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Construction d'une version Arakelov d'un groupe faible de cobordisme arithmétique

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Abstract

In this thesis we construct a weak group of arithmetic cobordism in the context of Arakelov geometry. We introduce weak versions of arithmetic K-theory and arithmetic Chow groups, that give rise to the notion of oriented homological theory of arithmetic type. We then build a universal such homological theory, and prove its main structural features.

Keywords Cobordism, Arakelov geometry, arithmetic intersection theory, arithmetic K-theory, oriented homological theory

Abstract

Dans cette thèse nous construisons un groupe faible de cobordisme arithmétique dans le contexte de la géométrie d'Arakelov. Nous introduisons des versions faibles des groupes de K-théorie arithmétique et de Chow arithmétique, et en dégageons une notion de théorie homologique orientée de type arithmétique. Nous construisons alors un groupe universel parmi ces théories homologiques et prouvons ses principales propriétés structurelles.

Mots-Clés Cobordisme, géométrie d'Arakelov, théorie de l'intersection arithmétique, K-théorie arithmétique, théorie cohomologique orientée.

Introduction

In this thesis we propose to give a generalization of the algebraic cobordism groups constructed by Levine and Morel in [LM07] in the context of Arakelov geometry.

The context

Arakelov theory is a refinement of arithmetic algebraic geometry destined to make it possible to use the tools of classical algebraic geometry in Diophantine problems. Perhaps the easiest way to grasp the essence of the theory is to look at the case of so-called arithmetic surfaces, which was the case studied first by Arakelov in [Ara75] and [Ara74].

The main idea is to compactify an arithmetic curve over $\operatorname{Spec}\mathbb{Z}$ (or $\operatorname{Spec}\mathcal{O}_k$ where k is a number field), which is a integral projective flat regular scheme over $\operatorname{Spec}\mathbb{Z}$ of relative dimension 1, by adding the data of the complex points $X(\mathbb{C})$ which is a compact Riemann surface. Arakelov proved that you could define an intersection pairing that satisfied all the properties that you could expect from a counterpart of the classical intersection pairing of divisors. In order to do this he added to the classical number of intersection of two divisors $D = \sum n_i[x_i]$ and $D' = \sum m_i[y_i]$ an analytic part defined by

$$-\sum \log g(D(\mathbb{C}), D'(\mathbb{C}))$$

where g is the so called Green-Arakelov function on $X(\mathbb{C}) \times X(\mathbb{C})$, and $D(\mathbb{C})$ (resp. $D'(\mathbb{C})$) the complex points of D (resp. D') through the different embeddings of k in $\mathbb{C}$. Arakelov also defined a notion of degree for such divisors with value in $\mathbb{R}$.

Later, Faltings extended to this context classical tools of intersection theory on algebraic curves such as the Hodge index theorem, Noether formula, or a Riemann-Roch formula, in his seminal paper [Fal84].

It was then natural to investigate how one could extend those results to higher dimensional varieties, but this posed tremendous technical difficulties.

In [GS], Gillet and Soulé define an arithmetic intersection theory for arbitrary arithmetic varieties i.e integral, projective, regular schemes flat over $\operatorname{Spec}\mathcal{O}_k$. They define an arithmetic cycle to be a pair $[Z, g]$ where Z in cycle over X and g is a Green current for $X(\mathbb{C})$, that is any real current g, of $(\bullet, \bullet)$ type[1], satisfying $F_\infty^*(g^{\{p\}}) = (-1)^p g^{\{p\}}$ where $g^{\{p\}}$ denotes the (p,p)-type part of g, such that

$$dd^c g + \delta_Z$$

is a smooth form. They define *arithmetic Chow groups*, $\widehat{\mathrm{CH}}(X)$ to be classes of arithmetic cycles modulo the principal divisors $\widehat{\mathrm{div}}(f)$ for any f rational function over a sub-variety V of X, and modulo $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$, where $\widehat{\mathrm{div}}(f) = [\mathrm{div}(f), -\log|f|^2]$, here $\log|f|^2$ denotes the current defined by integration over $V^{\mathrm{ns}}(\mathbb{C})$ of the locally integrable function $\log|f|^2$ against any smooth compactly supported form of appropriate type.

These arithmetic Chow groups contain Arakelov Chow groups, $\mathrm{CH}(\overline{X})$ as a direct factor. They are defined via the choice of a Kähler metric over X, invariant under conjugation. One can define an ω map, determined by the following equality

$$\omega(Z, g) = dd^c g + \delta_Z$$

which is a closed real smooth form, and set $\mathrm{CH}(\overline{X}) = \bigoplus_p \omega^{-1}(\mathcal{H}^{p,p}(X))$ where $\mathcal{H}^{p,p}(X)$ is the space of real harmonic forms of (p,p) type, satisfying $F_\infty^*(w) =$

[1] meaning that it is the sum of real currents of type (p,p) for various p's

$(-1)^p w$. These groups are the natural generalization of the initial theory constructed by Arakelov, on Riemann surfaces, but have the fundamental flaw of not being a sub-algebra of $\widehat{\mathrm{CH}}(X)$ which explains why we need to define those bigger groups $\widehat{\mathrm{CH}}$.

Gillet and Soulé define an intersection pairing

$$\widehat{\mathrm{CH}}(X)_{\mathbb{Q}} \otimes \widehat{\mathrm{CH}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}(X)_{\mathbb{Q}}$$

by defining the *star-product* of two Green currents for two arithmetic cycles.

A key difference between the geometric and arithmetic case lies in the functoriality properties of the $\widehat{\mathrm{CH}}$ ring, as the datum of a Green current for a cycle Z depends notably on the ambient variety, it is not possible to define a direct image morphism for closed immersions, therefore arithmetic Chow groups are covariant only with respect to projective morphisms that are generically smooth i.e the induced morphism $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ need to be smooth. This makes it possible to define the *arithmetic degree* of an arithmetic cycle with value in $\widehat{\mathrm{CH}}(\operatorname{Spec}\mathbb{Z}) \simeq \mathbb{R}$.

On the other hand Gillet and Soulé developed the notion of arithmetic K-theory in [GS86] and [GS92], let us describe their construction.

Recall that a hermitian vector bundle $\overline{E}$ over an arithmetic variety X, is a coherent locally free sheaf over X, such that E^{an} the analytified vector bundle over $X(\mathbb{C})$ is equipped with a hermitian metric invariant under complex conjugation. An exact sequence of such hermitian bundles

$$0 \to \overline{E}' \xrightarrow{f} \overline{E} \xrightarrow{g} \overline{E}'' \to 0$$

is simply an exact sequence of the locally free sheaves over X. We say that this sequence is ortho-split if

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \to 0$$

is split and if the section g^{-1} of g maps E''^{an} isometrically as the orthogonal complement of $f(E'^{\mathrm{an}})$ in E'^{an}. In general it is possible to associate to any exact sequence of hermitian vector bundle, say $\mathcal{E}$, a smooth form up to $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$, called the secondary Bott-Chern form of $\mathcal{E}$ satisfying

$$dd^c\,\widetilde{\mathrm{ch}}(\mathcal{E}) = \mathrm{ch}(\overline{E}) - \mathrm{ch}(\overline{E}') - \mathrm{ch}(\overline{E}'')$$

which is natural with respect to holomorphic map and that vanishes when $\mathcal{E}$ is ortho-split, the construction of that form can be found for instance in [GS90], but is due to Bott and Chern [BC65].

Let X be an arithmetic variety, Gillet and Soulé set $\widehat{K}_0(X)$ to be the free abelian group built on symbols $[\overline{E}, \omega]$ where $\overline{E}$ is a (isometry class of) hermitian vector bundle over $X$and ω a real smooth form up to an element in $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$, modulo the relations

$$[E, 0] = [\overline{E}', 0] + [\overline{E}'', 0] + \widetilde{\mathrm{ch}}(\mathcal{E})$$

for

$$\mathcal{E} : 0 \to \overline{E}' \xrightarrow{f} \overline{E} \xrightarrow{g} \overline{E}'' \to 0$$

every short exact sequence of hermitian bundles.

These arithmetic $\widehat{K}$-theory groups are contravariant with respect to flat equidimensional morphism, but it turned out to be much harder to define a direct image for projective morphisms, even when restricted to the class of generically smooth ones.

We will describe their construction at length in the thesis, but let's already give the gist of it.

In the case of a generically smooth submersion, $\pi : X \to B$, endowed with the structure of a Kähler fibration, and a hermitian bundle, $\overline{E}$ that is π_*-acyclic, they set $\pi_*[E, 0]$ to be

$$[\pi_* \overline{E}^{L^2}, T(\overline{E}, \pi_* E^{L^2}, h_{X/B})]$$

where $T(\overline{E}, \pi_* E^{L^2}, h_{X/Y})$ is the higher analytic torsion form defined by Bismut and Köhler in [BK92] satisfying the Riemann-Roch equation

$$\frac{i}{2\pi} \partial\overline{\partial} T(\overline{E}, \pi_* E^{L^2}, h_{X/Y}) = -\operatorname{ch}(\pi_* \overline{E}^{L^2}) + \int_{X(\mathbb{C})/B(\mathbb{C})} \operatorname{ch}(\overline{E}) \operatorname{Td}(\overline{T_{X/B}})$$

and whose 0-th component is the classical holomorphic torsion defined by Ray and Singer, [RS73] and studied by Quillen [Qui85] among others.

An important feature of the theory and a critical discrepancy from the classical geometric case is the fact that this direct image depends on some of the choices made for the construction, namely the Kähler structure on the submersion. This dependency was made explicit by Bismut and Zhang who proved the so-called anomaly formulas in [BZ92].

Continuing the analogy with the geometrical case, Gillet and Soulé built a theory of arithmetic characteristic classes in [GS90], the importance of the role of a *first arithmetic Chern class* and of a Riemann-Roch formula for it had already been highlighted by the works of Faltings [Fal84] and Vojta [Voj91].

The main step in that construction is to construct a splitting principle for hermitian vector bundles, Gillet and Soulé manage to do so, by computing the Arakelov Chow groups of the Grassmanian over $\operatorname{Spec} \mathbb{Z}$, and by imposing several natural conditions on the putative arithmetic characteristic classes, they later show that this determines uniquely these arithmetic characteristic classes.

Once this general framework was set up it was natural to ask for a general Riemann-Roch formula, computing the difference between

$$\widehat{\operatorname{ch}}(\pi_* \overline{E})$$

and

$$\pi_*(\widehat{\operatorname{ch}}(\overline{E}) \widehat{\operatorname{Td}}(\overline{T_\pi}))$$

for a smooth submersion endowed with a Kähler fibration structure.

The question of addressing the case of a closed immersion has been treated in numerous ways, since then. In [BGS], Bismut, Gillet and Soulé study the behavior of the direct image in $\widehat{K}$-theory for a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\quad i \quad} & X \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle g} & \\ & B & \end{array}$$

where f and g are both generically smooth, and i is a regular immersion.

A more direct attempt has been made by Zha [Zha99], and then later by Burgos, Freixas and Litcanu [BGFiML14] and [BGL10]. In every case, they had to modify the groups $\widehat{\mathrm{CH}}$ and $\widehat{K}$ to *weaker* versions in order to be able to define a general push forward for projective morphisms, and in any case a thorough study of the behavior of the analytic torsion through immersions had to be done, and this was made possible by deep results of Bismut such as [BL91].

The situation was totally clarified by the landmark papers of Burgos, Freixas and Litcanu [BGFiML14, BGFiML12, GiML12a].

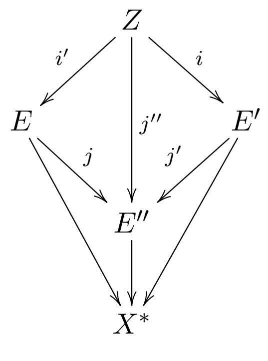
They show that we can define an arithmetic analog of derived categories by defining the notion of hermitian structure on the derived category of coherent sheaves on a projective algebraic variety, the arithmetic derived category on X consisting complexes of metric hermitian coherent sheaves together with an element of the *Deligne complex* on X. They show that one can define a general push forward in this setting, and that this push-forwards naturally gives rise to a push forward in a modified version of $\widehat{K}$-theory, with forms taken in the Deligne complex via an Euler-Poincaré map, in happy concord with the geometric case. Moreover they classify all the possible choices for these direct images and show that they're parametrized by the choice of the additive R-genus appearing in the Riemann-Roch formula.

The goal of this thesis was to add up the cobordism to this picture.

The notion of (unoriented) cobordism was studied mainly by Thom. Two smooth real compact manifolds of dimension n without boundaries are said to be *cobordant* if their disjoint union is the boundary of a compact manifold of dimension $n+1$. This defines an equivalence relation on the class of real compact manifolds and the quotient is denoted by $N_\bullet$. The disjoint sum, the Cartesian product, and the dimension confer a structure of graded ring to $N_\bullet$.

The structure of this ring was computed by Thom in [Tho54]. He proved that $N_\bullet$ is isomorphic to a polynomial algebra over $\mathbb{F}_2$, and also that the N_p where isomorphic to the p-th stable homotopy group of the *Thom Spectrum MO*. Several versions of the cobordism were defined since then.

We will be interested firstly in complex cobordism. For the obvious reason that a complex manifold is always even dimensional as e real one, it was not possible to adopt the previous definition to the complex case. Milnor in [Mil60], proposed the following definition for the complex cobordism ring $U_\bullet$. Recall that an oriented complex structure on a proper morphism of complex manifolds $f : Z \to X$ is a decomposition of $f^* : Z \to X^*$ into an immersion of Z into a complex bundle E over X^* equipped with a complex structure on $N_{Z/E}$ and followed by the structural morphism $E \to X^*$ where $X^* = X$ is the relative dimension of f is even, and $X^* = X \times \mathbb{R}$ if f is of odd relative dimension. Two such decompositions (Z, X, E, N) and (Z, X, E', N') are equivalent if we can find a "roof" (Z, X, E'', N'') such that we have a commutative diagram



such that j and j' are isotopic and that N'' induces the the complex structure on both N and N'.

The ring $U_\bullet$ is then defined to be the quotient of all classes of oriented complex proper morphisms modulo the cobording relation defined to be the following: two oriented morphism $g_0 : Z \to X$ and $g_1 : Z' \to X$ of complex manifolds are said to be cobordant if we can find a proper oriented complex morphism $V \xrightarrow{f} X \times \mathbb{R}$ transverse to the inclusions of X as both the fiber over 0 and 1 of f, and such that $f_0 : V_0 \to X$ and $f_1 : V_1 \to X$ are equivalent to g_0 and g_1 as oriented complex morphisms. In other words if g_0 and g_1 are both fibers of a family of proper oriented complex maps parametrized by the real line.

The set $U_\bullet$ can be given a natural structure of graded ring, and Milnor in [Mil60] proved that $U_\bullet(pt)$ is isomorphic to a polynomial algebra over $\mathbb{Z}$ and that after tensorization by $\mathbb{Q}$ that algebra was isomorphic to $\mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, ...]$ the polynomial algebra generated by the classes of projective spaces.

Inspired by some of the ideas of Grothendieck, Quillen refined Milnor's computations, and proved in [Qui71] that $U_\bullet(pt)$ was isomorphic to the Lazard ring $\mathbb{L}$ classifying the commutative 1-dimensional formal group laws. He proves that the complex cobordism ring is the universal (co)homological oriented theory over the category of differentiable manifolds.

This version of complex cobordism made it possible for Levine and Morel to define the analog of cobordism in algebraic geometry in [LM07]. In this paper they define the notion of Borel-Moore functor over an admissible category of schemes, that we will take to be the category of smooth quasi-projective schemes over a field k, roughly speaking this is an additive functor endowed with projective push forwards, smooth equidimensional pull-backs and first Chern operators that satisfy compatibility conditions modeled on those satisfied by CH-theory (see further on for the precise definitions), among such functors they distinguish the ones of *geometric type* satisfying three additional properties

1. nilpotence of the action of the first Chern class operator.

2. compatibility of the direct image via a smooth section of a line bundle and action of the first Chern operator.

3. the formula for the action of the first Chern class of a tensor product should be given by a formal group law.

With this in mind they build up a universal such functor of geometric type, denoted Ω, and prove that it is indeed a (co)homological oriented theory, at least when the base field k admits a resolution of singularities[2] by proving that it satisfies a Leray-Hirsh type formula for $\Omega(\mathbb{P}(E))$, that Ω is homotopy invariant, and that for a line bundle, the first Chern operator factors through the image of $\Omega(D)$ where D is the divisor of a global smooth section of L (provided it exists). These conditions define a weak homology theory.

More precisely they prove that $\Omega(\mathbb{P}(E))$ is a free $\Omega(X)$-module generated by the powers of the first Chern class of the tautological line bundle over $\mathbb{P}(E)$, that the natural pull-back $\Omega(X) \xrightarrow{q^*} \Omega(V)$ induces an isomorphism for any torsor V over a vector bundle over X, and that there exists a fundamental localization exact sequence

$$\Omega(Z) \to \Omega(X) \to \Omega(U) \to 0$$

for any closed immersion $Z \to X$ of a quasi-projective algebraic schemes. This last point being the main technical tool of all the construction. The proof of this last

[2] meaning that all reduced algebraic schemes over k admit a resolution of singularities

fact is highly non trivial and relies essentially on the resolution of singularities and the weak factorization theorem for such a resolution.

Levine and Morel then set up to prove several comparison theorems with other (co)homological oriented theories such as Chow theory, or K-theory, or rather a graded version of the latter, and prove that $X \mapsto \mathrm{CH}(X)$ is the universal oriented homology theory with additive group law and that $X \mapsto K_0(X)[\beta, \beta^{-1}]$ is the universal oriented homology theory with multiplicative group law.

Moreover, they compute the structure of $\Omega(k)$ and just like for complex cobordism prove that

$$\Omega_\bullet(k) \simeq \mathbb{L}_\bullet$$

at least for fields admitting a resolution of singularities.

Let us quickly mention, although we won't be using those facts in the following, that Morel and Levine go much further in their theory.

They construct pull-backs for local complete intersection morphisms, generalizing the intersection product in both Chow and K-theory.

They give a different proof of Rost degree formula.

They complete the analogy between algebraic topology and algebraic geometry, by establishing the existence of a morphism

$$\Omega^\bullet(X) \to \mathrm{MGL}^{2\bullet,\bullet}(X)$$

which they conjecture to be an isomorphism, where MGL is the oriented homology theory associated to the *algebraic* Thom spectrum which is the analog in motivic homotopy theory of the Thom Spectrum MU of complex cobordism.

All these questions transposed in the Arakelov context deserve to be adequately treated, a task which is yet to be done[3].

For a more detailed discussion on these topics see [Loe03].

Let us now turn to a more detailed description of the contents of the paper.

Contents of the paper

In the first section, we introduce some specializations of certain notions defined by Burgos, Freixas and Litcanu, mostly coming from [BGFiML14]. The notion of metrized sheaf, and secondary forms associated to it was already existent in the literature for instance in [GS92], although we give a slightly different version of it, using the language of [BGFiML14], notably the notion of meager complex and of quasi-isometry. The reader familiar with [BGFiML14] won't find anything new, although some of our proofs are different. This language will make it easy for us to introduce the different notions of weak arithmetic homological theories.

The second section contains the cobordism theory *per se*. In order to mimic the functorial construction of the cobordism group of Levine and Morel, we need to have good functorial properties for our arithmetic objects. Therefore we introduce a weak version of arithmetic Chow groups $\widetilde{CH}(\overline{X})$ for an arithmetic variety $\overline{X}$, that is an algebraic variety together with a Kähler metric on its tangent bundle invariant by complex conjugation. Those groups were introduced by Zha, Burgos and Moriwaki independently, we prove that these groups are the prototype of what we call an *oriented Borel Moore functor of arithmetic type.*

[3] Let us hope that this paper will convince the reader of the richness of this still vastly unexplored subject and encourage them to explore those topics

We then review the theory of Bott-Chern singular currents, and of the Analytic torsion forms both essentially to Bismut and his collaborators, we make heavy use of the language defined in [BGFiML14] which makes the analogy between those two objects clear. We then introduce the notion of weak arithmetic $\breve{K}$-theory and prove that it is also an oriented Borel Moore of arithmetic type.

The parallel between Chow and K-groups show the particular place occupied by the Todd form. It appears that both those theories have a Todd form, but the Todd form in the case of arithmetic Chow theory is just 1, therefore it disappears from the classical presentation of the theory and the usual Todd form appears to be a specificity of $\widehat{K}$-theory.

We then proceed to construct a universal Borel Moore functor of arithmetic type. For this, we will need a universal Todd form, for various reasons we define a universal inverse Todd form which we denote $\mathfrak{g}$, we also introduce secondary forms associated to it.

This $\mathfrak{g}$ class will enable us to construct a *universal Bott-Chern singular current* for the immersion of a smooth divisor. The crucial observation is that in the case of Chow theory we have the following relation[4] relating the first Chern class and the direct image via the immersion of a divisor

$$i_*(1_{\overline{Z}}) = \widehat{c_1}(L)(1_{\overline{X}}) + a(\log \|s\|^2)$$

whereas in $\breve{K}$ theory this relations become

$$i_*(1_{\overline{Z}}) = \widehat{c_1}(L)(1_{\overline{X}}) + a(\log \|s\|^2 \operatorname{Td}(\overline{L})^{-1})$$

It is therefore natural to replace the Td^{-1} form by the most general form $\mathfrak{g}(\overline{L})$.

The formal group law giving the action of $\widehat{c_1}(\overline{L} \otimes \overline{M})$ in function of $\widehat{c_1}(\overline{L})$ and $\widehat{c_1}(\overline{M})$ imposes relations between the coefficient of $\mathfrak{g}$ and those of $F_{\mathbb{L}}$ the universal law group on Lazard ring. We show that this enables to relate $\mathfrak{g}$ to the universal logarithmic class defined by Hirzebruch in $U_\bullet$. In other words, the formal group law imposed what the Todd form should be and *vice versa*. This sheds lights on various constructions of classical Arakelov theory and especially explains why it is possible to define covariant arithmetic Chow groups on the category of algebraic varieties but that it is only possible to define covariant arithmetic $\breve{K}$ groups on arithmetic varieties, the difference being explained by the triviality of the Todd form in Chow theory but not in $\breve{K}$-theory.

We then proceed to a technical discussion about projective Borel-Moore functor, essentially destined to prove that there are no surprises in passing from the quasi-projective to the projective case for Borel-Moore functors of geometric type.

We can now prove the fundamental exact sequence

$$\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) \xrightarrow{a} \breve{\Omega}(X) \xrightarrow{\zeta} \Omega(X) \to 0$$

which fulfills the goal of building arithmetic cobordism as an extension of the geometric cobordism by the space of currents.

We then prove that an analog of the star product lies in the groups $\breve{\Omega}$ that gives back the star-product of Gillet-Soulé when mapped to arithmetic Chow theory, and we prove a universal anomaly formula that, here again, explains the differences between arithmetic Chow and K-theory (the anomaly term being 0 in Chow theory).

This enables us to compute the structure of $\breve{\Omega}(k)$. It can be given the structure of a commutative ring because, over a point, strong object and weak objects should

[4] under a technical meager condition

coincide. In doing so we prove some kind of universal Hirzebruch-Riemann-Roch formula for the $\mathfrak{g}$ class, which is a reflexion of the isomorphism

$$U \simeq \mathbb{L}$$

this formula is the key fact that ensures that the groups $\breve{\Omega}(X)$ have a natural $\breve{\Omega}(k)$-module structure. The explicit description that we then give of $\breve{\Omega}(k)$ seems to fit perfectly in the general framework of Arakelov theory.

Finally we prove the existence of different arrows form $\Omega_{\mathbb{Z}}$ to $\widetilde{CH}$ and $\breve{K}$ and make explicit the notion of Borel-Moore functor of arithmetic type.

Notations and conventions

Throughout all the paper k will be a number field. If X is a complex manifold we set $A_{\mathbb{R}}^{(p,p)}(X)$ to be the set of smooth real forms over X of (p,p) type satisfying $F_\infty^*(w) = (-1)^p w$, the notation $D_{\mathbb{R}}^{(p,p)}(X)$ will represent the space of real currents of (p,p)-type, satisfying $F_\infty^*(\eta) = (-1)^p \eta$, and $\widetilde{A}_{\mathbb{R}}^{(p,p)}(X)$ will be $A_{\mathbb{R}}^{(p,p)}(X)/(\operatorname{im}\partial + \operatorname{im}\overline{\partial})$, in the same way $D_{\mathbb{R}}^{(p,p)}(X)/(\operatorname{im}\partial + \operatorname{im}\overline{\partial})$ is to be denoted $\widetilde{D}_{\mathbb{R}}^{p,p}(X)$.

When X is a complex quasi-projective variety, we will use the same notations to denote the corresponding for $X(\mathbb{C})$ seen as a complex manifold consisting of the disjoint union of the complex points of $X_{\sigma(\mathbb{C})} = X \times_{\sigma,k} \operatorname{Spec}\mathbb{C}$ where σ runs through the embeddings of k in the complex numbers.

The suggestion to differentiate weak objects and strong objects by capping them with a "check" for the former and a "hat" for the latter had been made to me by C.Soulé.

Contents

1 Metric Sheaves

1.1 Meager Complexes

1.1.1 Resolutions

We introduce here a theory of metrized coherent sheaves, that is a specialization of the general theory of metrized structures on derived categories see [BGFiML12] as we only need to metrize sheaves, and not complexes of sheaves we present the general theory in the context.

In this section X will denote a projective complex manifold (smooth over $\mathbb{C}$).

Definition 1.1.1. *Let $\mathcal{F}$ be a coherent sheaf over X, a metric structure (or sometimes just metric) on $\mathcal{F}$, will consist in the datum of a (finite) resolution of $\mathcal{F}$ by algebraic vector bundles, endowed with hermitian metrics.*

The following lemma is common knowledge

Lemma 1.1.2. *Let X a smooth projective variety over $\mathbb{C}$, every coherent sheaf $\mathcal{F}$, over X, can be resolved (on the left) by a finite complex of locally free coherent sheaves.*

Proof. Recall that Gaga enables us to assume that $\mathcal{F}$ can be written $j^*\mathcal{F}^a$ where $\mathcal{F}^a$ is a coherent sheaf on an algebraic variety over $\mathbb{C}$ and j is the inclusion of the closed points of this variety in X. As j^* is exact (in the category of ringed spaces), it is sufficient to prove the proposition for an *algebraic* projective variety over $\mathbb{C}$ and a coherent sheaf over it.

By Serre's lemma we may then assume that a certain twist, say $\mathcal{F}(n)$ is generated by a finite number of global sections, and we thus have a surjection

$$\mathcal{O}_X^d \to \mathcal{F}(n) \to 0$$

It follows that $\mathcal{F}$ is the quotient of a locally free sheaf.

Recall that the homological dimension of a coherent sheaf, $\mathrm{hd}(\mathcal{F})$ is the length of the smallest left resolution of $\mathcal{F}$ by locally free sheaves, moreover $\mathrm{hd}(\mathcal{F}) \leq n$ if and only if $\mathscr{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{G}) = 0$ for any $\mathcal{G}$ quasi-coherent, and all $i > n$, on the one hand as $\mathcal{F}$ is coherent, $\mathscr{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{G})_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$, on the other hand as $\mathcal{O}_{X,x}$ is local, noetherian, and regular $\mathrm{proj.dim}(\mathcal{F}_x) \leq \dim \mathcal{O}_{X,x}$, and over a local noetherian ring, free modules and projective modules coincide so we deduce $\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) = 0$ for all $i > \dim X$, thus $\mathrm{hd}(\mathcal{F}) < \infty$. And the proof is complete. □

Remark 1.1.3. This result is actually true for every separated regular noetherian scheme [BGI, 2.2.7.1.]

We will need some facts about complexes of vector bundles on a smooth manifold, that we recall here. Let us state some conventions, following [BGFiML12], we'll denote $\mathrm{V}^{\mathrm{b}}(X)$ (resp. $\overline{\mathrm{V}}^{\mathrm{b}}(X)$) the category of complexes of vector bundles (resp. hermitian vector bundles) over X; such a complex will be written homologically

$$0 \to E_n \xrightarrow{d_n} ... \to E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} 0$$

If we have a complex $E_\bullet$ resolving a coherent sheaf $\mathcal{F}$, we will label the resolution

$$.. \to E_1 \to E_0 \to \mathcal{F}$$

with $\mathcal{F}$ in degree -1.

Proposition 1.1.4. *Let $E_\bullet$, $F_\bullet$ and $G_\bullet$ be three complexes of vector bundles over X such that we have a diagram of quasi-isomorphisms*

$$\begin{array}{ccccc} F_\bullet & & & & E_\bullet \\ & \searrow^{f} & & \swarrow^{g} & \\ & & G_\bullet & & \end{array}$$

Then there exists a complex of vector bundles $H_\bullet$ such that we have a diagram, commuting up to homotopy

$$\begin{array}{ccccc} & & H_\bullet & & \\ & \swarrow & & \searrow & \\ F_\bullet & & & & E_\bullet \\ & \searrow & & \swarrow & \\ & & G_\bullet & & \end{array}$$

where the top arrows are quasi-isomorphisms.

Proof. We have a map from $E_\bullet$ to $\mathrm{cone}(f)$ given by sending x to $(0, g(x))$, we set $H = \mathrm{cone}(E, \mathrm{cone}(f))[-1]$, we have $H_n = E_n \oplus F_n \oplus G_{n-1}$ and we get diagram

$$\begin{array}{ccccc} & & H_\bullet & & \\ & \swarrow & & \searrow & \\ F_\bullet & & & & E_\bullet \\ & \searrow & & \swarrow & \\ & & G_\bullet & & \end{array}$$

as well as a homotopy $h : H_n \to G_{n-1}$ that makes the diagram commutes up to homotopy. □

Recall that in the general formalism of derived category, a basic observation is that a resolution of a coherent sheaf should be defined up to a "roof", we mimic this situation in the metric case, the analog of quasi-isomorphisms will be called quasi-isometries (a small discrepancy of [BGFiML14]'s vocabulary, but i believe it is a nice "visual" name). In order to define them we first define the notion of meager complex which is already introduced in [BGFiML14] this is the analog (and a refinement) of the notion of acyclic complex is the context of hermitian sheaves.
We have the following lemma.

Lemma 1.1.5. *Let $E_\bullet \to \mathcal{F}$ be a resolution of a coherent sheaf and $\mathcal{G} \to \mathcal{F}$ a morphism of sheaves, we can find a resolution $H_\bullet$ of $\mathcal{G}$ such that we have a commutative diagram*

$$\begin{array}{ccc} H_\bullet & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ E_\bullet & \longrightarrow & \mathcal{F} \end{array}$$

Moreover if $\mathcal{G} \to \mathcal{F}$ is onto, then we can choose the $H_i \to E_i$ to be onto too.

Proof. We build the resolution $H_\bullet$ by induction. Set π the morphism from $\mathcal{G}$ to $\mathcal{F}$, we set K the kernel of the diagonal embedding $E_0 \oplus \mathcal{G} \xrightarrow{\pi - d_0} F$, as K is a coherent sheaf, it is a quotient of some locally free sheaf, say H_1. We thus get a commutative diagram

$$\begin{CD}
H_1 @>>> \mathcal{G} @>>> 0 \\
@VVV @VVV \\
E_1 @>>> \mathcal{F} @>>> 0
\end{CD}$$

On the other hand if π is surjective, it is obvious that, by construction $\pi_1 : H_1 \to E_1$ is too.

Now, assume that we have built the bundles H_i, and the morphisms π_i for $i = 1...n$. If we introduce the kernels of the differentials $H_n \to H_{n-1}$ and $E_n \to E_{n-1}$ and by re-iterating the procedure described in the previous paragraph we build a commutative diagram

$$\begin{CD}
H_{n+1} @>>> \ker(H_n \to H_{n-1}) @>>> 0 \\
@VVV @VVV \\
E_{n+1} @>>> \ker d_{n-1} @>>> 0
\end{CD}$$

which enable us to construct H_{n+1}. the arrow from H_{n+1} to E_{n+1} being surjective by construction if the one form H_n to E_n is, because in that case, the arrow induced on the level of kernels will be surjective.

After a (finite!) number of steps, we're left with the following commutative diagram

$$\begin{CD}
0 @>>> \ker(H_{p+1} \to H_p) @>>> H_\bullet @>>> \mathcal{G} @>>> 0 \\
@VVV @VVV @VVV @VVV \\
0 @>>> 0 @>>> E_\bullet @>>> \mathcal{F} @>>> 0
\end{CD}$$

As $\ker(H_{p+1} \to H_p)$ is a coherent sheaf, it will be enough to replace it by a resolution by locally free sheaves to prove the proposition. $\square$

1.1.2 Acyclic calculus of Burgos, Freixas and Litcanu

Let X be a complex algebraic variety (the $\mathbb{C}$-valued points of an algebraic variety to be precise), we review here the theory of acyclic calculus developed in [BGFiML12]

Definition 1.1.6. *The class of* meager complexes*, denoted $\mathcal{M}(X)$ is the smallest class of complexes of hermitian bundles over X satisfying,*

1. *Every ortho-split complex is meager.*

2. *Every complex isometric to $\overline{F}_\bullet \oplus \overline{F}_\bullet[1]$, for $F_\bullet$ an acyclic complex endowed with any metric, i.e the cone of the zero map from an acyclic complex to itself is meager.*

3. *The cone of the identity of a complex $\overline{F}_\bullet$ is meager.*

4. *For every morphism of complex $f : \overline{E}_\bullet \to \overline{F}_\bullet$, if any two of the following complexes $\overline{E}_\bullet$, $\overline{F}_\bullet$, $\overline{\mathrm{cone}}(f)$ are meager, so is the third.*

5. *Every shift of a meager complex, is meager.*

Remark 1.1.7. We will call a family of complexes of hermitian vector bundles satisfying the above conditions, a hermitian admissible class. It is easy to see that the intersection of a family of hermitian admissible classes is still an hermitian admissible class, thus the class of meager complexes is the intersection of all hermitian admissible classes.

Proposition 1.1.8. *Let $\overline{E}_\bullet$ be a meager complex, then $E_\bullet$ is acyclic.*

Proof. Set $\mathcal{ACL}(X)$ for the class of all complexes of hermitian bundles, $\overline{E}_\bullet$ such that the underlying complex, $E_\bullet$ is acyclic. It will be sufficient to prove that $\mathcal{ACL}(X)$ is an hermitian admissible class.
It is clear that ortho-split complexes are acyclic, just as clear as that for every complex $F_\bullet$, the cone of the zero map of an acyclic complex and of the identity map of any complex, is acyclic. Now let's consider $f : E_\bullet \to F_\bullet$ a morphism of complexes, we have a long exact sequence in cohomology

$$... \to H_{i-1}(\text{cone}(f)) \to H_i(E_\bullet) \to H_i(F_\bullet) \to H_i(\text{cone}(f)) \to ...$$

that ensures that if any two of the three complexes, $E_\bullet, F_\bullet, \text{cone}(f)$ are acyclic, the the third is too.
Of course the shift of any acyclic complex is still acyclic. □

In the same way that a quasi-isomorphism has an acyclic cone, we define an equivalence relation between metric resolutions by declaring equivalent two resolutions differing by a meager cone.

Definition 1.1.9. *A morphism $f : \overline{E}_\bullet \to \overline{F}_\bullet$ is said to be tight iff $\overline{\text{cone}}(f)$ is meager.*

The preceding proposition admits the following translation

Corollary 1.1.10. *A tight morphism between two complexes of hermitian vector bundles is a quasi-isomorphism.*

Let us state

Definition 1.1.11. *We will say that two complexes of hermitian vector bundles, $\overline{E}_\bullet$ and $\overline{F}_\bullet$, are quasi-isometric*[5] *iff there exists a complex of hermitian bundles $\overline{H}_\bullet$ such that we have a diagram*

$$\overline{F}_\bullet \longleftarrow \overline{H}_\bullet \longrightarrow \overline{E}_\bullet$$

where the two arrows are tight morphisms.

We have the following characterization of the quasi-isometry relation

Proposition 1.1.12. *Two complexes of hermitian vector bundles, $\overline{E}_\bullet$ and $\overline{F}_\bullet$ are quasi-isometric iff we can find a complex of hermitian vector bundles $\overline{H}_\bullet$ such that we have a diagram*

$$\overline{E}_\bullet \xleftarrow{\;g\;} \overline{H}_\bullet \xrightarrow{\;f\;} \overline{F}_\bullet$$

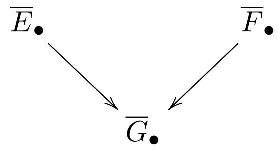
with g a quasi-isomorphism and such that complex $\overline{\text{cone}}(f) \oplus \overline{\text{cone}}(g)[1]$ is meager.

[5] In [BGFiML12] the term used is tightly related.

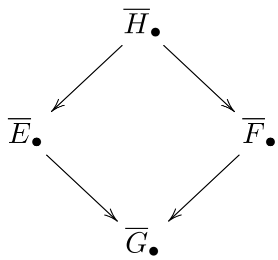
Proof. This is [BGFiML12, Lemma 2.20] □

Moreover we have

Proposition 1.1.13. *Any diagram of tight morphisms of the form*



can be completed as



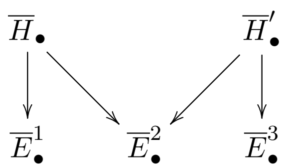
where all the arrows are tight.

Proof. This is [BGFiML12, Lemma 2.21] □

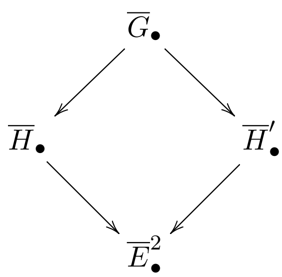
It is important to note that

Proposition 1.1.14. *The quasi-isometry is an equivalence relation.*

Proof. The only part that is not obvious is the transitivity of this relation.
Let us consider $\overline{E}^i_\bullet$, for $i = 1, 2, 3$ three complexes of hermitian vector bundles, we assume that $\overline{E}^1_\bullet$ and $\overline{E}^2_\bullet$ are quasi-isometric and that $\overline{E}^2_\bullet$ and $\overline{E}^3_\bullet$ are also quasi-isometric. We thus have a diagram



We can complete this diagram into a diagram which is commutative up to homotopy



where all the arrows are tight thanks to 1.1.13, as the composition of tight morphisms is tight, we deduce the result.

□

Let us set $\overline{\mathrm{V}}^{\mathrm{b}}(X)/\mathcal{M}(X)$ to be the class of hermitian vector bundle modulo the quasi-isometry relation and $\mathcal{KA}(X)$ be the subset of $\overline{\mathrm{V}}^{\mathrm{b}}(X)/\mathcal{M}(X)$ corresponding to the image of complexes of hermitian vector bundles such that the underlying complex is acyclic.

One can endow $\overline{\mathrm{V}}^{\mathrm{b}}(X)/\mathcal{M}(X)$ with a structure of monoid using the orthogonal sum as the addition, the image of a complex $\overline{E}_\bullet$ in $\overline{\mathrm{V}}^{\mathrm{b}}(X)/\mathcal{M}(X)$ will be denoted $[\overline{E}_\bullet]$.

This object inherits several properties summing up some diagram constructions, and that make proofs much less cumbersome that we list in the following proposition

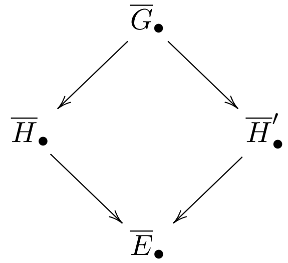
Proposition 1.1.15. *In $\overline{\mathrm{V}}^{\mathrm{b}}(X)/\mathcal{M}(X)$ we have*

1. *A complex $[\overline{E}_\bullet]$ is invertible iff it is acyclic and then its inverse is given by the shift $[\overline{E}_\bullet[1]]$.*
2. *For every arrow $\overline{E}_\bullet \to \overline{F}_\bullet$, if $\overline{E}_\bullet$ is acyclic (resp. $\overline{F}_\bullet$ acyclic) then*

$$[\overline{\mathrm{cone}}(E,F)_\bullet] = [\overline{E}_\bullet] - [\overline{F}_\bullet]$$

$$(\ \textit{resp.}\ [\overline{\mathrm{cone}}(E,F)_\bullet] = [\overline{E}_\bullet] + [\overline{F}[1]_\bullet])$$

3. *For every diagram*



which is commutative up to homotopy, we have

$$[\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(G,H),\overline{\mathrm{cone}}(H',E))] = [\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(G,H'),\overline{\mathrm{cone}}(H,E))]$$

4. *If $f : \overline{E}_\bullet \to \overline{F}_\bullet; g : \overline{F}_\bullet \to \overline{G}_\bullet$ are two morphism between metrized complexes then we have*

$$[\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(g \circ f), \overline{\mathrm{cone}}(g))] = [\overline{\mathrm{cone}}(f)[1]]$$

$$[\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(f), \overline{\mathrm{cone}}(g \circ f))] = [\overline{\mathrm{cone}}(g)]$$

Moreover if g or f is a quasi-isomorphism (resp. if $g \circ f$ is a quasi-isomorphism) then

$$[(\overline{\mathrm{cone}}(g \circ f)] = [\overline{\mathrm{cone}}(g))] + [\overline{\mathrm{cone}}(f)]$$

$$(\textit{resp.}\ [\overline{\mathrm{cone}}(g \circ f)] + [\overline{\mathrm{cone}}(f)[1]] = [\overline{\mathrm{cone}}(g)])$$

Proof. This is [BGFiML12, Theorem 2.27]. □

1.1.3 Metric resolutions

In their article [BGFiML14], Burgos, Freixas et Litcanu, define a notion of equivalence for hermitian structure on the derived category of coherent sheaves, here we will simply restrict their definition to the case of a single coherent sheaf over a projective complex variety X.

Definition 1.1.16. *We say that two hermitian structures $\overline{E}_\bullet \to \mathcal{F}$ and $\overline{F}_\bullet \to \mathcal{F}$ on a coherent sheaf are quasi-isometric if there exists a complex of hermitian vector bundles $\overline{H}_\bullet$ and a diagram commutative up to homotopy*

$$\begin{array}{ccccc} & & \overline{H}_\bullet & & \\ & \swarrow^{g} & & \searrow^{f} & \\ \overline{E}_\bullet & & & & \overline{F}_\bullet \\ & \searrow & & \swarrow & \\ & & \mathcal{F} & & \end{array}$$

such that f and g are tight morphisms.

Notice that as f and g are tight, they're quasi-isomorphisms and therefore $H_\bullet$ is a resolution of $\mathcal{F}$ (in two different homotopic ways).

We will need the following lemma

Lemma 1.1.17. *Assume that we have a diagram of complex of hermitian vector bundles*

$$\begin{array}{ccc} \overline{E}_\bullet & \xrightarrow{f} & \overline{E}_\bullet' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g'} \\ \overline{F}_\bullet & \xrightarrow{f'} & \overline{F'}_\bullet \end{array}$$

that commutes up to a homotopy, say h.

Then h induces two morphisms of complex

$$\psi : \operatorname{cone}(f) \to \operatorname{cone}(f')$$

and

$$\varphi : \operatorname{cone}(-g) \to \operatorname{cone}(g')$$

and we have a natural isometry

$$\operatorname{cone}(\psi) \simeq \operatorname{cone}(\varphi)$$

Proof. This is [BGFiML12, Lemma 2.3] □

Proposition 1.1.18. *Let $\mathcal{F}$ be a coherent sheaf on a complex algebraic variety and let $\overline{E}_\bullet \to \mathcal{F}$ and $\overline{G}_\bullet \to \mathcal{F}$ be two metric resolutions of $\mathcal{F}$, the following conditions are equivalent*

i) The two metric structures $\overline{E}_\bullet \to \mathcal{F}$ and $\overline{G}_\bullet \to \mathcal{F}$ are quasi-isometric.

ii) There exists $\overline{H}_\bullet \to \mathcal{F}$ *a metric resolution such that we have a diagram that commutes up to homotopy*

$$\begin{array}{ccccc} & & \overline{H}_\bullet & & \\ & \swarrow^{g} & & \searrow^{f} & \\ \overline{E}_\bullet & & & & \overline{F}_\bullet \\ & \searrow & & \swarrow & \\ & & \mathcal{F} & & \end{array}$$

with g *a quasi-isomorphism and such that complex* $\overline{\mathrm{cone}}(f) \oplus \overline{\mathrm{cone}}(g)[1]$ *is meager.*

iii) For any metric resolution $\overline{H}_\bullet \to \mathcal{F}$ *such that we have a diagram that commutes up to homotopy*

$$\begin{array}{ccccc} & & \overline{H}_\bullet & & \\ & \swarrow^{g} & & \searrow^{f} & \\ \overline{E}_\bullet & & & & \overline{F}_\bullet \\ & \searrow & & \swarrow & \\ & & \mathcal{F} & & \end{array}$$

where g *is a quasi-isomorphism, the complex* $\overline{\mathrm{cone}}(f) \oplus \overline{\mathrm{cone}}(g)[1]$ *is meager.*

Proof. The fact that *i)* implies *ii)* is obvious as the orthogonal sum of two meager complexes is meager.

Let's prove that *ii)* implies *iii)*, so assume that there exists a metric resolution $\overline{H}$ of $\mathcal{F}$ like the one in the proposition.

Now let us consider $\overline{H}'_\bullet$ any other metric resolution giving a diagram which is commutative up to homotopy just like the one in the proposition. We can, using proposition 1.1.4, find a complex of vector bundles, say $G_\bullet$ such that we have a commutative diagram up to homotopy

$$\begin{array}{ccccc} & & G_\bullet & & \\ & \swarrow^{\alpha} & & \searrow^{\beta} & \\ H_\bullet & & & & H'_\bullet \\ \downarrow g & & \overset{f}{\searrow}\!\!\!\!\underset{\delta}{\swarrow} & & \downarrow \delta' \\ E_\bullet & & & & F_\bullet \end{array}$$

with α and β being quasi-isomorphisms. Let us endow $G_\bullet$ with any metric.

By the previous lemma, we have $\mathrm{cone}(\mathrm{cone}(\alpha), \mathrm{cone}(\delta))$ isometric to $\mathrm{cone}(\mathrm{cone}(\beta), \mathrm{cone}(f))$ and $\mathrm{cone}(\mathrm{cone}(\alpha), \mathrm{cone}(\delta'))$ isometric to $\mathrm{cone}(\mathrm{cone}(\beta), \mathrm{cone}(g))$

Therefore, in $\overline{\mathrm{V}}^{\mathrm{b}}(X)/\mathcal{M}(X)$, using that α and β, as well as g and δ are quasi-isomorphisms and 1.1.15 we have

$$-[\mathrm{cone}(\alpha)] + [\mathrm{cone}(\delta)] = -[\mathrm{cone}(\beta)] + [\mathrm{cone}(g)]$$

$$-[\mathrm{cone}(\alpha)] + [\mathrm{cone}(\delta')] = -[\mathrm{cone}(\beta)] + [\mathrm{cone}(f)]$$

by subtracting the first equation to the second (which is possible because all the cones appearting in the first equation are acyclic), we get

$$[\mathrm{cone}(\delta')] - [\mathrm{cone}(\delta)] = [\mathrm{cone}(f)] - [\mathrm{cone}(g)] = [\mathrm{cone}(g)[1]] + [\mathrm{cone}(f)] = 0$$

and we are done.

For the implication *iii*) $\Rightarrow$ *i*) it results from 1.1.12 □

The following proposition is already proved because two resolutions define the same metric structure if the complexes of hermitian bundles obtained by truncating the $\mathcal{F}$ are quasi-isometric.

Proposition 1.1.19. *The relation of quasi-isometry is an equivalence relation over the set of metrized locally free finite resolutions of a given sheaf* $\mathcal{F}$

Remark 1.1.20. The group $\mathcal{KA}(X)$ is identified with the metric structures on the zero sheaf over X, we shall call this group, the group of universal secondary characteristic classes, or the group of acyclic K-theory.

1.1.4 Fitness

Observe now that for vector bundles there are two notion of metrics, the classical one, which we will refer to as a *hermitian metric*, and the one given by a resolution by hermitian vector bundles which we will refer to as a *hermitian structure*. It is natural to wonder whether every hermitian structure is a hermitian metric. To make this assertion precise, notice that every hermitian metric is a hermitian structure in a canonical way, namely $\overline{E} \to E$ is the natural hermitian structure associated to the hermitian metric on E. The question being: is every hermitian structure on a vector bundle quasi-isometric to one coming from a hermitian metric?

I do not know a general answer for this question, but we can give at least a sufficient condition.

Let X be a smooth projective complex variety over $\mathbb{C}$, we say that X satisfies a fitness lemma (resp. strong fitness lemma) iff for every acyclic complex of vector bundle

$$0 \to E_n \to E_{n-1} \to ... \to E_1 \to E_0 \to 0$$

(resp. for every acyclic complex of vector bundle with all the E_i's equipped with a metric except E_0) there exists a choice of metrics on the E_i's (resp. E_0) such that

$$0 \to \overline{E}_n \to \overline{E}_{n-1} \to ... \to \overline{E}_1 \to \overline{E}_0 \to 0$$

is meager.

The following proposition follows directly from the definitions.

Proposition 1.1.21. *Assume that a complex projective manifold X satisfies a strong fitness lemma then every hermitian structure on a vector bundle is quasi-isometric to a hermitian metric.*

In the case of line bundles we can say a bit more about that question.

Recall that $\widehat{\mathrm{Pic}}(X)$ is the group of hermitian line bundles over X up to isometry, with the tensor product as group law. We will denote $\widetilde{\mathrm{Pic}}(X)$ the monoïd of line bundles over X equipped with a hermitian structure, it is easy to see that it is indeed a monoïd for the tensor product of resolutions.

Lemma 1.1.22. *(Weak Fitness Lemma)*

For every projective smooth variety over $\mathbb{C}$, *we have a map* $\widecheck{Pic}(X) \to \widehat{\mathrm{Pic}}(X)$, *let us denote* $\mathrm{FD}(X)$ *its kernel, there is a natural isomorphism of monoïds*

$$\widecheck{Pic}(X) \simeq \widehat{\mathrm{Pic}}(X) \oplus \mathrm{FD}(X)$$

Proof. Assume that we have a resolution of a line bundle L given by

$$0 \to E_n \to ... \to E_1 \to L \to 0$$

we see that the line bundle

$$\det(E_n)^{(-1)^n} \otimes ... \otimes \det(E_1)^{\vee} \otimes L$$

is trivial.

Let us endow the E_i's with arbitrary metrics and L with the metrics rendering isometric the isomorphism

$$\det(E_n)^{(-1)^{n+1}} \otimes ... \otimes \det(E_1) \simeq L$$

where each E_i is of course equipped with the determinant metric.

This metric does not depend on the class of quasi isometry of the hermitian structure on L, to see this, let

$$0 \to \overline{H}_n \to \overline{H}_{n-1} \to ... \to \overline{H}_1 \to \overline{H}_0 \to 0$$

be a meager complex over X then

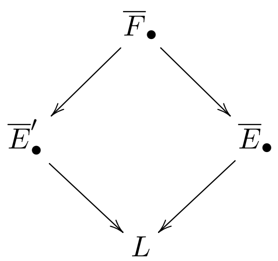
$$\bigotimes_{i \geq 0} \det \overline{H}_i^{(-1)^i}$$

is a trivial hermitian line bundle (that is trivial, and equipped with the trivial metric), because

$$\widehat{c}_1(\bigotimes_{i \geq 0} \det \overline{H}_i^{(-1)^i}) = \sum_{i \geq 0} (-1)^i \, \widehat{c}_1(\det \overline{H}_i) = \sum_i (-1)^i \, \widehat{c}_1(\overline{H}_i) = \widetilde{\mathrm{ch}}^{\{1\}}[\overline{H}_\bullet] = 0$$

where the last equality follows[6] from 1.1.24 and the fact that $\overline{H}_\bullet$ is meager, now as we have an isomorphism $\widehat{\mathrm{CH}}^1(X) \simeq \widehat{\mathrm{Pic}}(X)$ given by the first arithmetic Chern class (see [ABKS94, 4.2 Prop 1]) the claim follows.

Apply this for $\overline{H}\bullet$ being the (meager) complex $\mathrm{cone}(\overline{F}, \overline{E})[1] \oplus \mathrm{cone}(\overline{F}, \overline{E}')$ for a diagram commutative up to homotopy



giving the quasi-isometry between two hermitian structures $\overline{E}_\bullet$ and $\overline{E}'_\bullet$ for L. This gives us an arrow from $\widehat{\mathrm{Pic}}(X)$ to $\widecheck{\mathrm{Pic}}(X)$, that is a section of the obvious arrow and we have the desired decomposition for $\mathrm{FD}(X)$ being the kernel of $\widecheck{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(x)$. □

[6] the reader can check that we do not use anything proved in this paragraph to prove the mentioned lemma

The proof of that proposition shows that we have a map from $\mathcal{KA}(X)$ to $\widetilde{\mathrm{Pic}}(X)$ given by

$$\overline{H}_\bullet \mapsto \left[\bigotimes_{i\geq 0} \det \overline{H}_i^{(-1)^i} \xrightarrow{\sim} \bigotimes_{i\geq 0} \det H_i^{(-1)^i} \right]$$

the monoïd $\mathrm{FD}(X)$ appears to be some kind of multiplicative version of meager complexes resolving line bundles, as the $\mathcal{KA}$ group remains very mysterious, for instance the only case where it has been computed it the case of a point, [BGFiML12], maybe $\mathrm{FD}(X)$ could be easier to study. We will not pursue this study here[7].

1.1.5 Secondary classes

In this section we will construct a notion of secondary characteristic classic fitted for our needs. This construction is an analog of the one already provided by Zha in his thesis [Zha99], but with a different definition of metric structure. Those classes are concrete realization of universal classes built up in [BGFiML14]. In this whole section, X will design a complex algebraic variety, which we may assume to be projective and smooth over $\mathbb{C}$.

Theorem 1.1.23. *Let $\overline{\mathcal{F}_\bullet}$ be an acyclic complex of hermitian coherent sheaves, that is hermitian sheaves equipped with a metric structure. There exists a unique way of attaching to every such complex a Bott-Chern secondary characteristic form, denoted $\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}_\bullet) \in \widetilde{A}^{\bullet,\bullet}(X)$, satisfying the following conditions.*

1. *(Compatibility with Bott-Chern forms) If $\mathcal{E} : 0 \to \overline{E}_1 \to \overline{E}_2 \to \overline{E}_3 \to 0$ is an exact sequence of hermitian vector bundles, then*

$$\widetilde{\mathrm{ch}}(\mathcal{E}) = \widetilde{\mathrm{ch}}^{BC}(\mathcal{E})$$

   where $\widetilde{\mathrm{ch}}^{BC}$ is the Bott-Chern form associated to the exact sequence (see [GS90])

2. *(Normalization) If $\overline{E}_\bullet \to \mathcal{F}$ is the metric resolution defining the hermitian structure over $\overline{\mathcal{F}}$, then $\widetilde{\mathrm{ch}}(\overline{E}_\bullet \to \overline{\mathcal{F}}) = 0$*

3. *(Devissage) If we have a complex of acyclic coherent metrized sheaves $\overline{\mathcal{F}}_\bullet$ that can be split up into exact sequences*

$$\mathcal{E}_i, 1 \leq i \leq n-1 : 0 \to \mathcal{G}_i \to \mathcal{F}_i \to \mathcal{G}_{i-1} \to 0$$

   with $\mathcal{G}_{-1} = \mathcal{F}_0$ et $\mathcal{G}_{n-1} = \mathcal{F}_n$. We have

$$\widetilde{\mathrm{ch}}(\overline{\mathcal{F}}_\bullet) + \sum_{i\geq 1} (-1)^i \widetilde{\mathrm{ch}}(\overline{\mathcal{E}_i}) = 0$$

   for every choice of metric structure on the sheaves $\mathcal{G}_i$ for $1 \leq i \leq n-2$

[7] These questions certainly require further investigations, especially because I believe that the $\mathcal{KA}$ group should play a key role in a strong version of the arithmetic cobordism. One indication in this direction is the fact that the natural map $\mathcal{KA}(X) \xrightarrow{\widetilde{\mathrm{ch}}} \widetilde{A}_{\mathbb{R}}^{\bullet,\bullet}(X)$ is onto where it should be (namely on forms of degree not exceeding $(p-1, p-1)$), this result is readily implied by a paper of Pingali and Takhtajan [PT14]

4. (Exactitude) If we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_{1,1} & \longrightarrow & \mathcal{F}_{1,2} & \longrightarrow & \mathcal{F}_{1,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_{2,1} & \longrightarrow & \mathcal{F}_{2,2} & \longrightarrow & \mathcal{F}_{2,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_{3,1} & \longrightarrow & \mathcal{F}_{3,2} & \longrightarrow & \mathcal{F}_{3,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

then we have the following equality $\widetilde{\mathrm{ch}}(\overline{\mathcal{L}_1})-\widetilde{\mathrm{ch}}(\overline{\mathcal{L}_2})+\widetilde{\mathrm{ch}}(\overline{\mathcal{L}_3}) = \widetilde{\mathrm{ch}}(\overline{\mathcal{C}_1})-\widetilde{\mathrm{ch}}(\overline{\mathcal{C}_2})+\widetilde{\mathrm{ch}}(\overline{\mathcal{C}_3})$ *where* $\mathcal{C}_i$ *(resp.* $\mathcal{L}_i$*) designs the i-th exact column (resp. the i-th exact row).*

Proof. This is proved in [Zha99]. □

Let us simply note that these Bott-Chern classes are in fact defined for hermitian sheaves up to quasi-isometry in the sense of Burgos, Freixas, Litcanu. Notice that, now that we have at our disposition the notion of hermitian structure for a sheaf it is easy to prove the analog of 1.1.17 where the complexes of vector bundles are replaced with complexes of hermitian sheaves.

Lemma 1.1.24. *Assume that we have a short exact sequence of the form* $0 \to (\mathcal{F}, h_1) \to (\mathcal{F}, h_2) \to 0$ *where the hermitian structures on both copies of* $\mathcal{F}$ *are quasi-isometric, then its secondary Bott-Chern form* $\widetilde{\mathrm{ch}}(\mathcal{F}, h^1, h^2)$ *vanishes.*

Proof. Set $\overline{E}^1_\bullet$ and $\overline{E}^2_\bullet$ two metric structures on $\mathcal{F}$, that are assumed to be quasi-isometric. Then there exists a metric resolution, say $\overline{H}_\bullet$, and a commutative square up to homotopy

$$\begin{array}{ccccc}
 & & \overline{H}_\bullet & & \\
 & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\
\overline{E}^1_\bullet & & & & \overline{E}^2_\bullet \\
 & \searrow & & \swarrow & \\
 & & \mathcal{F} & &
\end{array}$$

such that $\overline{\mathrm{cone}}(f) \oplus \overline{\mathrm{cone}}(g)[1]$ is meager, it will be sufficient to prove that for every meager complex $\overline{M}_\bullet$, we have $\widetilde{\mathrm{ch}}(\overline{M}_\bullet) = 0$. Indeed, if such a result is satisfied, we have

$$\begin{aligned}
0 &= \widetilde{\mathrm{ch}}(\overline{\mathrm{cone}}(f) \oplus \overline{\mathrm{cone}}(g)[1]) \\
&= \widetilde{\mathrm{ch}}(\overline{E}^1_\bullet \to (\mathcal{F}, h^1)) - \widetilde{\mathrm{ch}}(\overline{E}^2_\bullet \to (\mathcal{F}, h^1)) \\
&= -\widetilde{\mathrm{ch}}(\overline{E}^1_\bullet \to (\mathcal{F}, h^2)) + \widetilde{\mathrm{ch}}(\overline{E}^2_\bullet \to (\mathcal{F}, h^2))
\end{aligned}$$

which will imply the result by the normalization condition. This also proves that in the general case, if $\overline{E}_\bullet \to \mathcal{F}$ is a metric structure on $\mathcal{F}$ and $\overline{E}'_\bullet$ another metric structure on $\mathcal{F}$, then as expected

$$\widetilde{\mathrm{ch}}(\overline{E}_\bullet \to (F,h')) = \widetilde{\mathrm{ch}}(\mathcal{F},h',h)$$

So let us first prove that a meager complex has a vanishing secondary class.

Let us first consider $0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$ a short exact sequence of hermitian bundles, that is orhto-split, then using the compatibility condition with traditional Bott-Chern classes we have $\widetilde{\mathrm{ch}}(0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0) = \widetilde{\mathrm{ch}}^{BC}(0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0) = 0$, so let us set $\mathcal{CSN}(X)$ the class of acyclic complex of hermitian bundles that have vanishing secondary classes.
This class certainly contains the cone of the identity map, and of the zero map of acyclic complexes, but also ortho-split complexes, and according to the preceding remark, it is also true that if any two of the three complexes $\overline{E}_\bullet$, $\overline{F}_\bullet$, $\overline{\mathrm{cone}}(f)$ (where, of course, f is an arrow from $E_\bullet$ to $F_\bullet$) have zero secondary classes, then so does the third, finally a shifting of an acyclic complex only changes the sign of the secondary classes of the complex in question; hence $\mathcal{CSN}(X)$ is an admissible class, and as such, contains the class of meager complexes, which make the proof complete. □

Corollary 1.1.25. *Secondary characteristic classes, only depend on the quasi-isometry class of the metric structure on the sheaves and not on the particular choice of a resolution within this quasi-isometry class.*

Proof. We readily see that it is enough to prove that for every exact sequence $0 \to \overline{\mathcal{F}_1} \to \overline{\mathcal{F}_2} \to \overline{\mathcal{F}_3} \to 0$ the associated secondary class does not depend on the quasi-isometry class of the hermitian sheaves, the general result will follow by devissage. Let us consider exact sequence $0 \to \overline{\mathcal{F}'_1} \to \overline{\mathcal{F}'_2} \to \overline{\mathcal{F}'_3} \to 0$ where the sheaves are the same, but where the hermitian structures on the $\mathcal{F}'_i$'s are quasi-isometric to the ones on the $\mathcal{F}_i$'s, then we certainly have a commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\mathcal{F}_1} & \longrightarrow & \overline{\mathcal{F}_2} & \longrightarrow & \overline{\mathcal{F}_3} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{\mathcal{F}'_1} & \longrightarrow & \overline{\mathcal{F}'_2} & \longrightarrow & \overline{\mathcal{F}'_3} & \longrightarrow & 0 \end{array}$$

So, using exactness, and the previous lemma, we get

$$\widetilde{\mathrm{ch}}(0 \to \overline{\mathcal{F}_1} \to \overline{\mathcal{F}_2} \to \overline{\mathcal{F}_3} \to 0) = \widetilde{\mathrm{ch}}(0 \to \overline{\mathcal{F}'_1} \to \overline{\mathcal{F}'_2} \to \overline{\mathcal{F}'_3} \to 0)$$

□

Theorem 1.1.26. *Let $\mathcal{F} : 0 \to \overline{\mathcal{F}_1} \to \overline{\mathcal{F}_2} \to \overline{\mathcal{F}_3} \to 0$ a short exact sequence of coherent sheaves, and let $\overline{E}$ be a hermitian vector bundle. We have*

1. $dd^c\widetilde{\mathrm{ch}}(\mathcal{F}) = \mathrm{ch}(\overline{\mathcal{F}_2}) - \mathrm{ch}(\overline{\mathcal{F}_1}) - \mathrm{ch}(\overline{\mathcal{F}_3})$
2. $\widetilde{\mathrm{ch}}(\mathcal{F} \otimes \overline{E}) = \widetilde{\mathrm{ch}}(\mathcal{F}).\,\mathrm{ch}(\overline{E})$

Proof. The first formula is immediate, it results from the fact that the formula is known to hold for complex of bundles, and from the the definitions we have given for secondary forms.

The second one is easy too, it follows from the fact that tensoring with a vector bundle is an exact functor and thus preserves dominating resolutions, and, there again, from the fact that the result is known to hold for complex of bundles for classical Bott-Chern forms. □

From the first formula one can deduce the following result, which is the one that interests us.

Corollary 1.1.27. *The Chern form associated to a metrized sheaf only depends on the quasi-isometry class of the metric structure on the sheaf.*

2 Weak arithmetic cobordism group

2.1 Weak arithmetic theories

2.1.1 Weak Arithmetic Chow Groups

Let X be an algebraic projective smooth variety over a number field k. Let V be a subvariety of X of dimension $d+1$, and let f be any rational function on V, recall that $\mathrm{div}(f)$ is the cycle on X defined as

$$\mathrm{div}(f) = \sum_{\text{irreducible } W \subset V; \mathrm{codim}_V(W)=1} \mathrm{ord}_W(f)[W]$$

We also set $\log|f|^2$ to be the current over X defined in the following manner; let ω be any real smooth compactly supported form over X, of type $(d+1, d+1)$, we set

$$\langle \log|f|^2, \omega \rangle = \int_{V^{\mathrm{ns}}} \log|f|^2 \omega$$

where V^{ns} denotes the open subset of $V(\mathbb{C})$ consisting of smooth points. As the singular locus of V is of codimension at least 1 in V, and as $\log|f|^2$ is a locally integrable function over V^{ns}, this is a well defined current over X, notice that $\log|f|^2$ is of type $(d_X - d_V, d_X - d_V)$. We could also define $\log|f|^2$ using the resolution of singularities of V.
We define the weak arithmetic Chow group in the following manner

Definition 2.1.1. *We call the arithmetic weak Chow group of X and we denote by $\widetilde{CH}(X)$ the group $\widehat{Z(X)}/\widehat{Rat(X)}$ where*

- *The group $\widehat{Z}(X)$ is the direct sum of the free abelian groups built on symbol $[Z]$ for every, Z, closed irreducible subset of X and the group $\widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X)$.*
- *The subgroup $\widehat{Rat}(X)$ is the subgroup of $\widehat{Z}(X)$ generated by $[div(f)] - \log|f|^2$ for every $f \in kV^*$ for every subvariety V of X.*

Remark 2.1.2. We have a natural grading over $\widetilde{CH}(X)$, where the homogenous piece of degree d is given by

$$\widehat{Z}_d(X) = \bigoplus_{\dim Z = d} \mathbb{Z}[V] \oplus \widetilde{D}_{\mathbb{R}}^{d_X-1-d, d_X-1-d}(X)$$

For any $d+1$-dimensional subvariety V, $div(f)$ is of degree d, and $\log|f|^2$ being of type $(d_X - (d+1), d_X - (d+1))$ is of degree d, thus $\widehat{\mathrm{Rat}}(X)$ is a homogenous subgroup of $\widehat{Z}_\bullet(X)$, and $\widetilde{CH}(X)$ inherits the grading.

From now on, we will simplify notations a bit, by writing $[Z, g]$, instead of $([Z], g)$, of course we have two natural maps $a : \widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X) \to \widetilde{CH}(X)$ sending g to $[0, g]$ and $\zeta : \widetilde{CH}(X) \to \mathrm{CH}(X)$ sending $[Z, g]$ to $[Z]$.

Let us briefly examine the different operations that we can define on such groups.

Definition 2.1.3. *Let $\pi : \overline{X} \to \overline{Y}$ be a projective morphism between arithmetic varieties, we define*

$$\pi_*[Z, g] = [\pi_*[Z], \pi_* g]$$

where $\pi_*[Z]$ is the push forward of geometric cycles defined in [Ful86]; and $\pi_*(g)$ is the push forward of current (which definition is recalled in 2.2.20).

It is a well known fact (see for instance [GS, Theorem 3.6.1]) that this push forward is well defined and gives a functorial map

$$\widetilde{CH}(X) \xrightarrow{\pi_*} \widetilde{CH}(Y)$$

this map is degree preserving.
In the same manner we can define a pull back-operation.

Definition 2.1.4. *(Pull-Back) Let $f : \overline{X}' \to \overline{X}$ be a smooth equidimensional morphism between arithmetic varieties, we define*

$$f^*[Z, g] = [f^*[Z], f^*g]$$

where $f^*[Z]$ is the cycle associated to the equidimensional scheme $X' \times_X Z$, see [Ful86]; and $f^*(g)$ is defined in 2.2.24

The proof that this map is well defined on the level of the $\widetilde{CH}$, and is functorial can be found in [GS, Theorem 3.6.1]

Remark 2.1.5. Here, the morphism f^*, for f equidimensional of relative dimension d increases degree by d, the relative dimension.

We can also define a first Chern class operator, but to do so let us fist notice that if $f \in k(V)$ is a rational function defined on a subvariety V of X, then for every closed subvariety Z of X generically transverse to V, we can restrict $\log|f|^2$ so a (locally integrable) function defined on $V \cap Z$, that defines a current on X by integration along the smooth locus of $V \cap Z$ (with the appropriate coefficient for each irreducible component of $V \cap Z$, namely its geometric multiplicity), we will denote such current as $\delta_Z \wedge \log|f|^2$, notice that we have a projection formula

$$i_*i^*(\log|f|^2) = \delta_Z \wedge \log|f|^2 = i_*i^*(1_Z) \wedge \log|f|^2$$

For every (regular) closed immersion $Z \xrightarrow{i} X$ it is a well known fact that we can find for every closed subvariety V of X, another variety, say W, rationally equivalent to V and transverse to i so that we can extend that procedure to arbitrary arithmetic cycles on X, in fact we can also extend this definition to rational sections of hermitian bundles, as locally such a section can be represented as a rational function via a holomorphic trivialization, for details see [GS, 1.3].

We can now define a *First Chern class operator*

Definition 2.1.6. *(First Chern class operator)*
Let $\overline{L} \in \widehat{\mathrm{Pic}}(X)$ be a hermitian line bundle over $\overline{X}$, we define $\widehat{c_1}(\overline{L})$ as an endomorphism of $\widetilde{CH}(X)$ by the following formula

$$\widehat{c_1}(\overline{L})[Z, g] = [\mathrm{div}(s).[Z], c_1(\overline{L}) \wedge g - \log\|s\|^2 \wedge \delta_Z]$$

where s is any rational section of L over Z, and $c_1(\overline{L})$ is the curvature of the bundle L, which can be defined locally as $(-2i\pi)^{-1}\partial\overline{\partial}\log\|s\|^2$ for any holomorphic local section of L.

Remark 2.1.7. Notice that $\widehat{c_1}(\overline{L})$ decreases degree by 1, and that on $\widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X)$, $\widehat{c_1}(\overline{L})$ only acts as $g \mapsto g \wedge c_1(\overline{L})$.

Let's sum up the fundamental properties of these operations in the following proposition

Proposition 2.1.8. *(Borel-Moore properties)*
Let X, Y, Y', S and S', be smooth projective varieties and let $\pi : X \to Y$ and $\pi' : Y \to Y'$ be projective morphisms and $f : S \to X$ and $f' : S' \to S$ be smooth equidimensional morphism. We also fix $\overline{M}$ (resp. $\overline{L}$ and $\overline{L}'$), a (resp. two) hermitian bundle on Y (resp. X), we have

1. *(Functoriality of the push forward)* $(\pi' \circ \pi)_* = \pi'_* \pi_*$
2. *(Functoriality of the pull back)* $(f \circ f')^* = f'^* f^*$
3. *(Naturality of the 1st Chern class)* $f^* \circ \widehat{c_1}(\overline{L}) = \widehat{c_1}(f^* \overline{L}) \circ f^*$.
4. *(Projection Formula)* $\pi_* \circ \widehat{c_1}(\pi^* \overline{M}) = \widehat{c_1}(L) \circ \pi_*$
5. *(Commutativity of the 1st Chern Classes)* $\widehat{c_1}(\overline{L}) \circ \widehat{c_1}(\overline{L}') = \widehat{c_1}(\overline{L}') \circ \widehat{c_1}(\overline{L})$
6. *(Grading) The degree of π_* is 0, the degree of f^* is d, the degree of $\widehat{c_1}(\overline{L})$ is -1.*

Proof. In each case we can evaluate the veracity of these statements on cycles of the form $[Z, 0]$ and $[0, g]$

1, 2. The functoriality on classes of the form $[Z, 0]$ is [Ful86, Thm 1.4, Thm 1.7], the result for currents results immediately form the functoriality of the pull-back (resp. of the integration over the fiber) of forms by proper (resp. submersive) maps between compact manifolds.

3 This is [Ful86, Prop 2.5.d] for cycles $[Z, 0]$ and the naturality of the Chern form for currents.

4 This is [Ful86, Prop 2.5.c] for cycles $[Z, 0]$ and the naturality of the Chern form for forms which implies this formula for currents by duality.

5 This is [Ful86, Prop 2.5.b] for cycles $[Z, 0]$ and a special case of [GS, Cor 2.2.9] for currents. Another proof can be given using 2.2.52.

6 This results from the definitions.

□

This properties give the functor $X \mapsto \widetilde{CH}(X)$ the properties of a Borel-Moore functor. The following ones illustrate the "arithmetic" nature of this functor.

Proposition 2.1.9. *(Arithmetic Type of $\widetilde{CH}$)*
Let X be a projective smooth variety over k of dimension d, we have

1. *For any hermitian line bundles over X, $\overline{L}_1, ..., \overline{L}_{d+2}$, we have*

$$\widehat{c_1}(\overline{L}_1) \circ ... \circ \widehat{c_1}(\overline{L}_{d+2}) = 0$$

as an endomorphism of $\widetilde{CH}(X)$.

2. *Let $\overline{L}$ be a hermitian line bundle over X, with s a global section of L that is transverse to the zero section. Let Z be the zero scheme of such a section, and $i : Z \to X$ the corresponding immersion. We have*

$$i_*(1_Z) = \widehat{c_1}(\overline{L})(1_X) + a(\log \|s\|^2)$$

3. *Given two hermitian bundles $\overline{L}$ and M over X we have*

$$\widehat{c_1}(\overline{L} \otimes \overline{M}) = \widehat{c_1}(L) + \widehat{c_1}(M)$$

Proof. 1. This results simply from the fact that we have a decomposition of abelian group

$$\widetilde{CH}(X) = \widetilde{CH}_d(X) \oplus ... \oplus \widetilde{CH}_0(X) \oplus \widetilde{CH}_{-1}(X)$$

and from the fact that the first Chern class operator is of degree -1.

2. Keeping the notation of the proposition we have $i_*(1_Z) = [Z, 0]$, and $\widehat{c_1}(\overline{L})(1_X) = [\mathrm{div}(s), -\log \|s\|^2] = [Z, 0] - a(\log \|s\|^2)$, and the result follows.

3. We have $c_1(L \otimes M) = c_1(L) + c_1(M)$ in $\mathrm{CH}(X)$ [Ful86], and by the very definition of the tensor product metric we have $\log \|s \otimes t\|^2 = \log[\|s\|^2 \|t\|^2]$, which implies the result.

□

To complete our description let us note that the weak arithmetic chow groups are an extension of classical geometric Chow groups by the space of real currents modulo $\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}$.

Proposition 2.1.10. *We have an exact sequence*

$$\widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(X) \xrightarrow{a} \widetilde{CH}(X) \xrightarrow{\zeta} \mathrm{CH}(X) \to 0$$

that breaks up into

$$\widetilde{D}^{d-1-p,d-1-p}_{\mathbb{R}}(X) \xrightarrow{a} \widetilde{CH}_p(X) \xrightarrow{\zeta} \mathrm{CH}_p(X) \to 0$$

Proof. Let $\alpha = \sum n_i [Z_i, g_i]$ be a weak arithmetic cycle. The fact that $\sum n_i [Z_i]$ is trivial in $\mathrm{CH}(X)$ is equivalent to the existence of subvarieties V_j of X and $f_j \in k(V_j)$ rational functions over V_j,such that $\sum [Z_i] = \sum \mathrm{div}(f_j)$ as cycles, we thus have $\alpha = \sum a(g_i) + \sum div(f_j) = \sum a(g_i) + \sum a(\log \|f_j\|^2)$ which is evidently in the image of a.

The fact that the first exact sequence implies the others is simply a reformulation of the fact that the maps a and ζ preserve the grading. □

Remark 2.1.11. The reader will compare this exact sequence to the one found in [ABKS94], and see that we have just replaced the space of real smooth forms by the space of general real currents, which has the advantage of having much better functoriality properties. This is why we have replaced the notion of arithmetic cycle presented in [GS] using a green current, by the notion of weak arithmetic cycle.

For a general subvariety Z of X we have a distinguished current, namely the current of integration over Z that we have mentioned earlier, as the push forward of current commutes with differentiation we have $\partial\overline{\partial}\delta_Z = \partial\overline{\partial} i_*(1_Z) = i_* \partial\overline{\partial}(1_Z) = 0$, we thus have a well defined map, called w, from $\widetilde{CH}_p(X) \to Z^{d-p}_{\mathbb{R}}(X) = \{g \in D^{d-p,d-p}_{\mathbb{R}}(X); \partial\overline{\partial} g = 0\}$ defined by $w[Z + a(g)] = \delta_Z + dd^c g$, which we call the double transgression map.

This map is an extension of the map ω defined for instance in [ABKS94], except that in this case the current obtained is a smooth closed form. If we denote by $\widetilde{CH}(X)^{w=\zeta}$ the kernel of the map $\widetilde{CH}_p(X) \xrightarrow{(w,\zeta)} Z^{d-p}_{\mathbb{R}}(X) \oplus \mathrm{CH}_p(X) \xrightarrow{[.]-\mathrm{cl}}$

$H^{d-p,d-p}(X)$, we can describe $\widetilde{CH}(X)^{w=\zeta}$ in terms of cycles using the "tame regulator", as it is the case in the "strong" construction of [GS], where we have an exact sequence

$$\mathrm{CH}^{p,p-1} \xrightarrow{\rho} H^{p-1,p-1}(X) \to \widehat{\mathrm{CH}}^p(X) \xrightarrow{(w,\zeta)} Z^{d-p}_{\mathbb{R}}(X) \oplus \mathrm{CH}_p(X) \xrightarrow{[.]-\mathrm{cl}} H^{d-p,d-p}(X) \to 0$$

Remark 2.1.12. It appears here that the arithmetic objects are a "double" transgression of the cohomological ones. What is meant by this is that what we associate with a geometric object, here a cycle, is a current that will give back the cohomological class of this object (here the image of the cycle through the cycle class map) after the application of $\partial\overline{\partial}$. We will later apply the same process to refine K-theory and give an arithmetic version of it.

Remark 2.1.13. The idea to double transgress cohomological theory to extract arithmetic information from a geometrical one is not at all obvious "at first glance", to see why we can apply Arakelov techniques to Diophantine geometry see [Fal91], [Voj91], [Lan88].

2.1.2 Higher Analytic Torsion of Bismut-Köhler

Recall the definition of arithmetic $\widehat{K}$-theory given by Gillet and Soulé in [GS90]

Definition 2.1.14. *Set $\widehat{K}_0(\overline{X})$ to be the free abelian group $\bigoplus \mathbb{Z}[\overline{E}] \times \widetilde{A}^{\bullet,\bullet}_{\mathbb{R}}(X)$ where $\overline{E}$ is an isometry class of hermitian vector bundle over X, subject to the following relations: for every exact sequence $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$,*

$$[\overline{E}, 0] = [\overline{E''}, 0] + [\overline{E'}, 0] + [0, \widetilde{\mathrm{ch}}(\mathcal{E})]$$

One of the most profound problem in Arakelov theory is to define a direct image for such groups and to compute it, firstly we need to fix a metric on $\pi_* E$, unfortunately *a priori* this is only a sheaf, and not a vector bundle, so Gillet and Soulé chose to examine a particular situation of utmost interest.

Consider a holomorphic proper submersion between complex manifolds, $\pi : M \to B$. Let g be a hermitian metric on the holomorphic relative tangent bundle to π, denoted $T_{M/B}$, and let J be the complex structure on the underlying real bundle to $T_{M/B}$ and $H_{M/B}$ the choice of a horizontal bundle i.e a smooth subbundle of TM such that we have $TM = T_{M/B} \oplus H_{M/B}$.

Definition 2.1.15. *(Kähler Fibration)*
We say that this data defines a Kähler Fibration is there exists a smooth $(1,1)$-real form, say ω over M such that

1. *The form ω is closed*
2. *The real bundles $H_{M/B_{\mathbb{R}}}$ and $T_{M/B_{\mathbb{R}}}$ are orthogonal with respect to ω*
3. *We have $\omega(X, Y) = g(X, JY)$ for X and Y vertical real vector fields.*

Let us take $\overline{E}$ a hermitian bundle, we will make the two following assumptions

(A1) Assume that π is a (proper) smooth submersion that is equipped with a structure of Kähler fibration.

(A2) Assume that E is π_*-acyclic, meaning that $R^q\pi_* E = 0$ as soon as $q > 0$.

In this case, the upper-semi-continuity theorem ensures that $\pi_* E$ is a vector bundle over Y, and for each (closed) point of y we have $j_y^* \pi_* E = H^0(X_y, E_{|X_y})$. Now, using the Kähler fibration structure on π, we get a smooth family of metrics over the relative tangent spaces to π, that give a Kähler structure to the fiber X_y. Now, we can identify the space $H^0(X_y, E_{|X_y})$ to the subspace of $A^0(X_y, E)$ of smooth forms with coefficients in E, that are holomorphic, i.e killed by $\overline{\partial}$. Now, on $A^0(X_y, E)$ we have a natural hermitian form given by the Kähler metric, defined by

$$\langle s, t \rangle = \int_{X_y} h^E(s(x), t(x))\omega$$

where ω is the volume form defined by the Kähler metric on X_y.

We have thus defined a (punctual) metric on each fiber of the vector bundle $\pi_* E$, which we will call *the L^2-metric associated to the Kähler fibration.*

Theorem 2.1.16. *Assuming the previous conditions, on X, Y, π and $\overline{E}$, the L^2 metric is smooth and thus define a hermitian vector bundle structure on $\pi_* E$.*

Proof. See [BGV92, p 278] □

We will denote $\overline{\pi_* E}^{L^2}$ the vector bundle $\pi_* E$ equipped with its L^2-metric subordinated to the choice of metrics on both X and Y. The problem now is to choose a form say Ξ such that

$$\pi_*[E, 0] = [\overline{\pi_* E}^{L^2}, \Xi]$$

A first step to understand what this Ξ should be, is to investigate the Riemann-Roch formula, let us recall the

Theorem 2.1.17. *(Grothendieck-Riemann-Roch) Let $\pi : X \to Y$ be a projective morphism between smooth projective varieties over a field k, and E be a vector bundle over X, then we have an equality of cycles in $\mathrm{CH}_{\mathbb{Q}}(Y)$,*

$$\mathrm{ch}(\pi_* E) = \pi_*(\mathrm{ch}(E)\,\mathrm{Td}(T_\pi))$$

Here, the class $\mathrm{Td}(T_\pi)$ denotes the Todd class of the virtual tangent bundle, defined as an element in K-theory, as

$$T_\pi = [j^* T_{P/Y}] - [N_{X/P}]$$

for any smooth variety P such that we have a factorization

$$X \xrightarrow{j} P \xrightarrow{p} Y$$

with j being a (regular) immersion and p being a smooth submersion, this virtual class does not depend on the choice of P.

Now, as we have equipped the bundle E mentioned in the previous theorem with a hermitian metric, we then have a canonical connection on E, defined by

1. preservation of the metric $d\langle s, t \rangle = \langle \nabla s, t \rangle + \langle s, \nabla t \rangle$.

2. compatibility with the $\overline{\partial}$ operator, $\nabla^{0,1} = \overline{\partial}$

this connection is called the Chern connection on $\overline{E}$. And we can define the Chern character form of such a connection as

$$\mathrm{ch}(\overline{E}) = \mathrm{tr}(e^{\frac{i}{2\pi}\nabla^2}) \in A_{\mathbb{R}}^{p,p}(X)$$

This is a closed form and if we change the metric on E the corresponding form is altered by a $\partial\overline{\partial}$-exact form, so its cohomology class does not depend on the different metrics involved in its definition. In fact we can relate the secondary Bott-Chern form to the infinitesimal generator of one-parameter family of metric on E (see for instance [Fal92, p.21-22])

Proposition 2.1.18. *Assume that we have a family of metrics over E, say h_t, depending smoothly on a real parameter t, let us denote by $\mathcal{E}_t$ the exact sequence $0 \to (E, h_0) \to (E, h_t) \to 0$, then we have*

$$\partial_t \,\mathrm{tr}(e^{\frac{i}{2\pi}\nabla_t^2}) = dd^c\,\mathrm{tr}(\frac{1}{2}N_t e^{\frac{i}{2\pi}\nabla_t^2}) = \frac{1}{2}dd^c\,\widetilde{\mathrm{ch}}(\mathcal{E}_t)$$

where N_t denotes the number-operator associated to h_t defined by $\partial_t h_t(a,b) = h_t(N_t a, b)$.

This results implies, by integration (and using the fact that the differential on X commutes with integration along $\mathbb{A}^1$), the fact that the class of the Chern character form of the connection does not depend on the metric, in fact we could simply prove a simple transgression formula much more easily (see [BGV92, Prop 1.41]) and deduce this result from it.

Now, using Grothendieck-Riemann-Roch Theorem we see that

$$\mathrm{ch}(\overline{\pi_* E}^{L^2}) - \pi_*[\mathrm{ch}(\overline{E})\,\mathrm{Td}(\overline{T_{X/Y}})]$$

must be the $\partial\overline{\partial}$ of some smooth form over Y, and our form Ξ should be one of those forms, of course there are many possible choices.

Bismut and Köhler were able to give a satisfying choice for Ξ. Let us show how they proceed.

Over each point y of Y, we have the relative Dolbeault complex, given by

$$0 \to A^0(X_y, E) \to A^{0,1}(X_y, E) \xrightarrow{\overline{\partial}} A^{0,2}(X_y, E) \to ...$$

where $A^{0,q}(X_y, E)$ denotes the space of smooth sections of the bundle $E \otimes \bigwedge^q T_{X_y}^{0,1}$. This complex, is equipped with a hermitian form extending the one defined of smooth sections of E, by the formula

$$\langle\, s\otimes\mu, t\otimes\nu\,\rangle = \int_{X_y} h^E(s(x), t(x))h^{X_y}(\mu,\nu)\omega$$

where we extend the Kähler metric on T_{X_y} to $\bigwedge^q T_{X_y}^{0,1}$. For this metric, the operator $\overline{\partial}$ has a formal adjoint, $\overline{\partial}^*$, and an associated Laplacian $\Delta_E = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. We need to define the notion of determinant of this Laplacian, for this, we use a ζ-regularization process. As the eigenvalues of Δ_q, the restriction of the Laplacian, on $(0,q)$-forms, form a discrete subset of the real positive numbers, we set for $\Re(s)$ large enough

$$\zeta_{\Delta_q}(s) = \sum_{\lambda_i > 0} \lambda_i^{-s}$$

this function extends to a meromorphic function over $\mathbb{C}$, which is holomorphic at the origin, so we can set

$$\det(\Delta_q) = e^{-\zeta'_{\Delta_q}(0)}$$

the analytic torsion (at the point y) is defined to be

$$T_y = \sum (-1)^q \frac{q}{2} \zeta'_{\Delta_q}(0)$$

this gives a smooth function on the base Y. This gives the 0-th degree part of the Higher analytic forms constructed by Bismut-Köhler, in general they defined (see [BK92, Def 1.7, Def 1.8, the paragraph before Thm 3.4 and Def 3.7] for the definition of the different terms, which we will not need)

Theorem 2.1.19. *(Bismut, Köhler)*
Let $\overline{E}$ be a hermitian bundle and $\pi : X \to Y$ a holomorphic submersion endowed with a Kähler fibration structure, set $T(\overline{T_{X/Y}}, \overline{E}) = \zeta'_E(0)$ where

$$\zeta_E(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^s}{u} \left[\varphi \operatorname{tr_s}(N_u e^{-B_u^2}) - \varphi \operatorname{tr_s}(N_V e^{-\nabla_\pi^2}) \right] du$$

then

$$\pi_*[\operatorname{ch}(\overline{E}) \operatorname{Td}(\overline{T_{X/Y}})] - \operatorname{ch}(\overline{\pi_* E}^{L^2}) = dd^c T(\overline{T_{X/Y}}, \overline{E})$$

The form $T(\overline{T_{X/Y}}, \overline{E})$ is called the Higher Analytic Torsion form *associated to $\overline{E}$ and $\overline{T_{X/Y}}$*

Proof. This is [BK92, Theorem 0.2] □

Remark 2.1.20. In the previous theorem the higher analytic torsion form $T(\overline{T_{X/Y}}, \overline{E})$ is associated to a Kähler fibration structure on π, note however that when T_X and T_Y are equipped with Kähler metrics, we have a natural structure of Kähler fibration on π (see [BGS88b, Thm 1.5]), in this case, we will then denote $T(\overline{T}_X, \overline{T}_Y, \overline{E})$ instead of $T(\overline{T_{X/Y}}, \overline{E})$ to mean the higher analytic torsion form associated to the Kähler fibration structure induced by the Kähler metrics over T_X and T_Y

We list here the fundamental properties of this higher analytic torsion that we may need.

Proposition 2.1.21. *Let $\pi : X \to Y$ be a smooth submersion equipped with a Kähler fibration structure, let $\overline{E}$ be a π_*-acyclic vector bundle, and let's endow $\pi_* E$ with its L^2-metric. The* analytic torsion *associated to this data, $T(\overline{T_{X/Y}}, \overline{E})$ is a smooth form in $\widetilde{A}_{\mathbb{R}}^{\bullet,\bullet}(Y)$ that satisfy*

1. *(Naturality) Let $g : Y' \to Y$ be projective morphism, then $X_{Y'} \to Y'$ is a Kähler fibration, and we have*

$$T(\overline{g^* T_{X/Y}}, g^* \overline{E}) = g^* T(\overline{T_{X/Y}}, \overline{E})$$

2. *(Additivity) For every pair $\overline{E}_1, \overline{E}_2$ of hermitian vector bundles on X, we have*

$$T(\overline{T_{X/Y}}, \overline{E}_1 \oplus^\perp \overline{E}_2) = T(\overline{T_{X/Y}}, \overline{E}_1) + T(\overline{T_{X/Y}}, \overline{E}_2)$$

3. *(Compatibility with the projection formula) For F a hermitian vector bundle on Y, we have*

$$T(\overline{T_{X/Y}}, \overline{E} \otimes \pi^* \overline{F}) = T(\overline{T_{X/Y}}, \overline{E}) \otimes \operatorname{ch}(\overline{F})$$

4. *(Transitivity) If $\pi : X \to Y$ and $\pi' : Y \to Z$ are two Kähler fibration structures and E is a bundle π_*-acyclic, such $\pi_* E$ is also π'_*-acyclic we have the following relation between the different analytic torsions*

$$\begin{aligned} T(\overline{T_{X/Z}}, \overline{E}) &= T(\overline{T_{Y/Z}}, \overline{\pi_* E}^{L^2}) + \pi'_*(T(\overline{T_{X/Y}}, \overline{E}) \operatorname{Td}(\overline{T_{Y/Z}})) \\ &\quad + \widetilde{\operatorname{ch}}(\overline{\pi' \circ \pi_* E}^{L^2}, \overline{\pi'_*(\overline{\pi_* E}^{L^2})}^{L^2}) \\ &\quad + \pi'_* \pi_* (\operatorname{ch}(\overline{E}) \widetilde{Td}(\mathcal{E}) \operatorname{Td}(\overline{T}_{X/Y}) \operatorname{Td}^{-1}(\overline{T}_X)) \end{aligned}$$

where $\mathcal{E}$ is the exact sequence

$$\mathcal{E} : 0 \to \overline{T_{X/Y}} \to \overline{T_{X/Z}} \to \pi^* \overline{T_{Y/Z}} \to 0$$

Proof. This is [BGFiML14, Cor 8.10, Cor 8.11]. □

2.1.3 Generalized Analytic Torsion of Burgos, Freixas, Litcanu

To investigate the situation for a general projective morphism, Burgos, Freixas and Litcanu have split the problem into two different ones. First one wants to construct direct images for projective spaces $\mathbb{P}^r_Y \to Y$ and for closed immersions, and ask for a compatibility condition that would ensure a general functoriality property.

That's why Burgos, Freixas and Litcanu defined

Definition 2.1.22. *(Generalized Theory of Analytic torsion for submersions) A theory of generalized analytic torsion forms for submersions is an assignment of a smooth real form, $T(\overline{T_X}, \overline{T_Y}, \overline{E})$ in $\widetilde{A}^{\bullet,\bullet}_{\mathbb{R}}(Y)$ to every smooth submersion $X \xrightarrow{\pi} Y$ and every hermitian bundle $\overline{E}$ over X, with T_X and T_Y equipped with a Kähler metric, and $\overline{E}$ being π_*-acyclic, satisfying*

$$\pi_*[\operatorname{ch}(\overline{E}) \operatorname{Td}(\overline{T_{X/Y}})] - \operatorname{ch}(\overline{\pi_* E}^{L^2}) = dd^c T(\overline{T}_X, \overline{T}_Y, \overline{E})$$

We say that a theory of generalized analytic torsion forms for submersions is well behaved, if it satisfies the following properties

1. *(Naturality) Let $g : Y' \to Y$ be projective morphism, then $X' = X_{Y'} \xrightarrow{g'} Y'$ is a also a smooth submersion, and for any choice of metrics over $T_{X'}$ and $T_{Y'}$ such that we have an isometry $\overline{T_{X'/Y'}} \simeq g'^* \overline{T_{X/Y}}$ we have*

$$T(\overline{T_{X'}}, \overline{T_{Y'}}, g'^* \overline{E}) = g^* T(\overline{T_X}, \overline{T_Y}, \overline{E})$$

2. *(Additivity) For every pair $\overline{E}_1, \overline{E}_2$ of hermitian vector bundles on X, we have*

$$T(\overline{T_X}, \overline{T_Y}, \overline{E}_1 \oplus^{\perp} \overline{E}_2) = T(\overline{T_X}, \overline{T_Y}, \overline{E}_1) + T(\overline{T_X}, \overline{T_Y}, \overline{E}_2)$$

3. *(Compatibility with the projection formula) For F a hermitian vector bundle on Y, we have*

$$T(\overline{T_X}, \overline{T_Y}, \overline{E} \otimes \pi^* \overline{F}) = T(\overline{T_X}, \overline{T_Y}, \overline{E}) \otimes \operatorname{ch}(\overline{F})$$

Remark 2.1.23. The theory of analytic torsion defined by Bismut and Köhler, is an example of such a well-behaved theory, we will denote it T^{BK}.

Let's now turn to the case of a closed immersion, so assume now, that we've been given i a (regular) closed immersion between projective smooth complex varieties. The geometric situation is somewhat more complicated given that in general i_*E will not be a vector bundle on Y, so we can just arbitrarily choose a hermitian structure on i_*E, given by a resolution of that sheaf on Y, let us choose such a resolution $E_\bullet \to i_*E \to 0$.

We thus have a current, represented by a smooth form $\mathrm{ch}(\overline{i_*E})$ which is equal to $\sum(-1)^i \mathrm{ch}(\overline{E}_i)$, we want to compare it to the current $i_*[\mathrm{ch}(\overline{E})\,\mathrm{Td}(\overline{N})^{-1}]$ for a choice of metric over the normal bundle to i.

Notice here that $\mathrm{ch}(\overline{E})\,\mathrm{Td}(\overline{N})^{-1}$ is a well-defined smooth form on X, but is only a current on Y, when we push it forward through i_*, however as the cohomology of currents coincide with that of forms ([GH94]), we do know that there exists a current $\Xi \in \widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(Y)$ depending *a priori* on the choice of the metric on N, and the metric structure on i_*E such that

$$i_*[\mathrm{ch}(\overline{E})\,\mathrm{Td}(\overline{N})^{-1}] - \sum(-1)^i \mathrm{ch}(\overline{E}_i) = \partial\overline{\partial}\Xi$$

we can, here again, try to give an explicit formula, let us examine the case of a smooth effective divisor.

Assume that we've been given a hermitian line bundle $\overline{L}$ over Y, such that X is the zero locus of a section s of L, transverse to the zero section. We have the following exact sequence

$$0 \to L^\vee \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0$$

that gives a resolution of $i_*\mathcal{O}_X$ over Y, if we equip $\mathcal{O}_X$ with the trivial metric, and $L^\vee$ with the dual metric, we get a hermitian structure on $i_*\mathcal{O}_X$. Moreover as i^*L is naturally isomorphic to $N_{X/Y}$ we also have a natural metric on the normal bundle to i. For this data, a current Ξ solving the Grothendieck-Riemann-Roch equation is easy to compute.

Proposition 2.1.24. *Let $\overline{L}$ be a hermitian vector bundle over a smooth complex variety, say Y, and let X be the zero scheme of a transverse section. We denote by j the corresponding regular immersion. We have*

$$\mathrm{ch}(\overline{j_*O_X}) = \mathrm{ch}([\overline{O}_Y] - [\overline{L}^\vee]) = j_*(\mathrm{ch}(\overline{O}_X)\,\mathrm{Td}(\overline{j^*L})^{-1}) - dd^c(\mathrm{Td}(\overline{L})^{-1} \wedge \log\|s\|^2)$$

Proof. First let us compute

$$\begin{aligned}
\mathrm{ch}(\overline{j_*O_X}) &= \mathrm{ch}([\overline{O}_Y] - [\overline{L}^\vee]) \\
&= e^{c_1(\overline{O}_Y)} - e^{c_1(\overline{L}^\vee)} \\
&= 1 - e^{-c_1(\overline{L})}
\end{aligned}$$

Let's now compute

$$\begin{aligned}
j_*(\mathrm{ch}(\overline{O}_X)\,\mathrm{Td}(\overline{j}^*L)^{-1}) &= j_*(\mathrm{Td}[j^*\overline{L}]^{-1}) \\
&= \mathrm{Td}(\overline{L})^{-1} j_*(1_X) \\
&= \mathrm{Td}(\overline{L})^{-1} \wedge \delta_X \\
&= \mathrm{Td}(\overline{L})^{-1} \wedge (dd^c \log\|s\|^2 + c_1(\overline{L})) \\
&= dd^c(\mathrm{Td}(\overline{L})^{-1} \wedge \log\|s\|^2) + c_1(\overline{L}) \wedge \mathrm{Td}(\overline{L})^{-1} \\
&= dd^c(\mathrm{Td}(\overline{L})^{-1} \wedge \log\|s\|^2) + (1 - e^{-c_1(\overline{L})})
\end{aligned}$$

where we have used the projection formula, the definition of δ_X, the Poincaré-Lelong formula, the fact that the Todd form is closed, and finally the definition of the Todd form. The proposition follows. □

Remark 2.1.25. The next case that we can explicitly compute is the case of the immersion of X into $\mathbb{P}(1+E)$ where E is a vector bundle over X. Here we have an explicit resolution given by the Koszul complex

$$0 \to \bigwedge^r Q \to ... \to \bigwedge^2 Q \to Q \xrightarrow{s} \mathcal{O}_{\mathbb{P}(1+E)} \to i_*\mathcal{O}_X \to 0$$

where s is a section having X for zero locus, given by the vanishing of the image of 1 in $Q^\vee$ in the exact sequence

$$0 \to \mathcal{O}(-1) \to q^*E \oplus 1 \to Q^\vee \to 0$$

Of course the general strategy to build a current Ξ is then to use the deformation to the normal cone process to deform an arbitrary resolution of i_*E into this explicit one.

To generalize this phenomenon, we need the following definition.

Definition 2.1.26. *(Singular Bott-Chern Current)*
Let $i : Z \to X$ be a (regular) immersion between smooth projective complex varieties and $\overline{E}$ a hermitian bundle over Z, we assume that we've been given a hermitian structure on i_*E and on $N = N_{Z/X}$.

A singular Bott-Chern current for this data, which we will denote $\mathrm{bc}(\overline{N}, \overline{i_*E})$ is current defined up to $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$ satisfying the following differential equation

$$\mathrm{ch}(\overline{i_*E}) = \sum_{i\geq 0}(-1)^i \mathrm{ch}(\overline{E}_i) = i_*(\mathrm{ch}\,\overline{E}\,\mathrm{Td}(\overline{N})^{-1}) - dd^c(\mathrm{bc}(\overline{N}, \overline{i_*E}))$$

Remark 2.1.27. Notice that we have chosen to compare $i_*[\mathrm{ch}(\overline{E})\,\mathrm{Td}(\overline{N})^{-1}]$ with $\sum(-1)^i \mathrm{ch}(\overline{E}_i)$ but there is another choice, just as natural, namely to compare $\sum(-1)^i \mathrm{ch}(\overline{E}_i)$ with $i_*[\mathrm{ch}(\overline{E})\,\mathrm{Td}(i^*\overline{T}_X)^{-1}\,\mathrm{Td}(\overline{T_Z})]$ as both classes are mapped to the same cohomology class.

We could of course give the same definition replacing $\mathrm{bc}(\overline{N}, \overline{i_*E})$ by $\mathrm{bc}(\overline{T}_Z, \overline{T}_X, \overline{i_*E})$ which would satisfy the equation

$$\mathrm{ch}(\overline{i_*E}) = \sum_{i\geq 0}(-1)^i \mathrm{ch}(\overline{E}_i) = i_*[\mathrm{ch}(\overline{E})\,\mathrm{Td}(i^*\overline{T}_X)^{-1}\,\mathrm{Td}(\overline{T_Z})] - dd^c(\mathrm{bc}(\overline{T}_Z, \overline{T}_X, \overline{i_*E}))$$

If we have a singular Bott-Chern current for one of these two choices, it is easy to find a singular Bott-Chern current for the other by the following formula

$$\mathrm{bc}(\overline{T}_Z, \overline{T}_X, \overline{i_*E}) = \mathrm{bc}(\overline{N}, \overline{i_*E}) + i_*[\mathrm{ch}(\overline{E})\widetilde{\mathrm{Td}}^{-1}(\mathcal{E})\,\mathrm{Td}(\overline{T}_Z)]$$

where $\mathcal{E}$ is the exact sequence

$$0 \to T_X \to i^*T_Y \to N \to 0$$

Of course if that exact sequence were to be meager for the different metrics chosen, then the two Bott-Chern singular currents would agree.

In the situations that will be of primal interest to us, we will have metrics on X and Y instead of N, so we will use the singular Bott-Chern determined by the tangent metrics rather than the normal one, nevertheless in the literature, the formulae for singular Bott-Chern currents are usually given for a choice of metric on the normal bundle, that's why we chose to follow this convention in the remainder of this section.

We hope that the reader will have no problem in making the occasional switch between properties for $\mathrm{bc}(\overline{N}, \overline{i_*E})$ and $\mathrm{bc}(\overline{T}_Z, \overline{T}_X, \overline{i_*E})$

The previous proposition gives us a useful characterization of a singular Bott-Chern current in the case of the immersion of a divisor.

Notice that, the fact that a singular Bott-Chern current always exists is a trivial consequence of the Grothendieck-Riemann-Roch theorem (and the $\partial\overline{\partial}$-lemma for currents), but notice also that there are many possible choices *a priori* for $\mathrm{bc}(\overline{N}, \overline{i_*E})$, all differing by a cohomology class.

The general case of a (regular) immersion is much more complicated but Bismut-Gillet-Soulé in [BGS] gave an explicit formula for a singular Bott-Chern current. Just like for the case of the analytic torsion a little bit of technology (which we will not explicitly describe) is needed to be able to state the result.

We can state the following formula (see [BGS] for the definition of the terms, we won't need those in the following)

Theorem 2.1.28. *(Bismut-Gillet-Soulé)*
Let $j : X \to Y$ be an immersion between compact complex manifold and $\overline{E}$ a hermitian vector bundle on X, let us endow $N_{X/Y}$ with a hermitian metric and i_*E with a hermitian structure $\overline{E}_i \to i_*E$ satisfying the condition (A) of Bismut, then $\zeta'_E(0)$ is a singular Bott-Chern Current for this data, where

$$\zeta_E(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^s}{u} \left[\mathrm{tr_s}(Ne^{-A_u^2}) - i_* \int_X \mathrm{tr_s}(Ne^{-B^2}) \right] du$$

This current agrees with $-\log \|s\|^2 \,\mathrm{Td}(\overline{L})^{-1}$ in the case of the immersion of a smooth divisor.

Proof. See [BGS, Theorem 1.9, Theorem 3.17] □

Here again, there are a priori many possible choices for a singular Bott-Chern current, and we wish to determine uniquely a choice for it that agrees with our explicit choice for divisors. Fortunately the classification of the theories of singular Bott-Chern currents has been accomplished by Burgos and Litcanu in [BGL10].

We now turn to a brief description of their theory, to do so we need to describe a paradigmatic situation in which we will state the analog of the properties of 2.1.21 for a singular Bott-Chern current.

Let $i : X \to Y$ and $j : Y \to Z$, be regular immersions of complex varieties, we have the following exact sequence

$$0 \to N_{X/Y} \to N_{X/Z} \to j^* N_{Y/Z} \to 0 \qquad (\dagger)$$

and assume that we have chose a hermitian structure on i_*E given by a complex $\overline{E}_\bullet$, as j_* is exact we have an exact sequence

$$0 \to j_*E_n \to ... \to j_*E_1 \to j_*i_*E \to 0$$

if we equip all the E_i's with hermitian structures, say $\overline{E_{i,\bullet}}$ we get a global resolution of j_*i_*E given by the total complex of the double complex $E_{\bullet,\bullet}$. This is the hermitian structure that j_*i_*E will be equipped with.

Definition 2.1.29. *A* theory of singular Bott-Chern classes *is an assignment of a current* $\mathrm{bc}(\overline{N},\overline{E},\overline{i_*E})$ *to each immersion* $i : X \to Y$ *between smooth projective complex varieties and a hermitian bundle* $\overline{E}$ *over* X*, equipped with a hermitian structure on both* $N_{X/Y}$ *and* i_*E*, satisfying*

$$\mathrm{ch}(\overline{i_*E}) = i_*(\mathrm{ch}\,\overline{E}\,\mathrm{Td}(\overline{N})^{-1}) - dd^c\,\mathrm{bc}(\overline{N},\overline{E},\overline{i_*E})$$

A theory of singular Bott-Chern Classes is said to be

1. *natural: if given* $g : Y' \to Y$ *a morphism transverse to* i *(e.g smooth), recall that the transversality condition implies* $g^*N_{X/Y} \simeq N_{X'/Y'}$*, we have*

$$\mathrm{bc}(\overline{g^*N_{X/Y}}, g^*\overline{E}, g^*\overline{i_*E}) = g^*\,\mathrm{bc}(\overline{N}_{X/Y},\overline{E},\overline{i_*E})$$

2. *additive: if*

$$\mathrm{bc}(\overline{N},\overline{E}_1 \oplus^{\perp} \overline{E}_2, \overline{i_*E_1} \oplus^{\perp} \overline{i_*E_2}) = \mathrm{bc}(\overline{N},\overline{E}_1,\overline{i_*E_1}) + \mathrm{bc}(\overline{N},\overline{E}_2,\overline{i_*E_2})$$

3. *compatible with the projection formula: if*

$$\mathrm{bc}(\overline{N},\overline{E}\otimes i^*\overline{F},\overline{i_*E}\otimes\overline{F}) = \mathrm{bc}(\overline{N},\overline{E}_1,\overline{i_*E_1})\,\mathrm{ch}(\overline{F})$$

   where $\overline{F}$ *is a hermitian vector bundle on* Y

4. *transitive: if it is additive and if for every composition of closed immersion* $i : X \to Y$*, and* $j : Y \to Z$*, and for every choice of metrics on the normal bundles, we have*

$$\begin{aligned}\mathrm{bc}(\overline{N}_{X/Z},\overline{E},\overline{j_*i_*E}) &= \sum_{r\geq 0}(-1)^r\,\mathrm{bc}(\overline{N}_{Y/Z},\overline{E}_r,\overline{j_*E_r})\\ &\quad + j_*[\mathrm{bc}(\overline{N}_{X/Y},E,\overline{i_*E})\,\mathrm{Td}(\overline{N}_{Y/Z})^{-1}] + j_*i_*[\mathrm{ch}(\overline{E})\widetilde{\mathrm{Td}}^{-1}(\dagger)]\end{aligned}$$

Remark 2.1.30. If the different conditions in the previous definition are only satisfied for a particular class of metric structure we will say that the corresponding theory of singular Bott-Chern is the corresponding adjective with respect to that particular choice of metrics.

If a theory of Bott-Chern singular currents satisfies all of the assumptions above, we will say that it is well-behaved.

We have the following proposition

Proposition 2.1.31. *The theory of singular Bott-Chern Classes defined by Bismut and given for a metric satisfying condition* (A) *by formula 2.1.28 is well-behaved.*

Proof. This is [BGL10, Prop 9.28] □

Therefore the singular Bott-Chern current constructed by Bismut-Gillet-Soulé (that we will denote bc^{BGS}) is an example of a well-behaved theory of singular Bott-Chern current, but it is far from being the only one. Indeed we have

Theorem 2.1.32. *For any choice of a real additive genus* S *there exists a unique theory of well behaved Bott-Chern singular currents satisfying*

$$\mathrm{bc}(\overline{E},\overline{i_*E},\overline{N}) = \mathrm{bc}^{BGS}(\overline{E},\overline{i_*E},\overline{N}) + i_*[\mathrm{ch}(E)\,\mathrm{Td}(N)S(N)]$$

Proof. This is [BGFiML14, 7.14] □

Remark 2.1.33. If Λ is a ring, a genus over Λ is simply a power series over Λ. We will say that a genus g is multiplicative (resp. additive) if we extend it as a power series given by

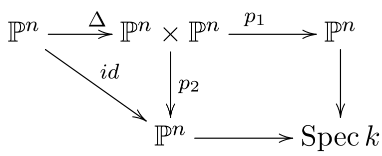
$$g(T_1, ..., T_n) = g(T_1)...g(T_n)(\text{resp. } g(T_1) + ... + g(T_n))$$

We can associate to such a genus over Λ a characteristic form with coefficients in Λ, which will be also called a genus.

Up to this point we have considered two kinds of secondary objects, the (higher) analytic torsion form for Kähler fibrations $\pi : X \to Y$, and the singular Bott-Chern classes for immersions $i : Y \to Z$, anticipating just a bit on the following section, we will see that these two secondary objects help us define a direct image in arithmetic weak $\widehat{K}$-theory, each construction will assure functoriality of this direct image with respect to the kind of morphism it is defined with, i.e we will have $(ij)_* = i_*j_*$ for composition of closed immersions, and we will have $(\pi\pi')_* = \pi_*\pi'_*$ for composition of Kähler fibrations.

In order to have a general functoriality property for arbitrary projective morphism, one needs to impose a compatibility condition between analytic torsion, and singular Bott-Chern classes. Burgos, Freixas, and Litcanu have studied that question in [BGFiML14] and it turns out that it can be done as soon as we have compatibility for them in a mild situation.

Let's consider the following diagram



If we want to achieve functoriality in $\widehat{K}$-theory, the least we can ask is that $p_{2,*}\Delta_* = \text{id}$, let us write down explicit equations for this condition to be true for the trivial bundle over $\mathbb{P}^n$.

We have an explicit resolution of $\Delta_*\mathcal{O}_{P_n}$ given by the Koszul complex

$$0 \to \bigwedge^r(p_2^*Q \otimes p_1^*\mathcal{O}(1)) \to ... \to \bigwedge^2(p_2^*Q \otimes p_1^*\mathcal{O}(1)) \to (p_2^*Q \otimes p_1^*\mathcal{O}(1)) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \Delta_*\mathcal{O}_X \to 0$$

where Q is the universal subbundle on $\mathbb{P}^n$. Now if we choose a trivial metric on the trivial bundle of rank $n+1$ on $\mathbb{P}^n$ we get a Fubini-Study metric on $\mathcal{O}(1)$, and on $T_{\mathbb{P}^n}$ and also on $T_{\mathbb{P}^n \times \mathbb{P}^n}$, moreover, the universal exact sequence

$$0 \to Q \to q^*\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}(1)$$

enables us to equip Q with a metric too.

Therefore we have a metric structure on $\Delta_*\mathcal{O}_{\mathbb{P}^n}$, now let us define $p_{2*}\overline{\Delta_*\mathcal{O}_{\mathbb{P}^n}}$ as $\sum(-1)^i \overline{p_{2*}\bigwedge^r(p_2^*\overline{Q} \otimes p_1^*\overline{\mathcal{O}(1)})}^{L^2}$ where we use the structure of Kähler fibration defined by the Fubini Study metrics over $\mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{P}^n$, of course as the normal bundle to the diagonal immersion is naturally isomorphic to the tangent bundle of $\mathbb{P}^n$, we also have a metric on it.

We need to compare this class in $\widehat{K}$-theory, with the class of $\mathcal{O}_{\mathbb{P}^n}$ with trivial metric; in order to have compatibility of the two kinds of secondary objects that have been added, the difference between them must be zero, and this justifies the following definition extracted form [BGFiML14, Def 6.2]

Definition 2.1.34. *We say that a well behaved theory of Bott-Chern singular class is compatible with a well behaved theory of analytic torsion if the following identity holds for the situation previously defined*

$$\begin{aligned} 0 &= \sum(-1)^r T(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{P}^n, \overline{p_{2*} \bigwedge^r p_2^*\overline{Q} \otimes p_1^*\overline{\mathcal{O}(1)})}^{L^2} \quad) \\ &\quad + p_{2*}[\mathrm{bc}(\overline{N}_{\mathbb{P}^n/\mathbb{P}^n\times\mathbb{P}^n}, \overline{i_*\mathcal{O}_{\mathbb{P}^n}}). \mathrm{Td}(\overline{T_{\mathbb{P}^n\times\mathbb{P}^n/\mathbb{P}^n}})] \end{aligned}$$

The following proposition is due to Burgos, Freixas and Litcanu

Theorem 2.1.35. *For any choice of well-behaved theory of singular Bott-Chern currents for closed immersions, there exists a well-behaved theory of higher analytic torsion classes compatible with it.*

Proof. This is [BGFiML14, Thm 7.7] □

Remark 2.1.36. In [BGFiML14] they use a different normalization of the singular Bott-Chern current of Bismut-Gillet-Soulé, due to the fact that they work with the Deligne complex, instead of general currents, therefore they have to multiply the singular Bott-Chern current by $-\frac{1}{2}$ in order to have compatibility.

Recall the the R-genus of Bismut-Gillet-Soul is the additive characteristic class determined by the following equation

$$R(L) = \sum_{m \text{ odd}} (2\zeta(-m) + \zeta'(-m)(1 + \frac{1}{2} + ... + \frac{1}{m}))\frac{c_1(L)^m}{m!}$$

Theorem 2.1.37. *(Bismut; Burgos-Freixas-Litcanu)*
The theory of analytic torsion for Kähler fibrations associated to the singular Bott-Chern current bc^{BGS} *is given by*

$$T(\overline{T_X}, \overline{T_Y}, \overline{E}) = T^{BK}(\overline{T_X}, \overline{T_Y}, \overline{E}) - \int_{X/Y} \mathrm{ch}(E)\,\mathrm{Td}(T_{X/Y})R(T_{X/Y})$$

where R is the R-genus of Bismut-Gillet-Soulé.

Proof. This is the conjunction of [BGFiML14, Thm 7.14] and [Bis97, Thm 0.1 and 0.2] □

Remark 2.1.38. This result which is absolutely essential to us, is remarkable in many ways. Not only is it, technically, a very impressive formula to prove, that relies on hard analysis, it illustrates that the two constructions that Bismut, Gillet and Soulé made for the analytic torsion and the singular Bott-Chern currents were corresponding to each other enough to ensure functoriality for direct images in arithmetic K-theory.

In fact, in their strong K-theory groups it was not possible to define a direct image morphism for a closed immersion, but Bismut studied the compatibility of analytic torsion for closed immersions in [Bis97] (see also [GRS08]) and his main result ensures that when one has a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & P \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

with p and g smooth, then in weak $\breve{K}$-theory, we have $p_*i_* = g_*$, which we will have to check later (see 2.1.57)

From now on we will only work with the theory of singular Bott-Chern current of Bismut-Gillet-Soulé, whose Bott-Chern current associated to the immersion of a divisor is given by $-\log\|s\|^2 \operatorname{Td}(\overline{L})^{-1}$, which we will simply denote bc and with the analytic torsion for submersions compatible to it, which we will simply denote T.

2.1.4 Weak $\breve{G}$ and $\breve{K}$-theory

We're now able to define a weak arithmetic analog of K-theory, and of G-theory. Recall that $G_0(X) \simeq K_0(X)$ for regular schemes, and *a fortiori* for smooth projective varieties over a number field. During this part, we will fix $\overline{X}$ an arithmetic variety.

Definition 2.1.39. *(Arithmetic variety)*
Let X be a smooth algebraic variety over a field k, such that its (holomorphic) tangent bundle is equipped with a hermitian Kähler metric $\overline{T_X}$, that is invariant under complex conjugation.

Remark 2.1.40. We will often denote d_X the dimension of X as a complex manifold, moreover, note that on the algebraic variety $\operatorname{Spec} k$ there is only one metric. We will thus simply denote $\operatorname{Spec} k$ for the arithmetic variety $\overline{\operatorname{Spec} k}$ equipped with that metric.

Definition 2.1.41. *(Weak $\breve{G}$-theory)*
We set $\breve{G}_0(\overline{X})$ to be the free abelian group $\bigoplus \mathbb{Z}[\overline{\mathcal{F}}] \times \widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X)$ where $\overline{\mathcal{F}}$ is a quasi-isometry class of hermitian coherent sheaves over X, subject to the following relations: for every exact sequence $\mathcal{E} : 0 \to \overline{\mathcal{F}''} \to \overline{\mathcal{F}} \to \overline{\mathcal{F}'} \to 0$,

$$\overline{\mathcal{F}} = \overline{\mathcal{F}''} + \overline{\mathcal{F}'} + \widetilde{\operatorname{ch}}(\mathcal{E})$$

Similarly we define

Definition 2.1.42. *(Weak $\breve{K}$-theory) We set $\breve{K}_0(\overline{X})$ to be the free abelian group $\bigoplus \mathbb{Z}[\overline{E}] \times \widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X)$ where $\overline{E}$ is an isometry class of hermitian vector bundle over X, subject to the following relations: for every exact sequence $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$,*

$$\overline{E} = \overline{E''} + \overline{E'} + \widetilde{\operatorname{ch}}(\mathcal{E})$$

For the case of smooth projective varieties it makes no difference to work with either one of them as the next proposition shows

Theorem 2.1.43. *Let X be a smooth variety, then we have a natural isomorphism*

$$\breve{G}_0(\overline{X}) \xrightarrow{\sim} \breve{K}_0(\overline{X})$$

Proof. We have an obvious map form $\breve{K}_0(\overline{X})$ to $\breve{G}_0(\overline{X})$ that maps a hermitian bundle $\overline{E}$ to the same bundle equipped with the hermitian structure $0 \to \overline{E} \xrightarrow{\text{id}} E \to 0$, and that maps $[0, g]$ to itself. Let us construct a map from $\breve{G}_0(\overline{X})$ to $\breve{K}_0(\overline{X})$, let $\overline{E}_\bullet \to \mathcal{F}$ be a hermitian coherent sheaf, we map $\overline{\mathcal{F}}$ to $\sum(-1)^i[\overline{E}_i]$.

This map is well defined, if $\overline{E}'_\bullet \to \mathcal{F}$ is another hermitian structure on $\mathcal{F}$ quasi-isometric to the first one, then we have a commutative up to homotopy diagram

$$\begin{array}{ccc} & \overline{H}_\bullet & \\ \swarrow & & \searrow \\ \overline{E}_\bullet & & \overline{E}'_\bullet \\ \searrow & & \swarrow \\ & \mathcal{F} & \end{array}$$

where the two top arrows are tight.

Now, by the very definition of $\breve{K}_0(\overline{X})$, if $\overline{M}_\bullet$ is a meager complex of vector bundles then $\sum(-1)^i\overline{M}_i = 0$, therefore the complex $\mathrm{cone}(\overline{H}_\bullet, \overline{E}_\bullet)[1] \oplus \mathrm{cone}(\overline{H}_\bullet, \overline{F}_\bullet)$ (which is a hermitian structure on the zero sheaf over X) being meager gives us the following identity in $\breve{K}_0(\overline{X})$,

$$\sum(-1)^i\overline{H}_i - \sum(-1)^i\overline{E}_i - \sum(-1)^i\overline{H}_i + \sum(-1)^i\overline{E}'_i = 0$$

which ensures that our map does not defined on the quasi-isometry class of the chosen metric structure.

Now if

$$\mathcal{E} : 0 \to \overline{\mathcal{F}''} \to \overline{\mathcal{F}} \to \overline{\mathcal{F}'} \to 0$$

is an exact sequence of hermitian sheaves we have to prove that

$$\sum(-1)^i\overline{F}''_i + \sum(-1)^i\overline{F}'_i - \sum(-1)^i\overline{F}_i + \widetilde{\mathrm{ch}}(\mathcal{E}) = 0 \qquad (\star)$$

Let us take another choice of resolutions of $\mathcal{F}$ and $\mathcal{F}''$, dominating the previous ones, such that we have a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G'_\bullet & \longrightarrow & G_\bullet & \longrightarrow & F'_\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

this is always possible as we have seen numerous times in the first part. Let us endow $G'_\bullet$ and $G_\bullet$ with arbitrary metrics. Now, by its very definition, we have

$$\widetilde{\mathrm{ch}}(0 \to \overline{G''_\bullet} \to \overline{G_\bullet} \to \overline{F'_\bullet} \to 0) + \widetilde{\mathrm{ch}}(\overline{G'_\bullet} \to \overline{F}'_\bullet) - \widetilde{\mathrm{ch}}(\overline{G_\bullet} \to \overline{F}_\bullet) = \widetilde{\mathrm{ch}}(\mathcal{E})$$

on the other hand we have in $\breve{K}_0(\overline{X})$,

$$\widetilde{\mathrm{ch}}(\overline{G_\bullet} \to \overline{F}_\bullet) = \sum(-1)^i\overline{F}_i - \sum(-1)^i\overline{G}_i$$

and similarly

$$\widetilde{\mathrm{ch}}(\overline{G''_\bullet} \to \overline{F}''_\bullet) = \sum(-1)^i\overline{F}''_i - \sum(-1)^i\overline{G}''_i$$

and of course

$$\widetilde{\mathrm{ch}}(0 \to \overline{G''_\bullet} \to \overline{G_\bullet} \to \overline{F'_\bullet} \to 0) = \sum(-1)^i\overline{G}_i - \sum(-1)^i\overline{G}''_i - \sum(-1)^i\overline{F}'_i$$

Putting all this together this yields $(\star)$, and our map is well defined.

This obviously gives a left inverse to the natural map from $\breve{K}_0(\overline{X})$ to $\breve{G}_0(\overline{X})$, it suffices thus to prove the surjectivity of this map, but this results immediately from the definition. □

Remark 2.1.44. One may wonder where the smoothness hypothesis intervene in the previous proof. It does not. By our very definition we have restricted ourselves to sheaves that admit finite locally free resolutions in $\breve{G}_0$ but the smoothness hypothesis implies that every coherent sheaf admits such resolutions, and this in turn will imply the surjectivity of the forgetful arrow $\zeta : \breve{G}_0(\overline{X}) \to G_0(X)$ which will be important for us, when given an arbitrary coherent sheaf, we want to equip it with a metric and view it as an element of the $\breve{K}_0$ as we will often do.

Remark 2.1.45. In the same spirit we could have defined an intermediate group where the vector bundles are equipped with hermitian structures given by resolutions instead of "classical metrics", we leave it to the reader to check that this group would have also been isomorphic to the $\breve{K}_0$ we defined.

Let's define a first Chern class operator and a pull-back operation.

Definition 2.1.46. *(Pull-Back)*
Let $f : \overline{X}' \to \overline{X}$ be a smooth equidimensional morphism between arithmetic varieties, we define $f^* : \breve{K}_0(\overline{X}) \to \breve{K}_0(\overline{X}')$ by the following formula

$$f^*[\overline{E}, g] = [f^*\overline{E}, f^*g]$$

where $f^*\overline{E}$ is the pull-back of E equipped with the hermitian metric that renders isometric the isomorphism $f^*E_x \simeq E_{f(x)}$; and $f^*(g)$ is the pull back of current (we recall its definition in 2.2.24).

The fact that this operation is well defined follows from the naturality of the secondary Bott-Chern classes.

Definition 2.1.47. *(First Chern class operator)*
Let $\overline{L} \in \widehat{\mathrm{Pic}}(X)$ be a hermitian line bundle over $\overline{X}$, we define

$$\widehat{c_1}(\overline{L})[\overline{E}, g] = [\overline{E}, g] - [\overline{E} \otimes \overline{L}^\vee, g \wedge \mathrm{ch}(\overline{L}^\vee)]$$

we will call this operator the first Chern class operator.

We have to check that this operator is well defined, namely that it sends a class (coming from an exact sequence $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$) of the form

$$\overline{E} - [\overline{E''} + \overline{E'} + \widetilde{\mathrm{ch}}(\mathcal{E})]$$

to zero, but this follows from the second point of 1.1.26, as the following sequence is exact

$$\mathcal{E} \otimes \overline{L}^\vee : 0 \to \overline{E'' \otimes L^\vee} \to \overline{E \otimes L^\vee} \to \overline{E' \otimes L^\vee} \to 0$$

therefore $\overline{E \otimes L^\vee} = \overline{E'' \otimes L^\vee} + \overline{E' \otimes L^\vee} + \widetilde{\mathrm{ch}}(\mathcal{E} \otimes L^\vee) = \overline{E'' \otimes L^\vee} + \overline{E' \otimes L^\vee} + \widetilde{\mathrm{ch}}(\mathcal{E})\,\mathrm{ch}(\overline{L}^\vee)$

Remark 2.1.48. Let's precise the action of $\widehat{c_1}(\overline{L})$ on classes of the form $[\overline{E}, 0]$ and $[0, g] = a(g)$, we see that

- $\widehat{c_1}(\overline{L})[\overline{E}, 0] = [\overline{E}, 0] - [\overline{E} \otimes \overline{L}^\vee, 0]$
- $\widehat{c_1}(\overline{L})a(g) = a(g \wedge (1 - \mathrm{ch}(\overline{L}^\vee))) = a(g \wedge c_1(\overline{L})\,\mathrm{Td}(\overline{L})^{-1})$

Remark 2.1.49. The attentive reader will have noticed that up to this point, nothing depends on the Kähler metric we have chosen on X. This dependency will play a part in the definition of the push-forward that we will give now.

Let us prove now that, as expected, arithmetic $\breve{K}$-theory is an extension of classical K-theory by the space of currents.

Proposition 2.1.50. *We have an exact sequence*

$$\widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X) \xrightarrow{a} \breve{K}_0(\overline{X}) \xrightarrow{\zeta} K_0(X) \to 0$$

Proof. Let $\alpha = \sum_i n_i[E_i, g_i]$ be any element in the kernel of ζ. We thus have $\sum_i n_i[E_i] = \sum_j m_j([F_j] - [F'_j] - [F''_j])$ for some exact sequences $\mathcal{F}_j : 0 \to F'_j \to F_j \to F''_j \to 0$, more precisely each E_i is isomorphic to an F_i. Therefore in $\breve{K}_0(\overline{X})$ we have

$$\alpha = \sum_j m_j([\overline{F}_j] - [\overline{F}'_j] - [\overline{F}''_j]) + a(g) = \sum_j a(\widetilde{\mathrm{ch}}(\overline{\mathcal{F}_j})) + a(g)$$

and we are done. □

It is much harder to define a push forward in arithmetic $\breve{K}$-theory, for two essential reasons.

The first one is geometric, we have to chose metrics on direct image of vector bundles, these images will in general be only coherent sheaves, but that's not a serious problem because of 2.1.43, however choosing a natural hermitian metric on the image sheaf is a much more serious question.

The second obstacle is due to the double-transgressive nature of Arakelov theory. We can extend the definition of characteristic form to $\breve{K}$-theory, by

$$\mathrm{ch}[\overline{E}, g] = \mathrm{ch}(\overline{E}) + dd^c g$$

Notice that this is the right choice because we obviously want

$$\mathrm{ch}\left(\overline{E} - [\overline{E''} + \overline{E'} + \widetilde{\mathrm{ch}}(\mathcal{E})]\right) = 0$$

and the secondary Bott-Chern forms satisfy

$$dd^c\widetilde{\mathrm{ch}}(\mathcal{E}) = \mathrm{ch}(\overline{E}) - \mathrm{ch}(\overline{E'}) - \mathrm{ch}(\overline{E''})$$

Therefore whichever definition we choose, and whichever metric structure on the direct image sheaf we pick out, say $\overline{f_*E}$ we must have

$$a(\widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(X)) \ni \delta = f_*[\overline{E}, 0] - [\overline{f_*E}, 0]$$

to be a higher generalized analytic torsion for $f, \overline{E}, \overline{f_*E}$, where $f_*[\overline{E}, 0]$ denotes the putative direct image in $\breve{K}$-theory.

Fortunately we can circumvent both difficulties, in order to do this let us state a lemma due to Quillen.

Lemma 2.1.51. *Let $f : X \to Y$ be a projective morphism between separated noetherian schemes, then $K_0(X)$ in generated as a group by f_*-acyclic vector bundles.*

Proof. This is [Qui73, p.41, paragraph 2.7] □

For f a projective morphism between arithmetic varieties, $\overline{X}$ and $\overline{Y}$, we set $\breve{K}_0^f(X)$ to be the free abelian group built on symbols $[\overline{E}]$ where E is a f_*-acyclic vector bundle, times $\widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(X)$, modulo $[\overline{E}] = [\overline{E''}] + [\overline{E'}] + \widetilde{\mathrm{ch}}(\mathcal{E})$ for every exact sequence $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$ where the bundles are f_*-acyclic. We can reformulate Quillen's lemma as

Lemma 2.1.52. *The natural map*

$$\breve{K}_0^f(X) \xrightarrow{c} \breve{K}_0(X)$$

is an isomorphism.

In view of the previous lemma, it will be sufficient to construct a direct image for f_*-acyclic vector bundles: we will give below a definition for a map $f_* : \breve{K}_0^f(X) \to \breve{K}_0(Y)$, the morphism $f_* : \breve{K}_0(X) \to \breve{K}_0(Y)$ will simply be $f_* \circ c^{-1}$.

We will examine separately the case of a smooth submersion and that of an immersion.

Definition 2.1.53. *(Direct image for a submersion)*
Let $\pi : \overline{X} \to \overline{Y}$ be a smooth submersion, the Kähler metrics on $\overline{X}$ and $\overline{Y}$ induce a structure of Kähler fibration on π, if $\overline{E}$ is a π_*-acyclic hermitian bundle on X we set

$$\pi_*[\overline{E}, g] = \left[\overline{\pi_* E}^{L^2}, T(\overline{T_{X/Y}}, \overline{E}) + \int_{X/Y} g \wedge \mathrm{Td}(T_{X/Y})\right]$$

We need to check that this definition makes sense, namely that if

$$\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$$

is an exact sequence of hermitian vector bundle that are π_*-acyclic, then as the exact sequence $\pi_*\mathcal{E}$ remains acyclic,

$$\pi_*[\overline{E}, 0] = \pi_*[\overline{E''} + \overline{E'}, \widetilde{\mathrm{ch}}(\mathcal{E})]$$

This is achieved by the following anomaly formula

Proposition 2.1.54. *(Anomaly Formula for the analytic torsion)*
Let $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$ be an exact sequence of π_*-acyclic vector bundles where π is a Kähler fibration from X to Y, we have

$$T(\overline{T_{X/Y}}, \overline{E}) - T(\overline{T_{X/Y}}, \overline{E}') - T(\overline{T_{X/Y}}, \overline{E}'') - \widetilde{\mathrm{ch}}(\pi_*\mathcal{E}) = \int_{X/Y} \widetilde{\mathrm{ch}}(\mathcal{E}) \,\mathrm{Td}(T_{X/Y})$$

Proof. This is the equation (47) of [GS91, p. 46] and 2.1.37 □

We thus obtain a well defined direct image for smooth submersions from X to Y. Let us now turn to the case of an immersion.

Definition 2.1.55. *(Direct image for an immersion)*
Let $i : \overline{X} \to \overline{Y}$ be a (regular) immersion between arithmetic varieties, for any hermitian vector bundle $\overline{E}$, we equip i_*E with any hermitian structure, we set

$$i_*[\overline{E}, g] = \left[\overline{i_*E}, \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, \overline{i_*E}') + i_*[g \wedge \mathrm{Td}(i^*\overline{T_X})^{-1} \,\mathrm{Td}(\overline{T_Z})]\right]$$

Here again we need to check that everything is well defined, we have to check that

$$[\overline{i_*E}, \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, \overline{i_*E}') + i_*[g \,\mathrm{Td}(i^*\overline{T_X})^{-1} \,\mathrm{Td}(\overline{T_Z})]$$

does not depend on the hermitian structure chosen on i_*E, and that for an exact sequence $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$, we have

$$i_*[\overline{E}, 0] = i_*[\overline{E''} + \overline{E'}, \widetilde{\mathrm{ch}}(\mathcal{E})]$$

We have the following anomaly formulae that ensure us that this is the case.

Proposition 2.1.56. *(Anomaly Formulae for the singular Bott-Chern current)*
Let $\mathcal{E} : 0 \to \overline{E''} \to \overline{E} \to \overline{E'} \to 0$ be an exact sequence vector bundles and i be a closed immersion from $\overline{X}$ to $\overline{Y}$, let us chose hermitian structures on i_*E, i_*E', and i_*E'', we have an exact sequence $i_*\mathcal{E} : 0 \to \overline{i_*E''} \to \overline{i_*E} \to \overline{i_*E'} \to 0$

1. *If $\overline{i_*E}$ and $\overline{i_*E}'$ denote two different hermitian structures on i_*E, we have*

$$\mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, \overline{i_*E}) - \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, \overline{i_*E}') = \widetilde{\mathrm{ch}}(0 \to \overline{i_*E}' \to \overline{i_*E} \to 0)$$

2. *Furthermore we have*

$$\begin{aligned} & \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, \overline{i_*E}) - \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}', \overline{i_*E'}) - \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}'', \overline{i_*E''}) \\ = \; & i_*[\widetilde{\mathrm{ch}}(\mathcal{E}) \, \mathrm{Td}(i^*\overline{T_X})^{-1} \, \mathrm{Td}(\overline{T_Z})] + \widetilde{\mathrm{ch}}(i_*\mathcal{E}) \end{aligned}$$

Proof. The first formula is a particular case of the second one for a very short exact sequence. The second formula is [BGS, Thm 2.9] □

This completes the definition of the direct image for a closed immersion, to have a fully fledged definition we need the following proposition

Theorem 2.1.57. *(Direct image in $\breve{K}$-theory)*
Let $f : \overline{X} \to \overline{Y}$ be a projective morphism between two arithmetic varieties, and let

$$\begin{array}{ccccc} & & \mathbb{P}^r_Y & & \\ & \overset{i}{\nearrow} & & \overset{p}{\searrow} & \\ X & & & & Y \\ & \underset{j}{\searrow} & & \underset{q}{\nearrow} & \\ & & \mathbb{P}^\ell_Y & & \end{array}$$

be two decompositions of f into an immersion followed by a smooth morphism[8] *(where the projective spaces are endowed with the Fubini-Study metric and $\mathbb{P}^\bullet_Y$ with the product metric). Then $p_*i_* = q_*j_*$ and this morphism only depends on the hermitian metrics on X and Y.*

Proof. The proof of this result can essentially be found in the literature, for instance in [BGFiML14, Theo 10.7] albeit using a slightly different language, or [GiML12b, Prop 5.8] □

Therefore the following definition makes sense

Definition 2.1.58. *Let $f : \overline{X} \to \overline{Y}$ be a projective morphism between arithmetic varieties, we set $f_* = p_*i_*$ for any choice*[9] *of factorization of f into $\overline{X} \xrightarrow{i} \overline{\mathbb{P}^\ell_Y} \xrightarrow{p} \overline{Y}$*

We see now that arithmetic $\breve{K}$-theory satisfies the same properties as arithmetic weak Chow theory except for the fact that the latter is graded whereas the former is not.

Proposition 2.1.59. *(Borel-Moore properties)*
Let X, Y, Y', S and S', be smooth projective varieties and let $\pi : X \to Y$ and $\pi' : Y \to Y'$ be projective morphisms and $f : S \to X$ and $f' : S' \to S$ be smooth equidimensional morphism. We also fix $\overline{M}$ (resp. $\overline{L}$ and $\overline{L}'$), a (resp. two) hermitian bundle on Y (resp. X), we have

1. *(Functoriality of the push forward) $(\pi' \circ \pi)_* = \pi'_* \pi_*$*

[8]In fact we would replace the projective spaces over Y by any variety smooth over Y equipped with a Kähler metric.

[9]we can choose $\ell = 0$ if f is an immersion

2. *(Functoriality of the pull back)* $(f \circ f')^* = f'^* f^*$

3. *(Naturality of the 1st Chern class)* $f^* \circ \widehat{c_1}(\overline{L}) = \widehat{c_1}(f^*\overline{L}) \circ f^*$.

4. *(Projection Formula)* $\pi_* \circ \widehat{c_1}(\pi^*\overline{M}) = \widehat{c_1}(L) \circ \pi_*$

5. *(Commutativity of the 1st Chern Classes)* $\widehat{c_1}(\overline{L}) \circ \widehat{c_1}(\overline{L}') = \widehat{c_1}(\overline{L}') \circ \widehat{c_1}(\overline{L})$

Proof. The first point is 2.1.57, the second point is obvious, so is the third using the naturality of the Chern character, and the last one is just as obvious. Let us prove the projection formula, we have

$$\pi_*(\mathrm{Td}(\overline{T}_\pi) \wedge g \wedge (1 - \mathrm{ch}(\pi^*\overline{L}^\vee))) = \pi_*(\mathrm{Td}(\overline{T}_\pi) \wedge g) \wedge (1 - \mathrm{ch}(\overline{L}^\vee)))$$

To prove the result on classes $[\overline{E}, 0]$ we prove it first for a closed immersion, we have

$$\begin{aligned} i_*[\widehat{c_1}(i^*\overline{L})[\overline{E}]] &= i_*([\overline{E}] - [\overline{E} \otimes i^*\overline{L}^\vee]) \\ &= [i_*\overline{E}] - [i_*(\overline{E} \otimes i^*\overline{L}^\vee)] + \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, i_*\overline{E}) - \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E} \otimes i^*\overline{L}^\vee, i_*\overline{E} \otimes \overline{L}^\vee) \\ &= [i_*\overline{E}] - [i_*\overline{E} \otimes \overline{L}^\vee] + \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, i_*\overline{E}) - \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, i_*\overline{E})\,\mathrm{ch}(\overline{L}^\vee) \\ &= \widehat{c_1}(\overline{L})\left[[i_*\overline{E}] + \mathrm{bc}(\overline{T_X}, \overline{T_Y}, \overline{E}, i_*\overline{E})\right] \\ &= \widehat{c_1}(\overline{L}) i_*[\overline{E}] \end{aligned}$$

where we have used the definition of the first Chern class, the definition of the direct image, the isometry of the chosen resolutions $i_*\overline{E} \otimes \overline{L}^\vee \simeq i_*(\overline{E} \otimes i^*\overline{L}^\vee)$ and the compatibility of the Bott-Chern singular current with the projection formula.

To prove the result for Kähler fibrations, we may assume that $\overline{E}$ is π_*-acyclic, and the same proof applies. □

As before, for arithmetic weak Chow groups, we have more, let us give now the properties that encodes the arithmetic nature of this functor

Proposition 2.1.60. *(Arithmetic Type of $\check{K}$)*
Let $\overline{X}$ be an arithmetic variety of dimension d, we have

1. *For any hermitian line bundles over X, $\overline{L}_1, ..., \overline{L}_{d+2}$, we have*

$$\widehat{c_1}(\overline{L}_1) \circ ... \circ \widehat{c_1}(\overline{L}_{d+2}) = 0$$

   as an endomorphism of $\check{K}(X)$.

2. *Let $\overline{L}$ be a hermitian line bundle over X, with s a global section of L that is transverse to the zero section. Let Z be the zero scheme of such a section, and $i : Z \to X$ the corresponding immersion. We have*

$$i_*(1_Z) = \widehat{c_1}(\overline{L})(1_X) + a(\log \|s\|^2 \,\mathrm{Td}(\overline{L})^{-1}) + i_*[\widetilde{\mathrm{Td}}^{-1}(\mathcal{E})\,\mathrm{Td}(\overline{T}_Z)]$$

   where $\mathcal{E}$ is the exact sequence $0 \to \overline{T_Z} \to i^*\overline{T_X} \to i^*\overline{L} \to 0$ associated to the immersion.

3. *Given two hermitian bundles $\overline{L}$ and $\overline{M}$ over X we have*

$$\widehat{c_1}(\overline{L} \otimes \overline{M}) = \widehat{c_1}(L) + \widehat{c_1}(M) - \widehat{c_1}(\overline{L})\,\widehat{c_1}(\overline{M})$$

Proof. Let us keep the notation of the proposition

1. Firstly, we see that $\widehat{c_1}(\overline{L}_1) \circ ... \circ \widehat{c_1}(\overline{L}_{d+2}).a(g) = 0$ as the action of the first Chern class increases the type by $(1,1)$.

   It is now a good time to notice that we can see $\breve{K}_0(X)$ as a module over $\widehat{K}_0(X)$ and that the action of the $\widehat{c_1}$ is given by multiplication by $[\mathcal{O}_X] - [\overline{L}^\vee]$, so the identity we want to prove is in fact an identity in $\widehat{K}_0(X)$, where a product is defined in a way that the composition of the actions of the first Chern classes is just the multiplication of the corresponding classes.

   Now since X is regular we have $\widehat{K}_0(X) \simeq \widehat{G}_0(X)$ ([GS92, Lem. 13]) and in $\widehat{G}_0(X)$ we have $[\mathcal{O}_X] - [\overline{L}^\vee] = [\overline{i_*\mathcal{O}_Z}]$ as soon as L is effective, where Z is the zero scheme of any global section of L (where the hermitian structure on $\overline{i_*\mathcal{O}_Z}$ is of course given by the obvious exact sequence).

   Let us assume for a moment, that all the L_i's are very ample, we want to prove $[i_{1*}\mathcal{O}_{Z_1}]...[i_{(d+2)*}\mathcal{O}_{Z_{d+2}}] = 0$, using Bertini's theorem [Mur94, Theo 2.3], we can choose global sections of each L_i such that $Z_i = \mathrm{div}(s_i)$ is generically transverse to $\bigcap_{j<i} Z_j$, but this is tantamount to saying that $\bigcap_{i=1...d+1} Z_i$ is empty, therefore $[i_{1*}\mathcal{O}_{Z_1}]...[i_{(d+2)*}\mathcal{O}_{Z_{d+1}}] = 0$.

   Let us now assume that all the bundles L_i are very ample, except one, which is anti-very-ample i.e its dual is very ample, and assume for simplicity that it is $\overline{L}_1 = \overline{L}$. As for any line bundles $\overline{M}$ and $\overline{M}'$ we have (in $\widehat{K}_0(X)$ or $\widehat{G}_0(X)$ where the product is defined)

   $$[\mathcal{O}_X]-[\overline{M}^\vee\otimes\overline{M}'] = ([\mathcal{O}_X]-[\overline{M}^\vee])+([\mathcal{O}_X]-[\overline{M}'])-([\mathcal{O}_X]-[\overline{M}^\vee]).([\mathcal{O}_X]-[\overline{M}'])$$

   (which incidentally proves the third point), we see, plugging $\overline{L} = \overline{M}_i = \overline{M}'_i$, that

   $$0 = [\mathcal{O}_X] - [\overline{L}^\vee \otimes \overline{L}] = ([\mathcal{O}_X] - [\overline{L}]) + ([\mathcal{O}_X] - [\overline{L}^\vee]) - ([\mathcal{O}_X] - [\overline{L}])([\mathcal{O}_X] - [\overline{L}^\vee])$$

   therefore we can replace $([\mathcal{O}_X]-[\overline{L}])$ by $([\mathcal{O}_X]-[\overline{L}])([\mathcal{O}_X]-[\overline{L}^\vee])-([\mathcal{O}_X]-[\overline{L}^\vee])$, and as $\overline{L}^\vee$ is very ample, the results follow from the previous case.

   Now let us consider the general case recall that each L_i can be written as $M_i \otimes M_i'^\vee$ with M_i and M_i' very ample, let's endow these two line bundles with any metric rendering the previous isomorphism, isometric. As

   $$[\mathcal{O}_X]-[\overline{M}_i^\vee\otimes\overline{M}_i'] = ([\mathcal{O}_X]-[\overline{M}_i^\vee])+([\mathcal{O}_X]-[\overline{M}_i'])-([\mathcal{O}_X]-[\overline{M}_i^\vee]).([\mathcal{O}_X]-[\overline{M}_i'])$$

   we're reduced to the case of ample and anti-ample line bundles which yields the result.

2. In view of the exact sequence

   $$0 \to L^\vee \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

   We have

   $$\begin{aligned} i_*(1_Z) &= [\mathcal{O}_X] - [\overline{L}^\vee] + \mathrm{bc}(\overline{T}_Z, \overline{T}_X, \mathcal{O}_Z, \overline{i_*O_Z}) \\ &= [\mathcal{O}_X] - [\overline{L}^\vee] + \mathrm{Td}(\overline{L})^{-1} \wedge \log\|s\|^2 + i_*[\widetilde{\mathrm{Td}}^{-1}(\mathcal{E})\,\mathrm{Td}(\overline{T}_Z)] \end{aligned}$$

   and on the other hand $\widehat{c_1}(\overline{L})(1_X) = [\mathcal{O}_X] - [\overline{L}^\vee]$, the result follows.

3. It has been proven in the course of the demonstration of the first point.

□

2.2 Weak Cobordism Group

We now proceed to the construction of a weak arithmetic cobordism group

2.2.1 Arithmetic Lazard Ring, Universal Todd class and secondary forms associated to it

I will define here a modified version of the Lazard ring.

Definition 2.2.1. *We set the* arithmetic Lazard ring *to be the ring*

$$\widehat{\mathbb{L}} = \mathbb{Z}[a^{ij}, t_k, (i,j) \in \mathbb{N} \times \mathbb{N}, k \in \mathbb{N}]$$

divided by the ideal I such that the following relations hold in $\widehat{\mathbb{L}}[[u, v, w]]$,

- $\sum_{i \geq 0} t_i (u+v)^{i+1} = \sum_{i \geq 0, j \geq 0} a^{i,j} u^i v^j \left(\sum_k t_k u^k\right)^i \left(\sum_r t_r u^r\right)^j$
- $\mathbb{F}(u, \mathbb{F}(v, w)) = \mathbb{F}(\mathbb{F}(u, v), w)$
- $\mathbb{F}(u, v) = u + v \ mod\ (u, v)^2$
- $\mathbb{F}(u, v) = \mathbb{F}(v, u)$
- $\mathbb{F}(0, u) = u$
- $t_0 = a^{1,0} = a^{0,1} = 1$

where $\mathbb{F}$ is the universal law group $\mathbb{F}(u, v) = \sum a^{i,j} u^i v^j$.

We need to check that this ring is not zero. To do so, we can build a quotient of that ring that is not zero. Let us consider the map $\{a^{i,j}, t_i\} \to \mathbb{Q}$ defined by $a^{1,1} \mapsto -1$ and $a^{i,j} \mapsto 0$ for $(i, j) \notin \{(1, 1), (1, 0), (0, 1)\}$ and $t_i \mapsto (-1)^i/(i + 1)!$ for $i > 0$. This map induces a map form $\widehat{\mathbb{L}}$ to $\mathbb{Q}$ that is not 0 ensuring that $\widehat{\mathbb{L}}$ is not trivial.

We will let $\mathfrak{g}(u)$ denote the universal power series over $\mathbb{Z}[\mathbf{t}]$,

$$\mathfrak{g}(u) = \sum_r t_r u^r$$

we can re-write the first axiom as

$$\mathbb{F}(u\mathfrak{g}(u), v\mathfrak{g}(v)) = (u + v)\mathfrak{g}(u + v)$$

Let us denote the unique power series $\mathfrak{h}$ over $\mathbb{Q}[\mathbf{t}]$, defined by $\mathfrak{h}(\mathfrak{g}(u)u) = u$, we see that

$$\mathfrak{h}(u) + \mathfrak{h}(v) = \mathfrak{h}(\mathbb{F}(u, v))$$

in other words, $\mathfrak{h}$ is a morphism of formal group laws, from the universal group law to the additive group law; what is surprising is that $\mathfrak{h}$ is in fact an isomorphism after tensorization by $\mathbb{Q}$.

Let us set up a bit of terminology

We can now prove

Proposition 2.2.2. *As rings the arithmetic Lazard ring and the Lazard ring are isomorphic after tensorization by $\mathbb{Q}$,*

$$\mathbb{L}_{\mathbb{Q}} \simeq \widehat{\mathbb{L}}_{\mathbb{Q}}$$

Proof. Let $\mathfrak{h}$ be the power series over $\mathbb{Q}[\mathbf{t}]$ previously defined, this power series define a formal law group on $\mathbb{Q}[\mathbf{t}]$ given by

$$\mathfrak{h}^{-1}(\mathfrak{h}(u)+\mathfrak{h}(v)) = F(u,v) \tag{$\star$}$$

and this defines a morphism

$$\mathbb{L}_{\mathbb{Q}} \to \mathbb{Q}[\mathbf{t}]$$

which is an isomorphism, because there is a natural bijection between

1. $\Lambda \mapsto \mathrm{Hom}_{\mathbb{Q}-\mathrm{algebras}}(\mathbb{L}_{\mathbb{Q}}, \Lambda)$.
2. $\Lambda \mapsto \mathrm{FGL}(\Lambda)$ (the set of formal group laws over the $\mathbb{Q}$-algebra Λ).
3. $\Lambda \mapsto \mathrm{Genera}(\Lambda)$ (the set of genera over Λ i.e satisfying $g(u) = u \mod u^2$)

where the bijection between 2 and 3 is given by ($\star$) (see [Och87]), let us denote ψ the isomorphism $\mathbb{L}_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}[\mathbf{t}]$ defined in this way.

With this in mind, we see that $\widehat{\mathbb{L}}_{\mathbb{Q}}$ is isomorphic (as a left $\mathbb{L}_{\mathbb{Q}}$-module) to $\mathbb{L}_{\mathbb{Q}} \otimes \mathbb{L}_{\mathbb{Q}}/I$ where I is the ideal generated by $a \otimes 1 - 1 \otimes \psi(a)$. Let us consider the arrow $m : \mathbb{L}_{\mathbb{Q}} \otimes \mathbb{L}_{\mathbb{Q}} \to \mathbb{L}_{\mathbb{Q}}$ given by multiplication $a \otimes b \mapsto a\psi(b)$, this map certainly factors through $\widehat{\mathbb{L}}_{\mathbb{Q}}$.

To see that it is an isomorphism we have to prove that the kernel of the multiplication map is exactly I, but this is easy, as we have a section $s : \mathbb{L}_{\mathbb{Q}} \to \mathbb{L}_{\mathbb{Q}} \otimes \mathbb{L}_{\mathbb{Q}}$ given by $a \mapsto a \otimes 1$, and the kernel of the multiplication if generated as a left $\mathbb{L}_{\mathbb{Q}}$-module by elements of the form $\sum 1 \otimes x_i$ with $\sum m(1 \otimes x_i) = 0$ thus

$$\sum 1 \otimes x_i = \sum 1 \otimes x_i - 0 = \sum 1 \otimes x_i - s \circ m(1 \otimes \psi(x_i)) = \sum 1 \otimes x_i - \psi(x_i) \otimes 1$$

and the proof is complete. $\square$

We see that $\widehat{\mathbb{L}}_{\mathbb{Q}}$ doesn't have a richer structure than $\mathbb{L}_{\mathbb{Q}}$, because it is equipped with a formal group law and a genus that corresponds to it, this is essentially the fact that in characteristic zero, there is only one formal group law. In a way $\widehat{\mathbb{L}}_{\mathbb{Q}}$ is just a different way of looking at $\mathbb{L}_{\mathbb{Q}}$.

Corollary 2.2.3. *(Mishenko[10], [Nov67, Appendix 1, p. 72])*
We have through the identification $\mathbb{L}_{\mathbb{Q}} \simeq \Omega(k)_{\mathbb{Q}} \simeq \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, ...]$,

$$\mathfrak{h}(u) = \sum_{i \geq 0} \frac{[\mathbb{P}^i]}{i+1} u^{i+1}$$

Remark 2.2.4. We have a natural grading for the Lazard ring given by $\deg(a^{i,j}) = i + j - 1$, if we set $\deg(t_i) = i$ then we have a natural grading on $\widehat{\mathbb{L}}$ given by the grading on the tensor product, namely $\deg(a^{i,j} t_k) = k + i + j - 1$

Remark 2.2.5. Before proceeding to the study of arithmetic cobordism let us introduce a couple notations defined below

$$\begin{array}{rclrcl}
\mathbb{Z}[t_0, t_1, ..] & = & \mathbb{Z}[\mathbf{t}] & & & \\
\mathbb{Z}[\mathbf{t}] \otimes \widetilde{A}^{\bullet;\bullet}_{\mathbb{R}}(X) & = & \widetilde{A}^{\bullet;\bullet}_{[\mathbf{t}]}(X) & \mathbb{Z}[\mathbf{t}] \otimes A^{\bullet;\bullet}_{\mathbb{R}}(X) & = & A^{\bullet;\bullet}_{[\mathbf{t}]}(X) \\
\mathbb{Z}[\mathbf{t}] \otimes \widetilde{D}^{\bullet;\bullet}_{\mathbb{R}}(X) & = & \widetilde{D}^{\bullet;\bullet}_{[\mathbf{t}]}(X) & \mathbb{Z}[\mathbf{t}] \otimes D^{\bullet;\bullet}_{\mathbb{R}}(X) & = & D^{\bullet;\bullet}_{[\mathbf{t}]}(X) \\
\mathbb{Z}[\mathbf{t}] \otimes Z^{\bullet;\bullet}_{\mathbb{R}}(X) & = & Z^{\bullet;\bullet}_{[\mathbf{t}]}(X) & & & \\
\widehat{\mathbb{L}} \otimes \widetilde{D}^{\bullet;\bullet}_{\mathbb{R}}(X) & = & \widetilde{D}^{\bullet;\bullet}_{\widehat{\mathbb{L}}}(X) & & &
\end{array}$$

[10] There's a typo in the first appearance of the formula in the paper, the correct formula is in its appendix

where we've extended the usual operations defined on $D^{\bullet,\bullet}$, such as $\partial, \overline{\partial}$, the pull back and push forward operations for suited maps etc..., by $\mathbb{Z}[\mathbf{t}]$-linearity. Notice that we still have a product

$$Z^{\bullet,\bullet}_{[\mathbf{t}]}(X) \otimes \widetilde{D}^{\bullet,\bullet}_{[\mathbf{t}]}(X) \to \widetilde{D}^{\bullet,\bullet}_{[\mathbf{t}]}(X)$$

that preserves $\widetilde{A}^{\bullet,\bullet}_{[\mathbf{t}]}(X)$

We now wish to construct both multiplicative characteristic forms associated to $\mathfrak{g}$ with value in $A^{\bullet,\bullet}_{[\mathbf{t}]}(X)$, and secondary Bott-Chern forms with value in $\widetilde{A}^{\bullet,\bullet}_{[\mathbf{t}]}(X)$. This can be done in a straightforward manner, let's quickly review the way to do so.

As we've seen before, if we have X a complex manifold, with $\overline{E}$ a hermitian vector bundle over it, $\overline{E}$ comes equipped with a natural Chern connection. Let us consider the power series $\varphi(T_1, ..., T_n) \in Z[\mathbf{t}][[T_1, ..., T_n]]$ defined by

$$\varphi(T_1, ..., T_n) = \prod_{i=1}^{n} \mathfrak{g}(T_i)$$

we can write φ as a sum of $\varphi^{(\ell)}$ with each $\varphi^{(\ell)}$ homogenous of degree ℓ (in $T_1, ..., T_n$). There exists a unique map, still denoted $\varphi^{(\ell)}$ defined on matrix with coefficients in $A^{1,1}(X)$ and invariant by conjugation such that

$$\varphi^{(\ell)}\left(\begin{pmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{pmatrix}\right) = \varphi^{(\ell)}(\omega_1, ..., \omega_n)$$

By identifying locally $\mathrm{End}(E)$ with the space of matrix with complex coefficients, we can define

$$\mathfrak{g}(\overline{E}) = \sum_k \varphi^{(k)}(\frac{i}{2\pi}\nabla^2) \in A^{\bullet,\bullet}_{[\mathbf{t}]}(X)$$

We get a closed form whose cohomology class (with coefficients in $\mathbb{Z}[\mathbf{t}]$) does not depend on the metric chosen on $\overline{E}$. Let's sum up the properties of this characteristic class.

Proposition 2.2.6. *The characteristic form $\mathfrak{g}(\overline{E}) \in A^{\bullet,\bullet}_{[\mathbf{t}]}(X)$ associated to a hermitian bundle on a complex manifold X, satisfies the following properties*

1. *(Naturality) For any holomorphic map of complex manifold $f : Y \to X$ we have $\mathfrak{g}(f^*\overline{E}) = f^*(\mathfrak{g}(\overline{E}))$.*
2. *(Definition for a line bundle) For a hermitian line bundle $\overline{L}$, we have $\mathfrak{g}(\overline{L}) = \sum_r t_r c_1(\overline{L})^r$.*
3. *(Mulitplicativity) If $0 \to \overline{E}'' \to \overline{E} \to \overline{E}' \to 0$ is an ortho-split exact sequence of hermitian bundles on X we have*

$$\mathfrak{g}(\overline{E}) = \mathfrak{g}(\overline{E}')\mathfrak{g}(\overline{E}'')$$

4. *(Closedness) The form $\mathfrak{g}(\overline{E})$ satisfies $d\mathfrak{g}(\overline{E}) = 0$*

Remark 2.2.7. The closedness property can easily be deduced from Bianchi's second identity using the fact that the Chern connection is torsion free on a Kähler manifold.

Remark 2.2.8. As $\mathfrak{g}(\overline{E})^{(0)} = 1$, the class $\mathfrak{g}(\overline{E})$ is invertible in $A^{\bullet,\bullet}_{[\mathbf{t}]}(X)$.

Remark 2.2.9. If $f : \overline{X} \to \overline{Y}$ is a morphism between arithmetic varieties we will denote $\mathfrak{g}(\overline{T_f})$ for $\mathfrak{g}(\overline{T}_X)\mathfrak{g}(f^*\overline{T_Y})^{-1}$

We can now construct secondary forms with value in $\widetilde{A}^{\bullet,\bullet}_{[\mathbf{t}]}(X)$ to measure the defect of multiplicativity in the case of an arbitrary exact sequence of hermitian bundles.

Proposition 2.2.10. *To each exact sequence*

$$\mathcal{E} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

of hermitian vector bundles on X we can associate a form in $\widetilde{A}^{\bullet,\bullet}_{[\mathbf{t}]}(X)$, denoted $\widetilde{\mathfrak{g}}(\mathcal{E})$ uniquely determined by the following properties

1. *(Naturality) For any holomorphic map of complex manifold $f : Y \to X$ we have $\widetilde{\mathfrak{g}}(f^*\mathcal{E}) = f^*(\widetilde{\mathfrak{g}}(\mathcal{E}))$.*
2. *(Differential equation) We have*

$$\mathfrak{g}(\overline{E}) = \mathfrak{g}(\overline{E}')\mathfrak{g}(\overline{E}'') + dd^c\widetilde{\mathfrak{g}}(\mathcal{E})$$

3. *(Vanishing) When $\mathcal{E}$ is orhto-split, $\widetilde{\mathfrak{g}}(\mathcal{E}) = 0$*

Proof. This is the same proof as [BGS88a, Theo 1.2.9], almost verbatim. □

Remark 2.2.11. We can give an explicit expression of $\widetilde{\mathfrak{g}}(\mathcal{E})$, if

$$0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

is a short exact sequence of hermitian bundles, we set $\widetilde{E}$ to be $(p_2^*E \oplus p_2^*E'(1))/p_2^*E'$ over $\mathbb{P}^1_X$ where $\mathcal{O}(1)$ is equipped with its Fubini-Study metric, and we endow $\widetilde{E}$ with any metric rendering isometric the isomorphisms over the fibers of $\widetilde{E}$ at 0 and ∞ with $\overline{E}$ and $\overline{E}' \oplus \overline{E}''$ respectively then we have

$$\widetilde{\mathfrak{g}}(\mathcal{E}) = -\int_{\mathbb{P}^1_X/X} \log|z|^2 \mathfrak{g}(\widetilde{E})$$

See [GS90, 1.2] for details.

Proposition 2.2.12. *(Naturality with respect to $\mathbf{t}$)*

Let R be a ring equipped with a morphism $\varphi : \mathbb{Z}[\mathbf{t}] \to R$ and let $\mathfrak{g}_R$ be the formal power series over R given by $\sum \varphi(t_i)u^i$ then we have

1. *In $A^{\bullet,\bullet}_R(X)$, $\mathfrak{g}_R(\overline{E}) = \varphi(\mathfrak{g}(\overline{E}))$ for every hermitian bundle $\overline{E}$ over a manifold X.*
2. *In $\widetilde{A}^{\bullet,\bullet}_R(X)_R$, $\widetilde{\mathfrak{g}}_R(\mathcal{E}) = \varphi(\widetilde{\mathfrak{g}}(\mathcal{E}))$ for every exact sequence of hermitian bundles over X. where $\mathfrak{g}(\overline{E})$ (resp. $\widetilde{\mathfrak{g}}(\mathcal{E})$) is obtained by the same process as above.*

Proof. The second point results from the first one as we have

$$\widetilde{\mathfrak{g}}_R(\overline{E}) = -\int_{\mathbb{P}^1_X/X} \log|z|^2 \mathfrak{g}_R(\widetilde{E})$$

and the integration over the fiber commutes with φ by definition.

Let us prove the first one, in fact the properties mentioned in 2.2.6, characterize the form $\mathfrak{g}_R$ over R, but using the fact that φ commutes with d and pull-back ensures us that $\varphi(\mathfrak{g}) = \mathfrak{g}_R$ and the proposition follows.

□

In every case that we will consider R will be a subring of $\mathbb{R}$. If

$$\mathcal{E} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

is a short exact sequence of hermitian vector bundles, we set

$$\widetilde{\mathfrak{g}^{-1}}(\mathcal{E}) = -\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{E}'')\mathfrak{g}^{-1}(\overline{E}')\mathfrak{g}^{-1}(\overline{E})$$

using the fact that $\mathfrak{g}^{-1}$ is closed and the differential equation satisfied by $\widetilde{\mathfrak{g}}$ we see that

$$dd^c\widetilde{\mathfrak{g}^{-1}}(\mathcal{E}) = \mathfrak{g}^{-1}(\overline{E}) - \mathfrak{g}^{-1}(\overline{E}')\mathfrak{g}^{-1}(\overline{E}'')$$

moreover $\widetilde{\mathfrak{g}^{-1}}(\mathcal{E}) = 0$ as soon as $(\mathcal{E})$ is ortho-split, and natural with respect to pull back, thus it is the secondary Bott-Chern form (with coefficient in $\mathbb{Z}[\mathbf{t}]$) associated to $\mathfrak{g}^{-1}$.

Remark 2.2.13. If α is a closed (p,p)-form (or current), we see that

$$\alpha\mathfrak{g}(\overline{E}) - \alpha\mathfrak{g}(\overline{E}')\mathfrak{g}(\overline{E}'') = \alpha dd^c\widetilde{\mathfrak{g}}(\mathcal{E}) = dd^c(\alpha\widetilde{\mathfrak{g}}(\mathcal{E}))$$

therefore if we work in $\widetilde{D}^{\bullet,\bullet}_{\hat{\mathbb{L}}}(X)$, we have $\alpha\mathfrak{g}(\overline{E}) = \alpha\mathfrak{g}(\overline{E}')\mathfrak{g}(\overline{E}'')$ for any closed form α, a fact that we will use in the following.

Let us finish by some basic comments about degrees in $\widetilde{D}^{\bullet,\bullet}_{\hat{\mathbb{L}}}(X)$, in the context of the weak cobordism group we will set (see 2.2.17)

$$\deg(\widetilde{D}^{p,p}_{\mathbb{R}}(X)) = d_X - (p+1)$$

that gives us a graded group structure on $\widetilde{D}^{\bullet,\bullet}_{\hat{\mathbb{L}}}(X)$, of course this grading is not compatible with the product of currents even when it is defined because our theory will be *homological* in nature. However we see that $\deg(\mathfrak{g}(\overline{E})\varphi(t)g) = \deg(\varphi(t).g)$ (and the same thing for $\mathfrak{g}^{-1}$), that $\deg(dd^c(\varphi(t).g)) = \deg(\varphi(t)g) - 1$ and therefore $\deg(\widetilde{\mathfrak{g}}(\mathcal{E})) = d_X$, these observations will ensure that the class

$$a[i_*[\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T}_Z)]] + a(\mathfrak{g}(\overline{L})\log\|s\|^2)$$

that will appear later, is homogenous of degree $d_X - 1$.

2.2.2 Construction of the Borel Moore Functor

We first construct a Borel-Moore functor on arithmetic varieties, we will only need a subclass of the traditionally defined arithmetic varieties (which are usually regular schemes over $\operatorname{Spec}\mathcal{O}_K$ the spectrum of the ring of integers of a number field, whereas we will restrict to the case of $\operatorname{Spec} k$)

Remark 2.2.14. In the following sections, many of our definitions could still make sense for arithmetic varieties over a Dedekind domain, or even a Dedekind scheme, however since we do not know how to prove even the geometric version of the properties of the arithmetic cobordism group I've chosen to remain in the context of varieties over a field, where the geometric theory is known to be well behaved.

Definition 2.2.15. *Let $\overline{X}$ be an arithmetic variety over k. We set $\mathcal{Z}(\overline{X})$ as the group*

$$\mathcal{Z}'(\overline{X})/\mathcal{R}'(\overline{X}) \times \left[\mathbb{Z}[\mathbf{t}] \otimes \widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(X)\right]$$

where $\mathcal{Z}'(\overline{X})$ denotes the free abelian group built on symbols

$$[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r]$$

with

- *The morphism f is a projective morphism between arithmetic varieties.*
- *The variety $\overline{Z}$ is integral (connected).*
- *The line bundle $\overline{L}_i$ is a hermitian line bundle over Z.*

The group $\mathcal{R}'(\overline{X})$ denotes the subgroup of $\mathcal{Z}'(\overline{X})$ generated by the classes

$$[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r] - [\overline{Z}' \xrightarrow{f'} \overline{X}, \overline{L}'_1, ..., \overline{L}'_r]$$

such that there exists h an X-isometry of $\overline{Z}$ on $\overline{Z}'$, that is to say an isomorphism

$$\begin{array}{ccc} Z & \xrightarrow{h} & Z' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

inducing an isometry from $\overline{Z}(\mathbb{C})$ to $\overline{Z}'(\mathbb{C})$; and such that there exists a permutation $\sigma \in \mathfrak{S}_r$ and isomorphisms of hermitian line bundles $\overline{L}_i \simeq \overline{L}'_{\sigma(i)}$, in other words, we allow re-indexing of the (classes of) hermitian line bundles.

Remark 2.2.16. In other, simpler, terms, we make no difference between two arithmetic varieties as long as they are isometric, ibidem for line bundles and we allow to permute the line bundles.

We naturally have a map

$$a : \begin{cases} \widetilde{D}^{\bullet,\bullet}_{[\mathbf{t}]}(X) & \to & \mathcal{Z}(\overline{X}) \\ \varphi(t).g & \mapsto & (0, \varphi(t).g) \end{cases}$$

We will sometimes write $[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, \varphi(t).g]$ for the element $[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r]+ a(\varphi(t).g)$.
The group $\mathcal{Z}(\overline{X})$ is equipped with a natural grading, defined in the following way.

Definition 2.2.17. *We set* $\deg([\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]) = d_Z - r$, $\deg(\widetilde{D}^{p,p}_{\mathbb{R}}(X)) = d_X - (p+1)$, *and* $\deg(t_i) = i$. *On set $\mathcal{Z}_d(\overline{X})$ the subgroup of $\mathcal{Z}(\overline{X})$ of d-degree, and we shall note $\mathcal{Z}_\bullet(\overline{X})$ the graded group.*

Remark 2.2.18. If $\overline{Z} \xrightarrow{i} \overline{X}$ is the closed immersion of a smooth divisor, then $[\overline{Z} \to \overline{X}]$ has degree $d-1$, where d is the dimension of X, and a Green current for Z is given by a current of $\widetilde{D}^{0,0}_{\mathbb{R}}(X)$, which has degree $d-1$, hence for such a current and for any $(1,1)$ closed smooth form ω, the class $[\overline{Z} \to \overline{X}] - a(\mathfrak{g}(\omega) \wedge g)$ is homogenous of degree $d_X - 1$, which should explain the different choices in the grading, that differ slightly from the usual ones used in Arakelov geometry where we tend to grade by the codimension, which is not possible here.

Remark 2.2.19. We will call a class of the form $[\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r] + a(g)$ a standard class, and we will refer to the term $[\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]$ as the geometric part of the class, and to the term $a(g)$ as the analytic part. A class $[\overline{Z} \to \overline{X}]$ will be called a purely geometric class.

2.2.3 Dynamics of the group $\mathcal{Z}(X)$

Let's have a closer look on the functoriality properties of the group $\mathcal{Z}(\overline{X})$.

Definition 2.2.20. *(Push-forward)*
Let $\pi : \overline{X} \to \overline{Y}$ be a projective morphism between arithmetic varieties, we define

$$\pi_*[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, g] = [\overline{Z} \xrightarrow{f} \overline{X} \xrightarrow{\pi} Y, \overline{L}_1, ..., \overline{L}_r, \pi_*(g \wedge \mathfrak{g}^{-1}(\overline{T_\pi}))]$$

where the current $\pi_*(g)$ is defined in the following way, for every smooth differential form ω compactly supported on Y and with appropriate degree $\langle \pi_*(g), \omega \rangle =< g, \pi^*(\omega) >$. We extend this morphism by linearity and we get a morphism

$$\pi_* : \mathcal{Z}(\overline{X}) \to \mathcal{Z}(\overline{Y})$$

whose functoriality is easy to verify.

Remark 2.2.21. Let us note that if π is a projective morphism between smooth equidimensional varieties, and if d designs the relative codimension of π, then π_* induces a morphism from $D^{p,p}_{\mathbb{R}}(X)$ to $D^{p-d,p-d}_{\mathbb{R}}(Y)$, as $\dim(Y) - \dim(X) = -d$, we have $\deg(\pi_*(g)) = \dim(Y) - p + d = \dim(X) - p = \deg(g)$, thus π_* is a graded morphism.

Remark 2.2.22. The equidimensionality hypothesis in the preceding remark is not really a restriction, in fact the smoothness hypothesis on the arithmetic varieties, forces their connected components over $\mathbb{C}$ to be their irreducible components, which means that over $\mathbb{C}$ such a variety is a disjoint union of integral varieties, and for such a variety $\mathcal{Z}(\overline{X}) = \oplus \mathcal{Z}(X_i)$ where the X_i's are the irreducible components of X. Hence the case of arbitrary varieties is easily reducible to the case of equidimensional varieties (or even integral ones).

Remark 2.2.23. Notice here, that we have been a bit sloppy and used the same notation for two different things: the natural push forward of currents and the "twisted" push-forward of currents are both denoted π_*.

It is also possible to define the pull back of any element in $\mathcal{Z}(\overline{X})$ along a smooth morphism.

Definition 2.2.24. *(Pull-back)*
Let $f : \overline{S} \to \overline{X}$ be a smooth equidimensional morphism between arithmetic varieties, we define

$$f^*[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, g] = [\overline{Z \times_X S} \xrightarrow{p}_2 \overline{S}, \overline{p_1^* L_1}, ..., \overline{p_1^* L_r}, f^*(g)]$$

The metric on $Z \times_X S$ is defined in the following way, as X/k is separated, we have a closed immersion $Z \times_X S \to Z \times_k S$, which gives an embedding $T_{Z \times_X S/k} \to T_{Z \times_k S/k} \simeq p_1^* T_{Z/k} \oplus p_2^* T_{S/k}$, this former bundle being equipped with a natural metric, we can induce this metric on $T_{Z \times_X S/k}$.

The current $f^*(g)$ is defined in the following way : as f is smooth, it induces an "integration along the fibers" morphism, $A^{p,p}_c(S) \to A^{p-d,p-d}_c(X)$, which in turn gives a dual morphism $D^{p,p}_{\mathbb{R}}(X) \to D^{p,p}_{\mathbb{R}}(S)$ We extend this morphism by linearity and we get a morphism

$$f^* : \mathcal{Z}(\overline{X}) \to \mathcal{Z}(\overline{Y})$$

whose functoriality is easy to verify.

Remark 2.2.25. Here again, for equidimensional varieties (e.g connected), this morphism is a graded morphism with degree the relative codimension $\delta = \dim(S) - \dim(X)$.

At last, it is also possible to define a first Chern class operator.

Definition 2.2.26. *(First Chern Class)*
Let $\overline{L} \in \widehat{\mathrm{Pic}}(X)$ be a hermitian line bundle over $\overline{X}$, we define

$$\widehat{c_1}(\overline{L})[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, g] = [\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, \overline{f^*L}, c_1(\overline{L}) \wedge \mathfrak{g}(\overline{L}) \wedge g]$$

We extend this morphism by linearity and we get a morphism

$$\widehat{c_1}(\overline{L}) : \mathcal{Z}_\bullet(\overline{X}) \to \mathcal{Z}_{\bullet-1}(\overline{X})$$

Remark 2.2.27. It will be useful to keep in mind the "different parts" of the action of $\widehat{c_1}(L)$, on geometric classes we have

$$\widehat{c_1}(\overline{L})[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r] = [\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, \overline{f^*L}]$$

whereas on analytic classes $\widehat{c_1}(\overline{L})$ acts by multiplication by $c_1(\overline{L})\mathfrak{g}(\overline{L})$, which we will sometimes denote $\mathfrak{h}^{-1}(\overline{L})$ because it is the (composition) inverse of the $\mathfrak{h}$ class we've defined earlier[11].

We list in the next proposition, the different compatibility properties between these morphisms.

Proposition 2.2.28. *Let $\overline{X}, \overline{Y}$ and $\overline{S}$ be arithmetic varieties, and let $\pi : \overline{Y} \to \overline{X}$ be a projective morphism, $f : \overline{X} \to \overline{S}$ a smooth equidimensional morphism, and $\overline{L}$ a hermitian line bundle over X.*

1. *Over $\mathcal{Z}(\overline{X})$, $\pi_* \circ \widehat{c_1}(\pi^*\overline{L}) = \widehat{c_1}(\overline{L}) \circ \pi_*$.*
2. *Over $\mathcal{Z}(\overline{X})$, $f^* \circ \widehat{c_1}(\overline{L}) = \widehat{c_1}(f^*\overline{L}) \circ f^*$.*
3. *Over $\mathcal{Z}(\overline{X})$, $\widehat{c_1}(\overline{L}) \circ \widehat{c_1}(\overline{M}) = \widehat{c_1}(\overline{M}) \circ \widehat{c_1}(\overline{L})$.*

Finally, if we have a fiber diagram

$$\begin{array}{ccc} X' & \xrightarrow{t'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{t} & S \end{array}$$

with π projective, and t smooth equidimensional, and $X' = X \times_S S'$ equipped with its natural metric, then

$$\pi'_* t'^* = t^* \pi_*$$

Proof. It suffices to check all assertions on standard classes, as any standard class $[Z \to S, \overline{L}_1, ..., \overline{L}_r, g]$ can be written as $[Z \to S, \overline{L}_1, .., \overline{L}_r] + a(g)$, it is enough to check the identities on both summands.

[11] A word of warning, $\mathfrak{g}^{-1}$ denotes the multiplicative inverse of $\mathfrak{g}$ whereas $\mathfrak{h}^{-1}$ denotes the composition inverse of $\mathfrak{h}$, it maybe unfortunate to use the same notation for two different things, but it shouldn't confuse the reader as $\mathfrak{h}$ does not have any multiplicative inverse, and $\mathfrak{g}$ doesn't have any composition one.

At the level of the analytic term, the identity $f^*\circ\widehat{c_1}(\overline{L})(a(g)) = \widehat{c_1}(f^*\overline{L})\circ f^*(a(g))$ is a consequence of the naturality of the $\mathfrak{g}$-class, the naturality of the action of Chern forms on differential smooth forms and the density of smooth forms in the space of currents.

This identity remains true for f projective if we replace the current g by a smooth form, and this, in turns, implies the first one by duality. The first three identities are evident enough for the geometric term.

Let's prove the last one, here again only the $a(g)$ term is not a priori clear, we need to prove that $\pi'_*t'^*a(g) = t^*\pi_*a(g)$, so in other words

$$\pi'_*(\mathfrak{g}^{-1}(\overline{T_{X'/S'}})t'^*g) = t^*(\pi_*(\mathfrak{g}^{-1}(\overline{T_{X/S}})g))$$

Now using the naturality of the $\mathfrak{g}$-class and the fact that for a Cartesian diagram such as the one in the proposition we have $\mathfrak{g}^{-1}(\overline{T}_{X'/S'}) = t'^*\mathfrak{g}^{-1}(\overline{T}_{X/S})$ we only need to prove that for any current η, we have

$$\pi'_*t'^*(\eta) = t^*\pi_*(\eta)$$

By duality, it is sufficient to prove that for any smooth compactly supported form ω on S' we have $\pi^*t_*\omega = t'_*\pi'^*\omega$. but this is tantamount to proving that

$$\int_{X'/X}\pi'^*\omega = \pi^*\int_{S'/S}\omega$$

.

Notice that S' being proper over k, and S being separated over k, t is proper, and thus closed, but it is also open because it is flat, we can thus assume that $t(S')$ is a connected component of S, and even surjective by making the base change with the respect to the connected component in question. By Ehresmann theorem [Kod86, Thm 2.4, p. 64], we can thus assume that $S' \to S$ is a proper fibration of typical fiber F.

Let (U_i) be an open cover of S, trivializing the fibration t, and let μ_i be a partition of unity associated with $U_i \times F$, which is an open cover of S'. We can choose U_i small enough so that it is isomorphic to an open subset of $\mathbb{C}^n$ As by its very definition, for any smooth form ω, $t_*(\omega) = \sum_i t_*(\mu_i\omega)$, and using the linearity of π^*, we may assume that ω is compactly supported in a open subset of the form $U_i \times F$, and can thus be written as a sum of $\alpha\wedge\beta$, where α (resp. β) is the pull-back of a smooth form on U_i (resp. F)

But then both sides of the identity we want to prove are equal to $\pi^*(\alpha)\wedge\int_F\beta$ □

2.2.4 Saturation of a subset of $\mathcal{Z}(\overline{X})$

Assume we've been given, for every arithmetic variety, $\overline{Y}$, an assignment $\overline{Y} \mapsto \mu(\overline{Y}) \subset \mathcal{Z}(\overline{Y})$.

Definition 2.2.29. *(Saturation of μ)*
We call the saturation of μ (if it exists) and we denote $\langle\,\mu\,\rangle$, the map $X \mapsto \langle\,\mu\,\rangle(X)$, where $\langle\,\mu\,\rangle(X)$ is the smallest class of subgroups of $\mathcal{Z}(\overline{X})$ satisfying, for every projective morphism $\pi : Y \to X$, for every smooth equidimensional morphism $f : X \to S$, and for every hermitian line bundle $\overline{L} \in \widehat{\mathrm{Pic}}(X)$,

$$\pi_*(\langle\,\mu\,\rangle(\overline{Y})) \subset \langle\,\mu\,\rangle(\overline{X});\, f^*(\langle\,\mu\,\rangle(\overline{S})) \subset \langle\,\mu\,\rangle(\overline{X});\, \widehat{c_1}(\overline{L})(\langle\,\mu\,\rangle(\overline{X})) \subset \langle\,\mu\,\rangle(\overline{X})$$

Proposition 2.2.30. *If the mapping μ is such that for every X, $\mu(X)$ consists of homogenous elements, then the saturation $\langle \mu \rangle$ exists, and the quotient $\mathcal{Z}_\mu(X)$ inherits a natural grading from $\mathcal{Z}(\overline{X})$.*

Proof. Let us notice that every standard class in $\mathcal{Z}(\overline{X})$ verifies

$$\begin{aligned}
[\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r, g] &= [\overline{Z} \xrightarrow{f} \overline{X}, \overline{L}_1, ..., \overline{L}_r] + a(g) \\
&= f_*[\overline{Z} \to \overline{Z}, \overline{L}_1, ..., \overline{L}_r] + a(g) \\
&= f_* \circ \widehat{c}_1(\overline{L}_r) \circ ... \circ \widehat{c}_1(\overline{L}_r)[\overline{Z} \to \overline{Z}] + a(g) \\
&= f_* \circ \widehat{c}_1(\overline{L}_r) \circ ... \circ \widehat{c}_1(\overline{L}_r)\pi^*_{\overline{Z}}(1_k) + a(g)
\end{aligned}$$

For every arithmetic variety $\overline{Y}$, set $\langle \mu \rangle(\overline{Y})$ the subgroup of $\mathcal{Z}(\overline{Y})$ generated by the set

$$A(\overline{Y}) = \{f_* \circ \widehat{c}_1(\overline{L}_r) \circ ... \circ \widehat{c}_1(\overline{L}_r)\pi^*(\alpha) | \alpha \in \mu(\overline{Z}), \pi : \overline{T} \to \overline{Z} \text{ projective}, f : \overline{T} \to \overline{Y} \text{smooth}, \overline{L}_i \in \widehat{\text{Pic}}(T)\}$$

we're left to check that the set $A(Y)$ is mapped to $A(Z)$ (resp. $A(S)$) under the action of a projective (resp. smooth equidimensional) morphism from $Y \to Z$ (resp. from S to Y). But this results simply from 2.2.28.

The fact that the quotient is naturally graded if μ takes only subset of homogenous elements in $\mathcal{Z}$ as value, and the fact that pull-backs, push-forwards and first Chern class operators, preserve the grading is immediate. □

If the saturation of μ exists, we shall denote $\mathcal{Z}_\mu(X)$ the (possibly graded) quotient $\mathcal{Z}(\overline{X})/\langle \mu \rangle(X)$.

2.2.5 The final construction

We will now impose the relations that'll turn our basic object $\mathcal{Z}(\overline{X})$ into an object with a real geometric and arithmetic significance, for that we need to impose the following three relations

$$\text{(DIM)} \quad [\overline{Y} \to \overline{X}, \overline{L}_1, ..., \overline{L}_{d+2}] = 0$$

for $d = \dim(Y)$.

$$\text{(SECT)} \quad [\overline{X} \to \overline{X}, \overline{L}] + a[i_*[\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T}_Z)]] = [\overline{Z} \to \overline{X}] - a(\mathfrak{g}(\overline{L}) \log \|s\|^2)$$

with s a section of $\overline{L}$ with smooth zero scheme, and $\| \cdot \|$ the norm induced by the norm on $\overline{L}$, where $\mathcal{E}$ is the exact sequence

$$\mathcal{E} : 0 \to \overline{T_Z} \to i^*\overline{T_X} \to i^*\overline{L} \to 0$$

and

$$\text{(FGL)} \quad \widehat{c}_1(\overline{L} \otimes \overline{M}) = \mathbb{F}(\widehat{c}_1(\overline{L}), \widehat{c}_1(\overline{M}))$$

where $\mathbb{F}$ is the universal formal law group.

In order to do this, we first need to impose the (DIM) condition, and to tensor over $\mathbb{Z}$ by $\mathbb{L}$ the Lazard ring, for the last relation to make any sense.

As the set $\text{SECT} + \text{DIM}(X) = \{[Y \to X, \overline{L}_1, ..., \overline{L}_r] | r > \dim(Y)\} \cup \{[X \to X, \overline{L}] - [Z \to X] + a(\log \|s\|^2) | s \text{ smooth section of } \overline{L}\}$ is made up of homogenous elements, we can consider the graded group $\mathcal{Z}_{\text{DIM,SECT},\bullet}(X)$.

Let's now return to the construction of arithmetic cobordism, we set $\breve{\mathcal{Z}}(X) = \mathbb{L} \otimes_{\mathbb{Z}} \mathcal{Z}(\overline{X})_{\text{DIM,SECT}}$, we can grade this group via the natural grading on both factors. It is naturally a $\mathbb{L}$-module, and we can extend all operations defined in the previous section, by linearity and we can prove the analog of 2.2.28 for $\mathbb{L}$-modules.

Definition 2.2.31. *(Arithmetic weak Cobordism)*
We set

$$\breve{\Omega}(X) = \breve{\mathcal{Z}}_{\mathrm{FGL}}(X)$$

where

$$\mathrm{FGL}(X) = \{\widehat{c_1}(\overline{L}\otimes\overline{M})(1_X) = \mathbb{F}(\widehat{c_1}(\overline{L}), \widehat{c_1}(\overline{M}))(1_X)\}\cup\{\widehat{c_1}(\overline{L}\otimes\overline{M})(a(g)) = \mathbb{F}(\widehat{c_1}(\overline{L}), \widehat{c_1}(\overline{M}))(a(g))\}$$

It is a graded $\mathbb{L}$*-module that we will call the* arithmetic weak cobordism group of X*.*

Remark 2.2.32. Notice that the operator $\widehat{c_1}(\overline{L})$ being locally nilpotent (i.e for every a, there exists $n > 0$, such that $\widehat{c_1}(\overline{L})^n(a) = 0$), the term $\mathbb{F}(\widehat{c_1}(\overline{L}), \widehat{c_1}(\overline{M}))$ does make sense.

Proposition 2.2.33. *Let* $\overline{X}$ *be an arithmetic variety, the map* $a : \mathbb{L} \otimes \mathbb{Z}[\mathbf{t}] \otimes \widetilde{D}_{\mathbb{R}}^{\bullet,\bullet}(X) \to \breve{\Omega}(\overline{X})$ *factors through* $\widetilde{D}_{\widehat{\mathbb{L}}}^{\bullet,\bullet}(X)$*, we will still denote by* a *this map* $\widetilde{D}_{\widehat{\mathbb{L}}}^{\bullet,\bullet}(X) \to \breve{\Omega}(\overline{X})$

Proof. The proof hinges on the following key remark

$$\int_{\mathbb{P}^r\times\mathbb{P}^\ell} c_1(p_1^*\mathcal{O}(1))^i c_1(p_2^*\mathcal{O}(1))^j = \delta_{ir}\delta_{jl}$$

To exploit this we will compute

$$I_{rl} = \int_{\mathbb{P}^r\times\mathbb{P}^\ell} \widehat{c_1}(p_1^*\mathcal{O}(1)\otimes p_2^*\mathcal{O}(1))a(1)$$

in two different ways. To ease notations we will simply write u for $c_1(p_1^*\mathcal{O}(1))$ and v for $c_1(p_2^*\mathcal{O}(1))$.

On the one hand, using FGL and the key remark we see that

$$I_{rl} = \sum_{i,j} a^{i,j} \int_{\mathbb{P}^r\times\mathbb{P}^\ell} (\mathfrak{h}^{-1}(u))^i(h^{-1}(v))^j = \sum_{i,j} a^{i,j}\left[(\mathfrak{h}^{-1}(u))^i(h^{-1}(v))^j\right]^{\{(r,l)\}}$$

where $\{(r,l)\}$ denotes the coefficient in front of $u^r v^l$.

On the other hand using the explicit expression of the action of the first Chern operator on $a(1)$ we see that

$$I_{rl} = (\mathfrak{h}^{-1}(u+v))^{\{(r,l)\}}$$

we thus have

$$(\mathfrak{h}^{-1}(u+v))^{\{(r,l)\}}a(1) = \sum_{i,j} a^{i,j}\left[(\mathfrak{h}^{-1}(u))^i(h^{-1}(v))^j\right]^{\{(r,l)\}} a(1)$$

in $\breve{\Omega}(k)$, and this pulls back to the same relation in $\breve{\Omega}(X)$ but those are exactly the relations between the t_i's and the $a^{i,j}$'s in $\widehat{\mathbb{L}}$, so the proof is complete. ☐

2.2.6 A remark on Borel-Moore Functors

We need to restrict the notion of Borel-Moore functor introduced in [LM07]. The reason for this is that the smallest class Levine and Morel consider to define a Borel-Moore functor is the class of quasi-projective smooth varieties over a field k whereas we are solely interested in the class of projective smooth varieties. We refer the reader to [LM07] for notations and vocabulary that we may not define.

Definition 2.2.34. *(Projective Borel-Moore functor, compare with [LM07, p.13])*

Let R be a graded ring, we call a (graded) projective R-Borel-Moore functor an assignment $X \to H_\bullet(X)$ for each X projective and smooth over k, such that

1. *$H_\bullet(X)$ is a (graded) R-module*
2. *(direct image homomorphisms) a homomorphism $f_* : H_\bullet(X) \to H_\bullet(Y)$ of degree zero for each projective morphism $f : X \to Y$,*
3. *(inverse image homomorphisms) a homomorphism $f^* : H_\bullet(Y) \to H_\bullet(X)$ of degree d for each smooth morphism $f : X \to Y$ of relative dimension d,*
4. *(first Chern class homomorphisms) a homomorphism $c_1(L) : H_\bullet(X) \to H_\bullet(X)$ of degree -1 for each line bundle L on X,*

satisfying the axioms

1. *the map $f \mapsto f_*$ is functorial;*
2. *the map $f \mapsto f^*$ is functorial;*
3. *if $f : X \to Z$ is a projective morphism, $g : Y \to Z$ a smooth equidimensional morphism, and the square*
$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$
is Cartesian, then one has
$$g^* \circ f_* = f'_* \circ g'^*$$
4. *if $f : Y \to X$ is projective and L is a line bundle on X, then one has*
$$f_* \circ c_1(f^*(L)) = c_1(L) \circ f_*$$
5. *if $f : Y \to X$ is a smooth equidimensional morphism and L is a line bundle on X, then one has*
$$c_1(f^*(L)) \circ f^* = f^* \circ c_1(L)$$
6. *if X is a projective smooth variety and L and M are line bundles on X, then one has*
$$c_1(L) \circ c_1(M) = c_1(M) \circ c_1(L)$$

We will only be interested in projective Borel-Moore $\mathbb{L}$-functor of a particular type.

Definition 2.2.35. *(Geometric type)*

A projective oriented Borel-Moore functor with product *is the data of a projective oriented Borel-Moore functor together with the data of*

1. *(external product) a bilinear graded multiplication map*
$$\times : H_\bullet(X) \times H_\bullet(Y) \to H_\bullet(X \times Y)$$
which is associative, commutative, and admits a unit $1_K \in H_0(\operatorname{Spec} k)$,

satisfying the axioms

1. *if $f : X \to Y$ and $g : X' \to Y'$ are projective morphisms, one has the equality*
$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times : H_\bullet(X) \times H_\bullet(X') \to H_\bullet(Y \times Y');$$
2. *if $f : X \to Y$ and $g : X' \to Y'$ are smooth equidimensional morphisms, one has the equality*
$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times : H_\bullet(Y) \times H_\bullet(Y') \to H_\bullet(X \times X');$$
3. *if L is a line bundle on X, and $\alpha \in H_\bullet(X)$, $\beta \in H_\bullet(Y)$, then one has the equality*
$$c_1(L)(\alpha) \times \beta = c_1(p^*(L))(\alpha \times \beta)$$
in $H_\bullet(X \times Y)$.

We will say that an $\mathbb{L}$ projective Borel-Moore functor with product, $H_\bullet$ is of geometric type if the following additional properties are satisfied

1. *(Dim) For X a smooth projective variety and $(L_1, \ldots, L_n)$ a family of line bundles on X with $n > \dim(X)$, one has*
$$c_1(L_1) \circ \cdots \circ c_1(L_n)(1_X) = 0$$
in $H_\bullet(X)$.
2. *(Sect) For X a smooth projective variety, L a line bundle on X, and s a section of L which is transverse to the zero section, one has the equality*
$$c_1(L)(1_X) = i_*(1_Z)$$
where $i : Z \to X$ is the closed immersion defined by the section s.
3. *(FGL) There exists a formal law group F_H on $\mathbb{L}$ such that, for X a smooth projective variety and L, M line bundles on X, one has the equality*
$$F_H(c_1(L), c_1(M))(1_Y) = c_1(L \otimes M)(1_Y)$$
where F_H acts on $H(X)$ via its $\mathbb{L}$-module structure. Moreover we require the different pull-backs and push-forward maps to preserve F_H.

Remark 2.2.36. Two classical examples of (projective) Borel-Moore functor of geometric type are given by CH and K_0 (the latter being non graded[12]). In fact one can show ([LM07, Thm 1.2.2 and Thm 1.2.3]) that CH is the universal additive[13] Borel-Moore functor of geometric type, while K_0 is the universal multiplicative unitary[14] Borel-Moore functor of geometric type, at least over fields of characteristic zero.

[12]It is possible to render it graded by considering $K_0(X) \otimes \mathbb{Z}[\beta, \beta^{-1}]$ where β is an indeterminate of degree 1.

[13]that means that the formal law group is given by the ordinary addition

[14]that means that the formal law group is given by $F(u, v) = u + v - uv$

Remark 2.2.37. To illustrate the depth of the fact that K_0 is a universal Borel-Moore functor, let us show how we can deduce Grothendieck-Riemann-Roch from this fact (this proof is given in [LM07]).

Let us define $\mathrm{CH}_!$ as the Borel-Moore functor determined by $\mathrm{CH}_!(X) = \mathrm{CH}(X)_{\mathbb{Q}}$, with pull-backs and products left unchanged, but with push-forwards and first Chern operator defined as

$$f_!(\alpha) = f_*(\alpha \operatorname{Td}(T_f)), \quad c_{1!}(L) = c_1(L) \operatorname{Td}(L)^{-1}$$

the axiom (*sect*) is still satisfied but $\mathrm{CH}_!$ is multiplicative unitary as a direct computation shows. Therefore the exists a natural map that we will call ch such that the following diagram commutes

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}} & \mathrm{CH}_!(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_!} \\ K_0(Y) & \xrightarrow[\mathrm{ch}]{} & \mathrm{CH}_!(Y) \end{array}$$

Now, as ch is a morphism of Borel-Moore functors it should preserve the first Chern operators, let $[E]$ be the class of a vector bundle on X, let us assume for a minute that E is a sum of line bundles, then

$$[E] = [L_1] + ... + [L_r] = r - ((1 - [L_1]) + ... + (1 - [L_r])) = r - (c_1(L_1^\vee) + ... + c_1(L_r^\vee))$$

therefore

$$\begin{aligned} \mathrm{ch}(E) &= r - (c_{1!}(L_1^\vee) + ... + c_{1!}(L_r^\vee)) \\ &= r - (c_1(L_1^\vee) \operatorname{Td}(L_1^\vee)^{-1} + ... + c_1(L_r^\vee) \operatorname{Td}(L_r^\vee)^{-1}) \\ &= r - (1 - e^{-c_1(L_1^\vee)} + ... + 1 - e^{-c_1(L_r^\vee)}) \\ &= e^{c_1(L_1)} + ... + e^{c_1(L_r)} \end{aligned}$$

and we see that ch is indeed the usual Chern character, at least for decomposable bundles, but we can reduce the general case to this case by using the splitting principle.

Rewriting the previous diagram in a more traditional manner we get that

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\operatorname{Td}(T_X)\,\mathrm{ch}} & \mathrm{CH}(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K_0(Y) & \xrightarrow[\operatorname{Td}(T_Y)\,\mathrm{ch}]{} & \mathrm{CH}(Y)_{\mathbb{Q}} \end{array}$$

is commutative and this is Grothendieck-Riemann-Roch.

Getting back to our problem, we can easily construct a universal projective Borel-Moore functor of geometric type by following the exact same procedure as in [LM07]. But what's a bit less obvious it that such a functor should coincide with the restriction of Ω to the category of smooth projective varieties.

The reason for this is that we may have some relationships in $\Omega(X)$ that may "come from quasi-projective varieties", that is to say that in $\Omega(X)$ we may observe the vanishing of classes of the form $\pi_*(a)$ (resp. $f^*(a)$) for a a vanishing class in $\Omega(Y)$ with Y quasi-projective.

The first case is totally innocent of course, because the composition of projective morphisms is projective, so no relation in $\Omega(X)$ with X projective can come from the cobordism ring of a quasi-projective variety. Let us take care of the second case.

Proposition 2.2.38. *The cobordism functor restricted to the category of projective smooth varieties is the universal projective Borel-Moore functor of geometric type.*

Proof. It is easy to give explicit generators for the saturation with respect to the relations we want to impose (see [LM07, Lemma 2.4.2; 2.4.7 and Remark 2.4.11], whose notations we will use).

More precisely $\Omega(X)$ can be constructed as the quotient of $\mathbb{L} \otimes \underline{\Omega}$ by the sub $\mathbb{L}$-module generated by the relations

$$f_* \circ c_1(L_1) \circ ... \circ c_1(L_p)\left([L \otimes M] - [\mathbb{F}(L, M)]\right)$$

where f is projective between smooth projective varieties.

Moreover $\underline{\Omega}$ is the quotient of $\underline{\mathcal{Z}}(X)$ by the subgroup generated by relations

$$[Z \to X, L_1, .., L_r] = [Z' \to X, i^*L_1, ..., i^*L_{r-1}]$$

where Z and Z' are (of course) projective and smooth.

And that $\underline{\mathcal{Z}}(X)$ is the quotient of $\mathcal{Z}(X)$ by the subgroup generated by relation of the form

$$[Y \to X, \pi^*L_1, ..\pi^*L_r, M_1, ..., M_d] \qquad (\star)$$

for every smooth equidimensional morphism $\pi : Y \to Z$ where where Z is a smooth quasi-projective variety of dimension strictly lower than r.

It will be sufficient to prove that we can replace the relations of the form $(\star)$ by the same relations but where Z is a smooth *projective* variety of dimension strictly lower than r.

To see this let $f : Y \to Z$ be a projective morphism from a projective smooth variety to a quasi-projective smooth variety, that we may assume to be embedded in some $\mathbb{P}^\ell$. Let us consider $\widetilde{Z} \xrightarrow{q} \overline{Z}$ a desingularization of the closure of Z in $\mathbb{P}^\ell$, which exists by [Hir64] and because that closure is of course reduced, as Z is isomorphic to an open subset of $\widetilde{Z}$ we have a (projective) morphism $\pi : Y \to \widetilde{Z}$ with $\widetilde{Z}$ projective, of the same dimension as Z, with $f^*L_i \simeq \pi^*q^*L_i$ which proves the claim and the proposition. ☐

From now on, we will only use the term Borel-Moore functor to mean a projective Borel-Moore functor.

2.2.7 An exact sequence

We will begin by a basic observation, notice that is X if a smooth projective variety, the choice of the metric on T_X doesn't change the structure of $\breve{\Omega}(\overline{X})$. To be precise

Proposition 2.2.39. *The natural map $\overline{X} \to \overline{X}'$ gives an isomorphism of $\mathbb{L}$-module*

$$\breve{\Omega}(\overline{X}) \to \breve{\Omega}(\overline{X}')$$

Proof. This is simply the functoriality of the push forward. ☐

Definition 2.2.40. *Let $\overline{X}$ be an arithmetic variety, we denote by $\Omega_{n.a}(\overline{X})$ the $\mathbb{L}$-module $\breve{\Omega}(\overline{X})/a(\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X))$ and we let $\omega(X)$ be the image of $\Omega_{n.a}(\overline{X})$ through ζ in $\Omega(X)$.*

Remark 2.2.41. Notice that, obviously $\omega(X)$ does not depend on the metric structure chosen on X, as a standard class $[\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]$ is mapped to $[Z \to X, L_1, ..., L_r]$, moreover $\zeta \circ a$ is of course trivial so our definition of $\omega(X)$ makes sense. We will still denote ζ the induced map from $\Omega_{n.a}(\overline{X})$ to $\omega(X)$.

We will usually denote $[\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]_{n.a}$ for the image of $[\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r, g]$ in $\Omega_{n.a}(\overline{X})$

With this definition it is not clear how $\Omega_{n.a}(\overline{X})$ depends on the choice of metric over X, even though its $\mathbb{L}$-module structure $\Omega_{n.a}(\overline{X})$ does not depend on it. In fact $\Omega_{n.a}(\overline{X})$ doesn't depend on the metric chosen on X at all. Let us prove that essential fact.

Firstly, let us notice that we have commutative diagrams (when they're defined)

$$
\begin{array}{ccc}
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) & \xrightarrow{a} & \breve{\Omega}(Y) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(Y) & \xrightarrow[a]{} & \breve{\Omega}(Y)
\end{array}
\ ,\quad
\begin{array}{ccc}
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) & \xrightarrow{a} & \breve{\Omega}(S) \\
{\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(S) & \xrightarrow[a]{} & \breve{\Omega}(S)
\end{array}
\ ,\quad
\begin{array}{ccc}
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) & \xrightarrow{a} & \breve{\Omega}(X) \\
{\scriptstyle \widehat{c_1}(\overline{L})}\downarrow & & \downarrow{\scriptstyle \widehat{c_1}(L)} \\
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) & \xrightarrow[a]{} & \breve{\Omega}(X)
\end{array}
$$

that ensure that the maps are well defined on the level of $\Omega_{n.a}$.

Lemma 2.2.42. *Let $\overline{X}$ and $\overline{X}'$ be two arithmetic variety structures on the same underlying algebraic variety, and let $\pi : \overline{X}' \to \overline{X}$ be the identity morphism. Then*

$$\pi_*[\overline{X}' \xrightarrow{\text{id}} \overline{X}'] = [\overline{X} \xrightarrow{\text{id}} \overline{X}] \text{ modulo } a(\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X))$$

Proof. Let $Y = \mathbb{P}^1_X$ be the projective line over X, and let $s_{[a:b]}$ be the section of $\mathcal{O}(1)$ over $\mathbb{P}^1$ defined *by* $-$ *ax*, if we see $\mathbb{P}^1$ as $\text{Proj}(k[X,Y])$, for any $(a,b) \in k^2$, $s_{[a:b]}$ is transverse to the zero section, and gives rise to an isomorphism

$$j^*_{[a,b]} T_{\mathbb{P}^1} \xrightarrow{\sim} j^*_{[a:b]} \mathcal{O}(-1)$$

if we look at the fiber square

$$
\begin{array}{ccc}
X & \xrightarrow{i_{[a,b]}} & Y \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
\text{Spec}\, k & \xrightarrow{j_{[a,b]}} & \mathbb{P}^1
\end{array}
$$

we have an exact sequence over X,

$$0 \to i^*_{[a,b]} p_2^* T_X \to i^*_{[a,b]} p_2^* T_X \oplus i^*_{[a,b]} p_1^* T_{\mathbb{P}^1} \to i^*_{[a,b]} p_1^* \mathcal{O}(1) \to 0$$

the term in the middle being isomorphic to $i^*_{[a,b]} T_Y$. From now on let us assume that both $T_{\mathbb{P}^1}$ and $\mathcal{O}(-1)$ are equipped with metric rendering isometric the isomorphisms given by s_0 and s_∞. Now as $T_Y \simeq p_2^* T_X \oplus p_1^* T_{\mathbb{P}^1}$, we can choose on T_Y a metric, say h (resp. h') such that we have an isometry $(T_Y, h) \simeq p_2^* \overline{T_X} \oplus p_1^* \overline{T_{\mathbb{P}^1}}$ (resp. $(T_Y, h') \simeq p_2^* \overline{T_{X'}} \oplus p_1^* \overline{T_{\mathbb{P}^1}}$). If φ is now any smooth function over $\mathbb{P}^1(\mathbb{C})$, such that $\varphi(0) = 1$ and $\varphi(\infty) = 1$, let us consider the metric $h'' = \varphi(t)h + \varphi(1/t)h'$ over Y, the two following exact sequences are meager (because they're ortho-split)

$$0 \to \overline{T_X} \to i_0^* p_2^* \overline{T_X} \oplus i_0^* p_1^* \overline{T_{\mathbb{P}^1}} \to i_0^* p_1^* \overline{\mathcal{O}(1)} \to 0$$

$$0 \to \overline{T_{X'}} \to i_\infty^* p_2^* \overline{T_{X'}} \oplus i_\infty^* p_1^* \overline{T_{\mathbb{P}^1}} \to i_\infty^* p_1^* \overline{\mathcal{O}(1)} \to 0$$

Let us now consider the class $[\overline{Y} \to \overline{Y}, p_1^*\overline{\mathcal{O}(1)}]$, where $\overline{Y}$ is equipped with the metric h'', by SECT we have, up to terms in $a(\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(\overline{Y}))$

$$\begin{aligned}[\overline{Y} \to \overline{Y}, p_1^*\overline{\mathcal{O}(1)}] &= [\overline{X} \to \overline{Y}] \\ &= [\overline{X}' \to \overline{Y}]\end{aligned}$$

the result follows from pushing-forward along $p_2 : \overline{Y} \to \overline{X}$. $\square$

Let us further investigate the independence on the metrics in $\Omega_{n.a}(\overline{X})$.

Lemma 2.2.43. *Let $\overline{X}$ and $\overline{Z}$ be two arithmetic varieties, and f any projective morphism between them, let us consider $L \in \mathrm{Pic}(Z)$ and let h and h' be two metrics on L, we have,*

$$[\overline{Z} \xrightarrow{f} \overline{X}, (L,h)] = [\overline{Z} \xrightarrow{f} \overline{X}, (L,h')] \text{ modulo } a(\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X))$$

Proof. This is the same idea as the previous lemma. Let us consider $\overline{Y} = \overline{\mathbb{P}^1_X}$ equipped with its "horizontal" metric that induces the metric on $\overline{X}$ over the fiber at 0 and ∞. Let us equip p_1^*L with the metric $h'' = \varphi(t)h + \varphi(1/t)h'$, where as before, φ is any smooth function over $\mathbb{P}^1(\mathbb{C})$, such that $\varphi(0) = 1$ and $\varphi(\infty) = 1$. Using SECT, we see that the class $[\overline{Y} \to \overline{Y}, \overline{p_1^*L}, \overline{p_2^*\mathcal{O}(1)}]$ equals both $[\overline{X} \to \overline{Y}, i_0^*\overline{p_1^*L}]$ and $[\overline{X} \to \overline{Y}, i_\infty^*\overline{p_1^*L}]$, up to an analytic class, this yields

$$[\overline{X} \to \overline{Y}, (L,h)] = [\overline{X} \to \overline{Y}, (L,h')]$$

and pushing forward along p_1 we get

$$[\overline{X} \to \overline{X}, (L,h)] = [\overline{X} \to \overline{X}, (L,h')]$$

but this is enough to prove the proposition as

$$[\overline{Z} \to \overline{X}, (L,h)] = f_*[\overline{Z} \to \overline{Z}, (L,h)] = f_*[\overline{Z} \to \overline{Z}, (L,h')] = [\overline{Z} \to \overline{X}, (L,h)]$$

and we are done. $\square$

Proposition 2.2.44. *Let $\overline{X}$ (resp. $\overline{Z}$) and $\overline{X}'$, (resp. $\overline{Z}'$) be two arithmetic variety structures on the same underlying algebraic variety, and let π be the identity morphism. Assume that we've been given $L_1, ..., L_r$, r line bundles over Z, which we will equip with two set of hermitian metric, $\overline{L}_i$ and $\overline{L}_i'$ for each i. We have*

$$\pi_*[\overline{Z}' \to \overline{X}', \overline{L}_1', ..., \overline{L}_r']_{n.a} = [\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]_{n.a}$$

Proof. Let us denote, for precision's sake, the different morphisms as in the following commutative diagram

$$\begin{array}{ccc} \overline{Z} & \xrightarrow{i} & Z' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \overline{X} & \xrightarrow{\pi} & X' \end{array}$$

where i the identity morphism from $\overline{Z}$ to $\overline{Z}'$. Of course we have $i^*\overline{L}_i = \overline{L}_i$ because the morphism i is the identity on the underlying variety. Using 2.2.42 and 2.2.43 we see that

$$\begin{aligned}
[\overline{Z} \to \overline{Z}', \overline{L}_1]_{n.a} &= \widehat{c_1}(\overline{L}_1)[\overline{Z} \to \overline{Z}']_{n.a} \\
&= \widehat{c_1}(\overline{L}_1)[\overline{Z}' \to \overline{Z}']_{n.a} \\
&= \widehat{c_1}(\overline{L}_1) i_*[\overline{Z}' \to \overline{Z}]_{n.a} \\
&= i_* \widehat{c_1}(\overline{L}_1)[\overline{Z}' \to \overline{Z}]_{n.a} \\
&= i_* \widehat{c_1}(\overline{L}_1')[\overline{Z}' \to \overline{Z}]_{n.a} \\
&= i_*[\overline{Z}' \to \overline{Z}, \overline{L}_1']_{n.a} \\
&= [\overline{Z}' \to \overline{Z}', \overline{L}_1']_{n.a}
\end{aligned}$$

By iterating, we see that

$$[\overline{Z} \to \overline{Z}', \overline{L}_1, ..., \overline{L}_r]_{n.a} = [\overline{Z}' \to \overline{Z}', \overline{L}_1', ..., \overline{L}_r']_{n.a}$$

now pushing forward along f' yields

$$[\overline{Z}' \to \overline{X}', \overline{L}_1', ..., \overline{L}_r']_{n.a} = [\overline{Z} \to \overline{X}', \overline{L}_1, ..., \overline{L}_r]_{n.a} = \pi_*[\overline{Z} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]_{n.a}$$

and the proof is complete. □

Corollary 2.2.45. *Let us fix a choice of metric on every (isomorphism class of) algebraic smooth projective variety. We have a Borel-Moore functor associated to this choice given by $X \to \Omega_{n.a}(\overline{X})$ for the specific choice of metric over X.*

If we take two of these Borel-Moore functors associated to two different choices of metrics, they're naturally isomorphic.

From now on we will denote $\Omega_{n.a}(X)$ instead of $\Omega_{n.a}(\overline{X})$ for this group, and we will omit the metrics when writing the elements of $\Omega_{n.a}(X)$. We shall now prove that we have in fact an isomorphism of Borel-Moore functor

$$\Omega_{n.a}(\bullet) \xrightarrow{\sim} \omega(\bullet) \xrightarrow{\sim} \Omega(\bullet)$$

In order to do this, we next show that in $\Omega_{n.a}(X)$ we have a stronger version of DIM

Lemma 2.2.46. *In $\Omega_{n.a}(X)$, we have*

$$[Y \to X, L_1, ..., L_r]_{n.a} = 0$$

as soon as $r > \dim(Y)$.

Proof. It is clear that we only need to prove $[Y \to Y, L_1, ..., L_r]_{n.a} = 0$, because pushing this formula will give the formula above.

Let us first show that it suffices to prove $[Y \to Y, L_1, ..., L_r]_{n.a} = 0$ where $L_1, ..., L_r$ are very ample line bundles to prove the general case. Indeed, every line bundle on a projective variety may be written as $M \otimes M'$ where M (resp. M') is very ample (resp. anti very ample), therefore if one of the bundles, say L_1 is not very ample we have[15]

$$[Y \to Y, L_1, ..., L_r]_{n.a} = \sum a^{i,j} c_1(M)^i \chi(c_1(M'^{\vee}))^j [Y \to Y, L_2, ..., L_r]$$

so it suffices to prove that $[Y \to Y, L_1, ..., L_r]_{n.a} = 0$ as soon as the L_i's are very ample to ensure that this class vanishes.

Now using SECT we see that

$$[Y \to Y, L_1, ..., L_r]_{n.a} = [\emptyset \to Y]_{n.a} = 0$$

and the result follows. □

[15] Here χ denotes the formal inverse characterized by $\mathbb{F}(u, \chi(u)) = 0$

Remark 2.2.47. We have a natural external product structure of $\Omega_{n.a}$ given by

$$[Y \to X, L_1, ..., L_r]_{n.a} \otimes [Z \to X', M_1, ..., M_k]_{n.a} \mapsto [Y \times Z \to X \times X', p_1^* L_1, ..., p_2^* M_k]_{n.a}$$

this endows $\Omega_{n.a}$ with the structure of a Borel-Moore functor with (external) products.

In happy concord with what happens for other weak homological theory we have

Proposition 2.2.48. *We have an exact sequence*

$$\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) \xrightarrow{a} \breve{\Omega}(X) \xrightarrow{\zeta} \Omega(X) \to 0$$

Proof. We will prove that ζ induces an isomorphism of Borel-Moore Functor of geometric type between $\Omega_{n.a}$ and Ω. We've already proven that $\Omega_{n.a}$ is a Borel-Moore functor with products and the fact that ζ is a morphism of Borel-Moore functor is obvious. The fact that $\Omega_{n.a}$ is of geometric type is easy as axioms (SECT) and (FGL) when ζ is applied to them, give the usual (SECT) and (FGL) axioms of a Borel-Moore functor of geometric type, as for (DIM) this is 2.2.46

So we get a map from Ω to $\Omega_{n.a}$, and a commutative diagram

$$\begin{array}{ccc} \breve{\Omega}(X) & \xrightarrow{\zeta} & \Omega_{n.a}(X) \\ & {\scriptstyle\zeta}\searrow \quad \nearrow & \\ & \Omega(X) & \end{array}$$

This map is an inverse of ζ, to check this we need to check that the standard classes $[X \to Y, L_1, ..., L_r]$ are left invariant by the application of $\Omega \to \Omega_{n.a} \to \Omega$, but that is obvious by construction.

The proof is then complete. □

Remark 2.2.49. It can be seen in the preceding proof that the exact sequence obtained is in fact an exact sequence of graded $\mathbb{L}$-modules, and it splits into exact sequences

$$\widetilde{D}_{\widehat{\mathbb{L}},p}(X) \xrightarrow{a} \breve{\Omega}_p(X) \xrightarrow{\zeta} \Omega_p(X) \to 0$$

where $\widetilde{D}_{\widehat{\mathbb{L}},p}(X)$ denotes the degree p part of $\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X)$ which is made up of

$$\widetilde{D}^{d_X-p-1,d_X-p-1}_{\mathbb{R}}(X) \oplus \widetilde{D}^{d_X-p,d_X-p}_{\mathbb{R}}(X) \otimes \widehat{\mathbb{L}}_1 \oplus ... \oplus \widetilde{D}^{d_X,d_X}_{\mathbb{R}}(X) \otimes \widehat{\mathbb{L}}_{p+1}$$

After tensorization by $\mathbb{Q}$ we can give a more explicit decomposition as

$$\widetilde{D}^{d_X-p-1,d_X-p-1}_{\mathbb{R}}(X) \oplus \mathbb{Q}[\mathbb{P}^1]\widetilde{D}^{d_X-p,d_X-p}_{\mathbb{R}}(X) \oplus ... \oplus \mathbb{Q}[\mathbb{P}^{p+1}]\widetilde{D}^{d_X,d_X}_{\mathbb{R}}(X)$$

Anticipating a little we get the result that served as a guideline during the construction of $\breve{\Omega}$

Corollary 2.2.50. *We have the following exact sequences (which we will refine in the following sections).*

$$\begin{array}{ccccccc} \widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X)_p & \xrightarrow{a} & \breve{\Omega}(\overline{X})_{\mathbb{Z},p} & \xrightarrow{\zeta} & \Omega(X)_{\mathbb{Z},p} & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \\ \widetilde{D}^{d_X-p+1,d_X-p+1}_{\mathbb{R}}(X)^a & \longrightarrow & \widetilde{CH}_p(\overline{X}) & \xrightarrow{\zeta} & \mathrm{CH}_p(X) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccc}
\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X) & \xrightarrow{a} & \breve{\Omega}(\overline{X})_{\mathbb{Z}} & \xrightarrow{\zeta} & \Omega(X)_{\mathbb{Z}} & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \\
\widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(X) & \xrightarrow{a} & \breve{K}_0(\overline{X}) & \xrightarrow{\zeta} & K_0(X) & \longrightarrow & 0
\end{array}$$

2.2.8 Some computations

In this section we will investigate more closely the different dependencies on the metric, by proving an anomaly formula.

According to 2.2.48 we see that the difference

$$[\overline{X} \to \overline{X}] - [\overline{X}' \to \overline{X}]$$

where $\overline{X}'$ and $\overline{X}$ are two different arithmetic structures on the same underlying variety should lie in the image of a, so it is a natural investigation to try and find an expression for that class. The answer is fairly simple and given by the

Proposition 2.2.51. *(Anomaly Formula)*

Let X be an algebraic projective smooth variety and let $\overline{X}, \overline{X}'$ and $\overline{X}''$ be three arithmetic structures on it, we have

$$[\overline{X}' \to \overline{X}] - [\overline{X}'' \to \overline{X}] = a(\mathfrak{g}(\overline{T}_X)\widetilde{\mathfrak{g}^{-1}}(T_X, h', h''))$$

and

$$[\overline{X}' \to \overline{X}] - [\overline{X}'' \to \overline{X}] = a(-\mathfrak{g}^{-1}(\overline{T}_X)\widetilde{\mathfrak{g}}(T_X, h', h''))$$

Proof. Let's use our usual trick consisting of endowing $Y = \mathbb{P}^1_X$ with a metric such that the fiber at 0 (resp. ∞) of T_Y is isometric to the orthogonal sum of $\overline{T'_X}$ (resp. $\overline{T''_X}$) and $\overline{T_{\mathbb{P}^1}}$ where $\mathbb{P}^1$ is equipped with its Fubini-Study metric. Let us compute

$$[\overline{Y} \to \overline{Y}, p_1^*\overline{\mathcal{O}(1)}] = [\overline{X}' \to \overline{Y}] - a(\mathfrak{g}(p_1^*\overline{\mathcal{O}(1)}) \log \|x\|^2) = [\overline{X}'' \to \overline{Y}] - a(\mathfrak{g}(p_1^*\overline{\mathcal{O}(1)}) \log \|y\|^2)$$

Therefore

$$[\overline{X}' \to \overline{X}] - [\overline{X}'' \to \overline{X}] \;=\; \int_{\mathbb{P}^1_X/X} \log|z|^2 \mathfrak{g}(p_1^*\overline{\mathcal{O}(1)})\mathfrak{g}^{-1}(\overline{T_{\mathbb{P}^1_X}})\mathfrak{g}(p_2^*\overline{T_X})$$

We obviously have $[\overline{X}' \to \overline{X}] - [\overline{X}'' \to \overline{X}] = 0$ as soon as X' and X'' are isometric, moreover we have

$$\begin{aligned}
dd^c([\overline{X}' \to \overline{X}] - [\overline{X}'' \to \overline{X}]) &= \mathfrak{g}(\overline{T_X}) \int_{\mathbb{P}^1_X/X} d_z d_z^c(\log|z|^2)\mathfrak{g}(p_1^*\overline{\mathcal{O}(1)})\mathfrak{g}^{-1}(\overline{T_{\mathbb{P}^1_X}}) \\
&= \mathfrak{g}(\overline{T_X})[i_0^*\mathfrak{g}(p_1^*\overline{\mathcal{O}(1)})\mathfrak{g}^{-1}(\overline{T_{\mathbb{P}^1_X}}) - i_\infty^*\mathfrak{g}(p_1^*\overline{\mathcal{O}(1)})\mathfrak{g}^{-1}(\overline{T_{\mathbb{P}^1_X}})] \\
&= \mathfrak{g}(\overline{T_X})[\mathfrak{g}(i_0^* p_1^*\overline{\mathcal{O}(1)})\mathfrak{g}^{-1}(i_0^*\overline{T_{\mathbb{P}^1_X}}) - \mathfrak{g}(i_\infty^* p_1^*\overline{\mathcal{O}(1)})\mathfrak{g}^{-1}(i_\infty^*\overline{T_{\mathbb{P}^1_X}})] \\
&= \mathfrak{g}(\overline{T_X})[\mathfrak{g}^{-1}(\overline{T_{X'}}) - \mathfrak{g}^{-1}(\overline{T_{X''}})]
\end{aligned}$$

and this suffices[16] to ensure that

$$[\overline{X}' \to \overline{X}] - [\overline{X}' \to \overline{X}] = \mathfrak{g}(\overline{T}_X)\widetilde{\mathfrak{g}^{-1}}(T_X, h', h'')$$

so the first formula holds and the second results directly form the first one. □

[16] One could also directly check that this is the formula that defined the secondary form associated to $\mathfrak{g}$

In [GS], Gillet and Soulé define a *star-product* operator on Green currents, to be able to define an intersection pairing for arithmetic cycles.

To this end, given two irreducible closed subsets of the ambient variety, say Z and Y, they need to select a specific green current for Z in the family of all admissible green current, that satisfies a "logarithmic-growth" condition. They prove that such a Green current always exists, and that we can multiply it with g_Y to get a green current for $[Z].[Y]$.

But if we look more closely at their construction, we see that we only need for a current to be of log-type singularities along the singular locus of another current to define a star-product between those two currents. This makes it possible to define an intersection pairing for divisors (or more precisely for classes of arithmetic divisors associated to hermitian line bundles), as in that case, we have a (family of) favored log-type current, namely $\log \|s\|^2$. It turns out that this construction is already embedded in the group $\breve{\Omega}(X)$.

Lemma 2.2.52. *Let $\overline{L}_1, .., \overline{L}_r$ be very ample hermitian line bundles over an arithmetic variety $\overline{X}$. Then*

$$\begin{aligned}[\overline{X} \to \overline{X}, \overline{L}_1, \overline{L}_2] &= [\overline{Z}' \to \overline{X}] - a(\log \|s_2\|^2 \mathfrak{g}(\overline{L}_2)\mathfrak{g}(\overline{L}_1)\delta_Z) + a(\mathfrak{h}^{-1}(\overline{L}_2) \log \|s_1\|^2 \mathfrak{g}(\overline{L}_1)) \\ &\quad - a(\mathfrak{g}(\overline{L}_2) j_*(\widetilde{\mathfrak{g}}(Z/X)\mathfrak{g}(\overline{T_Z}))) + a(j_*(i_*\widetilde{\mathfrak{g}^{-1}}(Z'/Z)\mathfrak{g}(\overline{N}_{Z'/X})\mathfrak{g}(j^*\overline{T}_X))))\end{aligned}$$

where $\overline{Z}'$ (resp $\overline{Z}$) is the smooth locus $\mathrm{div}(s_1) \cap \mathrm{div}(s_2)$ *(resp.* $\mathrm{div}(s_1)$*) endowed with any metric.*

Proof. let us compute

$$\widehat{c_1}(\overline{L})[Z \xrightarrow{j} X, g]$$

where $j : Z \to X$ is a regular immersion of smooth integral varieties, $\overline{L}$ is a very ample hermitian line bundle on X and g is any current on X, and where Z' is the smooth zero locus of a global section of j^*L over Z.

First, let us note that, by Bertini's theorem, such a section always exists. We have

$$\begin{aligned}\widehat{c_1}(\overline{L})[\overline{Z} \to \overline{X}, g] &= \widehat{c_1}(\overline{L})[\overline{Z} \to X] + a(\mathfrak{h}^{-1}(\overline{L}) \wedge g) \\ &= \widehat{c_1}(\overline{L}) j_*[\overline{Z} \to \overline{Z}] + a(\mathfrak{h}^{-1}(\overline{L}) \wedge g) \\ &= j_*[[\overline{Z}' \xrightarrow{i} \overline{Z}] - a(\mathfrak{g}(j^*\overline{L}) \log \|j^*s\|^2) - a(i_*[\widetilde{\mathfrak{g}}(Z'/Z)\mathfrak{g}^{-1}(\overline{T}_{Z'})])] + a(\mathfrak{h}^{-1}(\overline{L}) \wedge g) \\ &= [\overline{Z}' \to \overline{X}] - \mathfrak{g}(\overline{L}) a(\log \|s\|^2 j_*(\mathfrak{g}^{-1}(\overline{T_Z})\mathfrak{g}(j^*\overline{T_X})))) \\ &\quad - a\left[j_*(i_*\widetilde{\mathfrak{g}}(Z'/Z)\mathfrak{g}^{-1}(\overline{T}_{Z'}))\mathfrak{g}^{-1}(\overline{T}_Z)\mathfrak{g}(j^*\overline{T_X})\right] + a(\mathfrak{h}^{-1}(\overline{L}) \wedge g) \\ &= [\overline{Z}' \to \overline{X}] - \mathfrak{g}(\overline{L}) a(\log \|s\|^2 j_*(\mathfrak{g}(\overline{N_{Z/X}}))) - \mathfrak{g}(\overline{L}) a(\log \|s\|^2 j_*(dd^c \widetilde{\mathfrak{g}}(Z/X)\mathfrak{g}^{-1}(\overline{T_Z}))) \\ &\quad + a(j_*(i_*\widetilde{\mathfrak{g}^{-1}}(Z'/Z)\mathfrak{g}(\overline{N}_{Z'/X})\mathfrak{g}(j^*\overline{T}_X)))) + a(\mathfrak{h}^{-1}(\overline{L}) \wedge g) \\ &= [\overline{Z}' \to \overline{X}] - \mathfrak{g}(\overline{L}) a(\log \|s\|^2 j_*(\mathfrak{g}(\overline{N_{Z/X}}))) \\ &\quad - a(\mathfrak{g}(\overline{L}) j_*(\widetilde{\mathfrak{g}}(Z/X)\mathfrak{g}^{-1}(\overline{T_Z}))) - a(\mathfrak{h}^{-1}(L) j_*(\widetilde{\mathfrak{g}}(Z/X)\mathfrak{g}^{-1}(\overline{T_Z}))) \\ &\quad + a(j_*(i_*\widetilde{\mathfrak{g}^{-1}}(Z'/Z)\mathfrak{g}(\overline{N}_{Z'/X})\mathfrak{g}(j^*\overline{T}_X)))) + a(\mathfrak{h}^{-1}(\overline{L}) \wedge g)\end{aligned}$$

Where we've used the fact that for the composition of regular immersion we have an exact sequence

$$0 \to N_{Z'/Z} \to N_{Z'/X} \to i^* N_{Z/X} \to 0$$

replacing g by the expression given by SECT and using that $N_{Z/X} = j^* L_1$ yields the desired formula. □

Remark 2.2.53. In the preceding formula we can see that there is a variable part depending on the metrics chosen on the different strata of $Z_1 \cap Z_2 \subset Z_1 \subset X$ and a fixed part depending only on the metrics chosen on the bundles. For two hermitian line bundles over X, we set

$$(\overline{L}_1, \overline{L}_2)_X = -a(\log \|s_2\|^2 \mathfrak{g}(\overline{L}_2)\mathfrak{g}(\overline{L}_1)\delta_Z) + a(\mathfrak{h}^{-1}(\overline{L}_2) \log \|s_1\|^2 \mathfrak{g}(\overline{L}_1)) \in \breve{\Omega}(X)$$

inductively we can define

$$(\overline{L}_1, ..., \overline{L}_p)_X = \mathfrak{h}^{-1}(\overline{L}_p)(\overline{L}_1, ..., \overline{L}_{p-1})_X + \log \|s_p\|^2 \mathfrak{g}(\overline{L}_p)\delta_{Z_1 \cap ... \cap Z_{p-1}}$$

for a family of ample line bundles over X, and s_p a transverse section of L_p over X.

This bracket is easily seen to be symmetric in the L_i's, moreover, if we can find metrics on the Z_i's such that the different strata of the intersection of divisors $\bigcap Z_i$ can be endowed with metrics rendering all the associated exact sequences meager then

$$[\overline{X} \to \overline{X}, \overline{L}_1, ..., \overline{L}_p] = [\overline{\bigcap Z_i} \to \overline{X}] + (\overline{L}_1, ..., \overline{L}_p)_X$$

If f is projective morphism from X to Y we will denote $(\overline{L}_1, ..., \overline{L}_p)_Y$ for $f_*(\overline{L}_1, ..., \overline{L}_p)_X$, and $(\overline{L}_1, ..., \overline{L}_p)$ for $(\overline{L}_1, ..., \overline{L}_p)_{\mathrm{Spec}\, k}$

We thus have a formula for the class of the inclusion of a smooth subscheme given as a local complete intersection of smooth divisors. We can easily reduce the case of arbitrary intersection of divisors to this case, of course the formula obtained is more complicated.

Proposition 2.2.54. *Let $\overline{L}'_1, ..., \overline{L}'_r$ be arbitrary hermitian line bundles over an arithmetic variety $\overline{X}$. For each $0 \le i \le r$ there exists very ample hermitian line bundles $\overline{L}_i, \overline{M}_i$ such that*

$$[\overline{X} \to \overline{X}, \overline{L}'_1, ..., \overline{L}'_r] = \sum_{i_1, j_1, ..., i_r, j_r} a_{i_1,j_1} ... a_{i_r,j_r} [\overline{X} \to \overline{X}, \overline{L}_1, ..., \overline{L}_1, ..., \overline{M}_r^\vee, ..., \overline{M}_r^\vee]$$

where the bundle $\overline{L}_k$ (resp. $\overline{M}_k$) is repeated i_k (resp j_k) times.

Proof. As X is projective over k, every bundle L'_i can be written as $L_i \otimes M_i^\vee$ where L_i and M_i are very ample line bundles. Let's equip either one of them, say L_i with an arbitrary metric and the other one, M_i with the metric that turns the isomorphism $M_i \simeq L_i \otimes L'^\vee_i$ into an isometry.

The proposition readily follows. □

Remark 2.2.55. In the previous formula, the class

$$[\overline{X} \to \overline{X}, \overline{L}_1, ..., \overline{L}_1, \overline{M}_1^\vee, ..., \overline{M}_1^\vee, ..., \overline{M}_r^\vee, ..., \overline{M}_r^\vee]$$

can be computed via 2.2.52 and the observation that $\overline{M}^\vee$ is the inverse of $\overline{M}$ for the formal law $\mathbb{F}$, and thus $[\overline{X} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r, \overline{M}^\vee] = \chi(\widehat{c_1}(M))[\overline{X} \to \overline{X}, \overline{L}_1, ..., \overline{L}_r]$ and this enables us to define the bracket $(\overline{L}_1, ..., \overline{L}_p)_X$ for any family of hermitian line bundles.

Corollary 2.2.56. *As an $\mathbb{L}$-module, the group $\breve{\Omega}(X)$ is generated by purely geometric classes and analytic classes.*

2.2.9 Structure of $\breve{\Omega}(k)$

Let's start by a basic observation

Proposition 2.2.57. *Let u be any element in k^*, in $\breve{\Omega}(k)$ we have $a(-\log|u|) = 0$*

Proof. It suffices to consider the trivial line bundle over $\operatorname{Spec} k$, with metric $|u|^2$ (i.e, if x and y are lying in the line $\mathcal{O}_k(\mathbb{C}); x.y = |u|^2 x\overline{y}$), which will be denoted by $\overline{\mathcal{O}_k^u}$. Multiplication by u induces an (algebraic) isometry for the trivial bundle with trivial metric to $\overline{\mathcal{O}_k^u}$, thus the classes $[\operatorname{Spec} k \to \operatorname{Spec} k, \mathcal{O}_k]$ and $[\operatorname{Spec} k \to \operatorname{Spec} k, \overline{\mathcal{O}_k^u}]$ are equal. The first one is equal to $[\emptyset \to \operatorname{Spec} k] - a(\log|1|^2) = 0$, and the second one is equal to $[\emptyset \to \operatorname{Spec} k] - a(\log|u|^2)$. □

Remark 2.2.58*.* Let us draw the attention of the reader on the fact that over a point it is always possible to render an exact sequence of vector space $0 \to V \to V' \to V'' \to 0$ meager by an appropriate choice of metrics because it is obviously holomorphically split! Moreover if two of the three vector spaces appearing in this exact sequence are already equipped with metrics, it is possible to endow the last one with a metric rendering the short exact sequence meager. In short, a strong fitness lemma is true for the point.

Corollary 2.2.59. *We have a surjective arrow of groups*

$$\prod_{\tau:k\hookrightarrow\mathbb{C}} \mathbb{R}/(\oplus_{f\in k^*}\mathbb{Q}\log|\tau f|) \to \breve{\Omega}(k)_{-1,\mathbb{Q}}$$

as well as a global exact sequence

$$\prod_{\tau:k\hookrightarrow\mathbb{C}} \mathbb{L}_{\mathbb{R}}/(\oplus_{f\in k^*}\mathbb{L}\log|\tau f|) \to \breve{\Omega}(k) \to \mathbb{L} \to 0$$

A word of warning what we denoted somehow sloppily $\prod_{\tau:k\hookrightarrow\mathbb{C}}$ designs in fact the product over all real embeddings of k in $\mathbb{C}$ as well as pair of complex conjugate ones for the non real ones. Every time this notation appears that's how it should be understood.

Remark 2.2.60*.* In the geometric case we have $\Omega(k) \simeq \mathbb{L}$, for any field k that admits a resolution of singularities, that is for instance any fields of characteristic zero. This is a fundamental difference with the arithmetic theory developed here. The fact that k is a number field is used in a crucial manner to obtain the preceding corollary.

Notice also that $\breve{\Omega}(k)$ already depends on the number field k in a manner that is common in Arakelov theory, in fact the presence of the $\log(f)$ is some kind of artifact due to the fact that we work over fields instead of ring of integers, if we were able to define $\breve{\Omega}(\mathbb{Z})$ then we would expect that the $\log f$'s in the preceding formula should disappear.

Remark 2.2.61*.* We will later see that

$$\prod_{\tau:k\hookrightarrow\mathbb{C}} \mathbb{R}/(\oplus_{f\in k^*}\mathbb{Q}\log|\tau f|) \to \breve{\Omega}(k)_{-1,\mathbb{Q}}$$

is in fact an isomorphism. I believe that the other exact sequences in higher degree are also exact on the left.

Due to the fact that we chose to work with weak groups we will not be able to define a ring structure on $\breve{\Omega}(X)$ in full generality, however it is possible to define a ring structure on $\breve{\Omega}(k)$ and to deduce a module structure over $\breve{\Omega}(k)$, on $\breve{\Omega}(X)$. To understand that ring structure we need the following proposition that is closely related to the Riemann-Roch theorem of Hirzebruch

Proposition 2.2.62. *(Hirzebruch-Riemann-Roch)*

Let X be a projective smooth variety over a field k, and assume that we have equipped X with an arithmetic variety structure, we have in $\breve{\Omega}(k)_{\mathbb{Q}}$,

$$\int_X \mathfrak{g}^{-1}(\overline{T}_X)a(1) = \ell(X)a(1)$$

here ℓ denotes the composition

$$\breve{\Omega}(k) \xrightarrow{\zeta} \Omega(k) \simeq \mathbb{L}$$

(recall that the isomorphism $\Omega(k) \to \mathbb{L}$ is canonical).

Proof. This is essentially a combinatorics proof. Notice that it will be sufficient to prove this result in $\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(k)_{\mathbb{Q}}$ and to push this identity forward in $\breve{\Omega}(k)_{\mathbb{Q}}$ via the map a, notice also that the left hand side does not depend on the metric on T_X because of Stokes formula. So we'll prove the identity

$$\int_X \mathfrak{g}^{-1}(T_X) = \ell(X)$$

in $\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(k)_{\mathbb{Q}}$. This is tantamount to showing that $\mathfrak{g}^{-1}(\overline{T}_X)^{\{d_X\}} = \ell(X)$ where $\{n\}$ denotes the degree n part.

Let us first prove the result for projective spaces, if $X = \mathbb{P}^r$ we have the following Euler exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(1)^{r+1} \to T_X \to 0$$

and thus $\mathfrak{g}^{-1}(T_X) = \mathfrak{g}^{-1}(\mathcal{O}(1))^{r+1}$.

Now in view of Mishenko's formula 2.2.3, it will be sufficient to prove that

$$\left(\mathfrak{g}^{-1}(\mathcal{O}(1))^{r+1}\right)^{\{r\}} = (r+1)\mathfrak{h}(\mathcal{O}(1))^{\{r+1\}}$$

and this results from the following

Lemma 2.2.63. *(Lagrange Inversion formula)*

We have

$$\left(\frac{1}{\mathfrak{g}(u)^{r+1}}\right)^{\{r\}} = (r+1)\mathfrak{h}(u)^{\{r+1\}}$$

Proof. This is [Sta12, Thm 5.4.2, p. 38] □

Let us turn to the case of the product of projective spaces $X = \mathbb{P}^{r_1} \times ... \times \mathbb{P}^{r_k}$. We have

$$\mathfrak{g}^{-1}(T_X) = \mathfrak{g}^{-1}\left(\bigoplus_{i=1}^{k} p_i^* T_{\mathbb{P}^{r_i}}\right) = \prod_{i=1}^{k} \mathfrak{g}^{-1}(p_i^* T_{\mathbb{P}^{r_i}}) = \prod_{i=1}^{k} p_i^* \mathfrak{g}^{-1}(T_{\mathbb{P}^{r_i}})$$

therefore

$$\int_X \mathfrak{g}^{-1}(T_X) = \int_X \prod_{i=1}^{k} p_i^* \mathfrak{g}^{-1}(T_{\mathbb{P}^{r_i}}) = \prod_{i=1}^{k} \int_{\mathbb{P}^{r_i}} \mathfrak{g}^{-1}(T_{\mathbb{P}^{r_i}}) = [\mathbb{P}^{r_1}]...[\mathbb{P}^{r_k}] = [\mathbb{P}^{r_1} \times ... \times \mathbb{P}^{r_k}]$$

thus the results holds for a product of projective spaces.

Now, $\mathbb{L}_{\mathbb{Q}}$ is a polynomial ring over $\mathbb{Q}$ generated by the projective spaces, as the right hand side of the formula we want to prove is $\mathbb{Q}$-linear, it will be sufficient to prove that

$$\int_X \mathfrak{g}^{-1}(T_X) = \sum \alpha_{(r_1,..,r_k)} \int_{\mathbb{P}^{r_1}\times...\times\mathbb{P}^{r_k}} \mathfrak{g}^{-1}(T_{\mathbb{P}^{r_1}\times...\times\mathbb{P}^{r_k}})$$

as soon as we have

$$[X] = \sum \alpha_{(r_1,..,r_k)}[\mathbb{P}^{r_1}]...[\mathbb{P}^{r_k}]$$

in $\breve{\Omega}(k)_{\mathbb{Q}}$.

Recall [Mil60] that two complex manifold are in the same cobording class if and only if they share the same Chern numbers, $C_I(X) = \int_X c_{i_1}(T_X)...c_{i_p}(T_X) = \int_X c_I(X)$ for $I = (i_1, ..., i_p)$ any partition $d_X = i_1 + ... + i_p$. Let $a_1, ..., a_d$ be the Chern roots of T_X, we see that

$$\mathfrak{g}^{-1}(T_X)^{\{d\}} = (\prod_i \mathfrak{g}^{-1}(a_i))^{\{d\}} = (\prod_i(\sum_k t_k a_i^k)^{-1})^{\{d\}} = \sum_I v_{d,I}(t)c_I(X)$$

where $v_{d,I}(t)$ is a universal polynomial in (a finite number of) the t_i's depending only on the dimension of X. Therefore

$$\int \mathfrak{g}^{-1}(T_X) = \int_X \sum_I v_{d,I}(t)c_I(X) = \sum_I v_{d,I}(t)C_I(X)$$

Now, as $C_I(X) = \sum_J \alpha_J C_I(\mathbb{P}^J)$ for J some multi-indices of length d, we get

$$\begin{aligned}\sum_I v_{d,I}(t)C_I(X) &= \sum_I v_{d,I}(t)\sum_J \alpha_J C_I(\mathbb{P}^J)\\ &= \sum_J \alpha_J \sum_I v_{d,I}(t)C_I(\mathbb{P}^J)\\ &= \sum_J \alpha_J \int_{\mathbb{P}^J} \mathfrak{g}^{-1}(T_{\mathbb{P}^J})\\ &= \sum_J \alpha_J \ell(\mathbb{P}^J) = \ell(X)\end{aligned}$$

and the results follows. □

This result is the key ingredient that will enable us to show that we have a $\breve{\Omega}(k)$-module structure on $\breve{\Omega}(\overline{X})$

Proposition 2.2.64. *(Ring and Module Structures)*

We have a commutative $\mathbb{L}$-algebra structure on $\breve{\Omega}(k)$ given by

$$[\overline{X} \to \operatorname{Spec} k, \varphi(t)\alpha]\otimes[\overline{Y} \to \operatorname{Spec} k, \psi(t)\beta] \mapsto [\overline{X}\times\overline{Y} \to \operatorname{Spec} k]+\ell(X)\psi(t)a(\beta)+\ell(Y)\varphi(t)a(\alpha)$$

We have a natural $\breve{\Omega}(k)$-module structure on $\breve{\Omega}(\overline{X})$ given by

$$\begin{aligned}[\overline{X} \to \operatorname{Spec} k, \varphi(t)\alpha] \otimes [\overline{Z} \xrightarrow{f} \overline{Y}, \overline{L}_1, ..., \overline{L}_r, \psi(t)g] \mapsto\ & [\overline{X} \times \overline{Z} \to \overline{Y}, p_2^*\overline{L}_1, ..., p_2^*\overline{L}_r]\\ &+\ell(X)\psi(t)a(g)\\ &+\varphi(t)f_*[\widehat{c}_1(\overline{L}_1) \circ ... \widehat{c}_1(\overline{L}_r)\pi_Z^*(\alpha)]\end{aligned}$$

where π_Z is the structural morphism of $\overline{Z}$ ($\pi_Z^*(\alpha)$ is simply the locally function function $\tau\alpha$ over each connected component $Z_\tau(\mathbb{C})$ of $Z(\mathbb{C})$).

Proof. We need to show that all these operations are well defined. We will simply denote $[\overline{Y}]$ the class $[\overline{Y} \to \operatorname{Spec} k]$

Firstly let us notice that, by the usual trick of writing a possibly non very ample line bundle as the difference of two very ample line bundles, every class in $\breve{\Omega}(k)$ can be written as a linear combination of classes of the form $[\overline{Y}] + a(g)$ with coefficients in $\mathbb{L}$. So we only need to prove that this multiplication structure is compatible with SECT, DIM and FGL.

The case of DIM is obvious because $[\overline{Z}, \overline{L}_1 ..., \overline{L}_d + 2] = [\emptyset] = 0$ by the previous proposition, the case of SECT is just as obvious for the very same reason and the remark following the previous proposition. Concerning the case of FGL... there is nothing to prove.

Let's turn to the case of the module structure. We now need to check that the pairing vanishes as soon as the class on the right hand side is on of the form $\mathrm{SECT}(\overline{Y})$, $\mathrm{DIM}(\overline{Y})$ or $\mathrm{FGL}(\overline{Y})$ (the fact that is vanishes on the left hand side when the class is of the form $\mathrm{SECT}(k)$, $\mathrm{DIM}(k)$ or $\mathrm{FGL}(k)$ is just a repetition of the previous argument).

Let's start by DIM, and let's do the multiplication by an element of the form $a(\alpha)$ first. We have

$$a(\alpha)[\overline{Z} \xrightarrow{f} \overline{Y}, \overline{L}_1, ..., \overline{L}_r] = f_*[\widehat{c_1}(\overline{L}_1) \circ ... \widehat{c_1}(\overline{L}_r) \pi_Z^*(\alpha)]$$

but this zero as soon as $r > d_Z + 1$ because the action of the first Chern class increases the type of the forms by $(1, 1)$. Now for the case of a product

$$[\overline{X}][\overline{Z} \xrightarrow{f} \overline{Y}, \overline{L}_1, ..., \overline{L}_r] = [\overline{X} \times \overline{Z} \to \overline{Y}, p_2^*\overline{L}_1, ..., p_2^*\overline{L}_r]$$

which is also zero as soon as $r > d_Z + 1$ because this is $f_* p_{2*} p_1^* [\overline{Z} \to \overline{Z}, \overline{L}_1, ..., \overline{L}_r]$ and $[\overline{Z} \to \overline{Z}, \overline{L}_1, ..., \overline{L}_r]$ is zero.

Concerning SECT, for the multiplication by an analytic class, as the multiplication by an analytic class vanishes on analytic classes we're left with checking that

$$\begin{aligned} 0 &= \alpha\left([\overline{Z} \to \overline{X}] - [X \to X, \overline{L}]\right) \\ &= \alpha i_*(1) - \widehat{c_1}(\overline{L})\alpha \\ &= \mathfrak{g}(\overline{L})\alpha(\delta_Z - c_1(\overline{L})) \end{aligned}$$

and Poincare-Lelong formula ensures that this vanishes up to an exact current.

On the other hand, let us examine

$$\mu = [\overline{Y}]\left([\overline{Z} \xrightarrow{i} \overline{X}] - [X \to X, \overline{L}] - \log\|s\|^2 \mathfrak{g}(\overline{L}) - i_*(\widetilde{\mathfrak{g}(\mathcal{E})}\mathfrak{g}^{-1}(\overline{T_Z}))\right)$$

we have

$$\begin{aligned} \mu &= [\overline{Y} \times \overline{Z} \to \overline{X}] - [\overline{Y} \times \overline{X} \to \overline{X}, p_2^*\overline{L}] - \ell(Y)\mathfrak{g}(\overline{L}) \log\|s\|^2 - \ell(Y) i_*(\widetilde{\mathfrak{g}(\mathcal{E})}\mathfrak{g}^{-1}(\overline{T_Z})) \\ &= p_{2*}\left[[\overline{Y} \times \overline{Z} \xrightarrow{j} \overline{Y} \times \overline{X}] - [\overline{Y} \times \overline{X} \to \overline{Y} \times \overline{X}, p_2^*\overline{L}]\right] \\ &\quad -\ell(Y) i_*(\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T_Z})) - \ell(Y)\mathfrak{g}(\overline{L}) \log\|s\|^2 \\ &= p_{2*}[\log\|p_2^*s\|^2 \mathfrak{g}(p_2^*\overline{L}) + j_*(\widetilde{\mathfrak{g}}(p_2^*\mathcal{E})\mathfrak{g}^{-1}(p_2^*\overline{T_Z})\mathfrak{g}^{-1}(p_1^*\overline{T_Y}))] \\ &\quad -\ell(Y) i_*(\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T_Z})) - \ell(Y)\mathfrak{g}(\overline{L}) \log\|s\|^2 \\ &= \mathfrak{g}(\overline{L}) \log\|s\|^2 [p_{2*}p_2^* a(1) - \ell(Y)a(1)] - i_*(\ell(Y)\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T_Z})) + i_* p_{2*}(\widetilde{\mathfrak{g}}(p_2^*\mathcal{E})\mathfrak{g}^{-1}(p_2^*\overline{T_Z})\mathfrak{g}^{-1}(p_1^*\overline{T_Y})) \\ &= \mathfrak{g}(\overline{L}) \log\|s\|^2 [p_{2*}p_2^* a(1) - \ell(Y)a(1)] - i_*[\ell(Y)\widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T_Z}) - \widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T_Z}) p_{2*}p_2^* a(1)] \\ &= \left(\mathfrak{g}(\overline{L}) \log\|s\|^2 - \widetilde{\mathfrak{g}}(\mathcal{E})\mathfrak{g}^{-1}(\overline{T_Z})\delta_Z\right) [p_{2*}p_2^* a(1) - \ell(Y)a(1)] \end{aligned}$$

and this is seen to be zero because of 2.2.62.

Let's now tackle the case of FGL, an easy computation shows that

$$[X].\,\widehat{c_1}(\overline{L}) = \widehat{c_1}(\overline{L})[X], \quad a(\alpha)\,\widehat{c_1}(\overline{L}) = \widehat{c_1}(\overline{L})a(\alpha)$$

and this readily implies that the multiplication by $[X] + a(\alpha)$ of a class in FGL vanishes, which completes the proof. □

Remark 2.2.65. It may appear surprising at first glance that classes of the form $a(\alpha).a(g)$ vanish, but this should not be so because in other (strong) arithmetic theories, the product of such classes is given by $a(\alpha\partial\overline{\partial}g)$ which is $a((\partial\overline{\partial}\alpha)g)$ up to something in $\operatorname{im}\partial + \operatorname{im}\overline{\partial}$, and this is of course zero.

Recall that a Milnor hypersurface is defined as a hypersurface in $\mathbb{P}^n \times \mathbb{P}^m$ defined by the vanishing of a section of $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) = \mathcal{O}(1,1)$, transverse to the zero section. In fact we've already used them without saying so explicitly in 2.2.33. They're the original reason for the appearance of formal group laws in cobordism, as was shown in a pioneering work of Milnor [Mil60]. So it should not be surprising that we may need to study them.

Proposition 2.2.66. *Milnor hypersurfaces don't depend on the choice of the section of $\mathcal{O}(1,1)$(provided it is transverse to the zero section), are smooth over k, and can be given the structure of a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^m$*

Proof. We may assume $m \leq n$, if $T_0, ..., T_n$ (resp. $S_0, ..., S_m$) design the standard homogenous coordinates over $\mathbb{P}^n$ (resp. P^m, then a generic section of $\mathcal{O}(1)$ over $\mathbb{P}^n$ is given by $a_0.T_0 + ... + a_n.T_n$, thus a generic section of $\mathcal{O}(1,1)$ is given by $\sum_{0\leq \ell \leq n} \sum_{0 \leq r \leq m} a_\ell b_r T_\ell S_r$, an easy computation yields the fact that the zero scheme of that section is smooth if and only if it can be written as $\sum_{0\leq \ell\leq m} T_\ell S_\ell$ after a suitable linear change of coordinates.

As a linear change of coordinates is a k-automorphism of $\mathbb{P}^n \times \mathbb{P}^m$, it follows that $H_{n,m}$ as a k-scheme, does not depend on the choice of the section of $\mathcal{O}(1,1)$. Moreover it suffices to prove that over $V_+(T_i S_j)$, $H_{n,m}$ can be written as $\mathbb{P}^{n-1} \times V_+(S_j)$ which follows immediately from the description above. □

In algebraic cobordism Milnor hypersurfaces are important because they give generators for the cobordism ring over $\mathbb{Z}$ as well as expressions for $a^{i,j}$ up to decomposable elements. Here it is, of course impossible to give such a combinatorial description for $\breve{\Omega}(k)$ because of uncountability of the analytic term, as we've mentioned before we have

$$\prod_{\tau: k \hookrightarrow \mathbb{C}} \mathbb{R}/(\oplus_{f\in k^*}\mathbb{Q}\log|\tau f|) \to \breve{\Omega}(k)_{-1,\mathbb{Q}}$$

Nevertheless we can hope that we will obtain a description of $\breve{\Omega}(k)$ with respect to the class of Milnor hypersurfaces equipped with different family of metrics.

Definition 2.2.67. *Let $H^{n,m}$ be a Milnor hypersurface and h a metric over $T_{H^{n,m}}$ we will denote $\langle\, H^{n,m}, h \,\rangle$ the class $[\overline{H^{n,m}}]$*

In the following all the projective spaces will be endowed with their Fubini-Study metric coming from the trivial metric on the trivial bundle of rank $n+1$ over $\mathbb{P}^n$. We can prove the following

Proposition 2.2.68. *We have the following relation in* $\breve{\Omega}(k)$,

$$\begin{aligned}&\langle\, H^{n,m}, h\,\rangle - \int_{H^{n,m}} \widetilde{\mathfrak{g}^{-1}}(\mathcal{E}_h)\mathfrak{g}(i^*\overline{\mathcal{O}}(1,1)) \\ &+ \int_{\mathbb{P}^n\times\mathbb{P}^m} \log(\sum \|x_i\|^2\|y_i\|^2)\mathfrak{g}(\mathcal{O}(1,1))\mathfrak{g}^{-1}(T_{\mathbb{P}^n\times\mathbb{P}^m}) \\ = &\sum_{i,j\geq 0} a^{i,j}[\overline{\mathbb{P}}^{n-i}][\overline{\mathbb{P}}^{m-j}] + (p_1^*\overline{\mathcal{O}(1)}^i, p_2^*\overline{\mathcal{O}(1)}^j)\end{aligned}$$

where $(p_1^*\overline{\mathcal{O}(1)}^i, p_2^*\overline{\mathcal{O}(1)}^j)$ *denotes the bracket* $(p_1^*\overline{\mathcal{O}(1)}, ..., p_1^*\overline{\mathcal{O}(1)}, p_2^*\overline{\mathcal{O}(1)}, ..., p_2^*\overline{\mathcal{O}(1)})$ *where the first (resp. second) bundle is repeated* i *times (resp.* j *times), and where* $\mathcal{E}_h$ *is the exact sequence*

$$0 \to \overline{T_{H^{n,m}}} \to i^*\overline{T_{\mathbb{P}^n\times\mathbb{P}^m}} \to i^*\overline{\mathcal{O}(1,1)} \to 0$$

Proof. For $s = \sum x_i y_i$ the global section of the bundle $p_1^*\mathcal{O}(1)\otimes p_2^*\mathcal{O}(1)$, we have

$$\begin{aligned}[\overline{\mathbb{P}}^n\times\overline{\mathbb{P}}^m \to \overline{\mathbb{P}}^n\times\overline{\mathbb{P}}^m, \overline{\mathcal{O}(1,1)}] &= [\overline{H^{n,m}} \to \mathbb{P}^n\times\mathbb{P}^m] + \\ &\quad a(\mathfrak{g}(\overline{\mathcal{O}}(1,1))\log(\sum \|x_i\|^2\|y_i\|^2)) + a(i_*\widetilde{\mathfrak{g}}(\mathcal{E}_h)\mathfrak{g}^{-1}(h)) \\ &= \sum_{i\geq 0, j\geq 0} a^{i,j}[\overline{\mathbb{P}}^n\times\overline{\mathbb{P}}^m \to \overline{\mathbb{P}}^n\times\overline{\mathbb{P}}^m, p_1^*\overline{\mathcal{O}(1)}^i, p_2^*\overline{\mathcal{O}(1)}^j] \\ &= \sum_{i\geq 0, j\geq 0} a^{i,j}[\overline{\mathbb{P}}^{n-i}\times\overline{\mathbb{P}}^{m-j} \to \overline{\mathbb{P}}^n\times\overline{\mathbb{P}}^m] + (p_1^*\overline{\mathcal{O}(1)}^i, p_2^*\overline{\mathcal{O}(1)}^j)_{\mathbb{P}^n\times\mathbb{P}^m}\end{aligned}$$

To prove the last equality it will be sufficient to show that for the Fubini metrics over the different terms, the following exact sequence

$$0 \to T_{\mathbb{P}^k\times\mathbb{P}^r} \to j_0^*T_{\mathbb{P}^{k+1}\times\mathbb{P}^r} \to j_0^*p_1^*\mathcal{O}(1) \to 0$$

is meager, where j_0 is the closed immersion defined by the vanishing of the first homogenous coordinate. But this reduces to proving that

$$0 \to T_{\mathbb{P}^k} \to j_0^*T_{\mathbb{P}^{k+1}} \to j_0^*\mathcal{O}(1) \to 0$$

is meager, which is clear as this exact sequence is dual to the one defined by

$$0 \to \mathcal{O}(-1) \to j_0^*\Omega^1_{\mathbb{P}^{k+1}} \to \Omega^1_{\mathbb{P}^k} \to 0$$

where the second arrow is (in the standard $D_+(x_j)$ affine chart)

$$(d(x_0/x_j), ..., d(x_{k+2}/x_j)) \mapsto (\widehat{d(x_0/x_j)}, ..., d(x_{k+2}/x_j))$$

which is ortho-split for the Fubini-Study metric.

Now pushing forward along $\mathbb{P}^n\times\mathbb{P}^m$ yields the formula. □

In this formula we notice that the term

$$\langle\, H^{n,m}, h\,\rangle - \int_{H^{n,m}} \widetilde{\mathfrak{g}^{-1}}(\mathcal{E}_h)\mathfrak{g}(i^*\overline{\mathcal{O}}(1,1))$$

is equal to a term that does not depend on the metric h chosen. Therefore

$$\langle\, H^{n,m}, h\,\rangle - \int_{H^{n,m}} \widetilde{\mathfrak{g}^{-1}}(\mathcal{E}_h)\mathfrak{g}(i^*\overline{\mathcal{O}}(1,1)) = \langle\, H^{n,m}, h'\,\rangle - \int_{H^{n,m}} \widetilde{\mathfrak{g}^{-1}}(\mathcal{E}_{h'})\mathfrak{g}(i^*\overline{\mathcal{O}}(1,1))$$

but we already know, thanks to the anomaly formula that

$$\langle H^{n,m}, h \rangle = \langle H^{n,m}, h' \rangle + \int_{H^{n,m}} \widetilde{\mathfrak{g}}^{-1}(h, h')$$

therefore we can sum up the previous proposition as

$$\langle H^{n,m}, h \rangle = \text{ a fixed term } + \int_{H^{n,m}} \widetilde{\mathfrak{g}}^{-1}(h, h')$$

where the fixed term corresponds to the choice of a specified fixed metric.

2.2.10 Arrows

We will now construct arrows from the group $\breve{\Omega}(X)_{\mathbb{Z}}$ to the groups $\breve{CH}(X)$ and $\breve{K}_0(X)$, these arrows will be compatible with the different maps we have defined between those groups. To define those we will introduce the notion of Borel-Moore functor of arithmetic type on the category of arithmetic varieties.

Definition 2.2.69. *(Hermitian Borel Moore Functor)*
We will call a (graded) hermitian Borel-Moore functor an assignment $\overline{X} \to \widehat{H}_\bullet(X)$ *for each arithmetic variety* $\overline{X}$*, such that we have*

1. $H_\bullet(\overline{X})$ *is a (graded)* $\mathbb{L}$*-module with a specified element denoted* $1_{\overline{X}}$*, and called the unit element,*
2. $H_\bullet(\overline{X})$ *is equipped with an action of* $\widetilde{D}^{\bullet,\bullet}_{\widehat{\mathbb{L}}}(X)$ *denoted by a,*
3. *(genus) a multiplicative genus* $\varphi \in \widehat{H}(k)[[u]]$
4. *(direct image homomorphisms) a homomorphism* $f_* : \widehat{H}_\bullet(\overline{X}) \to \widehat{H}_\bullet(\overline{Y})$ *of degree zero for each projective morphism* $f : \overline{X} \to \overline{Y}$*,*
5. *(inverse image homomorphisms) a homomorphism* $f^* : \widehat{H}_\bullet(\overline{Y}) \to \widehat{H}_\bullet(\overline{X})$ *of degree d for each smooth equidimensional morphism* $f : \overline{X} \to \overline{Y}$ *of relative dimension d that preserves the unit element,*
6. *(first Chern class homomorphisms) a homomorphism* $\widehat{c}_1(\overline{L}) : \widehat{H}_\bullet(\overline{X}) \to \widehat{H}_\bullet(\overline{X})$ *of degree -1 for each hermitian line bundle* $\overline{L}$ *on* $\overline{X}$*,*

satisfying the axioms

1. *the map* $f \mapsto f_*$ *is functorial;*
2. *the map* $f \mapsto f^*$ *is functorial;*
3. *if* $f : \overline{X} \to \overline{Z}$ *is a projective morphism,* $g : \overline{Y} \to \overline{Z}$ *a smooth equidimensional morphism, and the square*

$$\begin{array}{ccc} \overline{W} & \xrightarrow{g'} & \overline{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{Y} & \xrightarrow{g} & \overline{Z} \end{array}$$

is Cartesian, then one has

$$g^* \circ f_* = f'_* \circ g'^*$$

4. *if $f : \overline{Y} \to \overline{X}$ is projective and $\overline{L}$ is a hermitian line bundle on $\overline{X}$, then one has*
$$f_* \circ \widehat{c_1}(f^*(\overline{L})) = \widehat{c_1}(\overline{L}) \circ f_*$$
5. *if $f : \overline{Y} \to \overline{X}$ is a smooth equidimensional morphism and $\overline{L}$ is a hermitian line bundle on $\overline{X}$, then one has*
$$\widehat{c_1}(f^*\overline{L}) \circ f^* = f^* \circ \widehat{c_1}(L)$$
6. *if $\overline{L}$ and $\overline{M}$ are hermitian line bundles on $\overline{X}$, then one has*
$$\widehat{c_1}(\overline{L}) \circ \widehat{c_1}(\overline{M}) = \widehat{c_1}(\overline{M}) \circ \widehat{c_1}(\overline{L})$$
7. *if $f : \overline{Y} \to \overline{X}$ is projective, then one has*
$$f_* \circ a(g) = a(g \wedge \varphi(\overline{T_f}))$$
8. *if $\overline{L}$ is a hermitian line bundle on $\overline{X}$, then one has*
$$\widehat{c_1}(\overline{L}) \circ a(g) = a(c_1(\overline{L})\varphi(\overline{L})g)$$

Just like for the geometric case we need to restrict the class of Borel Moore functors we'll be interested in, in order to give them an arithmetic significance.

Definition 2.2.70. *(Arithmetic Type)*
A Hermitian Borel-Moore functor with weak product *is the data of a hermitian Borel-Moore functor together with the data of*

1. *a commutative $\mathbb{L}$-algebra structure on $\widehat{H}(k)$,*
2. *a $\widehat{H}(k)$-module structure on $\widehat{H}(\overline{X})$ compatible with its $\mathbb{L}$ structure.*

We will say that a hermitian Borel-Moore functor with weak product, $\widehat{H}_\bullet$ is of arithmetic type if the following additional properties are satisfied

1. *(Dim) For $\overline{X}$ an arithmetic variety and $(\overline{L}_1, \ldots, \overline{L}_n)$ a family of hermitian line bundles on X with $n > \dim(X) + 1$, one has*
$$\widehat{c_1}(\overline{L}_1) \circ \cdots \circ \widehat{c_1}(\overline{L}_n)(1_{\overline{X}}) = 0$$
in $\widehat{H}_\bullet(X)$.
2. *(Sect) For $\overline{X}$ an arithmetic variety, $\overline{L}$ a hermitian line bundle on X, and s a section of L which is transverse to the zero section, one has the equality*
$$\widehat{c_1}(L)(1_{\overline{X}}) + a(i_*[\widetilde{\varphi}(\mathcal{E})\varphi^{-1}(\overline{T}_Z)]) + a(\varphi(\overline{L}) \log \|s\|^2) = i_*(1_{\overline{Z}})$$
where $i : Z \to X$ is the closed immersion defined by the section s and $\mathcal{E}$ is the exact sequence
$$0 \to \overline{T_Z} \to i^*\overline{T_X} \to i^*\overline{L} \to 0$$
3. *(FGL) If F_H is the formal group law defined by $\mathbb{L} \to \widehat{H}(\operatorname{Spec} k)$, then for $\overline{X}$ an arithmetic variety and $\overline{L}, \overline{M}$ hermitian line bundles on $\overline{X}$, one has the equality*
$$F_H(\widehat{c_1}(\overline{L}), \widehat{c_1}(M)) = \widehat{c_1}(\overline{L} \otimes \overline{M})$$
where F_H acts on $\widehat{H}(X)$ via its $\mathbb{L}$-module structure. Moreover we require the different pull-backs and push-forward maps to preserve F_H.

The following theorem is a tautology

Theorem 2.2.71. *The assignment* $\overline{X} \mapsto \breve{\Omega}(\overline{X})$ *is the universal (weak) Borel-Moore functor of arithmetic type.*

Remark 2.2.72. In fact in view of (sect) the genus of an arithmetic Borel-Moore functor is completely determined by its formal group law, we can prove it from the axiom sect, but as it is already the case with $\breve{\Omega}$, it will be automatically the case in every such functor also.

Remark 2.2.73. We've proven in 2.1.8 and 2.1.9 that $\overline{X} \mapsto \widetilde{CH}(\overline{X})$ is a (weak) Borel-Moore functor of arithmetic type, its formal group law is additive, and its genus is given by 1, which explains that $\widetilde{CH}(\overline{X})$ does not depend on the choice of hermitian structure on X.

We've also proven in 2.1.59 and 2.1.60 that $\overline{X} \mapsto \breve{K}(\overline{X})$ is a (weak) Borel-Moore functor of arithmetic type with multiplicative unitary law, and the usual Todd-genus as genus.

Corollary 2.2.74. *We have natural arrows*

$$\breve{\Omega}(\overline{X})_{\mathbb{Z}} \to \widetilde{CH}(X) \quad \breve{\Omega}(X)_{\mathbb{Z}} \to \breve{K}_0(X)$$

that make the following diagrams commute

$$\begin{array}{ccccccc}
\widetilde{D}^{\bullet,\bullet}_{\hat{\mathbb{L}}}(X)_p & \xrightarrow{a} & \breve{\Omega}(\overline{X})_{\mathbb{Z},p} & \xrightarrow{\zeta} & \Omega(X)_{\mathbb{Z},p} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\widetilde{D}^{d_X-p+1,d_X-p+1}_{\mathbb{R}}(X) & \xrightarrow{a} & \widetilde{CH}_p(X) & \xrightarrow{\zeta} & \mathrm{CH}_p(X) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccc}
\widetilde{D}^{\bullet,\bullet}_{\hat{\mathbb{L}}}(X) & \xrightarrow{a} & \breve{\Omega}(\overline{X})_{\mathbb{Z}} & \xrightarrow{\zeta} & \Omega(X)_{\mathbb{Z}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\widetilde{D}^{\bullet,\bullet}_{\mathbb{R}}(X) & \xrightarrow{a} & \breve{K}_0(\overline{X}) & \xrightarrow{\zeta} & K_0(X) & \longrightarrow & 0
\end{array}$$

Proof. Both arrows are given by the choices of the formal group law, if we chose the additive one, we get a map from $\breve{\Omega}(\overline{X})_{\mathbb{Z}}$ to $\widetilde{CH}(X)$, notice that by the anomaly formula and the fact that the arithmetic cobordism group is generated by purely analytic and purely geometric classes we have that $\breve{\Omega}(\overline{X})_{\mathbb{Z}}$ does not depend on the metric on X.

The arrow $\breve{\Omega}(\overline{X})_{\mathbb{Z}} \to \breve{K}_0(\overline{X})$ is given by the choice of the multiplicative unitary law. □

Corollary 2.2.75. *We have an isomorphism*

$$\prod_{\tau:k\hookrightarrow\mathbb{C}} \mathbb{R}/(\oplus_{f\in k^*}\mathbb{Q}\log|\tau f|) \to \breve{\Omega}(k)_{-1,\mathbb{Q}}$$

Proof. We already know that the map is surjective it remains to show that it is injective, but

$$\widetilde{CH}_{-1}(X) \simeq \prod_{\tau:k\hookrightarrow\mathbb{C}} \mathbb{R}/(\oplus_{f\in k^*}\mathbb{Z}\log|\tau f|)$$

therefore if the image of any element in $\prod_{\tau:k\hookrightarrow\mathbb{C}}\mathbb{R}/(\oplus_{f\in k^*}\mathbb{Q}\log|\tau f|)$ would be zero in $\breve{\Omega}(k)_{-1,\mathbb{Q}}$ then a multiple of it would be mapped to zero in $\widetilde{CH}_{-1}(X)$ and

$$\prod_{\tau:k\hookrightarrow\mathbb{C}}\mathbb{R}/(\oplus_{f\in k^*}\mathbb{Q}\log|\tau f|)\to\widetilde{CH}_{-1}(X)_{\mathbb{Q}}$$

would not be injective, a contradiction. □

These results shed some light on different constructions in Arakelov theory.

It explains why the direct image in K-theory depend on a choice of metric on the varieties whereas it is possible to construct a push forward for arithmetic Chow groups without specifying any metric. This is because the Todd class of the Chow theory is 1, and therefore the secondary forms associated to it are 0.

It also explains why the star product of [GS] is what it is, because the computation of the bracket $(\overline{L},\overline{M})_X$ reduces to the computation of the star-product $-\log\|s\|^2\star-\log\|t\|^2$

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