}\mathbb{S}_P,[W_{\tau_P}]) \otimes_{\mathcal{O}_v} \mathbb{C}_p \end{array}$$

More precisely, and more usefully, Hida explains how to construct an *integral* map

$$\mathcal{V} \xrightarrow{F.J.^P} \hat{\bigoplus_{\beta\in U^*}} H^0({}_{K_P(\infty)}\mathbb{S}_P, \mathcal{O}_{\mathbb{S}_P}) \tag{2.3.2}$$

which yields the bottom line of (2.3.1) upon tensoring with $\mathbb{C}_p$.[6]

Now we can state

[6] Actually Hida only considered the case of level prime to p; the general case is treated in [SU].

Theorem 2.3.3 (q-expansion principle, [Hi04]).
(a) The map $F.J.^P$ of (2.3.2) is injective and its cokernel has no p-torsion.
(b) Let $f \in H^0({}_{K(U,\infty)}Sh(2V), [W_\tau])$ and suppose f is defined over $\overline{\mathbb{Q}}$, viewed as a subfield of $\mathbb{C}$ or of $\mathbb{C}_p$. Then the expansions $F.J.^P(f)$, defined via (2.3.2) or (1.5.6.8), coincide, and the following are equivalent:

(i) $Ig(f) \in H^0(S_\infty, \rho_\kappa) \otimes \mathcal{O}_{\mathbb{C}_p}$
(ii) $F.J.^P(f)$ *has coefficients in* $\mathcal{O}_{\mathbb{C}_p}$.

Here, as in (1.5)*, the coefficients of $F.J.^P(f)$ can be viewed as functions on $L_P(\mathbf{A}_f)$, and to test their integrality it suffices to consider their values on $L_P(\mathbf{A}_f)$.*

When $n = 1$ and $E = \mathbb{Q}$, this theorem, or rather the corrected version of this theorem incorporating a growth condition at the cusps, is essentially due to Katz; for general E, still with $n = 1$, it is due to Ribet. The principal ingredient in the proof is the irreducibility theorem 2.1.10.

(2.4) The case of definite groups.

We end our discussion of p-adic modular forms with a naive description when V is definite. The comparison of this naive description, which is useful, for calculations, and the geometric description of the previous section is made in (2.4.7). We will need it to understand how the restriction of a p-adic modular form on $U(2V)$ to $U(V, -V)$ can be described in the naive sense.

Throughout this section we assume that $< \bullet, \bullet >_\sigma$ is positive definite for all $\sigma \in \Sigma$ (so $a_\sigma = n$ for all σ).

(2.4.1) *Spaces of forms and rational structures.*

For applications to definite unitary groups, we can avoid similitude factors, so for the moment we let G denote $U(V)$ or $U(-V)$ (since these are canonically identified, the distinction is made primarily for ease of subsequent notation). In what follows, we consider only compact subgroups $K \subset G(\mathbf{A}^f)$ of the form $K = \prod_v K_v$, the product being over finite places of $\mathbb{Q}$, with K_v a subgroup of G_v. We fix a rational prime p such that all places of E dividing p split in $\mathcal{K}$ and let $K^p = K \cap G(\mathbf{A}^{f,p}) \cong \prod_{v \neq p} K_v$.

Let ρ be a complex algebraic character of G. Via the fixed isomorphism $\mathbb{C}_p \cong \mathbb{C}$ we view ρ as an algebraic character over $\mathbb{C}_p$. Then ρ has a model over some finite extension F of $\mathbb{Q}_p$. We fix such an F. For each finite place v of $\mathbb{Q}$ let $\mathfrak{s}_v : K_v \to \mathrm{GL}(\mathfrak{L}_v)$ be a finite-dimensional F-representation of K_v factoring through a finite quotient of K_v and such that $\mathfrak{s}_v$ and $\mathfrak{L}_v$ are trivial for almost all v and for $v = p$. Let $\mathfrak{s} = \otimes_{v,F}\mathfrak{s}_v$ and $\mathfrak{L} = \otimes_{v,F}\mathfrak{L}_v$. The product $G_\infty \times K$ acts on $\mathbb{C} \otimes_F \mathfrak{L}$ via $\rho \otimes \mathfrak{s}$.

For a finite set S of places of $\mathbb{Q}$ and a finite-dimensional complex vector space H let $C^\infty(G(\mathbf{A}^S), H)$ denote the space of functions from $G(\mathbf{A}^S)$ to $W_\sigma(\mathbb{C})$ that are smooth as functions of the infinite component of $G(\mathbf{A}^S)$ and locally constant as functions of the finite component. If S contains ∞, $G' \subset G(\mathbf{A}^S)$ is an open subgroup, and M is any set, then we write $C^\infty(G', M)$ for the set of locally constant functions from G' to M.

Let

$$\mathcal{A}_0(G, K, \rho, \mathfrak{s}) = \{f \in C^\infty(G(\mathbf{A}), \mathbb{C} \otimes_F \mathfrak{L})) \mid f(\gamma g \cdot g_\infty k) = (\rho \otimes \mathfrak{s})(g_\infty \times k)^{-1} f(g)\},$$

where $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbf{A})$, $g_\infty \in G_\infty$, and $k \in K$. For any F-algebra R let

$$\mathcal{A}_f(G,K,\rho,\mathfrak{s})(R) = \{f \in C^\infty(G(\mathbf{A}^f), R \otimes_F \mathfrak{L}) \mid f(\gamma \cdot gk) = (\rho \otimes \mathfrak{s})(\gamma \times k^{-1}) f(g)\},$$

where $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbf{A}^f)$, and $k \in K$. Note that there is a canonical isomorphism

$$\mathcal{A}_f(G,K,\rho,\mathfrak{s})(R) = \mathcal{A}_f(G,K,\rho,\mathfrak{s})(F) \otimes_F R. \tag{2.4.1.1}$$

Restriction to $G(\mathbf{A}^f)$ defines a natural isomorphism

$$\mathrm{res} : \mathcal{A}_0(G,K,\rho,\mathfrak{s}) \xrightarrow{\sim} \mathcal{A}_f(G,K,\rho,\mathfrak{s})(\mathbb{C}), \tag{2.4.1.2}$$

and hence, by (2.4.1.1), $\mathcal{A}_f(G,K,\rho,\mathfrak{s})(F)$ defines an F-structure on $\mathcal{A}_0(G,K,\rho,\mathfrak{s})$.

When ρ or $\mathfrak{s}$ is the trivial one-dimensional representation, we drop it from our notation.

(2.4.2) *Integral structures.*

Let R be a commutative ring. For any $R[K]$-module M let

$$\mathcal{A}(G,K,M) = \{f \in C^\infty(G(\mathbf{A}^f), M) \mid f(\gamma g k) = k^{-1} \cdot f(g), \gamma \in G(\mathbb{Q}), k \in K\}.$$

If $K' \subseteq K$ is an open subgroup then $\mathcal{A}(G,K,M) \subseteq \mathcal{A}(G,K',M)$ and there is a trace map $\mathrm{tr}_{K',K} : \mathcal{A}(G,K',M) \to \mathcal{A}(G,K,M)$ defined by $\mathrm{tr}_{K',K} f(x) = \sum_{y \in K/K'} y f(xy)$. These maps are clearly functorial in M and R and they satisfy

$$\mathrm{tr}_{K'',K} = \mathrm{tr}_{K',K} \circ \mathrm{tr}_{K'',K'}, \quad K'' \subseteq K' \subseteq K. \tag{2.4.2.1}$$

Let A be the ring of integers of F. We choose a K_v-stable A-lattice $\Lambda_{\mathfrak{s}_v}$ in each $\mathfrak{L}_v$ and let $\Lambda_{\mathfrak{s}} = \otimes_{v,A} \Lambda_{\mathfrak{s}_v}$. Clearly $\mathcal{A}(G,K,\Lambda_{\mathfrak{s}})$ is an A-lattice in $A_f(G,K,\mathfrak{s})(F)$.

Let $\Gamma_K = G(\mathbb{Q}) \cap K_p$. For χ an $R^\times$-valued character of Γ_K and M an $R[K]$-module let

$$\mathcal{A}_f(G,K,\chi,M) = \{f \in C^\infty(G(\mathbf{A}^{f,p}) \times K_p, M \mid f(\gamma g k) = \chi(\gamma^{-1}) \cdot k^{-1} f(g), \gamma \in \Gamma_K, k \in K\}.$$

Weak approximation shows that restriction to $G(\mathbf{A}^{f,p}) \times K_p$ yields an isomorphism

$$\mathcal{A}(G,K,M) \xrightarrow{\sim} \mathcal{A}_f(G,K,\mathbf{1},M). \tag{2.4.2.2}$$

Similarly, when R is an F-algebra, restriction to $G(\mathbf{A}^{f,p}) \times K_p$ yields an isomorphism

$$\mathcal{A}_f(G,K,\rho,\mathfrak{s})(R) \xrightarrow{\sim} \mathcal{A}_f(G,K,\rho,\mathfrak{L} \otimes_F R). \tag{2.4.2.3}$$

It follows from (2.4.2.3) that to define an A-lattice in $\mathcal{A}_f(G,K,\rho,\mathfrak{s})(F)$ it suffices to define an A-lattice in $\mathfrak{L}$. In particular, $\mathcal{A}_f(G,K,\rho,\Lambda_{\mathfrak{s}})$ defines an A-lattice in $\mathcal{A}_f(G,K,\rho,\mathfrak{s})$.

For $K' \subseteq K$ we define a trace map $\mathrm{tr}_{K',K} : \mathcal{A}_f(G, K', \chi, M) \to \mathcal{A}_f(G, K, \chi, M)$ just as we did above. These maps also satisfy (2.4.2.1) and are functorial in M, and R and agree with our previous definitions via (2.4.2.2) when $\chi = \mathbf{1}$.

(2.4.3) *p-adic forms*

For a topological space X and a group $G' = H \times H'$ with $H \subseteq G(\mathbb{Q}_p)$ and $H' \subseteq G(\mathbf{A}^{f,p})$ open sets, we write $C^p(G', X)$ for the space of maps from G' to X that are continuous on H (for the p-adic topology) and locally constant on H'.

Let G_1 denote the group scheme $R_{\mathcal{O}_{E,p}/\mathbb{Z}_p}\mathrm{GL}(n)$ over $\mathbb{Z}_p$ and fix an identification of G with G_1 over $\mathbb{Q}_p$. Let $B \subseteq G_1$ be its upper-triangular Borel. Let $P \supseteq B$ be a standard parabolic of G_1. Let L be its standard Levi subgroup and U_P its unipotent radical. Upon fixing an identification $\mathcal{O}_{E,p} = \prod_{w|p} \mathcal{O}_{E,w}$ we have $G_1(\mathbb{Z}_p) = \prod_{w|p} \mathrm{GL}(n, \mathcal{O}_{E,w})$, $P(\mathbb{Z}_p) = \prod_{w|p} P_w(\mathcal{O}_{E,w})$ where $P_w \subseteq \mathrm{GL}(n)$ is a standard parabolic corresponding to a partition $p_w : n = n_{1,w} + \cdots + n_{l_w,w}$ of n, and $L(\mathbb{Z}_p) = \prod_{w|p} L_w$ where L_w is the set of block diagonal matrices $\mathrm{diag}(A_1, ..., A_{l_w})$ with $A_i \in \mathrm{GL}(n_{i,w}, \mathcal{O}_{E,w})$. Let $L_1 \subseteq L(\mathbb{Z}_p)$ be the subgroup $\prod_{w|p} L_{w,1}$ where $L_{w,1}$ is the subgroup defined by $\det(A_i) = 1$. Let $P_1 = L_1 U_P(\mathbb{Z}_p)$. For $m \geq 0$ let $U_{P,m} = \{x \in G_1(\mathbb{Z}_p) \mid x \mod p^m \in (P_1 \mod p^m)\}$. So $\cap U_{P,m} = P_1$. Let $I_{P,m} = \{x \in G_1(\mathbb{Z}_p) \mid x \mod p^m \in P(\mathbb{Z}_p/p^m)\}$.

Assume that $K = G_1(\mathbb{Z}_p) \times K^p$. Let $K_{P,m} = U_{P,m} \times K^p$ and let $K_P = P_1 \times K^p$. Then $\cap K_{P,m} = K_P$. Let R be a p-adic ring and M any finite R-module that is also an $R[K]$-module on which K_p acts trivially. Let

$$\mathcal{A}_p(G, K_P, M) = \{f \in C^p(G(\mathbf{A}^f), M) \mid f(\gamma g k) = k^{-1} \cdot f(g), \gamma \in G(\mathbb{Q}), k \in K_P\}.$$

Since $M/p^r M$ is discrete, the canonical projections $M \twoheadrightarrow M/p^r M$ together with (2.4.2.2) induce a canonical isomorphism

$$\mathcal{A}_p(G, K_P, M) \xrightarrow{\sim} \varprojlim_r \varinjlim_m \mathcal{A}_f(G, K_{P,m}, M/p^r M). \tag{2.4.3.1}$$

Let A and $\Lambda_{\mathfrak{s}}$ be as in (2.4.2) and take $R = A$. Then $\Lambda_{\mathfrak{s}}$ provides an important example of an M as above. We call $\mathcal{A}_p(G, K_P, \Lambda_{\mathfrak{s}})$ the space of ($\Lambda_{\mathfrak{s}}$-valued) p-adic modular forms on G relative to P (and K). When P is understood then we just call this the space of p-adic modular forms.

(2.4.4) *Characters*

The group $L(\mathbb{Z}_p)$ normalizes each $K_{P,m}$, $m > 0$, and so acts on $\mathcal{A}_p(G, K_P, M)$ via right translation, determining an action of

$$Z_P = L(\mathbb{Z}_p)/L_1 = P(\mathbb{Z}_p)/P_1 \xrightarrow{\sim} \varprojlim_m I_{P,m}/U_{P,m}.$$

For any $R^\times$-valued character χ of Z_P we define $\mathcal{A}_p(G, K_P, M)[\chi]$ to be the submodule on which Z_P acts via χ. Note that

$$Z_P \xrightarrow{\sim} \prod_{w|p} (\mathcal{O}_{E,w}^\times)^{l_w}, \quad \mathrm{diag}(A_1, ..., A_{l_w})) \mapsto (\det(A_1), ..., \det(A_{l_w})). \tag{2.4.4.1}$$

By an *arithmetic* character of $Z(\mathbb{Z}_p)$ we will mean a character χ such that $\chi = \chi_0 \rho$ with χ_0 a finite-order character and ρ arising from the restriction of an algebraic character of G as in (2.4.1). For an arithmetic character χ let m_χ be the smallest integer such that χ_0 is trivial on $I_{P,m_\chi}/U_{P,m_\chi}$. For $m \geq m_\chi$ we can extend χ to a character of $I_{P,m}$ by setting $\chi(x) = \chi(z)$ where $z \in Z_p$ is such that z has the same image as x in $I_{P,m}/U_{P,m}$. We also extend χ to a character of the center of $L(\mathbb{Q}_p)$ as follows. We fix a uniformizer ξ_w of $\mathcal{O}_{E,w}$ for each $w|p$. Then we put

$$\chi(\mathrm{diag}(\xi_w^{r_1} 1_{n_{1,w}}, ..., \xi_w^{r_{l_w}} 1_{n_{l_w,w}})) = \rho(\mathrm{diag}(\xi_w^{r_1} 1_{n_{1,w}}, ..., \xi_w^{r_{l_w}} 1_{n_{l_w,w}})).$$

Since any element of the center of $L(\mathbb{Q}_p)$ can be uniquely written as a product of a diagonal element as above and an element in $L(\mathbb{Z}_p)$ this is enough to define the desired extension.

For any $R^\times$-valued arithmetic character $\chi = \chi_0\rho$ of Z_P we have injective maps

$$\begin{gathered} r_\chi : \mathcal{A}_f(G, K^0_{P,m}, \chi, M) \hookrightarrow \mathcal{A}_p(G, K_P, M)[\chi], \quad m \geq m_\chi, \\ r_\chi(f)(g) = \chi(x_p) f(x), \\ g = \gamma x, \gamma \in G(\mathbb{Q}), x \in G(\mathbf{A}^{f,p}) \times I_{P,m}, \end{gathered} \tag{2.4.4.2}$$

where $K^0_{P,m} = K^p \times I_{P,m}$. A product decomposition of g as in (2.4.4.2) always exists by weak approximation.

An important observation is that the r_χ's induce an isomorphism

$$\varinjlim_{m \geq n} \mathcal{A}_f(G, K^0_{P,m}, \chi, M/p^r M) \xrightarrow{\sim} \mathcal{A}_p(G, K_P, M/p^r M)[\chi]. \tag{2.4.4.3}$$

For the surjectivity we note that for any $f \in \mathcal{A}_p(G, K_P, M/p^r M)[\chi]$ if m is sufficiently large then f belongs to $\mathcal{A}(G, K_{P,m}, M/p^r)$. For $g \in G(\mathbf{A}^{f,p}) \times I_{P,m}$ let $s_\chi(f)(g) = \chi(g_p^{-1}) f(g)$. Then $s_\chi(f) \in \mathcal{A}_f(G, K^0_{P,m}, \chi, M/p^r)$ and $r_\chi(s_\chi(f)) = f$.

(2.4.5) *Hecke actions.*

Let K be an open compact subgroup of $G(\mathbf{A}^f)$. Suppose $H \subseteq G(\mathbf{A})$ is a subgroup containing K and M is a $\mathbb{Z}[K]$-module on which K_v acts trivially for all v not in some finite set Σ_M. For an open subgroup $K' \subseteq K$ let $C(H, K', M)$ be the space of functions $f : H \to M$ such that $f(gk) = k^{-1} f(g)$ for all $k \in K'$. Then for any $g \in H \cap G(\mathbf{A}^f)$ such that $g_v = 1$ if $v \in \Sigma_M$ and any two open subgroups $K', K'' \subseteq K$, the double coset $K'gK''$ determines a map from $C(H, K', M)$ to $C(H, K'', M)$ by

$$[K'gK'']f(x) = \sum_i f(xg_i^{-1}), \quad K'gK'' = \sqcup K'g_i. \tag{2.4.5.1}$$

This map is obviously functorial in M. It is easy to see that from (2.4.5.1) we get actions of double cosets on the various modules of functions defined in the preceding sections; one need only observe that these actions preserve the requisite topological properties. These actions are compatible with all the various comparisons and isomorphisms described so far.

One important observation is that if g is such that the $K_{P,m}gK_{P,m}$ have the same left-coset representatives for all m, then from (2.4.3.1) we get an action of $T(g) = \varinjlim_m [K_{P,m}gK_{P,m}]$ on $\mathcal{A}_p(G, K_P, M)$. If we further assume that g_p is in the center of $L(\mathbb{Q}_p)$, then $T(g)$ commutes with the action of $P(\mathbb{Z}_p)$ and hence stabilizes each $\mathcal{A}_p(G, K_P, M)[\chi]$, χ a character of Z_P.

Let $C_P \subset G_1(\mathbb{Q}_p)$ be those elements g in the center of $L(\mathbb{Q}_p)$ such that

$$g^{-1}U_P(\mathbb{Z}_p)g \subseteq U_P(\mathbb{Z}_p). \tag{2.4.5.2}$$

For such g we also have

$$I_{P,m}gI_{P,m} = \sqcup I_{P,m}gu_i \quad \text{and} \quad U_{P,m}gU_{P,m} = \sqcup U_{P,m}gu_i, \quad u_i \in U_P(\mathbb{Z}_p). \tag{2.4.5.3}$$

Also, for $g, g' \in C_P$,

$$\begin{array}{c} I_{P,m}gI_{P,m} \cdot I_{P,m}g'I_{P,m} = I_{P,m}gg'I_{P,m} \\ U_{P,m}gU_{P,m} \cdot U_{P,m}g'U_{P,m} = U_{P,m}gg'U_{P,m} \end{array}, \tag{2.4.5.4}$$

where the multiplications are the usual double-coset multiplications.

Let $M_1 = R_{\mathcal{O}_{E,p}/\mathbb{Z}_p} M_{n\times n}$. Suppose $K_p \subseteq G_1(\mathbb{Z}_p)$ and let Δ_K be the semigroup in $M_1(\mathbb{Q}_p)$ generated by K_p and those g such that $g^{-1} \in C_P$. Let M be an $A[K]$-module for which there exists a finite set of places Σ_M, $p \notin \Sigma_M$, such that K_v acts trivially on M if $v \notin \Sigma_M$. Let $g \in G(\mathbf{A}^f)$ be such that $g_v = 1$ for all $v \in \Sigma_M$, $g_p \in C_P$, and suppose that

$$KgK = \sqcup Kg_i, \quad g_{i,p}^{-1} \in \Delta_K. \tag{2.4.5.5}$$

Under this assumption we define an action of KgK on $\mathcal{A}_f(G, K, \chi, M)$ by

$$\begin{array}{c} (KgK)f(x) = \sum_i \chi(\gamma_i)f(x_i), \\ \gamma_i \in G(\mathbb{Q}), \gamma_i x g_i^{-1} = x_i \in G(\mathbf{A}^{f,p}) \times K_p; \end{array} \tag{2.4.5.6}$$

the assumption (2.4.5.5) ensures that $\gamma_i^{-1} \in \Delta_K$.

Let χ be an $A^\times$-valued arithmetic character of Z_P. If $g_p \in C_P$ and $g_v = 1$ for $v \in \Sigma_M$ then (2.4.5.3) implies that (2.4.5.5) holds with K replaced by $K^0_{P,m}$ for any $m \geq m_\chi$. In particular, (2.4.5.6) defines an action of $\tilde{T}(g) = (K^0_{P,m}gK^0_{P,m})$ on $A_f(G, K^0_{P,m}, \chi, M)$, $m \geq m_\chi$, which is multiplicative in such g by (2.4.5.4). Moreover, viewing $\mathcal{A}_f(G, K^0_{P,m}, \chi, \Lambda_\mathfrak{s})$ as an A-submodule of $\mathcal{A}_f(G, K_{P,m}, \rho, \sigma)(F)$ we find that

$$\tilde{T}(g) = \chi^{-1}(g_p)[K_{P,m}gK_{P,m}]. \tag{2.4.5.6}$$

Additionally, it is clear from the definitions that

$$r_\chi \circ \tilde{T}(g) = T(g) \circ r_\chi, \tag{2.4.5.7}$$

where r_χ is as in (2.4.4.2).

(2.4.6) *Pairings.*

For $K \subseteq G(\mathbf{A}^f)$ an open compact subgroup, let

$$_K S(G) = G(\mathbb{Q})\backslash G(\mathbf{A}^f)/K.$$

This is a finite set. Let R be a commutative ring and let M, M' be $R[K]$-modules on which K_p acts trivially. Suppose $(\bullet, \bullet) : M \times M' \to R$ is a K-equivariant R-pairing. Given an $R^\times$-valued character χ of Γ_K we define an R-pairing

$$< \bullet, \bullet >_K : \mathcal{A}_f(G, K, \chi, M) \times \mathcal{A}_f(G, K, \chi^{-1}, M') \to R,$$

(2.4.6.1) $$< f, g >_K = \sum_{[x]\in_K S(G)} (f(x), g(x)), \quad x \in G(\mathbf{A}^{f,p}) \times K_p.$$

These pairings (integration with respect to the measure $d\mu_K(g)$ of (0.2.4)) are clearly functorial in R, M, M'. The following lemma records some basic but important properties of these pairings. For simplicity we will assume that

(2.4.6.2) $$\gamma x k = x, \gamma \in G(\mathbb{Q}), x \in G(\mathbf{A}^f), k \in K \implies k = 1.$$

This holds for sufficiently small K.

(2.4.6.3) Lemma. *Assume* (2.4.6.2).

(i) *If* $(\bullet, \bullet)$ *is a perfect pairing, then so is* $< \bullet, \bullet >_K$.

(ii) *Let* $K' \subseteq K$ *be an open subgroup. Then*

(2.4.6.4) $$\begin{gathered} < f, \mathrm{tr}_{K',K}(h) >_K = < f, h >_{K'}, \\ f \in \mathcal{A}_f(G, K, \chi, M), \ h \in \mathcal{A}_f(G, K', \chi^{-1}, M'). \end{gathered}$$

(iii) *Suppose there exists a finite set of places* Σ_M *such that* K_v *acts trivially on* M *if* $v \notin \Sigma_M$. *Let* $K', K'' \subseteq K$ *be open subgroups and let* $g \in G(\mathbf{A}^f)$ *be such that* $g_p = 1$ *and* $g_v = 1$ *for all* $v \in \Sigma_M$. *Then*

(2.4.6.5) $$\begin{gathered} < [K''gK']f, h >_{K'} = < f, [K'g^{-1}K'']h >_{K''}, \\ f \in \mathcal{A}_f(G, K'', \chi, M), \ h \in \mathcal{A}_f(G, K', \chi^{-1}, M'). \end{gathered}$$

Part (i) holds because $\mathcal{A}_f(G, K, \chi, M)$ is spanned by the functions $\delta_{x,m}$, $x \in G(\mathbf{A}^{f,p}) \times K_p$, $m \in M$, defined by

$$\delta_{x,m}(y) = \begin{cases} \chi(\gamma^{-1}) \cdot k^{-1}m & y = \gamma x k, \gamma \in \Gamma_K, k \in K, \\ 0 & \text{otherwise.} \end{cases}$$

The assumption (2.4.6.2) ensures that these functions are well-defined. Part (ii) is also clear from (2.4.6.2). Part (iii) follows from part (ii) and the observation that

$$[K''gK'](f(x)) = \mathrm{tr}_{K' \cap g^{-1}K''g, K'}(f(xg^{-1})).$$

For our purposes, the most important situation to which we will apply Lemma (2.4.6.3) is when R is the integer ring of some finite extension of F, χ comes from an arithmetic character of Z_P, and $M = \Lambda_{\mathfrak{s}} \otimes_A R$. In this case we let $M' = \mathrm{Hom}_A(\Lambda_{\mathfrak{s}}, R)$, the latter being an $R[K]$-module with the usual action, and let $(\bullet, \bullet)$ be the canonical pairing between M and M'. Let

(2.4.6.6) $$< \bullet, \bullet >_{m,\chi,\sigma} = < \bullet, \bullet >_{K^0_{P,m}}, \quad m \geq m_\chi,$$

where the right-hand side is defined by (2.4.6.1) with our current choices of M, M', etc. Assuming that (2.4.6.2) holds for $K^0_{P,m}$, then all the conclusions of Lemma (2.4.6.3) hold for $< \bullet, \bullet >_{m,\chi,\sigma}$.

(2.4.7) *Comparison with the geometric picture.*

Previously, we defined spaces of p-adic modular form for $GU(V)$ from a geometric perspective. We now compare these to the spaces in (2.4.3). For simplicity we will assume that the similitude character maps K onto $\widehat{\mathbb{Z}}^{\times}$.

In the definite situation the geometric constructions of (2.2) are simple. The varieties ${}_{K(U,m)}Sh(V)$ clearly all have models over $\mathcal{O}_v$; the base change to $\mathcal{O}_v/p^r$ is just $\mathcal{T}_{r,m}$. From this it is easily deduced that $\mathcal{V}^U_{r,m} = \Gamma(\mathcal{T}_{r,m}, \mathcal{O}_{\mathcal{T}_{r,m}})$ is naturally identified with the set of $\mathcal{O}_v/p^r$-valued functions on ${}_{K(U,m)}Sh(V)$ and so, under our hypotheses on K, with $\mathcal{A}_f(G, K(U,m), \mathcal{O}_v/p^m)$ (in particular, these identifications are compatible with varying r and m). Thus we have that

$$\mathcal{V} = \varprojlim_r \varinjlim_m \mathcal{V}^U_{r,m} = \varprojlim_r \varinjlim_m \mathcal{A}_f(G, K(U,m), \mathcal{O}_v/p^m). \tag{2.4.7.1}$$

Then (2.4.3.1) identifies $\mathcal{V}$ with $\mathcal{A}_p(G, K_B, \mathcal{O}_v)$. The spaces of p-adic modular forms for other parabolics are obtained by taking U_P-invariants.

The restriction on K can be dropped; then $\mathcal{V}$ is identified with a direct sum of copies of $\mathcal{A}_p(G, K_B, \mathcal{O}_v)$.

3. Fourier coefficients of Siegel Eisenstein series on unitary groups

(3.0) Conventions for automorphic forms on unitary groups.

We let Σ_E denote the set of archimedean places of E. Let W be any hermitian space over $\mathcal{K}$ of dimension n, and define $-W$ and $2W = W \oplus (-W)$ as in §1. Set

$$W^d = \{(v,v) \mid v \in W\}, \qquad W_d = \{(v,-v) \mid v \in W\}$$

These are totally isotropic subspaces of $2W$. Let P be the stablizer of W^d in $U(2W)$. As a Levi component of P we take the subgroup $M \subset U(2W)$ which is stablizer of both W^d and W_d. Then $M \simeq GL(W^d)$. We let U denote the unipotent radical of P.

The decomposition $2W = W^d \oplus W_d$ is a complete polarization. Choose a basis $(u_1,\ldots,u_m)$ for W, so that (u_i,u_i) is a basis for W^d. Let $(-v_j,v_j)$, $j=1,\ldots,m$, be the dual basis of W_d. For any $A \in GL(n)_{\mathcal{K}}$, we define $m(A)$ to be the element of $U(2W)$ with matrix $\begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix}$ in the basis $\{(u_i,u_i)\} \cup \{(-v_j,v_j)\}$, where $\bar{A}$ is the image of A under the non-trivial Galois automorphism of $\mathcal{K}/E$. We will let

$$w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

in the same basis; then $P\backslash PwP$ is the big cell in the Bruhat decomposition of $P\backslash U(2W)$.

All automorphic forms will be assumed K_∞-finite, where K_∞ will be a maximal compact modulo center subgroup of either $U(2W)(\mathbb{R})$ or $U(W)(\mathbb{R})$, as appropriate. Conventions are as in §1.5; in particular K_∞ will be associated to a CM point, except where otherwise indicated.

We let $GU(2W)$ be the group of rational similitudes, as in §1. Let $GP \subset GU(2W)$ denote the stabilizer of W^d, and let GM be the normalizer of M in GP. We can identify $GM \xrightarrow{\sim} M \times \mathbb{G}_m$ where M acts as $GL(W^d)$ and $\mathbb{G}_m$ acts via the center of $GL(W_d)$. Here and below $\mathbb{G}_m$ designates $\mathbb{G}_{m,\mathbb{Q}}$. In other words, writing GP in standard form:

$$GP = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\} \tag{3.0.1}$$

with $D = d \cdot {}^tc(A)^{-1}$ for some scalar d, we can identify the factor $\mathbb{G}_m \subset GM$ with the group of matrices

$$\{d(t) = \begin{pmatrix} 1_n & 0 \\ 0 & t1_n \end{pmatrix}\} \subset GU(2W). \tag{3.0.2}$$

Let v be any place of E, $|\cdot|_v$ the corresponding absolute value on $\mathbb{Q}_v$, and let

$$\delta_v(p) = |N_{\mathcal{K}/E} \circ \det(A(p))|_v^{\frac{n}{2}} |\nu(p)|^{-\frac{1}{2}n^2}, \ p \in GP(E_v). \tag{3.0.3}$$

This is the local modulus character of $GP(E_v)$. The adelic modulus character of $GP(\mathbf{A})$, defined analogously, is denoted $\delta_{\mathbf{A}}$. Let χ be a Hecke character of $\mathcal{K}$. We view χ as a character of $M(\mathbb{A}_E) \xrightarrow{\sim} GL(W^d)$ via composition with det. For any complex number s, define

$$\delta^0_{P,\mathbf{A}}(p,\chi,s) = \chi(\det(A(p))) \cdot |N_{\mathcal{K}/E} \circ \det(A(p))|^s_v |\nu(p)|^{-ns}$$

$$\delta_{\mathbf{A}}(p,\chi,s) = \delta_{\mathbf{A}}(p)\delta^0_{P,\mathbf{A}}(p,\chi,s) = \chi(\det(A(p)))\cdot|N_{\mathcal{K}/E}\circ\det(A(p))|_v^{\frac{n}{2}+s}|\nu(p)|^{-\frac{1}{2}n^2-ns}.$$

The local characters $\delta_{P,v}(\cdot,\chi,s)$ and $\delta^0_{P,v}(\cdot,\chi,s)$ are defined analogously. The restrictions to M of the characters $\delta_{P,v}$, $\delta^0_{P,v}$, and so on are denoted by the same notation.

As in (2.2), the symmetric domain $X(2W)$ is isomorphic to the $X^d_{n,n}$ of tube domains. Let $\tau_0 \in X(2W)$ be a fixed point of K_∞, X^+ the connected component of $X(2W)$ containing τ_0, $GU(n,n)^+ \subset GU(2W)(\mathbb{R})$ the stabilizer of X^+. Thus $X^+ \xrightarrow{\sim} \prod_{\sigma\in\Sigma_E} X^+_{n,n;\sigma}$ with $X^+_{n,n;\sigma}$ the symmetric space associated to $U(n,n) = U(E_\sigma)$. Let $GK_\infty \subset GU(n,n)^+$ be the stabilizer of τ_0; thus GK_∞ contains K_∞ as well as the center of $GU(n,n)$.

In the tube domain realization, the canonical holomorphic automorphy factor associated to GP and GK_∞ is given as follows. Let $\tau = (\tau_\sigma)_{\sigma\in\Sigma_E} \in X^+$ and $h = \left(\begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix}\right)_{\sigma\in\Sigma_E} \in GU(n,n)^+$. Then the triple

$$J(h,\tau) = (C_\sigma\tau_\sigma + D_\sigma)_{\sigma\in\Sigma_E},\ J'(h,\tau) = (\bar{C}^t_\sigma\tau_\sigma + \bar{D}_\sigma)_{\sigma\in\Sigma_E}, \nu(h) \tag{3.0.4}$$

defines a canonical automorphy factor with values in $(GL(n,\mathbb{C}) \times GL(n,\mathbb{C}))^d \times GL(1,\mathbb{R})$ (note the misprint in [H3, 3.3]). Write $J(h) = J(h,\tau_0) = (J_\sigma(h))_{\sigma\in\Sigma_E}$ and define $J'(h)$ and $J'_\sigma(h)$ analogously. Given a pair of integers (μ,κ), we define a complex valued function on $GU(n,n)^+$:

$$\mathbf{J}_{\mu,\kappa}((h_\sigma)_{\sigma\in\Sigma_E}) = \prod_{\sigma\in\Sigma_E} \det J_\sigma(h)^{-\mu} \cdot \det(J'_\sigma(h))^{-\mu-\kappa} \cdot \nu(h)^{n(\mu+\kappa)} \tag{3.0.5}$$

For purposes of calculation, we let $\tau_0 = (\sigma(\daleth))_{\sigma\in\Sigma}$, where $\daleth$ is the trace zero element of $\mathcal{K}$ chosen in (1.4). We also write $\daleth_\sigma = \sigma(\daleth)$. Then the stabilizer GK_∞ is rational over the reflex field $E(GU(2W), X(2W)) = E(\mathcal{K},\Sigma)$, and the map $h \mapsto J(h)$ is a rational function on the algebraic group $GU(2W)$ with values in GK_∞, rational over $E(\mathcal{K},\Sigma)$.

(3.1) The Siegel Eisenstein series and doubling.

In this section we let G denote $U(W)$, $H = U(2W)$, viewed alternatively as groups over E or, by restriction of scalars, as groups over $\mathbb{Q}$. Identifying G with $U(-W)$, we obtain a natural embedding $G \times G \subset H$. We choose maximal compact subgroups $K_{\infty,G} = \prod_{v\in\Sigma_E} K_{v,G} \subset G(\mathbb{R})$ and $K_\infty = \prod_{v\in\Sigma_E} K_v \subset H(\mathbb{R})$ – as at the end of the previous subsection – such that

$$K_\infty \cap (G\times G)(\mathbb{R}) = K_{\infty,G} \times K_{\infty,G}.$$

We will be more precise about these choices in (4.3).

(3.1.1) *Formulas for the Eisenstein series*

Let χ be a unitary Hecke character of $\mathcal{K}$. We view χ as a character of $M(\mathbb{A}_E) \xrightarrow{\sim} GL(W^d)$ via composition with det. Consider the induced representation

(3.1.1.1) $$I(\chi, s) = \mathrm{Ind}(\chi|\cdot|_{\mathcal{K}}^{s}) \xrightarrow{\sim} \otimes_v I_v(\chi_v|\cdot|_v^{s}),$$

the induction being normalized; the local factors I_v, as v runs over places of E, are likewise defined by normalized induction. At archimedean places we assume our sections to be K_∞-finite. For a section $f(h;\chi,s) \in I(\chi,s)$ (cf. [H4, I.1]) we form the Eisenstein series

(3.1.1.2) $$E_f(h;\chi,s) = \sum_{\gamma \in P(k)\backslash U(2V)(k)} f(\gamma h;\chi,s)$$

This series is convergent for $\mathrm{Re}(s) > n/2$, and it can be continued to a meromorphic function on the entire plane. We now fix an integer $m \geq n$ and assume

(3.1.1.3) $$\chi|_{\mathbf{A}} = \varepsilon_{\mathcal{K}}^{m}$$

Then the main result of [T] states that the possible poles of $E_f(g;\chi,s)$ are all simple, and can only occur at the points in the set

(3.1.1.4) $$\frac{n-\delta-2r}{2},\ r = 0,\ldots,[\frac{n-\delta-1}{2}],$$

where $\delta = 0$ if m is even and $\delta = 1$ if m is odd.

(3.1.2) The standard L-function via doubling. Let (π, H_π) be a cuspidal automorphic representation of G, $(\pi^\vee, H_{\pi^\vee})$ its contragredient, which we assume given with compatible isomorphisms of $G(\mathbf{A})$-modules

(3.1.2.1) $$\pi \xrightarrow{\sim} \otimes_v \pi_v,\ \pi^\vee \xrightarrow{\sim} \otimes_v \pi_v^\vee.$$

The tensor products in (3.1.2.1) are taken over places v of the totally real field E, and at archimedean places π_v is a admissible $(\mathfrak{g}_v, K_{v,G})$-module, which we assume to be of cohomological type, with lowest $K_{v,G}$-type (cf., e.g., [L1]) τ_v. For each v we let $(\bullet,\bullet)_{\pi_v}$ denote the canonical bilinear pairing $\pi_v \otimes \pi_v^\vee \to \mathbb{C}$.

Let $f(h;\chi,s)$ be a section, as above, $\varphi \in H_\pi$, $\varphi' \in H_{\tilde{\pi}}$, and let $\varphi'_\chi(g) = \varphi'(g)\chi^{-1}(\det g')$. We define the zeta integral:

(3.1.2.2) $$Z(s,\varphi,\varphi',f,\chi) = \int_{G\times G)(\mathbb{Q})\backslash(G\times G)(\mathbf{A})} E_f((g,g');\chi,s)\varphi(g)\varphi'_\chi(g')dgdg'.$$

The Haar measures $dg = dg'$ on $G(\mathbf{A})$ are normalized as in (0.2.2). The relation to the integral in terms of Tamagawa measure is determined by (0.2.3).

The theory of this function, due to Piatetski-Shapiro and Rallis [PSR], was worked out (for trivial χ) by Li [L2] and more generally in [HKS,§6]. We make the following hypotheses:

(3.1.2.4) Hypotheses

(a) There is a finite set of finite places S_f of E such that, for any non-archimedean $v \notin S_f$, the representations π_v, the characters χ_v, and the fields $\mathcal{K}_w$, for w dividing v, are all unramified;

(b) The section f admits a factorization $f = \otimes_v f_v$ with respect to (3.1.1.1).
(c) The functions φ, φ' admit factorizations $\varphi = \varphi_{S_f} \otimes \otimes_{v \notin S_f} \varphi_v$, $\varphi' = \varphi_{S_f} \otimes \otimes_{v \notin S_f} \varphi'_v$, with respect to (3.1.2.1)
(d) For $v \notin S_f$ non-archimedean, the local vectors f_v, φ_v, and φ'_v, are the normalized spherical vectors in their respective representations, with $(\varphi_v, \varphi'_v)_{\pi_v} = 1$.
(e) For v archimedean, the vector φ_v (resp. φ'_v) is a non-zero highest (resp. lowest) weight vector in τ_v (resp. in $\tau_v^\vee$), such that $(\varphi_v, \varphi'_v)_{\pi_v} = 1$.

We let $S = \Sigma_E \cup S_f$. Define

$$d_n(s,\chi) = \prod_{r=0}^{n-1} L(2s+n-r, \varepsilon_{\mathcal{K}}^{n-1+r}) = \prod_v d_{n,v}(s,\chi), \tag{3.1.2.5}$$

the Euler product on the right being taken only over finite places;

$$Q_W^0(\varphi,\varphi') = \int_{G(\mathbb{Q})\backslash G(\mathbf{A})} \varphi(g)\varphi'(g)dg; \tag{3.1.2.6}$$

$$Z_S(s,\varphi,\varphi',f,\chi) = \int_{\prod_{v\in S} G(E_v)} f_v((g_v,1);\chi,s)(\pi_v(g_v)\varphi,\varphi')dg_v; \tag{3.1.2.7}$$

$$\tilde{Z}_S(s,\varphi,\varphi',f,\chi) = \prod_{v\in S}[d_{n,v}(s,\chi)] Z_S(s,\varphi,\varphi',f,\chi).$$

The integral in (3.1.2.7) converges absolutely in a right halfplane and admits a meromorphic continuation to all s.[7] We have the following identity of meromorphic functions on $\mathbb{C}$:

(3.1.2.8) Basic Identity of Piatetski-Shapiro and Rallis.

$$d_n(s,\chi)Z(s,\varphi,\varphi',f,\chi) = \tilde{Z}_S(s,\varphi,\varphi',f,\chi)L^S(s+\frac{1}{2},\pi,\chi,St).$$

Here $L^S(s+\frac{1}{2},\pi,\chi,St) = \prod_{v\notin S} L_v(s+\frac{1}{2},\pi_v,\chi_v,St)$, *where* $L_v(s+\frac{1}{2},\pi_v,\chi_v,St)$ *is the local Langlands Euler factor attached to the unramified representations* π_v *and* χ_v *and the standard representation of the* L*-group of* $G \times R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}}$.[8]

[7] For non-archimedean places this is worked out in detail in [HKS]. There is no published reference for unitary groups at archimedean places in general. Shimura [S97] calculates the archimedean integrals explicitly for holomorphic automorphic forms of scalar weight. For general π_∞ meromorphic continuation is established by Kudla and Rallis [KR] for symplectic groups by reduction to principal series. The same technique applies to unitary groups, bearing in mind that not all unitary groups are quasi-split. For the special values we have in mind we appeal to the explicit calculations of Garrett [G].

[8] As in the previous footnote, there is no published reference for the meromorphic continuation and functional equation of standard L-functions of unitary groups, although the results of Kudla and Rallis for symplectic groups adapt to the case of unitary groups. In the applications we will restrict attention to π admitting base change to automorphic representations of $GL(n,\mathcal{K})$, which immediately implies the analytic continuation of the standard L-functions.

For any place $v \notin S$, there is a formal (unramified) base change from π_v to a representation $BC(\pi_v)$ of $G(\mathcal{K} \otimes_E E_v) \xrightarrow{\sim} GL(m, \mathcal{K} \otimes_E E_v)$, and

$$L_v(s, \pi_v, \chi_v, St) = L(s, BC(\pi_v) \otimes \chi_v \circ \det),$$

where the right-hand term is the standard Godement-Jacquet Euler factor (cf. [H4,I.1] for a further discussion).

If we assume φ_{S_f} and φ'_{S_f} to be factorizable over the $v \in S_f$, with respect to the isomorphisms (3.1.2.1), then the integral Z_S also breaks up as a product of local integrals multiplied by the factor Q^0_W, as in [H3,H4], as well as [PSR,Li92]. To treat congruences it seems preferable not to impose factorizability at this stage. However, under special hypotheses on the local data we can obtain a factorization, as follows. Write $G_v = G(E_v)$, and let $K_v \subset G(E_v)$ be a compact open subgroup fixing φ. The natural map $G \times G \to P\backslash H$ defines an isomorphism between $G \times 1$ and the open $G \times G$ orbit in the flag variety $P\backslash H$ [PSR, p. 4]. In particular, $P \cdot (G \times 1)$ is open in H and $P \cap (G \times 1) = \{1\}$. It follows that, if Y is any locally constant compactly supported function on G_v, there is a unique section $f_Y(h, \chi, s) \in I_v(\chi_v, s)$ such that $f_Y((g,1), \chi, s) = Y(g)$ for all $g \in G_v$, $s \in \mathbb{C}$. Let $f_{K_v}(h, \chi, s) = f_{Y(K_v)}(h, \chi, s)$, where $Y(K_v)$ is the characteristic function of the open compact subgroup K_v chosen above. With this choice, we have

$$\int_{G_v} f_{K_v}((g_v, 1); \chi, s)(\pi_v(g_v)\varphi, \varphi')dg_v = vol(K_v) \tag{3.1.2.9}$$

for any s. If we choose $f_v = f_{K_v}$ for all $v \in S_f$, the basic identity becomes

$$d_n(s,\chi)Z(s, \varphi, \varphi', f, \chi) = d_{n,S}(s,\chi) \cdot vol(K_{S_f}) Z_\infty(s, \varphi, \varphi', f, \chi) L^S(s + \frac{1}{2}, \pi, \chi, St),$$

where $d_{n,S}(s,\chi) = \prod_{v \in S_f} [d_{n,v}(s,\chi)]$, $K_{S_f} = \prod K_v$, and

$$Z_\infty(s, \varphi, \varphi', f, \chi) = \int_{\prod_{v \in \Sigma_E} G(E_v)} f_v((g_v, 1); \chi, s)(\pi_v(g_v)\varphi, \varphi')dg_v. \tag{3.1.2.10}$$

The integrals in (3.1.2.10) are purely local in the following sense. For any archimedean place v we can define a local analogue of (3.1.2.7) by

$$Z_v(s, \varphi_v, f_v, \chi_v) = \int_{G(E_v)} f_v((g_v, 1); \chi_v, s)\pi_v(g_v)\varphi_v dg_v. \tag{3.1.2.11}$$

This is a function of s with values in the $K_{\infty,v}$-finite vectors of π_v, absolutely convergent and holomorphic in a right half-plane, and admitting a meromorphic continuation to $\mathbb{C}$ (see note 5). Let $\tau_v^+ \subset \tau_v$ denote the line spanned by the highest weight vector φ_v, let $p_v^+ : \pi_v \to \tau_v^+$ denote orthogonal projection. Define the meromorphic function $Z_v(s, f_v, \chi_v)$ by

$$p_v^+(Z_v(s, \varphi_v, f_v, \chi_v)) = Z_v(s, f_v, \chi_v) \cdot \varphi_v.$$

This is well-defined, because τ_v^+ is a line, and does not depend on the choice of φ_v because both sides are linear functions of φ_v. Let $Z_\infty(s, f, \chi) = \prod_{v \in \Sigma_E} Z_v(s, f_v, \chi_v)$ It then follows that

$$Z_\infty(s, \varphi, \varphi', f, \chi) = Z_\infty(s, f, \chi) Q^0_W(\varphi, \varphi'), \tag{3.1.2.12}$$

hence that

(3.1.2.13) $d_n(s,\chi)Z(s,\varphi,\varphi',f,\chi)$

$$= d_{m,S}(s,\chi)\cdot vol(K_{S_f})Z_\infty(s,f,\chi)L^S(s+\frac{1}{2},\pi,\chi,St)Q^0_W(\varphi,\varphi')$$

We note the following consequence of the basic identity in the form (3.1.2.13). Let $K_f = K_{S_f}\times K^S$, where $K^S = \prod_{w\notin S} K_w$ is a product of hyperspecial maximal compact subgroups fixing φ and φ'.

(3.1.2.14) Hypothesis. *We assume f, $s = s_0$, and χ can be chosen so that*

$$d_{n,S}(s_0,\chi)Z_\infty(s_0,f,\chi)\neq 0.$$

Thus we are staying away from poles of the local Euler factors in $d_{n,S}(s,\chi)$) and the global Euler products $d_n(s,\chi)$ and $L^S(s+\frac{1}{2},\pi,\chi,St)$ have neither zeros nor poles at $s = s_0$. This hypothesis is easy to verify in practice, e.g. in the situation of [H3]; the only subtle point is the non-vanishing of $Z_\infty(s_0,f,\chi)$ when ϕ_v is holomorphic and the Eisenstein series defined by f_v is nearly holomorphic, and in this case the non-vanishing follows from the arguments of Garrett [G].

Let $\mathcal{A}_0(\pi,S)$, resp. $\mathcal{A}_0(\pi^\vee,S)$ denote the space spanned by K_f-invariant cusp forms on G, that generate irreducible automorphic representations whose v-component is isomorphic to π_v (resp. to $\pi_v^\vee$) for all $v\notin S_f$, and belonging to the highest weight subspace τ_v^+ of τ_v (resp. to the lowest weight subspace of $\tau_v^\vee$ for all $v\in\Sigma_E$. Then (3.1.2.12) asserts that the bilinear forms $Z(s_0,\varphi,\varphi',f,\chi)$ and Q^0_W on $\mathcal{A}_0(\pi,S)$ are proportional. (If π occurs with multiplicity one in $\mathcal{A}_0(G)$, then this is automatic.) This simplifies the arguments of §3 of [H3], proving, when $E=\mathbb{Q}$, that critical values of $L(s,\pi,\chi,St)$ are $\mathcal{K}$-multiples of a basic period equal to an elementary expression multiplied by a square norm of the form $Q^0_W(\varphi,\varphi')$, where φ and φ' are arithmetic holomorphic modular forms of the given type.[9] In particular, this gives a somewhat more natural proof of Corollary 3.5.12 of [H3], to the effect that, under the hypotheses of loc. cit. (existence of sufficiently many critical values) $Q^0_W(\varphi,\varphi')$ depends up to arithmetic factors only on the abstract representation π^S.

(3.1.2.15) Remark. Local Euler factors $L_v(s,\pi_v,\chi_v,St)$ are defined in [HKS] for all finite places, by the method of Piatetski-Shapiro and Rallis. It should not be difficult to prove by global methods that these factors coincide with $L(s,BC(\pi_v)\otimes\chi_v\circ\det)$, at least when π_v is a local component of an automorphic cuspidal representation for a definite unitary group. A complete proof would require local functional equations at archimedean primes. When $n=2$ the unitary group can be compared simply to the multiplicative group of a quaternion algebra, and the result can be proved easily in that case directly.

(3.1.3) *Eisenstein series and zeta integrals on similitude groups.*

We now return to the situation of (3.1). Let $GH = GU(2W)$, and consider the subgroup $GU(W,-W) = G(U(W)\times U(-W)\subset GH$. The induced representation

[9] In [H3] only values of s in the absolutely convergent range are considered, but the argument remains valid in general under hypothesis (3.1.2.14). See [H5] for a more extended discussion of this point.

$I(\chi, s)$ and the Eisenstein series $E_f((g,g');\chi,s)$ can be extended in various ways to automorphic forms on GH. Let $GP \subset GH$ denote the Siegel parabolic defined in (3.2.5). Global characters of $GM = M \times \mathbb{G}_m$ are given by pairs (χ, υ) where χ is a Hecke character of $M^{ab} = R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathbb{Q}}$, lifted to a character of M by composition with the determinant, and υ is a Hecke character of $\mathbf{A}^\times/\mathbb{Q}^\times$. Let

(3.1.3.1) $$I(\chi,\upsilon,s) = \mathrm{Ind}_{GP}^{GH}((\chi|\cdot|_{\mathcal{K}}^{s}) \circ \det \cdot \upsilon \circ \nu).$$

For any section $f(h;\chi,\upsilon,s) \in I(\chi,\upsilon,s)$ we form the Eisenstein series $E_f(h,\chi,\upsilon,s)$ by the analogue of the formula (3.1.1.2). The character υ factors through a character of GH and does not affect convergence.

Let π, π' be automorphic representations of $GU(W)$, with central characters ξ, ξ', respectively. Let $\varphi \in \pi$, $\varphi' \in \pi'$, and consider $\varphi \otimes \varphi'$ by restriction as an automorphic form on $GU(W,-W)$. Let Z be the identity component of the center of $GU(W,-W)$, which we may also view as a central subgroup of GH, or (via projection) as a central subgroup of $GU(W)$. We assume

(3.1.3.2) $$\xi \cdot \xi' \cdot \xi_{\chi,\upsilon} = 1;$$

here $\xi_{\chi,\upsilon}$ is the central character of $I(\chi,\upsilon,s)$. We can then define the zeta integral

(3.1.3.3) $$Z(s,\varphi,\varphi',f,\chi,\upsilon) = \int_{Z(\mathbf{A})GU(W,-W))(\mathbb{Q})\backslash(GU(W,-W)(\mathbf{A})} E_f((g,g');\chi,s)\varphi(g)\varphi'_\chi(g')dgdg'.$$

The basic identity (3.1.2.8) then takes the following form (cf. [H3,(3.2.4)]):

(3.1.3.4) $$d_n(s,\chi)Z(s,\varphi,\varphi',f,\chi,\upsilon) = Q_W(\varphi,\varphi')\tilde{Z}_S(s,\varphi,\varphi',f,\chi)L^S(s+\frac{1}{2},\pi,\chi,St).$$

where

(3.1.3.5) $$Q_W(\varphi,\varphi') = \int_{Z(\mathbf{A})GU(W)(\mathbb{Q})\backslash GU(W)(\mathbf{A})} \varphi(g)\varphi'(g)\xi_{\chi,\upsilon}^{-1}dg$$

and the remaining terms are as in (3.1.2). The period $Q_W(\varphi,\varphi')$ is slightly more natural from the standpoint of Shimura varieties.

(3.1.4) *Holomorphic Eisenstein series.*

Fix (μ,κ) as in (3.0.1). Define

$$\chi^* = \chi \cdot |N_{\mathcal{K}/E}|^{\frac{\kappa}{2}}.$$

Suppose the character χ has the property that

(3.1.4.1) $$\chi^*_\sigma(z) = z^\kappa, \ \chi^*_{c\sigma}(z) = 1 \ \forall \sigma \in \Sigma_E$$

Then the function $\mathbf{J}_{\mu,\kappa}$, defined in (3.0.5), belongs to

(3.1.4.2) $$I_n(\mu - \frac{n}{2}, \chi^*)_\infty = I_n(\mu + \frac{\kappa - n}{2}, \chi)_\infty \otimes |\nu|_\infty^{\frac{n\kappa}{2}}$$

(cf. [H3,(3.3.1)]). More generally, define

(3.1.4.3) $$\mathbf{J}_{\mu,\kappa}(h, s+\mu-\frac{n}{2}) = \mathbf{J}_{\mu,\kappa}(h)|\det(J(h)\cdot J'(h))|^{-s} \in I_n(s,\chi^*)_\infty$$

When $E=\mathbb{Q}$, these formulas just reduce to the formulas in [H3].

Let $f_\infty(h,\chi,s) = \mathbf{J}_{\mu,\kappa}(h, s+\mu-\frac{n}{2})$, and suppose the Eisenstein series $E_f(h;\chi,s)$ is holomorphic at $s=0$. Since $\mathbf{J}(\mu,\kappa)$ is a holomorphic vector in the corresponding induced representation, $E_f(h;\chi,0)$ is then a holomorphic automorphic form. As in [H3,(3.3.4)] we can identify $E_f(h;\chi,0)$ with an element of $H^0(Sh(2W),\mathcal{E}_{\mu,\kappa})$ where $\mathcal{E}_{\mu,\kappa}$ is the automorphic vector bundle defined in [H3,(3.3)]. The identification is as in (1.3.6) and depends on a choice of canonical trivialization of the fiber of $\mathcal{E}_{\mu,\kappa}$ at τ_0.

The center of symmetry $s=\frac{1}{2}$ for $L(s,\pi,\chi,St)$ in the unitary normalization corresponds via (3.1.2.8) to a zeta integral with the Eisenstein series at $s=s_0=0$. Since χ is by (3.1.1.3) a unitary character, this corresponds in turn to the relation to $s_0=\mu+\frac{\kappa-n}{2}=0$. More generally, the value of the motivically normalized L-function

$$L^{mot}(s,\pi,\chi*,St) \stackrel{def}{=} L(s-\frac{n-\kappa-1}{2},\pi,\chi,St)$$

at $s=s_0+\frac{n-\kappa}{2}$ corresponds as above to the Eisenstein series at $s_0=\mu+\frac{\kappa-n}{2})$, i.e. at $s=\mu$, as in [H3] (where μ was called m). It follows from (3.1.4.1) that we can choose m in (3.1.1.3) so that

(3.1.4.4) $$m = n+2s_0 = 2\mu+\kappa;$$

the assumption $m\geq n$ translates to $s_0\geq 0$, so the Eisenstein series is always to the right of the center of symmetry.

(3.2) Fourier coefficients of Eisenstein series: General considerations.

(3.2.1) *Notation and preliminaries.*

We let V, $2V=V\oplus -V$, and $H=U(2V)$ be as in (3.1) with $n=\dim V$. Let $\daleth$ be as in (1.4). We fix an orthogonal basis $u_1,\cdots,u_n$ of V, and set

(3.2.1.1) $$e_j=(u_j,u_j),\qquad f_j=\delta_j\cdot(-u_j,u_j)$$

where

(3.2.1.2) $$\delta_j = \frac{1}{2\daleth < u_j,u_j >_V}$$

With respect to this basis, the matrix of the skew-hermitian form $<,>_{2V,\daleth}$ is given by

$$\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

Let $g\in GL(V)$. When no confusion is possible, we use the same letter g to denote the $n\times n$ matrix (g_{ij}) given by

$$g(u_i)=\sum_{j=1}^n g_{ji}u_j$$

Write $\delta = \mathrm{diag}(\delta_1, \cdots, \delta_n)$. Then $g \in U(V)$ if and only if ${}^t\bar{g}\delta^{-1}g = \delta^{-1}$, or equivalently $g\delta^t\bar{g} = \delta$. With respect to the basis $\{e_i, f_j\}$, the matrix corresponding to $(g,1) \in U(V) \times U(V) \subseteq U(2V)$ is

$$(g,1) = \begin{pmatrix} \frac{1}{2}(1_n + g) & \frac{1}{2}(1_n - g)\delta \\ \frac{1}{2}\delta^{-1}(1_n - g) & \frac{1}{2}\delta^{-1}(1_n + g)\delta \end{pmatrix} \tag{3.2.1.3}$$

We let

$$w' = \mathrm{diag}(-1_V, 1_V) = \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix} \quad \text{with respect to } \ 2V = V \oplus -V$$

Then with respect to the basis e_i, f_j we have

$$w' = \begin{pmatrix} \delta & 0 \\ 0 & {}^t\bar{\delta}^{-1} \end{pmatrix} \cdot w = m(\delta) \cdot w, \qquad w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \tag{3.2.1.4}$$

This amounts to taking $g = -1_n$ in (3.2.1.3). In other words, the coset $P \cdot (-1_n, 1_n) \subset P \cdot G \times \{1\}$ belongs to the big cell PwP, and indeed

$$P \cdot (-1_n, 1_n) = Pw \cdot 1 \tag{3.2.1.5}$$

More generally, for any positive integer $r \leq n$ let V_r be the subspace of V spanned by $u_1, \cdots, u_r$. Let $V_r^{\perp}$ be the othogonal complement of V_r in V. We define

$$w'_r = \mathrm{diag}(-1_{V_r}, 1_{V_r^{\perp}}, 1_{V_r}, 1_{V_r^{\perp}}) \in U(2V) \tag{3.2.1.6}$$

Then $w' = w'_n$. With respect to the basis e_i, f_j we have

$$w'_r = \begin{pmatrix} {}_r\delta & 0 & 0 & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 0 & 0 & {}^t_r\bar{\delta}^{-1} & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix} \cdot w_r, \tag{3.2.1.7}$$

with ${}_r\delta = \mathrm{diag}(\delta_1, \cdots, \delta_r)$, where

$$w_r = \begin{pmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ -1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix}. \tag{3.2.1.8}$$

By means of the basis $\{e_i, f_j\}$ we identify elements of H as $2n \times 2n$ matrices. Then if v is any finite place of E we define $H(\mathcal{O}_v)$ to be the subgroup of $H(E_v)$ consisting of matrices whose entries are in $\mathcal{O}_{\mathcal{K}} \otimes \mathcal{O}_v$. Let B be the stablizer of the flag

$$[e_1] \subset [e_1, e_2] \subset \cdots \subset [e_1, \cdots, e_n]$$

where $[e_1, \cdots, e_r]$ denotes the linear span of $e_1, \cdots, e_r$. Then B is a Borel subgroup, and we have the Iwasawa decomposition $H(E_v) = B(E_v)H(\mathcal{O}_v)$.

In what follows we fix a non-trivial character $\psi = \prod \psi_v$ of $\mathbf{A}/E$, as follows. Let $\mathbf{e}^0 = \prod \mathbf{e}^0_p$ be the unique character of $\mathbf{A}_{\mathbb{Q}}/\mathbb{Q}$ such that

$$\mathbf{e}^0_\infty(x) = e^{2\pi i x} \qquad (x \in \mathbb{R}),$$

and that $\mathbf{e}^0_p$ has conductor $\mathbb{Z}_p$ for every finite p. Let $\mathbf{e} = \prod \mathbf{e}_v$ be the character of $\mathbf{A}/E$ defined by

(3.2.1.9) $$\mathbf{e}(x) = \mathbf{e}^0(\mathrm{Tr}_{E/Q}(x)) \qquad (x \in \mathbf{A})$$

Alternatively, we may characterize $\mathbf{e}$ as the unique character of $\mathbf{A}/E$ such that for every archimedean place v we have

$$\mathbf{e}_v(x) = e^{2\pi\sqrt{-1}x} \qquad (x \in E_v = \mathbb{R}).$$

An arbitrary character of $\mathbf{A}/E$ is given by $x \mapsto \mathbf{e}(ax)$, with a some element of E. We let ψ be one such character with $a \in E$ totally positive, fixed henceforward. We can and will always assume a to be a unit at all primes dividing p. Thus

(3.2.1.10) $$\psi(x) = \mathbf{e}(ax) = \mathbf{e}^0(\mathrm{Tr}_{E/Q}(ax)) \qquad (x \in \mathbf{A})$$

In paricular, for every archimedean place v we have

$$\psi_v(x) = e^{2\pi a\sqrt{-1}x} \qquad (x \in E_v = \mathbb{R}).$$

(3.2.2) *Formulas for Fourier coefficients.*

We start with a general Siegel Eisenstein series $F = E^f(h,\chi,s)$ with $f \in I(\chi,s)$. Here we have written E^f instead of E_f, in order to leave space for a subscript to denote Fourier coefficients. Let Her_n be the space of all $n \times n$ hermitian matrices. For $\beta \in \mathrm{Her}_n(E)$ we define the character ψ_β of $U(\mathbb{Q})\backslash U(\mathbf{A})$ by

$$\psi_\beta(n(b)) = \psi(\mathrm{tr}(\beta b))$$

Note that we have $\mathrm{tr}(\beta\beta') \in E$ for any $\beta, \beta' \in \mathrm{Her}_n(E)$.

We now fix a Haar measure dx on $U(\mathbf{A}) \simeq \mathrm{Her}_n(\mathbf{A})$ as follows. First we take counting measure on the discrete subgroup $\mathrm{Her}_n(E) \subseteq \mathrm{Her}_n(\mathbf{A})$. We choose dx, so that the quotient mesaure on $U(E)\backslash U(\mathbf{A}) = \mathrm{Her}_n(E)\backslash\mathrm{Her}_n(\mathbf{A})$ is normalized, with total volume 1. Consider the lattice $\Lambda \subseteq \mathrm{Her}_n(E)$ consisting of all hermitian matrices with entries in $\mathcal{O}_{\mathcal{K}}$. We shall also need the dual lattice Λ^*, defined by

$$\Lambda^* = \{\beta \in \mathrm{Her}_n(E) \,|\, \mathrm{tr}(\beta\xi) \in \mathcal{O}_E \ \ \forall \ \xi \in \Lambda\}$$

For each finite place v of E we set

$$\Lambda_v = \mathrm{Her}_n(\mathcal{O}_v) = \Lambda \otimes \mathcal{O}_v,$$

Define Λ^*_v similarly. Then $\Lambda_v = \Lambda^*_v$ unless v ramifies in $\mathcal{K}$. Let dx_v be the Haar measure of $\mathrm{Her}_n(E_v)$ normalized by $\int_{\Lambda_v} dx_v = 1$. For any archimedean place v we set

$$dx_v = |\bigwedge_{j=1}^{n} dx_{jj} \bigwedge_{j<k} (2^{-1} dx_{jk} \wedge d\bar{x}_{jk})|$$

where x_{jk} is the (j,k)-entry of x_v. There is a constant $c(n,E,\mathcal{K})$ so that

$$dx = c(n,E,\mathcal{K}) \cdot \prod_v dx_v.$$

Since $\mathrm{Her}_n(\mathbf{A})$ is the product of n copies of $\mathbf{A}$ and $n(n-1)/2$ copies of $\mathbf{A}_{\mathcal{K}}$, we obtain (say from [Tate])

$$c(n,E,\mathcal{K}) = 2^{n(n-1)[E:\mathbb{Q}]/2}|\delta(E)|^{-n/2}|\delta(K)|^{-n(n-1)/4}, \tag{3.2.2.1}$$

where $\delta(E)$ and $\delta(\mathcal{K})$ are the discriminants of E and $\mathcal{K}$. This is the same as [S97], p. 153.

For $\beta \in \mathrm{Her}_n(E)$ we define the β-th Fourier coefficient

$$F_\beta(h) = \int_{U(\mathbb{Q})\backslash U(\mathbf{A})} F(uh)\psi_{-\beta}(u)du$$

as in (1.5.6).

We now assume that f is factorizable, and write $f = \otimes f_v$. If β has full rank n then a familiar calculation gives

(3.2.2.2)

$$E^f_\beta(h,\chi,s) = c(n,E,\mathcal{K}) \cdot \prod_v \int_{U(E_v)} f_v(wn_vh_v,\chi_v,s)\psi_{-\beta}(n_v)dn_v, \qquad (\det\beta \neq 0)$$

the product being over all places of E. Here w is the Weyl group element given by (3.2.1.4).

Remark 3.2.2.3. Suppose that for at least one place v the function $f_v(\bullet,\chi_v,s)$ is supported on the big cell $P(E_v)wP(E_v)$. Then (3.2.2.2) is valid for $h \in P(\mathbb{A})$ and *any* β. Indeed for $h \in P(\mathbb{A})$ we have

$$f(\gamma h,\chi,s) \neq 0 \Longrightarrow \gamma \in P(E)wP(E) = P(E)wU(E)$$

So that

$$E^f(h,\chi,s) = \sum_{\delta\in U(E)} f(w\delta h,\chi,s)$$

and (3.2.2.1) follows immediately for any β, not necessarily of full rank.

Write

$$W_{\beta,v}(h_v,f_v,s) = \int_{U(E_v)} f_v(wn_vh_v,\chi_v,s)\psi_{-\beta}(n_v)dn_v. \tag{3.2.2.4}$$

This function satisfy a transformation law as follows. Suppose

$$m = m(A) = \begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix} \in M(E_v)$$

Then

$$\begin{aligned} W_{\beta,v}(mh_v,f_v,s) &= |N\circ\det A|_v^{n/2-s}\chi_v(\det A)\cdot W_{{}^t\bar{A}\beta A,v}(h_v,f_v,s) \\ &= |N\circ\det A|_v^{\frac{n-\kappa}{2}-s}\chi_v^*(\det A)\cdot W_{{}^t\bar{A}\beta A,v}(h_v,f_v,s) \end{aligned} \tag{3.2.2.5}$$

where $N = N_{\mathcal{K}/E}$.

We now recall a calculation of Shimura. In what follows, a is the totally positive element of E, prime to p, fixed in (3.2.1.10).

(3.2.2.6) Lemma. ([S97], 19.2) *Suppose β is of full rank n. Let v be a finite place of E. Let $f_v(\bullet, \chi_v, s)$ be the unique section which is invariant under $H(\mathcal{O}_v)$, and such that $f_v(1, \chi_v, s) = 1$. Let $m = m(A) \in M(E_v)$. Then $W_{\beta,v}(m, f_v, s) = 0$ unless ${}^t\bar{A}\beta A \in a^{-1}\mathfrak{D}(E/\mathbb{Q})_v^{-1}\Lambda_v^*$, where $\mathfrak{D}(E/\mathbb{Q})_v$ is the different of E_v relative to $\mathbb{Q}_p$ (p being the rational prime lying below v). In this case, one has*

$$W_{\beta,v}(m, f_v, s) = |N{\circ}\det A|_v^{n/2-s}\chi_v(\det A)g_{\beta,m,v}(\chi(\varpi_v)q_v^{-2s-n})\cdot\prod_{j=1}^{n} L_v(2s+j, \chi\varepsilon_{\mathcal{K}/E}^{n-j})^{-1}$$

Here $L_v(\bullet,\bullet)$ is the local abelian L-factor at v, with χ viewed as a character for $\mathbf{A}_E^\times$ by restriction, and $g_{\beta,m,v}$ is a polynomial with constant term 1 and coefficients in $\mathbb{Z}$. Let $\mathfrak{D}(E/\mathbb{Q})_v = \delta_v\mathcal{O}_v$ for some $\delta_v \in E_v$. If v is unramified in $\mathcal{K}$, and

$$\det(a\delta_v{}^t\bar{A}\beta A) \in \mathcal{O}_v^\times,$$

then $g_{\beta,m,v}(t) \equiv 1$.

Example. Let $n = 1$. Then $\beta \in E_v^\times$ and A is a scalar. Let $r \geq 0$ be the integer determined by

$$|a\delta_v\bar{A}A\beta|_v = q_v^{-r}$$

Then

$$g_{\beta,m,v}(t) = \frac{(1-t)[1-(qt)^{r+1}]}{1-qt}$$

(3.2.2.7) Corollary. *For any finite place v we let T_v be the characteristic function of $\mathfrak{D}(E/\mathbb{Q})_v^{-1}\Lambda_v^*$. Suppose that $\beta \in Her_n(E)$ is of full rank n. Let S be a finite set of places including all the archimedean ones and all places ramified in $\mathcal{K}$, and large enough so that the conditions of Lemma 3.2.2.6 are satisfied at any place $v \notin S$. Let $m = m(A) \in M(\mathbf{A})$. Then*

$$E_\beta^f(m,\chi,s) = c(n,E,\mathcal{K})\cdot|\det A|_{\mathcal{K}}^{n-s}\chi(\det A)\cdot(\prod_{v\in S} W_{{}^t\bar{A}\beta A,v}(1,f_v,s))$$
$$\cdot\prod_{v\notin S}[T_v(a{}^t\bar{A}\beta A)g_{\beta,m,v}(\chi(\varpi_v)q_v^{-2s-n})]\cdot\prod_{j=1}^{n} L^S(2s+j,\chi\varepsilon_{\mathcal{K}/E}^{n-j})^{-1} \tag{3.2.2.8}$$

Here $L^S(\bullet,\bullet)$ is the partial L-function, with χ viewed as a character for $\mathbf{A}_E^\times$ by restriction.

(3.2.2.9) Remarks.

(i) In the subsequent sections we will always assume S contains all primes of residue characteristic p. Suppose this is the case and $v \notin S$. Then the local factor $T_v(a{}^t\bar{A}\beta A)g_{\beta,m,v}(\chi(\varpi_v)q_v^{-2s_0-n})$ is p-adically integral for any half-integer s_0. In particular, the p-adic denominators of the Fourier coefficients $E_\beta^f(m,\chi,s_0)$, normalized by the product of the partial L-functions, are determined by the local factors at $v \in S$ and by the global factors.

(ii) Let $\beta \in \mathrm{Her}_n(E)$ be of full rank n. We say β is *S-primitive* if $\det(a\beta) \in \mathcal{O}_v^\times$ for all $v \notin S$. The condition depends implicitly on a. Since S contains the ramified primes, the local different factors can be ignored. It follows from (3.2.2.7) that for S-primitive β, the product of local coefficients satisfies

$$\prod_{j=1}^{n} L^S(2s_0+j,\chi\varepsilon_{\mathcal{K}/E}^{n-j})\cdot\prod_{v\notin S} W_{\beta,v}(1,f_v,s_0)=1$$

and in particular is a p-adic unit.

(iii) On the other hand, the factors $g_{\beta,m,v}(\chi(\varpi_v)q_v^{-2s-n})$ are p-units at half-integer values of s, provided v is prime to p. Our local data at primes v dividing p will guarantee the vanishing of coefficients $W_{\beta,v}$ unless $\det(a\delta_v{}^t\bar{A}\beta A)\in\mathcal{O}_v^\times$, and we will only evaluate the coefficients at points $m=m(A)$ with $A_v\in GL(n,\mathcal{O}_v)$). Thus we will always have the local factors $g_{\beta,m,v}(t)\equiv 1$ for v dividing p, and the product

$$T^0(\beta,m(A),s)=\prod_{v\notin S}[T_v(a{}^t\bar{A}\beta A)g_{\beta,m,v}(\chi(\varpi_v)q_v^{-2s-n})] \qquad (3.2.2.10)$$

will always be a p-adic unit when $s\in\frac{1}{2}\mathbb{Z}$.

(iv) In other words, the p-adic behavior of the Eisenstein series is completely determined by the global normalizing factor $\prod_{j=1}^n L^S(2s+j,\chi\varepsilon_{\mathcal{K}/E}^{n-j})^{-1}$ and by the local factors at $v\in S$. Calculation of the local factors will occupy most of the rest of this section.

(3.3) Local coefficients of holomorphic Eisenstein series.

In this section we consider a finite set S of places as in (3.2.2.7), containing all archimedean places, all places ramified in $\mathcal{K}/E$, all places dividing p, and all places at which the character χ_v is ramified. We also include in S a collection of finite places where, to guarantee non-vanishing of local zeta integrals for ramified π_v, f_v cannot be the unramified vector $f_v^{unr}\in I(\chi,s)$, i.e., the vector invariant under $H(\mathcal{O}_v)$. At the archimedean places we will take specific local data. Otherwise the data will vary according to circumstances to be defined later. The resulting calculation (3.3.1.5, 3.3.2.1) of the local Fourier coefficients at ramified finite primes is less precise than at unramified places.

We treat non-split places, split places, and archimedean places separately.

(3.3.1) *Finite non-split places.*

Let v be a finite place in S. Suppose first that v does not split in $\mathcal{K}$. We let w be the unique place of $\mathcal{K}$ dividing v. We define a special section in $I(\chi_v,s)$ as follows. Let u_v be a Schwartz function on $\mathrm{Her}_n(E_v)$. Define a section $f_v(h;\chi_v,s)\overset{def}{=} f_{u_v}(h;\chi_v,s)\in I(\chi_v,s)$ by the condition that it is supported in the big cell $P(E_v)wP(E_v)$, and

$$f_v(wn(b);\chi_v,s)=u_v(b)\qquad (b\in\mathrm{Her}_n(E_v)) \qquad (3.3.1.2)$$

It is easy to see that $W_{\beta,v}(1,f_v,s)=\hat{u}_v(\beta)$. Together with the transformation law (3.2.2.5), we find that

$$W_{\beta,v}(m(A)\tilde{f}_v,s)=|\det A|_v^{n/2-s}\chi_v(\det A)\cdot\hat{u}_v({}^t\bar{A}\beta A) \qquad (3.3.1.3)$$

We now choose a lattice $L_v\subset\mathrm{Her}_n(E_v)$, and make the following assumption:

(3.3.1.4) Hypothesis. *u_v is the characteristic function of L_v.*

Let $L_v^\vee$ be the dual lattice defined by

$$L_v^\vee = \{\beta \mid \psi(\mathrm{tr}\beta x) = 1 \text{ for all } x \in L_v\}$$

Then we have

$$W_{\beta,v}(m(A), f_{u_v}, s) = T_v({}^t\bar{A}\beta A)|\det A|_v^{n/2-s}\chi_v(\det A)\cdot vol(L_v), \tag{3.3.1.5}$$

(3.3.2)*Finite split places.*

Next we consider the case where v is finite and splits in $\mathcal{K}$, of residue characteristic different from p. Let u_v be a Schwartz function on $\mathrm{Her}_n(E_v) \simeq M_{n,n}(E_v)$ ($n \times n$ matrices with entries in E_v). Then there is a section $f_{u_v}(h;\chi_v,s)$ such that $f_v(\bullet;\chi_v,0)$ has support in $P(E_v)wP(E_v)$, and $f_v(wn(b);\chi_v,0) = u_v(b)$. Formula (3.3.1.3) remains valid for all β. If u_v is chosen as in (3.3.1.4), then we write f_{L_v} instead of f_{u_v}. In what follows, $A \in GL(n,\mathcal{K}_v)$ can be written as a pair $(\mathbf{A}_v,\mathbf{B}_v)$ with $\mathbf{A}_v,\mathbf{B}_v \in GL(n,E_v)$, and $|\det(A)|_v = |\det(\mathbf{A}_v\cdot\mathbf{B}_v^{-1})|_v$, with conventions as in (3.3.4) below.

(3.3.2.1) Lemma. *With $f_v = f_{L_v}$, formula (3.3.1.5) is valid for all β.*

At split places other choices might be more convenient. For example, let $U_v \subset GL(n,E_v)$ be a compact open subgroup and τ_v a finite-dimensional irreducible representation of U_v. Let u_v be a matrix coefficient of τ_v, viewed as a function on $U_v \subset GL(n,E_v)$ and extended by zero to $M(n,E_v)$. Then u_v takes values in the integers of some cyclotomic field. It then follows immediately from (3.3.1.3) that:

(3.3.2.2) Lemma. *The functions $\hat{u}_v$ and $W_{\beta,v}(m(A),f_v,0)$ are locally constant, compactly supported, not identically zero, and takes values in $\mathbb{Q}^{ab}$ with denominators bounded p-adically independently of τ_v.*

Indeed, the integral defining $\hat{u}_v$ is a finite sum of terms, each of which is an algebraic integer multiplied by a volume. The volume lies in $\mathbb{Q}$ and the denominators are bounded in terms of the orders of finite subgroups of $GL(n,E_v)$, independently of τ_v. The remaining factors in (3.3.1.5) are p-units.

(3.3.2.3) Remark 3.3.2.3 Alternatively, we can let u_v be a matrix valued function, namely the function τ_v, with values in $End(\tau_v)$, extended to zero off K_v. The Eisenstein series and its Fourier coefficients will then have values in $End(\tau_v)$. This will allow us to pair the Eisenstein series with forms taking values in the space of τ_v and its dual. The local zeta integral will be essentially a volume.

(3.3.3) *Archimedean places.*

Let $v \in S_\infty$ be a real place of E. We shall regard elements of $H(E_v) \simeq U(n,n)$ as $2n\times 2n$ matrices by means of the basis $\{e_i, f_j\}$ chosen in (3.2.1). Let $j = \sigma_v(\beth)$. We let $K_v \subset H(E_v)$ be the maximal compact subgroup consisting of those matrices k with ${}^t\bar{k}\ \mathrm{diag}(j^2I_n, -I_n)k = \ \mathrm{diag}(j^2I_n,-I_n)$, where I_n denote the identity matrix of size n. Then $K_v \simeq U(n)\times U(n)$. We make this isomorphism explicit as follows. Set

$$\gamma = \begin{pmatrix} 1_n & 1_n \\ j^{-1}1_n & -j^{-1}1_n \end{pmatrix} \in GU(n,n)$$

Then for any $A, B \in U(n)$ one has

$$k(A,B) \equiv= \gamma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \gamma^{-1} \in K_v$$

The map $(A,B) \mapsto k(A,B)$ is an isomorphism from $U(n) \times U(n)$ onto K_v.

Let $x \in \mathrm{Her}_n(\mathbb{R})$. One easily checks that the Iwasawa decomposition of $wn(x)$ is given by

$$wn(x) = \begin{pmatrix} 1_n & -\frac{x}{-j^2+x^2} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{-j^2+x^2}} & 0 \\ 0 & \sqrt{-j^2+x^2} \end{pmatrix} k(-\frac{x+j1_n}{\sqrt{-j^2+x^2}}, -\frac{x-j1_n}{\sqrt{-j^2+x^2}})$$

Let $m = 2\mu + \kappa$ as in (3.1.4.4), so that $\chi_v(-1) = (-1)^m$ (3.1.1.3). We follow Shimura [S82] and take f_v to be (up to sign) the v component of the canonical automorphy factor denoted $\mathbf{J}_{\mu,\kappa}(h, s+\frac{\kappa}{2})$ in (3.1.4.3); thus f_v is holomorphic for $s = s_0 = \frac{m-n}{2}$. More precisely,

(3.3.3.1)
$$\begin{aligned} f_v(wn(x), \chi_v, s) &= \det(-j^2+x^2)^{-s-n/2} \det(\frac{-j1_n - x}{\sqrt{-j^2+x^2}})^m \\ &= (-1)^{mn} \cdot \delta(x - j1_n)^{-s-\frac{m+n}{2}} \delta(x+j1_n)^{-s+\frac{m-n}{2}} \end{aligned}$$

In subsequent articles we will identify f_v with a Siegel-Weil section for the theta lift of the trivial representation of $U(m)$. Continuing the calculation, and making the simple change of variables, $x \mapsto x/\alpha$, where $\alpha = -j/i > 0$, we find

$$\begin{aligned} W_{\beta,v}(1, f_v, s) &= (-1)^{mn}(-j/i)^{-2ns} \int_{\mathrm{Her}_n(\mathbb{R})} \delta(x+i1_n)^{-s-\frac{m+n}{2}} \delta(x-i1_n)^{-s+\frac{m-n}{2}} e^{-2\pi i \mathrm{tr}(\beta x)} dx \\ &= (-1)^{mn}(-j/i)^{-2ns} \xi(1_n, \beta; s + \frac{n+m}{2}, s + \frac{n-m}{2}) \end{aligned}$$

([S82], p. 274, (1.25)). By ([S82], p. 275, (1.29)), this is equal to

$$(-i)^{mn} 2^n \pi^{n^2} (-j/i)^{-2ns} \Gamma_n(s+\frac{n+m}{2})^{-1} \Gamma_n(s+\frac{n-m}{2})^{-1} \eta(21_n, \pi\beta; s+\frac{n+m}{2}, s+\frac{n-m}{2})$$

Choose $A \in GL(n, \mathbb{C})$ with $AA^* = \pi\beta$, where $A^* = {}^t\bar{A}$. By ([S82], p.280-281), we have

$$\begin{aligned} \eta(21_n, \pi\beta; s + \frac{n+m}{2}, s + \frac{n-m}{2}) &= \delta(\pi\beta)^{2s} \cdot \eta(2A^*A, 1_n; s + \frac{n+m}{2}, s + \frac{n-m}{2}) \\ &= (2\pi)^{2ns} \delta(\beta)^{2s} e^{-2\pi \mathrm{tr}(\beta)} \zeta(4A^*A; s + \frac{n+m}{2}, s + \frac{n-m}{2}) \end{aligned}$$

Thus

$$\begin{aligned} W_{\beta,v}(1, f_v, s) = (-i)^{mn} 2^{n(m-n+1)} \pi^{ns+n(m+n)/2} (-j/i)^- 2ns \delta(\beta)^{s-\frac{n-m}{2}} e^{-2\pi \mathrm{tr}(\beta)} \\ \Gamma_n(s + \frac{n+m}{2})^{-1} \omega(4A^*A; s + \frac{n+m}{2}, s + \frac{n-m}{2}) \end{aligned}$$

The function $\omega(z;\mu,\lambda)$ is analytic in μ,λ and satisfies the functional equation

$$\omega(z;n-\lambda,n-\mu)=\omega(z;\mu,\lambda)$$

By (3.15) of Shimura we know $\omega(z;\mu,0)=1$. So at $s=(m-n)/2$ we obtain
(3.3.3.2)

$$\begin{aligned}W_{\beta,v}(1,f_v,\frac{m-n}{2}) &= (\daleth_v)^{-mn+n^2}(-i)^{-n^2}2^{n(m-n+1)}\pi^{mn}\det(\beta)^{m-n}e^{-2\pi\mathrm{tr}(\beta)}\Gamma_n(m)^{-1}\\ &= \frac{(\daleth_v)^{-mn+n^2}(-i)^{-n^2}2^{n(m-n+1)}\pi^{mn-n(n-1)/2}\det(\beta)^{m-n}}{\prod_{j=1}^{n}(m-j)!}\cdot e^{-2\pi\mathrm{tr}(\beta)}.\end{aligned}$$

The factor $e^{-2\pi\mathrm{tr}(\beta)}$ at the end is the value at $h_\infty=1$ of the function denoted q^β in §(1.5.6); more precisely, q^β factors over the archimedean primes, and $e^{-2\pi\mathrm{tr}(\beta)}$ is the factor at v. The coefficient preceding this factor is the local contribution at v to the Fourier coefficient f_β.

(3.3.4) *Local results at primes dividing p (choice of special functions at p)*

First we fix some notation.

(3.3.4.1) Notation. Let v be a place of E dividing p. Then v splits in $\mathcal{K}$ according to our assumptions. Throughout we shall identify E_v with $\mathcal{K}_w$, where w is the divisor of v with $w\in\Sigma_p$ (see (1.1.4)). We denote by $\mathcal{O}_v$ the ring of integers of E_v, and by $\mathfrak{p}_v$ the prime ideal in $\mathcal{O}_v$. For any pair of positive integers a,b we denote by $M_{a,b}$ or $M(a,b)$ the space of $a\times b$ matrices. Let dZ be the normalized Haar measure on $M_{n,n}(E_v)$ that assigns measure 1 to $M_{n,n}(\mathcal{O}_v)$. We write $d^\times Z=dZ/|\det Z|^n$. Let dg_v be the normalized Haar measure on $GL(n,E_v)$ that assigns measure 1 to $GL(n,\mathcal{O}_v)$. Then $d^\times Z=A(n)\cdot dg_v$, where

$$A(n)=\int_{GL(n,\mathcal{O}_v)}d^\times Z=\prod_{j=1}^{n}(1-q^{-j})=q^{-n^2}\#GL(n,\mathbb{F}_q)$$

This is just the right hand side of (3.2.2.6). Thus we may assume that $d^\times Z=L_v(1,\varepsilon_{\mathcal{K}})^{-1}d^\tau g_v$ in the notation of (0.2).

Let χ be the character of $\mathbb{A}_{\mathcal{K}}^\times$ that goes into the definition of our Siegel Eisenstein series. At the place v which splits in $\mathcal{K}$, χ is given by the pair of characters $(\chi_{1v},\chi_{2v}^{-1})$.

For the rest of section (3.3.4) we drop the subscript v from our notation, writing χ_1 for χ_{1v}, etc. On $H(E_v)\simeq GL(2n,E_v)$, the inducing character is

$$\begin{pmatrix}A & *\\ 0 & B\end{pmatrix}\mapsto\chi_1(\det A)\chi_2(\det B)\cdot|\det(AB^{-1})|^{s+\rho},$$

with $\rho=n/2$.

Fix a partition

$$n=n_1+\cdots+n_l$$

Let $P=LU$ be the standard parabolic subgroup of $GL(n)$ corresponding to the above partition. Let $\mathcal{I}\subseteq GL(n,\mathcal{O}_v)$ be the paraholic subgroup corresponding to P. Thus $\mathcal{I}$ consists of matrices $Z=(Z_{ij})$ (written in blocks with respect to the above partition of n), such that

- $Z_{jj} \in GL(n_j, \mathcal{O}_v)$ for $1 \le j \le l$.
- Z_{ij} has entries in $\mathcal{O}_v$ for $1 \le i < j \le l$.
- Z_{ij} has entries in $\mathfrak{p}_v$ for $i > j$.

Note that $\mathcal{I}$ is an open set in the space $M(n,n)$ of all $n \times n$ matrices with entries in E_v. Consider l characters $\nu = (\nu_1, \cdots, \nu_l)$ of $E_v^\times$. We define our Schwartz function ϕ_ν by the formula

$$\phi_\nu(Z) = \begin{cases} \nu_1(\det Z_{11}) \cdots \nu_l(\det Z_{ll}), & Z \in \mathcal{I} \\ 0, & \text{otherwise} \end{cases} \tag{3.3.4.2}$$

We use the same letter ν to denote the character of $L(\mathcal{O})$ given by

$$\nu(\mathrm{diag}(A_1, \cdots, A_l)) = \nu_1(\det A_1) \cdots \nu_l(\det A_l)$$

It is easy to see that the function ϕ_ν satisfies the relation

$$\phi_\nu(mZ) = \phi_\nu(Zm) = \nu(m)\phi_\nu(Z) \qquad (m \in L(\mathcal{O}), \text{any } Z)$$

Define Fourier transform by

$$\mathcal{F}(\phi)(x) = \int \phi(z)\overline{\psi_v(\mathrm{tr}(z^t x))}dz \tag{3.3.4.3}$$

The function $\mathcal{F}(\phi_\nu)$ satisfies the (obvious) condition

$$\mathcal{F}(\phi_\nu)(mx) = \mathcal{F}(\phi_\nu)(xm) = \nu^{-1}(m)\mathcal{F}(\phi_\nu)(x) \qquad (m \in L(\mathcal{O}), \text{any } x) \tag{3.3.4.4}$$

The explicit formula for $\mathcal{F}(\phi_\nu)$ is given in Part II, Appendix B.

Consider another l-tuple of characters $\mu = (\mu_1, \cdots, \mu_l)$. We can define ϕ_μ as above. Take any integer t which is large enough — say larger than the conductors of all the characters μ_j. Let

$$\Gamma = \Gamma(\mathfrak{p}^t) \subseteq GL(n, \mathcal{O})$$

be the subgroup of $GL(n, \mathcal{O})$ consisting of matrices whose off diagonal blocks are divisible by $\mathfrak{p}^t$.

Note that the restriction of ϕ_μ to $\Gamma(\mathfrak{p}^t)$ is a character. We have

$$\phi_\mu(\gamma x) = \phi_\mu(x\gamma) = \phi_\mu(\gamma)\phi_\mu(x) \qquad (\gamma \in \Gamma(\mathfrak{p}^t), \text{any } x) \tag{3.3.4.5}$$

Define a related function $\tilde{\phi}_\mu$ by

$$\tilde{\phi}_\mu(x) = \begin{cases} \mathrm{Vol}(\Gamma(\mathfrak{p}^t); d^\times Z)^{-1} \cdot \phi_\mu(x), & \text{if } x \in \Gamma(\mathfrak{p}^t) \\ 0, & \text{otherwise} \end{cases}$$

Here $\mathrm{Vol}(\Gamma(\mathfrak{p}^t); d^\times Z)$ is the volume of $\Gamma(\mathfrak{p}^t)$ with respect to the measure $d^\times Z$. We have

$$\mathrm{Vol}(\Gamma(\mathfrak{p}^t); d^\times Z)^{-1} = A(n)^{-1}[GL(n, \mathcal{O}) : \Gamma(\mathfrak{p}^t)] = (\prod_{j=1}^{l} A(n_j)^{-1})(\prod_{1 \le i < j \le l} q^{2tn_in_j})$$

Later on, we shall identify various spaces with $M_{n,n}$, and ϕ_μ, etc, will be viewed as a function on these spaces.

We define a Schwartz function Φ_1 on $M(n,n)$ by

$$\Phi_1(u,v) = \tilde{\phi}_\mu(\frac{u-v}{2}) \cdot \mathcal{F}(\phi_\nu)(u+v) \tag{3.3.4.6}$$

Recall that we have identified $U(2V)(E_v)$ with $GL(2n, E_v)$. Thus it acts on $M(n,2n)$ by right multiplications. We take a global section

$$f(h;\chi,s) = \otimes f_u(h;\chi,s) \in \mathrm{Ind}(\chi|\cdot|^s)$$

with u running through all places of E. At the place v we choose the local section by the following formula:

$$f_v(h;\chi,s) = f_{v,\mu}(h;\chi,s) \overset{def}{=} \chi_1(\det h)\cdot|\det h|^{s+\rho} \tag{3.3.4.7}$$

$$\cdot \int_{GL(n,E_v)} \Phi_1((Z,Z)h)\chi_1\chi_2^{-1}(\det Z)|\det Z|^{2(s+\rho)} d^\times Z.$$

Recall that we have the decomposition

$$2V = V^d \oplus V_d$$

of the doubled space $2V$ into totally isotropic subspaces. For the moment write matrices in blocks corresponding to the above decomposition. Set

$$w = w_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

This is ambiguous as we have not specified a basis. We will make a precise definition of w_n by writing it as a block matrix corresponding to the decomposition

$$2V = V \oplus (-V)$$

Then we simply take

$$w = w_n = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$$

For each index j with $1 \le j \le n$ we also define

$$w_j = \begin{pmatrix} 1_j & 0 & 0 & 0 \\ 0 & 1_{n-j} & 0 & 0 \\ 0 & 0 & -1_j & 0 \\ 0 & 0 & 0 & 1_{n-j} \end{pmatrix}$$

(Really, 1_n is the identity on V. But the definition of 1_j for $0 < j < n$ implies an implicit choice of an orthogonal basis for V).

(3.3.4.8) Lemma. *Let $P = P^d$ be the stabilizer of V^d in $U(2V)$. Then as a function of h the local section $f_v(h;\chi,s)$ is supported on the "big cell" $P(E_v)w_nP(E_v)$.*

Proof. We know that $U(2V)$ is the disjoint union of the double cosets Pw_jP. Since f_v is a section, it suffices to show that

$$f_v(w_jp;\chi,s) = 0, \qquad \text{for any } p \in P(E_v),\ j < n$$

As remarked above, the definition of w_j involves an implicit choice of a basis, and therefore a decomposition

$$V = V_j \oplus V^j$$

where V_j is of dimension j. Recall that $U(2V)(E_v) \simeq GL(2n, E_v)$. Under this identification, a typical element of $P(E_v)$, written in blocks with respect to the decomposition $2V = V \oplus -V$, is of the form

$$p = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices, and

$$A + C = B + D$$

In accordance with the decomposition $V = V_j \oplus V^j$, we may write an $n \times n$ matrix as $Z = (X, Y)$ where X is $n \times j$ and Y is $n \times (n - j)$. Then we find

$$(Z, Z)w_jp = (u, v)$$

with

$$u = (X, Y)A + (-X, Y)C, \qquad v = ((X, Y)B + (-X, Y)D$$

Consequently

$$u - v = (X, 0)(A + D - B - C)$$

(Here we have used the condition $A+C = B+D$). The right hand side is a singular matrix unless $j = n$. Since

$$\Phi_1(u, v) = \tilde{\phi}_\mu(\frac{u - v}{2})\mathcal{F}(\phi_\nu)(u + v)$$

and $\tilde{\phi}_\mu$ is supported on invertible matrices, we find

$$\Phi_1((Z, Z)w_jp) = 0 \qquad \text{for all } Z$$

if $j < n$. Hence $f_v(w_jp;\chi,s) = 0$ for $j < n$ and $p \in P(E_v)$.

We define the Eisenstein series $E^f(h;\chi,s) = E^{f_\mu}(h;\chi,s)$ and its Fourier coefficients as before. Let $P = MN$ be a Levi decomposition. We assume that M is normalized by w. We will calculate the v-component of $E^f_\beta(h;\chi,s)$ under the condition that $h_v \in P(E_v)$. In view of the above lemma and Remark 3.2.2.3, we know that the factorization (3.2.2.2) is valid for any β (full rank or otherwise), provided $h_v \in P(E_v)$. However, in (3.3.4.9) we will see that our choice of local data at primes dividing p forces $E^f_\beta(h,\chi,s) = 0$ for $\mathrm{rank}(\beta) < n$, provided $h_v \in P(E_v)$ for at least one place v dividing p.

For the remainder of this section we shall calculate

$$W_{\beta,v}(h_v, f_v, s) = \int_{N(E_v)} f_v(wn_vh_v;\chi_v,s)\psi_{-\beta}(n_v)dn_v$$

The group N can be identified with the space Herm_n of $n\times n$ hermitian matrices. We write this isomorphism as

$$\mathrm{Herm}_n \longrightarrow N, \qquad R \mapsto n(R)$$

If $R \in \mathrm{Herm}_n(\mathbf{A})$ then

$$\psi_\beta(n(R)) = \psi(\mathrm{tr}(\beta^t R))$$

where tr denotes trace of the matrix, followed by $\mathrm{tr}_{\mathcal{K}/E}$. We need to explain what this means at the split place v. We have the isomorphism

$$\mathcal{K}\otimes E_v \simeq E_v \oplus E_v$$

where the first summand E_v is identified with $\mathcal{K}_w$, with w the place of $\mathcal{K}$ dividing v, such that $w \in \Sigma_p$. The second summand is then identified with $\mathcal{K}_{w^c}$. Also, on the right hand side the trace map is identified with the summation of the two coordinates. This gives rise to

$$2V \otimes E_v = (2V)_1 \oplus (2V)_2$$

etc. Now any $R \in \mathrm{Herm}_n(E_v)$ is identified with an arbitrary $n\times n$ matrix with coefficients in E_v, as follows. We consider

$$M_{n,n}(\mathcal{K}) \subset M_{n,n}(\mathcal{K}_w) = M_{n,n}(E_v)$$

Then the embedding

$$M_{n,n}(\mathcal{K}) \longrightarrow M_{n,n}(E_v)\oplus M_{n,n}(E_v), \qquad \gamma \mapsto (\gamma,\bar\gamma)$$

extends to an isomorphism

$$M_{n,n}(\mathcal{K})\otimes E_v \longrightarrow M_{n,n}(E_v)\oplus M_{n,n}(E_v)$$

Since $\bar\gamma = {}^t\gamma$ for $\gamma \in \mathrm{Herm}_n(E)$, we see that the image of

$$\mathrm{Herm}_n(E_v) = \mathrm{Herm}_n(E)\otimes E_v \subset M_{n,n}(\mathcal{K})\otimes E_v$$

under the above isomorphism is precisely

$$\{(R, {}^tR) \mid R \in M_{n,n}(E_v)\}$$

Thus we get the identification $\mathrm{Herm}_n(E_v) = M_{n,n}(E_v)$ by the map $(R, {}^tR) \mapsto R$.

Now if a matrix $R \in M_{n,n}(E_v)$ is identified with an element of $\mathrm{Herm}_n(E_v)$ as above then a simple calculation gives

$$\psi_\beta(n(R)) = \psi(2 \cdot \mathrm{tr}_E(\beta^t R))$$

This time, on the right hand side $\mathrm{tr}_E(\beta^t R)$ is the trace of $\beta^t R$ viewed as a matrix with coefficients in E_v.

We may assume $h_v \in M(E_v)$. Then h_v preserves both the diagonal and the anti-diagonal. So there are $n \times n$ invertible matrices A and B such that

$$(Z, -Z)h_v = (ZA, -ZA), \qquad (Z, Z)h_v = (ZB, ZB)$$

for any Z. Suppose $n = n(R)$. A simple calculation gives

$$(Z, Z)wnh_v = (Z(RB + A), Z(RB - A))$$

Recalling the definition of Φ_1 we obtain

$$\Phi_1((Z, Z)wnh_v) = \tilde{\phi}_\mu(ZA)\mathcal{F}(\phi_\nu)(2ZRB) \tag{3.3.4.9}$$

We already know that f_v is supported on the big cell. In the integral expression for f_v given by (3.3.4.7) we may translate the variable Z by any element of $L(\mathcal{O})$ and then integrate over $L(\mathcal{O}) \subseteq GL(n, E_v)$. By formula (3.3.4.9) and the transformation properties of ϕ_μ and $\mathcal{F}(\phi_\nu)$ given by (3.3.4.4)-(3.3.4.5), we see immediately that f_v would be identically 0 unless the following conditions are satisfied:

$$\mu_j = \nu_j \chi_2 \chi_1^{-1} \ \text{ on } \ \mathcal{O}_v^\times, \quad \text{for} \ \ 1 \leq j \leq l \tag{3.3.4.10}$$

We assume this from now on. Then

$$f_v(wnh_v; \chi, s) = \chi_1(\det B)\chi_2(\det A)|\det BA^{-1}|^{s+\rho}\mathcal{F}(\phi_\nu)(2A^{-1}RB)$$

By Fourier inversion we obtain

(3.3.4.11) Lemma. *For $h_v \in M(E_v)$ as above, the v-component of the β-th Fourier coefficient $E^f_\beta(h, \chi, s) = E^{f_\mu}_\beta(h, \chi, s)$ is given by*

$$W_{\beta,v}(h_v, f_v, s) = \chi_1(\det B)\chi_2(\det A)|\det AB^{-1}|^{-s+\rho}\phi_\nu({}^t A\beta B^{-1}), \tag{3.3.4.12}$$

where ν is defined in terms of μ and χ by (3.3.4.10).

In particular, the β-th Fourier coefficient vanishes unless β is of full rank.

The last assertion of the lemma follows from the fact that ϕ_ν is supported on $\mathcal{I}$.

(3.3.4.13) Remark. From the above calculations, we see that we may change the definition of $\tilde{\phi}_\mu$ as we wish, in the following manner. Let R be any open neighborhood of 1 contained in $\Gamma(\mathfrak{p}^t)$. Let ϕ^R be the restriction of μ to R (which would be trivial for R sufficiently small), divided by the volume of R. It is then clear that we may replace $\tilde{\phi}_\nu$ by any such ϕ^R, and obtain the *same* section f_v.

(3.3.5) *Summary.*

Recall that $m = n + 2s_0$. Define

$$C_\infty(n,m,\mathcal{K}) = \prod_{v\in\Sigma}(\daleth_v)^{-mn+n^2}\cdot\left(\frac{(-i)^{-n^2}2^{n(m-n+1)}\pi^{mn-n(n-1)/2}}{\prod_{j=1}^n(m-j)!}\right)^{[E:\mathbb{Q}]},$$

(3.3.5.1) $$C^S(n,m,\mathcal{K}) = c(n,E,\mathcal{K})\prod_{j=0}^{n-1}L^S(m+j,\chi\varepsilon^j)^{-1}C_\infty(n,m,\mathcal{K});$$

We choose a global section

(3.3.5.2) $$f = f_\mu(h,\chi,s) = \bigotimes_{v\notin S}f_v^{unr}\otimes\bigotimes_{v\in S_\infty}f_v\otimes\bigotimes_{v\in S_f^p}f_{u_v}\otimes_{v|p}f_v(h;\chi,s)$$

in accordance with the preceding sections. The functions f_v for $v\mid\infty$, resp. $v\mid p$, are defined by (3.3.3.1), resp. (3.3.4.7), the characters μ_j being determined by ν_j and χ by (3.3.4.10). Finally, for $v\notin S$, f_v^{unr} is the unramified vector in $I(s,\chi)$ normalized to take value 1 at 1.

Let

$$\mathbb{E}(h,\chi,m,f) = \mathbb{E}(h,\chi,m,f_\mu) \stackrel{def}{=} C^S(n,\mathcal{K})^{-1}E^f(h,\chi,s_0).$$

We define the factor $T^0(\beta,m(A),s_0)$ by (3.2.2.10). When $h = m(A)\in M(\mathbf{A}_f)$, we write $m(A) = m(A^p)\cdot\prod_{v|p}h_v$, and let A_v be the local component of A at v for v prime to p. The preceding calculations show that the β-Fourier coefficient of $\mathbb{E}(m(A),\chi,m,f)$ equals zero if $rank(\beta) < n$. Otherwise, the Fourier coefficient is given by the following formula, in which χ has been replaced by the (motivic) Hecke character $\chi^* = \chi\cdot N_{\mathcal{K}/E}^{\kappa/2}$ and where for split v in S_f^p we write $\chi_v^*(\det(A_v))$ as an abbreviation for $\chi_v^*(\det(\mathbf{A}_v\cdot\mathbf{B}_v^{-1})$ as in (3.3.2):

(3.3.5.3)
$$\begin{aligned}\mathbb{E}_\beta(m(A),\chi,m,f) &= \mathbb{E}_\beta(m(A),\chi,m,f_\mu)\\ &= T^0(\beta,m(A),s_0)\det(\beta)^{(m-n)[E:\mathbb{Q}]}|\det A|_{\mathbf{A}}^{\frac{n-\kappa}{2}-s_0}\times\end{aligned}$$
$$\prod_{v\in\Sigma_p}\chi_1^*(\det B(h_v))\chi_2^*(\det A(h_v))\phi_\nu({}^tA(h_v)\beta B(h_v)^{-1})\times\prod_{v\in S_f^p}\chi_v^*(\det(A_v))\hat{u}_v({}^t\bar{A}_v\beta A_v)$$

We have dropped the term q^β of (1.5.6). The complete arithmetic Fourier expansion is

(3.3.5.4) $$\mathbb{E}(h_\infty m(A),\chi,m,f) = \mathbb{E}(h_\infty m(A),\chi,m,f_\mu) = \sum_\beta\mathbb{E}_\beta(m(A),\chi,m,f)q^\beta$$

with $m(A)\in M(\mathbf{A}_f)$ as before.

Remarks

(3.3.5.5) By (3.1.4.4) the exponent in the absolute value factor $|\det A|_{\mathbf{A}}^{\frac{n-\kappa}{2}-s_0}$ is an integer. Thus these factors are always integers, and in fact are p-units under our standing hypothesis that $A(h_v)$ and $B(h_v)$ are in $GL(n,\mathcal{O}_v)$ for all v dividing p. Similarly, since $m\geq n$, the factor $\det(\beta)^{m-n}$ is p-adically integral provided β is, and this is guaranteed by our hypothesis on $A(h_v)$ and $B(h_v)$ and the definition of $T^0(\beta, m(A))$.

(3.3.5.6) With u_v chosen as in (3.3.1) and (3.3.2) at places in S_f^p, the coefficients are then p-adic integers, and in fact are p-adic units where they are non-zero. Better control of the local theta correspondence at places in S will require different choices of f_v at S_f^p.

(3.3.5.7) In applications to the zeta function we will want to work with finite sums of Siegel-Weil Eisenstein series attached to hermitian spaces V' that differ locally at non-split primes in S, since at such primes we are forced to take the local sections denoted $\tilde{f}_v$ of (3.3.1.2), which are not generally Siegel-Weil sections. These Fourier coefficients of these sums remain p-adically integral and since the different V' represent different β, they are also p-adically primitive.

(3.4) Review of abstract p-adic distributions and measures.

Let T be a torus over $\mathbb{Z}_p$, and let R be a complete $\mathbb{Z}_p$-algebra, assumed $\mathbb{Z}_p$-flat and compact, $R[\frac{1}{p}] = R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For instance, we can take $R = \mathcal{O}_{\mathbb{C}_p}$, so that $R[\frac{1}{p}] = \mathbb{C}_p$. Let $\mathcal{B}$ denote an $R[\frac{1}{p}]$-Banach space, $\mathcal{M} \subset \mathcal{B}$ the unit ball of elements of norm ≤ 1. If $A = R, R[\frac{1}{p}]$, $\mathcal{B}$, or $\mathcal{M}$, let $C(T(\mathbb{Z}_p), A)$ denote the R-module of continuous A-valued functions on $T(\mathbb{Z}_p)$. Since $T(\mathbb{Z}_p)$ is compact, $C(T(\mathbb{Z}_p), \mathcal{B}) = C(T(\mathbb{Z}_p), \mathcal{M}) \otimes_R R[\frac{1}{p}])$, and this is true in particular for $\mathcal{M} = R$ itself. The sup norm makes $C(T(\mathbb{Z}_p, R[\frac{1}{p}]$ into an $R[\frac{1}{p}]$-Banach space. The locally constant functions in $C(T(\mathbb{Z}_p), A)$ are denoted $C^\infty(T(\mathbb{Z}_p), A)$.

A *p-adic distribution* on $T(\mathbb{Z}_p)$ with values in an $R[\frac{1}{p}]$-vector space $\mathcal{V}$ is a homomorphism of R-modules

$$\lambda : C^\infty(T(\mathbb{Z}_p, R) \to \mathcal{V}.$$

To define a distribution $\mathcal{V}$ need not be a Banach space. A $\mathcal{B}$-valued *p-adic measure* on $T(\mathbb{Z}_p)$ is a continuous homomorphism of $R[\frac{1}{p}]$-Banach spaces

$$\mu : C(T(\mathbb{Z}_p), R[\frac{1}{p}]) \to \mathcal{B}.$$

Let $X_{fin}(T)$ denote the set of characters of finite order of $T(\mathbb{Z}_p)$, viewed as a subset of $C^\infty(T(\mathbb{Z}_p), R)$ for any sufficiently large p-adic ring R, e.g. $R = \mathcal{O}_{\mathbb{C}_p}$. The set $X_{fin}(T)$ forms a *basis* for the $R[\frac{1}{p}]$-vector space $C^\infty(T(\mathbb{Z}_p), R[\frac{1}{p}])$, hence any function $\chi \mapsto v_\chi$ from $X_{fin}(T)$ to $\mathcal{V}$ determines a $\mathcal{V}$-valued distribution on $T(\mathbb{Z}_p)$ by linearity.

(3.4.1) Lemma. *Let $\chi \mapsto m_\chi$ be a function from $X_{fin}(T)$ to $\mathcal{M}$, and let $\lambda(m)$ denote the corresponding $\mathcal{B}$-valued distribution. Then $\lambda(m)$ extends to a p-adic measure if and only if, for every integer n and for any finite sum $\sum_j \alpha_j \chi_j$ with $\alpha_j \in R[\frac{1}{p}]$ and $\chi_j \in X_{fin}(T)$ such that $\sum_j \alpha_j \chi_j(t) \in p^n R$ for all $t \in T(\mathbb{Z}_p)$, we have*

$$\sum_j \alpha_j m_{\chi_j} \in p^n \mathcal{M}. \tag{3.4.2}$$

This is a version of the abstract Kummer congruences stated as Proposition 5.0.6 of [K].

(3.4.3) Corollary. *In Lemma (3.4.1) above, it actually suffices to check (3.4.2) with $n = 0$.*

Indeed, the case $n = 0$ of (3.4.2) implies that $\lambda(m)$ is a bounded distribution with values in a Banach space, hence a measure.

In the next section we will be constructing measures with values in the Banach space of p-adic modular forms on the Shimura variety $Sh(2V)$. Let $R = \mathcal{O}_{\mathbb{C}_p}$, so that $R[\frac{1}{p}] = \mathbb{C}_p$. Let $\mathcal{V}$ denote the algebra of p-adic modular forms, as in (2.2), and let $\mathcal{B} = \mathcal{V} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p$. Let $\mathcal{M}$ denote the right-hand side $\hat{\bigoplus}_{\alpha \in U^*} H^0({}_{K_P(\infty)}S(G_P, X_P), \mathcal{O}_{\mathbb{S}_P})$ of (2.3.2), and let $\mathcal{Q} = \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p$. The $\mathbb{C}_p$-vector space $\mathcal{B}$ is a Banach space via the sup norm, whereas $\mathcal{Q}$ can be viewed as a ring of formal series over the Banach space $H^0({}_{K_P(\infty)}S(G_P, X_P), \mathcal{O}_{\mathbb{S}_P})$, hence again becomes a Banach space via the sup norm. The q-expansion map $F.J.^P$ (2.3.2) is a continuous homomorphism of Banach spaces.

The following proposition follows from the q-expansion principle, as in [DR] or [K], and represents the primary application of the q-expansion principle to our project:

(3.4.4) Proposition. *Let T be a torus over $\mathbb{Z}_p$, and let μ be a p-adic measure on $T(\mathbb{Z}_p)$ with values in $\mathcal{Q}$. Suppose that $\mu(\chi) = \int_{T(\mathbb{Z}_p)} \chi d\mu$ lies in the image of $F.J.^P$ for all $\chi \in X_{fin}(T)$. Then μ is the image, under $F.J.^P$, of a measure with values in $\mathcal{B}$.*

(3.5) Construction of Eisenstein measures.

Let ℓ be a positive integer and let $T(\ell)^0$ denote the torus over $\mathbb{Z}_p$ given by $(R_{\mathcal{O}_E/\mathbb{Z}_p}\mathbb{G}_{m,\mathcal{O}_E})^\ell$. Thus $T(\ell)^0(\mathbb{Z}_p)$ is canonically isomorphic to $\prod_{w|p} \mathcal{O}_w^{\times,\ell}$, where w runs through places of E. This can also be identified with the product of ℓ copies of $\prod_{v\in\Sigma_p} \mathcal{O}_v^\times$, where now v are places of $\mathcal{K}$. The latter form will be the most useful for us. For brevity we write $\mathcal{O}_{\Sigma_p}^\times$ for $\prod_{v\in\Sigma_p} \mathcal{O}_v^\times$. We let

$$T(\ell) = T(\ell)^0 \times (R_{\mathcal{O}_\mathcal{K}/\mathbb{Z}_p}\mathbb{G}_{m,\mathcal{O}_\mathcal{K}}).$$

Then the set $X_{fin}(T(\ell))$ of finite order characters of $T(\ell)$ can be parametrized by $(\ell+1)$-tuples $(\nu_1, \dots, \nu_\ell, \chi)$, where each ν_i is a character of finite order of $\mathcal{O}_{\Sigma_p}^\times$, and χ is a character of finite order of $\prod_{v|p} \mathcal{O}_v^\times$ where now v runs over all places of $\mathcal{K}$ dividing p. We will further write $\chi = (\chi_1, \chi_2)$, where χ_1 is the restriction of χ to $\prod_{v\in\Sigma_p} \mathcal{O}_v^\times$ and χ_2 is a second character of the same group $\prod_{v\in\Sigma_p} \mathcal{O}_v^\times$ obtained by restricting χ^{-1} to $\prod_{v\in c\Sigma_p} \mathcal{O}_v^\times$ and then composing with c. So in the end, $X_{fin}(T)$ can be viewed as the set of $(\ell+2)$-tuples of characters of $\mathcal{O}_{\Sigma_p}^\times$. The character χ will in practice be the restriction to $\mathcal{O}_{\mathcal{K},p}^\times$ of a character of $\mathcal{K}_p^\times = \prod_{v|p} \mathcal{K}_v^\times$, which in turn will most commonly be the p-adic component of a global Hecke character.

We introduce additional notation: for $j = 1, \dots, \ell$, we let $\mu_j = \nu_j \cdot \chi_2 \cdot \chi_1^{-1}$. Let m, n, and s_0 be as in (3.3.5). Let $n = n_1 + \dots + n_\ell$ be a partition of n and Q the corresponding standard parabolic subgroup of $GL(n)$.

(3.5.1) Theorem. *There is a $\mathcal{B}$-valued measure λ_Q^m on $T(\ell)$ with the property that, for any $\ell+2$-tuple $(\mu, \chi) = (\mu_1, \dots, \mu_\ell, \chi_1, \chi_2)$ of characters of finite order of $\mathcal{O}_{\Sigma_p}^\times$.*

$$F.J.^P \circ \int_{T(\ell)} (\mu_1, \dots, \mu_\ell, \chi) d\lambda_Q^m = \mathbb{E}(\bullet, \chi, m, f_\mu) \tag{3.5.2}$$

where the right hand side is the q-expansion of (3.3.5.4).

Proof. The right-hand side of (3.5.2) defines the value at (μ, χ) of a $\mathcal{Q}$-valued distribution on $T(t)$. To show that this distribution is in fact a $\mathcal{Q}$-valued p-adic measure, it suffices, by Corollary (3.4.3), to show that the right-hand side of (3.5.2) satisfies the abstract Kummer congruences (3.4.2) for $n = 0$. In other words, for any $\beta \in U^* \cap C$, the Fourier coefficients $\mathbb{E}_\beta(m(A), \chi, m, f_\mu)$ as (ν, χ) vary, satisfy the abstract Kummer congruences as functions of $m(A) \in L_P(\mathbf{A}_f)$, with the coefficients $A_v \in GL(n, \mathcal{O}_v)$ for $v \mid p$. Bearing in mind the relation (3.3.4.10) between ν and μ, this follows immediately from (3.3.5.3) and Remarks (3.3.5.5) and (3.3.5.6).

Now the theorem follows from Proposition 3.4.4 and from the fact that $\mathbb{E}(\bullet, \chi, m, f_\mu)$ is a classical modular form for $(\mu, \chi) \in X_{fin}(T(\ell))$.

References

[BS] S. Böcherer, C.-G. Schmidt, p-adic measures attached to Siegel modular forms, *Ann. Inst. Fourier*, **50** (2000) 1375-1443.

[Co] J. Coates, Motivic p-adic L-functions, in J. Coates and M. J. Taylor, eds., *L-functions and Arithmetic* London Mathematical Society Lecture Note Series, **153** Cambridge: Cambridge University Press (1991), 141-172.

[D] P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, in *Automorphic forms, representations and L-functions*, *Proc. Symp. Pure Math.*, **33**, Part 2, (1979) 247–289.

[DR] P. Deligne and K. Ribet, Values of Abelian L-functions at negative integers over totally real fields, *Inv. Math.*, **59**, (1980), 227-286.

[F] K. Fujiwara, Arithmetic compactifications of Shimura varieties (I), unpublished manuscript (1992).

[G] P. Garrett, Archimedean zeta integrals for unitary groups, (handwritten notes 1991-1992; revised manuscript, July 19, 2005).

[H1] M. Harris, Arithmetic vector bundles and automorphic forms on Shimura varieties: I, *Invent. Math.*, **82** (1985)-151-189; II. *Compositio Math.*, **60** (1986), 323-378.

[H2] M. Harris, L-functions of 2 by 2 unitary groups and factorization of periods of Hilbert modular forms. *JAMS*, **6** (1993), 637-719.

[H3] M. Harris, L-functions and periods of polarized regular motives, *J.Reine Angew. Math.*, **483**, (1997) 75-161.

[H4] M. Harris, Cohomological automorphic forms on unitary groups, I: rationality of the theta correspondence, *Proc. Symp. Pure Math*, **66.2**, (1999) 103-200.

[H5] M. Harris, Cohomological automorphic forms on unitary groups, II: in preparation.

[HK] M. Harris, S. Kudla, On a conjecture of Jacquet, in H. Hida, D. Ramakrishnan, F. Shahidi, eds., *Contributions to automorphic forms, geometry, and number theory* (volume in honor of J. Shalika), 355-371 (2004)

[HKS] M. Harris, S. Kudla, W. J. Sweet, Theta dichotomy for unitary groups, *JAMS*, **9** (1996) 941-1004.

[HL] M. Harris and J.-P. Labesse, Conditional base change for unitary groups, *Asian J. Math.* (2005).

[HLS] M. Harris, J.-S. Li, C. Skinner, The Rallis inner product formula and p-adic L-functions, volume in honor of S. Rallis, in press.

[Hi02] H. Hida, Control theorems of coherent sheaves on Shimura varieties of PEL type, *J. Inst. Math. Jussieu*, **1** (2002) 1-76.

[Hi04] H. Hida, *p-adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics (2004).

[Hi05] H. Hida, p-adic automorphic forms on reductive groups, Notes of a course at the IHP in 2000, *Astérisque*, **296** (2005) 147-254.

[K] N. Katz, p-adic L-functions for CM fields, *Invent. Math.*, **49** (1978) 199-297.

[Ko] R. Kottwitz, , Points on some Shimura varieties over finite fields, *Jour. of the AMS* **5** (1992) 373-444.

[KR] S. S. Kudla, S. Rallis, Poles of Eisenstein series and L-functions, in Festschrift in Honor of I. I. Piatetski-Shapiro (Part II), *Israel Math. Conf. Proceedings*, **3** (1990) 81-110.

[L1] J.-S. Li, Theta liftings for unitary representations with non-zero cohomology, *Duke Math. J.*, **61** (1990) 913-937.

[L2] J.-S. Li, Non-vanishing theorems for the cohomology of certain arithmetic quotients, *J. reine angew. Math.*, **428** (1992) 177-217.

[Pa] A. A. Panchishkin, Admissible non-archimedean standard zeta functions associated with Siegel modular forms, in *Motives*, Proc. Symp. Pure Math., **55** (1994), Part 2, pp. 251-292.

[PSR] I. I. Piatetski-Shapiro, S. Rallis, L-functions for the classical groups, in S. Gelbart, I. Piatetski-Shapiro, and S. Rallis, Explicit constructions of automorphic L-functions, *Lecture Notes in Math.*, **1254** (1987).

[P] R. Pink, Arithmetical compactification of mixed Shimura varieties *Bonner Math. Schriften*, **209** (1990).

[Pr] K. Prasanna, Integrality of a ratio of Petersson norms and level-lowering congruences, *Annals of Math.*, in press.

[RZ] M. Rapoport and T. Zink: *Period Spaces for p-divisible Groups*, Princeton: Annals of Mathematics Studies **141** (1996).

[S82] G. Shimura, Confluent hypergeometric functions on tube domains, Math. Ann. **260** (1982) 269-302.

[S97] G. Shimura, Euler products and Eisenstein series, *CBMS Regional Conference Series in Mathematics*, **93**, Providence, R.I.: American Mathematical Society (1997).

[S00] G. Shimura, Arithmeticity in the theory of automorphic forms, *Mathematical Series and Monographs*, **82**, Providence, R.I.: American Mathematical Society (2000).

[SU] C. Skinner and E. Urban, to appear

[T] V. Tan, Poles of Siegel Eisenstein series on $U(n, n)$, *Can. J. Math.*, **51** (1999) 164-175.

[Tate] J. Tate, *Fourier analysis in number fields and Hecke's Zeta-functions* in Algebraic number theory J. W. S. Cassels and A Fröhlich, eds, Academic Press, (1967) 305-347.

[Wed] T. Wedhorn, Congruence Relations on some Shimura varieties *J. f.d. reine und angew. Math.*, **524**, (2000), 43–71.