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Déformations équivariantes (dérivées) de schémas algébriques et de variétés complexes compactes

dirigée par Julien Grivaux

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Craindre l'erreur et craindre la vérité est une seule et même chose. Celui qui craint de se tromper est impuissant à découvrir. C'est quand nous craignons de nous tromper que l'erreur qui est en nous se fait immuable comme un roc. Car dans notre peur, nous nous accrochons à ce que nous avons décrété "vrai" un jour, ou à ce qui depuis toujours nous a été présenté comme tel. Quand nous sommes mûs, non par la peur de voir s'évanouir une illusoire sécurité, mais par une soif de connaître, alors l'erreur, comme la souffrance ou la tristesse, nous traverse sans se figer jamais, et la trace de son passage est une connaissance renouvelée.

Alexandre Grothendieck, *Récoltes et semailles*

Remerciements

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Déformations équivariantes (dérivées) de schémas algébriques et de variétés complexes compactes

Résumé

Cette thèse est dédiée à une étude complète des déformations G-équivariantes de schémas algébriques (resp. variétés compactes complexes) dans le cadre classique ainsi que dans celui qui est dérivé où G est un groupe algébrique linéaire défini sur un corp de caractéristique 0 (resp. un groupe de Lie complexe). Quant à l'aspect classique, les points centraux sont l'existence d'une déformation semi-universelle G-équivariante où G est réductif et la non-existence de telles déformations au cas où G est non-réductif, tandis qu'à l'égard de l'aspect dérivé, la semi-proreprésentabilité du problème de modules formel associé est prise en compte.

Mots-clés

Théorie des déformations, Structure equivariante, Problèmes de modules formels, Semi-universalité, Semi-proreprésentabilité.

Equivariant (derived) deformations of algebraic schemes and of complex compact manifolds

Abstract

This thesis is dedicated to a complete study of G-equivariant deformations of algebraic schemes (resp. complex compact manifolds) in the classical setting as well as the derived one where G is a linear algebraic group defined over a field of characteristic 0 (resp. a complex Lie group). In the classical aspect, the central points are the existence of a G-equivariant semi-universal deformation where G is a reductive group and the non-existence of such deformations in the non-reductive case, while in the derived one, the semi-prorepresentability of the associated formal moduli problem is taken into account.

Keywords

Deformation theory, Equivariant structure, Formal moduli problem, Semi-universality, Semi-prorepresentability.

Contents

Introduction - Version française

Le contenu principal de cette thèse concerne le développement moderne de la théorie des déformations de variété complexes compactes et de celles qui sont algébriques, initiée par K. Kodaira, D. C. Spencer, M. Artin, A. Grothendieck, M. Schlessinger et d'autres dans les années 60. Nous nous concentrons surtout sur les façons de déformer un objet géométrique en présence de symétries. Ce problème est bien plus difficile car les symétries ne sont pas toujours préservées par petit déplacement, et il faut comprendre comment les propager.

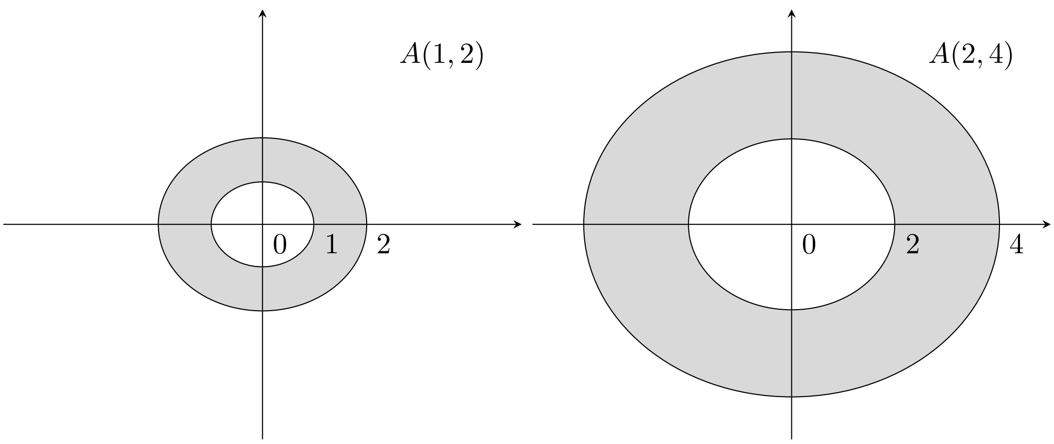
Tout d'abord, nous commençons par introduire ce qu'est la théorie des déformations sur un exemple simple. Nous considérons tous les anneaux de type

$$A(r, R) = \{z \in \mathbb{C} \mid r < |z| < R\}$$

dans le plan complexe. Il est bien connu que la condition nécessaire et suffisante pour que deux anneaux $A_1(r_1, R_1) = \{z \in \mathbb{C} \mid r_1 < |z| < R_1\}$ et $A_2(r_2, R_2) = \{z \in \mathbb{C} \mid r_2 < |z| < R_2\}$ de ce type soient biholomorphes est

$$\frac{R_1}{r_1} = \frac{R_2}{r_2}.$$

Par exemple, deux anneaux $A(1, 2)$ et $A(2, 4)$, comme illustré ci-dessous, sont biholomorphes. Un biholomorphisme explicite entre eux est donné par



$$\gamma : A(1,2) \to A(2,4)$$
$$z \mapsto 2z.$$

La quantité $\ln(R) - \ln(r)$ est appelée le module de l'anneau, elle varie dans $(0, +\infty)$ et classifie toutes les classes de déformations. Le but de la théorie des déformations est de comprendre en quelque sorte de combien de paramètres dépend une figure géométrique munie d'une certaine structure, ici une structure de variété complexe ou bien une structure conforme. Dans l'exemple ci-dessus, à première vue, chaque anneau est apparemment déterminé de manière unique par r et R. Cependant, sa classe d'isomorphismes ne dépend en fait que d'un paramètre : le module.

Un point essentiel de cette théorie est la capacité à linéariser le problème : on peut dériver une déformation de façon appropriée et il s'avère que la variation infinitésimale de l'objet géométrique considéré est assez souvent encodée dans un objet algébrique beaucoup plus simple : un espace vectoriel de dimension finie. Dans l'aspect formel, la théorie de la déformation est un analogue de la théorie des développements limités lorsque nous remplaçons des fonctions par des objets géométriques : les coefficients du développement limité sont des classes de cohomologie vivant dans certains espaces vectoriels de dimension finie familiers. De plus, l'analogue d'une fonction constante est une structure géométrique rigide : une déformation dont la dérivée s'annule est constante. Une illustration vivante pour cela est le travail fondateur de Kodaira-Spencer à propos des déformations de variétés complexes compactes. Soit X_0 une variété complexe compacte. Une déformation de X_0 selon eux est une application holomorphe, propre et submersive $\nu : \mathcal{X} \to (B, 0)$ où $B, \mathcal{X}$ sont des variétés complexes et 0 est un point dans B tel que $\nu^{-1}(0) = X_0$. Étonnamment, chaque déformation de X_0 correspond à une classe de cohomologie dans le premier groupe de cohomologie $H^1(X_0, \Theta_{X_0})$ du fibré tangent holomorphe Θ_{X_0}. Ce groupe de cohomologie est en fait un $\mathbb{C}$-espace vectoriel de dimension finie. De plus, trouver une déformation de X_0 équivaut à trouver une série convergente $\phi(t)$ à coefficients dans $H^1(X_0, \Theta_{X_0})$, satisfaisant l'équation de Maurer-Cartan, c'est-à-dire

$$\overline{\partial}\phi(t) = \frac{1}{2}[\phi(t), \phi(t)].$$

En résolvant cette équation, Kodaira et Spencer ont prouvé l'existence d'une variété complexe (plus précisément, un voisinage ouvert de l'origine dans $\mathbb{C}^n$ où $n = \dim_{\mathbb{C}} H^1(X_0, \Theta_{X_0})$) qui contient toutes les petites déformations de X_0, sous l'hypothèse que

$$H^2(X_0, \Theta_{X_0}) = 0 \tag{0.0.1}$$

(cf. [17]). La famille correspondant à cette variété a la propriété que toute autre déformation de X_0 est obtenue par le tiré en arrière de cette famille par une application holomorphe. Cette application n'est pas unique en général mais sa différentielle au point de référence est unique. Plus tard, Kuranishi a pu supprimer l'hypothèse (0.0.1) (cf. [19]). Cependant, le prix à payer est un assouplissement de la définition des déformations. Une déformation de X_0 est maintenant un morphisme holomorphe plat et propre d'espaces analytiques $\nu : \mathcal{X} \to (B, 0)$ avec le diagramme cartésien suivant

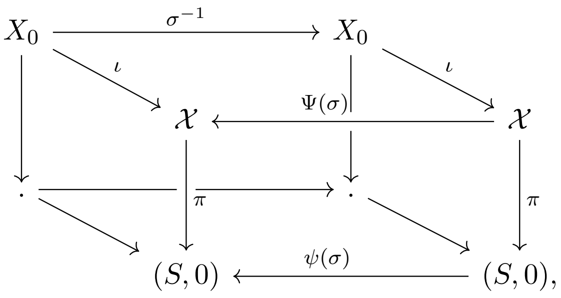
$$\begin{array}{ccc} X_0 & \xrightarrow{\iota} & \mathcal{X} \\ \downarrow & & \downarrow \nu \\ \cdot & \longrightarrow & (B, 0). \end{array}$$

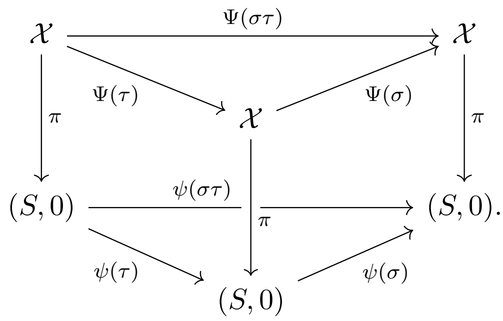
"L'espace de modules local" obtenu par Kuranishi (souvent appelé l'espace de Kuranishi) est singulier en général et en fait bien défini seulement près du point de référence. La famille de variétés complexes compactes associée est appelée la déformation semi-universelle (ou la famille de Kuranishi) de X_0. Dans le cadre algébrique, lorsque X_0 est un schéma affine avec au plus des singularités isolées ou un schéma complet défini sur un corps de caractéristique zéro, le même résultat est obtenu par M. Schlessinger au moyen du langage des foncteurs artiniens (voir [31] or [32]).

Le problème est beaucoup plus compliqué lorsque la symétrie entre en jeu : si X_0 est en outre doté d'une action d'un certain groupe G, G n'agit plus sur les petites déformations de X_0. L'exemple le plus simple de ce phénomène est lorsqu'on déforme une courbe hyperelliptique de genre au moins 3 : l'involution hyperelliptique ρ ne se propage pas à travers des déformations. Plus précisément, soit $\gamma : \mathcal{X} \to (B, 0)$ une déformation verselle de X_0 (c'est-à-dire une déformation dont l'application de Kodaira-Spencer est surjective). Si l'action s'étendait, alors l'image de l'application de Kodaira-Spencer serait fixée par ρ. Par conséquent, ρ agirait trivialement sur $H^1(X_0, \Theta_{X_0})$, ce qui n'est évidemment pas le cas. Cependant, il y a encore un peu d'espoir que G agisse sur l'espace de Kuranishi : de petites déformations de X_0 sont permutées entre elles sous l'action de G. Concrètement, soit $\pi : \mathcal{X} \to (S, 0)$ la déformation semi-universelle de X_0 où X_0 est muni d'une G-action. On considère alors le problème suivant

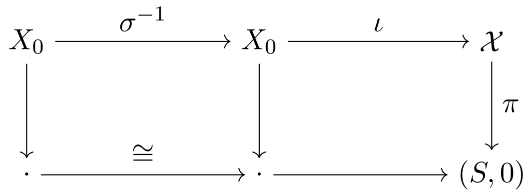
Problème I. *Existe-t-il une G-action sur S et une G-action sur X prolongeant la G-action initiale sur X_0 de telle sorte que π soit G-equivariant par rapport à ces G-actions ?*

En d'autres termes, on se demande s'il existe des morphismes de groupes $\Psi : G \to \mathrm{Aut}(\mathcal{X})$ et $\psi : G \to \mathrm{Aut}(S)$ tel que pour tout $\sigma, \tau \in G$, on ait les diagrammes commutatifs suivants





Comme on pouvait s'y attendre, si la famille $\pi : \mathcal{X} \to S$ est universelle, c'est-à-dire que l'application du tiré en arrière est en fait unique, la réponse au problème ci-dessus est sans aucun doute positive. Ceci résulte du fait qu'à chaque fois que l'on change la fibre centrale de la famille localement universelle par un automorphisme de X_0, on obtient une autre famille universelle de X_0



qui est canoniquement isomorphe à l'ancienne. Malheureusement, une telle famille existe rarement en raison de l'existence d'automorphismes non triviaux de X_0.

Le problème I a d'abord été considéré dans un travail pionnier de Pinkham dans lequel il a donné une réponse affirmative lorsque X_0 est un cône affine avec $\mathbb{G}_m$-action (cf. [26]). Plus tard, Rim a obtenu un résultat plus poussé dont la preuve est basée sur le langage des catégories et sur des représentations rationnelles des groupes algébriques (cf. [30]).

Théorème I. *Si G est un groupe algébrique linéairement réductif et X_0 est un schéma affine avec au plus des singularités isolées ou un schéma complet. Alors, il existe une déformation semi-universelle G-équivariante de X_0, unique à isomorphismes G-équivariants non-canonicaux près .*

L'objectif principal de notre travail est de traiter les problèmes qui se posent naturellement autour de ce théorème. La structure de cette thèse contient trois chapitres correspondant à nos trois articles [7], [8] and [9]. Dans le chapitre 1, nous montrons que dans le théorème I, l'hypothèse que G est réductif est vraiment optimale. Pour le dire autrement, nous proposons un schéma projectif X_0 sur lequel un groupe non-réductif G agit algébriquement de telle sorte que l'action de G ne s'étende pas à sa déformation semi-universelle formelle. À savoir, soit $\mathbb{F}_2$ la deuxième surface de Hirzebruch

$$\mathbb{F}_2 := \{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 | yv^2 = zu^2\}$$

et considérons son groupe d'automorphismes

$$G := \mathrm{Aut}(\mathbb{F}_2) \cong (\mathbb{C}^3 \rtimes GL(2, \mathbb{C}))/I$$

où

$$I = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu \in \mathbb{C}, \mu^2 = 1 \right\}.$$

Il est facile de voir que G est non réductif. Puis nous commençons notre histoire par le résultat suivant (cf. Theorem 1.5.1).

Théorème A. *L'action de G sur $\mathbb{F}_2$ ne s'étend pas à la déformation semi-universelle formelle de $\mathbb{F}_2$*

Intuitivement, lorsque X_0 est une variété complexe compacte, une version appropriée du théorème I devrait toujours être valable, mais cela n'apparaît pas dans la littérature existante. De plus, il existe une différence cruciale entre le monde algébrique et le monde analytique. Dans le cadre algébrique, la déformation semi-universelle de X_0 et les G-actions étendues, construites par Rim dans le théorème I, ne sont que formelles. Néanmoins, dans le cadre analytique, sa déformation semi-universelle est une vraie déformation (une déformation convergente). Ainsi, si X_0 est une variété complexe compacte projective, une application du résultat de Rim nous donne une famille de Kuranishi G-équivariante dont les G-actions étendues ne sont que formelles, c'est-à-dire ce sont des séries formelles dont la convergence n'est pas garantie. Alors, cela nous motive à prouver, au chapitre 2, les deux résultats suivants dont l'ingrédient principal des preuves est une combinaison délicate d'une version G-équivariante de la construction classique de l'espace de Kuranishi de variétés complexes compactes (cf. [18]) et des représentations des groupes de Lie complexes réductifs (cf. Corollaire 2.4.1 et Théorème 2.5.2, respectivement).

Théorème B. *Soit X_0 une variété complexe compacte X_0 munie d'une action K, où K est un groupe de Lie réel compact. Alors il existe une déformation semi-universelle K-équivariante de X_0.*

Théorème C. *Soit X_0 une variété complexe compacte munie d'une action holomorphe d'un groupe de Lie complexe réductif G. Alors il existe une déformation semi-universelle G-équivariante locale de X_0.*

Le langage des foncteurs artiniens développé par M. Schlessinger (voir [31]) permet de réécrire le théorème I comme suit. Soit $\mathbf{Art}_k$ la catégorie des k-algèbres artiniennes locales de corps résiduel k. Le foncteur $F_{X_0} : \mathbf{Art}_k \to \mathbf{Sets}$ qui associe à chaque k-algèbre artinienne locale A, l'ensemble des morphismes plats de schémas $X \to \mathrm{Spec}(A)$ avec un isomorphisme $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong X_0$ a un élément semi-universel formel qui peut encore être rendu équivariant si l'hypothèse qu'un groupe algébrique linéairement réductif

G agit algébriquement sur X_0 est ajoutée. Le chapitre 3 traite la semi-proreprésentabilité du problème de modules formel étendu Def_{X_0} de F_{X_0} dans le contexte des théories de déformations dérivées (rappelons qu'un problème de modules formel est un ∞-foncteur de $\mathbf{dgArt}_k \to \mathbf{SEns}$ satisfaisant certaines conditions d'exactitude, où $\mathbf{dgArt}_k$ est la catégorie des k-algèbres artiniennes différentielles graduées augmentées sur k et $\mathbf{SEns}$ est la ∞-catégorie des ensembles simpliciaux). Pour être plus précis, nous introduisons d'abord la notion de semi-proreprésentabilité des problèmes de modules formels qui ne semble pas exister dans le cadre dérivé. Cette nouvelle notion est la version plus faible de la proreprésentabilité et une généralisation naturelle de la notion de semi-universalité au sens de M. Schlessinger (cf. Définition 3.3.2 ci-dessous). Ensuite, au moyen de l'équivalence bien connue entre la ∞-catégorie des problèmes de modules formels et celle des algèbres de Lie différentielles graduées, un critère simple pour qu'un problème de modules formels soit semi-proreprésentable est fourni (cf. Théorème 3.3.2).

Théorème D. *Soit F un problème de modules formel dont l'algèbre de Lie différentielle graduée associée $\mathfrak{g}_*$ est cohomologiquement concentrée dans $[0, +\infty)$. Supposons en outre que $H^i(\mathfrak{g}_*)$ soit un espace vectoriel de dimension finie pour chaque $i \geq 0$. Alors F est semi-proreprésentable.*

Le foncteur $\mathrm{Def}_{X_0} : \mathbf{dgArt}_k \to \mathbf{SEns}$, qui associe à chaque k-algèbre artinienne différentielle graduée augmentée, le nerf de la catégorie des morphismes plats de schémas dérivés sur $\mathrm{Spec}(A)$, dont la fibre homotopique au point k de $\mathrm{Spec}(A)$ est X_0, est en fait un problème de modules formel qui étend F_{X_0}, c'est-à-dire

$$\pi_0(\mathrm{Def}_{X_0}) = F_{X_0}.$$

Il est bien connu que l'algèbre de Lie différentielle graduée associée à Def_{X_0} est

$$\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$$

où $\mathbb{T}_{X_0/k}$ est le complexe tangent de X_0 et $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ est l'espace des sections globales dérivées de $\mathbb{T}_{X_0/k}$ (cf. Théorème 3.4.2). Lorsque X_0 est un schéma affine avec au plus des singularités isolées ou un schéma projectif défini sur k, la semi-proreprésentabilité de Def_{X_0} découle du fait que $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ est cohomologiquement concentrée dans $[0, +\infty)$ et que $H^i(\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k}))$ est un espace vectoriel de dimension finie pour chaque $i \geq 0$. Autrement dit, Def_{X_0} possède un "élément semi-universel dérivé" dont les composantes connexes restituent l'élément semi-universel classique. Dans l'esprit du problème I, il est naturel de se demander si cet élément semi-universel dérivé peut être rendu G-équivariant dans un certain sens, où G est un groupe algébrique linéairement réductif agissant algébriquement sur X_0 . La réponse est donnée par le théorème suivant qui est une généralisation naturelle du théorème I dans le cadre dérivé. (cf. Théorème 3.4.6).

Théorème E. *Il existe une structure G-équivariante sur l'élément semi-universel dérivé de* Def_{X_0}. *Par conséquent, le foncteur artinien classique* $F_{X_0} = \pi_0(\mathrm{Def}_{X_0})$ *a un élément semi-universel G-équivariant.*

Enfin, lorsque X_0 est une variété complexe compacte sur laquelle un groupe de Lie complexe réductif agit de façon holomorphe, nous voudrions fournir une version formelle du théorème C en utilisant l'approche purement algébrique que nous développons plus tôt dans le même chapitre. Soit $\mathfrak{Def}_{X_0} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets}$ (resp. $\mathfrak{Def}^G_{X_0} : \mathbf{Art}^G_{\mathbb{C}} \to \mathbf{Sets}$) le foncteur qui associe à chaque k-algèbre artinienne locale A, la classe d'isomorphismes de morphismes (resp. morphismes G-équivariants) propres et plats d'espaces analytiques $X \to \mathrm{Spec}(A)$ avec un isomorphisme (resp. un isomorphisme G-équivariant)

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{C}) \cong X_0.$$

L'algèbre de Lie différentielle graduée contrôlant les déformations de X_0 est

$$\mathfrak{g}_* := \Gamma(X_0, \mathcal{A}^{0,0}(\mathcal{T}_{X_0})) \xrightarrow{\bar{\partial}} \Gamma(X_0, \mathcal{A}^{0,1}(\mathcal{T}_{X_0})) \xrightarrow{\bar{\partial}} \Gamma(X_0, \mathcal{A}^{0,2}(\mathcal{T}_{X_0})) \xrightarrow{\bar{\partial}} \cdots$$

avec le crochet de Lie défini par

$$[\phi d\bar{z}_I, \psi d\bar{z}_J] = [\phi, \psi]' d\bar{z}_I \wedge \bar{z}_J$$

où $\phi, \psi \in \mathcal{A}^{0,0}(\mathcal{T}_{X_0})$ sont des champs de vecteurs sur X_0, $[-,-]'$ est le crochet de Lie des champs de vecteurs habituel, $I, J \subset \{1, \ldots, n\}$ et $z_1, \ldots, z_n$ sont des coordonnées holomorphes locales. Par conséquent, le problème des modules formel qui étend $\mathfrak{Def}_{X_0}$ est

$$\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*)$$

où D est la dualité de Koszul (cf. Proposition 3.2.4 et Théorème 3.2.3). Il est évident que $\mathfrak{g}_*$ est cohomologiquement concentrée en degrés positifs. De plus, $\mathfrak{g}_*$ reçoit une G-action naturelle induite par celle sur X_0. Puis nous concluons ce dernier chapitre par le résultat suivant (cf. Théorème 3.4.9).

Théorème F. *Il existe une structure G-équivariante sur l'objet semi-proreprésentable de* $\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*)$ *par rapport à l'action de G prescrite sur $\mathfrak{g}_*$. Par conséquent, le foncteur classique des déformations G-équivariantes $\mathfrak{Def}^G_{X_0}$ de X_0 a un élément semi-universel G-équivariant formel.*

Introduction - English version

The main content of this thesis concerns the modern development of the theory of deformations of compact complex manifolds and algebraic varieties, initiated by K. Kodaira, Spencer, M. Artin, A. Grothendieck, M. Schlessinger and other people in the 1960s. We mainly focus on understanding how one can deform a geometric object with an additional symmetry. This problem is somehow difficult due to the fact that the symmetry is not always preserved through small deformations.

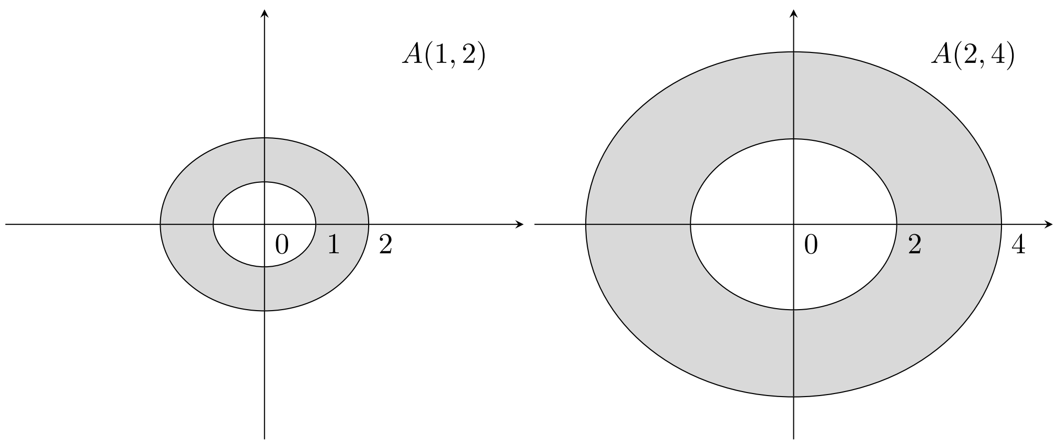
First of all, we introduce briefly what the deformation theory is by a simple example. We consider all rings of type

$$A(r, R) = \{z \in \mathbb{C} \mid r < |z| < R\}$$

in the complex plan. It is well-known that the necessary and sufficient condition for two rings $A_1(r_1, R_1) = \{z \in \mathbb{C} \mid r_1 < |z| < R_1\}$ and $A_2(r_2, R_2) = \{z \in \mathbb{C} \mid r_2 < |z| < R_2\}$ of this type to be biholomorphic to each other is

$$\frac{R_1}{r_1} = \frac{R_2}{r_2}.$$

For example, two rings $A(1, 2)$ and $A(2, 4)$, as pictured below are biholomorphic. An



explicit biholomorphism between them is given by

$$\gamma : A(1,2) \to A(2,4)$$
$$z \mapsto 2z.$$

The quantity $\ln(R) - \ln(r)$ is called the moduli of the ring which varies in $(0, +\infty)$ and which classifies all the classes of deformations. The principal purpose of the deformation theory is the study of how many parameters a geometric object equipped with some structure, in this case a complex structure or a conformal structure, depends on. In the above example, at first glance, each ring is apparently uniquely determined by r and R. However, its isomorphism class in fact depends only on one parameter which is their moduli.

One essential point of this theory is the capability of linearizing the problem: one can derive a deformation in an appropriate way and it turns out quite often that the infinitesimal variation of the considered geometric object is encoded in a much more simple algebraic object - a finite dimensional vector space. In the formal aspect, deformation theory is an analog of Taylor's expansion when we replace functions by geometric objects: the coefficients of the power series are cohomology classes lying in some familiar finite dimensional vector spaces. Moreover, the similarity of a constant function is a rigid geometric structure: a deformation whose derivative vanishes is constant. One vivid illustration for this is the foundational work of Kodaira-Spencer on deformations of compact complex manifolds. Let X_0 be a compact complex manifold. A deformation of X_0 in their sense is a proper and submersive holomorphic map $\nu : \mathcal{X} \to (B, 0)$ where $B, \mathcal{X}$ are complex manifolds and 0 is a point in B such that $\nu^{-1}(0) = X_0$. Surprisingly, each deformation of X_0 corresponds to a cohomology class in the first cohomology group $H^1(X_0, \Theta_{X_0})$ of the holomorphic tangent bundle Θ_{X_0}. This cohomology group is actually finite dimensional as a $\mathbb{C}$-vector space. Furthermore, finding a deformation of X_0 is equivalent to finding a convergent power series $\phi(t)$ with coefficients in $H^1(X_0, \Theta_{X_0})$, satisfying the Maurer-Cartan equation, i.e.

$$\overline{\partial}\phi(t) = \frac{1}{2}[\phi(t), \phi(t)].$$

By solving this equation, Kodaira and Spencer proved the existence of a complex manifold (more precisely, an open neighborhood of the origin in $\mathbb{C}^n$ where $n = \dim_{\mathbb{C}} H^1(X_0, \Theta_{X_0})$) which contains all small deformations of X_0, under the assumption that

$$H^2(X_0, \Theta_{X_0}) = 0 \tag{0.0.2}$$

(cf. [17]). The family corresponding to this manifold has the property that any other deformation of X is obtained by the pullback of this family by a holomorphic map. This map is not unique in general but its differential at the reference point is unique. Later, Kuranishi was able to remove the hypothesis (0.0.2) (cf. [19]). However, the price to pay is a loosening of the definition of deformations. A deformation of X_0 now is a proper flat holomorphic map $\nu : \mathcal{X} \to (B, 0)$ between complex spaces with the following cartesian diagram

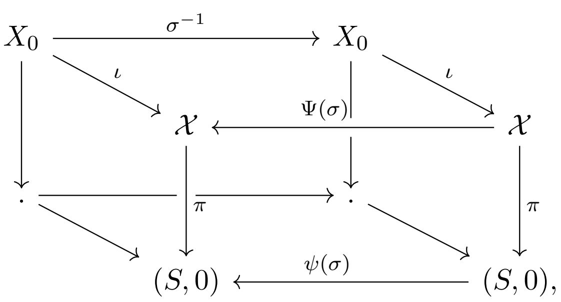
$$\begin{array}{ccc} X_0 & \xrightarrow{\iota} & \mathcal{X} \\ \downarrow & & \downarrow \nu \\ \cdot & \longrightarrow & (B,0). \end{array}$$

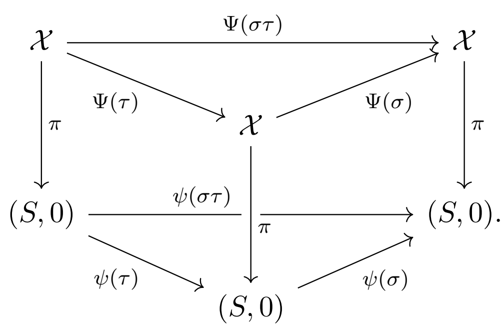
The "local moduli space" obtained by Kuranishi (often called Kuranishi space) is singular in general and actually only well-defined near the reference point. The associated family of compact complex manifolds is called the semi-universal deformation (or Kuranishi family) of X_0. In the algebraic setting, when X_0 is an affine scheme with at most isolated singularities or a complete scheme defined over a field of characteristic zero, the same result is obtained by M. Schlessinger by means of the language of Artinian functors (see [31] or [32]).

The problem is much more complicated when symmetry enters the game: if X_0 is further equipped with an action of some group G, G no longer acts on small deformations of X_0. The most simple example of this phenomenon is when one deforms a hyperelliptic curve of genus at least 3: the hyperelliptic involution ρ does not propagate through deformations. More precisely, let $\gamma : \mathcal{X} \to (B, 0)$ be a versal deformation of X_0 (i.e. a deformation whose Kodaira-Spencer map is surjective). If the action extended, then the image of the Kodaira-spencer map would be fixed by ρ. Therefore, ρ would act trivially on $H^1(X_0, \Theta_{X_0})$ which is obviously not the case. However, there is still some hope that G acts on Kuranishi space: small deformations of X_0 are permuted among them under the G-action. Specifically, let $\pi : \mathcal{X} \to (S, 0)$ be the semi-universal deformation of X_0 where X_0 is equipped with a G-action. Then we consider the following problem.

Problem I. *Does there exist a G-action on S and a G-action on $\mathcal{X}$ extending the initial G-action on X_0 such that π is G-equivariant with respect to these G-actions?*

In other words, we ask whether there exist group homomorphisms $\Psi : G \to \mathrm{Aut}(\mathcal{X})$ and $\psi : G \to \mathrm{Aut}(S)$ such that for all $\sigma, \tau \in G$, we have the following commutative diagrams





As might be expected, if the family $\pi : \mathcal{X} \to S$ is universal i.e. the pullback map is actually unique then the answer to the above problem is undoubtedly yes. This follows from the fact that each time we change the central fiber of the locally universal family by an automorphism of X_0, we obtain another universal family of X_0

$$\begin{array}{ccccc} X_0 & \xrightarrow{\sigma^{-1}} & X_0 & \xrightarrow{\iota} & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \pi \\ \cdot & \xrightarrow{\cong} & \cdot & \longrightarrow & (S,0) \end{array}$$

which is canonically isomorphic to the old one. Unfortunately, hardly does such a family exist due to the existence of non-trivial automorphisms of X_0.

Problem I was first considered in a pioneering work of Pinkham in which he gave an affirmative answer when X_0 is an affine cone with $\mathbb{G}_m$-action (cf. [26]). Later on, Rim obtained a more far-reaching result whose proof is based on the language of categories and rational representations of algebraic groups (cf. [30]).

Theorem I. *Let G be a linearly reductive algebraic group and X_0 an affine scheme with at most isolated singularities or a complete scheme. Then, there exists a G-equivariant semi-universal deformation of X_0, unique up to G-equivariant morphism.*

The main focus of our work is to deal with problems naturally arising around this theorem. The structure of this thesis contains three chapters corresponding to our three papers [7], [8] and [9].

In Chapter 1, we show that in Theorem I, the assumption that G is reductive is really optimal. To put it another way, we provide a projective scheme X_0 on which a non-reductive group G acts algebraically such that the G-action does not extend to its formal semi-universal deformation. Namely, let $\mathbb{F}_2$ be the second Hirzebruch surface

$$\mathbb{F}_2 := \{([x:y:z],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 | yv^2 = zu^2\}$$

and consider its automorphism group

$$G := \mathrm{Aut}(\mathbb{F}_2) \cong (\mathbb{C}^3 \rtimes GL(2,\mathbb{C}))/I$$

where

$$I = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu \in \mathbb{C}, \mu^2 = 1 \right\}.$$

It is easy to see that G is non-reductive. Then we start our story by the following result (cf. Theorem 1.5.1).

Theorem A. *The action of G on $\mathbb{F}_2$ does not extend to the formal semi-universal deformation of $\mathbb{F}_2$.*

Intuitively, when X_0 is a complex compact manifold, an appropriate version of Theorem I should still hold, but this is not contained in the existing literature. Moreover, there is a crucial difference between the algebraic world and the analytic one. In the algebraic setting, the semi-universal deformation of X_0 and the extended G-actions, constructed by Rim in Theorem I, are just formal. Nevertheless, in the analytic setting, its semi-universal deformation is a true deformation (a convergent deformation). So, if X_0 is a projective complex compact manifold, an application of Rim's result gives us a G-equivariant Kuranishi family whose the extended G-actions are only formal, i.e. they are formal power series whose convergence is not guaranteed. Thus, it motivates us to prove, in Chapter 2, the following two results of which the main ingredient of the proofs is a delicate combination of a G-equivariant version of Kuranishi's classical construction of semi-universal deformations of complex compact manifolds (cf. [18]) and representations of reductive complex Lie groups (cf. Corollary 2.4.1 and Theorem 2.5.2, respectively).

Theorem B. *Let X_0 be a complex compact manifold X_0 with a K-action, where K is a compact real Lie group. Then there exists a K-equivariant semi-universal deformation of X_0.*

Theorem C. *Let X_0 be a complex compact manifold with a holomorphic action of a complex reductive Lie group G. Then there exists a local G-equivariant semi-universal deformation of X_0.*

The language of Artinian functors developed by M. Schlessinger (see [31]) allows us to rewrite Theorem I as follows. Let $\mathbf{Art}_k$ denote the category of local artinian k-algebras with residue field k. The functor $F_{X_0}\colon \mathbf{Art}_k \to \mathbf{Sets}$ which associates to each local artinian k-algebra A, the set of flat morphisms of schemes $X \to \mathrm{Spec}(A)$ with an isomorphism $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong X_0$ has a formal semi-universal element which can be further made equivariant if an assumption that a linearly reductive algebraic group G acts algebraically on X_0 is added. Chapter 3 deals with the semi-prorepresentability of the extended formal moduli problem Def_{X_0} of F_{X_0} in the context of derived deformation theories (recall that a formal moduli problem is an ∞-functor from $\mathbf{dgArt}_k \to \mathbf{SEns}$ satisfying certain exactness conditions, where $\mathbf{dgArt}_k$ is the category of differential graded commutative artinian augmented k-algebras and $\mathbf{SEns}$ is the ∞-category of simplicial sets). To be more exact, we first introduce the notion of semi-prorepresentability of formal moduli problems which does not seem to exist in the derived setting. This new notion is the weaker version of prorepresentability and a natural generalization of the notion of semi-universality in Schlessinger's sense (cf. Definition 3.3.2 below). Afterward by means of the well-known equivalence between the ∞-category of formal moduli problems and that of differential graded Lie algebras, a simple criterion for a formal moduli problem to be semi-prorepresentable is provided (cf. Theorem 3.3.2).

Theorem D. *Let F be a formal moduli problem whose associated differential graded Lie algebra $\mathfrak{g}_*$ is cohomologically concentrated in $[0,+\infty)$. Assume further that $H^i(\mathfrak{g}_*)$ is a finite dimensional vector space for each $i \geq 0$. Then F is semi-prorepresentable.*

The functor $\mathrm{Def}_{X_0} : \mathbf{dgArt}_k \to \mathbf{SEns}$, which associates to each differential graded commutative artinian augmented k-algebra A, the nerve of the category of flat morphisms of derived schemes over $\mathrm{Spec}(A)$, whose homotopy fiber at the k-point of $\mathrm{Spec}(A)$ is X_0, is actually a formal moduli problem which extends F_{X_0}, i.e.

$$\pi_0(\mathrm{Def}_{X_0}) = F_{X_0}.$$

It is well-known that the differential graded Lie algebra associated to Def_{X_0} is

$$\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$$

where $\mathbb{T}_{X_0/k}$ is the tangent complex of X_0 and $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ is the derived global section of $\mathbb{T}_{X_0/k}$ (cf. Theorem 3.4.2). When X_0 is an affine scheme with at most isolated singularities or a projective scheme defined over k, the semi-prorepresentability of Def_{X_0} follows from the fact that $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ is cohomologically concentrated in $[0,+\infty)$ and that $H^i(\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k}))$ is a finite dimensional vector space for each $i \geq 0$. In other words, Def_{X_0} has a "derived semi-universal element" whose connected components give back the classical semi-universal one. In the spirit of Problem I, it is natural to wonder whether this derived semi-universal element can be made G-equivariant in some sense, where G is a linearly reductive group acting algebraically on X_0. The answer is the content of the following theorem which is a natural generalization of Theorem I in the derived literature. (cf. Theorem 3.4.6).

Theorem E. *There exists a G-equivariant structure on the derived semi-universal element of Def_{X_0}. Consequently, the classical deformation functor $F_{X_0} = \pi_0(\mathrm{Def}_{X_0})$ of X_0 has a G-equivariant semi-universal element.*

Finally, when X_0 is a complex compact manifold on which a reductive complex Lie group acts holomorphically, we would like to provide a formal version of Theorem C by using the purely algebraic approach that we develop earlier in the same chapter. Let $\mathfrak{Def}_{X_0} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets}$ (resp. $\mathfrak{Def}^G_{X_0} : \mathbf{Art}^G_{\mathbb{C}} \to \mathbf{Sets}$) be the functor which associates to each local artinian $\mathbb{C}$-algebra A, the isomorphism class of (resp. G-equivariant) flat proper morphisms of analytic spaces $X \to \mathrm{Spec}(A)$ with an isomorphism (resp. a G-equivariant isomorphism)

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{C}) \cong X_0.$$

The dgla controlling deformations of X_0 is

$$\mathfrak{g}_* := \Gamma(X_0, \mathcal{A}^{0,0}(\mathcal{T}_{X_0})) \xrightarrow{\bar\partial} \Gamma(X_0, \mathcal{A}^{0,1}(\mathcal{T}_{X_0})) \xrightarrow{\bar\partial} \Gamma(X_0, \mathcal{A}^{0,2}(\mathcal{T}_{X_0})) \xrightarrow{\bar\partial} \cdots$$

with the Lie bracket defined by

$$[\phi d\bar{z}_I, \psi d\bar{z}_J] = [\phi, \psi]' d\bar{z}_I \wedge \bar{z}_J$$

where $\phi, \psi \in \mathcal{A}^{0,0}(\mathcal{T}_{X_0})$ are vector fields on X_0, $[-,-]'$ is the usual Lie bracket of vector fields, $I, J \subset \{1, \dots, n\}$ and $z_1, \dots, z_n$ are local holomorphic coordinates. Therefore, the formal moduli problem which extends $\mathfrak{Def}_{X_0}$ is

$$\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*)$$

where D is the Koszul duality (cf. Proposition 3.2.4 and Theorem 3.2.3). It is obvious that $\mathfrak{g}_*$ is cohomologically concentrated in positive degrees. Besides, $\mathfrak{g}_*$ receives a natural G-action induced from the one on X_0. Then we conclude this final chapter by the following result (cf. Theorem 3.4.9).

Theorem F. *There exists a G-equivariant structure on the semi-prorepresentable object of $\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*)$ with respect to the prescribed G-action on $\mathfrak{g}_*$. Consequently, the classical functor of G-equivariant deformations $\mathfrak{Def}^G_{X_0}$ of X_0 has a formal G-equivariant semi-universal element.*

Chapter 1

A counter-example to the equivariance structure on semi-universal deformation

1.1 Introduction

Let X_0 be an algebraic variety defined over a field k of characteristic zero. Due to Schlessinger's work in [31], the existence of a formal semi-universal deformation (unique up to non-canonical isomorphism), which contains all the information of small deformations of X_0, is assured provided that $H^1(X_0, \mathcal{T}_{X_0})$ and $H^2(X_0, \mathcal{T}_{X_0})$ are finite dimensional vector spaces. These conditions realize for example, if X_0 is a complete scheme over k or an affine scheme with at most isolated singularities (see [32, Corollary 2.4.2]). Now, we equipe X_0 with an action of an algebraic group G defined over k. One question arising naturally is whether there exists a formal semi-universal deformation $\pi : \mathcal{X} \to S$ of X_0, on which we can provide a G-action extending the given one on X. The answer is positive in the case that G satisfies some vanishing conditions on its cohomology groups, i.e. $H^1(G, -) = 0$ and $H^2(G, -) = 0$ for a class of G-modules determined by X_0. In particular, these vanishing conditions hold for linearly reductive groups (see [30] or Theorem I above). However, we do not know if there exists a non-reductive group whose action on X_0 does not extend to the formal semi-universal deformation of X_0. Therefore, we wish to give an example which illustrates this phenomenon. More precisely, we prove that the action of the automorphism group of the second Hirzebruch surface $\mathbb{F}_2$ does not extend to its formal semi-universal deformation.

Our proof goes as follows. First, we find a nice presentation of $G := \mathrm{Aut}(\mathbb{F}_2)$. Then we construct a formal semi-universal deformation $\widehat{\mathcal{X}}$ of $\mathbb{F}_2$. It turns out that G is non-reductive and that the Lie algebra of G is a 7-dimensional vector space. As a matter of fact, we obtain seven vector fields on $\mathbb{F}_2$ with Lie bracket relations induced by those in $\mathrm{Lie}(G)$.

Next, we describe the general form of formal vector fields on $\widehat{\mathcal{X}}$. Finally, we conclude the chapter by means of contradiction. Suppose that the G-action on $\mathbb{F}_2$ does extend to a G-action on $\widehat{\mathcal{X}}$ then we also have seven formal vector fields on $\widehat{\mathcal{X}}$ whose restrictions on the central fiber are nothing but our initial ones on $\mathbb{F}_2$. By manipulating these vector fields with a filtration F given by the vanishing order at 0, we obtain the existence of a 3-dimensional abelian Lie subalgebra in $\mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K)$, where $\mathfrak{sl}_2(K)$ is the special linear group and K is the field of formal Laurent power series $\mathbb{C}[[t, t^{-1}]]$, which is not the case. A remark is in order. Since the semi-universal family of $\mathbb{F}_2$ is in fact not universal, another possible way to obtain a contradiction is to use Wavrik's criterion (see [35, Theorem 4.1]) but the calculations are rather complicated.

1.2 Formal schemes and formal deformations

In this section, by k, we always mean a field of characteristic zero. We begin by recalling the definition of formal schemes. For more details, the readers are referred to [13, Chapter III. 9].

Definition 1.2.1. *Let X be a noetherian scheme and let Y be a closed subscheme defined by a sheaf of ideals $\mathcal{I}$. Then we define the formal completion of X along Y, denoted $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ (sometimes just $\widehat{X}$), to be the following ringed space. We take the topological space Y, and on it the sheaf of rings $\mathcal{O}_{\widehat{X}} = \varprojlim \mathcal{O}_X/\mathcal{I}^n$. Here we consider each $\mathcal{O}_X/\mathcal{I}^n$ as sheaf of rings on Y*

Remark 1.2.1. For each n, let $X_n = (X, \mathcal{O}_X/\mathcal{I}^n)$. Then we obtain a sequence of closed immersions of schemes

$$X_1 \to X_2 \to \cdots \to X_n \to \cdots .$$

This expression is helpful in the sequel.

Definition 1.2.2. *A noetherian formal scheme is a locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ which has a finite open cover $\{\mathfrak{U}_i\}$ such that for each i, the pair $(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}} |_{\mathfrak{U}_i})$ is isomorphic, as a locally ringed space, to the completion of some noetherian scheme X_i along a closed subscheme Y_i. A morphism of noetherian formal schemes is a morphism as locally ringed spaces.*

Example 1.2.1. *If X is any noetherian scheme, and Y is a closed subscheme then the formal completion $\widehat{X}$ of X along Y is a formal scheme.*

Example 1.2.2. *For $X = \mathbb{C}^1 = \mathrm{Spec}(\mathbb{C}[t])$ and $Y = \{0\}$, the formal scheme $\widehat{X}$ is the locally ringed space $(Y, \mathcal{O}_{\widehat{X}})$, where the structure sheaf $\mathcal{O}_{\widehat{X}}$ is $\mathbb{C}[[t]]$. We denote* $\mathrm{Specf}(\mathbb{C}[[t]]) := (Y, \mathcal{O}_{\widehat{X}})$.

Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a noetherian formal scheme. We would like to define *formal vector fields* on $\mathfrak{X}$. Let $\{\mathfrak{U}_i\}$ be a finite open cover of $\mathfrak{X}$ such that for each i, the pair $(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}} |_{\mathfrak{U}_i})$ is the formal completion $(\widehat{X_i}, \mathcal{O}_{\widehat{X}_i})$ of some noetherian scheme X_i along a closed subscheme Y_i. By Remark 1.2.1, for each i we have a sequence of closed immersions of schemes

$$X_{i1} \to X_{i2} \to \cdots \to X_{in} \to \cdots .$$

Definition 1.2.3. *A formal vector field on a noetherian formal scheme $\mathfrak{X}$ is a sequence of vector fields $\{v_{i,n}\}$ such that*

(i) Each $v_{i,n}$ is a usual vector field on the scheme $X_{i,n}$,

(ii) $v_{i,n}$ induces $v_{i,n-1}$ via the natural inclusion $X_{i,n-1} \to X_{i,n}$,

(iii) $v_{i,n} |_{X_{i,n} \cap X_{j,n}} = v_{j,n} |_{X_{j,n} \cap X_{i,n}}$.

Next, we turn to the notion of infinitesimal deformations and that of formal deformations. Let X_0 be an algebraic scheme and let A be an artinian local k-algebra with residue field k. An infinitesimal deformation of X_0 is a deformation of X_0 over the scheme $\mathrm{Spec}(A)$, i.e. a commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\ i\ } & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

where $\pi : \mathcal{X} \to \mathrm{Spec}(A)$ is a flat surjective morphism of schemes.

Now, let A be a complete local noetherian k-algebra with the unique maximal ideal $\mathfrak{m}$ and with residue k.

Definition 1.2.4. *A formal deformation of X_0 over A is a sequence $\{\nu_n\}$ of infinitesimal deformations of X, in which ν_n is represented by a deformation*

$$\begin{array}{ccc} X_0 & \xrightarrow{\ f_n\ } & \mathcal{X}_n \\ \downarrow & & \downarrow \pi_n \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A_n) \end{array}$$

where $A_n = A/\mathfrak{m}^{n+1}$, such that for all $n \geq 1$, ν_n induces ν_{n-1} by pullback under the natural inclusion $\mathrm{Spec}(A_{n-1}) \to \mathrm{Spec}(A_n)$*, i.e. ν_{n-1} is also represented by the deformation*

$$\begin{array}{ccc} X_0 & \xrightarrow{f_{n-1}} & \mathcal{X}_n \times_{\mathrm{Spec}(A_n)} \mathrm{Spec}(A_{n-1}) \\ \downarrow & & \downarrow \pi_{n-1} \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A_{n-1}) \end{array}$$

In the language of formal schemes, we can write $\{\nu_n\}$ as the morphism of formal schemes

$$\widehat{\pi} : \widehat{\mathcal{X}} \to \mathrm{Specf}(A)$$

where

$$\widehat{\mathcal{X}} = (X, \lim_{\leftarrow} \mathcal{O}_{\mathcal{X}_n}) \text{ and } \widehat{\pi} = \lim_{\leftarrow} \pi_n.$$

Here, $\mathcal{O}_{\mathcal{X}_n}$ is the structure sheaf on $\mathcal{X}_n$ and $\mathrm{Specf}(A)$ is the formal scheme obtained by completing $\mathrm{Spec}(A)$ along its closed point, which corresponds to the unique maximal ideal of A. The easiest way to construct formal deformations is to build out of usual ones. This leads to the definition of *formal deformation associated to a given deformation.* Let X_0 be a projective scheme and let ν be a deformation represented by

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \longrightarrow & (S, s) \end{array}$$

where $S = \mathrm{Spec}(B)$ for some k-algebra of finite type B and s is a k-rational point of S.

Definition 1.2.5. *The formal deformation associated to ν is defined to be the sequence of deformations $\{\nu_n\}$ where each ν_n is the pullback of ν under the natural closed embedding*

$$S_n := \mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}) \to S$$

where $\mathfrak{m}_s$ is the unique maximal ideal of the local ring $\mathcal{O}_{S,s}$.

Remark 1.2.2. Note that $\{\nu_n\}$ is formal because of the isomorphism

$$\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1} \cong \widehat{\mathcal{O}}_{S,s}/\widehat{\mathfrak{m}}_s^{n+1}$$

for all n.

To end this section, we introduce a very interesting kind of (formal) deformations, namely, the kind of G-equivariant (formal) ones, which is of central interest of this chapter. Let G be a k-algebraic group acting algebraically on a projective variety X_0 and A an artinian local k-algebra.

Definition 1.2.6. *A G-equivariant infinitesimal deformation of X_0 over* $\mathrm{Spec}(A)$ *is a usual deformation of X_0, i.e. a commutative diagram*

$$\begin{array}{ccc} X_0 & \xrightarrow{\quad i \quad} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

where $\mathcal{X}$ and $\mathrm{Spec}(A)$ *are equipped with G-actions in a way that any map appearing in the above diagram is G-equivariant. In particular, the restriction of the G-action on $\mathcal{X}$ on the central fiber is nothing but the initial G-action on X_0.*

Finally, we give the definition of G-equivariant formal deformations and then we show how to produce formal vector fields from G-equivariant formal deformations.

Definition 1.2.7. *A G-equivariant formal deformation of X_0 over a complete local noetherian k-algebra A with the unique maximal ideal $\mathfrak{m}$ is a formal deformation of X_0, i.e. a sequence $\{\nu_n\}$ of infinitesimal deformations of X_0, in which ν_n is represented by a G-equivariant infinitesimal deformation*

$$\begin{array}{ccc} X_0 & \xrightarrow{\quad f_n \quad} & \mathcal{X}_n \\ \downarrow & & \downarrow \pi_n \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A_n) \end{array}$$

where $A_n = A/\mathfrak{m}^{n+1}$, such that for all $n \geq 1$, the G-equivariant deformation ν_n induces the G-equivariant deformation ν_{n-1} by pullback under the natural inclusion $\mathrm{Spec}(A_{n-1}) \to \mathrm{Spec}(A_n)$.

As before, we can write $\{\nu_n\}$ as the G-equivariant morphism of formal schemes

$$\widehat{\pi} : \widehat{\mathcal{X}} \to \mathrm{Specf}(A)$$

where

$$\widehat{\mathcal{X}} = (X, \lim_{\leftarrow} \mathcal{O}_{\mathcal{X}_n}) \text{ and } \widehat{\pi} = \lim_{\leftarrow} \pi_n.$$

Here, the G-equivariance of $\widehat{\pi}$ means that $\widehat{\pi}$ is an inverse limit of G-equivariant morphisms of schemes π_n. On one hand, on each n^{th}-infinitesimal neighborhood, G-actions on $\mathcal{X}_n$ and on $\mathrm{Spec}(A_n)$ induce vector fields on $\mathcal{X}_n$ and on $\mathrm{Spec}(A_n)$, respectively. They are related by the fact that the differential of π_n always maps the former ones to the latter ones. On the other hand, these induced vector fields on $\mathcal{X}_n$ and on $\mathrm{Spec}(A_n)$ are also induced by those on $\mathcal{X}_{n+1}$ and on $\mathrm{Spec}(A_{n+1})$, respectively, via the natural inclusion $\mathrm{Spec}(A_n) \to \mathrm{Spec}(A_{n+1})$.

Therefore, we obtain *formal vector fields*, induced by the G-actions, on $\widehat{\mathcal{X}}$ and on $\mathrm{Specf}(A)$, respectively.

1.3 The second Hirzebruch surface and its automorphism group

For the rest of the paper, we assume that k is the field of complex numbers $\mathbb{C}$. The general linear group $\mathrm{GL}(2,\mathbb{C})$ has an obvious linear action on $\mathbb{C}^2$. This induces an action on the $\mathbb{C}$-vector space of polynomials in two variables $\mathbb{C}[X,Y]$. Since the subspace of homogeneous polynomials of degree 2, denoted by $\mathbb{C}[X,Y]_2$, is $\mathrm{GL}(2,\mathbb{C})$-invariant then we have a $\mathrm{GL}(2,\mathbb{C})$-action on $\mathbb{C}[X,Y]_2$. More precisely, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ and $f = a_0X^2 + a_1XY + a_2Y^2 \in \mathbb{C}[X,Y]_2$, the action of g on f is given by the linear substitution

$$\begin{pmatrix} X \\ Y \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

i.e.

$$\begin{aligned} g.f &= a_0\,(aX+bY)^2 + a_1\,(aX+bY)\,(cX+dY) + a_2\,(cX+dY)^2 \\ &= \left(a^2a_0 + aca_1 + c^2a_2\right)X^2 + \left(2aba_0 + (ad+bc)a_1 + 2cda_2\right)XY + \left(b^2a_0 + bda_1 + d^2a_2\right)Y^2. \end{aligned}$$

Identifying $\mathbb{C}[X,Y]_2$ with $\mathbb{C}^3$, the corresponding action on $\mathbb{C}^3$ can be written as

$$g.(a_0,a_1,a_2) = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad+bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

This action gives rise to an algebraic group H which is the semi-product of $\mathbb{C}^3$ and $\mathrm{GL}(2,\mathbb{C})$, i.e.

$$H := \mathbb{C}^3 \rtimes \mathrm{GL}(2,\mathbb{C}).$$

This is a non-reductive linear group. Recall that an algebraic group K is reductive if the greatest connected normal subgroup $R_u(K)$ of K is trivial. In our case, $R_u(H) = \mathbb{C}^3$.

Next, we recall the definition of the second Hirzebruch surface. Let $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the projectivization of $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$, where $\mathcal{O}_{\mathbb{P}^1}$ is the structure sheaf of the projective space $\mathbb{P}^1$.

Definition 1.3.1. *The second Hirzebruch surface is defined to be* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$.

Proposition 1.3.1. *The second Hirzebruch surface is isomorphic to the variety*

$$\mathbb{F}_2 := \{([x:y:z],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 | yv^2 = zu^2\}.$$

Proof. Let $\sigma\colon \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$ be the canonical projection of the projectivization $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1})$, let $U = \mathrm{Spec}(\mathbb{C}[v])$ and $U' = \mathrm{Spec}(\mathbb{C}[v'])$ such that $v'v = 1$ on $U \cap U'$. Then $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1})$ has the following presentation

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}) = \sigma^{-1}(U)\cup\sigma^{-1}(U') = (U\times\mathbb{P}^1)\cup(U'\times\mathbb{P}^1).$$

such that on the intersection of the affine open sets $V = \mathrm{Spec}(\mathbb{C}[v,y]) \subset U\times\mathbb{P}^1$ and $V' = \mathrm{Spec}(\mathbb{C}[v',y']) \subset U\times\mathbb{P}^1$, we have

$$\begin{cases} vv' = 1 \\ y' = yv^2 \end{cases}.$$

So, an open covering of $\mathbb{F}_2$ is given by the open embeddings

$$\begin{aligned} \rho_1 : U\times\mathbb{P}^1 &\to \mathbb{F}_2 \\ (v,[x:y]) &\mapsto ([x:y:yv^2],[1:v]) \end{aligned}$$

and

$$\begin{aligned} \rho_2 : U'\times\mathbb{P}^1 &\to \mathbb{F}_2 \\ (v',[x':y']) &\mapsto ([x':y'v'^2:y'],[v':1]), \end{aligned}$$

which glue to give an isomorphism $\rho :\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}) \to \mathbb{F}_2$. □

Now, the algebraic group H acts on the second Hirzebruch surface

$$\mathbb{F}_2 = \{([x:y:z],[u:v]) \in \mathbb{P}^2\times\mathbb{P}^1 | yv^2 = zu^2\}$$

in the following manner: for $p = ([x:y:z],[u:v]) \in \mathbb{F}_2$ and $g = \left((a_0,a_1,a_2)^t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in H$,

$$g.p = \begin{cases} \left(\left[xu^2 + y(a_0v^2+a_1uv+a_2u^2) : y(au+bv)^2 : y(cu+dv)^2\right], [au+bv : cu+dv]\right) & \text{if } u\neq 0 \\ \left(\left[xv^2 + z(a_0v^2+a_1uv+a_2u^2) : z(au+bv)^2 : z(cu+dv)^2\right], [au+bv : cu+dv]\right) & \text{if } v\neq 0 \end{cases}.$$

The following theorem is well-known (see [2, Section 6.1]).

Theorem 1.3.1. *The group of automorphisms of $\mathbb{F}_2$ is exactly the quotient of H by the subgroup I consisting of diagonal matrices of the form $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ where $\mu \in \mathbb{C}$ such that $\mu^2 = 1$.*

1.4 A formal semi-universal deformation of $\mathbb{F}_2$ and formal vector fields on it

1.4.1 Construction of the semi-universal deformation of $\mathbb{F}_2$

We shall follow the construction given in [32, Example 1.2.2.(iii)]. Consider two copies of $\mathbb{C} \times \mathbb{C} \times \mathbb{P}^1$ given by $W := \mathrm{Proj}(\mathbb{C}[t, v, x, y])$ and $W' := \mathrm{Proj}(\mathbb{C}[t', v', x', y'])$ (note that these two rings are graded with respect to x, y and x', y', respectively). Consider the affine subsets $\mathrm{Spec}(\mathbb{C}[t, v, y]) \subset W$, $\mathrm{Spec}(\mathbb{C}[t', v', y']) \subset W'$ and then glue them along the open subsets

$$\mathrm{Spec}(\mathbb{C}[t, v, v^{-1}, y]) \subset \mathrm{Spec}(\mathbb{C}[t, v, y])$$

and

$$\mathrm{Spec}(\mathbb{C}[t', v', v'^{-1}, y']) \subset \mathrm{Spec}(\mathbb{C}[t', v', y'])$$

by the rules

$$\begin{cases} vv' = 1 \\ x' = x \\ y' = yv^2 - tvx \\ t' = t. \end{cases} \tag{1.4.1}$$

This gives a gluing of W and W' along

$$\mathrm{Proj}(\mathbb{C}[t, v, v^{-1}, x, y]) \text{ and } \mathrm{Proj}(\mathbb{C}[t', v', v'^{-1}, x', y']).$$

We denote the resulting scheme by $\mathcal{W}$. In other words, if we let $(t, v, [x : y])$ and $(t', v', [x' : y'])$ be the coordinates on $W = \mathbb{C} \times \mathbb{C} \times \mathbb{P}^1$ and on $W' = \mathbb{C} \times \mathbb{C} \times \mathbb{P}^1$, respectively. Then $\mathcal{W}$ is obtained by glue W and W' according to the rules (1.4.1). Now, let $\pi : \mathcal{W} \to \mathbb{C}$ be the morphism induced by the projections.

Theorem 1.4.1. *The family $\pi : \mathcal{W} \to \mathbb{C} = \mathrm{Spec}(\mathbb{C}[t])$ is a semi-universal deformation of $\mathbb{F}_2$. Moreover,*

$$\pi^{-1}(t) = \begin{cases} \mathbb{F}_2 \text{ if } t = 0 \\ \mathbb{P}^1 \times \mathbb{P}^1 \text{ otherwise.} \end{cases}$$

Proof. The map π is obviously surjective by construction. Since π is locally a projection, it is a flat morphism. Moreover, by Proposition 1.3.1, $\mathcal{W}_0 = \pi^{-1}(0) = \mathbb{F}_2$. Then $\pi : \mathcal{W} \to \mathbb{C}$ is a deformation of $\mathbb{F}_2$. Next, let $\mathcal{W}^* = \pi^{-1}(\mathbb{C}^*)$ and $\pi^* : \mathcal{W}^* \to \mathbb{C}^*$ is the restriction of π on $\mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[t, t^{-1}])$. We shall prove that $\mathcal{W}^*$ is in fact isomorphic to $\mathbb{C}^* \times \mathbb{P}^1 \times \mathbb{P}^1$.

Indeed, consider the following open embeddings

$$\begin{aligned} \phi : \mathbb{C}^* \times \mathbb{C} \times \mathbb{P}^1 &\to \mathbb{C}^* \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (t, v, [x : y]) &\mapsto (t, [1 : v], [ty : vy - tx]) \end{aligned}$$

and

$$\begin{aligned} \phi' : \mathbb{C}^* \times \mathbb{C} \times \mathbb{P}^1 &\to \mathbb{C}^* \times \mathbb{P}^1 \times \mathbb{P}^1 \\ (t', v', [x' : y']) &\mapsto (t', [v' : 1], [t'v'y' + t'^2x' : y']). \end{aligned}$$

By the gluing condition (1.4.1), we have that

$$\begin{aligned} (t', [v' : 1], [t'v'y' + t'^2x' : y']) &= (t', [v' : 1], [t'v'(yv^2 - tvx) + t'^2x' : yv^2 - tvx]) \\ &= (t', [v' : 1], [t'yv : yv^2 - tvx]) \\ &= (t, [1 : v], [ty : yv - tx]). \end{aligned}$$

Hence, the above two morphisms glue to give an isomorphism

$$\begin{array}{ccc} \mathcal{W}^* & \xrightarrow{\cong} & \mathbb{C}^* \times \mathbb{P}^1 \times \mathbb{P}^1 \\ & {\scriptstyle \pi^*} \searrow & \downarrow {\scriptstyle \mathrm{pr}_1} \\ & & \mathbb{C}^*, \end{array}$$

which means precisely that $\pi^* : \mathcal{W}^* \to \mathbb{C}^*$ is the trivial family whose fibers are all isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. In particular, for $t \in \mathbb{C}^*$, $\pi^{-1}(t) = (\pi^*)^{-1}(t) = \mathbb{P}_1 \times \mathbb{P}_1$.

It remains to prove that the family $\pi : \mathcal{W} \to \mathbb{C}$ is actually semi-universal. One way to see it is to compute the Kodaira-Spencer map $\mathcal{K}_{\pi,0}$ of π at 0. This map is uniquely determined by the element $\mathcal{K}_{\pi,0}(\frac{d}{dt})$ in $H^1(\mathbb{F}_2, \mathcal{T}_{\mathbb{F}_2})$. By definition, $\mathcal{K}_{\pi,0}(\frac{d}{dt})$ represents the first order deformation of $\mathbb{F}_2$, obtained by gluing $W_0 := \mathrm{Proj}(\mathbb{C}[\epsilon, v, x, y])$ and $W_0' := \mathrm{Proj}(\mathbb{C}[\epsilon, v', x', y'])$ along $\mathrm{Proj}(\mathbb{C}[\epsilon, v, v^{-1}, x, y])$ and $\mathrm{Proj}(\mathbb{C}[\epsilon, v', v'^{-1}, x', y'])$ by the rules

$$\begin{cases} vv' = 1 \\ y' = yv^2 - \epsilon v \end{cases},$$

where $\mathbb{C}[\epsilon]$ is the ring of complex dual numbers. Hence, $\mathcal{K}_{\pi,0}(\frac{d}{dt}) \in H^1(\mathcal{U}, \mathcal{T}_{\mathbb{F}_2})$ is the 1-cocycle which corresponds to the vector field $\{-v\frac{\partial}{\partial y}\}$ on $W_0 \cap W_0'$, where $\mathcal{U}$ is the covering $\{W_0, W_0'\}$. By [32, Example B.11(iii)], we see that $\{-v\frac{\partial}{\partial y}\}$ is nonzero and $\dim_{\mathbb{C}} H^1(\mathbb{F}_2, \mathcal{T}_{\mathbb{F}_2}) = 1$. Thus, the Kodaira-Spencer map is an isomorphism and so $\pi : \mathcal{W} \to \mathbb{C}$ is semi-universal. $\square$

Another useful presentation of $\mathcal{W}$ is given as follows.

Proposition 1.4.1. *The scheme $\mathcal{W}$ is isomorphic to the surface*

$$\mathcal{X} := \{([x:y:z],[u:v],t) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{C} \mid yv^2 - zu^2 - txuv = 0\}.$$

Proof. We have an open covering of $\mathcal{X}$ given by the open embeddings

$$\begin{aligned} \rho_1 : \mathbb{C} \times \mathbb{C} \times \mathbb{P}^1 &\to \mathcal{X} \\ (t, v, [x:y]) &\mapsto ([x:y:yv^2 - tv], [1:v], t) \end{aligned}$$

and

$$\begin{aligned} \rho_2 : \mathbb{C} \times \mathbb{C} \times \mathbb{P}^1 &\to \mathcal{X} \\ (t', v', [x':y']) &\mapsto ([x':y'v'^2 + t'v':y'], [v':1], t) \end{aligned}$$

which glue to give an isomorphism $\mathcal{W} \overset{\cong}{\to} \mathcal{X}$. □

Remark 1.4.1. By Proposition 1.3.1 and by Proposition 1.4.1, from now on, we use interchangeably between $\mathbb{F}_2, \mathcal{X}$ and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}), \mathcal{W}$, respectively.

1.4.2 Formal vector fields on the formal semi-universal deformation of $\mathbb{F}_2$

The formal deformation associated to $\mathcal{W}$, $\widehat{\pi} : \widehat{\mathcal{W}} \to \mathrm{Specf}(\mathbb{C}[[t]])$ is a formal semi-universal deformation of $\mathbb{F}_2$ (here $\mathbb{C}[[t]]$ is the ring of formal power series in the variable t). We will give explicit descriptions of formal vector fields on $\widehat{\mathcal{W}}$. Consider the covering $\{W, W'\}$ where $W := \mathrm{Proj}(\mathbb{C}[t,v,x,y])$ and $W' := \mathrm{Proj}(\mathbb{C}[t',v',x',y'])$, as before. A formal vector field on W is of the form

$$g_1(v,t)\frac{\partial}{\partial v} + (\alpha_1(v,t)y^2 + \beta_1(v,t)y + \gamma_1(v,t))\frac{\partial}{\partial y} + k_1(t)\frac{\partial}{\partial t} \tag{1.4.2}$$

where $g_1, \alpha_1, \beta_1, \gamma_1, k_1$ are formal power series in the variable t. Likewise, a formal vector field on W' is of the form

$$g_2(v',t')\frac{\partial}{\partial v'} + (\alpha_2(v',t')y'^2 + \beta_2(v',t')y' + \gamma_2(v',t'))\frac{\partial}{\partial y'} + k_2(t')\frac{\partial}{\partial t'} \tag{1.4.3}$$

where $g_2, \alpha_2, \beta_2, \gamma_2, k_2$ are formal power series in the variable t'. Therefore, a vector field on $\mathcal{W}$ which is of the form (1.4.2) on W and of the form (1.4.3) on W' must satisfy the relation

$$\begin{aligned} g_1(v,t)\tfrac{\partial}{\partial v} + (\alpha_1(v,t)y^2 + \beta_1(v,t)y + \gamma_1(v,t))\tfrac{\partial}{\partial y} + k_1(t)\tfrac{\partial}{\partial t} \\ = g_2(v',t')\tfrac{\partial}{\partial v'} + (\alpha_2(v',t')y'^2 + \beta_2(v',t')y' + \gamma_2(v',t'))\tfrac{\partial}{\partial y'} + k_2(t')\tfrac{\partial}{\partial t'} \end{aligned} \tag{1.4.4}$$

on the overlapping open set $W \cap W'$.

Lemma 1.4.1. *A global formal vector field on* $\widehat{\mathcal{W}}$ *whose restriction on* W *is*

$$g_1(v,t)\frac{\partial}{\partial v} + (\alpha_1(v,t)y^2 + \beta_1(v,t)y + \gamma_1(v,t))\frac{\partial}{\partial y} + k_1(t)\frac{\partial}{\partial t}$$

must satisfy the following

$$\begin{cases} g_1(v,t) = A(t)v^2 + B(t)v + C(t) \\ \alpha_1(v,t) = a(t)v^2 + b(t)v + c(t) \\ \beta_1(v,t) = -2[a(t)t + A(t)]v + e(t) \\ \gamma_1(v,t) = t^2 a(t) + tA(t) \end{cases} \tag{1.4.5}$$

where A, B, C, a, b, c, e, k_1 *are formal power series in the variable* t *with a relation*

$$b(t)t^2 + e(t)t + B(t)t - k_1(t) = 0. \tag{1.4.6}$$

Proof. By (1.4.1), we have

$$\begin{cases} y = v'^2 y' + tv' \\ v = \frac{1}{v'} \\ t = t' \\ \partial_v = -v'^2\partial_{v'} + (2y'v' + t)\partial_{y'} \\ \partial_y = \frac{1}{v'^2}\partial_{y'} \\ \partial_{t'} = -\frac{1}{v'}\partial_{y'} + \partial_{t'}. \end{cases} \tag{1.4.7}$$

Substituting (1.4.7) into the left hand side of (1.4.4) and equalizing, we get that

$$\begin{cases} g_2(v',t') = -v'^2 g_1(\frac{1}{v'},t') \\ \alpha_2(v',t') = v'^2\alpha_1(\frac{1}{v'},t') \\ \beta_2(v',t') = 2t'v'\alpha_1(\frac{1}{v'},t') + \beta_1(\frac{1}{v'},t') + 2v'g_1(\frac{1}{v'},t') \\ \gamma_2(v',t') = t'^2\alpha_1(\frac{1}{v'},t') + \frac{t'}{v'}\beta_1(\frac{1}{v'},t') + \frac{1}{v'^2}\gamma_1(v',t') + t'g_1(\frac{1}{v'},t') - \frac{k_1(t')}{v'}, \end{cases} \tag{1.4.8}$$

which implies that

$$\begin{cases} g_1(v,t) = A(t)v^2 + B(t)v + C(t) \\ \alpha_1(v,t) = a(t)v^2 + b(t)v + c(t) \\ \beta_1(v,t) = -2[a(t)t + A(t)]v + e(t) \\ \gamma_1(v,t) = t^2 a(t) + tA(t), \end{cases}$$

where A, B, C, a, b, c, e are formal power series in the variable t with a relation

$$b(t)t^2 + e(t)t + B(t)t - k_1(t) = 0.$$

This constraint comes from the coefficient of $\frac{1}{v'}$ in the fourth equation in (1.4.8). $\square$

Remark 1.4.2. If $t = 0$ then (1.4.5) becomes

$$\begin{cases} g_1(v) = Av^2 + Bv + C \\ \alpha_1(v) = av^2 + bv + c \\ \beta_1(v) = -2Av + e \\ \gamma_1(v,t) = 0 \end{cases}$$

which agrees with Kodaira's calculations of vector fields on $\mathcal{W}_0 = \mathbb{F}_2$ (see [17, Page 75]). In particular, we have seven linearly independent vector fields on $\mathbb{F}_2$. If t is non-zero and fixed then we have six linearly independent vector fields on the fiber $\mathcal{W}_t$, which is due to the existence of the relation (1.4.6).

1.5 The non-existence of G-equivariant structure on the formal semi-universal deformation

The Lie algebra of $G := \mathrm{Aut}(\mathbb{F}_2)$ is $\mathbb{C}^3 \times M(2,\mathbb{C})$, which is evidently 7-dimensional . A $\mathbb{C}$-basis of Lie(G) is given by the following elements

$$\begin{cases} e_1 = (1,0,0) \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ e_2 = (0,0,1) \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ e_3 = (0,0,0) \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ e_4 = (0,1,0) \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ e_5 = (0,0,0) \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_6 = (0,0,0) \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ e_7 = (0,0,0) \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

Then the G-action gives us 7 vector fields $E'_1, \dots, E'_7$ on $\mathbb{F}_2$ with the relations

$$\begin{cases} [E'_1, E'_2] = 0 \\ [E'_1, E'_3] = -2E'_4 \\ [E'_1, E'_4] = 0 \\ [E'_1, E'_5] = 0 \\ [E'_1, E'_6] = -2E'_1 \\ [E'_1, E'_7] = 0, \end{cases} \quad \begin{cases} [E'_2, E'_3] = 0 \\ [E'_2, E'_4] = 0 \\ [E'_2, E'_5] = -2E'_2 \\ [E'_2, E'_6] = 0 \\ [E'_2, E'_7] = -2E'_4, \end{cases} \quad \begin{cases} [E'_3, E'_4] = E'_2 \\ [E'_3, E'_5] = -E'_3 \\ [E'_3, E'_6] = E'_3 \\ [E'_3, E'_7] = E'_5 - E'_6, \end{cases}$$

$$\begin{cases} [E'_4, E'_5] = -E'_4 \\ [E'_4, E'_6] = -E'_4 \\ [E'_4, E'_7] = -E'_1, \end{cases} \quad \begin{cases} [E'_5, E'_6] = 0 \\ [E'_5, E'_7] = -E'_7, \end{cases} \quad [E'_6, E'_7] = E'_7.$$

Now, we are in the position to prove the main result of this chapter. Suppose that the G-action extends on $\widehat{\mathcal{W}}$. This implies that we also have 7 formal vector fields $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ on $\widehat{\mathcal{W}}$ with the following Lie bracket constraints

$$\begin{cases} [E_1, E_2] = 0 \\ [E_1, E_3] = -2E_4 \\ [E_1, E_4] = 0 \\ [E_1, E_5] = 0 \\ [E_1, E_6] = -2E_1 \\ [E_1, E_7] = 0, \end{cases} \quad \begin{cases} [E_2, E_3] = 0 \\ [E_2, E_4] = 0 \\ [E_2, E_5] = -2E_2 \\ [E_2, E_6] = 0 \\ [E_2, E_7] = -2E_4, \end{cases} \quad \begin{cases} [E_3, E_4] = E_2 \\ [E_3, E_5] = -E_3 \\ [E_3, E_6] = E_3 \\ [E_3, E_7] = E_5 - E_6, \end{cases}$$

$$\begin{cases} [E_4, E_5] = -E_4 \\ [E_4, E_6] = -E_4 \\ [E_4, E_7] = -E_1, \end{cases} \quad \begin{cases} [E_5, E_6] = 0 \\ [E_5, E_7] = -E_7, \end{cases} \quad [E_6, E_7] = E_7.$$

These vector fields form a Lie sub-algebra, denoted by $\mathfrak{g}$, of the Lie algebra of formal vector fields on $\widehat{\mathcal{W}}$. Of course, the restriction of E_i on the central fiber is nothing but E'_i $(i = 1, \dots, 7)$.

From the previous section, we can assume that our seven vector fields are of the form

$$E_i = g_i(v,t)\frac{\partial}{\partial v} + (\alpha_i(v,t)y^2 + \beta_i(v,t)y + \gamma_i(v,t))\frac{\partial}{\partial y} + k_i(t)\frac{\partial}{\partial t},$$

(cf. Lemma 1.4.1) where A, B, C, a, b, c, e are formal power series in t $(i = 1, \dots, 7)$.

Theorem 1.5.1. *The action of G on $\mathbb{F}_2$ does not extend to the formal semi-universal* *deformation $\widehat{\mathcal{W}}$, where G is the automorphism group of $\mathbb{F}_2$.*

Proof. We denote by $\mathfrak{v}$ the Lie algebra of formal vector fields in one variable t. Let $\delta : \mathfrak{g} \to \mathfrak{v}$ be the map which sends

$$g_i(v,t)\frac{\partial}{\partial v} + (\alpha_i(v,t)y^2 + \beta_i(v,t)y + \gamma_i(v,t))\frac{\partial}{\partial y} + k_i(t)\frac{\partial}{\partial t}$$

to

$$k_i(t)\frac{\partial}{\partial t},$$

for $i = 1, \dots, 7$. Since, the first two components $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y}$ contribute nothing to the component $\frac{\partial}{\partial t}$ in the Lie bracket then δ is a well-defined Lie homomorphism. Set $F_i := \delta(E_i) = k_i(t)\frac{\partial}{\partial t}$ $(i = 1, \dots, 7)$. Note that the seven formal vector fields F_i $(i = 1, \dots, 7)$ are nothing but those induced by the G-action on the base Specf($\mathbb{C}[[t]]$) (cf. the last paragraph of Section 1). Observe also that $\mathfrak{v}$ can be equipped with a filtration F given by the vanishing order at 0 and we have two well-known facts

$$[F^p\mathfrak{v}, F^p\mathfrak{v}] \subset F^{2p}\mathfrak{v}, \text{ and } [F^p\mathfrak{v}, F^q\mathfrak{v}] \subset F^{p+q-1}\mathfrak{v},$$

for $p, q \geq 1$. Furthermore, the vanishing order of all k_i at 0 is at least 1. Let $k_i(t) = \sum_{j=1}^{\infty} a_j^i t^j$ $(i = 1, 2, 4, 5)$. Using the first fact and the Lie relations induced by δ:

$$\begin{cases} [F_1, F_6] = -2F_1 \\ [F_2, F_5] = -2F_2 \\ [F_1, F_3] = -2F_4, \end{cases}$$

we obtain $a_1^1 = a_1^2 = a_1^4 = 0$. Suppose that $k_4(t)$ is not identically zero, then there exists $j^* \geq 2$ such that $a_{j^*}^4$ is nonzero. By computing explicitly the Lie relation $[F_4, F_5] = -F_4$ in terms of power series in t and then by equalizing coefficients, we get that

$$a_j^4[(j-1)a_1^5 - 1] = 0,$$

for all $j \geq 2$. Thus, $a_1^5 = \frac{1}{j^*-1}$, which is clearly nonzero. A similar computation for the relation $[F_1, F_5] = 0$ gives

$$(j-1)a_j^1 a_1^5 = 0,$$

for all $j \geq 2$. Hence, all $a_j^1 = 0$ so that $k_1(t) = 0$. By the relation $[F_1, F_3] = -2F_4$, we deduce that $k_4(t) = 0$, a contradiction. Therefore, $k_4(t) = 0$. From the relations $[F_3, F_4] = F_2$ and $[F_4, F_7] = -F_1$, we obtain that $k_2(t) = 0$ and $k_1(t) = 0$. As a sequence, E_1, E_2, and E_4 do not have the component $\frac{\partial}{\partial t}$.

In addition, by the proof of Theorem 1.4.1, as a scheme over $\mathbb{C}$,

$$\mathcal{W}^* \cong \mathbb{C}^* \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Then,

$$\mathcal{W}^* \cong \mathbb{P}^1_L \times \mathbb{P}^1_L,$$

as a scheme over L, where $L := \mathbb{C}[t, t^{-1}]$ and $\mathbb{P}^1_L$ is the 1-projective space over L. Therefore, the generic fiber $\widehat{\mathcal{W}}^*$ of $\widehat{\mathcal{W}}$ is isomorphic to $\mathcal{W}^* \times_{\mathrm{Spec}(\mathbb{C}[t,t^{-1}])} \mathrm{Spec}(\mathbb{C}[[t, t^{-1}]]) = \mathbb{P}^1_K \times \mathbb{P}^1_K$, as a scheme over K, where K is the field of Laurent formal power series $\mathbb{C}[[t, t^{-1}]]$. Now, by restricting on the generic fiber of $\widehat{\mathcal{W}}$, we obtain that E_1, E_2, and E_4 are formal vector fields on $\widehat{\mathcal{W}}^*$, considered as a $\mathbb{C}$-scheme. However, by the first paragraph, we have proved that there is no component $\frac{\partial}{\partial t}$ in the expression of E_i $(i = 1, 2, 4)$. So, if we think of E_1, E_2 and E_4 as vector fields with coefficients in K, then they are definitely vector fields on $\widehat{\mathcal{W}}^*$, regarded as a scheme over K. Note that the Lie algebra of vector fields on $\mathbb{P}^1_K \times \mathbb{P}^1_K$ is isomorphic to $\mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K)$, where $\mathfrak{sl}_2(K)$ is the special linear group. This means that there exists a 3-dimensional abelian Lie subalgebra of $\mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K)$. The image of that sub-algebra under one of the two canonical projections of the product $\mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K)$ provides a 2-dimensional abelian Lie sub-algebra in $\mathfrak{sl}_2(K)$. This is a contradiction since $\mathrm{rank}(\mathfrak{sl}_2(K))$ is only 1. $\square$

Remark 1.5.1. A naturally posed question is if the G-action extends to $\mathcal{W}_n$ over the base $\mathrm{Spec}(\mathbb{C}[t]/(t^{n+1}))$ for small value n. Although the above proof does not give any clue to reply to this question, the answer is yes for $n = 1$. More general, if G is an algebraic group acting algebraically on a projective variety X_0 and $\pi : \mathcal{X} \to S$ is the semi-universal deformation of X then the G-action on X_0 certainly extends up to the first infinitesimal deformation $\mathcal{X}_1$ over S_1. This follows easily from the semi-universality of the family $\pi : \mathcal{X} \to S$. Unfortunately, our example turns out to be the worst case. More precisely, we can even show that the G-action on $\mathbb{F}_2$ can not extend to the second infinitesimal $\mathcal{W}_2$ over $\mathrm{Spec}(\mathbb{C}[t]/(t^3))$ by extending E'_i $(i = 1, \ldots, 7)$ together with their Lie bracket relations, order by order with respect to t (cf. Theorem 1.6.1). This is the content of the next sub-section.

1.6 Another proof of Theorem 1.5.1

In this section, we denote by $\widehat{\mathcal{X}}$ the formal deformation associated to the semi-universal deformation of $\mathbb{F}_2$ given by the explicit equation in Proposition 1.4.1. On the intersection of two standard open sets $U_x = \{x = 1\}$ and $U = \{u = 1\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. the proof of

Proposition 1.3.1), the action of $G := \mathrm{Aut}(\mathbb{F}_2)$ on $\mathbb{F}_2$ given by

$$g.(v,[1:y]) = \left(\frac{c+dv}{a+bv}, \left[1 : \frac{y(a+bv)^2}{1+y(a_0v^2+a_1v+a_2)}\right]\right)$$

where $g = \left((a_0,a_1,a_2)^t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in G$ and $(v,[1:y]) \in \mathbb{F}_2$ under the identification given in Proposition 1.3.1. The Lie algebra of G is 7-dimensional then we obtain 7 vector fields on $\mathbb{F}_2$

$$\begin{cases} E_1' = -v^2y^2\partial_y \\ E_2' = -y^2\partial_y \\ E_3' = \partial_v \\ E_4' = -vy^2\partial_y \\ E_5' = 2y\partial_y - v\partial_v \\ E_6' = v\partial_v \\ E_7' = 2yv\partial_y - v^2\partial_v \end{cases}$$

on $\mathbb{F}_2$ with the relations

$$\begin{cases} [E_1',E_2'] = 0 \\ [E_1',E_3'] = -2E_4' \\ [E_1',E_4'] = 0 \\ [E_1',E_5'] = 0 \\ [E_1',E_6'] = -2E_1' \\ [E_1',E_7'] = 0 \end{cases}, \begin{cases} [E_2',E_3'] = 0 \\ [E_2',E_4'] = 0 \\ [E_2',E_5'] = -2E_2' \\ [E_2',E_6'] = 0 \\ [E_2',E_7'] = -2E_4' \end{cases}, \begin{cases} [E_3',E_4'] = E_2' \\ [E_3',E_5'] = -E_3' \\ [E_3',E_6'] = E_3 \\ [E_3',E_7'] = E_5' - E_6' \end{cases},$$

$$\begin{cases} [E_4',E_5'] = -E_4' \\ [E_4',E_6'] = -E_4' \\ [E_4',E_7'] = -E_1' \end{cases}, \begin{cases} [E_5',E_6'] = 0 \\ [E_5',E_7'] = -E_7' \end{cases}, [E_6',E_7'] = E_7'$$

These vector fields correspond to a basis of $\mathbb{C}^3 \times M(2,\mathbb{C})$ given at the beginning of the previous sub-section. Suppose that the G-action extends on $\hat{\mathcal{X}}$. This implies that we also have 7 formal vector fields $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ on $\hat{\mathcal{X}}$ with relations

$$\begin{cases} [E_1, E_2] = 0 \\ [E_1, E_3] = -2E_4 \\ [E_1, E_4] = 0 \\ [E_1, E_5] = 0 \\ [E_1, E_6] = -2E_1 \\ [E_1, E_7] = 0 \end{cases}, \begin{cases} [E_2, E_3] = 0 \\ [E_2, E_4] = 0 \\ [E_2, E_5] = -2E_2 \\ [E_2, E_6] = 0 \\ [E_2, E_7] = -2E_4 \end{cases}, \begin{cases} [E_3, E_4] = E_2 \\ [E_3, E_5] = -E_3 \\ [E_3, E_6] = E_3 \\ [E_3, E_7] = E_5 - E_6 \end{cases},$$

$$\begin{cases} [E_4, E_5] = -E_4 \\ [E_4, E_6] = -E_4 \\ [E_4, E_7] = -E_1 \end{cases}, \begin{cases} [E_5, E_6] = 0 \\ [E_5, E_7] = -E_7 \end{cases}, [E_6, E_7] = E_7$$

and the restriction of these vectors on the central fiber are nothing but $E_1', E_2', E_3', E_4', E_5', E_6', E_7'$.

By the discussion of the previous sub-section, we can assume that our seven vector fields are of the form (up to the first order with respect to t)

$$\begin{cases} E_1 = \left(-v^2y^2 + p_1(v,y)t\right)\partial_y + q_1(v)t\partial_y + k_1t\partial_t \\ E_2 = \left(-y^2 + p_2(v,y)t\right)\partial_y + q_2(v)t\partial_y + k_2t\partial_t \\ E_3 = p_3(v,y)t\partial_y + (1 + q_3(v)t)\,\partial_y + k_3t\partial_t \\ E_4 = \left(-vy^2 + p_4(v,y)t\right)\partial_y + q_4(v)t\partial_y + k_4t\partial_t \\ E_5 = (2y + p_5(v,y)t)\,\partial_y + (-v + q_5(v)t)\,\partial_y + k_5t\partial_t \\ E_6 = p_6(v,y)t\partial_y + (v + q_6(v)t)\,\partial_y + k_6t\partial_t \\ E_7 = (2yv + p_7(v,y)t)\,\partial_y + \left(-v^2 + q_7(v)t\right)\partial_y + k_7t\partial_t \end{cases}$$

where p_i, q_i are polynomial whose degree with respect to each variable does not exceed 2.

1.6.1 Form of E_i's on $\hat{\mathcal{X}}$

Since E_i's are formal vectors fields on $\hat{\mathcal{X}}$ then by Lemma 1.4.1,

$$E_i = g_i(v,t)\frac{\partial}{\partial v} + (\alpha_i(v,t)y^2 + \beta_i(v,t)y + \gamma_i(v,t))\frac{\partial}{\partial y} + k_i(t)\frac{\partial}{\partial t}$$

where

$$\begin{cases} g_i(v,t) = A_i(t)v^2 + B_i(t)v + C_i(t) \\ \alpha_i(v,t) = a_i(t)v^2 + b_i(t)v + c_i(t) \\ \beta_i(v,t) = -2(a_i(t)t + A_i(t))v + e_i(t) \\ \gamma_i(v,t) = t^2a_i(t) + tA_i(t) \end{cases}$$

and $A_i, B_i, C_i, a_i, b_i, c_i, e_i$ are formal power series in t with a relation

$$b_i(t)t^2 + e_i(t)t + B_i(t) - k_i(t) = 0$$

for $i = 1, \dots, 7$. Since the restriction of E_i on the central fiber are E_i', respectively, we have

- $E_1 \mid_{\mathcal{X}_0} = E_1'$:

$$\begin{cases} (a_1(0)v^2 + b_1(0)v + c_1(0))y^2 + (-2(a_1(0)0 + A_1(0))v + e_1(0))\, y = -v^2y^2 \\ A_1(0)v^2 + B_1(0)v + C_1(0) = 0, \end{cases}$$

  which implies that

$$\begin{cases} a_1(0) = -1 \\ b_1(0) = 0 \\ c_1(0) = 0 \\ A_1(0) = 0 \\ B_1(0) = 0 \\ C_1(0) = 0 \\ e_1(0) = 0. \end{cases}$$

  So, up to the first order, we have that

$$\begin{cases} a_1(t) = -1 + a_1 t \\ b_1(0) = b_1 t \\ c_1(t) = c_1 t \\ A_1(t) = A_1 t \\ B_1(t) = B_1 t \\ C_1(t) = C_1 t \\ e_1(t) = e_1 t \end{cases}$$

  where $a_1, b_1, c_1, A_1, B_1, C_1, e_1$ are constants. Moreover,

$$\begin{aligned} \beta_1(v,t) &= -2(a_1(t)t + A_1(t))v + e_1(t) \\ &= -2((-1 + a_1 t)vt + A_1 t) + e_1 t \\ &= 2tv(1 - A_1) + e_1 t (\mod t^2). \end{aligned}$$

Hence,

$$E_1 = \{-v^2y^2 + t[(a_1v^2 + b_1v + c_1)y^2 + (2v(1 - A_1) + e_1)y]\}\partial_y + (A_1v^2 + B_1v^2 + C_1)t\partial_v.$$

This implies that (up to the first order)

$$\begin{cases} p_1(v, y) = (a_1v^2 + b_1v + c_1)y^2 + (2v(1 - A_1) + e_1)y \\ q_1(v) = A_1v^2 + B_1v + C_1. \end{cases}$$

- $E_2 \mid_{\mathcal{X}_0} = E_2'$:

$$\begin{cases} (a_2(0)v^2 + b_2(0)v + c_2(0))y^2 + (-2(a_2(0)0 + A_2(0))v + e_2(0))\, y = -y^2 \\ A_2(0)v^2 + B_2(0)v + C_2(0) = 0, \end{cases}$$

which deduces that

$$\begin{cases} a_2(0) = 0 \\ b_1(0) = 0 \\ c_2(0) = -1 \\ A_2(0) = 0 \\ B_2(0) = 0 \\ C_2(0) = 0 \\ e_2(0) = 0. \end{cases}$$

This gives, up to the first order,

$$\begin{cases} a_2(t) = a_2t \\ b_2(t) = b_2t \\ c_2(t) = -1 + c_2t \\ A_2(t) = A_2t \\ B_2(t) = B_2t \\ C_2(t) = C_2t \\ e_2(t) = e_2t \end{cases}$$

where $a_2, b_2, c_2, A_2, B_2, C_2, e_2$ are constants. Furthermore,

$$\begin{aligned} \beta_2(v, t &= -2(a_2(t)t + A_2(t))v + e_2(t) \\ &= -2(a_2t.t + A_2t)v + e_2t \\ &= -2tvA_2 + e_2t(\mod t^2). \end{aligned}$$

As a sequence,

$$E_2 = \{-y^2 + t[(a_2v^2 + b_2v + c_2)y^2 + (-2vA_2 + e_2)y]\}\partial_y + (A_2v^2 + B_2v^2 + C_2)t\partial_v.$$

Thus, (up to first order)

$$\begin{cases} p_2(v,y) = (a_2v^2 + b_2v + c_2)y^2 + (-2vA_2 + e_2)y \\ q_2(v) = A_2v^2 + B_2v + C_2. \end{cases}$$

1.6.2 Extension of E_i's on $\hat{\mathcal{X}}$ up to the first order

We would like to see if there exists such E_i ($i = 1, \dots, 7$) by solving the system of equations given by the Lie relations between E_i's order by order with respect to t. An important remark is in order.

Remark 1.6.1. Note that there is no component $c\frac{\partial}{\partial t}$ in all E_i (c is a complex number). So, the Lie bracket between E_i's are well-defined when we take modulo t^n for any $n \in \mathbb{N}$.

Now, we wish to find q_1, q_2 and q_4 by explicit computations based on the Lie relations:

-
$$\begin{aligned}[E_1, E_6] &= \left[2v^2y^2 + t\left(-v^2y^2\partial_y p_6 + 2v^2yp_6 + 2vy^2q_6 - v\partial_v p_1 + k_1p_6 - k_6p_1\right)\right]\partial_y \\ &\quad + t\left(q_1 - v\partial_v q_1 + k_1q_6 - k_6q_1\right)\partial_v \\ &= -2E_1\end{aligned}$$

  so that $k_1 = 0$.

-
$$\begin{aligned}[E_3, E_4] &= \left[-y^2 + t\left(-2vyp_3 + vy^2\partial_y p_3 + \partial_v p_4 - y^2q_3 + k_3p_4 - k_4p_3\right)\right]\partial_y \\ &\quad + t\left(\partial_v q_4 + k_3q_4 - k_4q_3\right)\partial_v \\ &= E_2.\end{aligned}$$

  provides $k_2 = 0$ and
$$\partial_v q_4 + k_3a_4 - k_4q_3 = q_2. \tag{1.6.1}$$

-
$$\begin{aligned}[E_3, E_5] &= t\left(2p_3 - 2y\partial_y p_3 + \partial_v p_5 + v\partial_v p_3 + k_3p_5 - k_5p_3\right)\partial_y \\ &\quad + \left[-1 + t\left(\partial_v q_5 - q_3 + v\partial_v q_3 + k_3q_5 - k_5q_3\right)\right]\partial_v \\ &= -E_3\end{aligned}$$

  gives $k_3 = 0$.

- $$\begin{aligned}[E_1,E_3] &= \left[2vy^2 + t\left(-v^2y^2\partial_y p_3 + 2v^2yp_3 + 2vy^2q_3 - \partial_v p_1 + k_1p_3 - k_3p_1\right)\right]\partial_y \\ &\quad + t\left(-\partial_v q_1 + k_1q_3 - k_3q_1\right)\partial_v \\ &= -2E_4\end{aligned}$$

  which implies $k_4 = 0$ and
  $$-\partial_v q_1 = -2q_4. \tag{1.6.2}$$

- $$\begin{aligned}[E_5,E_7] &= \big[-2vy + t\big(-2y\partial_y p_7 + 2vp_5 - 2vy\partial_y p_5 - 2p_7 + 2yq_5 - v\partial_v p_7 + v^2\partial_v p_5 \\ &\quad + k_5p_7 - k_7p_5\big)\big]\partial_y + \left[v^2 + t\left(-2vq_5 - v\partial_v q_7 + v^2\partial_v q_5 + q_7 + k_5q_7 - k_7q_5\right)\right]\partial_v \\ &= -E_7\end{aligned}$$

  and thus $k_7 = 0$.

- $$\begin{aligned}[E_1,E_2] &= t\left(-v^2y^2\partial_y p_2 - 2yp_1 + y^2\partial_y p_1 + 2v^2yp_2 + 2vy^2q_2\right)\partial_y \\ &= 0.\end{aligned}$$

  So, we have
  $$-v^2y^2\partial_y p_2 - 2yp_1 + y^2\partial_y p_1 + 2v^2yp_2 + 2vy^2q_2 = 0 \tag{1.6.3}$$

- $$\begin{aligned}[E_1,E_7] &= t\left(-v^2y^2\partial_y p_7 + 2vp_1 - 2vy\partial_y p_1 + 2v^2yp_7 + 2yq_1 + v^2\partial_v p_1 + 2vy^2q_7\right)\partial_y \\ &\quad + t(-2vq_1 + v^2\partial_v q_1 + k_1q_7 - k_7q_1)\partial_v \\ &= 0.\end{aligned}$$

  Hence,
  $$-2vq_1 + v^2\partial_v q_1 = 0. \tag{1.6.4}$$

Combining (1.6.4), (1.6.2) and (1.6.1), we have

$$\begin{cases} q_1 = \mu v^2 \\ q_4 = \mu v \\ q_2 = \mu. \end{cases}$$

for some $\mu \in \mathbb{C}$. By the general form of formal vector fields on $\mathcal{X}$, we must have $\mu = A_1$. By (1.6.3), we have

$$
\begin{aligned}
0 &= -v^2y^2\partial_y p_2 - 2yp_1 + y^2\partial_y p_1 + 2v^2yp_2 + 2vy^2q_2 \\
&= v^2y(2p_2 - y\partial_y p_2) - y(2p_1 - y\partial_y p_1) + 2vy^2A_1(\text{ since } q_2 = A_1) \\
&= v^2y(-2vA_2 + e_2)y - y(2v(1-A_1) + e_1)y + 2vy^2A_1 \\
&= y^2\left(v^2(-2vA_2 + e_2) - (2v(1-A_1) + e_1) + 2A_1v\right) \\
&= y^2\left(-2v^3A_2 + e_2v^2 - 2v(1-2A_1) - e_1\right)
\end{aligned}
$$

which implies that

$$
\begin{cases}
A_2 = 0 \\
e_1 = e_2 = 0 \\
A_1 = \frac{1}{2}
\end{cases}.
$$

Hence,

$$
\begin{cases}
q_1 = \frac{1}{2}v^2 \\
q_2 = \frac{1}{2} \\
q_4 = \frac{1}{2}v.
\end{cases}
$$

Thus, we have

$$
\begin{cases}
E_1 = \left(-v^2y^2 + p_1(v,y)t\right)\partial_y + \frac{1}{2}tv^2\partial_v := E_1^{(1)} \\
E_2 = \left(-y^2 + p_2(v,y)t\right)\partial_y + \frac{1}{2}t\partial_v := E_2^{(1)}
\end{cases}
$$

up to the first order.

Theorem 1.6.1. *The action of G on $\mathbb{F}_2$ extends to the formally semi-universal deformation $\hat{\mathcal{X}}$ only up to order* 1.

Proof. By assumption, the vector fields $E_1^{(1)}, E_2^{(1)}$ (of the first order) can be extended to the vector fields $E_1^{(2)}, E_2^{(2)}$ of the second order. In other words, there exists $m_i(v,y), n_i(v), l_i$ $(i = 1,2)$ such that

$$
\begin{cases}
E_1^{(2)} = \left[-v^2y^2 + p_1(v,y)t + m_1(v,y)t^2\right]\partial_y + \left[\frac{1}{2}tv^2 + n_1(v)t^2\right]\partial_y + l_1t^2\partial_t \\
E_2^{(2)} = \left[-y^2 + p_2(v,y)t + m_2(v,y)t^2\right]\partial_y + \left[\frac{1}{2}t + n_2(v)t^2\right]\partial_y + l_2t^2\partial_t
\end{cases}
$$

and the relation

$$
[E_1^{(2)}, E_2^{(2)}] = 0(\mod t^3)
$$

holds true where m_i, n_i are polynomial in variables v, y whose degrees with respect to v and y do not exceeds 2 and l_i's are complex numbers. However, by direct inspection we have that the coefficient of the ∂_v-component of $[E_1^{(2)}, E_2^{(2)}]$ is

$$\left[-v^2y^2 + p_1(v,y)t + m_1(v,y)t^2\right]\partial_y\left[\frac{1}{2}t + n_1(v)t^2\right] - \left[-y^2 + p_2(v,y)t + m_2(v,y)t^2\right]\times$$
$$\partial_y\left[\frac{1}{2}tv^2 + n_1(v)t^2\right] + \left[\frac{1}{2}tv^2 + n_1(v)t^2\right]\partial_v\left[\frac{1}{2}t + n_1(v)t^2\right] - \left[\frac{1}{2}t + n_1(v)t^2\right]\times$$
$$\partial_v\left[\frac{1}{2}tv^2 + n_1(v)t^2\right] + l_1t^2\partial_t\left[\frac{1}{2}t + n_1(v)t^2\right] - l_2t^2\partial_t\left[\frac{1}{2}tv^2 + n_1(v)t^2\right]$$

which after simplification is

$$-\frac{1}{2}t^2v + \frac{l_1}{2}t^2 - \frac{l_2}{2}t^2v^2(\mod t^3).$$

This quantity is never zero ($\mod t^3$) for any choice of l_1 and l_2. Thus, the Lie relation $[E_1^{(1)}, E_2^{(1)}] = 0(\mod t^2)$ is not preserved when we extend them to the second order which is a contradiction. This finishes the proof. □

Chapter 2

Equivariant Kuranishi family of complex compact manifolds

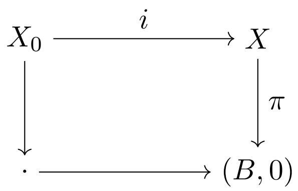
2.1 Introduction

In this chapter, we would like to reproduce Theorem I when X_0 is a complex compact manifold on which a compact Lie group G acts holomorphically and then try to address the case that G is a complex reductive Lie group (see Corollary 2.4.1 and Theorem 2.5.2 below). A remark should be in order. The main different point here is that in the algebraic setting, the semi-universal deformation of X_0 and the extended G-actions, constructed by Rim, are just formal. However, in the analytic setting, its semi-universal deformation (often called Kuranishi family) is a true deformation (a convergent deformation). So, an application of Rim's result gives us a G-equivariant Kuranishi family whose extended G-actions are only formal, i.e. they are formal power series whose convergence is not guaranteed. Actually, this way of using Rim's theorem keeps being repeated several times for example in the proof of Theorem 4.20 in [12] and in the proof of Proposition 7.1 in [24], where the convergence is supposedly needed to carry out. Moreover, an extension of the G-action on the Kuranishi space is immediate if the Kuranishi family is locally universal. This follows from the fact that each time we change the central fiber of the locally universal family by a biholomorphism of X_0, we obtain another locally universal family of X_0 which is canonically isomorphic to the old one. However, in the proof of Lemma 3.4 in [6], the author produced a G-action on the base by claiming that there exists a local universal deformation, which is not true in general even for the type of complex compact manifolds considered therein. Thus, a very natural wish is to have a convergent G-extension.

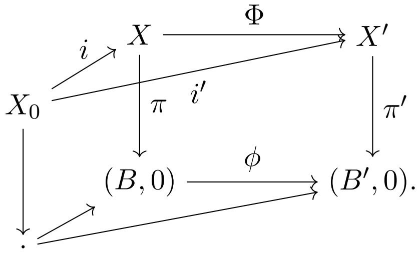
Let us now outline the organization of this chapter. First, we give a general picture of deformations of complex compact manifolds in §2.2. The most important result on the existence of semi-universal deformation (Kuranishi family) is also included. Next, we attack the problem by giving a useful existence criterion in §2.3, which turns out to be deduced from an elementary lemma on complex structures of real vector spaces. In §2.4, we treat the case that G is compact, in advance. The key point here is that in place of imposing an arbitrary Hermitian metric on the holomorphic tangent bundle, we can impose a G-invariant one for the sake of the compactness of G. In fact, this idea is already contained in Catanese's lecture note (see [5, Lecture III, §7]). However, the author uses it to treat only the case that the actions are required to be trivial on the base. If we take the set of fixed points by the G-action in the base constructed in our case then the restriction of our G-equivariant family on this set is nothing but Catanese's family. Afterward, in §2.5, we deal with the complex reductive case by means of complexification of compact groups. Finally, an explicit example of equivariant Kuranishi family of complex compact manifolds is given in §2.6.

2.2 Deformations of complex compact manifolds

We first recall some basic definitions in deformation theory of compact complex manifolds. Let $\mathfrak{B}$ be the category of germs of pointed complex space $(B, 0)$ (a complex space with a reference point) whose associated reduced complex space is a point and let X_0 be a complex compact manifold. An infinitesimal deformation of X_0 is a deformation of X_0 over a germ of complex space $(B, 0) \in \mathfrak{B}$, i.e. a commutative diagram



where $\pi : X \to (B, 0)$ is a flat proper morphism of complex spaces. For simplicity, we denote such a deformation by $\pi \colon X \to (B, 0)$ (or sometimes just X/B). If $\pi \colon X \to (B, 0)$ and $\pi' \colon X' \to (B', 0)$ are two infinitesimal deformations of X_0, a morphism of infinitesimal deformations is a pair (Φ, ϕ) of two morphisms of complex spaces $\Phi : X \to X'$ and $\phi : (B, 0) \to (B', 0)$ such that the following diagram commutes



Kuranishi proves the existence of a semi-universal deformation $\pi \colon X \to (S, 0)$, called Kuranishi family, which contains all the information of small deformations of X_0 (cf. [18]

or [19]). Semi-universality here means that any other deformation $\rho\colon Y \to (T,0)$ of X_0 is defined by the pullback of the Kuranishi family under a holomorphic map from $(T,0)$ to $(S,0)$, whose differential at the reference point is unique.

Next, let us take a moment to recall the definition of group actions on complex spaces. For the sake of completeness, we recall first that a mapping α from a real analytic (resp. complex) manifold W to a Fréchet space F over $\mathbb{C}$ is called *real analytic* (reps. *holomorphic*) if for each point $w_0 \in W$ there exists an open coordinate neighborhood N_{w_0} and a real analytic (resp. holomorphic) coordinate system $t_1, \ldots, t_n$ in N_{w_0} such that $t_i(w_0) = 0$ and for all $w \in N_{w_0}$, we have that

$$\alpha(w) = \sum a_{i_1,\ldots,i_n} t_1^{i_1}(w) \ldots t_n^{i_n}(w)$$

where $a_{i_1,\ldots,i_n} \in F$ and the convergence is absolute with respect to any continuous seminorm on F. Furthermore, by a C^p-map, we insinuate a p-times continuously differentiable function. Let G be a real (resp. complex) Lie group and X a complex space. A G-action on X is given by a group homomorphism $\Phi:\ G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of biholomorphisms of X.

Definition 2.2.1. *The G-action determined by Φ is said to be real analytic (resp. holomorphic) if for each open relatively compact $U \Subset X$ and for each open $V \subset X$, the following conditions are satisfied*

(i) $W := W_{\overline{U},V} := \{g \in G \mid g \cdot \overline{U} \subset V\}$ *is open in G,*

(ii) the map

$$\begin{aligned} * : W &\to \mathcal{O}(U) \\ g &\mapsto f \circ g\,|_U \end{aligned}$$

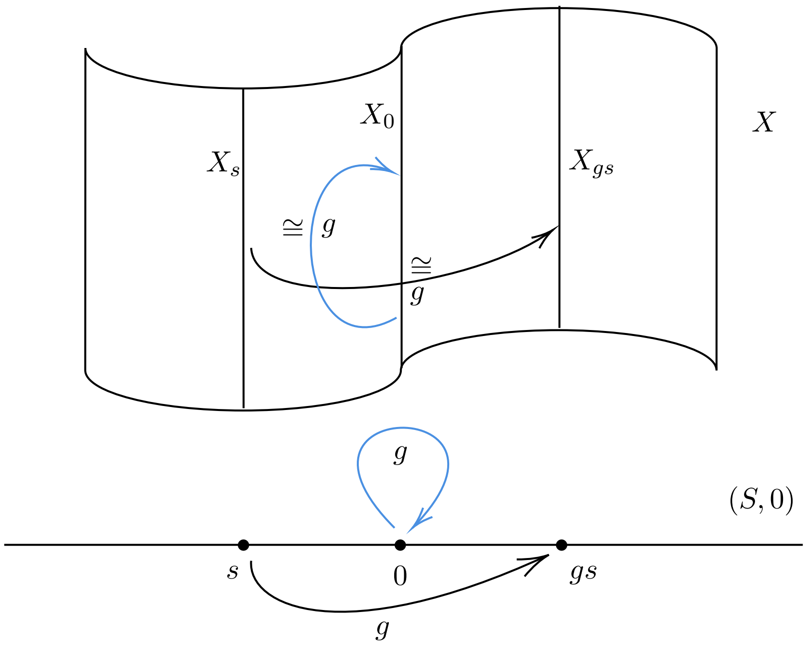
is real analytic (resp. holomorphic) for all $f \in \mathcal{O}(V)$,

where $\overline{U}$ is the closure of U and $\mathcal{O}(P)$ is the set of holomorphic functions on P for any open subset P of X ($\mathcal{O}(P)$ is equipped with the canonical Fréchet topology).

To end this section, we introduce a very interesting kind of deformations-the kind of G-equivariant ones, which is of central interest of this chapter. As before, let X_0 be a complex compact manifold equipped with a real analytic (resp. holomorphic) G-action.

Definition 2.2.2. *A real analytic (resp. holomorphic) G-equivariant deformation of X_0 is a usual deformation of X_0 $\pi\colon X \to B$ equipped with a real analytic (resp. holomorphic) G-action on X extending the given (resp. holomorphic) G-action on X_0 and a real analytic (resp. holomorphic) G-action on B in a way that π is a G-equivariant map with respect to these actions. We call these extended actions a real analytic (resp. holomorphic) G-equivariant structure on $\pi\colon X \to B$.*

Therefore, we can rephrase our objective as finding a real analytic (resp. holomorphic) G-equivariant semi-universal deformation of a given compact complex manifold with a real analytic (resp. holomorphic) G-action. Intuitively, the expected extended G-action on the Kuranishi space permutes the nearby complex structures and keeps the central one untouched.



Remark 2.2.1. For simplicity, by G-actions (resp. G-equivariant deformations), we really mean real analytic G-actions (resp. real analytic G-equivariant deformations).

2.3 A sufficient condition for the existence of equivariant structure

In this section, we give a criterion for a complex compact manifold X_0 with a G-action to have a G-equivariant semi-universal deformation. From now on, by complex compact manifold, we really mean a complex compact connected manifold. First, we recall a technical result concerning the holomorphicity of real analytic functions defined on complex spaces (cf. [18, Proposition 2.1]).

Proposition 2.3.1. *If V is a complex space and v is a point of V, there exists an integer α satisfying the following condition: If $f : V \to V'$ is a C^α-map, where V' is another complex space, such that f is holomorphic at each non-singular point of V then there is an open neighborhood $\overline{V}$ of v in V such that the restriction of f on $\overline{V}$ is holomorphic.*

Denote by $\mathrm{Diff}(\underline{X_0})$ the group of diffeomorphisms of $\underline{X_0}$ where $\underline{X_0}$ is the underlying differentiable manifold of X_0. For S a complex space, a map $\gamma : S \to \mathrm{Diff}(\underline{X_0})$ is said to be of class C^k when the map

$$\begin{aligned}\Gamma : \underline{X_0} \times S &\to \underline{X_0}\\ (p,s) &\mapsto \gamma(s)(p)\end{aligned}$$

is of class C^k. If this is indeed the case, then for each $s_0 \in S$ the map

$$\begin{aligned}\Gamma_{s_0} : \underline{X_0} \times S &\to \underline{X_0}\\ (p,s) &\mapsto \gamma(s) \circ (\gamma(s_0))^{-1}(p)\end{aligned}$$

is a C^k-family of deformations of the identity map of $\underline{X_0}$ with a parameter in (S, s_0). In particular, for each $p \in \underline{X_0}$, we obtain a C^k-map

$$\begin{aligned}\Gamma_{s_0,p} : S &\to \underline{X_0}\\ s &\mapsto \gamma(s) \circ (\gamma(s_0))^{-1}(p).\end{aligned}$$

Therefore, if we suppose further that s_0 is a non-singular point then each $L \in T^{\mathrm{Zar}}_{s_0} S$ will give rise to a vector $d(\Gamma_{s_0,p})_{s_0}(L)$, in $T_p\underline{X_0}$, where $d(\Gamma_{s_0,p})_{s_0}$ is the differential of $\Gamma_{s_0,p}$ at s_0. Thus, the map

$$\begin{aligned}\underline{X_0} &\to T\underline{X_0}\\ p &\mapsto d(\Gamma_{s_0,p})_{s_0}(L)\end{aligned}$$

defines a C^k-vector field on $\underline{X_0}$, which we shall denote by $L\sharp^{s_0}\gamma$.

Finally, before stating the main result, given a complex compact manifold X_0, let us bring back a celebrated characterization of its deformations and in particular of its semi-universal deformation (see [18, Theorem 8.1]).

Theorem 2.3.1. *A deformation of X_0 is entirely encoded by a real analytic map $\phi : S \to A^{0,1}(\Theta)$ which varies holomorphically in S such that*

(i) $\phi(0) = 0$*,*

(ii) $\overline{\partial}\phi(s) - \frac{1}{2}[\phi(s), \phi(s)] = 0$ *for all* $s \in S$*,*

where $A^{0,1}(\Theta)$ is the space of $(0,1)$-forms with values in the holomorphic tangent bundle Θ of X_0 and S is a complex space with a reference point 0. Moreover, this deformation is semi-universal if and only if

(iii) The Kodaira-Spencer map induced by ϕ is an isomorphism,

(iv) We can find an open neighborhood S' of 0 in S such that the following conditions hold true: for any complex space B and for any real analytic map $\psi : B \to A^{0,1}(\Theta)$, which varies holomorphically in B, such that $\psi(b_1) = \phi(s_1)$ for a point $(b_1, s_1) \in B \times S'$,

we can find a neighborhood B' of b_1, a holomorphic map $\tau: (B', b_1) \to (S', s_1)$ and a C^α-map $\gamma: B' \to \mathrm{Diff}(\underline{X_0})$ such that

(a) *$\phi(\tau(b)) = \psi(b) \circ \gamma(b)$ for all $b \in B'$. Here, $\psi(b) \circ \gamma(b)$ is the complex structure induced by the complex structure $\psi(b)$ and the diffeomorphism $\gamma(b)$,*

(b) *For each regular point $b \in B'$ and for all $L \in T_b^{0,1}B \subset T_b^{Zar}B = T_b^{1,0}B \oplus T_b^{0,1}B$, we have that $L\sharp^b\gamma^{-1} + \phi(\tau(b)) \circ \overline{\overline{L}\sharp^b(\gamma^{-1})} = 0$ where α is the integer in Proposition 2.3.1 for $(\mathbb{C}^{\dim_\mathbb{C} X_0} \times B, 0 \times b_1)$ and γ^{-1} is the map $B' \to \mathrm{Diff}(\underline{X_0})$ which to $b \in B'$, associates $(\gamma(b))^{-1}$.*

Now, coming back to our case where the group action joins the game, we claim the following.

Theorem 2.3.2. *If the map ϕ can also be made G-equivariant with respect to some G-action on S and the G-action on $A^{0,1}(\Theta)$, induced by the one on X_0, then a G-equivariant semi-universal deformation of X_0 exists.*

In order to prove this, let us introduce a lemma on complex structures of real vector spaces. Let V be a real vector space of even dimension imposed with three different complex structures J, J_m, J_n and $V^\mathbb{C}$ be its complexification then we have three complex vector spaces $(V, J), (V, J_m), (V, J_n)$ and decompositions

$$V^\mathbb{C} = V_J^{1,0} \oplus V_J^{0,1}, V^\mathbb{C} = V_{J_m}^{1,0} \oplus V_{J_m}^{0,1}, \text{ and } V^\mathbb{C} = V_{J_n}^{1,0} \oplus V_{J_n}^{0,1}$$

where $V_\cdot^{1,0}$ and $V_\cdot^{0,1}$ are eigenspaces attached to the eigenvalues i and $-i$, respectively. Let $\pi^{1,0} : V^\mathbb{C} \to V_J^{1,0}$ and $\pi^{0,1} : V^\mathbb{C} \to V_J^{0,1}$ be the canonical projections.

Now, suppose that the restrictions of $\pi^{0,1}$ on $V_{J_m}^{0,1}$ and on $V_{J_n}^{0,1}$ are isomorphisms. Define $m, n : V_J^{0,1} \to V_J^{1,0}$ by $m = \pi^{1,0} \circ (\pi^{0,1} |_{V_{J_m}^{0,1}})^{-1}$ and $n = \pi^{1,0} \circ (\pi^{0,1} |_{V_{J_n}^{0,1}})^{-1}$. It is well-known that

$$V_{J_m}^{0,1} = \left\{ u + m(u) \mid u \in V_J^{0,1} \right\} \text{ and } V_{J_n}^{0,1} = \left\{ u + n(u) \mid u \in V_J^{0,1} \right\}.$$

Lemma 2.3.1. *Let φ: $V \to V$ be an $\mathbb{R}$-linear map such that its complexification $\varphi^\mathbb{C}$ is a $\mathbb{C}$-linear map from (V, J) to (V, J). Then φ is $\mathbb{C}$-linear as a map from (V, J_m) to (V, J_n) if $\varphi^\mathbb{C} \circ m = n \circ \varphi^\mathbb{C}$.*

Proof. We claim that $\varphi^\mathbb{C}(V_{J_m}^{0,1}) \subseteq V_{J_n}^{0,1}$. Indeed, let $v \in V_{J_m}^{0,1}$ then $v = u + m(u)$ for some $u \in V_J^{0,1}$. So,

$$\begin{aligned} \varphi^\mathbb{C}(v) &= \varphi^\mathbb{C}(u + m(u)) \\ &= \varphi^\mathbb{C}(u) + \varphi^\mathbb{C} \circ m(u) \\ &= \varphi^\mathbb{C}(u) + n \circ \varphi^\mathbb{C}(u). \end{aligned}$$

Moreover, since $\varphi^{\mathbb{C}}$ is a $\mathbb{C}$-linear map from (V, J) to (V, J) then

$$\begin{aligned} J\varphi^{\mathbb{C}}(u) &= \varphi^{\mathbb{C}}J(u) \\ &= \varphi^{\mathbb{C}}(-iu) \text{ since } u \in V_J^{0,1} \\ &= -i\varphi^{\mathbb{C}}(u), \end{aligned}$$

which implies that $\varphi^{\mathbb{C}}(u) \in V_J^{0,1}$. Hence, $\varphi^{\mathbb{C}}(u) + n \circ \varphi(u^{\mathbb{C}}) \in V_{J_n}^{0,1}$ then so is $\varphi^{\mathbb{C}}(v)$, which proves the claim.

Now, let $v \in V_{J_m}^{0,1}$, then

$$\begin{aligned} J_n\varphi^{\mathbb{C}}(v) &= -i\varphi^{\mathbb{C}}(v) \text{ by the claim,} \\ &= \varphi^{\mathbb{C}}(-iv) \\ &= \varphi^{\mathbb{C}}J_m(v). \end{aligned}$$

Making use of the linear complex conjugation, we also get that

$$J_n\varphi^{\mathbb{C}}(v) = \varphi^{\mathbb{C}}J_m(v)$$

for all $v \in V_{J_m}^{1,0}$. This ends the proof. □

Finally, it is the time for us to prove Theorem 2.3.2.

Proof of Theorem 2.3.2. First of all, by the discussion at the very beginning of this section, we have a semi-universal deformation $\pi : X \to S$ of X_0, associated to ϕ. Let $\underline{X_0}$ be the underlying differentiable manifold of X_0. By [5, Theorem 4.5] after shrinking S if necessary, there exists a real analytic diffeomorphism $\gamma : \underline{X_0} \times S \to X$ with $\pi \circ \gamma$ being the projection on the second factor of $\underline{X_0} \times S$, and such that γ is holomorphic in the second set of variables. Thus, for a point $(x, s) \in \underline{X_0} \times S$, we have a decomposition of the tangent space

$$T_x X_0 \oplus T_s^{\mathrm{Zar}} S \cong T_{\gamma(x,s)}^{\mathrm{Zar}} X.$$

We claim that $\pi : X \to S$ carries a G-equivariant structure. Indeed, for $g \in G$ and $(x, s) \in \underline{X_0} \times S$, define

$$g.(x, s) = (g.x, g.s)$$

in which we think of g as just a diffeomorphism of $\underline{X_0}$. This gives clearly an action of G on $\underline{X_0} \times S$. We shall prove that in fact if we think of X as $\underline{X_0} \times S$ with the complex structure $\phi(-)$ then G acts on X by biholomorphisms. This is equivalent to showing that the differential of g at the point (x, s)

$$dg_{(x,s)} : T_{(x,s)}^{\mathrm{Zar}} X = T_x\underline{X_0} \oplus T_s^{\mathrm{Zar}} \to T_{g.(x,s)}^{\mathrm{Zar}} X = T_{g.x}\underline{X_0} \oplus T_{gs}^{\mathrm{Zar}}$$

is $\mathbb{C}$-linear with respect to the complex structure induced by ϕ on the tangent space $T^{\mathrm{Zar}}_{(x,s)}X$. Since $dg_{(x,s)} = (dg_x, dg_s)$ is a diagonal map and g acts holomorphically on S. Then it is sufficient to check that

$$dg_x : \ (T_x\underline{X_0}, J_{(x,s)}) \to (T_{gx}\underline{X_0}, J_{(gx,gs)})$$

is $\mathbb{C}$-complex linear where $J_{(x,s)}$ and $J_{(gx,gs)}$ are complex structures induced by maps $\phi(s)_x : T^{0,1}_x\underline{X_0} \to T^{1,0}_x\underline{X_0}$ and $\phi(gs)_{gx} : T^{0,1}_{gx}\underline{X_0} \to T^{1,0}_{gx}\underline{X_0}$, respectively. On the other hand, as ϕ is G-equivariant then we have

$$g\phi(s) = \phi(gs)$$

for any $s \in S$. This is equivalent to

$$dg\phi(s)dg^{-1} = \phi(gs),$$

by definition of the action of a diffeomorphism g on a complex structure $\phi(s)$. Thus, for each $x \in \underline{X_0}$,

$$dg_x\phi(s)_x = \phi(gs)_{gx}dg_x.$$

Making use of Lemma 2.3.1 for $m = \phi(s)$, $n = \phi(gs)$ and $\varphi = dg_x$, we deduce that dg_x is $\mathbb{C}$-complex linear so that g is in fact holomorphic. Thus, we have just extended the G-action on the central fiber X_0 to a G-action the total space X. This action together with the given G-action on S makes π G-equivariant, which completes the proof. □

2.4 The case that G is a compact Lie group

We treat compact group actions first. Let X_0 be an n-dimensional complex compact manifold equipped with a real analytic K-action, where K is a compact real Lie group. The main result of this section is the following.

Theorem 2.4.1. *There exists a complex space $(S, 0)$ and a real analytic map $\phi : (S, 0) \to A^{0,1}(\Theta)$ which varies holomorphically in S such that the conditions $(i), (ii), (iii)$ and (iv), listed in Theorem 2.3.1, are fulfilled. Furthermore, ϕ is K-equivariant with respect to some K-action on S and the K-action on $A^{0,1}(\Theta)$, induced by the one on X_0.*

Corollary 2.4.1. *Let X_0 be a complex compact manifold X_0 with a K-action, where K is a compact real Lie group. Then there exists a K-equivariant semi-universal deformation of X_0.*

Proof. It follows immediately from Theorem 2.4.1 above and Theorem 2.3.2. □

In order to prove Theorem 2.4.1, we shall follow Kuranishi's method in [18] with some appropriate modification. First of all, note that we have a natural linear K-action on

$A^{0,1}(\Theta)$ and then on $H^1(X_0, \Theta)$. Moreover, since K is compact, instead of imposing an arbitrary hermitian metric on Θ as Kuranishi did, we can impose a K-invariant Hermitian metric $\langle \cdot, \cdot \rangle$ on Θ by means of Weyl's trick. Therefore, we have a K-invariant metric on $A^{0,1}(\Theta)$. As usual, we find the formal adjoint $\overline{\partial}^*$ of $\overline{\partial}$. Since K acts on X_0 by biholomorphisms then the operator $\overline{\partial}$ is K-equivariant. By the adjoint property together with the fact that the imposed metric is K-invariant, we also have that $\overline{\partial}^*$ is K-equivariant. Hence, so is the Laplacian $\square := \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$. In addition, it is well-known that $\square$ is an elliptic operator of second order. As a matter of fact, Hodge theory provides us a famous orthogonal decomposition.

$$A^{0,1}(\Theta) = \mathcal{H}^{0,1} \bigoplus \square A^{0,1}(\Theta) \tag{2.4.1}$$

and two linear operators:

(a) The Green operator $G : A^{0,1}(\Theta) \to \square A^{0,1}(\Theta)$,

(b) The harmonic projection operator $H : A^{0,1}(\Theta) \to \mathcal{H}^{0,1}$,

where $\mathcal{H}^{0,1}$ is the vector space of all harmonic vector $(0,1)$-form on X_0 (this space can also be canonically identified with $H^1(X_0, \Theta)$), such that for all $v \in A^{0,1}(\Theta)$, we have

$$v = Hv + \square Gv. \tag{2.4.2}$$

Lemma 2.4.1. *The linear operators G and H are K-equivariant.*

Proof. For any $v \in \square A^{0,1}(\Theta)$ and $g \in K$, gv is also in $\square A^{0,1}(\Theta)$ for the sake of K-invariance of $\square A^{0,1}(\Theta)$. Thus, by (2.4.2) we have that

$$v = \square Gv \text{ and } gv = \square Ggv.$$

So, the K-equivariance of $\square$ gives us

$$\square (gGv) = g\square (Gv) = gv.$$

Hence,

$$\square (Ggv - gGv) = 0$$

so that $Ggv - gGv \in \mathcal{H}^{0,1}$. On the other hand, $gGv \in \square A^{0,1}(\Theta)$, and so is $Ggv - gGv$. Consequently,

$$Ggv - gGv \in \mathcal{H}^{0,1} \cap \square A^{0,1}(\Theta) = \{0\}$$

so that

$$Ggv = gGv$$

for any $v \in \square A^{0,1}(\Theta)$ and $g \in G$.

Now for any $v \in A^{0,1}(\Theta)$ and $g \in K$, we have that

$$
\begin{aligned}
gGv &= gG(Hv + \Box Gv) \\
&= gGHv + gG\left(\Box Gv\right) \\
&= gG\left(\Box Gv\right) \text{ since } GH = 0, \\
&= G\left(g\Box Gv\right) \text{ by the above case,} \\
&= Gg\left(v - Hv\right) \text{ by the decomposition (2.4.2),} \\
&= Ggv - GgHv \\
&= Ggv \text{ since } \mathcal{H}^{0,1} \text{ is also } K\text{-invariant.}
\end{aligned}
$$

Thus, the K-equivariance of G follows.

For the K-equivariance of H, we have that

$$
\begin{aligned}
gHv &= g(v - \Box Gv) \\
&= gv - g\Box Gv \\
&= gv - \Box Ggv \text{ since } \Box, G \text{ are } K\text{-equivariant,} \\
&= Hgv.
\end{aligned}
$$

This ends the lemma. □

Next, Kuranishi would like to parametrize the set

$$
\Phi := \left\{\phi \in A^{0,1}(\Theta) \mid \overline{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0, \overline{\partial}^*\phi = 0\right\}
$$

which actually forms an effective and complete family. We shall repeat briefly his argument. For any $\phi \in \Phi$, we have that

$$
\Box\phi - \frac{1}{2}\overline{\partial}^*[\phi, \phi] = 0.
$$

Applying Green's operator on this, we get

$$
\phi - \frac{1}{2}G\overline{\partial}^*[\phi, \phi] = H\phi.
$$

Thus, Φ is a subset of

$$
\Psi := \left\{\phi \in A^{0,1}(\Theta) \mid \phi - \frac{1}{2}G\overline{\partial}^*[\phi, \phi] \in \mathcal{H}^{0,1}\right\}.
$$

Therefore, it is natural to parametrize Ψ first. Let $\{U_\sigma\}$ be a finite covering of X_0 and $x_\sigma = (x_{1\sigma}, \cdots, x_{n\sigma})$ be a local chart of X_0 on U_σ. Let $\{f_\sigma\}$ be a smooth partition of unity with respect to the covering $\{U_\sigma\}$ of X_0. We introduce another norm in $A^{0,1}(\Theta)$. For $l = (l_1, \cdots, l_n)$, where l_j is some non-negative integer ($j \in [1, \cdots, n]$), we denote by D^l_σ, the partial derivative

$$\left(\frac{\partial}{\partial x_{1_\sigma}}\right)^{l_1} \cdots \left(\frac{\partial}{\partial x_{n_\sigma}}\right)^{l_n}$$

and set $|l| = l_1 + \cdots + l_n$. For $u \in A^{0,1}(\Theta)$ and for an integer $k \geq 0$, we set

$$\|u\|^2_k = \sum_\sigma \sum_{|l| \leq k} \int \left\langle D^l_\sigma f_\sigma u(x_\sigma), D^l_\sigma f_\sigma u(x_\sigma) \right\rangle dv$$

where dv is the volume element of X_0. This norm is called Sobolev k-norm. From now on, we fix once for all a sufficiently large integer k. Let $\mathfrak{H}^k(\Theta)$ be the Hilbert space obtained by completing $A^{0,1}(\Theta)$ with respect to this Sobolev k-norm. Making use of Inverse Mapping Theorem for Banach manifolds to the map

$$\begin{aligned} F : A^{0,1}(\Theta) &\to A^{0,1}(\Theta) \\ \phi &\mapsto \phi - \frac{1}{2} G\overline{\partial}^* [\phi, \phi], \end{aligned}$$

there exists a complex Banach analytic map $\phi : W \to \mathfrak{H}^k(\Theta)$ such that

$$s = F\phi(s) = \phi(s) - \frac{1}{2} G\overline{\partial}^* [\phi(s), \phi(s)]$$

for all $s \in W$, where

$$W := \left\{ s \in \mathcal{H}^{0,1} \mid \|s\|_k < \epsilon \right\}$$

and ϵ is sufficiently small. Hence, for $s \in W$, we have that

$$\Box\phi(s) - \frac{1}{2}\overline{\partial}^* [\phi(s), \phi(s)] = 0,$$

which follows from the fact that $\Box G\overline{\partial}^* = \overline{\partial}^*$ and that s is harmonic. By the regularity of elliptic differential operators, we deduce that ϕ is holomorphic and that the image of ϕ is actually in $A^{0,1}(\Theta)$. In other words, we obtain a holomorphic map

$$\phi : W \to A^{0,1}(\Theta) \tag{2.4.3}$$

whose image, by construction, covers a neighborhood of 0 in Ψ and so, a neighborhood of 0 in Φ.

Finally, a necessary and sufficient condition on s for $\phi(s)$ to be in Φ is that $H[\phi(s), \phi(s)] = 0$. Set $S' := \{s \in W \mid H[\phi(s), \phi(s)] = 0\}$. Restricting on S', we obtain a holomorphic map

$$\phi : S' \to A^{0,1}(\Theta) \tag{2.4.4}$$

which satisfies the conditions (i), (ii), (iii) and (iv) in Theorem 2.3.1.

Now, we add the K-action. Recall that the Lie bracket $[\cdot,\cdot]$ on $A^{0,1}(\Theta)$ is defined as follows. For two element $\alpha, \beta \in A^{0,1}(\Theta)$ given in local coordinates

$$\alpha = \sum m_i^u d\bar{z}^i \bigotimes \frac{\partial}{\partial z^u} \text{ and } \beta = \sum n_j^v d\bar{z}^j \bigotimes \frac{\partial}{\partial z^v}$$

then

$$[\alpha,\beta] := \sum d\bar{z}^i \wedge d\bar{z}^j \bigotimes \left[m_i^u \frac{\partial}{\partial z^u}, n_j^v \frac{\partial}{\partial z^v} \right]'$$

where $[\cdot,\cdot]'$ is the usual Lie bracket for the Lie algebra of vector fields on X_0. Let $g \in K$ then

$$g.\alpha := \sum g^* \left(d\bar{z}^i\right) \bigotimes g_* \left(g_i^u \frac{\partial}{\partial z^u} \right)$$

where g^* and g_* are the pull-back of differential forms and the push-forward of vector fields, respectively. With this definition, the G-action clearly commutes with the Lie bracket, i.e.

$$g[\cdot,\cdot] = [g\cdot, g\cdot]$$

because the wedge product $\wedge$ and the Lie bracket $[\cdot,\cdot]'$ do. Moreover, G and $\overline{\partial}^*$ are K-equivariant. Thus, F is also K-equivariant.

Lemma 2.4.2. *There exists an open neighborhood U of 0 contained in W such that U is K-invariant.*

Proof. For each $g \in K$, there exists a neighborhood V_g of g and K_g of 0 such that $V_g.K_g \in W$. By the compactness of K, there exists a finite set $I \subset K$ such that $K = \bigcup_{g\in I} V_g$. Let $P = \bigcap_{g\in I} K_g$ then P is an open neighborhood of 0 in $\mathcal{H}^{0,1}$. Thus,

$$K.P = \left(\bigcup_{g\in I} V_g \right) \cdot \left(\bigcap_{g\in I} K_g \right) \subseteq W.$$

Finally, set $U := \bigcup_{g\in K} V_g K$. This is the desired K-invariant open neighborhood of 0 contained in W. ☐

Now, restricting the map (2.4.3) on this U, we obtain a map

$$\phi : U \to \phi(U) \subseteq A^{0,1}(\Theta) \tag{2.4.5}$$

which is K-equivariant because it is the inverse of the K-equivariant map F on U. Finally, set $S := S' \cap U$.

Lemma 2.4.3. *S is K-invariant and this K-action is real analytic.*

Proof. Let $s \in S' \cap U$ and $g \in K$ then we have

$$\begin{aligned} H[\phi(g.s), \phi(g.s)] &= H[g.\phi(s), g.\phi(s)] \text{ since } \phi \text{ is } K\text{-equivariant on } U, \\ &= Hg.[\phi(s), \phi(s)] \text{ since the action commutes with the bracket,} \\ &= gH[\phi(s), \phi(s)] \text{ by Lemma 4.1,} \\ &= g.0 \text{ since } s \in S', \\ &= 0. \end{aligned}$$

Thus, $g.s \in S'$. Moreover, $g.s \in U$ by the construction of U. Hence, $g.s \in S' \cap U$ so that S is K-invariant. The part that this K-action on S is real analytic follows from the fact that it is the restriction of a linear K-action on U. □

Proof of Theorem 2.4.1. The restriction of the map ϕ in (2.4.5) on S gives us a map

$$\phi : S \to A^{0,1}(\Theta)$$

which satisfies all the conditions given in the theorem. □

2.5 The case that G is a complex reductive Lie group

In this final section, we would like to extend Corollary 2.4.1 to the case that G is a complex reductive Lie group.

We begin by introducing the definition of holomorphic local (G, K)-action on a complex space X where K is a compact subgroup of G. Denote by $\prod_X$ the collection of all pair $\pi = (U_\pi, V_\pi)$, where U_π and V_π are open subsets in X such that $U_\pi \Subset V_\pi$. Suppose that for each $\pi \in \prod_X$ we have an open neighborhood G_π of K and a mapping $\Phi_\pi : G_\pi \to \mathrm{Ho}(U_\pi, V_\pi)$ where $\mathrm{Ho}(U_\pi, V_\pi)$ is the set of all holomorphic functions from U_π to V_π.

Definition 2.5.1. *One says that the system $\{\Phi_\pi\}$ defines a local (G, K)-action on X if the following conditions are satisfied.*

(a) For all $g, h \in G$ such that $k := gh \in G_\pi$, we have

$$\Phi_\pi(g) \circ \Phi_\pi(h) \mid_{U_{\pi,h}} = \Phi_\pi(k) \mid_{U_{\pi,h}}$$

where $U_{\pi,h} := \{x \in U_\pi \mid \Phi_\pi(h)(x) \in U_\pi\}$;

(b) $\Phi_\pi(\mathbf{1}_G) = \mathbf{id}$;

(c) for all $\pi, \rho \in \prod_X$ and $g \in G_\pi \cap G_\rho$ we have

$$\Phi_\pi(g) \mid_{U_\pi \cap U_\rho} = \Phi_\rho(g) \mid_{U_\pi \cap U_\rho}$$

so that $gx := \Phi_\pi(g)x$ is independent of the choice of π with $x \in U_\pi, g \in G_\pi$;

(d) for any two open sets $U \Subset U_\pi$ and $V \Subset V_\pi$, the set

$$W := W_{\overline{U},V} := \{g \in G_\pi \mid g \cdot \overline{U} \subset V\}$$

is open in G_π and the map

$$\begin{aligned} * : W &\to \mathcal{O}(U) \\ g &\mapsto f \circ g \,|_U \end{aligned}$$

is continuous for all $f \in \mathcal{O}(V)$ where $\overline{U}$ is the closure of U and $\mathcal{O}(P)$ is the set of holomorphic functions on P for any open subset P of X;

(e) The restriction of the system $\{\Phi_\pi\}$ on K gives a global K-action on X, i.e. a homomorphism of topological groups $\Phi : K \to \mathrm{Aut}(X)$.

Moreover, if G is a real (resp. complex) Lie group and if $*$ and Φ are real analytic (resp. holomorphic), then the local (G, K)-action is called real analytic (resp. holomorphic). Two local (G, K)-actions defined by two systems $\{\Phi_\pi\}$ and $\{\Phi'_\pi\}$ are said to be equivalent if for all $\pi \in \prod_X$, the mappings $\Phi_\pi : G_\pi \to \mathrm{Ho}(U_\pi, V_\pi)$ and $\Phi'_\pi : G'_\pi \to \mathrm{Ho}(U_\pi, V_\pi)$ coincide on a sub-domain $G_\pi \cap G'_\pi$ containing K and their restrictions on K give the same global K-action.

As before, by local G-action, we really mean real analytic local G-action. If we let K be the identity element of G in Definition 2.5.1 then we recover the usual definition of (holomorphic) local G-action on complex spaces (see [1, Section 1.2] for more details). In this case, we have the following theorem ([1, Page 25, Corollary]).

Theorem 2.5.1. *Let G be a (complex) Lie group, $\mathfrak{g}$ the Lie algebra of G, and S a complex space. Then we have two bijections*

$$\left\{\begin{matrix}\text{equivalence classes of}\\ \text{local } G\text{-actions on } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Lie algebra homomorphisms}\\ \mathfrak{g} \to \mathcal{T}_S(S)\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{equivalence classes of}\\ \text{holomorphic local}\\ G\text{-actions on } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{complex Lie algebra}\\ \text{homomorphisms } \mathfrak{g} \to \mathcal{T}_S(S)\end{matrix}\right\}$$

where $\mathcal{T}_S(S)$ is the set of holomorphic vector fields on S.

Corollary 2.5.1. *Let K be a connected compact real Lie group acting on a complex space X and G be the complexification of K. There exists a holomorphic local (G, K)-action on X extending the initial global K-action.*

Proof. By Theorem 2.5.1, the initial K-action gives us a Lie algebra homomorphism $\varphi : \mathbf{Lie}(K) \to \mathcal{T}_X(X)$. Since $\mathcal{T}_X(X)$ is a complex Lie algebra, the $\mathbb{C}$-linear extension of φ

gives us a complex Lie algebra homomorphism $\varphi^{\mathbb{C}} : \mathbf{Lie}(K)^{\mathbb{C}} = \mathbf{Lie}(G) \to \mathcal{T}_X(X)$. An application of Theorem 2.5.1 again provides a holomorphic local G-action on X. Note that the restriction of this holomorphic local G-action on K gives a local K-action on X, which in fact is equivalent to the initial global one on X. This follows from the fact that they correspond to the same Lie algebra homomorphism $\varphi : \mathbf{Lie}(K) \to \mathcal{T}_X(X)$. Thus, it allows us to define a holomorphic local (G, K)-action on X as follows. If $g \in K$ then the action of g is determined by the initial global K-action. If $g \in G \setminus K$ then the action of g is determined by the extended holomorphic local G-action. This ends the proof. □

The two following lemmas are helpful in the sequel.

Lemma 2.5.1. *Let $f : X \to Y$ be a proper surjective flat map of complex spaces whose geometric fibers are all connected complex compact manifolds. Then the natural maps $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof. For $y \in Y$, we have that $H^0(X_y, \mathcal{O}_y) = \mathbb{C}$ since X_y is a compact complex manifold. So, the base change morphism

$$\phi^0(s) : (f_*\mathcal{O}_X)_x \otimes_{\mathcal{O}_{Y,y}} \mathbb{C} \to H^0(X_y, \mathcal{O}_y) = \mathbb{C}$$

is clearly surjective. By [3, Chapter III, Theorem 3.4], $\phi^0(s)$ is an isomorphism. Note that $\phi^{-1}(s)$ is trivially surjective. So, an easy application of [3, Chapter III, Corollary 3.7] gives us the freeness of the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ in a neighborhood of y. As $\phi^0(y)$ is an isomorphism then $f_*\mathcal{O}_X$ is free of rank 1 in a neighborhood of y. But this holds for any $y \in Y$. Thus, $f_*\mathcal{O}_X$ is locally free of rank 1 and then the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ turns out to be an isomorphism. This completes the proof. □

Lemma 2.5.2. *Let G be a complex reductive group and let K be a connected real maximal compact subgroup such that $K^{\mathbb{C}} = G$. Let Q be open subset of G. Let g be a point in G such that the K-orbit $K.g$ intersects every connected component of Q. Then if f is a holomorphic function on Q such that $f|_{K.g \cap Q} = 0$ then $f \equiv 0$ on Q.*

Proof. See [14, Page 634, Identity Theorem]. □

Now, we are ready to state the second main result of this paper.

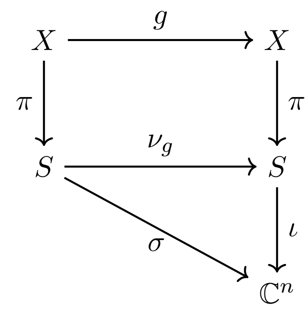
Theorem 2.5.2. *Let X/S be the Kuranishi family of a complex compact manifold X_0 with a holomorphic action of a complex reductive Lie group G. Then we can provide holomorphic local G-actions on X/S extending the holomorphic G-action on X_0.*

Proof. Let K be a connected real maximal compact subgroup whose complexification is exactly G. By Corollary 2.4.1, we obtain a K-equivariant Kuranishi family $\pi : X \to S$. If we can extend the K-actions on X and on S to holomorphic local (G, K)-actions such that π is G-equivariant with respect to these holomorphic local (G, K)-actions then our result follows immediately since any local (G, K)-action is obviously a local G-action. By Corollary 2.5.1, we obtain a holomorphic local (G, K)-action on X. Note that the restriction on K of this local (G, K)-action is nothing but the initial global K-action on X.

Let $g \in N(K) \setminus K$ where $N(K)$ is a neighborhood of K. We shall prove that g, as a biholomorphism on X, swaps fibers of π. Indeed, recall that by construction, S is an analytic subset defined in a open subset $U \subset \mathbb{C}^n$ where $n := \dim_{\mathbb{C}} H^1(X_0, \Theta)$. Consider the following holomorphic function

$$\rho_i : \; X \xrightarrow{g} X \xrightarrow{\pi} S \xrightarrow{\iota} \mathbb{C}^n \xrightarrow{\pi_i} \mathbb{C}$$

where ι is the inclusion and π_i is the i^{th}-projection. Lemma 2.5.1 tells us that $\pi_* \mathcal{O}_X = \mathcal{O}_S$ which means precisely that any holomorphic function from X to $\mathbb{C}$ factors through π. So, for each i, there exists a holomorphic function $\sigma_i : \; S \to \mathbb{C}$ such that $\rho_i = \sigma_i \circ \pi$. So, σ_i's together form a holomorphic function $\sigma : \; S \to \mathbb{C}^n$ which then is lifted to a holomorphic function $\nu_g : S \to S$. More precisely, we have the following commutative diagram



which means in particular that g exchanges fibers of π. Since g is a biholomorphism then so is ν_g. On one hand, ν_g is uniquely determined by g. This follows from the fact that X is constructed from $\underline{X_0} \times S$, as the underlying differentiable manifold, and the fact that g swaps fibers of π. On the other hand, since the local (G, K)-action on X is holomorphic then $\nu_g(-)$ varies holomorphically with respect to the variable g. Hence, the map $g \mapsto \nu_g$ defines a holomorphic local (G, K)-action on S, which extends the initial K-action on S.

Finally, we shall prove that the restriction of the holomorphic local (G, K)-action of X on the central fiber X_0 is the initial G-action on X_0. In order to do it, we first show that the holomorphic local (G, K)-action on S fixes the reference point 0. Let $N(K)$ be a connected open neighborhood of K. Note that the holomorphic function

$$\begin{aligned} \chi : G &\to (S, 0) \\ g &\mapsto \nu_g(0) \end{aligned}$$

is constant on K, i.e. $\chi(k) = 0$ for all $k \in K$. Consider the holomorphic function

$$\mu_i : \ G \xrightarrow{\chi} (S, 0) \xrightarrow{\iota} (\mathbb{C}^n, 0) \xrightarrow{\pi_i} \mathbb{C}$$

where ι is the inclusion and π_i is the i^{th}-projection. Hence, we also have $\mu_i(k) = 0$ for all $k \in K$. Applying Lemma 2.5.2 with $g = \mathbf{1}_G$ and $Q = N(K)$, we obtain that μ_i is zero on $N(K)$. But this holds for any i and so $\chi(g) = 0$ for all $g \in N(K)$. This justifies the claim. Therefore, the local (G, K)-action on X preserves the central fiber X_0, i.e. $gX_0 \subset X_0$ for $g \in G$ whenever it is defined. Consequently, we have a holomorphic local (G, K)-action on X_0, which is the restriction on X_0 of the one on X. Because X_0 is compact then this action turns out to be global and it contains the initial K-action on X_0. As a matter of fact, it must coincide with the initial G-action on X_0 because the action of G on a complex compact manifold is uniquely determined by the one of K.

In summary, what we have just done is to equip holomorphic local (G, K)-actions on X and on S in a way that the map $\pi : X \to S$ is G-equivariant with respect to these holomorphic local (G, K)-actions and that the restriction on the central fiber X_0 of the holomorphic local (G, K)-action on X is nothing but the initial holomorphic G-action on X_0. This finishes the proof. □

2.6 An example of G-equivariant Kuranishi family

In this final section, we shall show that the local actions in Theorem 2.5.2 can not be global in general by giving a detailed example.

The Hirzebruch surface $\mathbb{F}_2$ and its Kuranishi familly $\pi : \mathcal{X} \to \mathbb{C}$, introduced in Chapter 1 are once again taken into account. We consider further an action of $G := \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{F}_2$ as follows. $\mathbb{C}^* \times \mathbb{C}^*$ can be embedded into $\mathrm{GL}(2, \mathbb{C})$ as a subgroup of invertible diagonal matrices. In particular, its action on $\mathbb{F}_2$ given by

$$g(p) = \begin{cases} \left(\left[xu^2 : ya^2u^2 : yd^2v^2\right], [au : dv]\right) & \text{if } u \neq 0 \\ \left(\left[xv^2 : za^2u^2 : zd^2v^2\right], [au : dv]\right) & \text{if } v \neq 0 \end{cases},$$

or equivalently

$$g(p) = \left(\left[x : ya^2 : zd^2\right], [au : dv]\right),$$

where $p = ([x : y : y], [u : v]) \in \mathbb{F}_2$.

An application of Theorem 2.5.2 gives us a local G-equivariant structure on $\pi : \mathcal{X} \to \mathbb{C}$. We shall first show that if there is any extended G-action on $\mathcal{X}$, then G can not act trivially on the Kuranishi space. Indeed, on the intersection of two standard open sets $U_x = \{x = 1\}$ and $U = \{u = 1\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. the proof of Proposition 1.3.1 , the action of G on $\mathbb{F}_2$

given by

$$g.(v,[1:y]) = \left(\frac{dv}{a},[1:ya^2]\right)$$

where $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G$ and $(v,[1:y]) \in \mathbb{F}_2$ under the identification given in Proposition 1.3.1. So, we obtain two commuting vector fields on $\mathbb{F}_2$

$$\begin{cases} E'_5 = 2y\partial_y - v\partial_v \\ E'_6 = v\partial_v. \end{cases}$$

Suppose for the moment that the G-action on $\mathbb{F}_2 = \mathcal{X}_0$ could be lifted to a G-action on $\mathcal{X}$, then we have two commuting vector fields on $\mathcal{X}$, say E_5 and E_6. We can assume that our two vector fields are of the form (up to the first order with respect to t)

$$\begin{cases} E_5 = (2y + p_5(v,y)t)\,\partial_y + (-v + q_5(v)t)\,\partial_y + k_5 t\partial_t \\ E_6 = p_6(v,y)t\partial_y + (v + q_6(v)t)\,\partial_y + k_6 t\partial_t, \end{cases}$$

where p_i, q_i are polynomials whose degree with respect to each variable does not exceed 2 (cf. Proposition 1.4.1). Note that the restriction of these above vector fields on the central fiber are nothing but E'_5, E'_6 respectively. In other words, we have $E_5\mid_{\mathcal{X}_0}= E'_5$ which implies

$$\begin{cases} (a_5(0)v^2 + b_5(0)v + c_5(0))y^2 + (-2(a_5(0)0 + A_5(0))v + e_5(0))\,y = 2y \\ A_5(0)v^2 + B_5(0)v + C_5(0) = -v. \end{cases}$$

In particular,

$$\begin{cases} b_5(0) = 0 \\ B_5(0) = -1 \\ e_5(0) = 2. \end{cases}$$

Expanding these functions in power series up to the first order provides

$$\begin{cases} b_5(0) = b_5 t \\ B_5(t) = -1 + B_5 t \\ e_5(t) = 2 + e_5 t. \end{cases}$$

where b_5, B_5, e_5 are constants. The obstruction of lifting vector fields (1.4.6) forces $k_5 = 1$.

Thus, E_5 is vertical. Likewise, we have $E_6 \mid_{\mathcal{X}_0} = E'_6$ which implies

$$\begin{cases} (a_6(0)v^2 + b_6(0)v + c_6(0))y^2 + (-2(a_6(0)0 + A_6(0))v + e_6(0))\, y = 0 \\ A_6(0)v^2 + B_6(0)v + C_6(0) = v. \end{cases}$$

In particular,

$$\begin{cases} b_6(0) = 0 \\ B_6(0) = 1 \\ e_6(0) = 0. \end{cases}$$

Expanding these functions in power series up to the first order provides

$$\begin{cases} b_6(0) = b_6 t \\ B_6(t) = 1 + B_6 t \\ e_6(t) = e_6 t, \end{cases}$$

where b_6, B_6, e_6 are constants. Once again, the obstruction of lifting vector fields (1.4.6) forces $k_6 = 1$. Thus, E_6 is not vertical as well. Therefore, if the G-action was extended then, G (more precisely, two subgroups $\mathbb{C}^* \times \{1\}$ and $\{1\} \times \mathbb{C}^*$) could not act trivially on the base.

A possible G-action on $\mathcal{X}$ that extends the initial G-action on $\mathbb{F}_2$ given by

$$g(p) = \left(\left[x : ya^2 : zd^2\right], [au : dv], adt\right)$$

for $p := ([x : y : z], [u : v], t) \in \mathcal{X}$ and $g = (a, d) \in \mathbb{C}^* \times \mathbb{C}^*$. For $\mathbb{C}^*$-action, we can restrict the $\mathbb{C}^* \times \mathbb{C}^*$-action on its subgroup $\{(\alpha, 1) \mid \alpha \in \mathbb{C}^*\}$. Evidently, this $\mathbb{C}^*$-action can not be global on the germ of complex space $(\mathbb{C}, 0)$.

Chapter 3

Semi-prorepresentability of formal moduli problems and equivariance structure

3.1 Introduction

The theory of deformations of algebraic schemes with algebraic group actions is first studied by the voluntary work of Pinkham (see [26]) in which affine cones with $\mathbb{G}_m$-actions are taken into account. Six years later, Rim obtains a far-reaching result which claims that if G is a linearly reductive group acting algebraically on an algebraic scheme X_0 where X_0 is supposed to be either an affine scheme with at most isolated singularities or a complete algebraic variety then a G-equivariant formal semi-universal deformation of X_0 exists, unique up to G-equivariant isomorphism (see [30]). In the language of functors of Artin rings, this result can be rephrased as follows. Let k be an algebraically closed field and $\mathbf{Art}_k$ (resp. $\widehat{\mathbf{Art}}_k$) be the category of local artinian k-algebras (resp. complete local noetherian k-algebras) with residue field k. The functor $F_{X_0}\colon \mathbf{Art}_k \to \mathbf{Sets}$ which associates to each local artinian k-algebra A, the set of flat morphisms of schemes $X \to \mathrm{Spec}(A)$ with an isomorphism $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong X_0$ has a formal semi-universal element, i.e. there exists a pro-object R in $\widehat{\mathbf{Art}}_k$ and an element $\hat{u} \in \widehat{F_{X_0}}(R)$ such that the morphism of functors

$$\mathrm{Hom}_{\widehat{\mathbf{Art}}_k}(R, -) \to F_{X_0}$$

defined by $\hat{u}$ is smooth and such that

$$\mathrm{Hom}_{\widehat{\mathbf{Art}}_k}(R, k[\epsilon]/(\epsilon^2)) \to F_{X_0}(k[\epsilon]/(\epsilon^2))$$

is bijective, where $\widehat{F_{X_0}}$ is the extension of F_{X_0} on $\widehat{\mathbf{Art}}_k$ (see [32, §2.2] for more details) and $k[\epsilon]/(\epsilon^2)$ is the ring of dual numbers. Furthermore, this formal semi-universal element can be made G-equivariant. A recently-constructed counter-example in [7] has shown that the reductiveness assumption on G turns out to be optimal. In general, F_{X_0} is hardly prorepresentable by a pro-object due to the existence of non-trivial automorphisms of X_0 as always. Therefore, the smooth morphism

$$\mathrm{Hom}_{\widehat{\mathbf{Art}_k}}(R,-) \to F_{X_0}$$

can be considered the best formal approximation of F_{X_0} that we can expect. A similar result on the existence of G-equivariant Kuranishi family of compact complex manifolds is obtained as well in Chapter 2 (see also [8]). The main difference here is that on the analytic side, all deformations are required to be convergent.

Besides, a well-known philosophy of Drindfeld states that: "If X is a moduli space over a field k of characteristic zero, then a formal neighborhood of any point $x \in X$ is controlled by a differential graded Lie algebra" of which Lurie's famous thesis (cf. [21]) has given a rigorous formulation. Namely, instead of working with $\mathbf{Art}_k$, he works with the category of differential graded commutative artinian augmented k-algebras, denoted by $\mathbf{dgArt}_k$ and a formal moduli problem in his sense is defined to be a functor from $\mathbf{dgArt}_k \to \mathbf{SEns}$ satisfying certain exactness conditions, where $\mathbf{SEns}$ is the ∞-category of simplicial sets. Then he proves that there is an equivalence of ∞-categories between the homotopy category of formal moduli problems and that of differential graded Lie algebras. Furthermore, the prorepresentability (which corresponds to the notion of universality in the classical sense) of a formal moduli problem is reduced to checking some cohomological conditions on its associated differential graded Lie algebra, which is feasible for most of natural formal moduli problems that we encounter in reality. This can be viewed as an extremely astonishing generalization of Schlessinger's work on functors of artinian rings (cf. [31]).

However, the notion of semi-universality apparently does not exist in the derived literature. Therefore, in this final chapter, our aim is to introduce such a notion which we shall call "semi-prorepresentability". This notion should generalize the notion of semi-universality given by M. Schlessinger. Then we prove the semi-prorepresentability for a class of formal moduli problems of which the formal moduli problem Def_{X_0} associated to derived deformations of algebraic schemes or to those of complex compact manifolds (which is a natural extension of the functor F_{X_0} in the derived literature) is a typical example. This gives us an algebraic way to recover the formal existence of semi-universal deformations in the classical setting. At last, we will prove a theorem of Rim's type. More precisely, we would like to provide a G-equivariant structure to the pro-object in $\mathbf{dgArt}_k$, which semi-prorepresents Def_{X_0}. Inspired by the spirit of Lurie's equivalance, we shall carry things out on the corresponding differential graded Lie algebra. Once again, Rim's result in the non-derived setting is just an immediate corollary of this.

Let us now outline the organization of this chapter. We first, in §3.2, give an overview of the ∞-equivalence between formal moduli problems and differential graded Lie algebras. The representations of differential graded Lie algebras, which are one of the essential tools for the rest of the chapter, is recalled as well. In §3.3, we shall introduce the notion of semi-prorepresentability and a criterion for a formal moduli problem to be semi-prorepresentable. If further the associated differential graded Lie algebra of this formal moduli problem is equipped with an action of some linearly reductive group G, we show that the corresponding semi-prorepresentable pro-object can be equipped with a versal compatible G-action (cf. Definition 3.3.3 below). What concerns us first in §3.4 is a folklore, in derived deformation theory, which says that the differential graded Lie algebra corresponding to the derived deformation functor Def_{X_0} of an algebraic scheme is the derived global section of $\mathbb{T}_{X_0/k}$ where $\mathbb{T}_{X_0/k}$ is the tangent complex of X_0 over k. It is well-known but we can not find a literature that contains a proof of it. Therefore, our aim is to give a detailed proof, with the help of Lurie's general results on representations of differential graded Lie algebras. Afterward, we give a characterization of G-equivariant derived deformations of X_0 in terms of this differential graded Lie algebra. Next, we recall also the famous differential graded Lie algebra which controls analytic deformations of a given complex compact manifold. Finally, the existence of (G-equivariant) formal semi-universal deformation of algebraic schemes and that of complex compact manifolds are just immediate consequences of what we have done in §3.3.

Conventions and notations:

- A field of characteristic 0 will be always denoted by k.
- dgla is the abbreviation of differential graded Lie k-algebra while cdga means commutative differential graded augmented k-algebra.
- Mod_k is the category of chain complexes of k-modules and $\mathbf{Mod}_k$ is the corresponding ∞-category.
- Lie_k is the category of differential graded Lie k-algebras and $\mathbf{Lie}_k$ is the corresponding ∞-category.
- cdga_k is the category of commutative differential graded augmented k-algebras and $\mathbf{cdga}_k$ is the corresponding ∞-category.
- $\mathbf{dgArt}_k$ denotes the full sub-category of cdga_k consisting of commutative differential graded artinian algebras cohomologically concentrated in non-positive degrees.
- $\mathbf{Art}_k$ denotes the category of local artinian k-algebras with residue field k.
- **SEns** is the category of simplicial sets.
- fmp is the abbreviation of formal moduli problem.
- $\mathcal{FMP}$ is the homotopy category of formal moduli problems.

3.2 Formal moduli problems revisited

3.2.1 Presentable ∞-categories

A glimpse on presentable ∞-categories is provided in this section. Let $\mathbf{\Delta}$ be the category of finite ordinal numbers with order-preserving maps between them. Concretely, the objects of $\mathbf{\Delta}$ are strings

$$\mathbf{n}: \ 0 \to 1 \to \cdots \to n$$

where n is a positive integer and morphisms of $\mathbf{\Delta}$ are order-preserving set functors $\mathbf{m} \to \mathbf{n}$. For each $\mathbf{n} \in \mathbf{\Delta}$, consider the following morphisms:

$$d^i : \mathbf{n-1} \to \mathbf{n}$$
$$(0 \to 1 \to \cdots \to n-1) \mapsto (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$$

and

$$s^j : \mathbf{n+1} \to \mathbf{n}$$
$$(0 \to 1 \to \cdots \to n+1) \mapsto (0 \to 1 \to \cdots \to j \to j \to \cdots \to n)\,.$$

The former ones are called cofaces while the latter ones are called codegeneracies. They satisfies the following cosimplicial identities

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = \mathrm{Id} = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j. \end{cases} \tag{3.2.1}$$

The maps d^i, s^j together with these relations constitute a set of generators and relations for $\mathbf{\Delta}$ (cf. [20]).

Definition 3.2.1. *A simplicial set is a contravariant functor $X : \mathbf{\Delta} \to$ **Sets**. A map of simplicial sets $f : X \to Y$ is simply a natural transformation of contravariant set-valued functors defined over $\mathbf{\Delta}$.*

Using the generators d^i, s^j and the relations (3.2.1), to give a simplicial set Y is equivalent to giving sets Y_n, $n \geq 0$ together with maps

$$\begin{cases} d_i : Y_n \to Y_{n-1}, & 0 \leq i \leq n \text{ (faces)} \\ s_j : Y_n \to Y_{n+1}, & 0 \leq j \leq n \text{ (degeneracies)} \end{cases}$$

satisfying the simplicial identities

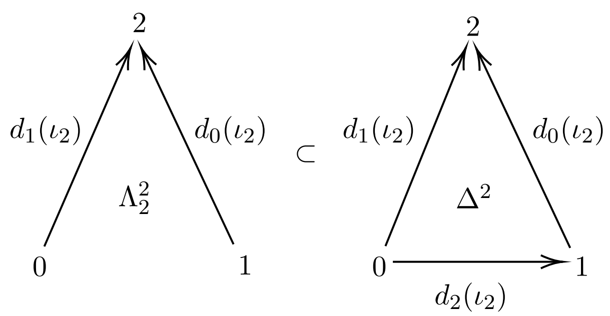
$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = \mathrm{Id} = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j. \end{cases}$$

We denote the category of simplicial sets by SEns and refer the reader to [11] for a complete study of this category.

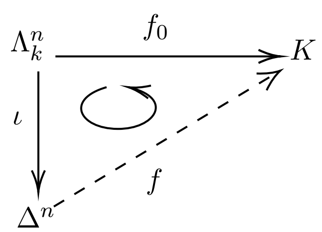
Definition 3.2.2. *(1) The standard n-simplex in the category* SEns *is defined by*

$$\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(\cdot, \mathbf{n}).$$

(2) Denote by ι_n the standard simplex $\mathrm{Id}_{\mathbf{n}} \in \mathrm{Hom}_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{n})$. For $0 \leq k \leq n$, the k-horn Λ_k^n of Δ^n is the union of all the faces $d_j(\iota_n)$ except $d_k(\iota_n)$.



Definition 3.2.3. *An ∞-category is a simplicial set K which has the following property: for any $0 < k < n$, any map $f_0 : \Lambda_k^n \to K$ admits an extension $f : \Delta^n \to K$*



(cf. [23, Definition 1.1.2.4]). A functor (often called ∞-functor) between two ∞-categories is simply a map of simplicial sets.

To end this section, we introduce the notion of presentability of ∞-categories. (cf. [23, Definition 5.4.2.1, Proposition 5.4.2.2 and Definition 5.5.0.1]).

Definition 3.2.4. *Let $\mathcal{C}$ be a category (or an ∞-category). We say that $\mathcal{C}$ is presentable if $\mathcal{C}$ admits small colimits and is generated under small colimits by a set of κ-compact objects, for some regular cardinal number κ. Here, an object $C \in \mathcal{C}$ is said to be κ-compact if the functor* $\mathrm{Hom}_{\mathcal{C}}(C, -)$ *preserves κ-filtered colimits*

Remark 3.2.1. We often omit the cardinal number κ and say simply "compact" and "filtered" for simplicity.

The following two lemmas concerning adjoint functors of ∞-categories are useful in the sequel (cf. [27, Corollary 2.1.65] and [23, Corollary 5.5.2.9], respectively).

Lemma 3.2.1. *Let $g : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories and $f : \mathcal{C} \to \mathcal{D}$ a right adjoint of g. Then f is an equivalence if and only if*

(1) f reflects equivalences,

(2) the unit transformation $\mathrm{Id}_{\mathcal{D}} \to f \circ g$ is an equivalence.

Lemma 3.2.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a ∞-functor between two presentable ∞-categories.*

(1) The functor F has a right adjoint if and only if it preserves small colimits.

(2) The functor F has a left adjoint if and only if it preserves small limits and filtered colimits.

There is a general effective method to construct presentable ∞-categories via combinatorial model categories (see [16] for the notion of combinatorial model category) and Dwyer-Kan simplicial localization ([10]), which we shall use several times in the sequel. We recall it here for completeness. Let $\mathcal{C}$ be a model category and $\mathrm{N}(\mathcal{C})$ its associated nerve category (cf. [23, Definition 1.1.5.5]). Concretely, the simplices of $\mathrm{N}(\mathcal{C})$ can be explicitly described as follows.

- 0-simplices are objects of $\mathcal{C}$,
- 1-simplices are morphisms of $\mathcal{C}$.

  ...

- n-simplices are strings of n composable morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

  which the face map d_i and the degeneracy map s_j carry to

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

  and

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_j} C_j \xrightarrow{\mathrm{Id}_{C_j}} C_j \xrightarrow{f_{j+1}} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n,$$

  respectively.

By formally inverting the class $W_{\mathcal{C}}$ of weak equivalences in $\mathcal{C}$, we obtain a category $\mathrm{N}(\mathcal{C})[W_{\mathcal{C}}^{-1}]$ which is the associated ∞-category of $\mathcal{C}$. The presentability of $\mathrm{N}(\mathcal{C})[W_{\mathcal{C}}^{-1}]$ follows immediately from the following theorem (cf. [22, Proposition 1.3.4.22]).

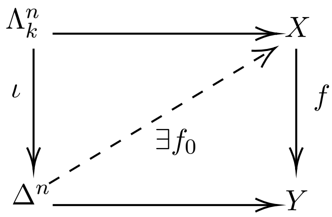
Theorem 3.2.1. *Let $\mathcal{C}$ be a combinatorial model category. Then the associated ∞-category of $\mathcal{C}$ is presentable.*

As a fundamental example, we shall mention the associated presentable ∞-category of the category SEns of simplicial sets.

Proposition 3.2.1. *The category* SEns *of simplicial sets admits a combinatorial model category structure where*

(W) A map of simplicial sets $f : X \to Y$ is a weak equivalence if and only if its geometric realization is a weak homotopy equivalence of topological spaces.

(F) A map of simplicial sets $f : X \to Y$ is a fibration if and only if it satisfies the Kan condition, i.e. for any $0 \leq k \leq n$ and any diagram



of maps of simplicial sets, there exists a map f_0 such that the above diagram commutes.

The reader is referred to [16, Chapter 3.3.2] for a detailed treatment of this proposition. We denote the associated presentable ∞-category of SEns by **SEns**.

3.2.2 Differential graded Lie algebras and its ∞-category

Definition 3.2.5. *A differential graded Lie algebra (or briefly dgla) over k is a chain complex $(\mathfrak{g}_*, d)$ of k-vector spaces equipped with a Lie bracket $[-,-] : \mathfrak{g}_p \otimes_k \mathfrak{g}_* \to \mathfrak{g}_{p+q}$ satisfying the following conditions:*

(1) For $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, we have $[x,y] + (-1)^{pq}[y,x] = 0$.

(2) For $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$ and $z \in \mathfrak{g}_r$, we have

$$(-1)^{pr}[x,[y,z]] + (-1)^{pq}[y,[z,x]] + (-1)^{qr}[z,[x,y]] = 0.$$

(3) The differential d is of degree 1 and is a derivation with respect to the Lie bracket. That is, for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$,

$$d[x,y] = [dx,y] + (-1)^p[x,dy].$$

Given a pair of dglas $(\mathfrak{g}_*, d)$ *and* $(\mathfrak{g}'_*, d')$*, a map of dglas from* $(\mathfrak{g}_*, d)$ *to* $(\mathfrak{g}'_*, d')$ *is a map of chain complexes* $F : (\mathfrak{g}_*, d) \to (\mathfrak{g}'_*, d')$ *such that*

$$F([x, y]) = [F(x), F(y)]$$

for $x \in \mathfrak{g}_p$ *and* $y \in \mathfrak{g}_q$*.*

The collection of all dglas over k *forms a category, which we shall denote by* Lie_k*.*

Proposition 3.2.2. *The category* Lie_k *of dglas over* k *admits a combinatorial model category structure where*

(W) A map of dglas $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ *is a weak equivalence if and only if it is a quasi-isomorphism of chain complexes.*

(F) A map of dglas $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ *is a fibration if and only if it is degree-wise surjective.*

Proof. See [21, Proposition 2.1.10]. □

By the construction mentioned at the end of the previous sub-section, we obtain an ∞-category $\mathrm{N}(\mathrm{Lie}_k)[W^{-1}]$, denoted simply by $\mathbf{Lie}_k$. As an immediate consequence, we have the following.

Corollary 3.2.1. *The* ∞*-category* $\mathbf{Lie}_k$ *is presentable.*

3.2.3 Commutative differential graded algebras and its ∞-category

Definition 3.2.6. *A commutative differential graded algebra (or briefly cdga) over* k *is a chain complex* (A, d) *equipped with a morphism of chain complexes (multiplication map)* $\mu : A \otimes_k A \to A$ *and with a* 0*-cocycle* 1 *(neutral element) such that*

(1) $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ *(associativity),*

(2) $\mu(a, b) = (-1)^{pq}\mu(b, a)$ *(commutativity),*

(3) $\mu(a, 1) = \mu(1, a) = a$*,*

for any $a \in A_p$ *and* $b \in A_q$*. A morphism of cdgas is a morphism of chain complexes commuting with multiplication maps. The collection of all cdgas over* k *forms a category, which we shall denote by* CAlg_k*.*

Proposition 3.2.3. *The category* CAlg_k *of dglas over* k *possesses a combinatorial model category structure where*

(W) A map of cdgas $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ *is a weak equivalence if and only if it is a quasi-isomorphism of chain complexes.*

(F) A map of cdgas $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ *is a fibration if and only if it is degree-wise surjective.*

The same construction as in the case of dglas gives us the associated ∞-category $\mathbf{CAlg}_k$ of CAlg_k. Let us denote by cdga_k the full sub-category of CAlg_k consisting of cdgas A with an additional augmented map $A \to k$. This sub-category inherits a combinatorial model category structure from CAlg_k, which permits us to talk about its corresponding ∞-category, denoted by $\mathbf{cdga}_k$. Finally, we introduce a sub-category of $\mathbf{cdga}_k$, on which formal moduli problems are defined.

Definition 3.2.7. *A commutative differential graded augmented k-algebra $A \in \mathbf{cdga}_k$ is said to be artinian if the three following conditions hold:*

(1) The cohomology groups $H^n(A) = 0$ for n positive and for n sufficiently negative.

(2) All cohomology groups $H^n(A)$ are of finite dimension over k.

(3) $H^0(A)$ is a local artinian ring with maximal ideal $\mathfrak{m}$ and the morphism

$$H^0(A)/\mathfrak{m} \to k$$

is an isomorphism.

We denote the full sub-category of $\mathbf{cdga}_k$ consisting of artinian commutative differential graded augmented k-algebras by $\mathbf{dgArt}_k$.

3.2.4 Chevalley-Eilenberg complex of dglas and Koszul duality

Definition 3.2.8. *Let $(\mathfrak{g}_*, d)$ be a differential graded Lie algebra over a field k. The cone of $\mathfrak{g}_*$, denoted by $\mathrm{Cn}(\mathfrak{g})_*$, is defined as follows:*

(1) For each $n \in \mathbb{Z}$, the vector space $\mathrm{Cn}(\mathfrak{g})_*$ is $\mathfrak{g}_n \oplus \mathfrak{g}_{n-1}$. A general element of $\mathrm{Cn}(\mathfrak{g})_n$ is of the form

$$x + \epsilon y,$$

where $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_{n-1}$ and ϵ is a formal symbol of degree 1 such that $\epsilon^2 = 0$.

(2) The differential of degree 1 on $\mathrm{Cn}(\mathfrak{g})_*$ is given by the formula

$$d(x + \epsilon y) = dx + y - \epsilon dy.$$

(3) The Lie bracket on $\mathrm{Cn}(\mathfrak{g})_*$ is given by

$$[x + \epsilon y, x' + \epsilon y'] = [x, y] + \epsilon([y, x'] + (-1)^p [x, y'])$$

where $x \in \mathfrak{g}_p$.

By definition, $\mathrm{Cn}(\mathfrak{g})_*$ is also a differential graded Lie algebra. Moreover, its underlying chain complex can be identified with the mapping cone of the identity: $\mathfrak{g}_* \to \mathfrak{g}_*$. In particular, $0 \to \mathrm{Cn}(\mathfrak{g})_*$ is a quasi-isomorphism of dglas. Note that the zero map $\mathfrak{g}_* \to 0$ induces a map of differential graded algebras $U(\mathfrak{g}_*) \to U(0) = k$, where $U(\mathfrak{g}_*)$ and $U(0)$

are the universal enveloping differential graded algebras of $\mathfrak{g}_*$ and that of 0, respectively. Another evident map of dglas is the inclusion $\mathfrak{g}_* \to \mathrm{Cn}(\mathfrak{g})_*$.

Definition 3.2.9. *The **cohomological Chevalley-Eilenberg complex** of $\mathfrak{g}_*$ is defined to be the linear dual of the tensor product*

$$U(\mathrm{Cn}(\mathfrak{g})_*) \otimes^{\mathbb{L}}_{U(\mathfrak{g}_*)} k,$$

which we shall denote by $C^*(\mathfrak{g}_*)$.

There is a natural multiplication on $C^*(\mathfrak{g}_*)$. More precisely, for $\lambda \in C^p(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*)$, we define $\lambda\mu \in C^{p+q}(\mathfrak{g}_*)$ by the formula

$$(\lambda\mu)(x_1 \cdots x_n) = \sum_{S,S'} \epsilon(S,S')\lambda(x_{i_1} \cdots x_{i_m})\mu(x_{j_1} \cdots x_{j_{n-m}}),$$

where $x_i \in \mathfrak{g}_{r_i}$, the sum is taken over all disjoint sets $S = \{i_1 < \cdots < i_m\}$ and $S' = \{j_1 < \cdots < j_{n-m}\}$ and $r_{i_1} + \cdots + r_{i_m} = p$, and $\epsilon(S,S') = \prod_{i\in S', j\in S, i<j}(-1)^{r_i r_j}$. This multiplication imposes a structure of cdga on $C^*(\mathfrak{g}_*)$.

Proposition 3.2.4. *With above notations, we have the followings:*

(1) The construction $\mathfrak{g}_* \mapsto C^*(\mathfrak{g}_*)$ sends quasi-isomorphisms of dglas to quasi-isomorphisms of cdgas. In particular, we obtain a functor between ∞-categories $\mathbf{Lie}_k \to \mathbf{cdga}_k^{op}$, which, by abuse of notation, we still denote by C^*.

(2) Let V_* be a chain complex of vector spaces and $\mathrm{Free}(V_*)$ be the free dgla generated by V_* then we have a map

$$C^*(\mathrm{Free}(V_*)) \to k \oplus V_*^\vee[-1]$$

which is a quasi-isomorphism of cdgas, here $V_*^\vee$ is the linear dual of V_*.

(3) The ∞-functor C^* preserves small co-limits. Thus, C^* admits a right adjoint D: $\mathbf{cdga}_k^{op} \to \mathbf{Lie}_k$ to which we refer as Koszul duality.

(4) The unit map

$$A \overset{\simeq}{\to} C^*D(A)$$

is an equivalence in $\mathbf{dgArt}_k$ and

$$DC^*D(A) \overset{\simeq}{\to} D(A)$$

in $\mathbf{Lie}_k$.

Proof. For the first three statements, see [21, Chapter 2, Proposition 2.2.6, Proposition 2.2.7, Proposition 2.2.17]. For the last one, see [27, Chapter 4, Proposition 4.3.5]. □

Definition 3.2.10. *We say that an object $\mathfrak{g}_*$ in $\mathbf{Lie}_k$ is good if it is cofibrant with respect to the model structure on $\mathbf{Lie}_k$ and there exists a graded vector subspace $V_* \subset \mathfrak{g}_*$ such that*

(1) For every integer n, V_n is of finite dimension.

(2) For every non-positive integer n, V_n is trivial.

(3) As a graded Lie algebra, $\mathfrak{g}_*$ is freely generated by V_*, i.e. $\mathfrak{g}_* = \mathrm{Free}(V_*)$.

Denote the full subcategory of $\mathbf{Lie}_k$ spanned by those good objects by $\mathcal{C}^\circ$.

3.2.5 Mapping spaces in $\mathbf{Lie}_k$ and in $\mathbf{cdga}_k$

For each $n \in \mathbb{N}$, the algebraic simplex Δ^n of dimension n is the sub-variety of the affine space $\mathbb{A}^{n+1}$, defined by the equation $\sum_i x_i = 1$. Let L and L' be two dglas then the simplicial set of morphisms from L to L' is the simplicial set

$$\underline{\mathrm{Hom}}^{\Delta}(L, L') : [n] \mapsto \mathrm{Hom}_{\mathbf{Lie}_k}(L, L' \otimes_k C^*(\Delta^n))$$

where $C^*(\Delta^n)$ is the de Rham differential graded algebra on the algebraic simplex Δ^n and $\mathrm{Hom}_{\mathbf{Lie}_k}(L, L' \otimes_k C^*(\Delta^n))$ is the usual set of morphisms between two dglas L and $L' \otimes_k C^*(\Delta^n)$.

Definition 3.2.11. *With the above notations, the mapping space $\mathrm{Map}_{\mathbf{Lie}_k}(L, L')$ between two dglas L and L' is the simplicial set $\underline{\mathrm{Hom}}^{\Delta}(QL, L')$ where QL is a cofibration replacement of L.*

Remark 3.2.2. In particular, $\pi_0(\mathrm{Map}_{\mathbf{Lie}_k}(L, L')) = \mathrm{Hom}_{\mathrm{Lie}_k}(QL, L')$.

The mapping space $\mathrm{Map}_{\mathbf{cdga}_k}(A, A')$ between two cdgas A and A' can be defined in a very similar way.

3.2.6 Formal moduli problems for $\mathbf{cdga}_k$

In this subsection, we shall work with the deformation context $(\mathbf{cdga}_k, \{k \oplus k[n]\}_{n \in \mathbb{Z}})$. Here, the cdga $k \oplus k[n]$ is the square extension of k by $k[n]$.

Definition 3.2.12. *A functor $X : \mathbf{dgArt}_k \to \mathbf{SEns}$ is called a formal moduli problem if the following conditions are fulfilled.*

(1) *The space $X(k)$ is contractible.*

(2) *For every pullback diagram*

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

in $\mathbf{dgArt}_k$, if $\pi_0(R_0) \to \pi_0(R_{01}) \leftarrow \pi_0(R_1)$ are surjective, then the diagram of spaces

$$\begin{array}{ccc} X(R) & \longrightarrow & X(R_0) \\ \downarrow & & \downarrow \\ X(R_1) & \longrightarrow & X(R_{01}) \end{array}$$

is also a pullback diagram.

Remark 3.2.3. We can equivalently replace the condition (2) in the above definition by the following condition: for every pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ R' & \longrightarrow & k \oplus k[n] \end{array}$$

in $\mathbf{dgArt}_k$, the diagram of spaces

$$\begin{array}{ccc} X(R) & \longrightarrow & X(k) \\ \downarrow & & \downarrow \\ X(R') & \longrightarrow & X(k \oplus k[n]) \end{array}$$

is also a pullback diagram for any $n \geq 1$ (see [4, Remark 1.5 and Corollary 1.6] for a proof).

We would like to study the full ∞-subcategory $\mathcal{FMP} \subset \mathbf{Fun}(\mathbf{dgArt}_k, \mathbf{SEns})$ spanned by formal moduli problems.

Theorem 3.2.2. *The functor $D : \mathbf{cdga}_k^{op} \to \mathbf{Lie}_k$ in Proposition 3.2.4 satisfies the following conditions*

(i) The ∞-category $\mathbf{Lie}_k$ is presentable.

(ii) The functor D admits a left adjoint $C^* : \mathbf{Lie}_k \to \mathbf{cdga}_k^{op}$.

(iii) The full subcategory $\mathcal{C}^\circ$ of $\mathbf{Lie}_k$ in Definition 3.2.10 fulfills the following conditions

(a) For every object $\mathfrak{g}_*$ in $\mathcal{C}^\circ$, the unit map $\mathfrak{g}_* \to DC^*(\mathfrak{g}_*)$ is an equivalence in $\mathbf{Lie}_k$.

(b) The initial object 0 of $\mathbf{Lie}_k$ is in $\mathcal{C}^\circ$.

(c) For every $n \in \mathbb{Z}$, let $K_n = \mathrm{Free}(k[-n-1]) \in \mathcal{C}^\circ$, then $C^*(K_n) \simeq k \oplus k[n]$ in $\mathbf{cdga}_k$.

(d) For every push-out diagram

$$\begin{array}{ccc} K_n & \longrightarrow & K \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K' \end{array}$$

if $K \in \mathcal{C}^\circ$ then so is K'.

Proof. (i) is essentially Corollary 3.2.1. (ii) follows from Proposition 3.2.4. For a detailed proof of (iii) see [21, Proposition 2.3.4]. □

Remark 3.2.4. In general the pair

$$D : \mathbf{cdga}_k \underset{\longrightarrow}{\longleftarrow} \mathbf{Lie}_k^{op} : C^*$$

does not induce an equivalence of categories. However, its restriction to the sub-categories $\mathbf{dgArt}_k$ and $\mathcal{C}^\circ$ really does, i.e. the following pair

$$D : \mathbf{dgArt}_k \underset{\longrightarrow}{\longleftarrow} \mathcal{C}^\circ : C^*$$

is indeed an equivalence for the sake of Proposition 3.2.4 and Theorem 3.2.2. In addition, $\mathcal{C}^\circ$ contains essentially compact objects of $\mathbf{Lie}_k$ (cf. Definition 3.2.4 for the notion of compact object).

Now, we are in a position to give a sketch for the proof of the following very well-known fundamental result in derived deformation theory, proved independently by Lurie in [21] and Pridham in [28].

Theorem 3.2.3. *The functor*

$$\begin{aligned} \Psi : \mathbf{Lie}_k &\to \mathcal{FMP} \\ \mathfrak{g}_* &\mapsto \mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*) \end{aligned}$$

induces an equivalence of ∞-categories between $\mathbf{Lie}_k$ and $\mathcal{FMP}$.

Proof. First, we verify that for each $\mathfrak{g}_* \in \mathbf{Lie}_k$, the functor $\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*)$ defines a formal moduli problem in the sense of Definition 3.2.12. Indeed, it is obvious that $\mathrm{Map}_{\mathbf{Lie}_k}(D(k), \mathfrak{g}_*)$ is contractible due to the fact that $D(k) \simeq 0$. It remains to verify the condition (2) in Definition 3.2.12. By Remark 3.2.3, we can consider the cartesian diagram

$$\begin{array}{ccc} N & \longrightarrow & k \\ \downarrow & & \downarrow \\ M & \longrightarrow & k \oplus k[n] \end{array}$$

in $\mathbf{dgArt}_k$. Applying the functor D, we get a cartesian diagram

$$\begin{array}{ccc} D(N) & \longleftarrow & D(k) \\ \uparrow & & \uparrow \\ D(M) & \longleftarrow & D(k \oplus k[n]) \end{array}$$

in $\mathbf{Lie}_k^{op}$ by Remark 3.2.4 and therefore a cartesian diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{Lie}_k}(D(N), \mathfrak{g}_*) & \longrightarrow & \mathrm{Map}_{\mathbf{Lie}_k}(D(k), \mathfrak{g}_*) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathbf{Lie}_k}(D(M), \mathfrak{g}_*) & \longrightarrow & \mathrm{Map}_{\mathbf{Lie}_k}(D(k \oplus k[n]), \mathfrak{g}_*). \end{array}$$

This justifies $\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*)$ being a formal moduli problem.

Next, Ψ preserves small limits by its definition. Moreover, for each $A \in \mathbf{dgArt}_k$, $D(A)$ is a compact object (cf. Definition 3.2.4) in $\mathbf{Lie}_k$ by Remark 3.2.4. Hence, Ψ preserves also filtered colimits in $\mathbf{Lie}_k$. Therefore, the adjoint functor theorem 3.2.2 guarantees the existence of a left adjoint Φ of Ψ. By Lemma 3.2.1, it is reduced to showing that

(1) Ψ reflects equivalences,

(2) the unit transformation $\mathrm{Id}_{\mathcal{FMP}} \to \Psi \circ \Phi$ is an equivalence.

To prove (1), let $f : \mathfrak{g}_* \to \mathfrak{h}_*$ be a morphism of dglas, inducing an equivalence $\Psi(\mathfrak{g}_*) \simeq \Psi(\mathfrak{h}_*)$ of formal moduli problems. In particular, for n positive,

$$\begin{aligned} & \mathrm{Map}_{\mathbf{Lie}_k}(D(k \oplus k[n]), \mathfrak{g}_*) \simeq \mathrm{Map}_{\mathbf{Lie}_k}(D(k \oplus k[n]), \mathfrak{h}_*) \\ \Leftrightarrow \quad & \mathrm{Map}_{\mathbf{Lie}_k}(DC^*(\mathrm{Free}(k[-n-1])), \mathfrak{g}_*) \simeq \mathrm{Map}_{\mathbf{Lie}_k}(DC^*(\mathrm{Free}(k[-n-1])), \mathfrak{h}_*) \\ \Leftrightarrow \quad & \mathrm{Map}_{\mathbf{Lie}_k}(\mathrm{Free}(k[-n-1]), \mathfrak{g}_*) \simeq \mathrm{Map}_{\mathbf{Lie}_k}(\mathrm{Free}(k[-n-1]), \mathfrak{h}_*) \\ \Leftrightarrow \quad & \mathrm{Map}_{\mathbf{Mod}_k}(k[-n-1], \mathfrak{g}_*) \simeq \mathrm{Map}_{\mathbf{Mod}_k}(k[-n-1], \mathfrak{h}_*) \\ \Leftrightarrow \quad & \mathrm{Map}_{\mathbf{Mod}_k}(k, \mathfrak{g}_*[n+1]) \simeq \mathrm{Map}_{\mathbf{Mod}_k}(k, \mathfrak{h}_*[n+1]) \end{aligned}$$

where the second and the third line follow from Theorem 3.2.2(iii)(c) and Remark 3.2.4, respectively (here, $\mathbf{Mod}_k$ is the ∞-category of chain complexes of k-vector spaces). As a sequence, we have a quasi-isomorphism of chain complexes $\mathfrak{g}_*[n+1] \simeq \mathfrak{h}_*[n+1]$, or equivalently, a quasi-isomorphism $\mathfrak{g}_* \simeq \mathfrak{h}_*$. Thus, (1) follows.

By a smooth hypercovering argument (see [21, Proposition 1.5.8]), it is sufficient to prove (2) for representable formal moduli problems, i.e. formal moduli problems of the form $\mathrm{Spec}(A) := \mathrm{Map}_{\mathbf{cdga}_k}(A, -)$ where $A \in \mathbf{dgArt}_k$. For representable fmps, Φ can be explicitly described. Indeed, for any $\mathfrak{g}_* \in \mathbf{Lie}_k$,

$$\begin{aligned}
\mathrm{Map}_{\mathbf{Lie}_k}(\Phi(\mathrm{Spec}(A)), \mathfrak{g}_*) &\simeq \mathrm{Map}_{\mathcal{FMP}}(\mathrm{Spec}(A), \Psi(\mathfrak{g}_*)) \\
&\simeq \mathrm{Map}_{\mathcal{FMP}}(\mathrm{Map}_{\mathbf{cdga}_k}(A, -), \mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*)) \\
&\simeq \mathrm{Map}_{\mathcal{FMP}}(\mathrm{Map}_{\mathbf{Lie}_k}(D(-), D(A)), \mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*)) \\
&\simeq \mathrm{Map}_{\mathbf{Lie}_k}(D(A), \mathfrak{g}_*)
\end{aligned}$$

which gives an equivalence

$$\Phi(\mathrm{Spec}(A)) \simeq D(A).$$

So, to finish the verification, we just need to show that the morphism $\mathrm{Spec}(A) \to \Psi(D(A))$ is an equivalence. This is indeed the case since for each $B \in \mathbf{dgArt}_k$, the following chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathbf{Lie}_k}(D(B), D(A)) &\simeq \mathrm{Map}_{\mathbf{cdga}_k}(A, C^* D(B)) \\
&\simeq \mathrm{Map}_{\mathbf{cdga}_k}(A, B) \\
&\simeq \mathrm{Spec}(A)(B)
\end{aligned}$$

is available again by Remark 3.2.4. □

3.2.7 Representations of dglas

Definition 3.2.13. *Let $\mathfrak{g}_*$ be a dgla over a field k. A **representation** of $\mathfrak{g}_*$ is a differential graded vector space V_*, equipped with a map*

$$\mathfrak{g}_* \otimes_k V_* \to V_*$$

such that $[x, y]v = x(yv) + (-1)^{pq} y(xv)$ for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$.

A morphism between two representations V_* and W_* of $\mathfrak{g}_*$ is a morphism of differential graded vector spaces $f : V_* \to W_*$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathfrak{g}_* \otimes_k V_* & \longrightarrow & V_* \\
{\scriptstyle \mathrm{Id}_{\mathfrak{g}_*} \otimes f} \downarrow & & \downarrow {\scriptstyle f} \\
\mathfrak{g}_* \otimes_k W_* & \longrightarrow & W_*
\end{array}$$

The representations of $\mathfrak{g}_*$ comprise a category which we will denote by $\mathbf{Rep}^{dg}_{\mathfrak{g}_*}$.

Proposition 3.2.5. *The category $\mathbf{Rep}^{dg}_{\mathfrak{g}_*}$ of representations of a dgla $\mathfrak{g}_*$ admits a combinatorial model structure, where:*

(1) A map $f : V_* \to W_*$ of representations of $\mathfrak{g}_*$ is a weak equivalence if and only if it is an isomorphism on cohomology.

(2) A map $f : V_* \to W_*$ of representations of $\mathfrak{g}_*$ is a fibration if and only if it is degreewise surjective.

We denote $\mathbf{Rep}_{\mathfrak{g}_*}$ to be the corresponding ∞-category of $\mathbf{Rep}^{dg}_{\mathfrak{g}_*}$ with respect to this model structure.

Proof. See [21, Chapter 2, Proposition 2.4.5.]. □

Definition 3.2.14. *Let $\mathfrak{g}_*$ be a dgla and $V_* \in \mathbf{Rep}^{dg}_{\mathfrak{g}_*}$. The cohomological Chevalley-Eilenberg complex of $\mathfrak{g}_*$ with coefficients in V_* is defined to be the differential graded vector space of $U(\mathfrak{g}_*)$-module maps from $U(\mathrm{Cn}(\mathfrak{g})_*)$ into V_*.*

Observe that $C^*(\mathfrak{g}_*, V_*)$ has the structure of a module over the differential graded algebra $C^*(\mathfrak{g}_*)$. The action is given by k-bilinear maps

$$C^p(\mathfrak{g}_*) \times C^q(\mathfrak{g}_*, V_*) \to C^{p+q}(\mathfrak{g}_*, V_*)$$

which send $\lambda \in C^*(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*, V_*)$ to the element $\lambda\mu \in C^{p+q}(\mathfrak{g}_*, V_*)$ provided by

$$(\lambda\mu)(x_1 \cdots x_n) = \sum_{S,S'} \epsilon(S, S')\lambda(x_{i_1} \cdots x_{i_m})\mu(x_{j_1} \cdots x_{j_{n-m}}),$$

as in the construction of multiplication on $C^*(\mathfrak{g}_*)$.

Let $\mathfrak{g}_*$ be a dgla and $\mathbf{Mod}^{dg}_{C^*(\mathfrak{g}_*)}$ be the category of differential graded modules over $C^*(\mathfrak{g}_*)$.

Theorem 3.2.4. *The functor*

$$\begin{aligned} C^*(\mathfrak{g}_*, -) : \mathbf{Rep}^{dg}_{\mathfrak{g}_*} &\to \mathbf{Mod}^{dg}_{C^*(\mathfrak{g}_*)} \\ V_* &\mapsto C^*(\mathfrak{g}_*, V_*) \end{aligned}$$

preserves weak equivalences and fibrations. Moreover, it has a left adjoint F given by

$$\begin{aligned} F : \mathbf{Mod}^{dg}_{C^*(\mathfrak{g}_*)} &\to \mathbf{Rep}^{dg}_{\mathfrak{g}_*} \\ M_* &\mapsto U(\mathrm{Cn}(\mathfrak{g})_*) \otimes_{C^*(\mathfrak{g}_*)} M_*. \end{aligned}$$

Thus, $C^*(\mathfrak{g}_*, -)$ is a right Quillen functor, which induces a map between ∞-categories $\mathbf{Rep}_{\mathfrak{g}_*}$ and $\mathbf{Mod}_{C^*(\mathfrak{g}_*)}$.

Proof. See [21, Chapter 2, Proposition 2.4.10 and Remark 2.4.11]. □

Definition 3.2.15. *Let $\mathfrak{g}_*$ be a dgla and V_* be a representation of $\mathfrak{g}_*$. V_* is said to be **connective** if the cohomology groups of the chain complex V_* are concentrated in non-positive degrees. Let $\mathbf{Mod}^{cn}_{\mathfrak{g}_*}$ denote the full subcategory of $\mathbf{Rep}_{\mathfrak{g}_*}$ spanned by the connective $\mathfrak{g}_*$-modules.*

Theorem 3.2.5. *Let f be the corresponding ∞-functor of F in Theorem 3.2.4, then f induces an equivalence of ∞-categories*

$$\mathbf{Mod}^{cn}_{C^*(\mathfrak{g}_*)} \to \mathbf{Mod}^{cn}_{\mathfrak{g}_*}$$

which sends M_* to

$$U(\mathrm{Cn}(\mathfrak{g})_*) \otimes^{\mathbb{L}}_{C^*(\mathfrak{g}_*)} M_*,$$

i.e. f is the left derived functor of F.

Proof. See [21, Chapter 2, Proposition 2.4.16]. □

To end this section, we recall a little bit about tensor products of representations.

Definition 3.2.16. *Let V_* and W_* be two representations of $\mathfrak{g}_*$, then tensor product $V_* \otimes_k W_*$ can be considered a representation of $\mathfrak{g}_*$ with action given by the formula*

$$x(v \otimes w) = (xv) \otimes w + (-1)^{pq} v \otimes (xw)$$

for homogeneous elements $x \in \mathfrak{g}_p, v \in V_q$ and $w \in W_r$.

By a general theorem of Lurie, we can prove that the construction

$$W_* \mapsto V_* \otimes_k W_*$$

preserves quasi-isomorphisms. Consequently, the ∞-category $\mathbf{Rep}_{\mathfrak{g}_*}$ inherits a symmetric monoidal structure.

3.2.8 Derived schemes

Let $\mathbf{sComm}_k$ be the ∞-category of simplicial commutative rings (some authors use the terminology "derived rings").

Definition 3.2.17. *A derived scheme is a data $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a stack of derived rings on X such that two following conditions are satisfied*

(1) The truncation $(X, \pi_0(\mathcal{O}_X))$ is a scheme.

(2) For all i the sheaf of $\pi_0(\mathcal{O}_X)$-modules $\pi_i(\mathcal{O}_X)$ is quasi-coherent.

We denote the ∞-category of derived schemes by $\mathbf{dSch}_k$. We let also $\mathbf{dAff}_k$ be the full ∞-sub-category of $\mathbf{dSch}_k$ consisting of derived schemes whose truncation $\pi_0(X)$ is an affine scheme.

In the world of derived schemes, we also have a derived version of the global section functor which we denote by $\mathbb{R}\Gamma(-,-)$. This functor takes a derived scheme $(X, \mathcal{O}_X)$ to the space of global functions $\mathbb{R}\Gamma(X, \mathcal{O}_X)$ on X. The following theorem is fundamental (see [33, Page 186]).

Theorem 3.2.6. *There is an equivalence of ∞-categories*

$$\mathbb{R}\Gamma(-,-): \ \mathbf{dAff}_k^{op} \to \mathbf{sComm}_k$$

whose inverse functor is denoted by $\mathrm{Spec}(-)$*. Moreover, for any derived scheme X and any derived affine scheme* $\mathrm{Spec}(A)$ *where $A \in \mathbf{sComm}_k$, we have an equivalence of simplicial sets*

$$\mathrm{Map}_{\mathbf{dSch}_k}(X, \mathrm{Spec}(A)) \xrightarrow{\simeq} \mathrm{Map}_{\mathbf{sComm}_k}(A, \mathbb{R}\Gamma(X, \mathcal{O}_X)).$$

3.3 Semi-prorepresentability of formal moduli problems

3.3.1 Smooth and étale morphisms of formal moduli problems

Definition 3.3.1. *Let X and Y be fmps and $u: \ X \to Y$ be a map between them.*

(i) u is said to be smooth if for every small map $\phi: \ A \to B$ in $\mathbf{dgArt}_k$, the natural map

$$X(A) \to X(B) \times_{Y(B)} Y(A)$$

is surjective on connected components.

(ii) u is étale if it is smooth and furthermore $\pi_0(X(k \oplus k)) \to \pi_0(Y(k \oplus k))$ is an isomorphism.

Remark 3.3.1. Let $\mathfrak{g}_*$ and $\mathfrak{h}_*$ be the dglas associated to X and Y, respectively. Then the condition that $\pi_0(X(k \oplus k)) \cong \pi_0(Y(k \oplus k))$ is equivalent to the more explicit condition that

$$\mathrm{Hom}_{\mathbf{Lie}_k}(D(k \oplus k), \mathfrak{g}_*) \cong \mathrm{Hom}_{\mathbf{Lie}_k}(D(k \oplus k), \mathfrak{h}_*),$$

on the side of dglas.

Proposition 3.3.1. *Using the same notations as in Definition 3.3.1. The following conditions are equivalent:*

(i) u is smooth.

(ii) for every $n > 0$, the homotopy fiber of $X(k \oplus k[n]) \to Y(k \oplus k[n])$ is connected.

Proof. See [21, Proposition 1.5.5]. □

The following statement gives an explicit criterion for a morphism of fmps to be étale, on the side of corresponding dglas.

Proposition 3.3.2. *Let X and Y be fmps whose associated dglas are $\mathfrak{g}_*$ and $\mathfrak{h}_*$, respectively and $u: \ X \to Y$ be a map between them, inducing a map $u^*: \ \mathfrak{g}_* \to \mathfrak{h}_*$ of dglas . If $H^i(\mathfrak{g}_*) \cong H^i(\mathfrak{h}_*)$ for any $i > 0$ then u is étale.*

Proof. Note that we always have that

$$\begin{cases} H^{n-i}(\mathfrak{g}_*) = \pi_i X(k \oplus k[n-1]) \\ H^{n-i}(\mathfrak{h}_*) = \pi_i Y(k \oplus k[n-1]) \end{cases}$$

for any $i, n \geq 0$. In particular,

$$\begin{cases} H^{n+1}(\mathfrak{g}_*) = \pi_0 X(k \oplus k[n]),\ H^{n+1}(\mathfrak{h}_*) = \pi_0 Y(k \oplus k[n]) & \text{if } n \geq 0 \\ H^{n}(\mathfrak{g}_*) = \pi_1 X(k \oplus k[n]),\ H^{n}(\mathfrak{h}_*) = \pi_1 Y(k \oplus k[n]) & \text{if } n > 0. \end{cases}$$

Consider the homotopy pull-back

$$\begin{array}{ccc} F & \longrightarrow & X(k \oplus k[n]) \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y(k \oplus k[n]) \end{array}$$

whose corresponding homotopy fiber sequence is

$$\cdots \to \pi_1(X(k \oplus k[n])) \to \pi_1(Y(k \oplus k[n])) \to \pi_0(F) \\ \to \pi_0(X(k \oplus k[n])) \to \pi_0(Y(k \oplus k[n])) \to 0.$$

By assumption we have that

$$\pi_1(X(k \oplus k[n])) \to \pi_1(Y(k \oplus k[n]))$$

and

$$\pi_0(X(k \oplus k[n])) \to \pi_0(Y(k \oplus k[n]))$$

are all isomorphisms for $n > 0$. Thus, $\pi_0(F) = 0$ and then F is connected so that u is smooth by Proposition 3.3.1. Besides,

$$\pi_0(X(k \oplus k)) = H^1(\mathfrak{g}_*) \cong H^1(\mathfrak{h}_*) = \pi_0(Y(k \oplus k)).$$

Hence, u is étale. □

Remark 3.3.2. The notion of smoothness and the one of étaleness are in fact a generalization of those introduced by M. Schlessinger (cf. [31])

3.3.2 Semi-prorepresentable formal moduli problems

One of the corollaries of Theorem 3.2.3 is the following criterion for a fmp to be prorepresentable (cf. [21, Corollary 2.3.6]).

Theorem 3.3.1. *A fmp F is prorepresentable by a pro-object in $\mathbf{dgArt}_k$ if and only if the corresponding dgla $\mathfrak{g}_*$ is cohomologically concentrated in degrees $[1, +\infty)$.*

However, in reality there are many fmps which are not prorepresentable due to the fact that their associated dglas have some components in negatives degrees. The typical example is the derived deformation functor Def_{X_0} of a given algebraic scheme X_0. The 0^{th}-cohomology group of the associated dgla of Def_{X_0} is nothing but the vector space of global vector fields on X_0, which is not vanishing in general (cf. Theorem 3.4.2). This leads us to a weaker notion of prorepresentability, which in fact generalizes that of semi-universality in the classical sense.

Definition 3.3.2. *A fmp F is said to be semi-prorepresentable if there exists a pro-object in $\mathbf{dgArt}_k$ and a morphism of fmps $u : \mathrm{Map}_{\mathbf{dgArt}_k}(A, -) \to F$ such that u is étale.*

Remark 3.3.3. In particular, if F is a semi-prorepresentable fmp in the sense of Definition 3.3.2 then the functor of artinian rings $E := \pi_0(F)$ is semi-prorepresentable by $H^0(QA)$ in Schlessinger's sense:

(a) the morphism of functors $\mathrm{Hom}_{\widehat{\mathbf{Art}}_k}(H^0(QA), -) \to E$ is smooth,

(b) $\mathrm{Hom}_{\widehat{\mathbf{Art}}_k}(H^0(QA), k[\epsilon]/(\epsilon^2)) \to E(k[\epsilon]/(\epsilon^2))$ is bijective

where QA is the cofibrant replacement of A (cf. [31] or [32] for more details).

3.3.3 A criterion for semi-prorepresentability

In this section we try to give a sufficient condition for a given fmp whose associated dgla is cohomologically concentrated in $[0, +\infty)$ to be semi-prorepresentable.

Theorem 3.3.2. *Let F be a fmp whose associated dgla $\mathfrak{g}_*$ is cohomologically concentrated in $[0, +\infty)$. Assume further that $H^i(\mathfrak{g}_*)$ is a finite dimensional vector space for each $i \geq 0$. Then F is semi-prorepresentable.*

Proof. We first treat the case when each $\mathfrak{g}_i$ is finite-dimensional. Denote $B^1(\mathfrak{g}_*)$ and $Z^1(\mathfrak{g}_*)$ to be the first space of boundaries and the one of cycles, respectively. Since, $\mathfrak{g}_1$ is finite-dimensional we can choose the following splittings:

$$\mathfrak{g}_1 = Z^1(\mathfrak{g}_*) \oplus E^1,\ Z^1(\mathfrak{g}_*) = B^1(\mathfrak{g}_*) \oplus H^1(\mathfrak{g}_*).$$

Define a new dgla $\mathfrak{k}_*$

$$\begin{cases} \mathfrak{k}_i = 0 & \text{if } i \leq 0 \\ \mathfrak{k}_1 = E^1 \oplus H^1(\mathfrak{g}_*) & \text{if } i = 1 \\ \mathfrak{k}_i = \mathfrak{g}_i & \text{if } i > 1, \end{cases}$$

whose Lie bracket and differential are induced by those of $\mathfrak{g}_*$. The natural inclusion $u: \mathfrak{k}_* \to \mathfrak{g}_*$ induces isomorphisms

$$H^i(\mathfrak{k}_*) \to H^i(\mathfrak{g}_*),$$

for $i > 0$ by construction. For the sake of Proposition 3.3.2, the corresponding map of fmps

$$\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}_*) \to \mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*) = F(-)$$

is étale. Moreover, $\mathfrak{k}_*$ is cohomological concentrated in $[1, +\infty)$, by construction. Thus, the fmp $\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}_*)$ is prorepresentable by a pro object in $\mathbf{dgArt}_k$, let's say K, i.e.

$$\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}_*) = \mathrm{Map}_{\mathbf{cdga}_k}(K, -)$$

by Theorem 3.3.1. Therefore, F is semi-prorepresentable, which finishes the proof of this case.

For the general case, we have that $\mathfrak{g}_* = \mathrm{colim}_i\ \mathfrak{g}(i)_*$ where each $\mathfrak{g}(i)_k$ is of finite dimension and $\mathfrak{g}(i)_*$ is cohomologically concentrated in $[0, +\infty)$. This fact will be proved in Lemma 3.3.1 below. Then for each dgla $\mathfrak{g}(i)_*$, we repeat the above procedure to obtain $\mathfrak{k}(i)_*$. Denote $\mathfrak{k}_* := \mathrm{colim}_i\ \mathfrak{k}(i)$. Then, the induced map

$$\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}_*) \to \mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*) = F(-)$$

is étale. Furthermore, since each $\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}(i)_*)$ is prorepresentable then so is $\mathrm{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}_*)$. □

Remark 3.3.4. The dgla $\mathfrak{k}_*$ constructed in Theorem 3.3.2 is unique up to quasi-isomorphism in $\mathbf{Lie}_k$.

Lemma 3.3.1. *Let $\mathfrak{g}_*$ be a dgla which is cohomologically concentrated in $[0, +\infty)$. If all the cohomology groups of $\mathfrak{g}_*$ are of finite dimension then*

$$\mathfrak{g}_* = \mathrm{colim}_i\ \mathfrak{g}(i)_*$$

where each $\mathfrak{g}(i)_k$ is finite-dimensional and $\mathfrak{g}(i)_*$ is cohomologically concentrated in $[0, +\infty)$

Proof. Dually, we can assume equivalently that the homology $H_i(\mathfrak{g}_*) \simeq 0$ for all $i \geq 1$. We aim to construct by induction a sequence of dglas

$$0 = \mathfrak{g}(0)_* \to \mathfrak{g}(1)_* \to \mathfrak{g}(i)_* \to \cdots$$

equipped with maps $\phi(i): \mathfrak{g}(i)_* \to \mathfrak{g}_*$ such that

$$\mathfrak{g}_* = \varinjlim \mathfrak{g}(i)_*$$

and that for all $i \geq 0$

$$H_n(\mathfrak{g}(i)_*) \simeq 0, \forall n \geq 1.$$

For each $n \in \mathbb{Z}$, we pick a finite-dimensional graded subspace $V_n \in \mathfrak{g}_n$ consisting of cycles which maps isomorphically onto the homology $H_n(\mathfrak{g}_*)$. We think of V_* as a differential graded vector space with the trivial differential. Let $\mathfrak{g}(1)_*$ denote the free differential graded Lie algebra generated by V_* and let $\phi(1): \mathfrak{g}(1)_* \to \mathfrak{g}_*$ be the canonical map. By construction, we have that

$$H_n(\mathfrak{g}(1)_*) \simeq 0, \forall n \geq 1$$

and that the inclusion $V_0 \to \mathfrak{g}(1)_0$ induces an isomorphism

$$V_0 \to H_0(\mathfrak{g}(1)_*).$$

Now, suppose that $i \geq 1$ and that we have built a map $\phi(i): \mathfrak{g}(i)_* \to \mathfrak{g}_*$ extending $\phi(1)$. Then $\phi(i)$ induces a surjection

$$\theta(i): H_*(\mathfrak{g}(i)_*) \to H_*(\mathfrak{g}_*).$$

Choose a collection of cycles $x_\alpha \in \mathfrak{g}(i)_{n_\alpha}$ whose images form a basis for $\ker(\theta)$. So, we can write

$$\phi(i)(x_\alpha) = dy_\alpha$$

for some $y_\alpha \in \mathfrak{g}_{n_\alpha+1}$. Let $\mathfrak{g}(i+1)_*$ be the differential graded Lie algebra obtained from $\mathfrak{g}(i)_*$ by freely adding elements Y_α (in degrees $n_\alpha + 1$) such that $dY_\alpha = x_\alpha$. We let $\phi(i+1): \mathfrak{g}(i+1)_* \to \mathfrak{g}_*$ denote the unique extension of $\phi(i)$ satisfying

$$\phi(i+1)(Y_\alpha) = y_\alpha.$$

We shall prove that by induction on i that

$$H_n(\mathfrak{g}(i)_*)) \simeq 0, \forall n \geq 1$$

and that the inclusion $V_0 \to \mathfrak{g}(i)_0$ induces an isomorphism

$$V_0 \to H_0(\mathfrak{g}(i)_*)$$

for each $i \geq 1$. The case $i = 1$ is obvious by the above explanation. Suppose that it holds for i, we must prove that it also holds for $i + 1$. Indeed, by construction, we have the following commutative diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\cong} & H_0(\mathfrak{g}(i)_*) \\ & \searrow^{\cong} & \Big\downarrow{\scriptstyle \theta(i)} \\ & & H_0(\mathfrak{g}_*) \end{array}$$

Hence, $\phi(i)$ is an isomorphism in degrees ≥ 0. Thus, $\mathfrak{g}(i+1)_*$ is obtained from $\mathfrak{g}(i)_*$ by freely adding generators Y_α in degree ≤ 0, which implies that

$$H_n(\mathfrak{g}(i+1)_*) \simeq 0$$

for all $n \geq 1$. Furthermore, we can write

$$\mathfrak{g}(i+1)_0 \simeq \mathfrak{g}(i)_0 \oplus W$$

where W is the subspace generated by the elements Y_α with $n_\alpha = -1$, constructed as above. Note that the differential on $\mathfrak{g}(i+1)_*$ induces an injective map

$$d: \; W \to \mathfrak{g}(i)_{-1}/d\mathfrak{g}(i)_0$$

because by construction the set of $dY_\alpha = x_\alpha$ form a basis for $\ker(\theta) \subset \mathfrak{g}(i)_{-1}/d\mathfrak{g}(i)_0$. Therefore,

$$H_0(\mathfrak{g}(i)_*) = H_0(\mathfrak{g}(i+1)_*)$$

so that the inclusion $V_0 \to \mathfrak{g}(i+1)_0$ induces an isomorphism

$$V_0 \to H_0(\mathfrak{g}(i+1)_*).$$

This finishes the induction argument.

Finally, we let $\mathfrak{g}'_*$ denote the colimit of the sequence $\{\mathfrak{g}(i)_*\}_{i\geq 0}$. The canonical map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is surjective on homology since the map $\mathfrak{g}(1)_* \to \mathfrak{g}_*$ is surjective on homology. Let $\eta \in \ker(H_*(\mathfrak{g}'_*) \to H_*(\mathfrak{g}_*))$ then η is represented by a class $\overline{\eta} \in \ker(H_*(\mathfrak{g}(i)_*) \to H_*(\mathfrak{g}_*))$ for i is sufficiently large. By construction, the image of $\overline{\eta}$ vanishes in $H_*(\mathfrak{g}(i+1)_*)$. Thus, $\eta = 0$ so that

$$\mathfrak{g}_* = \underrightarrow{\operatorname{colim}}\, \mathfrak{g}(i)_*.$$

This ends the proof. □

Remark 3.3.5. The finiteness condition on the cohomology groups of $\mathfrak{g}_*$ can be seen as a generalization of Schlessinger's finiteness condition on the tangent space of a classical functor of artinian rings.

3.3.4 Semi-prorepresentability and G-equivariant structure

In this subsection, we intend to generalize the notion of G-equivariant structure on versal deformations initiated by D. S. Rim in [30] (see also Introduction), in the world of formal moduli problems.

Let F be a fmp and let $\mathfrak{g}_*$ be its corresponding dgla. Suppose that F is semi-prorepresentable and that $\mathfrak{g}_*$ is prescribed an action of some group G.

Definition 3.3.3. *F is said to have a G-equivariant structure if there exists a pro-object K in $\mathbf{dgArt}_k$ such that the following conditions are satisfied.*

(i) F is semi-prorepresentable by K,

(ii) Denote the associated dgla of K by $\mathfrak{k}_*$. Then we can equip $\mathfrak{k}_*$ with a compatible G-action such that

(a) the natural morphism of dglas $\Phi : \mathfrak{k}_* \to \mathfrak{g}_*$ is G-equivariant with respect to the prescribed G-action on $\mathfrak{g}_*$,

(b) $\mathfrak{k}_*$ is versal in the following sense: for any $A \in \mathbf{dgArt}_k$ and any G-equivariant map $\phi : QD(A) \to \mathfrak{g}_*$ with respect to the given G-action on $\mathfrak{g}_*$, there exists a G-equivariant map $\tau : QD(A) \to \mathfrak{k}_*$ such that the following diagram commutes

$$\begin{array}{ccc} QD(A) & \overset{\tau}{\dashrightarrow} & \mathfrak{k}_* \\ & \underset{\phi}{\searrow} & \downarrow{\scriptstyle \Phi} \\ & & \mathfrak{g}_* \end{array}$$

where $QD(A)$ is a cofibrant replacement of $D(A)$,

(c) the construction in (b) is a bijection on the tangent level. In other words,

$$\mathrm{Hom}^G_{\mathbf{Lie}_k}(D(k \oplus k), \mathfrak{k}_*) \cong \mathrm{Hom}^G_{\mathbf{Lie}_k}(D(k \oplus k), \mathfrak{g}_*)$$

where $\mathrm{Hom}^G_{\mathbf{Lie}_k}(D(k \oplus k), \mathfrak{k}_*)$ and $\mathrm{Hom}^G_{\mathbf{Lie}_k}(D(k \oplus k), \mathfrak{k}_*)$ are sets of G-equivariant maps of dglas into $\mathfrak{g}_*$ and $\mathfrak{k}_*$ with the prescribed G-actions, respectively.

Remark 3.3.6. The G-equivariant structure on F with respect to a fixed G-action on its corresponding dgla is unique up to G-quasi-isomorphisms.

Remark 3.3.7. If F has a G-equivariant structure then K in the above definition will naturally carry a G-action. So, the map $\tau : QD(A) \to \mathfrak{k}_*$ in (b) will correspond to a G-equivariant map of cdgas: $QK \to A$, as well.

A criterion for a semi-prorepresentable formal moduli problem to have a G-equivariant structure will be given by the following.

Theorem 3.3.3. *Let F be a fmp whose associated dgla $\mathfrak{g}_*$ is cohomologically concentrated in $[0, +\infty)$ and G be a linearly reductive algebraic group defined over k, acting on $\mathfrak{g}_*$.*

Assume further that $H^i(\mathfrak{g}_*)$ is a finite-dimensional vector space for each $i \geq 0$ and that the following colimit is available

$$\mathfrak{g}_* = \operatorname{colim}_i \mathfrak{g}(i)_* \tag{3.3.1}$$

where

(i) each $\mathfrak{g}(i)_k$ is finite-dimensional,

(ii) $\mathfrak{g}(i)_*$ is cohomologically concentrated in $[0, +\infty)$,

(iii) each $\mathfrak{g}(i)_*$ carries an algebraic G-action and the colimit of these G-actions gives back the initial G-action on $\mathfrak{g}_*$.

Then F admits a G-equivariant structure.

Proof. As usual, we first deal with the case where each $\mathfrak{g}_i$ is finite-dimensional. Denote $B^1(\mathfrak{g}_*)$ and $Z^1(\mathfrak{g}_*)$ to be the first space of boundaries and the one of cycles, respectively. Note that $B^1(\mathfrak{g}_*)$ and $Z^1(\mathfrak{g}_*)$ are also G-invariant. Since $\mathfrak{g}_1$ is a finite-dimensional G-module and G is reductive, we can choose the following splittings:

$$\mathfrak{g}_1 = Z^1(\mathfrak{g}_*) \oplus E^1,\ Z^1(\mathfrak{g}_*) = B^1(\mathfrak{g}_*) \oplus H^1(\mathfrak{g}_*)$$

as G-modules. Define a new dgla $\mathfrak{k}_*$

$$\begin{cases} \mathfrak{k}_i = 0 & \text{if } i \leq 0 \\ \mathfrak{k}_1 = E^1 \oplus H^1(\mathfrak{g}) & \text{if } i = 1 \\ \mathfrak{k}_i = \mathfrak{g}_i & \text{if } i > 1, \end{cases}$$

whose Lie bracket and differential are induced by those of $\mathfrak{g}_*$. It is clear that $\mathfrak{k}_*$ inherits an algebraic G-action. By the proof of Theorem 3.3.2, the fmp $F = \operatorname{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*)$ is semi-propresentable by a pro-object K whose associated dgla is exactly $\mathfrak{k}_*$. Moreover, the natural map of dglas $\Phi : \mathfrak{k}_* \to \mathfrak{g}_*$ is G-equivariant, by construction. It is left to verify the versality of $\mathfrak{k}_*$. However, this follows immediately from the étaleness of the map

$$\operatorname{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{k}_*) \to \operatorname{Map}_{\mathbf{Lie}_k}(D(-), \mathfrak{g}_*) = F$$

and the injectivity of the natural map $\Phi : \mathfrak{k}_* \to \mathfrak{g}_*$.

To deal with the general case, we shall make use of the assumption (3.3.1). For each dgla $\mathfrak{g}(i)_*$, we repeat the above procedure to obtain $\mathfrak{k}(i)_*$. Finally, the desired $\mathfrak{k}_*$ is nothing but $\operatorname{colim}_i \mathfrak{k}(i)_*$. □

Remark 3.3.8. The approximation (3.3.1) in fact can be done in several specific situations, for example, if we make a condition that each G-module $\mathfrak{g}_i$ is rational G-module (this will be proved in Lemma 3.3.2 below) or when $\mathfrak{g}_*$ is the Kodaira-Spencer dgla that controls deformations of compact complex manifolds equipped with an appropriate holomorphic action of a reductive complex Lie group (cf. Lemma 3.4.1 below). These two cases cover all the deformation functors that we would like to treat in this chapter.

Lemma 3.3.2. *Let $\mathfrak{g}_*$ be a dgla which has no component in degrees ≤ -1. Suppose that all the cohomology groups of $\mathfrak{g}_*$ are finite-dimensional and that each component $\mathfrak{g}_i$ is a rational G-module. Then*

$$\mathfrak{g}_* = \operatorname{colim}_i \mathfrak{g}(i)_*$$

where

- *(i) each $\mathfrak{g}(i)_k$ is finite-dimensional,*
- *(ii) $\mathfrak{g}(i)_*$ is cohomologically concentrated in $[0, +\infty)$,*
- *(iii) each $\mathfrak{g}(i)_*$ carries an algebraic G-action and the colimit of these G-actions gives back the initial G-action on $\mathfrak{g}_*$.*

Proof. We aim to construct by induction a sequence of dglas with G-actions

$$0 = \mathfrak{g}(0)_* \to \mathfrak{g}(1)_* \to \mathfrak{g}(i)_* \to \cdots$$

equiped with G-equivariant maps $\phi(i) : \mathfrak{g}(i)_* \to \mathfrak{g}_*$ such that

$$\mathfrak{g}_* = \underrightarrow{\operatorname{colim}}\, \mathfrak{g}(i)_*$$

and that for all $i \geq 0$

$$H^n(\mathfrak{g}(i)_*) \simeq 0, \forall n \leq -1.$$

For each $n \in \mathbb{Z}$, we pick a finite-dimensional graded subspace $\tilde{V}_n \in \mathfrak{g}_n$ consisting of cocycles which maps isomorphically onto the cohomology $H^n(\mathfrak{g}_*)$. Let V_n be the subrepresentation of $\mathfrak{g}_n$ generated by $\tilde{V}_n$ under the G-action. Since G acts rationally on $\mathfrak{g}_n$ then V_n is finite-dimensional. Note that $\tilde{V}_0$ is nothing but the space of cocycles of $\mathfrak{g}_*$ due to the fact that $\mathfrak{g}_*$ has no components in degrees ≤ -1. As a matter of fact, $\tilde{V}_0$ is already G-invariant and $V_0 = \tilde{V}_0$. We think of V_* as a differential graded vector space with the trivial differential. Let $\mathfrak{g}(1)_*$ denote the free differential graded Lie algebra generated by V_* and let $\phi(1) : \mathfrak{g}(1)_* \to \mathfrak{g}_*$ be the canonical map. By construction, we have that $\phi(1)$ is G-equivariant and that

$$H^n(\mathfrak{g}(1)_*) \simeq 0, \forall n \leq -1$$

and that the inclusion $V_0 \to \mathfrak{g}(1)_0$ induces an isomorphism

$$V_0 \to H^0(\mathfrak{g}(1)_*).$$

Now, suppose that $i \geq 1$ and that we have built a G-equivariant map $\phi(i) : \mathfrak{g}(i)_* \to \mathfrak{g}_*$

extending $\phi(1)$. Then $\phi(i)$ induces a surjection

$$\theta(i):\ H^*(\mathfrak{g}(i)_*) \to H^*(\mathfrak{g}_*).$$

For each n, since n^{th}-component $\ker(\theta)_n$ of $\ker(\theta)$ is a G-invariant finite-dimensional sub-vector space of $H^n(\mathfrak{g}(i)_*)$, we choose a collection of cocycles $\{x^n_\alpha\}_{\alpha\in A_n} \subset \mathfrak{g}(i)_n$ whose images form a basis for $\ker(\theta)_n$, where A_n is a finite index set. For each $\alpha \in A_n$ and $g \in G$, we write

$$\begin{cases} g.x^n_\alpha = \sum_{\beta\in A_n} \lambda^g_{\alpha\beta} x^n_\beta, \\ \phi(i)(x^n_\alpha) = dy^{n-1}_\alpha \end{cases}$$

for some $y^{n-1}_\alpha \in \mathfrak{g}_{n-1}$. On one hand, we have that

$$\begin{aligned} d(g.y^n_\alpha) &= gdy^{n-1}_\alpha, \text{ since } d \text{ is equivariant}, \\ &= g\phi(i)(x^n_\alpha) \text{ by construction}, \\ &= \phi(i)(gx^n_\alpha) \text{ by the equivariance of } \phi(i), \\ &= \phi(i)(\sum_{\beta\in A_n} \lambda^g_{\alpha\beta} x^n_\beta) \\ &= \sum_{\beta\in A_n} \lambda^g_{\alpha\beta} \phi(i)(x_\beta) \\ &= \sum_{\beta\in A_n} \lambda^g_{\alpha\beta} d(y^{n-1}_\beta) \\ &= d\left(\sum_{\beta\in A_n} \lambda^g_{\alpha\beta} y^{n-1}_\beta\right). \end{aligned}$$

Denote $z^{n-1}_{\alpha,g} = g.y^{n-1}_\alpha - \sum_{\beta\in A_n} \lambda^g_{\alpha\beta} y^{n-1}_\beta$ then

$$dz^{n-1}_{\alpha,g} = 0 \tag{3.3.2}$$

Let T_{n-1} be the vector space generated by y^{n-1}_α's and $z^{n-1}_{\alpha,g}$'s. On the other hand, since $\mathfrak{g}_{n-1}$ is a rational G-module then the sub-representation of G generated by y^{n-1}_α's is a finite-dimensional vector space, which we shall call T'_{n-1}. Clearly, T_{n-1} is included in T'_{n-1}. Hence, T_{n-1} is also finite-dimensional. Moreover, it is easy to see that T_{n-1} is also G-invariant, by construction. Let W_{n-1} be a vector space identical to T_{n-1}, as G-representations. Let Y^{n-1}_α's and $Z^{n-1}_{\alpha,g}$'s be elements in W_{n-1} corresponding to y^{n-1}_α's and $z^{n-1}_{\alpha,g}$'s, respectively. Finally, for each n, define $\mathfrak{g}(i+1)_*$ to be the differential graded Lie algebra obtained from $\mathfrak{g}(i)_*$ by freely adding a basis of W_{n-1} (in degrees $n-1$) such that

$$\begin{cases} d(Y^{n-1}_\alpha) = x^n_\alpha \\ d(Z^{n-1}_{\alpha,g}) = 0 \end{cases} \tag{3.3.3}$$

and let $\phi(i+1):\ \mathfrak{g}(i+1)_* \to \mathfrak{g}_*$ denote the unique extension of $\phi(i)$ satisfying

$$\begin{cases} \phi(i+1)(Y_\alpha^{n-1}) = y_\alpha^{n-1} \\ \phi(i+1)(Z_{\alpha,g}^{n-1}) = z_{\alpha,g}^{n-1}. \end{cases} \tag{3.3.4}$$

It is not difficult to see that d and $\phi(i+1)$ defined in this way are G-equivariant.

We shall prove by induction on i that

$$H^n(\mathfrak{g}(i)_*)) \simeq 0, \forall n \leq -1$$

and that the inclusion $V_0 \to H^0(\mathfrak{g}(i)_*)$ induces an isomorphism

$$V_0 \to H^0(\mathfrak{g}(i)_*)$$

for each $i \geq 1$. The case $i = 1$ is obvious by the above explanation. Suppose that it holds for i, we must prove that it also holds for $i+1$. Indeed, by construction, we have the following commutative diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{\cong} & H_0(\mathfrak{g}(i)_*) \\ & \underset{\cong}{\searrow} & \downarrow{\scriptstyle \theta(i)} \\ & & H_0(\mathfrak{g}_*) \end{array}$$

Hence, $\theta(i)$ is an isomorphism in degrees ≤ 0. Thus, $\mathfrak{g}(i+1)_*$ is obtained from $\mathfrak{g}(i)_*$ by freely adding generators Y_α^{n+1} and $Z_{\alpha,g}^{n+1}$ in degree ≥ 0, which implies that

$$H^n(\mathfrak{g}(i+1)_*) \simeq 0$$

for all $n \leq -1$. Furthermore, we can write

$$\mathfrak{g}(i+1)_0 \simeq \mathfrak{g}(i)_0 \oplus Y_0 \oplus Z_0$$

where Y_0 and Z_0 are the subspaces generated by the elements Y_α^0 and $Z_{\alpha,g}^0$, constructed as above. Note that the differential on $\mathfrak{g}(i+1)_*$ induces an injective map

$$d:\ Y_0 \to \mathfrak{g}(i)_1/d\mathfrak{g}(i)_0$$

because by construction the set of $dY_\alpha^0 = x_\alpha^1$ form a basis for

$$\ker(\theta)_1 \subset \mathfrak{g}(i)_1/d\mathfrak{g}(i)_0 \subseteq \mathfrak{g}(i+1)_1/d\mathfrak{g}(i+1)_0.$$

This guarantees that there are no new cocycles coming from Y_0. However, the space Z_0

consists merely of new cocycles on $\mathfrak{g}(i+1)_0$ by (3.3.3). Therefore, in general,

$$\boxed{V_0 \cong H^0(\mathfrak{g}(i)_*) \neq H^0(\mathfrak{g}(i+1)_*).}$$

In order to remedy this situation, we note that there is a canonical isomorphism

$$\theta(i): \; H^0(\mathfrak{g}(i)_* \to H^0(\mathfrak{g}_*)$$

as G-representations. By (3.3.2), $z^0_{\alpha,g} \in \mathfrak{g}_0$ is a cocycle. Let $\overline{z}^0_{\alpha,g}$ denote its cohomology class in $H^0(\mathfrak{g}_*)$. So, there exists a unique cohomology class $\overline{z'}^0_{\alpha,g} \in H^0(\mathfrak{g}(i)_*)$ such that

$$\theta(i)(\overline{z'}^0_{\alpha,g}) = \overline{z}^0_{\alpha,g}.$$

On the other hand, we have a decomposition of

$$\mathfrak{g}(i)_0 = E_0 \oplus \mathcal{Z}^0(\mathfrak{g}(i)_*)$$

where E_0 is some subspace and $\mathcal{Z}^0(\mathfrak{g}(i)_*)$ is the space of cocycles. Thus,

$$\mathfrak{g}(i+1)_0 \simeq E_0 \oplus \mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0 \oplus Y_0$$

Let $\pi: \; \mathcal{Z}^0(\mathfrak{g}(i)_*) \to H^0(\mathfrak{g}(i)_*)$ denote the canonical projection. We define a linear map

$$\Phi: \; \mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0 \to H^0(\mathfrak{g}(i)_*)$$

as follows:

$$\begin{cases} \Phi(x) = \pi(x) & \text{if } x \in \mathcal{Z}^0(\mathfrak{g}(i)_*), \\ \Phi(Z^0_{\alpha,g}) = \overline{z'}^0_{\alpha,g} & \text{for any } Z^0_{\alpha,g} \in Z_0. \end{cases}$$

Therefore, we have a decomposition

$$\mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0 = \ker(\Phi) \oplus \overline{\mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0}$$

where $\overline{\mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0}$ is isomorphic to the quotient $(\mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0)/\ker\Phi$. If we denote by $\overline{\mathfrak{g}(i+1)_0}$ the minimal G-stable sub-vector space generated by $E_0 \oplus \overline{\mathcal{Z}^0(\mathfrak{g}(i)_*) \oplus Z_0} \oplus Y_0$ under the G-action on $\mathfrak{g}(i+1)_0$ then $\overline{\mathfrak{g}(i+1)_0}$ is in general not a Lie sub-algebra of $\mathfrak{g}(i+1)_0$. Let $\widetilde{\mathfrak{g}(i+1)_0}$ be the Lie sub-algebra generated by $\overline{\mathfrak{g}(i+1)_0}$ under the Lie bracket of $\mathfrak{g}(i+1)_0$. Replace the 0^{th}-component $\mathfrak{g}(i+1)_0$ of $\mathfrak{g}(i+1)_*$ by $\widetilde{\mathfrak{g}(i+1)_0}$ with the induced differential map and the induced map of $\phi(i+1)$. Note that these induced maps are well-defined by construction (3.3.3) and (3.3.4). Now, with this new dgla $\mathfrak{g}(i+1)_*$, we have at last that

$$H^0(\mathfrak{g}(i)_*) \cong H^0(\mathfrak{g}(i+1)_*)$$

because on the cohomological level, each new cocycle in Z_0 is eventually some cocycle coming from $\mathfrak{g}(i)_0$, by construction. In other words, there is no new cohomology class created by Z_0 in $H^0(\mathfrak{g}(i+1)_*)$. This finishes the induction argument.

Finally, we let $\mathfrak{g}'_*$ denote the colimit of the sequence $\{\mathfrak{g}(i)_*\}_{i\geq 0}$. The canonical map $\mathfrak{g}'_* \to \mathfrak{g}_*$ is surjective on homology since the map $\mathfrak{g}(1)_* \to \mathfrak{g}_*$ is surjective on homology. Let $\eta \in \ker(H^*(\mathfrak{g}'_*) \to H^*(\mathfrak{g}_*))$ then η is represented by a class $\overline{\eta} \in \ker(H^*(\mathfrak{g}(i)_*) \to H^*(\mathfrak{g}_*))$ for i sufficiently large. By construction, the image of $\overline{\eta}$ vanishes in $H^*(\mathfrak{g}(i+1)_*)$. Thus, $\eta = 0$ so that

$$\mathfrak{g}_* = \underset{\longrightarrow}{\operatorname{colim}}\, \mathfrak{g}(i)_*.$$

This ends the proof. □

3.4 Applications: Derived deformations of some geometric objects

3.4.1 Semi-universal deformation of algebraic schemes

Let X_0 be an algebraic scheme defined over k. For each $A \in \mathbf{dgArt}_k$, denote C_A the category of flat morphisms of derived schemes $X \to \operatorname{Spec}(A)$. A morphism between two objects $X \to \operatorname{Spec}(A)$ and $Y \to \operatorname{Spec}(A)$ in C_A is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

in $\mathbf{dSch}_k$. Consider the functor

$$\begin{aligned} \operatorname{Def} : \mathbf{dgArt}_k &\to \mathbf{SEns} \\ A &\mapsto \mathcal{N}(C_A/\text{quasi-isomorphisms}) \end{aligned}$$

where $\mathcal{N}$ is the nerve of the category C_A. Let $\phi\colon A \to A'$ be a morphism in $\mathbf{dgArt}_k$, then we have an induced morphism

$$\begin{aligned} \operatorname{Def}(\phi) : \operatorname{Def}(A) &\to \operatorname{Def}(A') \\ (X \to \operatorname{Spec}(A)) &\mapsto (X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A') \to \operatorname{Spec}(A')) \end{aligned}$$

which clearly preserves the quasi-isomorphisms. The fact that $X_0 \in \operatorname{Def}(k)$ allows us to define a new functor

$$\operatorname{Def}_{X_0} : \ \mathbf{dgArt}_k \to \mathbf{SEns}$$

which sends $(A \stackrel{\phi_A}{\to} k)$ to the homotopy fiber at X_0, i.e. $\mathrm{Def}(A) \times_{\mathrm{Def}(k)} X_0$ which is equivalent to the following cartesian diagram

$$\begin{array}{ccc} \mathrm{Def}(A) \times_{\mathrm{Def}(k)} X_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow i \\ \mathrm{Def}(A) & \xrightarrow[\mathrm{Def}(\phi_A)]{} & \mathrm{Def}(k) \end{array}$$

Thus, Def_{X_0} is the derived deformation functor of X_0 and $\mathrm{Def}_{X_0} \in \mathcal{FMP}$. If $X_0 = \mathrm{Spec}(B_0)$ is an affine scheme, $\mathrm{Def}_{X_0}(A)$ is simply the set of cofibrant flat commutative A-dg-algebras B such that we have the following cartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \uparrow & & \uparrow i \\ A & \xrightarrow[\phi_A]{} & k \end{array}$$

in $\mathbf{cdga}_k$.

Remark 3.4.1. The formal moduli problem Def_{X_0} defined as above is the natural extension of the functor of artinian rings F_{X_0} discussed in the introduction of this chapter.

Using the setting of Section 3.2, we now compute the dgla associated to Def_{X_0}. The case that $X_0 = \mathrm{Spec}(B_0)$ is an affine scheme shall be treated in advance. We follow strictly the sketch of proof given by B. Toën in [34, Page 1111-30]. Theorem 3.2.5 turns out to be the key tool. Let $\mathfrak{g}_*^A := D(A)$ for each $A \in \mathbf{dgArt}_k$, and f be the ∞-functor defined as in Theorem 3.2.5.

Theorem 3.4.1. *The dgla corresponding to the derived deformation functor of an affine scheme* $X_0 = \mathrm{Spec}(B_0)$ *is*

$$\mathrm{Der}_k(B_0', B_0')$$

the dg-derivations of B_0'*, where* B_0' *is a cofibration replacement of* $(k \to B_0)$*.*

Proof. Observe that by definition all the elements of $\mathbf{dgArt}_k$ are connective (cf. Definition 3.2.15) and then so are those of $\mathrm{Def}_{B_0}(A)$ for each $A \in \mathbf{dgArt}_k$ (by flatness). Let $B \in \mathrm{Def}_{B_0}(A)$ then $f(B)$ is a connective $\mathfrak{g}_*^A$-module in $\mathbf{Mod}^{cn}_{\mathfrak{g}_*^A}$. Recall again that $\mathbf{Rep}_{\mathfrak{g}_*^A}$ has a symmetric monoidal structure. Thus, saying that $f(B)$ is a cdga in $\mathbf{Rep}_{\mathfrak{g}_*^A}$ is the same as saying that B is a representation of $\mathfrak{g}_*^A$ and the "multiplication" map

$$B \otimes_k B \to B$$

is a morphism of representations. However, by Definition 3.2.16 about tensor product of two representations, the multiplication map

$$B \otimes_k B \to B$$

being a morphism of representations $\mathfrak{g}_*^A$ means exactly that each $l \in \mathfrak{g}_*^A$ acts on $f(B)$ by derivations. Equivalently, there exists a morphism of dglas:

$$\mathfrak{g}_*^A \to \mathrm{Der}_k(f(B), f(B)).$$

In brief, what we have just done is to associate to each element of $\mathrm{Def}_{B_0}(A)$, an element of $\mathrm{Map}_{\mathbf{Lie}_k}(\mathfrak{g}_*^A, \mathrm{Der}_k(f(B), f(B)))$. Finally, since f is an equivalence of ∞-categories, this correspondence is an equivalence of simplicial sets.

Now, unwinding the definition of f, we have

$$f(B) = U(\mathrm{Cn}(\mathfrak{g}_*^A)) \otimes^{\mathbb{L}}_{C^*(\mathfrak{g}_*^A)} B$$

The cone $\mathrm{Cn}(\mathfrak{g}_*^A)$ of $\mathfrak{g}_*^A$ is a contractible chain complex since its underlying chain complex can be identified with the mapping cone of the identity $\mathfrak{g}_*^A \to \mathfrak{g}_*^A$. In particular, $0 \to \mathrm{Cn}({\mathfrak{g}_*}^A)$ is a quasi-isomorphism of dglas. Because the universal enveloping algebra construction preserves quasi-isomorphisms, $U(0) = k \to U(\mathrm{Cn}({\mathfrak{g}_*}^A))$ is also a weak equivalence. Thus,

$$U(\mathrm{Cn}(\mathfrak{g}_*^A)) \otimes^{\mathbb{L}}_{C^*(\mathfrak{g}_*^A)} B \simeq k \otimes^{\mathbb{L}}_{C^*(\mathfrak{g}_*^A)} B.$$

Moreover, by Proposition 3.2.4, we have an equivalence in $\mathbf{dgArt}_k$

$$A \xrightarrow{\simeq} C^*(\mathfrak{g}_*^A).$$

As a consequence,

$$f(B) \simeq U(\mathrm{Cn}(\mathfrak{g}_*^A)) \otimes^{\mathbb{L}}_{C^*(\mathfrak{g}_*^A)} B \simeq k \otimes^{\mathbb{L}}_A B.$$

By the definition of B, this is just the image B_0 in the homotopy category of cdgas. In other words, if we take B_0' a cofibrant replacement of B_0 then

$$f(B) \simeq B_0'.$$

Therefore, we have an equivalence

$$\mathrm{Def}_{B_0}(A) \simeq \mathrm{Map}_{\mathbf{Lie}_k}(\mathfrak{g}_*^A, \mathrm{Der}_k(B_0', B_0'))$$

as simplicial sets. This completes the proof. □

Now, we deal with the general case where X_0 is an arbitrary scheme.

Theorem 3.4.2. *The dgla corresponding to the derived deformation functor* Def_{X_0} *of a scheme* X_0 *is*

$$\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$$

where $\mathbb{T}_{X_0/k}$ *is the tangent complex of* X_0 *over* k*.*

Proof. For $A \in \mathbf{dgArt}_k$, an object in $\mathrm{Def}_{X_0}(A)$ is a flat morphism of derived schemes $X \to \mathrm{Spec}(A)$. By Theorem 3.2.6, it corresponds to a flat morphism

$$A \to \mathbb{R}\Gamma(X, \mathcal{O}_X).$$

By the proof of affine case, it is equivalent to a morphism of dglas:

$$\mathfrak{g}_*^A \to \mathrm{Der}(f(\mathbb{R}\Gamma(X, \mathcal{O}_X)), f(\mathbb{R}\Gamma(X, \mathcal{O}_X))) \simeq \mathrm{Der}(\mathbb{R}\Gamma(X_0, \mathcal{O}_{X_0}), \mathbb{R}\Gamma(X_0, \mathcal{O}_{X_0})).$$

On the other hand,

$$\begin{aligned}
\mathrm{Der}(\mathbb{R}\Gamma(X_0, \mathcal{O}_{X_0}), \mathbb{R}\Gamma(X_0, \mathcal{O}_{X_0})) &= \mathbb{R}\Gamma(X_0, \mathbb{D}\mathrm{er}(\mathcal{O}_{X_0}, \mathcal{O}_{X_0})) \\
&= \mathbb{R}\Gamma(X_0, \mathbb{D}\mathrm{er}(\mathcal{O}_{X_0}, \mathcal{O}_{X_0})) \\
&= \mathbb{R}\Gamma(X_0, \mathrm{Map}_{L_{qcoh}(X)}(\mathbb{L}_{X_0}, \mathcal{O}_{X_0})) \\
&= \mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k}).
\end{aligned}$$

where $L_{qcoh}(X)$ is the ∞-category of derived quasi-coherent sheaves of X_0. Thus, we have just proved that

$$\mathrm{Def}_{X_0}(A) \simeq \mathrm{Map}_{\mathbf{Lie}_k}(\mathfrak{g}_*^A, \mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$$

as simplicial sets. This tells us that the dgla corresponding to Def_{X_0} is $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$. □

Theorem 3.4.3. *If* X_0 *is either an affine scheme with at most isolated singularities or a complete algebraic variety then* Def_{X_0} *is semi-prorepresentable. Consequently, the classical functor of deformations* $\pi_0(\mathrm{Def}_{X_0})$ *of* X_0 *has a semi-universal element.*

Proof. Since X_0 is either an affine scheme with at most isolated singularities or a complete algebraic variety then all the cohomology groups of $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ are finite-dimensional vector spaces. Moreover, $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ is cohomologically concentrated in $[0, +\infty)$. Therefore, Def_{X_0} is semi-prorepresentable by Theorem 3.3.2. The last statement follows immediately by Remark 3.3.3. □

3.4.2 Equivariant semi-universal deformation of algebraic schemes

Now, suppose further that there is an algebraic group G acting algebraically on X_0. For $A \in \mathbf{dgArt}_k$, we consider a special type of derived deformations of X_0 over $\mathrm{Spec}(A)$.

Definition 3.4.1. *An element* $\pi : X \to \mathrm{Spec}(A)$ *of* $\mathrm{Def}_{X_0}(A)$ *is said to be G-equivariant if the following conditions are satisfied*

(i) X *and* $\mathrm{Spec}(A)$ *can be equipped with some G-actions with respect to which* π *is G-equivariant,*

(ii) *The isomorphism* $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \xrightarrow{\cong} X_0$ *is G-equivariant.*

Remark 3.4.2. For each $A \in \mathbf{dgArt}_k$, we can define a G-action on $\mathrm{Def}_{X_0}(A)$ by the central-fiber-changing trick as follows. For each $g \in G$ and each $(X \to \mathrm{Spec}(A)) \in \mathrm{Def}_{X_0}(A)$, $g.(X \to \mathrm{Spec}(A))$ is the following deformation

$$\begin{array}{ccccc} X_0 & \xrightarrow{g^{-1}} & X_0 & \xrightarrow{\iota} & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ \cdot & \xrightarrow{\cong} & \cdot & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

Hence, we obtain a G-action on Def_{X_0}, which then gives a G-action on the associated dgla $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$, by Theorem 3.2.3. Moreover, the initial G-action of X_0 induces also a G-action on the derived global section $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ of its tangent complex $\mathbb{T}_{X_0/k}$. It can be seen that this G-action coincides with the one induced by the central-fiber-changing trick.

We would like to give a characterization of G-equivariant derived deformation in terms of dglas. As usual, we deal with the affine case first. Let $X_0 = \mathrm{Spec}(B_0)$ be an affine scheme equipped with an action of some algebraic group G. If B'_0 is a cofibrant replacement of $k \to B_0$ then by the functoriality of the cofibrant replacement functor, we have an induced G-action on B'_0 and then a G-action on $\mathrm{Der}_k(B'_0, B'_0)$ is given by conjugations, i.e. for $g \in G$ and $d \in \mathrm{Der}_k(B'_0, B'_0)$, we have that $g.d = g \circ d \circ g^{-1}$. Hence, $\mathrm{Der}_k(B'_0, B'_0)$ is a G-object in $\mathbf{Lie}_k$.

Theorem 3.4.4. *For* $A \in \mathbf{dgArt}_k$*, a G-equivariant derived deformation* $X \to \mathrm{Spec}(A)$ *of* $X_0 = \mathrm{Spec}(B_0)$ *corresponds homotopically to a G-equivariant maps of dglas:* $D(A) \to \mathrm{Der}_k(B'_0, B'_0)$

Proof. For $A \in \mathbf{dgArt}_k$, let $\mathfrak{g}^A_* := D(A)$ and let $\phi_A : A \to B$ be an object in $\mathrm{Def}^G_{X_0}(A)$. First, by Theorem 3.4.1, it corresponds to a morphism of dglas

$$\Phi_A : \mathfrak{g}^A_* \to \mathrm{Der}_k(B'_0, B'_0).$$

We shall prove that Φ_A is G-equivariant with respect to the fixed G-action on $\mathrm{Der}_k(B'_0, B'_0)$ and the G-action on $\mathfrak{g}^A_*$, induced from the G-action on A by the functor D. Indeed, let

$$\begin{array}{ccc} A' & \xrightarrow{\phi_{A'}} & B \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h} \\ A & \xrightarrow[\phi_A]{} & B \end{array}$$

be a commutative diagram in $\mathbf{cdga}_k$ where g and h are isomorphisms. Let also f_A and $f_{A'}$ be the functor f corresponding to $\mathfrak{g}_*^A$ and $\mathfrak{g}_*^{A'}$, respectively in Theorem 3.2.5. By Koszul duality, we have a morphism

$$D(g) : \mathfrak{g}_*^{A'} = D(A') \to \mathfrak{g}_*^A = D(A).$$

Note as well that we have a canonical morphism

$$A' \to A' \otimes_A B.$$

Thus, $A' \otimes_A B \in \mathbf{Mod}_{A'}^{cn}$ so that $f_{A'}(A' \otimes_A B)$ is a representation of $\mathfrak{g}_*^{A'}$. The functoriality of $f_{A'}$ tells us exactly that the differential graded vector space of $f_{A'}(A' \otimes_A B)$ is nothing but $f_A(B)$ with action of $\mathfrak{g}_*^{A'}$ given by the morphism

$$D(g) : \mathfrak{g}_*^{A'} \to \mathfrak{g}_*^A.$$

More precisely, if we let

$$\alpha : \ \mathfrak{g}_*^A \otimes_k f_A(B) \to f_A(B)$$

be the representation of $\mathfrak{g}_*^A$ corresponding to the arrow $\phi_A : \ A \to B$ then

$$\begin{aligned} \beta : \mathfrak{g}_*^{A'} \otimes_k f_A(B) &\to f_A(B) \\ x \otimes v &\mapsto \alpha(D(g)(x) \otimes v) \end{aligned}$$

is the representation of $\mathfrak{g}_*^{A'}$ corresponding to $f_{A'}(A' \otimes_A B)$. Now, let

$$\alpha' : \ \mathfrak{g}_*^{A'} \otimes_k f_{A'}(B) \to f_{A'}(B)$$

be the representation of $\mathfrak{g}_*^{A'}$ corresponding to the arrow $\phi_{A'} : \ A' \to B$. Then h corresponds exactly to a morphism of two representations α' and β of $\mathfrak{g}_*^{A'}$, which will be denoted by $\rho_h : \ \alpha' \to \beta$. In other words, for $x \in \mathfrak{g}_*^{A'}$ and $v \in f_A(B)$, we have

$$\rho_h(\alpha'(x \otimes v)) = \beta(x \otimes \rho_h(v))$$

which is the same as

$$\rho_h(\alpha'(x \otimes v)) = \alpha(D(g)(x) \otimes \rho_h(v)).$$

Now, let $\mu : G \to \mathrm{Aut}_k(A)$ and $\nu : G \to \mathrm{Aut}_k(B)$ be the G-actions on A and B, respectively. By the functoriality of D and that of f_A, we have induced actions on $\mathfrak{g}_*^A$ and $f_A(B)$ given by

$$\begin{aligned} \overline{\mu} : G &\to \mathrm{Aut}_k(\mathfrak{g}_*^A) \\ g &\mapsto D(\mu(g)) \end{aligned}$$

and

$$\begin{aligned} \overline{\mu} : G &\to \mathrm{Aut}_k(f_A(B)) \\ g &\mapsto \rho_{\nu(g)}, \end{aligned}$$

respectively. Consider the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\phi_A} & B \\ {\scriptstyle \mu(g)}\uparrow & & \uparrow{\scriptstyle \nu(g)} \\ A & \xrightarrow[\phi_A]{} & B \end{array}$$

By the previous paragraph, we have

$$\rho_{\nu(g)}\big(\alpha(x \otimes v)\big) = \alpha\Big(D\big(\mu(g)\big)(x) \otimes \rho_{\nu(g)}(v)\Big),$$

for $x \in \mathfrak{g}_*^A$ and $v \in f_A(B)$. Or equivalently,

$$\rho_{\nu(g)} \circ \Phi_A(x) = \Phi_A(D(\mu(g))x) \circ \rho_{\nu(g)}$$

which is the same as

$$\Phi_A(D(\mu(g))x) = \rho_{\nu(g)} \circ \Phi_A(x) \circ \rho_{\nu(g)}^{-1}$$

But $f_A(B)$ is nothing but B_0' so that

$$\rho_{\nu(g)} = g$$

for all $g \in G$. So,

$$\Phi_A(D(\mu(g)(x)) = g.\Phi_A(x).$$

This precisely means the following diagram

$$\begin{array}{ccc} \mathfrak{g}_*^A & \xrightarrow{\Phi_A} & \mathrm{Der}_k(B_0', B_0') \\ {\scriptstyle D(\mu(g))}\downarrow & & \downarrow{\scriptstyle g.} \\ \mathfrak{g}_*^A & \xrightarrow[\Phi_A]{} & \mathrm{Der}_k(B_0', B_0') \end{array} .$$

Therefore, Φ_A is G-equivariant. This ends the proof. □

Finally we deal with X_0 a general algebraic scheme. Let $\mathbb{T}_{X_0/k}$ be its tangent complex and $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ be its derived global section equipped with the G-action in Remark 3.4.2.

Theorem 3.4.5. *For $A \in \mathbf{dgArt}_k$, a G-equivariant derived deformation $X \to \mathrm{Spec}(A)$ of X_0 corresponds homotopically to a G-equivariant maps of dglas: $D(A) \to \mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$*

Proof. The proof is just an adaptation of the one of Theorem 3.4.4 and that of Theorem 3.4.2, in the equivariant case. □

The following theorem generalizes the result of the existence of equivariance G-structure on versal deformations of algebraic schemes, obtained by D.S. Rim, in the derived setting.

Theorem 3.4.6. *If X_0 is either an affine scheme with at most isolated singularities or a complete algebraic variety and G is a linearly reductive group acting algebraically on X_0, there exists a G-equivariant structure on the semi-prorepresentable dg-object of Def_{X_0}. Consequently, the classical functor of G-equivariant deformations $\pi_0(\mathrm{Def}_{X_0})$ of X_0 has a G-equivariant semi-universal element.*

Proof. Since G acts algebraically on X_0 then the tangent complex $\mathbb{T}_{X_0/k}$ of X_0 is a complex of G-equivariant quasi-coherent $\mathcal{O}_{X_0}$-modules (cf. [29] for the notion of G-equivariant sheaves). Let us denote the category of G-equivariant quasi-coherent $\mathcal{O}_{X_0}$-modules by $\mathrm{QCoh}^G_{X_0}$ and the corresponding derived category by $\mathbf{D}(\mathrm{QCoh}^G_{X_0})$. By [29, Lemma 2.13], the derived global section of $\mathbb{T}_{X_0/k}$ can be calculated in $\mathbf{D}(\mathrm{QCoh}^G_{X_0})$. Using [29, Proposition 2.16] for the structure morphism $X_0 \to k$, we have that

$$\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k}) \in \mathbf{D}(\mathrm{QCoh}^G_k) = \mathbf{D}(\mathrm{Rep}_k(G))$$

where $\mathrm{Rep}_k(G)$ is the category of rational representations of G and $\mathbf{D}(\mathrm{Rep}_k(G))$ is its associated derived category. Hence each component of $\mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ is a rational G-module.

For the sake of Theorem 3.3.2, Def_{X_0} is semi-prorepresentable by a pro-object K in $\mathbf{dgArt}_k$. Let $\mathfrak{k}_*$ be the corresponding dgla of K. By Lemma 3.3.2 and Theorem 3.3.3, there exists a compatible G-action on $\mathfrak{k}_*$ which is also versal in the sense mentioned therein. Equivalently, there exists a compatible G-action on K which is versal in the following sense. By Theorem 3.4.5, any G-equivariant derived deformation $X \to \mathrm{Spec}(A)$ of X_0 corresponds to a (non-homotopic) G-equivariant map of dglas: $QD(A) \to \mathbb{R}\Gamma(X_0, \mathbb{T}_{X_0/k})$ which then corresponds to a G-equivariant map of dglas $QD(A) \to \mathfrak{k}_*$. Finally, the last map gives rise to a G-equivariant map of cdgas $QK \to A$.

For the last statement, restricting our fmp on the category of local artinian rings $\mathbf{Art}_k$ and unwinding the definition of versality mentioned in the previous paragraph, we can see that $H^0(QK)$ is nothing but the base space of the G-equivariant semi-universal constructed by Rim in Theorem I. □

3.4.3 Equivariant deformations of complex compact manifolds

Let X_0 be a complex complex manifold and $\mathcal{T}_{X_0}$ be its holomorphic tangent bundle. Denote by $\mathcal{A}^{p,q}$ the sheaf of differential forms of type (p,q) and by $\mathcal{A}^{p,q}(\mathcal{T}_{X_0})$ the sheaf of differential forms of type (p,q) with values in $\mathcal{T}_{X_0}$. Let $\mathfrak{g}_*$ be the following differential graded Lie algebra

$$\Gamma(X_0, \mathcal{A}^{0,0}(\mathcal{T}_{X_0})) \xrightarrow{\bar{\partial}} \Gamma(X_0, \mathcal{A}^{0,1}(\mathcal{T}_{X_0})) \xrightarrow{\bar{\partial}} \Gamma(X_0, \mathcal{A}^{0,2}(\mathcal{T}_{X_0})) \xrightarrow{\bar{\partial}} \cdots$$

with the Lie bracket defined by

$$[\phi d\bar{z}_I, \psi d\bar{z}_J] = [\phi, \psi]' d\bar{z}_I \wedge \bar{z}_J$$

where $\phi, \psi \in \mathcal{A}^{0,0}(\mathcal{T}_{X_0})$ are vector fields on X_0, $[-,-]'$ is the usual Lie bracket of vector fields, $I, J \subset \{1, \ldots, n\}$ and $z_1, \ldots, z_n$ are local holomorphic coordinates. Note that $\mathfrak{g}_*$ is concentrated in degrees ≥ 0. It is well-known that deformations of X_0 is governed by this $\mathfrak{g}_*$. Furthermore if there is a reductive Lie group acting holomorphically on X_0, then $\mathfrak{g}_*$ receives naturally an induced linear G-action and any G-equivariant deformation of X_0 is controlled by $\mathfrak{g}_*$ equipped with this induced G-action (for a quick review of (equivariant) deformations of complex compact manifolds, we refer the reader to Chapter 2).

Now, we would like to recall the classical deformation functor $\mathrm{MC}_{\mathfrak{g}_*}$ associated to $\mathfrak{g}_*$, defined via the Maurer-Cartan equation (see [25, §6] for more details). We have two functors:

(1) The Gauge functor

$$G_{\mathfrak{g}_*} : \mathbf{Art}_{\mathbb{C}} \to \mathrm{Grp}$$
$$A \mapsto \exp(\mathfrak{g}_0 \otimes \mathrm{m}_A)$$

where m_A is the unique maximal ideal of A and Grp is the category of groupoids.

(2) The Maurer-Cartan functor $MC_{\mathfrak{g}_*} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets}$ defined by

$$MC_{\mathfrak{g}_*} : \mathbf{Art}_{\mathbb{C}} \to \mathrm{Grp}$$

$$A \mapsto \left\{ x \in \mathfrak{g}_1 \otimes \mathrm{m}_A \mid \overline{\partial} x + \frac{1}{2}[x, x] = 0 \right\}.$$

For each A, the gauge action of $G_{\mathfrak{g}_*}(A)$ on the set $MC_{\mathfrak{g}_*}(A)$ is functorial in A and gives an action of the group functor $G_{\mathfrak{g}_*}$ on $MC_{\mathfrak{g}_*}$. This allows us to define the quotient functor

$$\begin{aligned} \mathrm{MC}_{\mathfrak{g}_*} : \ & \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets} \\ & A \mapsto MC_{\mathfrak{g}_*}(A)/G_{\mathfrak{g}_*}(A), \end{aligned}$$

Let $\mathfrak{Def}_{X_0} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets}$ (resp. $\mathfrak{Def}^G_{X_0} : \mathbf{Art}^G_{\mathbb{C}} \to \mathbf{Sets}$) be the functor which associates to each local artinian k-algebra (resp. G-local artinian k-algebra) A, the isomorphism (resp. G-equivariant isomorphism) classes of flat proper morphisms of analytic spaces $X \to \mathrm{Spec}(A)$ with an isomorphism (resp. G-equivariant isomorphism)

$$X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{C}) \cong X_0.$$

The following is fundamental (cf. [25, Theorem V.55]).

Theorem 3.4.7. *There is an isomorphism*

$$\mathfrak{Def}_{X_0} \cong \mathrm{MC}_{\mathfrak{g}_*}$$

as functors of Artin rings.

On one hand, the classical deformation functor $\mathrm{MC}_{\mathfrak{g}_*}$ can be naturally extended to a formal moduli problem in Lurie's sense (cf. §3.2.6) via a simplicial version of the Maurer-Cartan equation (see [15] for such a construction). In other words, we have a fmp

$$\mathfrak{MC}_{\mathfrak{g}_*} : \mathbf{dgArt}_{\mathbb{C}} \to \mathbf{SEns}$$

such that

$$\pi_0(\mathfrak{MC}_{\mathfrak{g}_*}) = \mathrm{MC}_{\mathfrak{g}_*}. \tag{3.4.1}$$

On the other hand, there is an equivalence

$$\mathfrak{MC}_{\mathfrak{g}_*} \to \mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*) \tag{3.4.2}$$

as fmps (cf. [21, §2]). Consequently, we can think of $\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*)$ as a natural extension of $\mathfrak{Def}_{X_0}$ in the derived world.

Theorem 3.4.8. *The fmp* $\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*)$ *is semi-prorepresentable. Consequently, the classical functor of deformations* $\mathfrak{Def}_{X_0}$ *has a formal semi-universal element.*

Proof. The first statement follows from the fact that $\mathfrak{g}_*$ is concentrated in degrees $[0, +\infty)$

and that all the cohomologies $H^i(\mathfrak{g}_*)$ are finite-dimensional vector spaces. The last statement is the immediate consequence of the following chain of isomorphisms

$$\mathfrak{Def}_{X_0} \cong \mathrm{MC}_{\mathfrak{g}_*} \cong \pi_0(\mathfrak{MC}_{\mathfrak{g}_*}) \cong \pi_0(\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(-), \mathfrak{g}_*))$$

and of Remark 3.3.3. □

Remark 3.4.3. The above theorem gives an algebraic approach to produce a formal solution to the deformation problem of complex compact manifolds. The base of the formal semi-universal element can be thought of as a formal Kuranishi space in the classical sense. However, the hardest part is always to ensure that among the formal solutions, there exists at least a convergent one.

Finally, we allow the group action to rejoin the game. The rest of this section is devoted to proving the existence of a formal G-equivariant semi-universal element for the functor $\mathfrak{Def}^G_{X_0}$. Recall that $\mathfrak{g}$ has naturally a G-action induced from the one on X_0.

Remark 3.4.4. In order to approximate $\mathfrak{g}_*$, we can not apply directly Lemma 3.3.2 as in the algebraic case since each component of $\mathfrak{g}_*$ is not a rational G-module, in general. This is the reason why we shall make use of a G-equivariant version of Hodge decomposition for complex compact manifolds.

Lemma 3.4.1.

$$\mathfrak{g}_* = \mathrm{colim}_i \; \mathfrak{g}(i)_*$$

where

(i) each $\mathfrak{g}(i)_k$ is finite-dimensional,

(ii) $\mathfrak{g}(i)_*$ is cohomologically concentrated in $[0, +\infty)$,

(iii) each $\mathfrak{g}(i)_*$ carries a G-action and the colimit of these G-actions gives back the inital G-action on $\mathfrak{g}_*$.

Proof. We treat the case when G is a compact Lie group first then the case when G is a reductive complex Lie group will be deduced by a complexification argument.

Since G is compact, we can impose a G-invariant Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{T}_{X_0}$ by means of Weyl's trick. Therefore, we have a G-invariant metric on $\mathfrak{g}_p = \Gamma(X_0, \mathcal{A}^{0,p}(\mathcal{T}_{X_0}))$. As usual, we find the formal adjoint $\overline{\partial}^*$ of $\overline{\partial}$. Since G acts on X_0 by biholomorphisms then the operator $\overline{\partial}$ is G-equivariant. By the adjoint property together with the fact that the imposed metric is G-invariant, we also have that $\overline{\partial}^*$ is G-equivariant. Hence, so is the Laplacian $\square := \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*$. As a matter of fact, Hodge theory provides us an orthogonal decomposition

$$\mathfrak{g}_p = \mathcal{H}^{0,p} \bigoplus \square \mathfrak{g}_p \tag{3.4.3}$$

as representations of G and two linear operators:

(a) The Green operator $\mathcal{G} : \mathfrak{g}_p \to \Box\mathfrak{g}_p$,

(b) The harmonic projection operator $H : \mathfrak{g}_p \to \mathcal{H}^{0,p}$,

where $\mathcal{H}^{0,p}$ is the vector space of all harmonic vector $(0,p)$-forms on X_0 (this space can also be canonically identified with $H^p(X_0, \mathcal{T}_{X_0})$ as G-modules, such that for all $v \in \mathfrak{g}_p$, we have

$$v = Hv + \Box\mathcal{G}v. \tag{3.4.4}$$

Therefore, we can deduce the following decomposition.

$$\mathfrak{g}_p = \mathcal{H}^{0,p} \oplus \bar{\partial}\mathfrak{g}_{p-1} \oplus \bar{\partial}^*\mathfrak{g}_{p+1} \tag{3.4.5}$$

as G-modules.

We aim to construct by induction a sequence of dglas with G-actions

$$0 = \mathfrak{g}(0)_* \to \mathfrak{g}(1)_* \to \mathfrak{g}(i)_* \to \cdots$$

equiped with G-equivariant maps $\phi(i) :\ \mathfrak{g}(i)_* \to \mathfrak{g}_*$ such that

$$\mathfrak{g}_* = \underrightarrow{\operatorname{colim}}\, \mathfrak{g}(i)_*$$

and that for all $i \geq 0$

$$H^n(\mathfrak{g}(i)_*) \simeq 0, \forall n \leq -1.$$

For each $n \in \mathbb{Z}$, set $V_n := \mathcal{H}^{0,n}$. We think of V_* as a differential graded vector space with the trivial differential. Let $\mathfrak{g}(1)_*$ denote the free differential graded Lie algebra generated by V_* and let $\phi(1) :\ \mathfrak{g}(1)_* \to \mathfrak{g}_*$ be the canonical map. By construction, we have that $\phi(1)$ is G-equivariant and that

$$H^n(\mathfrak{g}(1)_*) \simeq 0, \forall n \leq -1$$

and that the inclusion $V_0 \to \mathfrak{g}(1)_0$ induces an isomorphism

$$V_0 \to H^0(\mathfrak{g}(1)_*).$$

Now, suppose that $i \geq 1$ and that we have built a G-equivariant map $\phi(i) :\ \mathfrak{g}(i)_* \to \mathfrak{g}_*$ extending $\phi(1)$. Then $\phi(i)$ induces a surjection on cohomologies

$$\theta(i) :\ H^*(\mathfrak{g}(i)_*) \to H^*(\mathfrak{g}_*).$$

For each n, since n^{th}-component $\ker(\theta)_n$ of $\ker(\theta)$ is a G-invariant finite-dimensional sub-vector space of $H_n(\mathfrak{g}(i)_*)$, we choose a collection of cycles $\{x_\alpha^n\}_{\alpha \in A_n} \subset \mathfrak{g}(i)_n$ whose images form a basis for $\ker(\theta)_n$, where A_n is a finite index set. For each x_α^n, we choose $z_\alpha \in \mathfrak{g}_{n-1}$

such that

$$\phi(i)(x_\alpha^n) = \bar{\partial} z_\alpha^{n-1}. \tag{3.4.6}$$

Now, setting $y_\alpha^{n-1} = \overline{\partial}^* \mathcal{G} \phi(i)(x_\alpha^n)$, we have that

$$\begin{aligned}
\phi(i)(x_\alpha^n) &= \Box \mathcal{G} \phi(i)(x_\alpha^n), \text{ since } \phi(i)(x_\alpha^n) \text{ has no hamornic part by (3.4.6)},\\
&= (\overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*) \mathcal{G} \phi(i)(x_\alpha^n)\\
&= \overline{\partial}^* \overline{\partial} \mathcal{G} \phi(i)(x_\alpha^n) + \overline{\partial}\overline{\partial}^* \mathcal{G} \phi(i)(x_\alpha^n)\\
&= \overline{\partial}^* \mathcal{G} \overline{\partial} \phi(i)(x_\alpha^n) + \overline{\partial}\overline{\partial}^* \mathcal{G} \phi(i)(x_\alpha^n), \text{ since } \mathcal{G} \text{ commutes with } \overline{\partial},\\
&= \overline{\partial}\overline{\partial}^* \mathcal{G} \phi(i)(x_\alpha^n) \text{ by (3.4.6)},\\
&= \overline{\partial} y_\alpha^{n-1}.
\end{aligned}$$

Let T_{n-1} be the vector space generated by y_α^{n-1}'s. Since both $\overline{\partial}^*$ and $\mathcal{G}$ are G-equivariant (see Lemma 2.4.1) then T_{n-1} is a finite-dimensional sub-representation of G. Let W_{n-1} be a vector space identical to T_{n-1}, as G-representations. Let Y_α^{n-1}'s be elements in W_{n-1} corresponding to y_α^{n-1}'s. Finally, for each n, define $\mathfrak{g}(i+1)_*$ to be the differential graded Lie algebra obtained from $\mathfrak{g}(i)_*$ by freely adding a basis of W_{n-1} (in degrees $n-1$) such that

$$\overline{\partial}(Y_\alpha^{n-1}) = x_\alpha^n$$

and let $\phi(i+1): \mathfrak{g}(i+1)_* \to \mathfrak{g}_*$ denote the unique extension of $\phi(i)$ satisfying

$$\phi(i+1)(Y_\alpha^{n-1}) = y_\alpha^{n-1}.$$

It is easy to see that $\overline{\partial}$ and $\phi(i+1)$ defined in this way are G-equivariant. The rest of the proof now is identical the the one given in Lemma 3.3.1.

Finally, for G a general reductive complex Lie group, let K be its maximal compact subgroup whose complexification is exactly G. Then by the case of compact groups, we have the limit

$$\mathfrak{g}_* = \operatorname{colim}_i \mathfrak{g}(i)_*$$

satisfying (i), (ii) and (iii) in Lemma 3.4.1 for K. Complexifying all the maps $\phi(i)$ will give the desired colimit for $\mathfrak{g}_*$. □

Theorem 3.4.9. *There exists a G-equivariant structure on the semi-prorepresentable object of* $\mathrm{Map}_{\mathbf{Lie}_\mathbb{C}}(D(-), \mathfrak{g}_*)$ *with respect to the action on $\mathfrak{g}_*$, induced by the fixed one on X_0. Consequently, the classical functor of G-equivariant deformations $\mathfrak{Def}_{X_0}^G$ of X_0 has a formal G-equivariant semi-universal element.*

Proof. For the sake of Theorem 3.3.2, $\mathrm{Map}_{\mathbf{Lie}_\mathbb{C}}(D(-), \mathfrak{g}_*)$ is semi-prorepresentable by a pro-object K in $\mathbf{dgArt}_k$. Let $\mathfrak{k}_*$ be the corresponding dgla of K. By Lemma 3.4.1 and Theorem 3.3.3, there exists a compatible G-action on $\mathfrak{k}_*$ which is also versal in the sense mentioned therein. Equivalently, there exists a compatible G-action on K which is versal in the following sense. For each $A \in \mathbf{dgArt}_{\mathbb{C}}$, denote by $Q(A)$ any cofibrant replacement of A. Then any (non-homotopic) G-equivariant map of dgla: $QD(A) \to \mathfrak{g}_*$ which then corresponds to a G-equivariant map of cdgas from $QK \to A$. Note also that $H^0(QK)$ is a pro-object in $\mathbf{Art}_{\mathbb{C}}^G$.

For the last statement, we claim that $\mathfrak{Def}_{X_0}^G$ is semi-prorepresentable by $H^0(QK)$ in the sense that

(a) the morphism of functors $\mathrm{Hom}_{\widehat{\mathbf{Art}}_k^G}(H^0(QK), -) \to \mathfrak{Def}_{X_0}^G$ defined by $\hat{u}$ is surjective where $\hat{u} \in \mathfrak{Def}_{X_0}^G(H^0(QK))$,

(b) $\mathrm{Hom}_{\widehat{\mathbf{Art}}_k^G}(H^0(QK), k[\epsilon]/(\epsilon^2)) \to \mathfrak{Def}_{X_0}^G(\mathbb{C}[\epsilon]/(\epsilon^2))$ is bijective

(cf. Remark 3.3.3 above). Let $X \to \mathrm{Spec}(A)$ be an element of $\mathfrak{Def}_{X_0}^G$ where $A \in \mathbf{Art}_{\mathbb{C}}^G$. By Theorem 2.3.2, it corresponds to a G-equivariant map $\Phi_A : \mathrm{Spec}(A) \to \mathfrak{g}_1$ with respect to the action on $\mathfrak{g}_*$, induced by the fixed one on X_0 such that the following conditions are satisfied:

(i) $\Phi_A(0) = 0$,

(ii) $\Phi_A(a) + \frac{1}{2}[\Phi_A(a), \Phi_A(a)] = 0$ for all $a \in \mathrm{Spec}(A)$.

This is equivalent to a G-equivariant map $\phi_A :\ QD(A) \to \mathfrak{g}_*$ by Theorem 3.4.7, isomorphisms 3.4.1, 3.4.2. Hence, by the previous paragraph, we have that ϕ_A corresponds to a G-equivariant map of cdgas $\sigma_A : QK \to A$. However, A is concentrated in degree 0. Thus, σ_A can be given as a G-equivariant map $H^0(QK) \to A$. Hence (a) is proved. Finally, (b) can be deduced from the fact that

$$\begin{aligned}\mathrm{Hom}_{\widehat{\mathbf{Art}}_k}(H^0(QK), k[\epsilon]/(\epsilon^2)) &= \pi_0(\mathrm{Map}_{\mathbf{Lie}_{\mathbb{C}}}(D(k[\epsilon]/(\epsilon^2)), \mathfrak{g}_*)) \\ &= \mathrm{Hom}_{\mathbf{Lie}_{\mathbb{C}}}(D(k[\epsilon]/(\epsilon^2)), \mathfrak{g}_*).\end{aligned}$$

This completes the proof. □

Remark 3.4.5. Once again a formal version of the existence G-equivariant Kuranishi space shown in Chapter 2 is given by a purely algebraic method except the step in which we used Hodge decomposition. This reflects a natural phenomenon when dealing with analytic deformations of geometric objects, i.e. a formal solution is always somewhat easy to produce.

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