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Sur l'arithmétique des systèmes de Bost-Connes

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Diese Arbeit widme ich meinen Eltern,
für all ihre Liebe und Unterstützung.

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fait ; on ne voit que ce qui reste à faire.
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Contents

Introduction

The central theme of the present thesis is concerned with arithmetic properties of Bost-Connes systems, which are of central importance in the slowly emerging field of mathematics shimmering at the horizon where Noncommutative Geometry and Number Theory meet and hopefully fertilize each other one day in both directions. For the time being it seems that mainly NCG is profiting as NT gives rise to interesting and rich objects in the world of NCG, and among the most prominent and interesting ones are Bost-Connes systems.

The history of BC-systems starts in '95 with the seminal paper of Bost and Connes [BC95] introducing the now called BC-system $\mathcal{A}_{\mathbb{Q}} = (A_{\mathbb{Q}}, \sigma_t)$ which is a C^*-dynamical system, or in physical terms a quantum statistical mechanical system, with many interesting properties. For example, its partition function is given by the Riemann zeta function, it obeys an interesting phase transition and, moreover, its dynamics realizes the class field theory of $\mathbb{Q}$.

It was a natural problem to generalize the BC-system to arbitrary number fields and this was achieved by Ha and Paugam [HP05] and Laca, Larsen and Neshveyev [LLN09] (building upon important earlier work of [BC95], Laca [Lac98], Neshveyev [Nes02], Connes and Marcolli [CM06] and Connes, Marcolli and Ramachandran [CMR05] and [CMR06]). They constructed [2] for every number field K a C^*-dynamical system (cf., sections 2.1 and 1.5.1)

$$\mathcal{A}_K = (A_K, \sigma_t) \tag{0.1}$$

with the following four properties generalizing the classical BC-system

(i) The partition function of $\mathcal{A}_K$ is given by the Dedekind zeta function $\zeta_K(\beta)$ of K.
(ii) The maximal abelian Galois group $Gal(K^{ab}/K)$ of K acts as symmetries on $\mathcal{A}_K$.
(iii) For each inverse temperature $\beta \in (0, 1]$ there is a unique KMS_β-state.
(iv) For each $\beta \in (1, \infty]$ the action of the symmetry group $Gal(K^{ab}/K)$ on the set of extremal KMS_β-states is free and transitive.

(0.2)

For this reason we call the system $\mathcal{A}_K$ the BC-system for K. The following question, whether BC-systems admit an arithmetic model, was, in general, an open problem before this thesis and goes back to [BC95]. Explicitely the problem was first stated in the paper [CMR05]. A BC-system $\mathcal{A}_K$ is said to admit an arithmetic model if there exists a K-rational subalgebra

$$A_K^{arith} \subset A_K$$

2. More precisely, Ha and Paugam constructed the systems $\mathcal{A}_K$ and showed property (i) and (ii). Laca, Larsen and Neshveyev showed that (iii) and (iv) hold by classifying the KMS_β-states of $\mathcal{A}_K$.

such that

(v) For every extremal KMS_∞-state ϱ and every $f \in A_K^{arith}$ we have

$$\varrho(f) \in K^{ab}$$

and further K^{ab} is generated over K by these values.

(vi) If we denote by ${}^{\nu}\varrho$ the action of a symmetry $\nu \in \mathrm{Gal}(K^{ab}/K)$ on an extremal KMS_∞-state ϱ (given by pull-back) we have for every element $f \in A_K^{arith}$ the compatibility relation

$${}^{\nu}\varrho(f) = \nu^{-1}(\varrho(f)).$$

(vii) The $\mathbb{C}$-algebra $A_K^{arith} \otimes_K \mathbb{C}$ is dense in A_K.

(0.3)

In this case the K-algebra A_K^{arith} is called an arithmetic subalgebra of $\mathcal{A}_K$. In particular, arithmetic models relate BC-systems to class field theory and potentially to Hilbert 12th problem, which asks for an explicit class field theory. The latter problem is widely open, except for the rational field, imaginary quadratic fields and partially for CM fields.
The existence of arithmetic models for BC-sytems, before this thesis, was known in the case of the rational number field, cf. [BC95], and in the case of imaginary quadratic fields, see [CMR05], based on the theory of Complex Multiplication on the modular curve and the GL_2-system of [CM06].

The main result achieved in this thesis is the construction of arithmetic models for BC-systems in complete generality.

Outline of our results

In the **first chapter** we generalize the work of Connes, Marcolli and Ramachandran [CMR05] and [CMR06] to the case of arbitrary CM fields. The main ingredients of our construction are the theory of Complex Multiplication on general Siegel modular varieties, e.g., [MS81], and the GSp_{2n}-systems of Ha and Paugam [HP05]. Due to the fact that for a CM field K which is not quadratic over $\mathbb{Q}$ the theory of Complex Multiplication does not generate the maximal abelian extension K^{ab} we obtain only partial, but still very interesting, arithmetic subalgebra in these cases. Moreover, due to the fact that our construction involves arithmetic modular functions and Shimura's reciprocity law our partial arithmetic subalgebra has a quite explicit flavour.

In the **second chapter** we prove the existence of arithmetic models in full generality. This is the **main result of the present thesis**. It was already remarked by Marcolli [Mar09] that the classical BC-system can be described in the framework of Endomotives and Λ-rings. We follow this route and show that the theory of Endomotives, introduced by Connes, Consani and Marcolli [CCM07], and a classification result of certain Λ-rings in terms of the Deligne-Ribet monoid by Borger and de Smit [BdS11] provide the correct ingredients to construct (in a non-explicit manner) arithmetic models of arbitrary BC-systems. Moreover, our construction shows that BC-systems are in general closely related to Witt-vectors, Λ-rings and Frobenius-lifts. In the case of the classical BC-system the relation to Witt vectors has been exploited very recently by Connes and Consani [CC11] who constructed p-adic representations of the BC-system. We expect that our results will provide p-adic representations of BC-systems for arbitrary number fields.

Further, the second chapter contains an appendix by Sergey Neshveyev who showed that our arithmetic model is essentially unique.

The **last chapter** is concerned with functoriality properties of BC-systems. More precisely, in the context of Endomotives, we will construct an algebraic refinement of a functor from the category of number fields to an appropriate category of BC-systems which was recently constructed by Laca, Neshveyev and Trifkovic [LNT]. For this we introduce a notion of base-change for certain algebraic endomotives and show that this gives rise naturally to the functor of [LNT].

The results of chapter one are based on [Yal10] and the results of chapter two and three are based on [Yal11].

Introduction française

Le thème principal de cette thèse est l'étude des propriétés arithmétiques des systèmes de Bost-Connes qui sont d'une importance centrale dans le domaine situé à l'interface entre la géométrie non-commutative et la théorie des nombres. On espère qu'un jour les deux théories vont s'enrichir mutuellement. Actuellement la géométrie non-commutative en profite davantage dans la mesure où la théorie des nombres donne naissance à des objets très riches et intéressants dans le monde de la géométrie non-commutative et parmi eux se trouvent les systèmes de Bost-Connes.

L'histoire des systèmes de Bost-Connes commence en 1995 avec le papier fondateur de Bost et Connes [BC95]. Ces dernièrs introduisent un C^*-système dynamique ou, en langage de la physique, un système mécanique statistique quantique $\mathcal{A}_{\mathbb{Q}} = (A_{\mathbb{Q}}, \sigma_t)$ qu'on appelle système de Bost-Connes avec des propriétés très remarquables. Par exemple sa fonction de partition est donnée par la fonction zêta de Riemann, il obéit à un phénomène de brisure spontanée de symétries très intéressant et de plus sa dynamique réalise la théorie de corps des classes sur $\mathbb{Q}$.

C'est un problème naturel et très intéressant de généraliser le système BC aux corps de nombres quelconques et cela a été réalisé par Ha et Paugam [HP05] et Laca, Larsen, Neshveyev [LLN09] (inspiré par les travaux importants de [BC95], Laca [Lac98], Nehveyev [Nes02], Connes et Marcolli [CM06] et Connes, Marcolli et Ramachandran [CMR05], [CMR06]). Ils ont construit [3], pour chaque corps de nombres K, un C^*-système dynamique (cf. sections 2.1 et 1.5.1)

$$\mathcal{A}_K = (A_K, \sigma_t),$$

qu'on appelle système de Bost-Connes associé à K généralisant le système BC classique et vérifiant les quatre propriétés suivantes:

(i) La fonction partition de $\mathcal{A}_K$ est la fonction zêta de Dedekind $\zeta_K(\beta)$ de K.
(ii) Le groupe abélien maximal $Gal(K^{ab}/K)$ de K agit par des symétries sur $\mathcal{A}_K$.
(iii) Pour chaque température inverse $\beta \in (0, 1]$ il existe un seul état KMS_β.
(iv) Pour chaque $\beta \in (1, \infty]$ l'action du groupe des symétries $Gal(K^{ab}/K)$ sur l'ensemble des états extrémaux KMS_β est libre et transitive.

La question de savoir si les systèmes de Bost-Connes possèdent un modèle arithmétique en général, introduit dans [BC95], était un problème ouvert avant cette thèse. Le problème explicite a été formulé pour la première fois dans [CMR05]. On dit qu'un système BC $\mathcal{A}_K$ possède un modèle arithmétique s'il existe une-sous algèbre K-rationnelle

$$A_K^{arith} \subset A_K$$

3. Plus précisément Ha et Paugam ont construit les systèmes $\mathcal{A}_K$ et ont affirmé les propriétés (i) et (ii). Laca, Larsen et Neshveyev ont démontré les propriétés (iii) et (iv) en classifiant les états KMS_β de $\mathcal{A}_K$.

tel que

(v) Pour tout état KMS_∞ extremal ϱ et tout element $f \in A_K^{arith}$ on a

$$\varrho(f) \in K^{ab}$$

et K^{ab} est engendré sur K par ces valeurs.

(vi) Notons ${}^\nu\varrho$ l'action d'une symétrie $\nu \in \mathrm{Gal}(K^{ab}/K)$ sur un état KMS_∞ extremal ϱ (par image inverse), pour tout element $f \in A_K^{arith}$ on a la relation suivante

$${}^\nu\varrho(f) = \nu^{-1}(\varrho(f)).$$

(vii) L'algèbre obtenue par extension des scalaires de A_K^{arith} sur $\mathbb{C}$ est dense dans A_K.

On appelle l'algèbre A_K^{arith} une sous-algèbre arithmétique de $\mathcal{A}_K$. En particulier les modèles arithmétiques relient les systèmes BC à la théorie du corps de classes et potentiellement au douzième problème de Hilbert qui demande une théorie du corps de classes explicite. Le douzième problème de Hilbert est toujours ouvert à ce jour sauf pour le corps des rationnels, les corps imaginaires quadratiques et partiellement pour les corps CM.
Avant cette thèse l'existence des modèles arithmétiques était établie uniquement dans le cas du corps rationnel [BC95] et le cas d'un corps imaginaire quadratique (cf. [CMR05]) basé sur la théorie de multiplication complexe sur la courbe modulaire et le système GL_2 de [CM06].

Le résultat principal de cette thèse est la construction d'un modèle arithmétique pour les systèmes BC en toute généralité.

Des résultats de cette thèse

Le **premier chapitre** consiste à généraliser les travaux de Connes, Marcolli et Ramachandran [CMR05] et [CMR06] aux cas des corps CM quelconques. Les ingrédients principaux de notre construction viennent de la théorie de la multiplication complexe sur les variétés modulaires de Siegel générales [MS81] et les systèmes GSp_{2n} de Ha et Paugam [HP05]. Comme la théorie de la multiplication complexe pour un corps CM K non-quadratique n'engendre pas l'extension abélienne maximale de K on obtient seulement une sous-algèbre arithmétique partielle qui est néanmoins très intéressante. Dû au fait que notre construction utilise des fonctions modulaires arithmétiques et la loi de réciprocité de Shimura, cette sous-algèbre partielle est définie *très explicitement.*

Dans le **deuxième chapitre** on démontre, le résultat principal de cette thèse, l'existence des modèles arithmétiques en toute généralité. Marcolli avait déjà remarqué [Mar09] qu'on peut décrire le système BC classique dans le cadre des endomotifs et Λ-anneaux. On suit cette route et démontre que la théorie des endomotifs introduite par Connes, Consani et Marcolli [CCM07] et la classification de certains Λ-anneaux en termes des monoides de Deligne-Ribet obtenue par Borger et de Smit [BdS11] fournissent les bons ingrédients pour la construction des modèles arithmétiques des systèmes BC arbitraires. (Notre construction n'est pas suffisamment explicite pour pouvoir résoudre le douzième problème de Hilbert). Mais en particulier cette construction montre qu'en général les systèmes BC sont étroitement liés aux vecteurs de Witt, Λ-anneaux et relèvements de Frobenius. Dans le cas du système BC classique la relation avec les vecteurs de Witt a été exploitée très

récemment par Connes et Consani [CC11] qui ont construit des représentations p-adiques du système BC classique. Nous pensons que ces résultats fournissent des représentations p-adiques pour les systèmes BC arbitraires.
En outre le deuxième chapitre contient une annexe de Sergey Neshveyev qui a démontré que notre modèle arithmétique est (essentiellement) unique.

Le **dernier chapitre** est concerné aux propriétés fonctorielles des systèmes BC. Plus précisément dans le contexte des endomotifs on va construire un raffinement algébrique d'un foncteur de la catégorie des corps de nombres vers une catégorie appropriée des systèmes BC construit récemment par Laca, Neshveyev et Trifkovic [LNT]. Pour cela on introduit une notion de changement de base pour certains endomotifs et démontre que cela donne lieu au foncteur de [LNT].

Le premier chapitre est basé sur [Yal10] et les deux autres chapitres sur [Yal11].

Chapter 1

On BC-systems and Complex Multiplication

In this chapter we generalize the work of Connes, Marcolli and Ramachandran [CMR05] in the case of imaginary quadratic fields to general CM fields by constructing partial arithmetic models of BC-systems $\mathcal{A}_K$ when K contains a CM field.

A CM field is a imaginary quadratic extension of a totally real number field. So for example cyclotomic number fields $\mathbb{Q}(\zeta_n)$, ζ_n being a primitive n-th root of unity, are seen to be CM fields. In nature a CM field arises as (rational) endomorphism ring of an Abelian variety with Complex Multiplication.

The construction of arithmetic models in [CMR05] is based on the theory of Complex Multiplication (see section 1.3), which is a part of the arithmetic theory of Shimura Varieties. The theory of Complex Multiplication allows to construct explicitly abelian extensions of CM fields which are obtained by evaluating arithmetic Modular functions on CM-points on a Siegel upper half plane (see section 1.3.1 for more information).
Except for the case of an imaginary quadratic field, it is unfortunately not possible to generate the maximal abelian extension E^{ab} of a CM field E in this way, but still a non-trivial abelian extension of infinite degree over E. We denote the latter abelian extension of E by

$$E^c \subset E^{ab}.$$

A characterization of E^c was given by Wei [Wei94], cf., Theorem 1.6.

Due to the lack of a general knowledge of the explicit class field theory of an arbitrary number field K we content ourself with the following weakening of an arithmetic model:

Definition 1.1. *Let $F \subset K^{ab}$ be an arbitrary abelian extension of K. A* ***partial arithmetic subalgebra*** *for the extension F/K of the BC-system $\mathcal{A}_K$ is a K-rational subalgebra*

$$A_K^F \subset A_K$$

such that

(v)' For every extremal KMS_∞-state ϱ and every $f \in A_K^F$ we have

$$\varrho(f) \in F$$

and further F is generated over K by these values.

(vi) If we denote by ${}^\nu\varrho$ the action of a symmetry $\nu \in \mathrm{Gal}(K^{ab}/K)$ on an extremal KMS_∞-state ϱ (given by pull-back) we have for every element $f \in A_K^F$ the following compatibility relation

$${}^\nu\varrho(f) = \nu^{-1}(\varrho(f))$$

(1.1)

The right framework for working with Shimura varieties and C^*-dynamical systems is provided by the theory of Ha and Paugam [HP05], which allows to attach a C^*-dynamical system to an arbitrary Shimura variety. Their approach is inspired by [CMR05] and [CMR06].

The main result of this chapter will be a proof of the following theorem.

Theorem 1.1. *Let K be a number field containing a CM field. Denote by E the maximal CM field contained in K and define the abelian extension K^c of K to be the compositum*

$$K^c = K \cdot E^c. \tag{1.2}$$

Then the BC-system $\mathcal{A}_K$ admits a partial arithmetic subalgebra $A_K^{K^c}$.

Idea of our construction

Let (G, X, h) be a Shimura datum and denote by $\mathrm{Sh}(G, X, h)$ the associated Shimura variety. In [HP05] the authors attach to the datum (G, X, h), the variety $\mathrm{Sh}(G, X, h)$ and some additional data a quotient map

$$U \longrightarrow Z \tag{1.3}$$

between a topological groupoid U and a quotient $Z = \Gamma\backslash U$ for a group Γ. Out of this data they construct a C^*-dynamical system $\mathcal{A} = (A, (\sigma_t)_{t\in\mathbb{R}})$. A dense $*$-subalgebra H of A is thereby given by the compactly supported, continuous functions

$$H = C_c(Z) \tag{1.4}$$

on Z, where the groupoid structure of U induces the $*$-algebra structure on H.

Moreover there are two variations of the above quotient map, called positive and adjoint, respectively, which are related by the following (commutative) diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \uparrow & & \uparrow \\ U^+ & \longrightarrow & Z^+ \\ \downarrow & & \downarrow \\ U^{ad} & \longrightarrow & Z^{ad}. \end{array} \tag{1.5}$$

All this is explained in section 1.4. We will apply this general procedure in two special cases. For this we fix a number field K together with a maximal CM subfield E.

I) The BC-system $\mathcal{A}_K$

The (0-dimensional) Shimura datum $\mathcal{S}_K = (T^K, X_K, h_K)$ gives rise to a quotient map denoted by

$$U_K \longrightarrow Z_K.$$

In this case the groupoid U_K is of the form (cf., (1.9))

$$U_K = T^K(\mathbb{A}_f) \boxplus (\widehat{\mathcal{O}}_K \times \mathrm{Sh}(T^K, X_K, h_K)).$$

The associated C^*-dynamical system is denoted by

$$\mathcal{A}_K = (A_K, (\sigma_t)_{r\in\mathbb{R}}). \tag{1.6}$$

It gives rise to the BC-system for K. For the precise definition of $\mathcal{A}_K$ and its properties we refer the reader to section 1.2.1 and 1.5.1. Moreover we denote by H_K the dense subalgebra of A_K given by

$$H_K = C_c(Z_K). \tag{1.7}$$

II) The Shimura system $\mathcal{A}_{Sh}$

To the CM field E we attach the Shimura datum

$$\mathcal{S}_{Sh} = (\mathrm{GSp}(V_E, \psi_E), \mathbb{H}_g^{\pm}, h_{cm})$$

where in fact the construction of the morphism h_{cm} takes some time (see 1.2.2). This is due to two difficulties that arise in the case of a general CM field E which are not visible in the case of imaginary quadratic fields. On the one hand one has to use the Serre group S^E and on the other hand in general the reflex field E^* of a CM field E is not anymore equal to E (see B.3). We denote the associated quotient map by

$$U_{Sh} \longrightarrow Z_{Sh}$$

and analogously its variations (see 1.5.2). Here the relevant groupoids are of the form

$$U_{Sh} = \mathrm{GSp}(\mathbb{A}_f) \boxplus (\Gamma_{Sh,M} \times \mathrm{Sh}(\mathrm{GSp}(V_E, \psi_E), \mathbb{H}_g^{\pm}, h_{cm}))$$

and

$$U_{Sh}^{ad} = \mathrm{GSp}^{ad}(\mathbb{Q})^+ \boxplus (\Gamma_{Sh,M}^{ad} \times \mathbb{H}_g).$$

We denote the resulting C^*-dynamical system by $\mathcal{A}_{Sh}$ and call it Shimura system (cf., section 1.4.3).

Remark 1.1. In the case of an imaginary quadratic field K the Shimura system $\mathcal{A}_{Sh}$ gives rise to the GL_2-system of Connes and Marcolli [CM08].

The second system is of great importance for us because of the following: Denote by $x_{cm} \in \mathbb{H}_g$ the CM-point associated with h_{cm} and denote by $\mathcal{M}^{cm}$ the ring of arithmetic Modular functions on $\mathbb{H}_g$ defined at x_{cm}. By the theory of Complex Multiplication we know that for every $f \in \mathcal{M}^{cm}$ we have (cf., (1.2) and 1.3.3)

$$f(x_{cm}) \in K^c \subset K^{ab}$$

and moreover K^c is generated in this way. Our idea is now that $\mathcal{M}^{cm}$ gives rise to the arithmetic subalgebra $A_K^{K^c}$. More precisely we will construct a (commutative) diagram (see 1.5.3)

$$\begin{array}{ccc} U_K & \longrightarrow & Z_K \\ \downarrow & & \downarrow \\ U_{Sh} & \longrightarrow & Z_{Sh} \end{array} \tag{1.8}$$

which is induced by a morphism of Shimura data $\mathcal{S}_K \to \mathcal{S}_{Sh}$ constructed in section 1.2.3. Then, using criterion 1.10, we see that the morphism $Z_{Sh}^+ \to Z_{Sh}$ (see (1.5)) is invertible and obtain in this way a continuous map

$$\Theta : Z_K \longrightarrow Z_{Sh} \longrightarrow Z_{Sh}^+ \longrightarrow Z_{Sh}^{ad}.$$

Using easy properties of the automorphism group of $\mathcal{M}^{cm}$, we can model each f in $\mathcal{M}^{cm}$ as a function $\widetilde{f}$ on the space Z_{Sh}^{ad} (which might have singularities). Nevertheless in Proposition 1.15 we see that for every $f \in \mathcal{M}^{cm}$ the pull back $\widetilde{f} \circ \Theta$ lies in $H_K = C_c(Z_K)$ and we can define $A_K^{K^c}$ as the K-algebra generated by these elements:

$$A_K^{K^c} = < \widetilde{f} \circ \Theta \mid f \in \mathcal{M}^{cm} >_K .$$

Now, using the classification of extremal KMS_∞-states of $\mathcal{A}_K$ (see section 1.5.1), the verification of property (v)' is an immediate consequence of our construction and property (vi) follows by using Shimura's reciprocity law and the observation made in Proposition 1.8 (see section 1.7 for the details).

Outline

This chapter is organized along the lines of the preceding section, recalling on the way the necessary background. In addition we put some effort in writing a long Appendix which covers hopefully enough information to make this chapter "readable" for a person which has little beforehand knowledge of the arithmetic theory of Shimura varieties.

Notations and conventions

If A denotes a ring or monoid, we denote its group of multiplicative units by $A^\times$. A number field is a finite extension of $\mathbb{Q}$. The ring of integers of a number field K is denoted by $\mathcal{O}_K$. We denote by $\mathbb{A}_K = \mathbb{A}_{K,f} \times \mathbb{A}_{K,\infty}$ the adele ring of K (with its usual topology), where $\mathbb{A}_{K,f}$ denotes the finite adeles and $\mathbb{A}_{K,\infty}$ the infinite adeles of K. $\mathbb{A}_K$ contains K by the usual diagonal embedding and by $\widehat{\mathcal{O}}_K$ we denote the closure of $\mathcal{O}_K$ in $\mathbb{A}_{K,f}$. Invertible adeles are called ideles. The idele class group $\mathbb{A}_K^\times/K^\times$ of K is denoted by C_K, its connected component of the identity by D_K.
We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Usually we think of a number field K as lying in $\mathbb{C}$ by an embedding $\tau : K \to \overline{\mathbb{Q}} \subset \mathbb{C}$. Complex conjugation on $\mathbb{C}$ is denoted by ι.

Sometimes we write z^ι for the complex conjugate of a complex number z.
Artin's reciprocity map $\mathbb{A}_K^\times \to Gal(K^{ab}/K) : \nu \mapsto [\nu]$ is normalized such that an uniformizing parameter maps to the arithmetic Frobenius element. Further given a group G acting partially on a set X we denote by

$$G \boxplus X = \{(g,x) \in G \times X \mid gx \in X\} \tag{1.9}$$

the corresponding groupoid (see p. 327 [LLN09]).
If X denotes a topological space we write $\pi_0(X)$ for its set of connected components.

1.1 On the arithmetic subalgebra for $\mathbb{Q}(i)$

Before we describe our general construction we will explain the easiest case $K = \mathbb{Q}(i)$, where many simplifications occur, in some detail and point out the modifications necessary for the general case. For the reminder of this section K always denotes $\mathbb{Q}(i)$, although many of the definitions work in general. For the convenience of the reader we will try to make the following section as self-contained as possible.

1.1.1 The quotient map $U_K \to Z_K$

We denote by T^K the $\mathbb{Q}$-algebraic torus given by the Weil restriction $T^K = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$ of the multiplicative group $\mathbb{G}_{m,K}$, i.e., for a $\mathbb{Q}$-algebra R the R-points of T^K are given by $T^K(R) = (R \otimes_{\mathbb{Q}} K)^\times$. In particular we see that $T^K(\mathbb{Q}) = K^\times$, $T^K(\mathbb{A}_f) = \mathbb{A}_{K,f}^\times$ and $T^K(\mathbb{R}) = \mathbb{A}_{K,\infty}^\times$. In our special case we obtain that, after extending scalars to $\mathbb{R}$, the $\mathbb{R}$-algebraic group $T_{\mathbb{R}}^K$ is isomorphic to $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{R}})$. Further the finite set $X_K = T^K(\mathbb{R})/T^K(\mathbb{R})^+ = \pi_0(T^K(\mathbb{R}))$ consists in our case of only one point. With this in mind we consider the 0-dimensional Shimura datum (see D.5)

$$\mathcal{S}_K = (T^K, X_K, h_K) \tag{1.10}$$

where the morphism $h_K : \mathbb{S} \to T_{\mathbb{R}}^K$ is simply given by the identity (thanks to $T_{\mathbb{R}}^K \cong \mathbb{S}$). (In the general case h_K is chosen accordingly to Lemma 1.2.)
The (0-dimensional) Shimura variety $\mathrm{Sh}(\mathcal{S}_K)$ is in our case of the simple form

$$\mathrm{Sh}(\mathcal{S}_K) = T^K(\mathbb{Q})\backslash(X_K \times T^K(\mathbb{A}_f)) = K^\times\backslash\mathbb{A}_{K,f}^\times. \tag{1.11}$$

We write $[z,l]$ for an element in $\mathrm{Sh}(\mathcal{S}_K)$ meaning that $z \in X_K$ and $l \in T^K(\mathbb{A}_f)$.
(For general number fields the description of $\mathrm{Sh}(\mathcal{S}_K)$ is less explicit but no difficulty occurs.)

Remark 1.2. The reader should notice that by class field theory we can identify $\mathrm{Sh}(\mathcal{S}_K) = K^\times\backslash\mathbb{A}_{K,f}^\times$ with the Galois group $Gal(K^{ab}/K)$ of the maximal abelian extension K^{ab} of K. This is true in general, see section 1.5.1.

The (topological) groupoid U_K underlying the BC-system $\mathcal{A}_K = (A_K, (\sigma_t)_{t\in\mathbb{R}})$ is now of the form (see (1.9) for the notation)

$$U_K = T^K(\mathbb{A}_f) \boxplus (\widehat{\mathcal{O}}_K \times \mathrm{Sh}(\mathcal{S}_K)) \tag{1.12}$$

with the natural action of $T^K(\mathbb{A}_f)$ on $\mathrm{Sh}(\mathcal{S}_K)$ (see D.2) and the partial action of $T^K(\mathbb{A}_f) = \mathbb{A}_{K,f}^\times$ on the multiplicative semigroup $\widehat{\mathcal{O}}_K \subset \mathbb{A}_{K,f}$ by multiplication. The group

$$\Gamma_K^2 = \widehat{\mathcal{O}}_K^\times \times \widehat{\mathcal{O}}_K^\times \tag{1.13}$$

is acting on U_K as follows

$$(\gamma_1, \gamma_2)(g, \rho, [z, l]) = (\gamma_1^{-1} g \gamma_2, \gamma_2 \rho, [z, l\gamma_2^{-1}]), \tag{1.14}$$

where $\gamma_1, \gamma_2 \in \widehat{\mathcal{O}}_K^{\times}$, $g, l \in T^K(\mathbb{A}_f)$, $\rho \in \widehat{\mathcal{O}}_K$ and $z \in X_K$, and we obtain the quotient map

$$U_K \longrightarrow Z_K = \Gamma_K^2 \backslash U_K. \tag{1.15}$$

In the end of this section we will construct the arithmetic subalgebra A_K^{Kc} of the BC system $\mathcal{A}_K$ which is contained in $H_K = C_c(Z_K) \subset A_K$. For this we will need

1.1.2 The quotient map $U_{Sh} \longrightarrow Z_{Sh}$

In our case of $K = \mathbb{Q}(i)$ the maximal CM subfield E of K is equal to K. The Shimura datum $\mathcal{S}_{Sh}$ associated with E is of the form (see 1.2.2)

$$\mathcal{S}_{Sh} = (\mathrm{GSp}(V_E, \psi_E), \mathbb{H}^{\pm}, h_{cm}). \tag{1.16}$$

Here GSp = GSp(V_E, ψ_E) is the general symplectic group (cf., D.3) associated with the symplectic vector space (V_E, ψ_E).
The latter is in general chosen accordingly to (1.51). Due to the fact that the reflex field E^* (cf., B.3) is equal to E and the Serre group S^E is equal to $T^E = T^K$ we can simply choose the $\mathbb{Q}$-vector space V_E to be the $\mathbb{Q}$-vector space E and the symplectic form $\psi_E : E \times E \to \mathbb{Q}$ to be the map $(x, y) \mapsto Tr_{E/\mathbb{Q}}(ixy^{\iota})$. A simple calculation shows that $\psi_E(f(x), f(y)) = det(f)\psi_E(x, y)$, for all $f \in End_{\mathbb{Q}}(V_E)$ and all $x, y \in V_E$, therefore we can identify GSp with $GL_2 = GL(V_E)$. Now again using the fact that the Serre group S^E equals T^E, we see that the general construction of $h_{cm} : \mathbb{S} = T^E_{\mathbb{R}} \to \mathrm{GSp}_{\mathbb{R}} = GL_{2,\mathbb{R}}$ (see (1.55)) is given on the $\mathbb{R}$-points by $a + ib \in \mathbb{C}^{\times} = \mathbb{S}(\mathbb{R}) \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in GL_2(\mathbb{R})$. Each $\alpha \in \mathrm{GSp}(\mathbb{R})$ defines a map $\alpha^{-1} h_{cm} \alpha : \mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$ given on the $\mathbb{R}$-points by $a+ib \in \mathbb{C}^{\times} \mapsto \alpha^{-1} h_{cm}(a+ib)\alpha \in GL_2(\mathbb{R})$ and the GSp$(\mathbb{R})$-conjugacy class $X = \{\alpha^{-1} h_{cm} \alpha \mid \alpha \in \mathrm{GSp}_{\mathbb{R}}(\mathbb{R})\}$ of h_{cm} can be identified with the Siegel upper lower half space $\mathbb{H}^{\pm} = \mathbb{C} - \mathbb{R}$ by the map

$$\alpha^{-1} h_{cm} \alpha \in X \mapsto (\alpha^{-1} h_{cm}(i) \alpha) \cdot i \in \mathbb{H}^{\pm},$$

where the latter action $\cdot$ denotes Moebius transformation. Under this identification, the morphism h_{cm} corresponds to the point $x_{cm} = i$ on the upper half plane $\mathbb{H}$. The point $x_{cm} \in \mathbb{H}$ is a so-called CM-point (see D.6).

Remark 1.3. The definition of a CM-point and the observation made in (1.54) explain the need of using the Serre group in the general construction of h_{cm}. The explanation given in 1.3.1 shows in particular why we have to define the vector space V_E in general accordingly to (1.51).

The Shimura variety Sh$(\mathcal{S}_{Sh})$ is of the nice form (cf., (D.1))

$$\mathrm{Sh}(\mathcal{S}_{Sh}) = \mathrm{GSp}(\mathbb{Q}) \backslash (\mathbb{H}^{\pm} \times \mathrm{GSp}(\mathbb{A}_f)).$$

Again we write elements as $[z, l] \in \mathrm{Sh}(\mathcal{S}_{Sh})$ with $z \in \mathbb{H}^{\pm}$ and $g \in \mathrm{GSp}(\mathbb{A}_f)$.
In our case the topological groupoid U_{Sh} underlying the Shimura system $\mathcal{A}_{Sh}$ is given by (cf., 1.5.2)

$$U_{Sh} = \mathrm{GSp}(\mathbb{A}_f) \boxplus (M_2(\widehat{\mathbb{Z}}) \times \mathrm{Sh}(\mathcal{S}_{Sh})), \tag{1.17}$$

where $\mathrm{GSp}(\mathbb{A}_f) = GL_2(\mathbb{A}_f)$ is acting in the natural way on $\mathrm{Sh}(\mathcal{S}_{Sh})$ and partially on the multiplicative monoid of 2×2-matrices $M_2(\widehat{\mathbb{Z}}) \subset M_2(\mathbb{A}_f)$ with entries in $\widehat{\mathbb{Z}}$. The group

$$\Gamma_{Sh}^2 = GL_2(\widehat{\mathbb{Z}}) \times GL_2(\widehat{\mathbb{Z}}) \tag{1.18}$$

is acting on U_{Sh} exactly like in (1.14) and induces the quotient map

$$U_{Sh} \longrightarrow Z_{Sh} = \Gamma_{Sh}^2 \backslash U_{Sh}. \tag{1.19}$$

Remark 1.4. Note that the quotient Z_{Sh} is not a groupoid anymore (see top of p. 251 [HP05]).

In our example it is sufficient to consider the positive groupoid U_{Sh}^+ (see 1.5.2) associated with U_{Sh}. (In the general case the adjoint groupoid U_{Sh}^{ad} seems to be more appropriate.) It is given by

$$U_{Sh}^+ = \mathrm{GSp}(\mathbb{Q})^+ \boxplus (M_2(\widehat{\mathbb{Z}}) \times \mathbb{H}) \tag{1.20}$$

together with the group

$$(\Gamma_{Sh}^+)^2 = GL_2(\mathbb{Z})^+ \times GL_2(\mathbb{Z})^+ = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \tag{1.21}$$

acting by

$$(\gamma_1, \gamma_2)(g, \rho, z) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z) \tag{1.22}$$

and inducing the quotient map

$$U_{Sh}^+ \longrightarrow Z_{Sh}^+ = (\Gamma_{Sh}^+)^2 \backslash U_{Sh}^+. \tag{1.23}$$

By construction $\mathrm{GSp}(\mathbb{R})$ is acting (free and transitively) on $\mathbb{H}^{\pm}$ and $\mathrm{GSp}(\mathbb{R})^+$, the connected component of the identity, can be thought of as stabilizer of the upper half plane $\mathbb{H}^+ = \mathbb{H}$ which explains the action of $\mathrm{GSp}(\mathbb{Q})^+ = \mathrm{GSp}(\mathbb{Q}) \cap \mathrm{GSp}(\mathbb{R})^+$ on $\mathbb{H}$. Thanks to criterion 1.10 we know that the natural (equivariant) morphism of topological groupoids $U_{Sh}^+ \to U_{Sh}$ given by $(g, \rho, z) \mapsto (g, \rho, [z, 1])$ induces a homeomorphism on the quotient spaces $Z_{Sh}^+ \longrightarrow Z_{Sh}$ in the commutative diagram

$$\begin{array}{ccc} U_{Sh} & \longrightarrow & Z_{Sh} \\ \uparrow & & \uparrow \cong \\ U_{Sh}^+ & \longrightarrow & Z_{Sh}^+ . \end{array} \tag{1.24}$$

Remark 1.5. The groupoid U_{Sh}^+ corresponds to the GL_2-system of Connes and Marcolli (see [CM08] and 5.8 [HP05])

1.1.3 A map relating U_K and U_{Sh}

We want to define an equivariant morphism of topological groupoids $U_K \longrightarrow U_{Sh}$, where equivariance is meant with respect to the actions of Γ_K on U_K and Γ_{Sh} on U_{Sh}. For this it is necessary (and more or less sufficient) to construct a morphism of Shimura data between $\mathcal{S}_K = (T^K, X_K, h_K)$ and $\mathcal{S}_{Sh} = (\mathrm{GSp}(V_E, \psi_E)), \mathbb{H}^{\pm}, h_{cm})$, which is given by a morphism of algebraic groups

$$\varphi : T^K \longrightarrow \mathrm{GSp} \tag{1.25}$$

such that $h_{cm} = \varphi_{\mathbb{R}} \circ h_K$. In our case the general construction of φ, stated in (1.56), reduces to the simple map (on the $\mathbb{Q}$-points)

$$a + ib \in K^\times = T^K(\mathbb{Q}) \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in GL_2(\mathbb{Q}) = \mathrm{GSp}(\mathbb{Q})$$

and we see in fact that after extending scalars to $\mathbb{R}$ the morphism $\varphi_{\mathbb{R}} : T^K_{\mathbb{R}} = \mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$ is already equal to $h_{cm} : \mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$. The simplicity of our example comes again from the fact that we don't have to bother about the Serre group, which makes things less explicit, although the map $\varphi : T^K \to \mathrm{GSp}$ still has a quite explicit description even in the general case thanks to Lemma 1.4.

Now by functoriality (see section D.5) we obtain a morphism of Shimura varieties

$$\mathrm{Sh}(\varphi) : \mathrm{Sh}(\mathcal{S}_K) \to \mathrm{Sh}(\mathcal{S}_{Sh})$$

which can be explicitly described by

$$[z, l] \in K^\times \backslash (X_K \times T^K(\mathbb{A}_f)) \mapsto [x_{cm}, \varphi(\mathbb{A}_f)(l)] \in \mathrm{GSp}(\mathbb{Q}) \backslash (\mathbb{H}^\pm \times \mathrm{GSp}(\mathbb{A}_f)).$$

In the general case we have essentially the same description (see (1.101)), the point being that every element z in X_K is mapped to $x_{cm} \in \mathbb{H}^\pm$, as in the general case. Using $\widehat{\mathcal{O}}_K = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_K$ we can continue the map $\varphi(\mathbb{A}_f)$ to a morphism of (topological) semigroups $M(\varphi)(\mathbb{A}_f) : \widehat{\mathcal{O}}_K \to M_2(\widehat{\mathbb{Z}})$ by setting $n \otimes (a + ib) \mapsto \begin{pmatrix} an & -bn \\ bn & an \end{pmatrix}$. By continuation we mean that $\varphi(\mathbb{A}_f)$ and $M(\varphi)(\mathbb{A}_f)$ agree on the intersection of $T^K(\mathbb{A}_f) \cap \widehat{\mathcal{O}}_K \subset \mathbb{A}_{K,f}$. In the general case the explicit description of φ given in (1.58) is used to continue φ to $M(\varphi)$ (see 1.5.3), the above example being a special case. Now it can easily be checked that

$$(g, \rho, [z, l]) \in U_K \mapsto (\varphi(\mathbb{A}_f)(g), M(\varphi)(\mathbb{A}_f)(\rho), [x_{cm}, \varphi(\mathbb{A}_f)(l)]) \in U_{Sh} \tag{1.26}$$

defines the desired equivariant morphism of topological groupoids

$$U_K \longrightarrow U_{Sh}. \tag{1.27}$$

Summarizing we obtain the following commutative diagram (using (1.24))

$$\begin{array}{ccc} U_K & \longrightarrow & Z_K \\ \downarrow & & \downarrow \\ U_{Sh} & \longrightarrow & Z_{Sh} \\ \uparrow & & \uparrow \cong \\ U^+_{Sh} & \longrightarrow & Z^+_{Sh} \end{array} \tag{1.28}$$

which gives us the desired morphism of topological spaces

$$\Theta : Z_K \longrightarrow Z_{Sh} \longrightarrow Z^+_{Sh}. \tag{1.29}$$

In general, we have to go one step further and use the adjoint groupoid Z^{ad}_{Sh} (cf., 1.5.2) but this only due to the general description of the automorphism group $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{M})$ of the field of arithmetic automorphic functions $\mathcal{M}$, see 1.3.3 and the next section for explanations.

1.1.4 Interlude: Theory of Complex Multiplication

In this section we will provide the number theoretic background which is necessary to understand the constructions done so far.
We are interested in constructing the maximal abelian extension E^{ab} of $E = K = \mathbb{Q}(i)$, and there are in general two known approaches to this problem.

I) The elliptic curve $A : y^2 = x^3 + x$

Let us denote by A the elliptic curve defined by the equation

$$A : y^2 = x^3 + x. \tag{1.30}$$

It is known that the field of definition of A and its torsion points generate the maximal abelian extension E^{ab} of E. (By field of definition of the torsion points of A, we mean the coordinates of the torsion points.)
Notice that in our case, the complex points of our elliptic curve A are given by $A(\mathbb{C}) = \mathbb{C}/\mathcal{O}_E$ and the rational ring of endomorphisms of A turns out to be $\mathrm{End}(A(\mathbb{C}))_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_E = E$. We say that A has complex multiplication by E. Remember that $\mathcal{O}_E = \mathbb{Z}[i]$.

Remark 1.6. To obtain the abelian extensions of K provided by A explicitly, one may use for example the Weierstrass $\mathfrak{p}$-function associated with A (see [Sil94]), but as we will use another approach, we don't want to dive into this beautiful part of explicit class field theory.

II) The (Siegel) Modular curve $\mathrm{Sh}(\mathrm{GSp}, \mathbb{H}^{\pm}, h_{cm})$

We want to interpret the Shimura variety $\mathrm{Sh}(\mathcal{S}_{Sh}) = \mathrm{Sh}(\mathrm{GSp}, \mathbb{H}^{\pm}, h_{cm})$ constructed in the last section as moduli space of elliptic curves with torsion data.

The moduli theoretic picture

For this, we consider the connected component $Sh^o = \mathrm{Sh}(\mathcal{S}_{Sh})^o$ of our Shimura variety which is described by the projective system (see D.4 and [Mil04])

$$Sh^o = \varprojlim_{N} \Gamma(N)\backslash\mathbb{H}, \tag{1.31}$$

where $\Gamma(N)$, for $N \geq 1$, denotes the subgroup of $\Gamma = \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ defined by $\Gamma(N) = \{g \in \Gamma \mid g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\}$. We can view the quotient $\Gamma(N)\backslash\mathbb{H}$ as a complex analytic space, but thanks to the work of Baily and Borel, it carries also a unique structure of an algebraic variety over $\mathbb{C}$ (see [Mil04]). We will use both viewpoints. Seen as an analytic space we write $\mathbb{H}(N) = \Gamma(N)\backslash\mathbb{H}$ and for the algebraic space we write $Sh^o_N = \Gamma(N)\backslash\mathbb{H}$.

Observe now that the space $\mathbb{H}(N)$ classifies isomorphism classes of pairs (A, t) given by an elliptic curve A over $\mathbb{C}$ together with a N-torsion point t of A. In particular $\mathbb{H}(1) = \Gamma\backslash\mathbb{H}$ classifies isomorphism classes of elliptic curves over $\mathbb{C}$.

In this picture, our CM-point $[x_{cm}]_1 = [i]_1 \in \mathbb{H}(1)$ corresponds to the isomorphism class of the elliptic curve A from (1.30). More generally, the points $[x_{cm}]_N \in \mathbb{H}(N)$ capture the field of definition of A and its various torsion points. In this way, they recover the maximal abelian extension E^{ab} of E! By $[z]_N$ we denote the image of $z \in \mathbb{H}$ in $\mathbb{H}(N)$ under the natural quotient map $\mathbb{H} \to \mathbb{H}(N)$.

Remark 1.7. For the relation between Sh and Sh^o we refer the reader to pp. 51 [Mil04].

The field of arithmetic Modular functions $\mathcal{M}$

To construct explicitly the abelian extensions provided by the various points $[x_{cm}]_N$ we proceed as follows: we consider the connected canonical model M^o of Sh^o (see D.7), which provides us with an algebraic model $M_N^o = \Gamma(N)\backslash M^o$ of the algebraic variety Sh_N^o over the cyclotomic field $\mathbb{Q}(\zeta_N)$. In general, we obtain algebraic models over subfields of $\mathbb{Q}^{ab}$. This means M_N^o is an algebraic variety defined over the cyclotomic field $\mathbb{Q}(\zeta_N)$, and after scalar extension to $\mathbb{C}$, it becomes isomorphic to the complex algebraic variety Sh_N^o. Let us denote by $k(M_N^o)$ the field of rational functions on M_N^o, in particular, this means elements in $k(M_N^o)$ are rational over $\mathbb{Q}(\zeta_N)$. It makes sense to view the point $[x_{cm}]_N$ as a point on M_N^o, and if a function $f \in k(M_N^o)$ is defined at $[x_{cm}]_N$, then we know (cf., 1.3.3) that

$$f([x_{cm}]_N) \in E^{ab}. \tag{1.32}$$

In particular varying over the various N and the rational functions in $k(M_N^o)$, the values $f([x_{cm}]_N)$ generate E^{ab} over E. The next step is to realize that the function field $k(M_N^o)$ can be seen as a subset of the field of rational functions $k(Sh_N^o)$ on the complex algebraic variety Sh_N^o (cf., 1.3.3). As rational functions in $k(Sh_N^o)$ correspond to meromorphic functions on $\mathbb{H}(N)$, and meromorphic functions on $\mathbb{H}(N)$ are nothing else than meromorphic functions on $\mathbb{H}$ that are invariant under the action of $\Gamma(N)$, we can view each rational function in $k(M_N^o)$ as a meromorphic function on $\mathbb{H}$ which is invariant under $\Gamma(N)$. If we denote by $k(M_N^o)_{cusp}$ the subfield of $k(M_N^o)$ consisting of functions $f \in k(M_N^o)$ that give rise to meromorphic functions on $\mathbb{H}$ that are meromorphic at the cusps (see 1.3.3), it makes sense to define the field of meromorphic functions $\mathcal{M}$ on $\mathbb{H}$ given by the union

$$\mathcal{M} = \bigcup_N k(M_N^o)_{cusp}. \tag{1.33}$$

Due to (1.32), we know furthermore that for every $f \in \mathcal{M}$, which is defined in x_{cm}, we have

$$f(x_{cm}) \in K^{ab} = E^{ab} \tag{1.34}$$

and K^{ab} is generated in this way. Therefore we call the field $\mathcal{M}$ the **field of arithmetic Modular functions**.
Explicitly $\mathcal{M}$ is described for example in [Shi00] or [CM08] (Def. 3.60). A very famous arithmetic Modular function is given by the j-function which generates the Hilbert class field of an arbitrary imaginary quadratic field.

Remark 1.8. 1) In light of 1.3.1 and 1.3 2), we mention that the field of definition $E(x_{cm})$ of x_{cm} is, in our example, equal to $E = \mathbb{Q}(i)$, which is the reason why our example is especially simple.
2) If we take a generic meromorphic function g on $\mathbb{H}(N)$ that is defined in $[x_{cm}]_N$, then the value $g([x_{cm}]_N) \in \mathbb{C}$ will not even be algebraic. This is the reason why we need the canonical model M^o which provides an arithmetic structure for the field of meromorphic functions on $\mathbb{H}(N)$.

In the general case, the construction of $\mathcal{M}$ is quite similar to the construction above (see section 1.3), the only main difference being that in general the theory is much less explicit (e.g. the description of $\mathcal{M}$).

Automorphisms of $\mathcal{M}$ and Shimura's reciprocity law

In our example $K = E = \mathbb{Q}(i)$, using the notation from 1.3.3, we have the equality $\overline{\mathcal{E}} = \frac{\mathrm{GSp}(\mathbb{A}_f)}{\mathbb{Q}^\times}$, and obtain a group homomorphism

$$\mathrm{GSp}(\mathbb{A}_f) \longrightarrow \overline{\mathcal{E}} \longrightarrow \mathrm{Aut}_{\mathbb{Q}}(\mathcal{M}), \tag{1.35}$$

where the first arrow is simply the projection, and the second arrow comes from 1.3.3. We denote the action of $\alpha \in \mathrm{GSp}(\mathbb{A}_f)$ on a function $f \in \mathcal{M}$ by ${}^{\alpha}f$. In particular we see that $\alpha \in \mathrm{GSp}(\mathbb{Q})^+ = \mathrm{SL}_2(\mathbb{Q})$ is acting by

$${}^{\alpha}f = f \circ \alpha^{-1}, \tag{1.36}$$

where α^{-1} acts on $\mathbb{H}$ by Mobius transformation. The adjoint system, which is well suited for the general case (cf., 1.5.2), is not needed in our special example. We note that the group $\mathrm{GSp}^{ad}(\mathbb{Q})^+$ occuring in (1.5.2) is given by $\mathrm{SL}_2(\mathbb{Q})/\{\pm I\}$, where I denotes the unit in $\mathrm{SL}_2(\mathbb{Q})$. Due to the fact that $\{\pm I\}$ acts trivially on $\mathbb{H}$, we can lift the action to $\mathrm{GSp}(\mathbb{Q})^+$. Further, there is a morphism of algebraic groups

$$\eta : T^K \to \mathrm{GSp} \tag{1.37}$$

which induces a group homomorphism denoted by (compare (1.71))

$$\overline{\eta} = \eta(\mathbb{A}_f) : T^K(\mathbb{A}_f) \to \mathrm{GSp}(\mathbb{A}_f). \tag{1.38}$$

The **reciprocity law of Shimura** can be stated in our special case as follows:
Let ν be in $\mathbb{A}_{K,f}^\times$, denote by $[\nu] \in Gal(K^{ab}/K)$ its image under Artin's reciprocity map and let $f \in \mathcal{M}$ be defined in $x_{cm} \in \mathbb{H}$. Then ${}^{\overline{\eta}(\nu)}f$ is also defined in $x_{cm} \in \mathbb{H}$ and

$${}^{\overline{\eta}(\nu)}f(x_{cm}) = [\nu]^{-1} f(x_{cm}) \in K^{ab}. \tag{1.39}$$

The formulation in the general case concentrates on the group $\overline{\mathcal{E}}$ (see 1.3.4).

1.1.5 Construction of the arithmetic subalgebra

Define $\mathcal{M}^{cm}$ to be the subring of functions in $\mathcal{M}$ which are defined at $x_{cm} \in \mathbb{H}$. For every $f \in \mathcal{M}^{cm}$, we define a function $\widetilde{f}$ on the groupoid U_{Sh}^+ by

$$\widetilde{f}(g, \rho, z) = \begin{cases} {}^{\rho}f(z) & \rho \in \mathrm{GSp}(\widehat{\mathbb{Z}}) \\ 0 & \rho \in M_2(\widehat{\mathbb{Z}}) - \mathrm{GSp}(\widehat{\mathbb{Z}}) \end{cases} \tag{1.40}$$

Thanks to (1.36), $\widetilde{f}$ is invariant under the action of $\Gamma_{Sh}^+ = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, and therefore $\widetilde{f}$ descends to the quotient Z_{Sh}^+ (cf., (1.23)). Proposition 1.15 shows further that the pull-back $\widetilde{f} \circ \Theta$ (see (1.29)) defines a compactly supported, continuous function on Z_K, i.e., we have $\widetilde{f} \circ \Theta \in H_K = C_c(Z_K) \subset A_K$. Therefore we can define a K-subalgebra A_K^{arith} of H_K by

$$A_K^{arith} = < \widetilde{f} \circ \Theta \mid f \in \mathcal{M}^{cm} >_K . \tag{1.41}$$

Now we want to show that A_K^{arith} is indeed an arithmetic subalgebra for $\mathcal{A}_K$ (see (0.3)). The set $\mathcal{E}_\infty$ of extremal KMS_∞-states of $\mathcal{A}_K$ is indexed by the set $\mathrm{Sh}(\mathcal{S}_K)$ and for every

$\omega \in \mathrm{Sh}(\mathcal{S}_K)$ the corresponding KMS_∞-state ϱ_ω is given on an element $f \in H_K = C_c(Z_K)$ by (cf., 1.5.1)

$$\varrho_\omega(f) = f(1, 1, \omega). \tag{1.42}$$

Using remark 1.2 we write $[\omega]$ for an element $\omega \in \mathrm{Sh}(\mathcal{S}_K)$ when regarded as element in $Gal(K^{ab}/K)$. Now, if we take a function $f \in \mathcal{M}^{cm}$ and $\omega \in \mathrm{Sh}(\mathcal{S}_K)$, we immediately see that

$$\varrho_\omega(\widetilde{f} \circ \Theta) = [\omega]^{-1}(f(x_{cm})) \in K^{ab} \tag{1.43}$$

and property (v) follows (in general we will only show property (v') of course). To show property (vi) we take a symmetry $\nu \in C_K = \mathbb{A}_K^\times / K^\times$ (see 1.5.1) of $\mathcal{A}_K$ and denote by $[\nu] \in Gal(K^{ab}/K)$ its image under Artin's reciprocity homomorphism and let f and ω be as above. We denote the action (pull-back) of ν on ϱ_ω by ${}^\nu\varrho_\omega$ and obtain

$${}^\nu\varrho_\omega(\widetilde{f} \circ \Theta) = {}^{\varphi(\mathbb{A}_f)(\nu)}f(x_{cm}). \tag{1.44}$$

But thanks to Proposition 1.8 we know that

$$\varphi(\mathbb{A}_f)(\nu) = \overline{\eta}(\nu) \in \mathrm{GSp}(\mathbb{A}_f) \tag{1.45}$$

and now thanks to Shimura's (1.39) we can conclude that

$${}^\nu\varrho_\omega(\widetilde{f} \circ \Theta) = {}^{\overline{\eta}(\nu)}f(x_{cm}) = [\nu]^{-1}(f(x_{cm})) = [\nu]^{-1}(\varrho_\omega(\widetilde{f} \circ \Theta)) \in K^{ab}, \tag{1.46}$$

which proves property (vi). For the general case and more details we refer the reader to 1.7.

Remark 1.9. Our arithmetic subalgebra $\mathcal{A}_K$ in the case of $K = \mathbb{Q}(i)$ is essentially the same as in [CMR05].

Remark 1.10. In a fancy (and very sketchy) way, we might say that the two different pictures, one concentrating on the single elliptic curve A the other on the moduli space of elliptic curves (see the beginning of section 1.1.4), are related via the Langlands correspondence. In terms of Langlands correspondence, the single elliptic curve A lives on the motivic side whereas the moduli space of elliptic curves lives (partly) on the automorphic side. As we used the second picture for our construction of an arithmetic subalgebra, we might say that our construction is automorphic in nature. This explains the fact that we have a "natural" action of the idele class group on our arithmetic subalgebra (see above). Using the recent theory of endomotives (see chapter 4 and in particular p. 551 [CM08]) one can recover the arithmetic subalgebra A_K^{arith} by only using the single elliptic curve A. In particular one obtains a natural action of the Galois group. This and more will be elaborated in chapter 2 and elsewhere.

We now concentrate on the general case.

1.2 Two Shimura data and a map

As throughout this chapter, let K denote a number field and E its maximal CM subfield. We fix an embedding $\tau : K \to \overline{\mathbb{Q}} \to \mathbb{C}$ and denote complex conjugation on $\mathbb{C}$ by ι.
To K, resp. E, we will attach a Shimura datum $\mathcal{S}_K$, resp. $\mathcal{S}_{Sh}$, and show how to construct a morphism $\varphi : \mathcal{S}_K \to \mathcal{S}_{Sh}$ between them. We will freely use the Appendix: every object not defined in the following can be found there or in the references given therein.
Recall that the Serre group attached to K is denoted by S^K (cf., C.1), it is a quotient of the algebraic torus T^K (defined below), the corresponding quotient map is denoted by $\pi^K : T^K \to S^K$.

1.2.1 Protagonist I: $\mathcal{S}_K$

The 0-dimensional Shimura datum $\mathcal{S}_K = (T^K, X_K, h_K)$ (see section D.5) is given by the Weil restriction $T^K = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$, the discrete and finite set $X_K = T^K(\mathbb{R})/T^K(\mathbb{R})^+ \cong \pi_0(\mathbb{A}^\times_{K,\infty})$ and a morphism $h_K : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to T^K_\mathbb{R}$ which is chosen accordingly to the next lemma,

Lemma 1.2. *There is a morphism of algebraic groups $h_K : \mathbb{S} \to T^K_\mathbb{R}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_K} & T^K_\mathbb{R} \\ & {\scriptstyle h^K}\searrow & \downarrow{\scriptstyle \pi^K_\mathbb{R}} \\ & & S^K_\mathbb{R} \end{array} \tag{1.47}$$

commutes.

Proof. Remember that $h^K : \mathbb{S} \to S^K_\mathbb{R}$ is defined as the composition

$$\mathbb{S} \xrightarrow{\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mu^K)} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(S^K_\mathbb{C}) \xrightarrow{\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}} S^K_\mathbb{R}, \tag{1.48}$$

where $\mu^K : \mathbb{G}_{m,\mathbb{C}} \to T^K_\mathbb{C}$ is defined by $\mu^K = \pi^K_\mathbb{C} \circ \mu_\tau$ (cf., C.2). Define $h_K : \mathbb{S} \to T^E_\mathbb{R}$ simply by

$$\mathbb{S} \xrightarrow{\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mu_\tau)} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(T^K_\mathbb{C}) \xrightarrow{\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}} T^K_\mathbb{R}. \tag{1.49}$$

For proving our claim it is enough to show that the following diagram

$$\begin{array}{ccccc} \mathbb{S} & \xrightarrow{\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mu_\tau)} & \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(T^K_\mathbb{C}) & \xrightarrow{\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}} & T^K_\mathbb{R} \\ & {\scriptstyle \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mu^K)}\searrow & \downarrow{\scriptstyle \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\pi^K_\mathbb{C})} & & \downarrow{\scriptstyle \pi^K_\mathbb{R}} \\ & & \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(S^K_\mathbb{C}) & \xrightarrow{\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}} & S^K_\mathbb{R} \end{array} \tag{1.50}$$

is everywhere commutative.
The triangle on the left is commutative, because $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ is a functor. Thanks to Theorem A.1 it is enough to show that rectangle on the right is commutative after applying the functor X^* (cf., A.3). Since $\pi^K : T^K \to S^K$ is defined to be the inclusion $X^*(S^K) \subset X^*(T^K)$ on the level of characters (cf., C.1), we see that $X^*(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\pi^K_\mathbb{C}))$ and $X^*(\pi^K_\mathbb{R})$ are inclusions as well, and the commutativity follows.

□

1.2.2 Protagonist II: $\mathcal{S}_{Sh}$

The construction of the Shimura datum $\mathcal{S}_{Sh}$ in this section goes back to Shimura [Shi00], see also [Wei94]. It is of the form $\mathcal{S}_{Sh} = (\mathrm{GSp}(V_E, \psi_E), \mathbb{H}^\pm_g, h_{cm})$. The symplectic $\mathbb{Q}$-vector space (V_E, ψ_E) is defined as follows.
Choose a finite collection of primitive CM types (E_i, Φ_i), $1 \le i \le r$, such that
i) for all i the reflex field E^*_i is contained in E, i.e. $\forall i\ E^*_i \subset E$, and
ii) the natural map (take (B.2) and apply the universal property from C.1)

$$S^E \xrightarrow{\prod N_{E/E^*_i}} \prod_{i=1}^r S^{E^*_i} \xrightarrow{\prod \rho_{\Phi_i}} \prod_{i=1}^r T^{E_i} \tag{1.51}$$

is injective. (Proposition 1.5.1 [Wei94] shows that this is allways possible).
For every $i \in \{1,..,r\}$, we define a symplectic form $\psi_i : E_i \times E_i \to \mathbb{Q}$ on E_i by choosing a totally imaginary generator ξ_i of E_i (over $\mathbb{Q}$) and setting

$$\psi_i(x,y) = Tr_{E_i/\mathbb{Q}}(\xi_i x y^\iota). \tag{1.52}$$

Now we define (V_E, ψ_E) as the direct sum of the symplectic spaces (E_i, ψ_i). Instead of $\mathrm{GSp}(V_E, \psi_E)$ we will sometimes simply write GSp.

To define the morphism h_{cm} the essential step is to observe (see Remark 9.2 [Mil98]) that the image of the map $\rho_{\Phi_i} \circ N_{E/E_i^*} : S^E \to T^{E_i}$ is contained in the subtorus $\mathcal{T}^{E_i}$ of T^{E_i}, which is defined on the level of $\mathbb{Q}$-points by

$$\mathcal{T}^{E_i}(\mathbb{Q}) = \{x \in E_i^\times \mid x x^\iota \in \mathbb{Q}^\times\} \tag{1.53}$$

and analogously, $T^{E_i}(R)$ is defined for an arbitrary $\mathbb{Q}$-algebra R. This is an important observation, because there is an obvious inclusion of algebraic groups (cf., A.2)

$$\mathfrak{i} : \prod_{i=1}^r \mathcal{T}^{E_i} \to \mathrm{GSp}(V_E, \psi_E), \tag{1.54}$$

whereas there is in general **no** embedding $\prod T^{E_i} \to \mathrm{GSp}$.
With this in mind, we define h_{cm} as the composition

$$\mathbb{S} \xrightarrow{h^E} S^E_{\mathbb{R}} \xrightarrow{\prod N_{E/E_i^*,\mathbb{R}}} \prod_{i=1}^r S^{E_i^*}_{\mathbb{R}} \xrightarrow{\prod \rho_{\Phi_i,\mathbb{R}}} \prod_{i=1}^r \mathcal{T}^{E_i}_{\mathbb{R}} \xrightarrow{\mathfrak{i}_{\mathbb{R}}} \mathrm{GSp}_{\mathbb{R}}. \tag{1.55}$$

Write $h'_{cm} : \mathbb{S} \to \prod_{i=1}^r \mathcal{T}^{E_i}_{\mathbb{R}}$ for the composition of the first three arrows.

Remark 1.11. 1) By construction h_{cm} is a CM point (cf., D.6) which is needed later to construct explicitly abelian extensions K. See 1.3.1.
2) Viewed as a point on the complex analytic space $\mathbb{H}_g^\pm$, we write x_{cm} instead of h_{cm}. Further, we denote the connected component of $\mathbb{H}_g^\pm$ containing x_{cm} by $\mathbb{H}_g$, i.e., $x_{cm} \in \mathbb{H}_g$.

Our CM point h_{cm} enjoys the following properties:

Lemma 1.3. *1) We have $h_{cm} = \mathfrak{i} \circ \prod_{i=1}^r h_{\Phi_i}$ (see (B.1)).*
2) The field of definition $E(x_{cm})$ of x_{cm} is equal to the composite of the reflex fields $\widetilde{E} = E_1^* \cdots E_r^* \subset E$, i.e. the associated cocharacter μ_{cm} of h_{cm} is defined over $\widetilde{E}$ (see D.6).
3) The $\mathrm{GSp}(\mathbb{R})$-conjugacy classes of h_{cm} can be identified with the Siegel upper-lower half plane $\mathbb{H}_g^\pm$, for some $g \in \mathbb{N}$ depending on E.

Proof. 1) This follows immediately from (C.9) and (C.13).
2) This follows from p. 105 [Mil04] and 1).
3) For this, we refer to the proof of lemma 3.11 [Wei94]. □

1.2.3 The map $\varphi : \mathcal{S}_K \to \mathcal{S}_{Sh}$

On the level of algebraic groups, $\varphi : T^K \to \mathrm{GSp}$ is simply defined as the composition

$$T^K \xrightarrow{\pi^K} S^K \xrightarrow{N_{K/E}} S^E \xrightarrow{\prod N_{E/E_i^*}} \prod_{i=1}^r S^{E_i^*} \xrightarrow{\prod \rho_{\Phi_i}} \prod_{i=1}^r \mathcal{T}^{E_i} \xrightarrow{\mathfrak{i}} \mathrm{GSp}. \tag{1.56}$$

For φ being a map between $\mathcal{S}_K$ and $\mathcal{S}_{Sh}$ we have to check that the diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h_K} & T^K_{\mathbb{R}} \\ & {\scriptstyle h_{cm}} \searrow & \downarrow {\scriptstyle \varphi_{\mathbb{R}}} \\ & & \mathrm{GSp}_{\mathbb{R}} \end{array} \tag{1.57}$$

commutes, but this compatibility is built into the construction of h_K. Using the reflex norm (cf., B.3 and (C.11)) we can describe φ as follows

Lemma 1.4. *The map* $\varphi : T^K \to \mathrm{GSp}$ *is equal to the composition*

$$T^K \xrightarrow{\prod N_{K/E_i^*}} \prod T^{E_i^*} \xrightarrow{\prod N_{\Phi_i}} \prod_{i=1}^r \mathcal{T}^{E_i} \xrightarrow{\mathfrak{i}} \mathrm{GSp}. \tag{1.58}$$

1.3 About arithmetic modular functions

1.3.1 Introduction

We follow closely our references [Del79], [MS81] and [Wei94]. See also [Hid04]. The reader should be aware of the fact that we are using a different normalization of Artin's reciprocity map than in [MS81] and have to correct a "sign error" in [Del79] as pointed out on p. 106 [Mil04].

As usual, we denote by K a number field containing a CM subfield, and denote by E the maximal CM subfield of K. In this section we want to explain how the theory of Complex Multiplication provides (explicit) abelian extensions of K. In general one looks at a CM-point $x \in X$ on a Shimura variety $\mathrm{Sh}(G, X)$ and by the theory of canonical models, one knows that the point $[x, 1]$ on the canonical model $M(G, X)$ of $\mathrm{Sh}(G, X)$ is rational over the maximal abelian extension $E(x)^{ab}$ of the field of definition $E(x)$ of x (see D.6).

In our case we look at Siegel modular varieties $\mathrm{Sh}(\mathrm{GSp}, \mathbb{H}_g^{\pm})$ which can be considered as (fine) moduli spaces of Abelian varieties over $\mathbb{C}$ with additional data (level structure, torsion data and polarization). See chap. 6 [Mil04] for an explanation of this. Each point $x \in \mathbb{H}_g$ corresponds to an Abelian variety A_x.
In opposite to the case of imaginary quadratic fields, in general, the field of definition $E(x)$ is not contained in E. Therefore, in order to construct abelian extensions of K, we have to find an Abelian variety A_x such that

$$E(x) \subset K. \tag{1.59}$$

This is exactly the reason for our choice of $x_{cm} \in \mathbb{H}_g$, because we know (see 1.3) that

$$E(x_{cm}) = \widetilde{E} = E_1^* \cdots E_r^* \subset E \subset K. \tag{1.60}$$

Here x_{cm} corresponds to a product $A_{cm} = A_1 \times \cdots \times A_r$ of simple Abelian varieties A_i, with complex multiplication given by E_i. This construction is the best one can do to generate abelian extensions of K using the theory of Complex Multiplication. The miracle here is again that the field of definition of A_{cm} and of its torsion points generate abelian extensions of $E(x_{cm})$. Now, to obtain these abelian extensions explicitly, one proceeds in complete analogy with the case of $\mathbb{Q}(i)$ explained in 1.1.4, namely rational functions on the connected canonical model M^o of the connected Shimura variety $\mathrm{Sh}(\mathrm{GSp}, \mathbb{H}_g^{\pm})^o$ give rise to arithmetic Modular functions on $\mathbb{H}_g$ which generate the desired abelian extensions when evaluated at x_{cm}. This will be explained in detail in the following.

1.3.2 Working over $\overline{\mathbb{Q}}$

The field $\mathcal{F}$ of arithmetic automorphic functions

We start with the remark that the reflex field of $(\mathrm{GSp}, \mathbb{H}_g^{\pm})$ (cf., D.6) is equal to $\mathbb{Q}$ (see remark D.3).

Remark 1.12. This is the second notion of "reflex field". But the reader shouldn't get confused.

Denote by Σ the set of arithmetic subgroups Γ of $\mathrm{GSp}^{ad}(\mathbb{Q})^+$ which contain the image of a congruence subgroup of $\mathrm{GSp}^{der}(\mathbb{Q})$. The connected component of the identity Sh^o of $\mathrm{Sh} = \mathrm{Sh}(\mathrm{GSp}, \mathbb{H}_g^{\pm})$ is then given by the inverse limit $Sh^o = \varprojlim_{\Gamma \in \Sigma} \Gamma \backslash \mathbb{H}_g$ (cf., D.4).
Denote by $M^o = M^o(\mathrm{GSp}, \mathbb{H}_g^{\pm})$ the canonical model of Sh^o in the sense of 2.7.10 [Del79], i.e. M^o is defined over $\overline{\mathbb{Q}}$. For every $\Gamma \in \Sigma$ the space $\Gamma \backslash \mathbb{H}_g$ is an algebraic variety over $\mathbb{C}$ and $\Gamma \backslash M^o$ a model over $\overline{\mathbb{Q}}$.
The field of rational functions $k(\Gamma \backslash M^o)$ on $\Gamma \backslash M^o$ is contained in the field of rational functions $k(\Gamma \backslash \mathbb{H}_g)$. Elements in the latter field correspond to meromorphic functions on $\mathbb{H}_g$ (now viewed as a complex analytic space) that are invariant under $\Gamma \in \Sigma$.
Following [MS81] we call the field $\mathcal{F} = \bigcup_{\Gamma \in \Sigma} k(\Gamma \backslash M^o)$ the field of **arithmetic automorphic functions** on $\mathbb{H}_g$.

About $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{F})$

The (topological) group $\mathcal{E}$ defined by the extension (see 2.5.9 [Del79])

$$1 \longrightarrow \overline{\mathrm{GSp}^{ad}(\mathbb{Q})^+} \longrightarrow \mathcal{E} \xrightarrow{\sigma} Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \qquad (1.61)$$

is acting continuously on M^o (2.7.10 [Del79]) and induces an action on $\mathcal{F}$ by

$$^{\alpha}f = \sigma(\alpha) \cdot (f \circ \alpha^{-1}) = (\sigma(\alpha)f) \circ (\sigma(\alpha)\alpha^{-1}) \qquad (1.62)$$

(see [MS81] 3.2). This is meaningful because f and α^{-1} are both defined over $\overline{\mathbb{Q}}$. Using this action one can prove

Theorem 1.5 (3.3 [MS81])**.** *The map $\mathcal{E} \to \mathrm{Aut}_{\mathbb{Q}}(\mathcal{F})$ given by (1.62) identifies $\mathcal{E}$ with an open subgroup of* $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{F})$*.*

1.3.3 Going down to $\mathbb{Q}^{ab}$

We said that M^o is defined over $\overline{\mathbb{Q}}$, but it is already defined over a subfield k of $\mathbb{Q}^{ab}$. More precisely, k is the fixed field of the kernel of the map $Gal(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} \to \overline{\pi}_0\pi(\mathrm{GSp})$ defined in 2.6.2.1 [Del79]. Therefore the action of $\mathcal{E}$ on M^o factors through the quotient $\overline{\mathcal{E}}$ of $\mathcal{E}$ defined by the following the commutative diagram with exact rows (see 4.2 and 4.12 [MS81] or 2.5.3 [Del79])

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{\mathrm{GSp}^{ad}(\mathbb{Q})^+} & \longrightarrow & \mathcal{E} & \xrightarrow{\sigma} & Gal(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle res} & & \\
1 & \longrightarrow & \overline{\mathrm{GSp}^{ad}(\mathbb{Q})^+} & \longrightarrow & \overline{\mathcal{E}} & \xrightarrow{\sigma} & Gal(k/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle l} & & \\
1 & \longrightarrow & \overline{\mathrm{GSp}^{ad}(\mathbb{Q})^+} & \longrightarrow & \frac{\mathrm{GSp}(\mathbb{A}_f)}{\overline{C(\mathbb{Q})}} & \longrightarrow & \overline{\pi}_0\pi(\mathrm{GSp}) & \longrightarrow & 1,
\end{array} \qquad (1.63)$$

here C denotes the center of GSp.

Remark 1.13. 1) We know that $C(\mathbb{Q}) = \mathbb{Q}^\times$ is discrete in $\mathrm{GSp}(\mathbb{A}_f)$ (cf., [Hid04]).
2) Because l is injective, τ is injective as well and we can identify $\overline{\mathcal{E}}$ with an (open) subgroup of $\frac{\mathrm{GSp}(\mathbb{A}_f)}{\mathbb{Q}^\times}$.
3) $\overline{\mathcal{E}}$ is of course depending on K, but we suppress this dependence in our notation.

The field $\mathcal{M}$ of arithmetic modular functions

Let $f \in \mathcal{F}$ be a rational function, i.e. f is a rational function on $\Gamma\backslash M^o$, for some $\Gamma \in \Sigma$. We call f an **arithmetic modular function** if it is rational over k, and meromorphic at the cusps (when viewed on the corresponding complex analytic space). Compare this to 3.4 [Wei94] or pp. 35 [Mil98].

Definition 1.3.1. The subfield of $\mathcal{F}$ generated by all arithmetic modular functions is denoted by $\mathcal{M}$. Further we denote by $\mathcal{M}^{cm}$ the subring of $\mathcal{M}$ of all arithmetic modular functions which are defined in x_{cm}.

The importance of $\mathcal{M}^{cm}$ for our purposes is explained (see 14.4 [Mil98] and 3.11 [Wei94]) by

Theorem 1.6. *Let A_{cm} denote the abelian variety corresponding to x_{cm} (cf., 1.3.1).*
Denote by $K^{A_{cm}}$ the field extension of K obtained by adjoining the field of definition of A_{cm} and all of its torsion points.
Further, denote by $K^{\mathcal{M}}$ the field extension of K obtained by adjoining the values $f(x_{cm})$, for $f \in \mathcal{M}^{cm}$.
Finally, denote by K^c the composition of K with the fixed field of the image of the Verlagerungsmap Ver : $Gal(F^{ab}/F) \to Gal(E^{ab}/E)$, *where F is the maximal totally real subfield of E. Then we have the equality*

$$K^c = K^{A_{cm}} = K^{\mathcal{M}} \tag{1.64}$$

Remark 1.14. Notice we are not simply using the field of arithmetic automorphic functions as considered by Shimura, see 4.8 [MS81] for his definition, because the exact size of the abelian extension obtained by using these functions is not clear (at least to the author). It is clear that the field of Shimura is contained in $K^{\mathcal{M}}$ and we guess that it should generate the same extension K^c of K.

About $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{M})$

It is clear that $\mathcal{M}$ is closed under the action of $\overline{\mathcal{E}}$ (see 3.2 or 4.4 [MS81]) and therefore we obtain a continuous map

$$\overline{\mathcal{E}} \to \mathrm{Aut}_{\mathbb{Q}}(\mathcal{M}) \tag{1.65}$$

given like above by

$$^{\alpha}f = \sigma(\alpha) \cdot (f \circ \alpha^{-1}) = (\sigma(\alpha)f) \circ (\sigma(\alpha)\alpha^{-1}). \tag{1.66}$$

In particular $\mathrm{GSp}^{ad}(\mathbb{Q})^+$ is acting on $\mathcal{M}$ by

$$^{\alpha}f = f \circ \alpha^{-1}. \tag{1.67}$$

1.3.4 The reciprocity law at x_{cm}

Write

$$\mu_{cm} : \mathbb{G}_{m,\mathbb{C}} \longrightarrow S^E_{\mathbb{C}} \xrightarrow{(h'_{cm})_{\mathbb{C}}} \prod_{i=1}^r \mathcal{T}^{E_i,\mathbb{C}} \xrightarrow{\mathfrak{i}} \mathrm{GSp}_{\mathbb{C}} \tag{1.68}$$

for the associated cocharacter of h_{cm} (cf., D.6). From Lemma 1.3 1) we know that $h_{cm} = \mathfrak{i} \circ \prod h_{\phi_i}$ and therefore $\mu_{cm} = \mathfrak{i} \circ \prod \mu_{\phi_i}$. Because μ_{ϕ_i} is defined over E_i^* the cocharacter $\mu'_{cm} = \prod \mu_{\phi_i}$ is defined over $\widetilde{E} = E_1^* \cdots E_r^* \subset E \subset K$. To simplify the notation set $\mathcal{T} = \prod_{i=1}^r \mathcal{T}^{E_i}$. Define the morphism

$$\eta : T^K \to \mathrm{GSp} \tag{1.69}$$

as composition of

$$T^K \xrightarrow{\mathrm{Res}_{K/\mathbb{Q}}(\mu'_{cm})} \mathrm{Res}_{K/\mathbb{Q}}(\mathcal{T}_K) \xrightarrow{\mathrm{Nm}_{K/\mathbb{Q}}} \mathcal{T} \xrightarrow{\mathfrak{i}} \mathrm{GSp}. \tag{1.70}$$

If we identify $\overline{\mathcal{E}}$ with an (open) subset of $\frac{\mathrm{GSp}(\mathbb{A}_f)}{\mathbb{Q}^\times}$ using τ (see (1.63)), we can show (see 4.5 [MS81] or 2.6.3 [Del79]) that $\eta(\mathbb{A}) : \mathbb{A}_K^\times \to \mathrm{GSp}(\mathbb{A})$ induces, by $\nu \mapsto \eta(\mathbb{A}_f)(\nu)$ mod $\mathbb{Q}^\times$, a group homomorphism

$$\overline{\eta} : \mathbb{A}_K^\times \to \overline{\mathcal{E}}. \tag{1.71}$$

If we denote by $[\nu] \in Gal(K^{ab}/K)$ the image of $\nu \in \mathbb{A}_K^\times$ under Artin's reciprocity map, we can show (cf., 4.5 [MS81] and (1.63)) that

$$\sigma(\overline{\eta}(\nu)) = [\nu]^{-1}{}_{|k}. \tag{1.72}$$

Remark 1.15. The careful reader will ask why it is allowed to define η using the extension K of the field of definition of the cocharacter μ_{cm} given by $\widetilde{E}$, because our reference [MS81] uses $\widetilde{E}$ to define η. The explanation for this is given by lemma 1.9 and standard class field theory.

Now we are able to state the **reciprocity law**

Theorem 1.7 (see 4.6 and 4.10 [MS81]). *Let $\nu \in \mathbb{A}_K^\times$ and $f \in \mathcal{M}^{cm}$. Then $f(x_{cm})$ is rational over K^{ab}. Further ${}^{\overline{\eta}(\nu)}f$ is defined in x_{cm} and*

$${}^{\overline{\eta}(\nu)}f(x_{cm}) = [\nu]^{-1}(f(x_{cm})). \tag{1.73}$$

Proof. We simply reproduce the argument given in the proof of Thm. 4.6 [MS81]. The first assertion is clear by the definition of the canonical model (cf., D.6) and the other two assertions follow from the following calculation.

Regard the special point x_{cm} as a point on the canonical model $[x_{cm}, 1] \in M^o$. The action of $\overline{\eta}(\nu)^{-1}$ is given by $\overline{\eta}(\nu)^{-1}[x_{cm}, 1] = \sigma(\overline{\eta}(\nu)^{-1})[x_{cm}, \eta(\nu)]$ and further we know $[x_{cm}, \eta(\nu)] = [\nu]^{-1}[x_{cm}, 1]$ (by (D.7)). Therefore we obtain

$$\begin{aligned}
{}^{\overline{\eta}(\nu)}f(x_{cm}) &= \sigma(\overline{\eta}(\nu)) \cdot (f \circ \overline{\eta}(\nu)^{-1})([x_{cm}, 1]) \\
&= (\sigma(\overline{\eta}(\nu) f) \circ (\sigma(\overline{\eta}(\nu)) \overline{\eta}(\nu)^{-1})([x_{cm}, 1]) \\
&= (\sigma(\overline{\eta}(\nu) f) \circ (\sigma(\overline{\eta}(\nu)) \sigma(\overline{\eta}(\nu))^{-1})([x_{cm}, \eta(\nu)]) \\
&\overset{(1.72)}{=} ([\nu]^{-1}|_k f)([\nu]^{-1}[x_{cm}, 1]) = [\nu]^{-1}(f([x_{cm}, 1])) \\
&= [\nu]^{-1}(f(x_{cm})).
\end{aligned}$$

□

The next observation is one of the key ingredients in our construction of the arithmetic subalgebra.

Proposition 1.8. *The two maps of algebraic groups φ and η are equal.*

Proof. This is an immediate corollary of Proposition C.1, Lemma 1.4 , the compatibility properties of the norm map and the next simple Lemma. □

Lemma 1.9. *Let (L, ϕ) a CM type and L' a finite extension of the reflex field L^*. Then the following diagram*

$$\begin{array}{ccccc} T^{L'} & \xrightarrow{\mathrm{Res}_{L'/\mathbb{Q}}\mu_{L'}} & \mathrm{Res}_{L'/\mathbb{Q}}(T^L_{L'}) & \xrightarrow{\mathrm{Nm}_{L'/\mathbb{Q}}} & T^L \\ \downarrow N_{L'/L^*} & & & \nearrow \mathrm{Nm}_{L^*/\mathbb{Q}} & \\ T^{L^*} & \xrightarrow{\mathrm{Res}_{L^*/\mathbb{Q}}\mu_{L^*}} & \mathrm{Res}_{L^*/\mathbb{Q}}(T^L_{L^*}) & & \end{array} \tag{1.74}$$

is commutative.

Now having all the number-theoretic ingredients we need in hand, we can move on to the "operator-theoretic" part of this chapter.

1.4 On Bost-Connes-Marcolli systems

We review very briefly the general construction of C^*-dynamical systems, named Bost-Connes-Marcolli systems, as given in [HP05].

1.4.1 BCM pairs

A BCM pair $(\mathcal{D}, \mathcal{L})$ is a pair consisting of a BCM datum $\mathcal{D} = (G, X, V, M)$ together with a level structure $\mathcal{L} = (L, \Gamma, \Gamma_M)$ of $\mathcal{D}$.
A BCM datum is a Shimura datum (G, X) together with an enveloping algebraic semigroup M and a faithful representation $\phi : G \to GL(V)$ such that $\phi(G) \subset M \subset End(V)$. Here V denotes a $\mathbb{Q}$-vector space of finite dimension.
A level structure $\mathcal{L}$ of $\mathcal{D}$ consists of a lattice $L \subset V$, a compact open subgroup $\Gamma \subset G(\mathbb{A}_f)$ and a compact open semigroup $\Gamma_M \subset M(\mathbb{A}_f)$ such that $\phi(\Gamma) \subset \Gamma_M$ and Γ_M stabilizes $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

Remark 1.16. In 3.1 [HP05], a more general notion of Shimura datum is allowed than ours given in the Appendix.

To every BCM datum $\mathcal{D}$ and lattice $L \subset V$, one can associate the following so-called **maximal level structure** to obtain a BCM pair by setting $\Gamma_M = M(\mathbb{A}_f) \cap End(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ and $\Gamma = \phi^{-1}(\Gamma_M^{\times})$.

The level structure $\mathcal{L}$ is called **fine** if Γ is acting freely on $G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f))$.

Remark 1.17. For the definition of the topology of $G(\mathbb{A}_f)$ and $M(\mathbb{A}_f)$ we refer the reader to [PR94]. Especially, one can show that $\phi(\mathbb{A}_f) : G(\mathbb{A}_f) \to M(\mathbb{A}_f)$ is a continuous map (cf., lemma 5.2 [PR94]).

1.4.2 Quotient maps attached to BCM pairs

Let $(\mathcal{D}, \mathcal{L})$ be a BCM pair.

The BCM groupoid

There is a partially defined action of $G(\mathbb{A}_f)$ on the direct product $\Gamma_M \times \mathrm{Sh}(G,X)$ given by

$$g(\rho,[z,l]) = (g\rho,[z,lg^{-1}]), \tag{1.75}$$

where we suppressed the mophism ϕ. Using this the BCM groupoid U is the topological groupoid (using the notation given in (1.9)) defined by

$$U = G(\mathbb{A}_f) \boxplus (\Gamma_M \times \mathrm{Sh}(G,X)). \tag{1.76}$$

There is an action of the group $\Gamma \times \Gamma$ on U given by

$$(\gamma_1,\gamma_2)(g,\rho,[z,l])(\gamma_1 g\gamma_2^{-1},\gamma_2\rho,[z,l\gamma_2^{-1}]). \tag{1.77}$$

We denote the quotient by $Z = (\Gamma \times \Gamma)\backslash U$ and obtain a natural quotient map

$$U \longrightarrow Z. \tag{1.78}$$

We denote elements in Z by $[g,\rho,[z,l]]$.

Remark 1.18. In general the quotient Z is not a groupoid anymore, see 4.2.1 [HP05].

The positive BCM groupoid

Assume that the Shimura datum (G,X) of our BCM pair satisfies $(SV5)$ (cf., D.1). Moreover we choose a connected component X^+ of X and set $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$, where $G(\mathbb{R})^+$ denotes the connected component of the identity of $G(\mathbb{R})$. Then $G(\mathbb{Q})^+$ is acting naturally on X^+, because X^+ can be regarded as a $G(\mathbb{R})^+$-conjugacy class (see D.4). Now we can consider the positive (BCM) groupoid U^+ which is the topological groupoid given by

$$U^+ = G(\mathbb{Q})^+ \boxplus (\Gamma_M \times X^+). \tag{1.79}$$

If we set $\Gamma^+ = \Gamma \cap G(\mathbb{Q})^+$ we see further that $\Gamma^+ \times \Gamma^+$ is acting on U^+ by

$$(\gamma_1,\gamma_2)(g,\rho,z) = (\gamma_1 g\gamma_2^{-1},\gamma_2\rho,\gamma_2 z). \tag{1.80}$$

We denote the quotient by $Z^+ = (\Gamma^+ \times \Gamma^+)\backslash U^+$ and obtain another quotient map

$$U^+ \longrightarrow Z^+. \tag{1.81}$$

There is a natural equivariant morphism of topological groupoids

$$U^+ \longrightarrow U \tag{1.82}$$

given by $(g,\rho,z) \mapsto (g,\rho,[z,1])$, inducing a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \uparrow & & \uparrow \\ U^+ & \longrightarrow & Z^+ \end{array} \tag{1.83}$$

The following criterion given in 5.1 of [HP05] will be crucial for our approach.

Criterion 1.10. *If the natural map* $G(\mathbb{Q})\cap\Gamma \to G(\mathbb{Q})/G(\mathbb{Q})^+$ *is surjective and* $|G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\Gamma| = 1$*, then the natural morphism (1.82) induces a homeomorphism of topological spaces*

$$Z^+ \longrightarrow Z. \tag{1.84}$$

Remark 1.19. 1) In other words the two conditions of the criterion simply mean that we have a decomposition of the form $G(\mathbb{A}_f) = G(\mathbb{Q})^+ \cdot \Gamma$.
2) The inverse $Z \longrightarrow Z^+$ of the above homeomorphism is given explicitly as follows. By using the first remark we can write every $l \in G(\mathbb{A}_f)$ as a product $l = \alpha\beta$ with $\alpha \in G(\mathbb{Q})^+$ and $\beta \in \Gamma$ (this decomposition is unique up to an element in $\Gamma^+ = \Gamma \cap G(\mathbb{Q})^+$). In particular every element $[g, \rho, [z, l]] \in Z$ can be written as $[g, \rho, [z, l]] = [g\beta^{-1}, \beta\rho, [\alpha^{-1}z, 1]]$ and, under the inverse of the above homeomorphism, this element is sent to $[g\beta^{-1}, \beta\rho, \alpha^{-1}z] \in Z^+$.

The adjoint BCM algebra

Let us denote by C the center of G and **assume** further that $\phi(\overline{C(\mathbb{Q})})$ is a normal subsemigroup of $M(\mathbb{A}_f)$. The adjoint group G^{ad} of G is the quotient of G by its center C (in the sense of algebraic groups, see [Wat79]). Let us define the semigroup Γ_M^{ad} to be the quotient of Γ_M by the normal subsemigroup $\phi(\overline{C(\mathbb{Q})})\cap\Gamma_M$ and remember that X^+ can be naturally regarded as a $G^{ad}(\mathbb{R})^+$-conjugacy class (see D.4). With this in hand we define the adjoint (BCM) groupoid U^{ad} to be the topological groupoid

$$U^{ad} = G^{ad}(\mathbb{Q})^+ \boxplus (\Gamma_M^{ad} \times X^+). \tag{1.85}$$

It is known that the projection $G \longrightarrow G^{ad}$ induces a surjective group homomorphism $\pi^{ad} : G(\mathbb{Q})^+ \longrightarrow G^{ad}(\mathbb{Q})^+$ (see 5.1 [Mil04]). Setting $\Gamma^{ad} = \pi^{ad}(\Gamma^+)$ we see immediately that $\Gamma^{ad} \times \Gamma^{ad}$ is acting on U^{ad} exactly as in 1.80. We obtain yet another quotient map

$$U^{ad} \longrightarrow Z^{ad}. \tag{1.86}$$

Using the two projections $\Gamma_M \to \Gamma_M^{ad}$ and π^{ad} there is by construction an obvious equivariant morphism of topological groupoids

$$U^+ \longrightarrow U^{ad} \tag{1.87}$$

which induces (together with (1.83)) a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \uparrow & & \uparrow \\ U^+ & \longrightarrow & Z^+ \\ \downarrow & & \downarrow \\ U^{ad} & \longrightarrow & Z^{ad}. \end{array} \tag{1.88}$$

1.4.3 BCM algebras and systems

Let $(\mathcal{D}, \mathcal{L})$ be a BCM pair. The **BCM algebra** $H = H_{\mathcal{D},\mathcal{L}}$ is defined to be the set of compactly supported, continuous function on the quotient $Z = (\Gamma \times \Gamma)\backslash U$ of the BCM groupoid U, i.e.

$$H = C_c(Z). \tag{1.89}$$

By viewing functions in H as $\Gamma \times \Gamma$-invariant functions on the groupoid U, we can equip H with the structure of a $*$-algebra by using the usual convolution and involution on U (like in the construction of groupoid C^*-algbras). We refer to 4.3.2 [HP05] for the details. After completing H in a suitable norm we obtain a C^*-algebra A (see 6.2 [HP05]). Further there is a time evolution $(\sigma_t)_{t\in\mathbb{R}}$ on H (resp. A) so that we end up with the **BCM system** $\mathcal{A} = \mathcal{A}_{\mathcal{D},\mathcal{L}}$ given by the C^*-dynamical system

$$\mathcal{A} = (A, (\sigma_t)_{t\in\mathbb{R}}) \tag{1.90}$$

associated with the BCM pair $(\mathcal{L}, \mathcal{D})$. For the general definition of the time evolution, we refer to 4.4 of [HP05]. We will state the **time evolution** $(\sigma_t)_{t\in\mathbb{R}}$ only in the case of our BC-systems $\mathcal{A}_K$ (cf., 1.5.1).

Remark 1.20. In complete analogy one, might construct a positive (respectively adjoint) BCM system, but we don't need this.

1.4.4 On Symmetries of BCM algebras

In section 4.5 [HP05], the authors define symmetries of BCM algebras, but for our purpose, we need to deviate from their definition in order to be in accordance with the definition of symmetries for BC-systems given in [LLN09].

Let $(\mathcal{D}, \mathcal{L})$ be a BCM pair with fine level structure (see 1.4.1), and recall (D.2) that there is a natural right action of $G(\mathbb{A}_f)$ on the Shimura variety $\mathrm{Sh}(G, X)$ which is denoted by $m[z, l] = [z, lm]$. Define the subgroup $G_\Gamma(\mathbb{A}_f) = \{g \in G(\mathbb{A}_f) \mid g\gamma = \gamma g \ \forall \gamma \in \Gamma\}$. Further, if we denote by C the center of G, the group $C(\mathbb{R})$ is acting on $\mathrm{Sh}(G, X)$ by $c[z, l] = [cz, l]$. We end up with a right action of $G_\Gamma(\mathbb{A}_f) \times C(\mathbb{R})$ as symmetries on the BCM algebra $H_{\mathcal{D},\mathcal{L}}$ given on a function $f \in C_c(Z)$ by

$$^{(m,c)}f(g, \rho, [z, l]) = f(g, \rho, [cz, lm]). \tag{1.91}$$

Remark 1.21. If $G(\mathbb{A}_f)$ is a commutative group and we have a decomposition $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot \Gamma$ then it is immediate that our symmetries agree with the ones defined in 4.5 of [HP05].

1.5 Two BCM pairs and a map

In this section we will apply the constructions from the last section to our Shimura data $\mathcal{S}_K = (T^K, X_K, h_K)$ and $\mathcal{S}_{Sh} = (\mathrm{GSp}(V_E, \psi_E), \mathbb{H}_g^{\pm}, h_{cm})$ from section 1.2, and show how the two resulting systems can be related.

1.5.1 Costume I: $(\mathcal{D}_K, \mathcal{L}_K)$ and $\mathcal{A}_K$

This section is valid for an arbitrary number field K.

The BCM groupoid $(\mathcal{D}_K, \mathcal{L}_K)$

Let us recall the BCM pair $(\mathcal{D}_K, \mathcal{L}_K)$ from 5.5 [HP05] attached to $\mathcal{S}_K$. It is given by

$$(\mathcal{D}_K, \mathcal{L}_K) = ((\mathcal{S}_K, K, M^K), (\mathcal{O}_K, \widehat{\mathcal{O}}_K^\times, \widehat{\mathcal{O}}_K)), \tag{1.92}$$

where the algebraic semigroup M^K is represented by the functor which assigns to a $\mathbb{Q}$-algebra R the semigroup of $\mathbb{Q}$-algebra homomorphisms $Hom(K[X], K \otimes_{\mathbb{Q}} R)$. By definition we have that $M^K(R)^\times = T^K(R)$ for every $\mathbb{Q}$-algebra R, which gives an embedding $\phi : T^K \to M^K$. Further, It will be convenient to set $\Gamma_K = \widehat{\mathcal{O}}_K^\times$.

The quotient map $U_K \to Z_K$

The corresponding BCM groupoid, denoted by U_K, is given by

$$U_K = T^K(\mathbb{A}_f) \boxplus (\widehat{\mathcal{O}}_K \times \mathrm{Sh}(\mathcal{S}_K)) \tag{1.93}$$

with $\Gamma_K^2 = \widehat{\mathcal{O}}_K^\times \times \widehat{\mathcal{O}}_K^\times$ acting as in (1.77). We denote the quotient of this action by

$$Z_K = \Gamma_K^2 \backslash U_K. \tag{1.94}$$

The time evolution

Following 7.1 of [HP05] the time evolution $(\sigma_t)_{t\in\mathbb{R}}$ on the BCM algebra $H_K = C_c(Z_K)$ is given as follows. Denote by $\mathcal{N} : \mathbb{A}_{K,f}^\times \to \mathbb{R}$ the usual idele norm. If $f \in H_K$ is a function, then we have

$$\sigma_t(f)(g, \rho, [z, l]) = \mathcal{N}(g)^{it} f(g, \rho, [z, l]). \tag{1.95}$$

On symmetries

First, from 2.2.3 [Del79], we know that there is an isomorphism between $\mathrm{Sh} = \mathrm{Sh}(T^K, X_K, h_K)$ and $\pi_0(C_K)$. By class field theory, the latter space $\pi_0(C_K) = C_K/D_K$ is identified with the Galois group $Gal(K^{ab}/K)$ of the maximal abelian extension of K using the Artin reciprocity homomorphism. Under this identification, the natural action of $T^K(\mathbb{A}_f) = \mathbb{A}_{K,f}^\times$ on Sh corresponds simply to the Artin reciprocity map, i.e. if ν is a finite idele in $T(\mathbb{A}_f)$ and $\omega_1 = [g, l] \in Sh$ corresponds to the identity in $Gal(K^{ab}/K)$, then $\nu\omega_1 = [g, l\nu]$ corresponds to the image $[\nu]$ in $Gal(K^{ab}/K)$ of ν under Artin's reciprocity map.

Now, because T^K is commutative, we see that $C(\mathbb{A}_f) \times C(\mathbb{R}) = T^K(\mathbb{A}) = \mathbb{A}_K^\times$ is acting by symmetries on H_K. By what we just said, this action is simply given by the natural map $\mathbb{A}_K^\times \to \pi_0(C_K) = C_K/D_K$ so that we obtain (tautologically) the desired action of $C_K/D_K \cong Gal(K^{ab}/K)$ on H_K.

Remark 1.22. The reader should notice that in the case of an imaginary quadratic number field K, our symmetries do not agree with the symmetries defined [CMR05] (except when the class number of K is equal to one, where the two definitions agree). For a short discussion on this matter we refer the reader to remark 1.24 and Appendix E.

About extremal KMS_∞-states of $\mathcal{A}_K$

We refer to pp. 445 [CM08] or [BR81] for the notion of extremal KMS_∞-states.

Let $\mathcal{A}_K = (A_K, (\sigma_t)_{t\in\mathbb{R}})$ denote the corresponding BCM system (cf., 1.4.3). In Theorem 2.1 (vi) [LLN09], it is shown that the set $\mathcal{E}_\infty$ of extremal KMS_∞-states of $\mathcal{A}_K$ is indexed by the set $\mathrm{Sh} = \mathrm{Sh}(T^K, X_K, h_K)$, and the extremal KMS_∞-state ϱ_ω associated with $\omega \in \mathrm{Sh}$ is given on a function $f \in H_K$ by evaluation, namely

$$\varrho_\omega(f) = f(1, 1, \omega). \tag{1.96}$$

Remark 1.23. It follows immediately that the symmetry group C_K/D_K is acting freely and transitively on the set of extremal KMS_∞-states.

All put together we get the following theorem.

Theorem 1.11 ([HP05] and [LLN09]). *Let K be an arbitrary number field. Then the BCM system $\mathcal{A}_K = (A_K, (\sigma_t)_{t\in\mathbb{R}})$ satisfies all four properties from* (0.2).

To follow the general convention, we call the systems $\mathcal{A}_K$ simply BC-systems.

1.5.2 Costume II: $(\mathcal{D}_{Sh}, \mathcal{L}_{Sh})$

The BCM groupoid $(\mathcal{D}_{Sh}, \mathcal{L}_{Sh})$

Recall the construction of the symplectic vector space (V_E, ψ_E) (see 1.2.2). We still have some freedom in specifying the totally imaginary generators ξ_i of the (primitive) CM fields E_i, which in turn define the symplectic form ψ_E (cf., (1.52)). Let us denote by L_E the lattice $L_E = \oplus_{i=1}^r \mathcal{O}_{E_i} \subset V_E = \oplus_{i=1}^r E_i$. We now fix generators ξ_i according to the following lemma.

Lemma 1.12. *For each $i \in \{1, .., r\}$, there exists a totally imaginary generator $\xi_i \in E_i$, such that the associated symplectic vector space (V_E, ψ_E) is integral with respect to L_E, i.e., there exists a symplectic basis $\{e_j\}$ for (V_E, ψ_E) such that $e_j \in L_E$, for each j.*

Proof. For each i, choose any totally imaginary generator $\widetilde{\xi}_i \in E_i$, and regard the associated symplectic form $\widetilde{\psi}_E$ on V_E (see (1.52)). It is known that there exists a symplectic basis $\{\widetilde{e}_j\}$ for $(V_E, \widetilde{\psi}_E)$. Now, for each j, there exists a $q_j \in \mathbb{N}$ such that $e_j = q_j \widetilde{e}_j \in L_E$, because $E_i = \mathcal{O}_{E_i} \otimes_{\mathbb{Z}} \mathbb{Q}$. Set $q = \prod_j q_j$, and define the symplectic form ψ_E on V_E by using the totally imaginary generators $\xi_i = q^{-2}\widetilde{\xi}_i \in E_i$, for every i. By construction it is now clear that $\{e_j\}$ is an integral symplectic basis for (V_E, ψ_E). $\square$

Now having fixed our Shimura datum $\mathcal{S}_{Sh} = (\mathrm{GSp}(V_E, \psi_E), \mathbb{H}_g^{\pm}, h_{cm})$, we define the BCM pair $(\mathcal{D}_{Sh}, \mathcal{L}_{Sh})$ equipped with the maximal level structure (cf., 1.4.1) with respect to the lattice L_E by

$$(\mathcal{D}_{Sh}, \mathcal{L}_{Sh}) = ((\mathcal{S}_{Sh}, V_E, \mathrm{MSp}), (L_E, \Gamma_{Sh}, \Gamma_{Sh,M})), \tag{1.97}$$

where the algebraic semigroup $\mathrm{MSp} = \mathrm{MSp}(V_E, \psi_E)$ is represented by the functor which assigns to a $\mathbb{Q}$-algebra R the semigroup

$$\mathrm{MSp}(R) =$$
$$\{f \in End_R(V_E \otimes_{\mathbb{Q}} R) \mid \exists\, \nu(f) \in R : \psi_{E,R}(f(x), f(y)) = \nu(f)\psi_{E,R}(x, y)\ \forall x, y\}.$$

It is clear by definition (compare A.2) that $\mathrm{MSp}(R)^\times = \mathrm{GSp}(R)$ which defines a natural injection $\phi : \mathrm{GSp} \to \mathrm{MSp}$.

Some quotient maps

We denote the corresponding BCM groupoid by

$$U_{Sh} = \mathrm{GSp}(\mathbb{A}_f) \boxplus (\Gamma_{Sh,M} \times \mathrm{Sh}(\mathcal{S}_{Sh})), \tag{1.98}$$

where the group $\Gamma_{Sh}^2 = \Gamma_{Sh} \times \Gamma_{Sh}$ is acting as usual. We denote the quotient of U_{Sh} by this action by

$$Z_{Sh} = \Gamma_{Sh}^2 \backslash U_{Sh}. \tag{1.99}$$

Thanks to D.3 and Remark 1.13 1) we are allowed to consider the positive and adjoint BCM groupoid, which we denote by U_{Sh}^+ and U_{Sh}^{ad} respectively. The corresponding quotients are denoted analogously by $Z_{Sh}^+ = (\Gamma_{Sh}^+)^2 \backslash U_{Sh}^+$ and $Z_{Sh}^{ad} = (\Gamma_{Sh}^{ad})^2 \backslash U_{Sh}^{ad}$.

1.5.3 The map $\Theta : Z_K \to Z_{Sh}^{ad}$

The aim in this section is to construct a continuous map $\Theta : Z_K \longrightarrow Z_{Sh}^{ad}$.

Relating Z_K and Z_{Sh}

Recall that the morphism of Shimura data $\varphi : \mathcal{S}_K \to \mathcal{S}_{Sh}$ constructed in 1.2.3 is a morphism of algebraic groups $\varphi : T^K \to \mathrm{GSp}$, inducing a morphism $\mathrm{Sh}(\varphi) : \mathrm{Sh}(\mathcal{S}_K) \to \mathrm{Sh}(\mathcal{S}_{Sh})$ of Shimura varieties. Moreover there is a natural continuation of φ to a morphism of algebraic semigroups $M(\varphi) : M^K \to \mathrm{MSp}$ due to the following. We know that $\varphi : T^K \to \mathrm{GSp}$ can be expressed in terms of reflex norms (see lemma 1.4 and B.3), which are given by determinants, and this definition still makes sense if we replace T^K and GSp by their enveloping semigroups M^K and MSp, respectively. Now we can define an equivariant morphism of topological groupoids

$$\Omega : U_K \to U_{Sh}, \tag{1.100}$$

by

$$(g, m, z) \in U_K \mapsto (\varphi(\mathbb{A}_f)(g), M(\varphi)(\mathbb{A}_f)(z), \mathrm{Sh}(\varphi)(z)) \in U_{Sh}. \tag{1.101}$$

To show the equivariance of Ω, use $\varphi(\mathbb{A}_f)(\Gamma_K) \subset \Gamma_{Sh}$ and the equivariance of $\mathrm{Sh}(\varphi)$ (cf., D.2). We obtain a continuous map

$$\overline{\Omega} : Z_K \to Z_{Sh}. \tag{1.102}$$

Relating Z_{Sh} and Z_{Sh}^{ad}

In order to relate Z_{Sh} and Z_{Sh}^{ad}, we will show that we are allowed to apply Criterion 1.10 by proving the following two lemmata.

Lemma 1.13. *We have* $\mathrm{GSp}(V_E, \psi_E)(\mathbb{A}_f) = \mathrm{GSp}(V_E, \psi_E)(\mathbb{Q}) \cdot \Gamma_{Sh}$.

Proof. Let $\{e_j\}$ be an integral symplectic basis of V_E with respect to L_E (cf., Lemma 1.12). Each $f \in \mathrm{GSp}(\mathbb{A}_f)$ is $\mathbb{A}_f$-linear and therefore determined by the values on $e_j \otimes 1 \otimes 1 \in L_E \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ given by

$$f(e_j \otimes 1 \otimes 1) = \sum_k a_{k,j} \otimes b_{k,j} \otimes c_{k,j} \in L_E \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}. \tag{1.103}$$

Let $d_{k,j} \in \mathbb{N}$ be the denominator of $b_{k,j}$, and define $c(f) = \prod_{k,j} d_{k,j} \in \mathbb{N}$.
Now observe that the map

$$M_f : e_i \mapsto c(f)e_i$$

is a map in $\mathrm{GSp}(\mathbb{Q}) \subset \mathrm{GSp}(\mathbb{A}_f)$, i.e., M_f is compatible with the symplectic struture ψ_E. Obviously,

$$M_f \circ f \in \Gamma_{Sh},$$

and we thus obtain the desired decomposition

$$f = M_f^{-1} \circ (M_f \circ f) \in \mathrm{GSp}(\mathbb{Q}) \cdot \Gamma_{Sh}.$$

□

Lemma 1.14. *The map* $\mathrm{GSp}(\mathbb{Q}) \cap \Gamma_{Sh} \to \mathrm{GSp}(\mathbb{Q})/\mathrm{GSp}(\mathbb{Q})^+$ *is surjective.*

Proof. We know that $\mathrm{GSp}(\mathbb{R})^+ = \{f \in \mathrm{GSp}(\mathbb{R}) | \nu(f) > 0\}$ (see A.2).
From this we get $\mathrm{GSp}(\mathbb{Q})^+ = \mathrm{GSp}(\mathbb{Q}) \cap \mathrm{GSp}(\mathbb{R})^+ = \{f \in \mathrm{GSp}(\mathbb{Q}) | \nu(f) > 0\}$.
Let f be an element in $\mathrm{GSp}(\mathbb{Q})$. If we define M_f exactly like in the proof above, we see that $M_f \in \mathrm{GSp}(\mathbb{Q})^+$ and can conclude $M_f \circ f \in \mathrm{GSp}(\mathbb{Q}) \cap \Gamma_{Sh}$. □

Thus we see that the natural morphism $Z_{Sh}^+ \to Z_{Sh}$ (see 1.82) is a homeomorphism, so that we can invert this map and compose it with the natural map $Z_{Sh}^+ \to Z_{Sh}^{ad}$ (cf., 1.87), to obtain a continuous map

$$Z_{Sh} \longrightarrow Z_{Sh}^{ad}. \tag{1.104}$$

Finally, if we compose the last map with $\overline{\Omega}$ from above, we obtain a continuous morphism denoted

$$\Theta : Z_K \longrightarrow Z_{Sh}^{ad}. \tag{1.105}$$

One crucial property of Θ is that every element $z \in Z_K$ will be sent to an element of the form

$$\Theta(z) = [g\beta^{-1}, \beta\rho, \alpha^{-1}x_{cm}] \in Z_{Sh}^{ad}, \tag{1.106}$$

where $g \in G^{ad}(\mathbb{Q})^+$, $\rho \in \Gamma_{Sh,M}^{ad}$, $\alpha \in G^{ad}(\mathbb{Q})$ and $\beta \in \Gamma_{Sh}^{ad}$, such that $\alpha\beta \in \pi^{ad}(\varphi(T^K(\mathbb{A}_f))) \subset G^{ad}(\mathbb{A}_f)$.

1.6 Construction of our partial arithmetic subalgebra

The idea of the construction to follow goes back to [CMR05] and [CMR06].

We constructed the ring $\mathcal{M}^{cm}$ of arithmetic Modular functions on $\mathbb{H}_g$ that are defined in x_{cm} (see 1.3.3). Further the group $\overline{\mathcal{E}}$ acts by automophisms on $\mathcal{M}^{cm}$ according to 1.3.3. (Recall that we use the notation ${}^{\alpha}f$ to denote the action of an automorphism α on a function $f \in \mathcal{M}^{cm}$.) Thanks to the embeddings (cf., 1.3.3)

$$\overline{\mathcal{E}} \longrightarrow \frac{\mathrm{GSp}(\mathbb{A}_f)}{\mathbb{Q}^\times} \longrightarrow \frac{\mathrm{MSp}(\mathbb{A}_f)}{\mathbb{Q}^\times} \longleftarrow \Gamma_{Sh,M}^{ad} \tag{1.107}$$

the intersection $\overline{\mathcal{E}} \cap \Gamma_{Sh,M}^{ad}$ is meaningful, and thus we can define, for each $f \in \mathcal{M}^{cm}$, a function $\widetilde{f}$ on U_{Sh}^{ad} by

$$\widetilde{f}(g,\rho,z) = \begin{cases} {}^{\rho}f(z), & \text{if } \rho \in \overline{\mathcal{E}} \cap \Gamma_{Sh,M}^{ad} \\ 0 & \text{else.} \end{cases} \tag{1.108}$$

By construction $\widetilde{f}$ is invariant under the action of $(\gamma_1, \gamma_2) \in \Gamma_{Sh}^{ad} \times \Gamma_{Sh}^{ad}$, because

$$\widetilde{f}(\gamma_1 g \gamma_2^{-1}, \gamma_2 m, \gamma_2 z) \overset{\text{def}}{=} {}^{\gamma_2 m}f(\gamma_2 z) \overset{(1.67)}{=} {}^{\gamma_2^{-1}\gamma_2 m}f(z) = \widetilde{f}(g,m,z). \tag{1.109}$$

Therefore, we can regard $\widetilde{f}$ as function a on the quotient $Z_{Sh}^{ad} = (\Gamma_{Sh}^{ad})^2 \backslash U_{Sh}^{ad}$.

We set $W_K = \widehat{\mathcal{O}}_K^\times \times \widehat{\mathcal{O}}_K \times \mathrm{Sh}(T^K, X_K)$, which is a compact and clopen subset of U_K, and invariant under the action of Γ_K^2, i.e., $\Gamma_K^2 \cdot W_K \subset W_K$. With these preliminaries we have the following.

Proposition 1.15. *Let f be a function in $\mathcal{M}^{cm}$. Then $\widetilde{f} \circ \Theta$ is contained in $C_c(Z_K)$, i.e.,*

$$\widetilde{f} \circ \Theta \in H_K \subset A_K. \tag{1.110}$$

Proof. As we already remarked in (1.106) the image of an element $z \in Z_K$ under Θ is of the form $[g\beta^{-1}, \beta\rho, \alpha^{-1}x_{cm}] \in Z^{ad}_{Sh}$. Therefore $f_K = \widetilde{f} \circ \Theta$ is continous, because Θ is continuous, the action of $\overline{\mathcal{E}}$ is continuous and does not produce singularities at special points (see Theorem 1.7).
Let us now regard f_K as a Γ^2_K-invariant function on U_K. Thanks to $\overline{\mathcal{E}} \subset \frac{\mathrm{GSp}(\mathbb{A}_f)}{\mathbb{Q}^\times}$ and (1.108), we see that the support of our function f_K is contained in the clopen subset $\widehat{\mathcal{O}}^\times_K \times \widehat{\mathcal{O}}^\times_K \times \mathrm{Sh}(T^K, X_K) \subset U_K$. Using the compact subset $W_K \subset U_K$, the next easy lemma finishes the proof. □

Lemma 1.16. *Let G be a topological group, X be a topological G-space and $Y \subset X$ a compact, clopen subset such that $GY \subset Y$. If we have a continuous, G-invariant function $f \in C^G(X)$ then $f|_Y \in C_c(G\backslash X)$.*

Now we can define our arithmetic subalgebra of $\mathcal{A}_K = (A_K, \sigma_t)$ (cf., 1.5.1).

Definition 1.6.1. Denote by $A^{K^c}_K$ the K-rational subalgebra of A_K generated by the set of functions $\{\widetilde{f} \circ \Theta \mid f \in \mathcal{M}^{cm}\}$.

1.7 Proof of Theorem 1.1

Let $f \in \mathcal{M}^{cm}$ and denote $f_K = \widetilde{f} \circ \Theta \in A^{K^c}_K$. Further denote by ϱ_ω the extremal KMS_∞-state of $\mathcal{A}_K$ corresponding to $\omega \in \mathrm{Sh} = \mathrm{Sh}(T^K, X_K, h_K)$ (see 1.5.1). Recall the isomorphism $Gal(K^{ab}/K) \cong \pi_0(C_K) = Sh$ given by Artin reciprocity. Considered as element in $Gal(K^{ab}/K)$ we write $[\omega]$ for ω.

Property (vi)

Let $\nu \in \mathbb{A}^\times_K$ be a symmetry of $\mathcal{A}_K$ (see 1.5.1). Thanks to Lemma 1.12 and 1.13, we can write $\varphi(\mathbb{A}_f)(\nu) = \alpha\beta \in \mathrm{GSp}(\mathbb{A}_f)$ with $\alpha \in \mathrm{GSp}(\mathbb{Q})^+$ and $\beta \in \Gamma_{Sh}$. By $\overline{\alpha}$ resp. $\overline{\beta}$ we will denote their images in $\mathrm{GSp}^{ad}(\mathbb{Q})^+$ resp. Γ^{ad}_{Sh} under the map $\pi^{ad} \circ \varphi(\mathbb{A}_f)$. Moreover, we denote the image of ν under Artin reciprocity by $[\nu] \in Gal(K^{ab}/K)$.
The action of the symmetries on the extremal KMS_∞-states is given by pull-back, and because this action is free and transitive, it is enough to restrict to the case of the extremal KMS_∞-state ϱ_1 corresponding to the identity in $Gal(K^{ab}/K)$.
Using Proposition 1.8 and the reciprocity law (1.73), we can calculate the action of ν on $\varrho_\omega(f_K)$ as follows

$$\begin{aligned} {}^{\nu}\varrho_1(f_K) &\overset{\mathrm{def}}{=} \varrho_1({}^{\nu}f_K) \overset{(1.106)}{=} \widetilde{f}(\overline{\beta}^{-1}, \overline{\beta}, \overline{\alpha}^{-1}x_{cm}) \overset{1.8}{=} {}^{\overline{\eta}(\beta)}f(\overline{\alpha}^{-1}x_{cm}) \\ &\overset{(1.67)}{=} {}^{\overline{\eta}(\alpha)\overline{\eta}(\beta)}f(x_{cm}) = {}^{\overline{\eta}(\nu)}f(x_{cm}) \overset{(1.73)}{=} [\nu]^{-1}(f(x_{cm})) \\ &= [\nu]^{-1}(\varrho_1(f_K)) \end{aligned}$$

This is precisely the intertwining property we wanted to prove.

Property (v)

Using the notation from above, we conclude immediately that by construction and Theorem 1.6 we have

$$\varrho_1(f_K) = \widetilde{f}(1, 1, x_{xm}) = f(x_{cm}) \in K^c \subset K^{ab} \tag{1.111}$$

and the above calculation shows further that

$$\varrho_\omega(f_K) = {}^{[\omega]}\varrho_1(f_K) = [\omega]^{-1}(f(x_{cm})) \in K^c \subset K^{ab}, \tag{1.112}$$

finishing our proof.

Remark 1.24. We want to conclude this chapter with a short discussion comparing our construction with the original construction of Connes, Marcolli and Ramachdran (see [CMR05]) in the case of an imaginary quadratic field K. Apart from the fact that we are not dealing with the "K-lattice" picture as done in [CMR05], the main difference lies in the different definitions of symmetries. If the class number h_K of K is equal to one, it is immediate that the two definitions agree, however for $h_K > 1$ their symmetries contain endomorphisms (see Prop. 2.17 [CMR05]) whereas our symmetries are always given by automorphisms. We want to mention that it is no problem to generalize (this is already contained in [CM08]) their definition to the context of a BC-system for an arbitrary number field and, without changing the definition of our arithmetic subalgebra, we could have proved Theorem 1.1 by using the new definition of symmetries (now containing endomorphisms).
This might look odd at first sight but is explained in Appendix E.

Chapter 2

On arithmetic models of BC-systems

In this chapter, we show the existence of arithmetic models of Bost-Connes systems for arbitrary number fields, which was an open problem before this thesis going back to the work of Bost and Connes [BC95] and has first been stated explicitely in the paper of Connes, Marcolli and Ramachandran [CMR05].

The starting point of our investigations was the observation that the classical BC-system $\mathcal{A}_{\mathbb{Q}}$ can be described in the context of endomotives, introduced by Connes, Consani and Marcolli [CCM07], and the theory of Λ-rings, i.e., rings with a commuting family of Frobenius lifts as extra structure. This was already observed by Marcolli in [Mar09]. We will show in this chapter that this approach is in fact the correct one for the general case. An elegant classification result of Borger and de Smit [BdS11] of certain Λ-rings in terms of the Deligne-Ribet monoid paves the way for the case of arbitrary number fields.

More precisely, for every number field K, the results of [BdS11] allow us to construct an algebraic endomotive (cf., 2.6.1)

$$\mathcal{E}_K = E_K \rtimes I_K$$

over K, where the K-algebra E_K is a direct limit $\varinjlim E_{\mathfrak{f}}$ of finite, étale K-algebras $E_{\mathfrak{f}}$ which come from a refined Grothendieck-Galois correspondence in terms of the Deligne-Ribet monoid DR_K (see Corollary 2.7). The monoid of (non-zero) integral ideals I_K of K is acting by Frobenius lifts on E_K.
In general, there is a functorial way of attaching to an algebraic endomotive $\mathcal{E}$ a C^*-algebra $\mathcal{E}^{an}$ containing $\mathcal{E}$, which is called the analytic endomotive of $\mathcal{E}$. Moreover, in good situations, $\mathcal{E}$ determines naturally a time evolution $\sigma : \mathbb{R} \to Aut(\mathcal{E}^{an})$ on $\mathcal{E}^{an}$ by means of Tomita-Takesaki theory, so we end up with a C^*-dynamical system

$$\mathcal{E}^{mean} = (\mathcal{E}^{an}, \sigma_t)$$

depending only on $\mathcal{E}$ called the measured analytic endomotive of $\mathcal{E}$ (cf., section 2.2). Our first main result will be

Theorem 2.1. *For every number field K, the measured analytic endomotive $\mathcal{E}_K^{mean}$ of the algebraic endomotive $\mathcal{E}_K$ exists, and is in fact naturally isomorphic to the BC-system $\mathcal{A}_K$.*

The key observations for proving this theorem are Proposition 2.11 which shows that the Deligne-Ribet monoid DR_K is naturally isomorphic to $\widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$ and Proposition 2.19 which shows that the time evolutions of both systems agree.

The most important result of this thesis is to show that all $\mathcal{A}_K$ possess an arithmetic model.

Theorem 2.2. *For all number fields K, the BC-systems $\mathcal{A}_K$ (resp. $\mathcal{E}_K^{mean}$) possess an arithmetic model, with arithmetic subalgebra given by the algebraic endomotive $\mathcal{E}_K = E_K \rtimes I_K$ (cf., (0.3)).*

The proof of this theorem relies on the fact that the algebras $E_{\mathfrak{f}}$ defining the algebraic endomotive $\mathcal{E}_K$ are finite products of strict ray class fields of K (cf., (2.24)). In particular our main result shows that the class field theory of an arbitrary number field can be realized through the dynamics of an operator algebra.

In the appendix 2.9, Sergey Neshveyev has shown moreover that under very natural conditions, satisfied by our arithmetic subalgebra, the arithmetic model of a BC-system is in fact unique, see Theorem 2.22 and 2.24.

Outline

Before we explain and perform our construction of arithmetic subalgebras in form of the algebraic endomotives $\mathcal{E}_K$, we will briefly recall the definition and properties of the systems $\mathcal{A}_K$ of Ha and Paugam, present the theory of endomotives to an extent sufficient for our applications and explain then in some detail the Deligne-Ribet monoid DR_K, which will be an object of central importance for the construction of the algebraic endomotives $\mathcal{E}_K$, and the classification result of Borger and de Smit.

Notations and Conventions

K will always denote a number field with ring of integers $\mathcal{O}_K$. Further, we fix an embedding $K \subset \mathbb{C}$, and consider the algebraic closure $\overline{K}$ of K in $\mathbb{C}$. The maximal abelian algebraic extension of K is denoted by K^{ab}. By I_K we denote the monoid of (non-zero) integral ideals of $\mathcal{O}_K$ and by J_K the group of fractional ideals of K. As usual, we write $\mathbb{A}_K = \mathbb{A}_{K,f} \times \mathbb{A}_{K,\infty}$ for the adele ring of K, with $\mathbb{A}_{K,f}$ the finite adeles, and $\mathbb{A}_{K,\infty}$ the infinite adeles. If R is a ring, we denote by $R^\times$ its group of invertible elements. Invertible adeles are called ideles. By $\widehat{\mathcal{O}}_K \subset \mathbb{A}_K$ we denote the finite, integral adeles of K, further we set $\widehat{\mathcal{O}}_K^\natural = \mathbb{A}_{K,f}^\times \cap \widehat{\mathcal{O}}_K$. We denote Artin's reciprocity map by $[\cdot]_K : \mathbb{A}_K^\times \to \mathrm{Gal}(K^{ab}/K)$. Usually we omit the subscript $_K$ and write only $[\cdot]$. Moreover we denote the idele norm by $N_{K/\mathbb{Q}} : \mathbb{A}_{K,f}^\times \to \mathbb{A}_{\mathbb{Q},f}^\times$ which induces in particular the norm maps $N_{K/\mathbb{Q}} : J_K = \mathbb{A}_{K,f}^\times/\widehat{\mathcal{O}}_K^\times \to \mathbb{Q}$ and $N_{K/\mathbb{Q}} : I_K = \widehat{\mathcal{O}}_K^\natural/\widehat{\mathcal{O}}_K^\times \to \mathbb{Z}$. Also, we use the delta function $\delta_{a,b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$. Finally, we denote the cardinality of a set X by $|X|$.

2.1 BC-systems

Let us recall the definition of the C^*-dynamical systems $\mathcal{A}_K$ and some of its properties, following [LLN09]. Consider the topological space

$$Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) \tag{2.1}$$

defined as the quotient space of the direct product $\widehat{\mathcal{O}}_K \times \mathrm{Gal}(K^{ab}/K)$ under the action of $\widehat{\mathcal{O}}_K^\times$ given by

$$s \cdot (\rho, \alpha) = (\rho s, [s]^{-1}\alpha)$$

There are two natural actions on Y_K. On the one hand, the monoid $I_K \cong \widehat{\mathcal{O}}_K^\natural / \widehat{\mathcal{O}}_K^\times$ of (non-zero) integral ideals of K acts by

$$s \cdot [\rho, \alpha] = [\rho s, [s]^{-1}\alpha],$$

and, on the other hand, the maximal abelian Galois group $\mathrm{Gal}(K^{ab}/K)$ acts by

$$\gamma \cdot [\rho, \alpha] = [\rho, \gamma\alpha].$$

The first action gives rise to the semigroup crossed product C^*-algebra

$$A_K = C(Y_K) \rtimes I_K, \tag{2.2}$$

and together with the time evolution defined by

$$\sigma_t(fu_s) = \mathcal{N}_{K/\mathbb{Q}}(s)^{it} f u_s, \tag{2.3}$$

where $f \in C(Y_K)$ and u_s the isometry encoding the action of $s \in I_K$, we end up with the BC-system of K in form of the C^*-dynamical system

$$\mathcal{A}_K = (A_K, \sigma_t). \tag{2.4}$$

Moreover, the action of the Galois group $\mathrm{Gal}(K^{ab}/K)$ on Y_K induces naturally a map

$$\mathrm{Gal}(K^{ab}/K) \longrightarrow \mathrm{Aut}(\mathcal{A}_K)$$

Later we will need the classification of extremal σ-KMS_β-states, as given elegantly in [LLN09], at $\beta = 1$ and $\beta = \infty$. The approach of [LLN09] relates KMS_β-states of $\mathcal{A}_K$ to measures on Y_K with certain properties. We recommend the reader to consult their paper.

2.1.1 Classification at $\beta = 1$

In the proof of [LLN09] Theorem 2.1 it is shown that the unique KMS_1-state of $\mathcal{A}_K$ corresponds to the measure μ_1 on Y_K which is given by the push-forward (under the natural projection) of the product measure

$$\prod_{\mathfrak{p}} \mu_{\mathfrak{p}} \times \mu_{\mathcal{G}}$$

on $\widehat{\mathcal{O}}_K \times \mathrm{Gal}(K^{ab}/K)$, where $\mu_{\mathcal{G}}$ is the normalized Haar measure on $\mathrm{Gal}(K^{ab}/K)$, and $\mu_{\mathfrak{p}}$ is the additive normalized Haar measure on $\mathcal{O}_{K_{\mathfrak{p}}}$. Equivalently, it is shown that μ_1 is the unique measure on Y_K satisfying $\mu_1(Y_K) = 1$, and the scaling condition

$$\mu_1(gZ) = \mathcal{N}_{K/\mathbb{Q}}(g)^{-1}\mu_1(Z) \tag{2.5}$$

for every Borel subset $Z \subset Y_K$ and $g \in J_K = \mathbb{A}_{K,f}^\times / \widehat{\mathcal{O}}_K^\times$ such that $gZ \subset Y_K$ [1].

1. The action of the group $\mathbb{A}_{K',f}^\times / \widehat{\mathcal{O}}_K^\times$ of fractional ideals of K on Y_K is the obvious one.

2.1.2 Classification at $\beta = \infty$

The set of extremal KMS_∞-states of $\mathcal{A}_K$ is parametrized by the subset $Y_K^\times = \widehat{\mathcal{O}}_K^\times \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$ of Y_K, and for $\omega \in Y_K^\times$ the corresponding extremal KMS_∞-state φ_ω is given by

$$\varphi_\omega(fu_s) = \delta_{s,1} f(\omega).$$

In other words, extremal KMS_∞-states of $\mathcal{A}_K$ correspond to Dirac measures on Y_K with support in $Y_K^\times$.

2.2 Endomotives

We will recall briefly the theory of endomotives, following our main reference [CM08]. Endomotives come in three different flavours: algebraic, analytic and measured analytic. Each aspect could be developed independently, but for our purposes, it is enough to concentrate on algebraic endomotives, and show how to associate an analytic and a measured analytic endomotive to it.

Recall that we fixed an embedding $K \subset \mathbb{C}$ and understand $\overline{K}$ to be the algebraic closure of K in $\mathbb{C}$.

2.2.1 Algebraic flavour

We denote by $\mathfrak{E}_K$ the category of finite dimensional, étale K-algebras with morphisms given by K-algebra homomorphisms. Let $((A_i)_{i\in I}, S)$ be a pair consisting of an inductive system $(A_i)_{i\in I}$ (with transition maps $\xi_{i,j}$ for $i \leq j$) in $\mathfrak{E}_K$ and an abelian semigroup S acting on the inductive limit $A = \varinjlim_i A_i$ by K-algebra endomorphisms. We don't require the action of S to respect the levels A_i or to be unital, so in general $e = \rho(1)$, for $\rho \in S$, will only be an idempotent, i.e., $e^2 = e$. Moreover, we assume that every $\rho \in S$ induces an isomorphism of K-algebras $\rho : A \xrightarrow{\cong} eAe = eA$.

Definition 2.2.1. Let $((A_i), S)$ be a pair like above. Then the associated algebraic endomotive $\mathcal{E}$ is defined to be the associative, unital K-algebra given by the crossed product

$$\mathcal{E} = A \rtimes S$$

The algebraic endomotive $\mathcal{E}$ can be described explicitly in terms of generators and relations by adjoining to A new generators U_ρ and U_ρ^*, for $\rho \in S$, and imposing the relations

$$\begin{array}{lll} U_\rho^* U_\rho = 1, & U_\rho U_\rho^* = \rho(1), & \forall\ \rho \in S \\ U_{\rho_1} U_{\rho_2} = U_{\rho_1\rho_2}, & U_{\rho_2\rho_1}^* = U_{\rho_1}^* U_{\rho_2}^*, & \forall\ \rho_1, \rho_2 \in S \\ U_\rho a = \rho(a) U_\rho, & a U_\rho^* = U_\rho^* \rho(a), & \forall\ \rho \in S, \forall\ a \in A \end{array} \tag{2.6}$$

Lemma 2.3 (Lemma 4.18 [CM08]). *1) The algebra $\mathcal{E}$ is the linear span of the monomials $U_{\rho_1}^* a U_{\rho_2}$, for $a \in A$ and $\rho_1, \rho_2 \in S$.*
2) The product $U_g = U_{\rho_2}^* U_{\rho_1}$ only depends on the ratio ρ_1/ρ_2 in the group completion $\widetilde{S}$ of S.
3) The algebra $\mathcal{E}$ is the linear span of the monomials aU_g, for $a \in A$ and $g \in \widetilde{S}$.

Remark 2.1. Equivalently one can rephrase the theory of algebraic endomotives in the language of Artin motives. Namely, every finite, étale K-algebra B gives rise to a zero-dimensional variety $\mathrm{Spec}(B)$, or in other words, to an Artin motive. This coined the term "endomotive".

2.2.2 Analytic flavour

Given an algebraic endomotive $((A_i), S)$, we obtain a topological space $\mathcal{X}$ defined by the projective limit

$$\mathcal{X} = \varprojlim_i Hom_{K\text{-alg}}(A_i, \overline{K}),$$

which is equipped with the profinite topology, i.e., $\mathcal{X}$ is a totally disconnected compact Hausdorff space [2]. Using $\mathcal{X} \cong Hom_{K\text{-alg}}(\varinjlim A_i, \overline{K}) = Hom_{K\text{-alg}}(A, \overline{K})$, we see in particular that each $\rho \in S$ induces a homeomorphism $\rho : \mathcal{X}^e = \mathrm{Hom}(eA, \overline{K}) \longrightarrow \mathcal{X}$ by $\chi \in \mathcal{X}^e \mapsto \chi \circ \rho \in \mathcal{X}$, where $e = \rho(1)$. In this way, we get an action of S on the abelian C^*-algebra $C(\mathcal{X})$ by endomorphisms

$$\phi(f)(x) = \begin{cases} 0 & \text{if } \chi(e) = 0 \\ f(\chi \circ \rho) & \text{if } \chi(e) = 1 \end{cases} \tag{2.7}$$

and we can consider the semigroup crossed product C^*-algebra (see, e.g., [Lac00] and [LR96])

$$\mathcal{E}^{an} = C(\mathcal{X}) \rtimes S, \tag{2.8}$$

which we define to be the **analytic endomotive** of the algebraic endomotive $((A_i), S)$. Using the embedding $\iota : K \to \mathbb{C}$ we obtain an embedding of commutative algebras $A \hookrightarrow C(\mathcal{X})$ by

$$a \mapsto ev_a : \chi \mapsto \chi(a),$$

and this induces an embedding of algebras

$$\mathcal{E} = A \rtimes S \hookrightarrow C(\mathcal{X}) \rtimes S. \tag{2.9}$$

The algebraic endomotive is said to give an *arithmetic structure* to the analytic endomotive $\mathcal{E}^{an}$.

Galois action

The natural action of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ on $\mathcal{X} = \mathrm{Hom}(A, \overline{K})$ induces an action of $\mathrm{Gal}(\overline{K}/K)$ on the analytic endomotive $\mathcal{E}^{an}$ by automorphisms preserving the abelian C^*-algebra $C(\mathcal{X})$ and fixing the U_ρ and U_ρ^*. Moreover, the action is compatible with pure states on $\mathcal{E}^{an}$ which do come from $C(\mathcal{X})$ in the following sense (see Prop. 4.29 [CM08]). For every $a \in A$, $\alpha \in \mathrm{Gal}(\overline{K}/K)$, and any pure state φ on $C(\mathcal{X})$, we have $\varphi(a) \in \overline{K}$ and

$$\alpha(\varphi(a)) = \varphi(\alpha^{-1}(a)).$$

Moreover, it is not difficult to show (see Prop. 4.30 [CM08]) that in case where all the A_i are finite products of abelian, normal field extensions of K, as in our applications later on, the action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathcal{E}^{an}$ descends to an action of the maximal abelian quotient $\mathrm{Gal}(K^{ab}/K)$.

2. In other words $\mathcal{X}$ is given by the $\overline{K}$-points of the provariety $\varprojlim Spec(A_i)$

2.2.3 Measured analytic flavour

Let us start again with an algebraic endomotive $((A_i), S)$. On every finite space $\mathcal{X}_i = \mathrm{Hom}(A_i, \overline{K})$, we can consider the normalized counting measure μ_i. We call our algebraic endomotive *uniform* if $\mu_i = (\xi_{i,j})_* \mu_j$ for all $i \leq j$. In this case the μ_i give rise to a projective system of measures and induce a propability measure μ, the so-called Prokhorov extension, on $\mathcal{X} = \varprojlim \mathcal{X}_i$ (compare p. 545 [CM08]).

A time evolution

Now, let us write $\varphi = \varphi_\mu$ for the corresponding state on the analytic endomotive $\mathcal{E}^{an} = C(\mathcal{X}) \rtimes S$ given by

$$\varphi(fu_s) = \delta_{s,1} \int_{\mathcal{X}} f d\mu$$

The GNS construction gives us a representation π_φ of $\mathcal{E}^{an}$ on a Hilbert space $\mathcal{H}_\varphi$ (depending only on φ). Further, we obtain a von Neumann algebra $\mathcal{M}_\varphi$ as the bicommutant of the image of π_φ, and, under certain technical assumptions on φ (see pp. 616 [CM08]), the theory of Tomita-Takesaki equips $\mathcal{M}_\varphi$ with a time evolution $\sigma^\varphi : \mathbb{R} \to Aut(\mathcal{M}_\varphi)$, the so-called modular automorphism group. Now, if we **assume** that π_φ is faithful, and moreover, the time evolution σ^φ respects the C^*-algebra $C(\mathcal{X}) \rtimes S \cong \pi_\varphi(C(\mathcal{X}) \rtimes S) \subset \mathcal{M}_\varphi$, we end up with a C^*-dynamical system

$$\mathcal{E}^{mean} = (C(\mathcal{X}) \rtimes S, \sigma^\varphi)$$

which we call a measured analytic endomotive. If it exists, it only depends on the (uniform) algebraic endomotive we started with.

2.3 The Deligne-Ribet monoid

We follow [DR80] and [BdS11] in this section. Recall that I_K denotes the monoid of (nonzero) integral ideals of our number field K. For every $\mathfrak{f} \in I_K$ we define an equivalence relation $\sim_\mathfrak{f}$ on I_K by

$$\mathfrak{a} \sim_\mathfrak{f} \mathfrak{b} :\Leftrightarrow \exists\, x \in K_+^\times \cap (1 + \mathfrak{f}\mathfrak{b}^{-1}) : (x) = \mathfrak{a}\mathfrak{b}^{-1},$$

where $K_+^\times$ denotes the subgroup of totally positive units in K and (x) the fractional ideal generated by x. The quotient

$$DR_\mathfrak{f} = I_K / \sim_\mathfrak{f}$$

is a finite monoid under the usual multiplication of ideals. Moreover, for every $\mathfrak{f} \mid \mathfrak{f}'$ we obtain a natural projection map $f_{\mathfrak{f},\mathfrak{f}'} : DR_{\mathfrak{f}'} \to DR_\mathfrak{f}$ and thus a projective system $(I_\mathfrak{f})_{\mathfrak{f} \in I_K}$ whose limit

$$DR_K = \varprojlim_{\mathfrak{f}} DR_\mathfrak{f} \tag{2.10}$$

is a (topological) monoid [3] that we call the **Deligne-Ribet monoid** of K.

3. We take the profinite topology.

2.3.1 Some properties of DR_K

First, we have to recall some notations. A cycle $\mathfrak{h}$ is given by a product $\prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$ running over all primes of K, where the $n_{\mathfrak{p}}$'s are non-negative integers, with only finitely many of them non zero. Further $n_{\mathfrak{p}} \in \{0, 1\}$ for real primes, and $n_{\mathfrak{p}} = 0$ for complex primes. The finite part $\prod_{\mathfrak{p}\nmid\infty} \mathfrak{p}^{n_{\mathfrak{p}}}$ can be viewed as an element in I_K. Moreover, we write (∞) for the cycle $\prod_{\mathfrak{p} \text{ real}} \mathfrak{p}$.

If we denote by $C_{\mathfrak{f}}$ the (strict) ray class group of K associated with the cycle $\mathfrak{f}(\infty)$, for $\mathfrak{f} \in I_K$, one can show that

$$DR_{\mathfrak{f}}^{\times} = C_{\mathfrak{f}}, \tag{2.11}$$

i.e., the group of invertible elements $DR_{\mathfrak{f}}^{\times}$ can be identified naturally with $C_{\mathfrak{f}}$ (see (2.6) [DR80]).
As an immediate corollary we obtain

$$DR_K^{\times} = \varprojlim_{\mathfrak{f}} C_{\mathfrak{f}} \cong \mathrm{Gal}(K^{ab}/K), \tag{2.12}$$

i.e., using class field theory, we can identify the invertible elements of DR_K with the maximal abelian Galois group of K. Moreover we have the following description

$$DR_{\mathfrak{f}} \cong \coprod_{\mathfrak{d}|\mathfrak{f}} C_{\mathfrak{f}/\mathfrak{d}}, \tag{2.13}$$

where an element $\mathfrak{a} \in C_{\mathfrak{f}/\mathfrak{d}}$ is sent to $\mathfrak{a}\mathfrak{d} \in DR_{\mathfrak{f}}$ (see the bottom of p. 239 [DR80] or [BdS11]).
There is an important map of topological monoids

$$\iota : \widehat{\mathcal{O}}_K \longrightarrow DR_K \tag{2.14}$$

given as follows: For $m_{\mathfrak{f}} \in \mathcal{O}_K/\mathfrak{f}$, we choose a lift $m_{\mathfrak{f}}^{+} \in \mathcal{O}_{K,+}$, and map this to the ideal $(m_{\mathfrak{f}}^{+}) \in DR_{\mathfrak{f}}$. The map ι is then defined by

$$(m_{\mathfrak{f}}) \in \varprojlim_{\mathfrak{f}} \mathcal{O}_K/\mathfrak{f} \cong \widehat{\mathcal{O}}_K \longmapsto ((m_{\mathfrak{f}}^{+})) \in \varprojlim_{\mathfrak{f}} DR_{\mathfrak{f}} = DR_K,$$

which can be shown to be independent of the choice of the liftings (Prop. 2.13 [DR80]).
Let us denote by U_K^{+} the closure of the totally positive units $\mathcal{O}_{K,+}^{\times} = \mathcal{O}_K^{\times} \cap K_{+}^{\times}$ in $\widehat{\mathcal{O}}_K^{\times}$.

Proposition 2.4 (Prop. 2.15 [DR80]). *Let $\rho, \rho' \in \widehat{\mathcal{O}}_K$. Then $\iota(\rho) = \iota(\rho')$ if and only if $\rho = u\rho'$ for some $u \in U_K^{+}$.*

Therefore, it makes sense to speak of ι having kernel U_K^{+}. Moreover, if we denote by $(\rho) \in I(\mathcal{O}_K)$ (resp. $[\rho] \in \mathrm{Gal}(K^{ab}/K)$) the ideal generated by an idele (resp. the image under Artin reciprocity's map), then we have the following:

Proposition 2.5 (Prop. 2.20 and 2.23 [DR80]). *For $\rho \in \widehat{\mathcal{O}}_K^{\natural}$, we have*

$$\iota(\rho) = (\rho)[\rho]^{-1} \in DR_K.$$

In particular, for $\rho \in \widehat{\mathcal{O}}_K^{\times}$, we obtain

$$\iota(\rho) = [\rho]^{-1} \in DR_K^{\times}.$$

Remark 2.2. The reader should keep in mind, that the intersection $I(\mathcal{O}_K) \cap DR_K^{\times}$ is trivial.

2.4 A classification result of Borger and de Smit

The results in this section are based on the unpublished preprint [BdS11] of Borger and de Smit. First we will fix again some notation.

For a prime ideal $\mathfrak{p} \in I_K$, we denote by $\kappa(\mathfrak{p})$ the finite residue field $\mathcal{O}_K/\mathfrak{p}$. The Frobenius endomorphism $Frob_{\mathfrak{p}}$ of a $\kappa(\mathfrak{p})$-algebra is defined by $x \mapsto x^{|\kappa(\mathfrak{p})|}$. An endomorphism f of a $\mathcal{O}_K$-algebra E is called a Frobenius lift (at $\mathfrak{p}$) if $f \otimes 1$ equals $Frob_{\mathfrak{p}}$ on $E \otimes_{\mathcal{O}_K} \kappa(\mathfrak{p})$.

Definition 2.4.1. Let E be a torsion-free $\mathcal{O}_K$-algebra. A Λ_K-structure on E is given by a family of endomorphisms $(f_{\mathfrak{p}})$ indexed by the (non-zero) prime ideals of K, such that for all $\mathfrak{p}, \mathfrak{q}$

1) $f_{\mathfrak{p}} \circ f_{\mathfrak{q}} = f_{\mathfrak{q}} \circ f_{\mathfrak{p}}$
2) $f_{\mathfrak{p}}$ is a Frobenius lift

Definition 2.4.2. A K-algebra E is said to have an *integral* Λ_K-structure if there exists a $\mathcal{O}_K$-algebra $\widetilde{E}$ with Λ_K-structure and an isomorphism $E \cong \widetilde{E} \otimes_{\mathcal{O}_K} K$. In this case, $\widetilde{E}$ is called an integral model of E.

Remark 2.3. The Frobenius-lift property is vacuous for K-algebras. This is why we need to ask for an integral structure.

In [BdS11], Borger and de Smit were able to classify finite, étale K-algebras with integral Λ_K-structure. Their result can be described as an arithmetic refinement of the classical *Grothendieck-Galois correspondence*, which says that the category $\mathfrak{E}_K$ of finite, étale K-algebras is antiequivalent to the category $\mathfrak{S}_{G_K}$ of finite sets equipped with a continuous action of the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ [4]. The equivalence is induced by the contravariant functor $A \mapsto \mathrm{Hom}(A, \overline{K})$.
The first observation is that giving a Λ_K-structure to a finite, étale K-algebra E is the same as giving a monoid map [5]

$$I_K \to End_{\mathfrak{S}_K}(\mathrm{Hom}(A, \overline{K})),$$

so that we end up with an action of the direct product $I_K \times G_K$ on $Hom_K(E, \overline{K})$.
Asking for an integral model of E is much more delicate and is answered beautifuly in [BdS11] by making extensive use of class field theory as follows.

Theorem 2.6 ([BdS11] Theorem 1.2)**.** *Let E be a finite, étale K-algebra with Λ_K-structure. Then E has an integral model if and only if there is an integral ideal $\mathfrak{f} \in I_K$ such that the action of $I_K \times G_K$ on $Hom_K(E, \overline{K})$ factors (necessarily uniquely) through the map $I_K \times \mathrm{Gal}(K^{ab}/K) \longrightarrow DR_{\mathfrak{f}}$ given by the natural projection on the first factor and by the Artin reciprocity map* [6] *on the second factor.*

In particular one obtains the following arithmetic refinement of the classical Grothendieck-Galois correspondence.

Corollary 2.7 ([BdS11])**.** *The functor $\mathfrak{H}_K : E \mapsto \mathrm{Hom}(E, \overline{K})$ induces an antiequivalence*

$$\mathfrak{H}_K : \mathfrak{E}_{\Lambda,K} \longrightarrow \mathfrak{S}_{DR_K} \tag{2.15}$$

4. The morphisms are given by K-algebra homomorphisms resp. G_K-equivariant maps of sets.
5. Recall that I_K is generated as a (multiplicative) monoid by its (non-zero) prime ideals.
6. $G_K \to G_k^{ab} \to C_{\mathfrak{f}} \subset DR_{\mathfrak{f}}$.

between the category $\mathfrak{E}_{\Lambda,K}$ *of finite, étale* K*-algebras with integral* Λ_K*-structure and the category* $\mathfrak{S}_{DR_K}$ *of finite sets equipped with a continuous action of the Deligne-Ribet monoid* DR_K [7]*.*

Note that we will use the same notation $\mathfrak{H}_K$ to denote the induced functor

$$\mathfrak{E}_{ind\text{-}\Lambda,K} \longrightarrow \mathfrak{S}_{pro\text{-}DR_K} \tag{2.16}$$

from the category of inductive systems in $\mathfrak{E}_{\Lambda,K}$ to projective systems in $\mathfrak{S}_{DR_K}$.

2.5 A simple decomposition of the Deligne-Ribet monoid

In this section, we describe an observation on the Deligne-Ribet monoid that will be used later on. First, notice (see (2.5) [DR80]) that for ideals $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{d}$ in I_K we have the simple fact

$$\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b} \Leftrightarrow \mathfrak{d}\mathfrak{a} \sim_{\mathfrak{d}\mathfrak{f}} \mathfrak{d}\mathfrak{b}. \tag{2.17}$$

This allows us to define a DR_K-equivariant embedding

$$\mathfrak{d}\cdot : DR_{\mathfrak{f}} \hookrightarrow DR_{\mathfrak{d}\mathfrak{f}} \ ; \ \mathfrak{a} \mapsto \mathfrak{d}\mathfrak{a}, \tag{2.18}$$

and we can identify $DR_{\mathfrak{f}}$ with its image $\mathfrak{d}DR_{\mathfrak{d}\mathfrak{f}}$. Now taking projective limits, we obtain an injective map

$$\varrho_{\mathfrak{d}} : DR_K \to DR_K \tag{2.19}$$

defined by

$$\varprojlim_{\mathfrak{f}} DR_{\mathfrak{f}} \xrightarrow[\mathfrak{d}\cdot]{\cong} \varprojlim_{\mathfrak{f}} \mathfrak{d}DR_{\mathfrak{d}\mathfrak{f}} \xrightarrow[inc]{} \varprojlim_{\mathfrak{f}} DR_{\mathfrak{d}\mathfrak{f}} \xrightarrow{\cong} \varprojlim_{\mathfrak{f}} DR_{\mathfrak{f}} \tag{2.20}$$

which is in fact just a complicated way of writing the multiplication map

$$\mathfrak{a} \in DR_K \longmapsto \mathfrak{d}\mathfrak{a} \in DR_K. \tag{2.21}$$

We profit from our reformulation in that we see immediately that the image $Im(\varrho_{\mathfrak{d}}) = \varprojlim_{\mathfrak{f}} \mathfrak{d}DR_{\mathfrak{d}\mathfrak{f}}$ is a closed subset of DR_K. Also, using (2.17), we see that the complement of $Im(\varrho_{\mathfrak{d}})$ in DR_K is closed, and therefore we obtain for every $\mathfrak{d} \in I_K$ a (topological) decomposition

$$DR_K = Im(\varrho_{\mathfrak{d}}) \sqcup Im(\varrho_{\mathfrak{d}})^c. \tag{2.22}$$

2.6 The endomotive $\mathcal{E}_K$

For every number field K, we want to construct an algebraic endomotive $\mathcal{E}_K$.
The correspondence (2.15) tells us that for every $\mathfrak{f} \in I_K$ there exists a finite, étale K-algebra $E_{\mathfrak{f}}$ with integral Λ_K-structure such that

$$DR_{\mathfrak{f}} \cong \mathrm{Hom}(E_{\mathfrak{f}}, \overline{K}) = \mathrm{Hom}(E_{\mathfrak{f}}, K^{ab}) \tag{2.23}$$

7. The morphisms are given by K-algebra homomorphisms respecting the integral Λ_K-structure resp. by DR_K-equivariant maps of finite sets.

More precisely the decomposition (2.13) shows that we have in fact

$$E_{\mathfrak{f}} \cong \prod_{\mathfrak{d}|\mathfrak{f}} K_{\delta}, \tag{2.24}$$

where K_δ denotes the (strict) ray class field associated with the cycle $\mathfrak{d}(\infty)$. Moreover, the transition maps of the projective system $(DR_{\mathfrak{f}})$ are equivariant with respect to the action of DR_K so that we obtain an inductive system $(E_{\mathfrak{f}})$ in $\mathfrak{E}_{\Lambda,K}$, i.e., we obtain a natural action of I_K on the commutative K-algebra

$$E_K = \varinjlim_{\mathfrak{f}} E_{\mathfrak{f}}, \tag{2.25}$$

given by Frobenius lifts which we denote, for $\mathfrak{d} \in I_K$, by

$$\sigma_{\mathfrak{d}} \in End_{ind\text{-}\mathfrak{E}_{\Lambda,K}}(E_K). \tag{2.26}$$

By construction we have, for every $\mathfrak{d} \in I_K$, the equality

$$\mathfrak{H}_K(\sigma_{\mathfrak{d}}) = \varrho_{\mathfrak{d}}.$$

On the other hand, the decomposition (2.22) shows the existence [8] of an idempotent element $\pi_{\mathfrak{d}}$ in E_K, for every $\mathfrak{d} \in I_K$, such that $Im(\varrho_{\mathfrak{d}}) = \mathrm{Hom}(\pi_{\mathfrak{d}} E_K, \overline{K})$, or in other words

$$E_K = \pi_{\mathfrak{d}} E_K \oplus (1 - \pi_{\mathfrak{d}}) E_K. \tag{2.27}$$

The projections satisfy the following basic properties.

Lemma 2.8. *For all $\mathfrak{d}, \mathfrak{e}$ in I_K we have*

$$\pi_{\mathfrak{d}} \pi_{\mathfrak{e}} = \pi_{lcm(\mathfrak{d},\mathfrak{e})} \tag{2.28}$$

Further, if $\mathfrak{d}$ divides $\mathfrak{f}$ we have

$$\pi_{\mathfrak{d}} \pi_{\mathfrak{f}} = \pi_{\mathfrak{d}} \tag{2.29}$$

Proof. The first assertion follows from the second together with Lemma 2.9. The second assertion follows from the fact that $\mathfrak{f} \cdot DR_K \subset \mathfrak{d} \cdot DR_K$ (see section 2.5). □

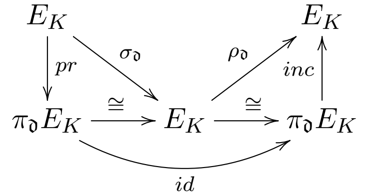
Invoking the refined Grothendieck-Galois correspondence, we define for every $\mathfrak{d} \in I_K$ the endomorphism $\rho_{\mathfrak{d}} \in End(E_K)$ by

$$\rho_{\mathfrak{d}} = i \circ \mathfrak{H}_K^{-1}(\varrho_{\mathfrak{d}}^{-1} : Im(\varrho_{\mathfrak{d}}) \xrightarrow{\cong} DR_K) \tag{2.30}$$

where $i : \pi_{\mathfrak{d}} E_K \to E_K$ denotes the natural inclusion.

Remark 2.4. The reader should be aware of the fact that the $\rho_{\mathfrak{d}}$ are not level preserving like the $\sigma_{\mathfrak{d}}$, in the sense that the latter restricts to a map $E_{\mathfrak{f}} \to E_{\mathfrak{f}}$.

Let us give a schematic overview in the form of the following everywhere commutative diagram



(2.31)

The following relations hold by construction.

8. Because $\mathrm{Hom}(A \oplus B, \overline{K}) = \mathrm{Hom}(A, \overline{K}) \sqcup \mathrm{Hom}(B, \overline{K})$.

Lemma 2.9. *For all $\mathfrak{d}, \mathfrak{e}$ in I_K and every $x \in E_K$ we have*

$$\begin{gathered}
\rho_{\mathfrak{d}}(1) = \pi_{\mathfrak{d}}, \\
\sigma_{\mathfrak{d}} \circ \sigma_{\mathfrak{e}} = \sigma_{\mathfrak{d}\mathfrak{e}}, \qquad \rho_{\mathfrak{d}} \circ \rho_{\mathfrak{e}} = \rho_{\mathfrak{d}\mathfrak{e}}, \\
\rho_{\mathfrak{d}} \circ \sigma_{\mathfrak{d}}(x) = \pi_{\mathfrak{d}} x, \quad \sigma_{\mathfrak{d}} \circ \rho_{\mathfrak{d}}(x) = x
\end{gathered}$$

Now we can define our desired algebraic endomotive.

Definition 2.6.1. The algebraic endomotive $\mathcal{E}_K$ is given by the inductive system $(E_{\mathfrak{f}})_{\mathfrak{f} \in I_K}$ together with the action of I_K on $E_K = \varinjlim E_{\mathfrak{f}}$ by means of the ρ_δ.

Remark 2.5. It might be interesting to construct an integral version of our endomotive, as done in [CCM09] in the case of $K = \mathbb{Q}$. The integrality of the $A_{\mathfrak{f}}$ should make this possible.

2.7 Proof of Theorem 2.1 and 2.2

Theorem 2.10. *The algebraic endomotive $\mathcal{E}_K$ gives rise to a C^*-dynamical system that is naturally isomorphic to the BC-system $\mathcal{A}_K$ (see* (2.4)*).*

We will prove the theorem in two steps.

2.7.1 Step One

For every number field K there is a natural map of topological monoids

$$\Psi : Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) \longrightarrow DR_K$$

given by

$$[\rho, \alpha] \longmapsto \iota(\rho)\alpha^{-1}.$$

This map is well defined due to the fact that $\iota(s) = [s]^{-1} \in \mathrm{Gal}(K^{ab}/K)$ for $s \in \widehat{\mathcal{O}}_K^\times$.

Proposition 2.11. *The map Ψ is an equivariant isomorphism of topological monoids with respect to the natural actions of I_K and* $\mathrm{Gal}(K^{ab}/K)$*.*

Proof. It is enough to show that the map

$$\Psi_{\mathfrak{f}} : \mathcal{O}_K/\mathfrak{f} \times_{(\mathcal{O}_K/\mathfrak{f})^\times} C_{\mathfrak{f}} \longmapsto DR_{\mathfrak{f}}, \tag{2.32}$$

given by

$$[\rho, \alpha] \mapsto \iota_{\mathfrak{f}}(\rho)\alpha^{-1},$$

is an isomorphism of finite monoids for every $\mathfrak{f} \in I_K$. This follows from the compactness of $Y_{K,\mathfrak{f}} = \mathcal{O}_K/\mathfrak{f} \times_{(\mathcal{O}_K/\mathfrak{f})^\times} C_{\mathfrak{f}}$ and the simple fact that $\varprojlim_{\mathfrak{f}} Y_{K,\mathfrak{f}} \cong Y_K$. Denote by π_0 the group of connected components of the infinite idele group $(\mathbb{A}_{K,\infty})^\times$ and consider, for every $\mathfrak{f} \in I_K$, the following everywhere commutative and exact diagram

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & \uparrow & & \\
\pi_0 \times (\mathcal{O}_K/\mathfrak{f})^\times & \longrightarrow & C_{\mathfrak{f}} & \longrightarrow & C_K & \longrightarrow & 1 \\
\uparrow & & \uparrow = & & \uparrow & & \\
(\mathcal{O}_K/\mathfrak{f})^\times & \xrightarrow{j_{\mathfrak{f}}} & C_{\mathfrak{f}} & \longrightarrow & C_1 & \longrightarrow & 1 \\
 & & & & \uparrow & & \\
 & & & & \pi_0 & &
\end{array} \tag{2.33}$$

as can be found for example in [Neu99]. From F.1 we know that $\mathcal{O}_K/\mathfrak{f}$ and $\coprod_{\mathfrak{d}|\mathfrak{f}}(\mathcal{O}_K/\mathfrak{d})^\times$ are isomorphic as sets, but they are in fact isomorphic as monoids:

Lemma 2.12. *There is an isomorphism of monoids $\sigma_\mathfrak{f} : \mathcal{O}_K/\mathfrak{f} \to \coprod_{\mathfrak{d}|\mathfrak{f}}(\mathcal{O}_K/\mathfrak{d})^\times$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{O}_K/\mathfrak{f} & \xrightarrow{\iota_\mathfrak{f}} & DR_\mathfrak{f} \\ {\scriptstyle \sigma_\mathfrak{f}}\downarrow & & \downarrow \\ \coprod_{\mathfrak{d}|\mathfrak{f}}(\mathcal{O}_K/(\mathfrak{f}/\mathfrak{d}))^\times & \xrightarrow{\coprod j_{\mathfrak{f}/\mathfrak{d}}} & \coprod_{\mathfrak{d}|\mathfrak{f}} C_{\mathfrak{f}/\mathfrak{d}} \end{array} \tag{2.34}$$

Proof. It is enough to consider the case $\mathfrak{f} = \mathfrak{p}^k$ where $\mathfrak{p}$ a prime ideal. The general case follows using the chinese reminder theorem. It is well known that $\mathcal{O}_K/\mathfrak{p}^k$ is a local ring with maximal ideal $\mathfrak{p}/\mathfrak{p}^k$, i.e. we have a disjoint union $\mathcal{O}_K/\mathfrak{p}^k = (\mathcal{O}_K/\mathfrak{p}^k)^\times \sqcup \mathfrak{p}/\mathfrak{p}^k$. Further, there is a filtration $\{0\} \subset \mathfrak{p}^{k-1}/\mathfrak{p}^k \subset \mathfrak{p}^{k-2}/\mathfrak{p}^k \subset \ldots \subset \mathfrak{p}/\mathfrak{p}^k$ and, for $x \in \mathfrak{p}/\mathfrak{p}^k$ and $x_+ \in \mathcal{O}_K$ a (positive) lift, we have

$$x \in \mathfrak{p}^{k-i}/\mathfrak{p}^k - \mathfrak{p}^{k-i+1}/\mathfrak{p}^k \Leftrightarrow \mathfrak{p}^{k-i} \mid\mid (x_+) \Leftrightarrow x_+ \in (\mathcal{O}_K/\mathfrak{p}^{k-i+1})^\times$$

A counting argument as in F.1, and recalling the definition of (2.13), finishes the proof. □

Now, we can conclude the **injectivity** of $\Psi_\mathfrak{f}$, because assuming $\iota_\mathfrak{f}(\rho)\alpha^{-1} = \iota_\mathfrak{f}(\sigma)\beta^{-1}$, for $\rho, \sigma \in \mathcal{O}_K/\mathfrak{f}$, $\alpha, \beta \in C_\mathfrak{f}$, we must have that α and β map to the same element in C_1. This is, because $\iota_\mathfrak{f}(\sigma)\alpha\beta^{-1}$ lies in the image of $\iota_\mathfrak{f}$ and is therefore mapped to the trivial element in C_1. But lying over the same element in C_1 means that there exists $s \in (\mathcal{O}_K/\mathfrak{f})^\times$ such that $\alpha\beta^{-1} = [s] = \iota_\mathfrak{f}(s)^{-1}$, and therefore, we get $[\rho, \alpha] = [\sigma, \beta] \in Y_{K,\mathfrak{f}}$.
To prove **surjectivity**, we use again the decomposition $DR_\mathfrak{f} = \coprod_{\mathfrak{d}|\mathfrak{f}} C_{\mathfrak{f}/\mathfrak{d}}$. We have to show that for every $\mathfrak{d} \mid \mathfrak{f}$ we have $C_\mathfrak{f} \cdot Im(j_\mathfrak{d}) = C_{\mathfrak{f}/\mathfrak{d}}$, where $\cdot$ denotes the multiplication in the monoid $DR_\mathfrak{f}$. One has to be careful because it is not true that $C_\mathfrak{f}$ acts transitively on $Im(j_\mathfrak{d})$.[9] Instead, we show that $C_\mathfrak{f} \cdot \mathfrak{d}$ intersects every fibre of $C_{\mathfrak{f}/\mathfrak{d}} \to C_1$ non-trivially. For every element $x \in C_1$ we find lifts $x_\mathfrak{f} \in C_\mathfrak{f}$ and $x_{\mathfrak{f}/\mathfrak{d}} \in C_{\mathfrak{f}/\mathfrak{d}}$ such that $x_\mathfrak{f}$ is mapped to $x_{\mathfrak{f}/\mathfrak{d}}$ under the natural projection $DR_\mathfrak{f} \to DR_{\mathfrak{f}/\mathfrak{d}}$. Our claim is equivalent to $x_\mathfrak{f}\mathfrak{d} \sim_\mathfrak{f} x_{\mathfrak{f}/\mathfrak{d}}\mathfrak{d}$, which is equivalent (see (2.17)) to $x_\mathfrak{f} \sim_{\mathfrak{f}/\mathfrak{d}} x_{\mathfrak{f}/\mathfrak{d}}$, which is is true by construction.
To finish the proof, we have to show that Ψ is compatible with the natural actions of I_K and $\mathrm{Gal}(K^{ab}/K)$ on Y_K and DR_K respectively. Let us recall that the action of $I_K \cong \widehat{\mathcal{O}}_K^\natural/\widehat{\mathcal{O}}_K^\times$ on Y_K is given by $s[\rho, \alpha] = [\rho s, [s]^{-1}\alpha]$, and $\mathrm{Gal}(K^{ab}/K)$ is acting by $\gamma[\rho, \alpha] = [\rho, \gamma\alpha]$. The equivariance of Ψ under the action of $\mathrm{Gal}(K^{ab}/K)$ is clear, and the equivariance under the action of I_K follows from Proposition 2.5, namely $\Psi(s[\rho, \alpha]) = \iota(\rho)\iota(s)[s]\alpha^{-1} \overset{2.5}{=} \iota(\rho)(s)\alpha^{-1} = (s)\Psi([\rho, \alpha])$. This shows that Ψ is an isomorphism of topological DR_K-monoids. □

Now we obtain immediately:

Corollary 2.13. *Let K be a number field. Then the isomorphism Ψ from above induces an isomorphism*

$$\Psi : A_K = C(Y_K) \rtimes I_K \longrightarrow \mathcal{E}_K^{an} = C(DR_K) \rtimes I_K \tag{2.35}$$

between the C^*-algebra A_K of the BC-system $\mathcal{A}_K$ and the analytic endomotive $\mathcal{E}_K^{an}$.

9. Consider for the example the case when $gcd(\mathfrak{d}, \mathfrak{f}/\mathfrak{d}) = 1$, then $\mathfrak{d}, \mathfrak{d}^2 \in Im(j_\mathfrak{d})$ but $\mathfrak{d}^2 \notin C_\mathfrak{f} \cdot \mathfrak{d}$

2.7.2 Step Two

It remains to show that $\mathcal{E}_K$ defines a measured analytic endomotive whose time evolution on $\mathcal{E}_K^{an}$ agrees with the time evolution of the BC-system $\mathcal{A}_K$ (see (2.4)).

First, we will show that $\mathcal{E}_K$ is a uniform endomotive, i.e., the normalized counting measures $\mu_\mathfrak{f}$ on $DR_\mathfrak{f}$ give rise to a measure $\mu_K = \varprojlim \mu_\mathfrak{f}$ on $DR_K = \mathrm{Hom}(E_K, \overline{K})$.
Then, in order to show that μ_K indeed defines a time evolution on $\mathcal{E}_K^{an}$ using the procedure described in section 2.2.3 which, in addition, agrees with the time evolution of $\mathcal{A}_K$, we only have to show that μ_K equals the measure μ_1 on Y_K characterizing the unique KMS_1-state of $\mathcal{A}_K$ (see section 2.1.1).
This follows from standard arguments in Tomita-Takesaki theory. Namely, if μ_K defines a time evolution σ_t on $\mathcal{E}_K^{an}$, then we know a priori that the corresponding state $\varphi_{\mu_K} : \mathcal{E}_K^{an} \to \mathbb{C}$ is a KMS_1-state characterizing the time evolution σ_t uniquely (cf., chapter 4 4.1 [CM08] and the references therein).

Lemma 2.14. *Let $\mathfrak{f}$ be an arbitrary ideal in I_K. Then we have*

$$|DR_\mathfrak{f}| = 2^{r_1} h_K N_{K/\mathbb{Q}}(\mathfrak{f}) \tag{2.36}$$

where h_K denotes the class number of K and r_1 is equal to the real embeddings of K.

Proof. Recall the fundamental exact sequence of groups (see e.g. [Neu99])

$$1 \longrightarrow U_\mathfrak{f} \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times \xrightarrow{j_\mathfrak{f}} C_\mathfrak{f} \longrightarrow C_1 \longrightarrow 1 \tag{2.37}$$

with notations as in (2.33) and $U_\mathfrak{f}$ making the sequence exact, from which we obtain immediately

$$|C_\mathfrak{f}| = \frac{2^{r_1}\varphi_K(\mathfrak{f}) h_K}{|U_\mathfrak{f}|} \tag{2.38}$$

where φ_K denotes the generalized Euler totient function from Appendix F. In order to count the elements of DR_K we notice (cf., Prop. 2.4) that the fibers of the natural projection $\mathcal{O}_K/\mathfrak{f} \times C_\mathfrak{f} \to \mathcal{O}_K/\mathfrak{f} \times_{(\mathcal{O}_K/\mathfrak{f})^\times} C_\mathfrak{f} \cong DR_\mathfrak{f}$ all have the same cardinality given by $\frac{\varphi_K(\mathfrak{f})}{|U_\mathfrak{f}|}$ and this finishes the proof. □

Lemma 2.15. *Let $\mathfrak{f}$ and $\mathfrak{g}$ be in I_K such that $\mathfrak{f}$ divides $\mathfrak{g}$. Then the cardinalities of all the fibres of the natural projection $DR_\mathfrak{g} \to DR_\mathfrak{f}$ are equal to $|DR_\mathfrak{g}|/|DR_\mathfrak{f}| = N_{K/\mathbb{Q}}(\mathfrak{g}/\mathfrak{f})$.*

Proof. To show that all the cardinalities of the fibers of the projection $DR_\mathfrak{g} \to DR_\mathfrak{f}$ are equal, we look at the following commutative diagram (with the obvious maps)

$$\begin{array}{ccc} \mathcal{O}_K/\mathfrak{g} \times C_\mathfrak{g} & \longrightarrow & \mathcal{O}_K/\mathfrak{g} \times_{(\mathcal{O}_K/\mathfrak{g})^\times} C_\mathfrak{g} \\ \downarrow & & \downarrow \xi \\ \mathcal{O}_K/\mathfrak{f} \times C_\mathfrak{f} & \longrightarrow & \mathcal{O}_K/\mathfrak{f} \times_{(\mathcal{O}_K/\mathfrak{f})^\times} C_\mathfrak{f} \end{array} \tag{2.39}$$

All the maps in the diagram are surjective, and in order to show that the cardinalities of all the fibers of ξ are equal, it is enough to show this property for the other three maps. In the proof of the preceding lemma, we have shown that the horizontal maps have this property, and for the remaining vertical map on the left, this property is trivial. Therefore, we conclude that the cardinalities of all the fibers of ξ are equal and, together with the isomorphism (2.32) and the preceding lemma, the assertion follows. □

Corollary 2.16. *The algebraic endomotive $\mathcal{E}_K$ is uniform.*

Proof. Let $\mathfrak{f}, \mathfrak{g} \in I_K$ with $\mathfrak{f} \mid \mathfrak{g}$, and denote by ξ the natural projection $DR_{\mathfrak{g}} \to DR_{\mathfrak{f}}$. In order to show that $\mathcal{E}_K$ is uniform, we have to show that $\xi_*(\mu_{\mathfrak{g}}) = \mu_{\mathfrak{f}}$, which follows directly from the preceding lemma. More precisely, if we take a subset $X \subset DR_K$, we obtain

$$\xi_*(\mu_{\mathfrak{g}})(X) = \mu_{\mathfrak{g}}(\xi^{-1}(X)) \overset{2.15}{=} |X| \cdot N_{K/\mathbb{Q}}(\mathfrak{g}/\mathfrak{f})/|DR_{\mathfrak{g}}| \overset{2.15}{=} |X|/|DR_{\mathfrak{f}}| = \mu_{\mathfrak{f}}(X)$$

□

Lemma 2.17. *Denote by $\widetilde{\mu}_{\mathfrak{f}}$ the push-forward of μ_1 under the projection $\pi_{\mathfrak{f}} : Y_K \xrightarrow{\Psi} DR_K \longrightarrow DR_{\mathfrak{f}}$. Then $\widetilde{\mu}_{\mathfrak{f}}$ is the normalized counting measure on $DR_{\mathfrak{f}}$.*

Proof. We only have to show that

$$\widetilde{\mu}_{\mathfrak{f}}(q) = \widetilde{\mu}_{\mathfrak{f}}(q') \text{ for all } q, q' \in DR_{\mathfrak{f}},$$

because by definition we have $1 = \widetilde{\mu}_{\mathfrak{f}}(DR_{\mathfrak{f}}) = \sum_q \widetilde{\mu}_{\mathfrak{f}}(q)$. Recall that μ_1 is defined to be the push forward of the product measure $\mu = \prod_{\mathfrak{p}} \mu_{\mathfrak{p}} \times \mu_{\mathcal{G}}$ on $\widehat{\mathcal{O}}_K \times \mathrm{Gal}(K^{ab}/K)$, where the $\mu_{\mathfrak{p}}$ and $\mu_{\mathcal{G}}$ are normalized Haar measures under the natural projection $\pi : \widehat{\mathcal{O}}_K \times \mathrm{Gal}(K^{ab}/K) \to Y_K$ (cf., section 2.1.1). It is immediate that for given q and q' in I_K, we find $m = m_{q,q'} \in I_K$ and $s = s_{q,q'} \in \mathrm{Gal}(K^{ab}/K)$, such that the translate of $X_q = \pi_{\mathfrak{f}}^{-1}(\pi^{-1}(q))$ under m and s equals $X_{q'}$, i.e.

$$mX_q s := \{(m + \rho, s\alpha) \mid (\rho, \alpha) \in X_q\} = X_{q'}.$$

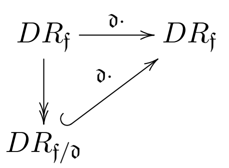
Due to translation invariance of Haar measures we can conclude $\mu(X_q) = \mu(mX_q s) = \mu(X_{q'})$ and therefore

$$\widetilde{\mu}_{\mathfrak{f}}(q) = \widetilde{\mu}_{\mathfrak{f}}(q').$$

□

Lemma 2.18. *The measure $\mu_K = \varprojlim \mu_{\mathfrak{f}}$ satisfies the scaling condition* (2.5)*.*

Proof. Let $\mathfrak{d}$ and $\mathfrak{f}$ be in I_K. Without loss of generality, we can assume that $\mathfrak{d}$ divides $\mathfrak{f}$, because we are looking at the limit measure. Recall further the commutative diagram



(2.40)

In order to show that μ_K satisfies the scaling condition, it is enough to show that the cardinalities of the (non-trivial) fibers of the multiplication map $\mathfrak{d}\cdot : DR_{\mathfrak{f}} \to DR_{\mathfrak{f}}$ are all equal to the norm $N_{K/\mathbb{Q}}(\mathfrak{d}) = |\mathcal{O}_K/\mathfrak{d}|$. By the commutativity of the last diagram, we only have to show that the fibres of the natural projection $DR_{\mathfrak{f}} \to DR_{\mathfrak{f}/\mathfrak{d}}$ all have cardinality $N_{K/\mathbb{Q}}(\mathfrak{d})$. This follows immediately from lemma 2.15. □

As corollary of the last two lemma we obtain the following.

Proposition 2.19. *We have the equality of measures*

$$\mu_K = \mu_1. \tag{2.41}$$

Proof. We have seen that μ_K satisfies the two defining properties of μ_1 (cf., section 2.1.1). □

Corollary 2.20. *The procedure described in 2.2.3 defines a time evolution* $(\sigma_t)_{t\in\mathbb{R}}$ *on* $\mathcal{E}_K^{an}$*, and the resulting measured analytic endomotive* $\mathcal{E}^{mean} = (\mathcal{E}_K^{an}, (\sigma_t)_{t\in\mathbb{R}})$ *is naturally isomorphic to* $\mathcal{A}_K$*, via* Ψ*.*

Next we will show that $\mathcal{E}_K = E_K \rtimes I_K$ provides $\mathcal{A}_K$ with an arithmetic subalgebra. This follows in fact directly from the construction.

Theorem 2.21. *For all number fields K the BC-systems* $\mathcal{A}_K$ *(resp.* $\mathcal{E}_K^{mean}$*) posses an arithmetic model with arithmetic subalgebra given by the algebraic endomotive* $\mathcal{E}_K = E_K \rtimes I_K$*.*

Proof. Recall from section 2.1.2 that extremal KMS_∞-states are indexed by $\mathrm{Gal}(K^{ab}/K) \overset{(2.12)}{\cong} DR_K^\times \subset \mathrm{Hom}(E_K, \overline{K})$, i.e., an extremal KMS_∞-state ϱ_ω for $\omega \in DR_K^\times$ is given on a function $f \in C(DR_K)$ simply by

$$\varrho_\omega(f) = f(\omega).$$

Now, if we an element $ev_a \in E_K \subset C(DR_K)$ which was defined by $ev_a : g \in \mathrm{Hom}(E_K, \overline{K}) \mapsto g(a) \in K^{ab}$ (see (2.23)), we find that

$$\varrho_\omega(ev_a) = ev_a(\omega) = \omega(a) \in K^{ab}, \tag{2.42}$$

and this shows together with the definition of E_K that property (v) from the list of axioms of a Bost-Connes system is valid. In order to show property (vi), we take a symmetry $\nu \in \mathrm{Gal}(K^{ab}/K)$ and simply calculate

$${}^\nu\varrho_\omega(ev_a) = \varrho_\omega({}^\nu ev_a) = {}^\nu ev_a(\omega) = ev_a(\nu^{-1}\circ\omega) = \nu^{-1}(\omega(a)) = \nu^{-1}(\varrho_\omega(ev_a)). \tag{2.43}$$

□

2.8 Outlook

We would like to state some questions and problems which might be interesting for further research.

- As already mentioned above it would be interesting to construct integral models $A_{\mathcal{O}_K}$ of our Bost-Connes systems $\mathcal{A}_K$ (by using integral models of our arithmetic subalgebras) as done in [CCM09] in the case of the classical BC-system for $K = \mathbb{Q}$. In particular, one could investigate whether general BC-systems can be defined over $\mathbb{F}_1$ or some (finite) extensions of $\mathbb{F}_1$ (depending maybe on the roots of unity contained in K).
- In a recent preprint [CC11] Connes and Consani construct p-adic representations of the classical Bost-Connes system $\mathcal{A}_\mathbb{Q}$ using its integral model $A_\mathbb{Z}$. One of the main tools is thereby the classical Witt functor which attaches to a ring its ring of Witt vectors. Borger [Bor08] has introduced a more general framework of Witt functors which are compatible with our arithmetic subalgebras. It might be interesting to construct analogous p-adic representations of general Bost-Connes systems.

- In particular, Connes and Consani [CC11] recover p-adic L-functions in the p-adic representations of $\mathcal{A}_{\mathbb{Q}}$. Using the results of [DR80], it would be interesting to try to recover p-adic L-functions of totally real number fields in the p-adic representations of BC-systems of totally real number fields.
- On the other hand, it seems interesting to ask whether p-adic BC-systems are related to Lubin-Tate theory.

2.9 On uniqueness of arithmetic models. Appendix by Sergey Neshveyev

The goal of this appendix is to show that the endomotive $\mathcal{E}_K$ constructed in this chapter is, in an appropriate sense, the unique endomotive that provides an arithmetic model for the BC-system $\mathcal{A}_K$. We will also give an alternative proof of the existence of $\mathcal{E}_K$.

Assume $\mathcal{E} = E \rtimes S$ is an algebraic endomotive such that the analytic endomotive $\mathcal{E}^{an}$ is $A_K = C(Y_K) \rtimes I_K$. By this we mean that $S = I_K$ and there exists a $\mathrm{Gal}(\overline{K}/K)$- and I_K-equivariant homeomorphism of $Hom_{K\text{-alg}}(E, \overline{K})$ onto $Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$. Then E considered as a K-subalgebra of $C(Y_K)$ has the following properties:

(a) every function in E is locally constant;
(b) E separates points of Y_K;
(c) E contains the idempotents $\rho_{\mathfrak{a}}^n(1)$ for all $\mathfrak{a} \in I_K$ and $n \in \mathbb{N}$;
(d) for every $f \in E$ we have $f(Y_K) \subset K^{ab}$ and the map $f\colon Y_K \to K^{ab}$ is $\mathrm{Gal}(K^{ab}/K)$-equivariant.

Recall that the endomorphism $\rho_{\mathfrak{a}}$ is defined by $\rho_{\mathfrak{a}}(f) = f(\mathfrak{a}^{-1}\cdot)$, with the convention that $\rho_{\mathfrak{a}}(f)(y) = 0$ if $y \notin \mathfrak{a}Y_K$.

Theorem 2.22. *The subalgebra $E_K = \varinjlim E_{\mathfrak{f}}$ of $C(Y_K)$ constructed in this chapter is the unique K-subalgebra of $C(Y_K)$ with properties (a)-(d). It is, therefore, the K-algebra of locally constant K^{ab}-valued $\mathrm{Gal}(K^{ab}/K)$-equivariant functions on Y_K.*

Proof. We have to show that if a K-subalgebra $E \subset C(Y_K)$ satisfies properties (a)-(d), then it contains every locally constant K^{ab}-valued $\mathrm{Gal}(K^{ab}/K)$-equivariant function f. Fix a point $y \in Y_K$. Let $L \subset K^{ab}$ be the field of elements fixed by the stabilizer G_y of y in $\mathrm{Gal}(K^{ab}/K)$. Then $f(y) \in L$ by equivariance.

Lemma 2.23. *The map $E \ni h \mapsto h(y) \in L$ is surjective.*

Proof. Let L' be the image of E under the map $h \mapsto h(y)$. Since E is a K-algebra, L' is a subfield of L. If $L' \neq L$ then there exists a nontrivial element of $\mathrm{Gal}(L/L') \subset \mathrm{Gal}(L/K) = \mathrm{Gal}(K^{ab}/K)/G_y$. Lift this element to an element g of $\mathrm{Gal}(K^{ab}/K)$. Then, on the one hand, $gy \neq y$, and, on the other hand, for every $h \in E$ we have $h(gy) = gh(y) = h(y)$. This contradicts property (b). ☐

Therefore there exists $h \in E$ such that $h(y) = f(y)$. Since the functions f and h are locally constant, there exists a neighbourhood W of y such that f and h coincide on W. We may assume that W is the image of an open set of the form

$$\left(\prod_{v\in F} W_v \times \widehat{\mathcal{O}}_{K,F}\right) \times W' \subset \widehat{\mathcal{O}}_K \times \mathrm{Gal}(K^{ab}/K)$$

in Y_K, where F is a finite set of finite places of K; here we use the notation $\widehat{\mathcal{O}}_K = \prod_{v\in V_{K,f}} \mathcal{O}_{K,v}$, $\widehat{\mathcal{O}}_{K,F} = \prod_{v\in V_{K,f}\setminus F} \mathcal{O}_{K,v}$. Furthermore, we may assume that $F = F' \sqcup F''$ and for $v \in F'$ we have $W_v \subset \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v}^\times$, while for $v \in F''$ we have $W_v = \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v}$. Since the functions f and h are equivariant, they coincide on the set $U = \mathrm{Gal}(K^{ab}/K)W$. The equality

$$\mathrm{Gal}(K^{ab}/K)W = \left(\prod_{v\in F'} \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v}^\times \times \prod_{v\in F''} \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v} \times \widehat{\mathcal{O}}_{K,F}\right) \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$$

shows that the characteristic function p of U belongs to E: it is the product of $\rho_{\mathfrak{p}_v}^{n_v}(1) - \rho_{\mathfrak{p}_v}^{n_v+1}(1)$, $v \in F'$, and $\rho_{\mathfrak{p}_v}^{n_v}(1)$, $v \in F''$. Therefore $fp = hp \in E$.
Thus we have proved that for every point $y \in Y_K$ there exists a neighbourhood U of y such that the characteristic function p of U belongs to E and $fp \in E$. By compactness we conclude that $f \in E$. □

The following consequence of the above theorem shows that the arithmetic subalgebra $\mathcal{E}_K = E_K \rtimes I_K$ of the BC-system is unique within a class of algebras not necessarily arising from endomotives.

Theorem 2.24. *The K-subalgebra $\mathcal{E}_K$ of A_K constructed in this chapter is the unique arithmetic subalgebra that is generated by some locally constant functions on Y_K and by the elements $U_\mathfrak{a}$ and $U_\mathfrak{a}^*$, $\mathfrak{a} \in I_K$.*

Proof. Assume $\mathcal{E}$ is such an arithmetic subalgebra. Consider the K-algebra $E = \mathcal{E} \cap C(Y_K)$. It satisfies properties (a)-(c), while (d) a priori holds only on the subset $Y_K^\times \subset Y_K$. However, the algebra E is invariant under the endomorphisms $\sigma_\mathfrak{a}$, $\mathfrak{a} \in I_K$, defined by $\sigma_\mathfrak{a}(f) = f(\mathfrak{a}\,\cdot) = U_\mathfrak{a}^* f U_\mathfrak{a}$. Hence property (d) holds on the subsets $\mathfrak{a}Y_K^\times$ of Y_K. Since $\cup_{\mathfrak{a}\in I_K}\mathfrak{a}Y_K^\times$ is dense in Y_K and the functions in E are locally constant, it follows that (d) holds on the whole set Y_K. Therefore $E = E_K$ by the previous theorem, and so $\mathcal{E} = \mathcal{E}_K$. □

Let E be the K-algebra of locally constant K^{ab}-valued $\mathrm{Gal}(K^{ab}/K)$-equivariant functions on Y_K. Let us now show directly that $E \rtimes I_K$ is an arithmetic subalgebra of A_K.
In order to prove the density of the $\mathbb{C}$-algebra generated by $E \rtimes I_K$ in A_K, by the Stone-Weierstrass theorem it suffices to check that E separates points of Y_K. Note that E is closed under complex conjugation, since complex conjugation defines an element of $\mathrm{Gal}(K^{ab}/K)$.
Consider two points $y', y'' \in Y_K$. We have a canonical projection $Y_K \to \widehat{\mathcal{O}}_K/\widehat{\mathcal{O}}_K^\times = \prod_{v\in V_{K,f}} \mathcal{O}_{K,v}/\mathcal{O}_{K,v}^\times$, so for $y \in Y_K$ it makes sense to talk about $\mathrm{ord}_v(y)$. Consider two cases:

1) Assume there exists $v \in V_{K,f}$ such that $\mathrm{ord}_v(y') \neq \mathrm{ord}_v(y'')$. We may assume that $\mathrm{ord}_v(y') \neq +\infty$. Then the characteristic function of the set of points $y \in Y_K$ such that $\mathrm{ord}_v(y) = \mathrm{ord}_v(y')$, is in E and separates the points y' and y''.

2) Assume $n_v := \mathrm{ord}_v(y') = \mathrm{ord}_v(y'')$ for all $v \in V_{K,f}$. There exists a finite (possibly empty) subset $F \subset V_{K,f}$ such that the projections of y' and y'' onto $\mathcal{O}_{K,F} \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$ are different, where $\mathcal{O}_{K,F} = \prod_{v\in F} \mathcal{O}_{K,v}$. Replacing F by a smaller set we may further assume that $n_v \neq +\infty$ for all $v \in F$. Consider the points

$$z' = \mathfrak{a}^{-1}y' \quad \text{and} \quad z'' = \mathfrak{a}^{-1}y'', \quad \text{where} \quad \mathfrak{a} = \prod_{v\in F} \mathfrak{p}_v^{n_v}.$$

If we could find a function f in E separating z' and z'', then $\rho_{\mathfrak{a}}(f)$ would separate y' and y''. Therefore we may assume that $z' = y'$, which means $n_v = 0$ for all $v \in F$. In other words, the projections of y' and y'' onto $\mathcal{O}_{K,F} \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$ lie in

$$\mathcal{O}_{K,F}^\times \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) \cong \mathrm{Gal}(K^{ab}/K)/[\widehat{\mathcal{O}}_{K,F}^\times]$$

and define two different points g' and g'' in the latter group. Let $L \subset K^{ab}$ be the subfield of elements fixed by $[\widehat{\mathcal{O}}_{K,F}^\times]$. Take a point $a \in L$ such that $g'a \neq g''a$. We now define a function f separating y' and y'' as follows: on the set $(\mathcal{O}_{K,F}^\times \times \widehat{\mathcal{O}}_{K,F}) \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$ it is the composition of the projection

$$(\mathcal{O}_{K,F}^\times \times \widehat{\mathcal{O}}_{K,F}) \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) \to \mathcal{O}_{K,F}^\times \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) = \mathrm{Gal}(L/K)$$

with the map $\mathrm{Gal}(L/K) \ni g \mapsto ga$, and on the complement it is zero.
The property that K^{ab} is generated by the values $f(y)$, $f \in E$, for any $y \in Y_K^\times$, follows now from Lemma 2.23 as $\mathrm{Gal}(K^{ab}/K)$ acts freely on $Y_K^\times$. It can also be proved by the same argument as in case 2) above, since any point $a \in K^{ab}$ is fixed by $[\widehat{\mathcal{O}}_{K,F}^\times]$ for sufficiently large F. Thus $E \rtimes I_K \subset A_K$ is indeed an arithmetic subalgebra. Furthermore, using that E consists of locally constant equivariant functions and separates points of Y_K, it is easy to show that E is an inductive limit of finite, étale K-algebras and $Hom_{K\text{-alg}}(E, \overline{K}) = Y_K$. Therefore $\mathcal{E} = E \rtimes I_K$ is, in fact, an endomotive and $\mathcal{E}^{an} = A_K$.

We finish by making a few remarks about general arithmetic subalgebras of the BC-system $\mathcal{A}_K$. Assume $\mathcal{E} \subset A_K$ is an arithmetic subalgebra. Also assume that it contains the elements $U_{\mathfrak{a}}$ and $U_{\mathfrak{a}}^*$ for all $\mathfrak{a} \in I_K$. Consider the image of $\mathcal{E}$ under the canonical conditional expectation $A_K \to C(Y_K)$, and let E be the K-algebra generated by this image. Then E satisfies the following properties:

- (a′) every function in E is continuous;
- (b′) the $\mathbb{C}$-algebra generated by E is dense in $C(Y_K)$; in particular, E separates points of Y_K;
- (c′) E is invariant under the endomorphisms $\rho_{\mathfrak{a}}$ and $\sigma_{\mathfrak{a}}$ for all $\mathfrak{a} \in I_K$;
- (d′) for every $f \in E$ we have $f(Y_K^\times) \subset K^{ab}$ and the map $f\colon Y_K^\times \to K^{ab}$ is $\mathrm{Gal}(K^{ab}/K)$-equivariant.

Conversely, if E is a unital K-algebra of functions on Y_K with properties (a′)-(d′), then $\mathcal{E} = E \rtimes I_K$ is an arithmetic subalgebra of A_K and the intersection $\mathcal{E} \cap C(Y_K)$, as well as the image of $\mathcal{E}$ under the conditional expectation onto $C(Y_K)$, coincides with E. Note again that the property that K^{ab} is generated by the values $f(y)$, $f \in E$, for any $y \in Y_K^\times$, follows from the proof of Lemma 2.23. The largest algebra satisfying properties (a′)-(d′) is the K-algebra of continuous functions such that their restrictions to $\mathfrak{a}Y_K^\times$ are K^{ab}-valued and $\mathrm{Gal}(K^{ab}/K)$-equivariant for all $\mathfrak{a} \in I_K$. This algebra is strictly larger than the algebra E_K. Indeed, it, for example, contains the functions of the form $\sum_{n=0}^\infty q_n \rho_{\mathfrak{p}_v}^n(1)$, where $\sum_n q_n$ is any convergent series of rational numbers. Such a function takes value $\sum_{n=0}^\infty q_n$, which can be any real number, at every point $y \in Y_K$ with $\mathrm{ord}_v(y) = +\infty$.

Chapter 3

On functoriality of BC-systems

In [LNT], Laca, Neshveyev and Trifkovic were able to construct a functor from the category of number fields to the category of BC-systems. In the latter, morphisms are given by correspondences in form of a Hilbert C^*-bimodule. More precisely, for an inclusion $\sigma : K \to L$ of number fields, they construct, quite naturally, an A_L-A_K correspondence $Z = Z^L_{K,\sigma}$, i.e., a right Hilbert A_K-module Z with a left action of A_L (cf., (2.4)). Unfortunately, the time evolutions of A_K and A_L are not compatible under Z, which is in fact not surprising. In order to remedy the situation, the authors of [LNT] introduce a normalized time evolution $\widetilde{\sigma}_t$ on the A_K given by

$$\widetilde{\sigma}_t(fu_s) = N_{K/\mathbb{Q}}(s)^{it/[K:\mathbb{Q}]} fu_s. \tag{3.1}$$

With this normalization, they obtain a functor $K \mapsto (A_K, \widetilde{\sigma}_t)$, where the correspondences are compatible with the time evolutions.

We will show that their functor arises naturally in the context of (algebraic) endomotives. However it doesn't seem likely that the normalized time evolution (3.1) can be recovered naturally in the framework of endomotives, at least not in a naive sense (see section 3.4.3).

The first obstacle in constructing an algebraic version of the functor constructed in [LNT] is that the different algebraic endomotives $\mathcal{E}_K$ are defined over different number fields, which means that they live in different categories.
To overcome this, we introduce the notion of "base-change" in this context. More precisely, one finds two natural ways of changing the base of $\mathcal{E}_K$, which correspond to the two fundamental functoriality properties of class field theory given by the Verlagerung and restriction map respectively. Although both procedures change the algebraic endomotive, the analytic endomotive of the initial and base-changed endomotive will remain the same.

Our strategy is then, first, to base-change all the $\mathcal{E}_K$ down to $\mathbb{Q}$ and then, second, construct a functor from the category of number fields to the category of algebraic endomotives over $\mathbb{Q}$. Finally, we will show that our functor recovers the functor constructed in [LNT] (except for the normalization (3.1)). More precisely, we prove:

Theorem 3.1. *The functor from the category of number fields to the category of algebraic endomotives over $\mathbb{Q}$ defined by $K \mapsto \mathcal{E}^{\mathbb{Q}}_K$ and $(K \to L) \mapsto \mathcal{Z}^L_K$, cf., 3.4.1, recovers, by passing to the analytic endomotive, the functor constructed by Laca, Neshveyev and Trifkovic [LNT].*

Notations and Conventions

In the following, when speaking about extensions of number fields, instead of saying $\sigma : K \to L$, we simply write L/K. Moreover we fix a tower $M/L/K$ of finite extensions of number fields (contained in $\mathbb{C}$). We denote the Artin reciprocity map by $[\cdot]_K : \mathbb{A}_K^\times \to Gal(K^{ab}/K)$.

3.1 Algebraic preliminaries

Recall the two fundamental functoriality properties of Artin's reciprocity map in form of the following two commutative diagrams (cf., [Neu99])

$$\begin{array}{ccc} \mathbb{A}_L^\times & \xrightarrow{[\cdot]_L} & \mathrm{Gal}(L^{ab}/L) \\ {\scriptstyle i_{K/L}}\uparrow & & \uparrow{\scriptstyle Ver} \\ \mathbb{A}_K^\times & \xrightarrow{[\cdot]_K} & \mathrm{Gal}(K^{ab}/K) \end{array} \qquad \begin{array}{ccc} \mathbb{A}_L^\times & \xrightarrow{[\cdot]_L} & \mathrm{Gal}(L^{ab}/L) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle Res} \\ \mathbb{A}_K^\times & \xrightarrow{[\cdot]_K} & \mathrm{Gal}(K^{ab}/K) \end{array} \tag{3.2}$$

Remark 3.1. Notice that the Verlagerung Ver is injective.

The diagrams allow one to define two maps of topological monoids (which are of central importance for everything that eventually follows)

$$\mathcal{V}_{L/K} : \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) \longrightarrow \widehat{\mathcal{O}}_L \times_{\widehat{\mathcal{O}}_L^\times} \mathrm{Gal}(L^{ab}/L) \; ; [\rho, \alpha] \mapsto [i_{K/L}(\rho), Ver(\alpha)] \tag{3.3}$$

and

$$\mathcal{N}_{L/K} : \widehat{\mathcal{O}}_L \times_{\widehat{\mathcal{O}}_L^\times} \mathrm{Gal}(L^{ab}/L) \longrightarrow \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K) \; ; [\gamma, \beta] \mapsto [N_{L/K}(\gamma), Res(\beta)]. \tag{3.4}$$

Remark 3.2. The first map is always injective [1], whereas the second map is in general neither injective nor surjective.

Now, using these two maps, we can define two "base-change" functors relating the categories $\mathfrak{S}_{DR_K}$ and $\mathfrak{S}_{DR_L}$ (cf., section 2.4).
The first functor

$$\mathfrak{V} = \mathfrak{V}_{L/K} : \mathfrak{S}_{DR_L} \longrightarrow \mathfrak{S}_{DR_K} \tag{3.5}$$

is given by sending a finite set S with action of DR_L to the set S with an action of DR_K given by restricting the action of DR_L via $\mathcal{V}_{L/K}$.
The second functor

$$\mathfrak{N} = \mathfrak{N}_{L/K} : \mathfrak{S}_{DR_K} \longrightarrow \mathfrak{S}_{DR_L} \tag{3.6}$$

is defined by sending a finite set S with its action by DR_K to the same set S with an action of DR_L defined by pulling back the action of DR_K via $\mathcal{N}_{L/K}$. Using the functorial equivalence $\mathfrak{S}_{DR_K} \to \mathfrak{E}_{\Lambda_K}$ (see (2.15)), we obtain corresponding functors on the algebraic side

$$\mathfrak{V}^{alg} : \mathfrak{E}_{\Lambda_L} \longrightarrow \mathfrak{E}_{\Lambda_K} \tag{3.7}$$

and

$$\mathfrak{N}^{alg} : \mathfrak{E}_{\Lambda_K} \longrightarrow \mathfrak{E}_{\Lambda_L}. \tag{3.8}$$

1. This follows from a Galois descent argument.

Lemma 3.2. *1) The functor $\mathfrak{V}^{alg}$ is determined by the fact that a finite abelian extension L' of L is sent to the direct product $\prod_{i=1}^{h} K'$, where the finite, abelian extension K' of K and the index h are specified below.*
2) The functor $\mathfrak{N}^{alg}$ is given by $E \mapsto E \otimes_K L$.

Proof. 1) Define the map ϕ to be the composition of the Verlagerung $Gal(K^{ab}/K) \to Gal(L^{ab}/L)$ and the projection $Gal(L^{ab}/L) \to Gal(\widetilde{L'}/L)$ where $\widetilde{L'}$ denotes the Galois closure of L'. We can identify the quotient $Gal(K^{ab}/K)/\mathrm{Ker}\phi$ with a finite, abelian Galois group $Gal(\widetilde{K}/K)$ sitting inside $Gal(\widetilde{L'}/L)$, i.e. $Gal(\widetilde{K}/K) \cong Gal(\widetilde{L'}/L^K)$ for a subfield $L^K \subset \widetilde{L'}$. We define K' to be the subfield of $\widetilde{K}$ corresponding to the subgroup $Gal(\widetilde{L'}/L') \cap Gal(\widetilde{L'}/L^K) \subset Gal(\widetilde{K}/K)$. Using again only basic Galois theory we see that the fraction

$$\frac{|Gal(\widetilde{L'}/L)| \cdot |Gal(\widetilde{L'}/L') \cap Gal(\widetilde{L'}/L^K)|}{|Gal(\widetilde{L'}/L^K)| \cdot |Gal(\widetilde{L'}/L')|}$$

is actually a natural number and this will be the index h. In particular we see that we have the equality $|\mathrm{Hom}_L(L', \overline{L})| = h \cdot |\mathrm{Hom}_K(K', \overline{K})| = |\mathrm{Hom}_K(\prod_{i=1}^{h} K', \overline{K})|$.
2) This is obvious. □

Remark 3.3. If L'/L is Galois then K'/K is also Galois.

Let us make the functor $\mathfrak{V}^{alg}$ more transparent in the context of strict ray class fields which occur in the definition of the $\mathcal{E}_K$. For this, let us first introduce the following notation. If $\mathfrak{d}$ denotes a non-zero, integral ideal in I_K, we denote by $\mathfrak{d}^L$ the corresponding ideal in I_L. For example, if $\mathfrak{d} = \mathfrak{p}$ is a prime ideal then $\mathfrak{p}^L = \mathfrak{p}\mathcal{O}_L$ is usually written in the form

$$\mathfrak{p}^L = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e(\mathfrak{P}|\mathfrak{p})}$$

where $\mathfrak{P}$ denotes a prime ideal of L and $e(\mathfrak{P}|\mathfrak{p})$ the ramification index of $\mathfrak{P}$ in $\mathfrak{p}$. Moreover, let us denote by $K_{\mathfrak{d}}$ and $L_{\mathfrak{d}^L}$ the corresponding strict ray class fields and by $K^{\mathfrak{d}^L}$ the field constructed from $L_{\mathfrak{d}^L}$ above. Then we have the following:

Lemma 3.3. *With the notations from above let $\mathfrak{d}$ be in I_K. Then we have*

$$K_{\mathfrak{d}} = K^{\mathfrak{d}^L} \subset L_{\mathfrak{d}^L}. \tag{3.9}$$

Proof. Using basic class field theory (cf., [Neu99]) the two assertions can be reformulated in the idelic language and are seen to be equivalent to

$$\iota_{L/K}^{-1}(C_L^{\mathfrak{d}^L}) = C_K^{\mathfrak{d}} \qquad \text{and} \qquad \iota_{L/K}(N_{L/K}(C_L^{\mathfrak{d}^L})) \subset C_L^{\mathfrak{d}^L},$$

where $C_K^{\mathfrak{d}}$ is the standard open subset of $C_K = \mathbb{A}_K^{\times}/K^{\times}$ such that $C_K/C_K^{\mathfrak{d}} \cong Gal(K_{\mathfrak{d}}/K)$ and analogously for $C_L^{\mathfrak{d}^L}$. Further, it is enough to consider the case $\mathfrak{d} = \mathfrak{p}^i$ for some $i \geq 1$. Let us recall the following fact from ramification theory. If $\mathfrak{P}$ divides $\mathfrak{p}$ with ramification index $e = e(\mathfrak{P}|\mathfrak{p})$ and if we denote by $\iota_{\mathfrak{P}} : K_{\mathfrak{p}} \to L_{\mathfrak{P}}$ the natural inclusion of local fields, we have

$$\iota_{\mathfrak{P}}^{-1}(\mathfrak{P}^{ei}\mathcal{O}_{L_{\mathfrak{P}}}) = \mathfrak{p}^i\mathcal{O}_{K_{\mathfrak{p}}}$$

This proves the first assertion, and the second assertion follows directly from the definition of the norm map $N_{L/K}$. □

As a first application, it is shown in the next proposition how one can relate the different algebras E_K (cf., (2.25)).

Proposition 3.4. *1) The functor $\mathfrak{V}^{alg}$ induces a K-algebra homomorphism $\mathfrak{V}^{alg}(E_L) \to E_K$ compatible with the Λ_K-structure.*
2) The functor $\mathfrak{N}^{alg}$ induces a L-algebra homomorphism $E_K \otimes_K L \to E_L$ compatible with the Λ_L-structure.
3) There exists an injective K-algebra homomorphism $E_K \to E_L$.

Proof. Using the two commutative diagrams

$$\begin{array}{ccc} DR_L & \xleftarrow{\mathcal{V}_{L/K}} & DR_K \\ \circlearrowright & & \circlearrowright \\ \mathcal{V}_{L/K} & & \\ DR_K & = & DR_K \end{array} \qquad \begin{array}{ccc} DR_K & \xleftarrow{\mathcal{N}_{L/K}} & DR_L \\ \circlearrowright & & \circlearrowright \\ \mathcal{N}_{L/K} & & \\ DR_L & = & DR_L \end{array} \tag{3.10}$$

the **first two** assertions follow immediately if we can show that $\mathcal{V}_{L/K} : DR_K \to DR_L$ and $\mathcal{N}_{L/K} : DR_L \to DR_K$ are compatible with the profinite structures of DR_K and DR_L. In this case we can simply apply the equivalence (2.16). The compatibility of $\mathcal{N}_{L/K}$ with the profinite structure of DR_L follows from the compatibility of $\mathcal{V}_{L/K}$ with the profinite structure of DR_K, and this would follow if $\mathcal{V}_{L/K}$ factors over

$$\mathcal{O}_K/\mathfrak{f} \times_{(\mathcal{O}_K/\mathfrak{f})^\times} C_\mathfrak{f} \to \mathcal{O}_L/\mathfrak{f}^L \times_{(\mathcal{O}_L/\mathfrak{f}^L)^\times} C_{\mathfrak{f}^L}$$

But this is true thanks to our previous lemma.
To prove the **third** assertion we define a surjective map

$$\omega_\mathfrak{f} : \{\mathfrak{D} \mid \mathfrak{D} \text{ divides } \mathfrak{f}^L\} \to \{\mathfrak{d} \mid \mathfrak{d} \text{ divides } \mathfrak{f}\}$$

by

$$\mathfrak{D} \longmapsto \prod_{\mathfrak{p}|\mathfrak{f}} \mathfrak{p}^{max\{j \,:\, \mathfrak{P}^{je(\mathfrak{P}|\mathfrak{p})}|\mathfrak{D} \;\forall\; \mathfrak{P}|\mathfrak{p}\}}.$$

Now we can define an embedding of K-algebras

$$E_{K,\mathfrak{f}} = \prod_{\mathfrak{d}|\mathfrak{f}} K_\mathfrak{d} \longrightarrow E_{L,\mathfrak{f}^L} = \prod_{\mathfrak{D}|\mathfrak{f}^L} L_\mathfrak{D}$$

by embedding $K_\mathfrak{d}$ into $L_\mathfrak{D}$ whenever $\omega_\mathfrak{f}(\mathfrak{D}) = \mathfrak{d}$. It is not very difficult to check that these maps induce a K-algebra embedding of the corresponding inductive systems. □

Remark 3.4. Due to the fact that $\mathfrak{f}^M = (\mathfrak{f}^L)^M$ we see that the third map of the last proposition is in fact functorial, i.e., the composition $E_K \to E_L \to E_M$ equals $E_K \to E_M$. But, on the other hand, the inclusion $E_K \to E_L$ is not compatible with any Λ-structure.

3.2 On correspondences of endomotives

In this section we will follow our main reference [CM08] pp. 594.

3.2.1 Algebraic correspondences

An algebraic endomotive $\mathcal{E} = A \rtimes S$ can be described equivalently as a groupoid $\mathcal{G}$ as follows. Let us introduce, for $s = \rho_1/\rho_2 \in \widetilde{S}$, two projections $E(s) = \rho_1^{-1}(\rho_2(1)\rho_1(1))$ and $F(s) = \rho_2^{-1}(\rho_2(1)\rho_1(1))$. They satisfy the relations $E(s^{-1}) = F(s) = s(E(s))$ and show up naturally in that they are the biggest projections such that $s : A_{E(s)} = E(s)A \to A_{F(s)}$ is an isomorphism. Now, as a set $\mathcal{G}$ is defined by

$$\mathcal{G} = \mathrm{Spec}(\bigoplus_{s\in\widetilde{S}} A_{F(s)}) = \sqcup_{s\in\widetilde{S}} \mathrm{Spec}(A_{F(s)}) \tag{3.11}$$

The range and source maps

$$r, s : \mathrm{Spec}(\bigoplus_{s\in\widetilde{S}} A_{F(s)}) \to \mathrm{Spec}(A) \tag{3.12}$$

are given by the natural projection $A \to A_{F(s)}$, and the natural projection composed with the antipode $P : \bigoplus A_{F(s)} \to \bigoplus A_{F(s)}$ given by

$$P(a)_s = s(a_{s^{-1}}), \quad \forall s \in \widetilde{S}.$$

An **algebraic correspondence** between two algebraic endomotives $\mathcal{E}' = A' \rtimes S'$ and $\mathcal{E} = A \rtimes S$ is given by a disjoint union of zero-dimensional pro-varietes $\mathcal{Z} = \mathrm{Spec}(C)$ together with compatible left and right actions of $\mathcal{G}'$ and $\mathcal{G}$ respectively. A right action of $\mathcal{G}$ on $\mathcal{Z}$ is given by a continuous map

$$g : \mathrm{Spec}(C) \to \mathrm{Spec}(A) \tag{3.13}$$

together with a family of partial isomorphisms

$$z \in g^{-1}(\mathrm{Spec}(A_{E(s)})) \mapsto z \cdot s \in g^{-1}(\mathrm{Spec}(A_{F(s)})) \quad \forall s \in \widetilde{S} \tag{3.14}$$

satisfying the obvious rules for a partial action of an abelian group (cf., [CM08] p. 597). Analogously, one defines a left action. It is straightforward to check that a left (resp. right) action of $\mathcal{G}$ on $\mathcal{Z}$ is equivalent to a left (resp. right) $\mathcal{E}$-module structure on C.
The composition of algebraic correspondences is given by the fibre product over a groupoid. On the algebraic side this corresponds to the tensor product over a ring.

Remark 3.5. The main advantage of using the groupoid language comes from the fact that it provides a natural framework for constructing so called analytic correspondences $\mathcal{Z}^{an}$ between $\mathcal{E}'^{an}$ and $\mathcal{E}^{an}$ out of algebraic correspondences. In fact, the procedure is functorial (see Thm. 4.34 [CM08])

In our reference [CM08], morphisms of the category of algebraic endomotives over K are defined in terms of étale correspondences, where $\mathcal{Z}$ is étale if it is finite, and projective as a right module. We shall eventually see that the finiteness condition is too restrictive for our applications. Nevertheless, the functorial assignment $\mathcal{Z} \mapsto \mathcal{Z}^{an}$ has a domain much larger than only étale correspondences, containing in particular the algebraic correspondences occurring in our applications. In summary, we enlarge tacitly the morphisms in the category of algebraic endomotives by allowing those contained in the domain of the assignment $\mathcal{Z} \mapsto \mathcal{Z}^{an}$.

3.2.2 Analytic correspondences

As we have already seen in the section 2.2.2 about analytic endomotives, the (functorial) transition between algebraic and analytic endomotives is based on the functor $X \mapsto X(\overline{K})$ taking $\overline{K}$-valued points.
Given an algebraic endomotive $\mathcal{E}$ with corresponding groupoid $\mathcal{G}$, we define the analytic endomotive $\mathcal{G}^{an}$ to be the totally disconnected locally compact space $\mathcal{G}(\overline{K})$ of $\overline{K}$-valued points of $\mathcal{G}$. An element of $\mathcal{G}^{an}$ is therefore given by a pair (χ, s) with $s \in \widetilde{S}$ and χ a character of the (reduced) algebra $A_{F(s)}$, i.e. $\chi(F(s)) = 1$. The range and source maps

$$r, s : \mathcal{G}^{an} \to \mathrm{Hom}(A, \overline{K}) \tag{3.15}$$

are given by

$$r(\chi, s) = \chi \qquad \text{and} \qquad s(\chi, s) = \chi \circ s. \tag{3.16}$$

One shows that $\mathcal{E}^{an} = C(\mathrm{Hom}(A, \overline{K}))$ is isomorphic to the groupoid C^*-algebra $C^*(\mathcal{G}^{an})$.

Now, given an algebraic correspondence $\mathcal{Z}$ between $\mathcal{E}'$ and $\mathcal{E}$, i.e., we have (for the right action) a continuous map

$$g : \mathcal{Z} \to \mathrm{Spec}(A),$$

together with partial isomorphisms, we obtain, by taking the $\overline{K}$-valued points, a continuous map of totally disconnected locally compact spaces

$$g_K = g(\overline{K}) : \mathcal{Z}(\overline{K}) = \mathrm{Hom}(C, \overline{K}) \to \mathrm{Hom}(A, \overline{K}), \tag{3.17}$$

together with partial isomorphisms

$$z \in g_K^{-1}(\mathrm{Hom}(A_{F(s)}, \overline{K})) \mapsto z \circ s \in g_K^{-1}(\mathrm{Hom}(A_{E(s)}, \overline{K})) \tag{3.18}$$

fulfilling again the obvious rules.
As in the algebraic case, this right action of $\mathcal{G}^{an}$ on $\mathcal{Z}(\overline{K})$ gives the space of continuous and compactly supported functions $C_c(\mathcal{Z}(\overline{K}))$ on $\mathcal{Z}(\overline{K})$ the structure of a right $C_c(\mathcal{G}^{an})$-module. Moreover, **if** the fibers of g_K are discrete (and countable) there is a natural way of defining a $C_c(\mathcal{G}^{an})$-valued inner product on $C_c(\mathcal{Z}(\overline{K}))$ by setting

$$\langle \xi, \eta \rangle (\chi, s) = \sum_{z \in g_K^{-1}(\chi)} \bar{\xi}(z) \eta(z \circ s). \tag{3.19}$$

In this case we obtain a right Hilbert-C^*-module $\mathcal{Z}^{an}$ over $C^*(\mathcal{G}^{an})$ by completion. Together with the left action $\mathcal{Z}^{an}$ becomes a $C^*(\mathcal{G}'^{an})$-$C^*(\mathcal{G}^{an})$ Hilbert-C^*-bimodule.

3.2.3 Examples

1) Every algebraic endomotive $\mathcal{E}$ is a correspondence over itself. In particular the inner product is given on $\mathcal{E}^{an} = C(\mathcal{X}) \rtimes S$ simply by

$$\langle \xi, \eta \rangle = \xi^* \eta \quad \forall \xi, \eta \in \mathcal{E}^{an}$$

2) Let $S \subset T$ be an inclusion of abelian semigroups. Then the algebraic endomotive $K[T] = K \rtimes T$ is naturally a $K[T]$-$K[S]$ correspondence with the obvious continuous map $g : \mathrm{Spec}(\bigoplus_{t \in \widetilde{T}} K) \to \mathrm{Spec}(K)$ and partial isomorphisms. If we denote the corresponding

analytic endomotives by $C^*(T)$ and $C^*(S)$, which are related by the natural conditional expectation $E : C^*(T) \to C^*(S)$ induced by $t \in T \mapsto \begin{cases} t & \text{if } t \in S \\ 0 & \text{otherwise} \end{cases}$, we see that the $C^*(S)$-valued inner product on $C^*(T)$ is given by

$$\langle \xi, \eta \rangle = E(\xi^* \eta), \quad \forall \xi, \eta \in \mathcal{C}^*(T).$$

3.3 On base-change

Let us start with the data defining our algebraic endomotive $\mathcal{E}_L$, namely the inductive system $(E_{\mathfrak{f}})_{\mathfrak{f} \in I_L}$ and the collection of "Frobenius lifts" $\sigma_{\mathfrak{d}}$ (cf., (2.30)), where the latter define of course the $\rho_{\mathfrak{d}}$ but are better suited for the functors $\mathfrak{V}^{alg}_{L/K}$ and $\mathfrak{N}^{alg}_{M/L}$ due to their level preserving property. Let us concentrate on the functor $\mathfrak{V} = \mathfrak{V}^{alg}_{L/K}$, the arguments for $\mathfrak{N}^{alg}_{M/L}$ are analogous. Define the K-algebras $\widetilde{E}_{\mathfrak{f}} = \mathfrak{V}(E_{\mathfrak{f}})$, $\widetilde{E}_L = \varinjlim \widetilde{E}_{\mathfrak{f}}$ and the K-algebra homomorphisms $\widetilde{\sigma}_{\mathfrak{d}} = \mathfrak{V}(\sigma_{\mathfrak{d}}) : \widetilde{E}_L \to \widetilde{E}_L$. Due to the fact that (cf., (2.16))

$$\mathfrak{H}_L(\widetilde{E}_L) = DR_L \tag{3.20}$$

and

$$\mathfrak{H}_L(\widetilde{\sigma}_{\mathfrak{d}}) = \sigma_{\mathfrak{d}} : DR_L \to DR_L \tag{3.21}$$

the same arguments as in section 2.6 show the existence of projections $\widetilde{\pi}_{\mathfrak{d}}$ and endomorphisms $\widetilde{\rho}_{\mathfrak{d}}$ of $\widetilde{E}_L$ such that

$$\mathcal{E}^K_L = \mathfrak{V}^{alg}_{L/K}(\mathcal{E}_L) = ((\widetilde{E}_{\mathfrak{f}}), \widetilde{I_L}) \tag{3.22}$$

is in fact an algebraic endomotive over K. Analogously we construct

$$\mathcal{E}^M_L = \mathfrak{N}^{alg}_{M/L}(\mathcal{E}_L) \tag{3.23}$$

and obtain in summary the following base-change properties of our algebraic endomotives $\mathcal{E}_L$.

Proposition 3.5. *With the notations from above we have that $\mathcal{E}^K_L$ and $\mathcal{E}^M_L$ are algebraic endomotives over K and M, respectively. Moreover, on the analytic level we have*

$$(\mathcal{E}^K_L)^{an} = \mathcal{E}^{an}_L = (\mathcal{E}^M_L)^{an}. \tag{3.24}$$

Remark 3.6. Both assignments are functorial.

3.4 A functor, a pseudo functor and proof of Theorem 3.1

3.4.1 Going down to $\mathbb{Q}$

The base-change mechanism from the last section enables us now to construct a functor from the category of number fields to the category of algebraic endomotives over $\mathbb{Q}$ which sends a number field K to $\mathcal{E}^{\mathbb{Q}}_K$. Unfortunately, it is not possible to construct an algebra homomorphism between $\mathcal{E}^{\mathbb{Q}}_K$ and $\mathcal{E}^{\mathbb{Q}}_L$ because the actions of I_K and I_L are not compatible. Instead, given an extension L/K we construct an algebraic $\mathcal{E}^{\mathbb{Q}}_L$-$\mathcal{E}^{\mathbb{Q}}_K$ correspondence $\mathcal{Z}^L_K$ as follows. Recall the examples 3.2.3. From the first one, we see that we can regard

$\mathcal{E}_K^{\mathbb{Q}}$ as a $\mathbb{Q}[I_K]$-$\mathcal{E}_k^{\mathbb{Q}}$ correspondence, because we have naturally the inclusion $\mathbb{Q}[I_K] \subset \mathcal{E}_K^{\mathbb{Q}}$. Using the second example in the case of the inclusion $I_K \subset I_L$ we obtain the $\mathbb{Q}[I_L]$-$\mathbb{Q}[I_K]$ correspondence $\mathbb{Q}[I_L]$. In performing the fibre product over $\mathbb{Q}[I_K]$ we obtain the $\mathbb{Q}[I_L]$-$\mathcal{E}_K^{\mathbb{Q}}$ correspondence $\mathcal{Z}_K^L = \mathbb{Q}[I_L] \times_{\mathbb{Q}[I_K]} \mathcal{E}_K^{\mathbb{Q}}$ which can be described algebraically by

$$\mathcal{Z}_K^L = \mathbb{Q}[I_L] \otimes_{\mathbb{Q}[I_K]} \mathcal{E}_K^{\mathbb{Q}}. \tag{3.25}$$

We want to show that there is a natural left action of $\mathcal{E}_L^{\mathbb{Q}}$ making $\mathcal{Z}_K^L$ the desired $\mathcal{E}_L^{\mathbb{Q}}$-$\mathcal{E}_K^{\mathbb{Q}}$ correspondence. Namely, using the same arguments as in Proposition 3.4, we obtain a $\mathbb{Q}$-algebra homomorphism $\phi : \mathfrak{V}_{L/\mathbb{Q}}^{alg}(E_L) \to \mathfrak{V}_{K/\mathbb{Q}}(E_K)$ which is furthermore compatible with the I_K-actions on both algebras induced by functoriality from the actions of DR_K and $\mathcal{V}_{L/K}(DR_K)$ on DR_K and DR_L, respectively. Thus, we see that

$$eU_s \cdot (U_t \otimes f) = U_{st} \otimes \phi(\widetilde{e}) f, \tag{3.26}$$

for $s, t \in I_L$, $e \in \mathfrak{V}_{L/\mathbb{Q}}^{alg}(E_L)$, $f \in \mathcal{E}_K^{\mathbb{Q}}$ and $\widetilde{e}$ defined by the equation $eU_{st} = U_{st}\widetilde{e} \in \mathcal{E}_L^{\mathbb{Q}}$, gives a well-defined left $\mathcal{E}_L^{\mathbb{Q}}$-module structure on $\mathcal{Z}_K^L$.

We can now prove the main result of this chapter.

Theorem 3.6. *1) The assignments $K \mapsto \mathcal{E}_K^{\mathbb{Q}}$ and $L/K \mapsto \mathcal{Z}_K^L$ define a (contravariant) functor from the category of number fields to the category of algebraic endomotives over $\mathbb{Q}$.*
2) The corresponding functor given by $K \mapsto (\mathcal{E}_K^L)^{an}$ and $L/K \mapsto (\mathcal{Z}_K^L)^{an}$ from the category of number fields to the category of analytic endomotives is equivalent to the functor constructed by Laca, Neshveyev and Trifkovic in Thm. 4.4 [LNT].

Proof. 1) One only has to show that $\mathcal{Z}_L^M \otimes_{\mathcal{E}_L^{\mathbb{Q}}} \mathcal{Z}_K^L \cong \mathcal{Z}_K^M$, which is obvious.
2) One can check without difficulties that $(\mathcal{Z}_K^L)^{an}$ is given as a Hilbert C^*-module by the inner tensor product of the right $C^*(I_K)$-module $C^*(I_L)$ and the right $\mathcal{E}_K^{an}$-module $\mathcal{E}_K^{an}$ with its natural left action of $C^*(I_K)$, i.e.,

$$(\mathcal{Z}_K^L)^{an} = C^*(I_L) \otimes_{C^*(I_K)} \mathcal{E}_K^{an}, \tag{3.27}$$

and this is exactly the same correspondence as constructed in Theorem 4.4 of [LNT]. □

Remark 3.7. We see that $\mathcal{Z}_K^L$ is not an étale correspondence because the complement of I_K in I_L is infinite. Nevertheless the definition of $\mathcal{Z}_K^L$ seems to be the most natural one under the circumstances that it is not possible to define interesting algebra homomorphisms between $\mathcal{E}_K^{(\mathbb{Q})}$ and $\mathcal{E}_L^{(\mathbb{Q})}$ which comes from the fact that Verlagerung and Restriction are not inverse to each other in general and therefore the actions of I_K and I_L are not compatible.

3.4.2 $\overline{\mathbb{Q}}$ is too big

In analogy with the last section, where we constructed algebraic correspondences using the base-change induced by the functor $\mathfrak{V}_{L/K}^{alg}$, one can also use the functor $\mathfrak{N}_{L/K}^{alg}$ to construct bimodules of algebraic endomotives.

Again, by Proposition 3.4, we see that

$$\mathcal{Y}_K^L = L[I_K] \otimes_{L[I_L]} \mathcal{E}_L \tag{3.28}$$

is an $\mathcal{E}_K^L$-$\mathcal{E}_L$ correspondence. The right $L[I_L]$-module structure of $L[I_K]$ is induced by the norm map $I_L \to I_K$.
But this time, we do not obtain a functor. Of course, one can check that for a tower $M/L/K$ of number fields we have an isomorphism of $\mathcal{E}_K^M$-$\mathcal{E}_M$ bimodules

$$(M[I_K] \otimes_{M[I_L]} \mathcal{E}_L^M) \otimes_{\mathcal{E}_L^M} (M[I_L] \otimes_{M[I_M]} \mathcal{E}_M) \cong M[I_K] \otimes_{M[I_M]} \mathcal{E}_M,$$

but in order to make this functorial for all number fields, we would have to make sense of a Λ-structure over $\overline{\mathbb{Q}}$ which is compatible with Λ-structures over number fields and this does not seem likely to the author.

3.4.3 On the time evolution

In this section we would like to make some remarks about the question of whether the normalized time evolution (3.1) introduced in [LNT] fits into the framework of endomotives.

Due to the fact that the analytic endomotive of the base-changed algebraic endomotive $\mathcal{E}_K^{\mathbb{Q}}$ is equal to $\mathcal{E}_K^{an}$ we see in particular that $\mathcal{E}_K^{\mathbb{Q}}$ is an uniform endomotive (over $\mathbb{Q}$) with the same measure μ_K as the natural measure of $\mathcal{E}_K$. So, in particular the base-changed endomotive $\mathcal{E}_K^{\mathbb{Q}}$ does not recover the normalized time evolution, if one tries to define the time evolution on $\mathcal{E}_K^{\mathbb{Q}}$ by means of normalized counting measures. This is clear, because the normalized norm $\widetilde{N} = N_{K/\mathbb{Q}}^{1/[K:\mathbb{Q}]}$ used in [LNT] is no longer rational-valued on ideals of K, so $\widetilde{N}$ cannot arise from a counting procedure as one can for the usual norm $N_{K/\mathbb{Q}}$. This shows that in order to extend the base-change $\mathcal{E}_K \mapsto \mathcal{E}_K^{\mathbb{Q}}$ in a way such that the normalized time evolution appears on $(\mathcal{E}_K^{\mathbb{Q}})^{an}$ one has to find a natural method of assigning to μ_K a measure $\mu_K^{\mathbb{Q}}$ which recovers the normalized time evolution [2]. We have argued that this cannot be done in the naive sense, but it would surely be interesting to find a natural method solving this problem.

2. The methods of [LLN09] show that such a measure should exist and is in fact determined by the normalized norm $\widetilde{N}$.

Appendix A

Algebraic Groups

Our references are [Wat79] and [Mil06]. Let k denote a field of characteristic zero, K a finite field extension of k and $\overline{k}$ an algebraic closure of k. Further we denote by R a k-algebra.

A.1 Functorial definition and basic constructions

An **(affine) algebraic group** G (over k) is a representable functor from (commutative, unital) k-algebras to groups. We denote by $k[G]$ its representing algebra, i.e. for any R we have $G(R) = Hom_{k-alg}(k[G], R)$.
A **homomorphism** $F : G \to H$ between two algebraic groups G and H (over k) is given by a natural transformation of functors.

Let G and H be two algebraic groups over k, then their **direct product** $G \times H$ is the algebraic group (over k) given by $R \mapsto G(R) \times H(R)$.
Let G be an algebraic group over k and K and extension of k. Then by **extension of scalars** we obtain an algebraic group G_K over K represented by $K \otimes_k k[G]$.
Now let G be an algebraic group over K. The **Weil restriction** $Res_{K/k}(G)$ is an algebraic group over k defined by $Res_{K/k}(G)(R) = G(K \otimes_k R)$.

Remark A.1. All three constructions are functorial.

A.2 Examples

1) The **multiplicative group** $\mathbb{G}_{m,k}$ (over k) is represented by $k[x, x^{-1}] = k[x, y]/(xy-1)$, i.e. $\mathbb{G}_{m,k}(R) = R^\times$.
2) Define $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. We have $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$. In particular we have $\mathbb{S}_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$.
3) More general an algebraic group T over k is called a **torus** if $T_{\overline{k}}$ is isomorphic to a product of copies of $\mathbb{G}_{m,\overline{k}}$.
4) The **general symplectic group** $GSp(V, \psi)$ attached to a symplectic $\mathbb{Q}$-vector space (V, ψ) is an algebraic group over $\mathbb{Q}$ defined on a $\mathbb{Q}$-algebra R by

$$GSp(R) =$$
$$\{f \in End_R(V \otimes_\mathbb{Q} R) \mid \exists\ \nu(f) \in R^\times : \psi_R(f(x), f(y)) = \nu(f)\psi_R(x, y)\ \forall x, y \in V \otimes_\mathbb{Q} R\}.$$

A.3 Characters

Let G be an algebraic group over k and set $\Lambda = \mathbb{Z}[Gal(\overline{k}/k)]$. The **character group** $X^*(G)$ of G is defined by $Hom(G_{\overline{k}}, \mathbb{G}_{m,\overline{k}})$. There is a natural action of $Gal(\overline{k}/k)$ on $X^*(G)$, i.e. $X^*(G)$ is a Λ-module. Analogously the **cocharacter group** $X_*(G)$ of G is the Λ-module $Hom(\mathbb{G}_{m,\overline{k}}, G_{\overline{k}})$. We denote the action of $\sigma \in \Lambda$ on a (co)character f by σf or f^σ. There is the following important

Theorem A.1 (7.3 [Wat79])**.** *The functor $G \mapsto X^*(G)$ is a contravariant equivalance from the category of algebraic groups of multiplicative type over k and the category of finitely generated abelian groups with a continuous action of $Gal(\overline{k}/k)$.*

Remark A.2. See 7.2 [Wat79] for the definition of groups of multiplicative type. We only have to know that algebraic tori are of multiplicative type.

There is a natural bi-additive and $Gal(\overline{k}/k)$-invariant pairing $< \cdot, \cdot >: X_*(G) \times X^*(G) \to \mathbb{Z}$ given by $< \chi, \mu >= \mu \circ \chi \in Hom(\mathbb{G}_{m,\overline{k}}, \mathbb{G}_{m,\overline{k}}) \cong \mathbb{Z}$. If G is of multiplicative type the pairing is perfect, i.e. there is an isomorphism of Λ-modules $X_*(G) \cong Hom(X^*(G), \mathbb{Z})$.

A.4 Norm maps

Let L be a finite field extension of K, i.e. we have a tower $k \subset K \subset L$, and let T be a torus over k. Then there are two types of morphisms of algebraic groups which we call **norm maps**. The first one

$$Nm_{L/K} : Res_{L/K}(T_L) \to T_K \tag{A.1}$$

is induced by the usual norm map of algebras $R \otimes_K L \to R$, for R a K-algebra. In applying the Weil restriction functor $Res_{K/k}$ we obtain the second one, namely

$$N_{L/K} : Res_{L/k}(T_L) \to Res_{K/k}(T_K).$$

A.5 The case of number fields

Let K be a number field. We are interested in the algebraic group $T^K = Res_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$ (over $\mathbb{Q}$). We have $T^K(R) = (K \otimes_{\mathbb{Q}} R)^\times$.
It is easy to see that the isomorphism $K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \prod_{\rho \in Hom(K,\overline{\mathbb{Q}})} \overline{\mathbb{Q}}$ induces an isomorphism of algebraic groups $T^K_{\overline{\mathbb{Q}}} \cong \prod_{\rho \in Hom(K,\overline{\mathbb{Q}})} \mathbb{G}_{m,\overline{\mathbb{Q}}}$. It follows that $X^*(T^K) \cong \mathbb{Z}^{Hom(K,\overline{\mathbb{Q}})}$ with $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acting as follows.
For $f = \sum_{\rho \in Hom(K,\overline{\mathbb{Q}})} a_\rho[\rho] \in \mathbb{Z}^{Hom(K,\overline{\mathbb{Q}})}$ and $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ we have

$$\sigma f = \sum_{\rho \in Hom(K,\overline{\mathbb{Q}})} a_\rho[\sigma \circ \rho] = \sum_{\rho \in Hom(K,\overline{\mathbb{Q}})} a_{\sigma^{-1} \circ \rho}[\rho].$$

For any inclusion $K \subset L$ of number fields the norm map $N_{L/K} : T^L \to T^K$ is defined by saying that a character $f = \sum_{\rho \in Hom(K,\overline{\mathbb{Q}})} a_\rho[\rho] \in X^*(T^K)$ is mapped to the character $f_L = \sum_{\rho' \in Hom(L,\overline{\mathbb{Q}})} (a_{\rho'|K})[\rho'] \in X^*(T^L)$.

Appendix B

CM fields

We follow [Mil06] and [Mil04]. By ι we denote the complex conjugation of $\mathbb{C}$.

B.1 CM fields and CM types

Let E denote a number field. If E is a totally imaginary quadratic extension of a totally real field, we call E a **CM field**. In particular the degree of a CM field is always even. A **CM type** (E, Φ) is a CM field E together with a subset $\Phi \subset Hom(E, \mathbb{C})$ such that $\Phi \cup \iota\Phi = Hom(E, \mathbb{C})$ and $\Phi \cap \iota\Phi = \emptyset$.

B.2 About h_ϕ and μ_ϕ

Let (E, Φ) be a CM type. Then there are natural isomorphisms $T^E_{\mathbb{R}} \cong \prod_{\phi\in\Phi} \mathbb{S}$ resp. $T^E_{\mathbb{C}} \cong \prod_{\phi\in\Phi} \mathbb{G}_{m,\mathbb{C}} \times \prod_{\overline{\phi}\in\iota\Phi} \mathbb{G}_{m,\mathbb{C}}$, where the first one is induced by $E \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\phi\in\Phi} \mathbb{C}$ and the second one by $E \otimes \mathbb{C} \cong \prod_{\phi\in\Phi} \mathbb{C} \times \prod_{\overline{\phi}\in\iota\Phi} \mathbb{C}$.
Thus we obtain natural morphisms

$$h_\Phi : \mathbb{S} \to T^E_{\mathbb{R}} \ ; \ z \mapsto (z)_{\phi\in\Phi} \tag{B.1}$$

and

$$\mu_\Phi : \mathbb{G}_{m,\mathbb{C}} \to T^E_{\mathbb{C}} \ ; \ z \mapsto (z)_{\phi\in\Phi} \times (1)_{\overline{\phi}\in\iota\Phi}. \tag{B.2}$$

If we take μ_Φ for granted we could have defined h_Φ by the composition

$$Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}(\mu_\Phi)} Res_{\mathbb{C}/\mathbb{R}}(T^E_{\mathbb{C}}) \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} T^E_{\mathbb{R}}. \tag{B.3}$$

In particular we see that h_Φ and μ_Φ are related by

$$h_{\Phi,\mathbb{C}}(z, 1) = \mu_\Phi(z). \tag{B.4}$$

Remark B.1. In the last two sections one might have replaced $\mathbb{C}$ by $\overline{\mathbb{Q}}$.

B.3 The reflex field and reflex norm

Let (E, Φ) be a CM type. The **reflex field** E^* of (E, Φ) is the subfield of $\overline{\mathbb{Q}}$ defined by any one of the following conditions:

a) $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ fixes E^* if and only if $\sigma\Phi = \Phi$;
b) E^* is the field generated over $\mathbb{Q}$ by the elements $\sum_{\phi\in\Phi} \phi(e)$, $e \in E$;
c) E^* is the smallest subfield of $\overline{\mathbb{Q}}$ such that there exists a $E \otimes_{\mathbb{Q}} E^*$-module V such that

$$Tr_{E^*}(e|V) = \sum_{\phi\in\Phi} \phi(e), \text{ for all } e \in E. \tag{B.5}$$

The **reflex norm** of (E, Φ) is the morphism of algebraic groups $N_\Phi : T^{E^*} \to T^E$ given, for R a $\mathbb{Q}$-algebra, by

$$a \in T^{E^*}(R) \mapsto det_{E\otimes_{\mathbb{Q}} R}(a|V \otimes_{\mathbb{Q}} R) \in T^E(R). \tag{B.6}$$

Appendix C

The Serre group

Our references are [Mil98], [Mil06] and [Wei94]. Let K be a number field. We fix an embedding $\tau : K \to \overline{\mathbb{Q}} \to \mathbb{C}$ and denote by ι complex conjugation on $\mathbb{C}$.

C.1 Definition of the Serre group

The following are equivalent:

(1) The **Serre group** attached to K is a pair (S^K, μ^K) consisting of a $\mathbb{Q}$-algebraic torus S^K and a cocharacter $\mu^K \in X_*(S^K)$ defined by the following universal property. For every pair (T, μ) consisting of a $\mathbb{Q}$-algebraic torus T and a cocharacter $\mu \in X_*(T)$ defined over K satisfying the Serre condition

$$(\iota + 1)(\sigma - 1)\mu = 0 = (\sigma - 1)(\iota + 1)\mu \quad \forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{C.1}$$

there exists a unique morphism $\rho_\mu : S^K \to T$ such that the diagram

$$\begin{array}{ccc} S^K_{\overline{\mathbb{Q}}} & \xrightarrow{\rho_{\mu,\overline{\mathbb{Q}}}} & T_{\overline{\mathbb{Q}}} \\ & {\scriptstyle \mu^K}\nwarrow \quad \nearrow{\scriptstyle \mu} & \\ & \mathbb{G}_{m,\overline{\mathbb{Q}}} & \end{array} \tag{C.2}$$

commutes.

(2) The **Serre group** S^K is defined to be the quotient of T^K such that $X^*(S^K)$ is the subgroup of $X^*(T^K)$ given by all elements $f \in X^*(T^K)$ which satisfy the Serre condition

$$(\sigma - 1)(\iota + 1)f = 0 = (\iota + 1)(\sigma - 1)f \quad \forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}).$$

The cocharacter μ^K is induced by the cocharacter $\mu_\tau \in X_*(T^K)$ defined by

$$< \mu_\tau, \Sigma n_\sigma[\sigma] >= n_\tau, \quad \forall \ \Sigma n_\sigma[\sigma] \in \mathbb{Z}^{Hom(K,\overline{\mathbb{Q}})} \cong X^*(T^K).$$

(3) If K does not contain a CM subfield, we set $E = \mathbb{Q}$, otherwise E denotes the maximal CM subfield of K and F the maximal totally real subfield of E. Then there is an exact sequence of $\mathbb{Q}$-algebraic groups

$$1 \longrightarrow ker(N_{F/\mathbb{Q}} : T^F \to T^{\mathbb{Q}}) \xrightarrow{\ i\ } T^K \xrightarrow{\pi^K} S^K \longrightarrow 1, \tag{C.3}$$

where i is the obvious inclusion. The cocharacter μ_τ of T^K, defined as in (2), induces μ^K, i.e. $\mu^K = \pi^K \circ \mu_\tau$.

Remark C.1. For $K = \mathbb{Q}$ or K an imaginary quadratic field there is the obvious equality $S^K = T^K$.

C.2 About μ^K and h^K

The cocharacter $\mu^K = \pi^K \circ \mu_\tau : \mathbb{G}_{m,\mathbb{C}} \to S^K_{\mathbb{C}}$ from the last section induces a natural morphism

$$h^K : \mathbb{S} \to S^K_{\mathbb{R}} \tag{C.4}$$

defined by

$$Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \xrightarrow{Res_{\mathbb{C}/\mathbb{R}}(\mu^K)} Res_{\mathbb{C}/\mathbb{R}}(S^K_{\mathbb{C}}) \xrightarrow{Nm_{\mathbb{C}/\mathbb{R}}} S^K_{\mathbb{R}}. \tag{C.5}$$

We see that μ^K and h^K are related by

$$h^K_{\mathbb{C}}(z, 1) = \mu^K(z) \tag{C.6}$$

or in other words, for $z \in \mathbb{C}^\times$, we have $h^K(z) = \mu^K(z)\mu^K(z)^\iota$.

C.3 About ρ_Φ and the reflex norm N_Φ

Let (E, Φ) be a CM type. The natural morphism $\mu_\Phi \in X_*(T^E)$ (cf., B.2) is defined over the reflex field E^* and an easy calculation shows that it satisfies the Serre condition (C.1). By the universal property of the Serre group we obtain a $\mathbb{Q}$-rational morphism

$$\rho_\Phi : S^{E^*} \longrightarrow T^E \tag{C.7}$$

such that

$$\mu_\Phi = \rho_{\Phi,\mathbb{C}} \circ \mu^{E^*}. \tag{C.8}$$

Also, we see immediately that

$$h_\Phi = \rho_{\Phi,\mathbb{R}} \circ h^{E^*}. \tag{C.9}$$

Moreover we can relate ρ_Φ and μ_Φ by the following commutative diagram

$$\begin{array}{ccccc} T^{E^*} & \xrightarrow{Res(\mu_{\Phi,E^*})} & Res_{E^*/\mathbb{Q}}(T^E_{E^*}) & \xrightarrow{N_{E^*/\mathbb{Q}}} & T^E \\ & {\scriptstyle \pi^{E^*}} \searrow & & \nearrow {\scriptstyle \rho_\Phi} & \\ & & S^{E^*} & & \end{array} \tag{C.10}$$

which can be seen on the level of characters. The relation with the reflex norm $N_\Phi : T^{E^*} \to T^E$ (see B.3) is given by the following important

Proposition C.1 ([Mil06]). *We have the equality*

$$N_\Phi = N_{E^*/\mathbb{Q}} \circ Res(\mu_{\Phi,E^*}). \tag{C.11}$$

C.4 More properties of the Serre group

The following properties are all taken from [Mil06].

Proposition C.2. *Let $E \subset K$ denote two number fields.*

1. *The norm map $N_{K/E} : T^K \to T^E$ induces a commutative diagram*

$$\begin{array}{ccc} T^K & \xrightarrow{N_{K/E}} & T^E \\ \downarrow{\scriptstyle \pi^K} & & \downarrow{\scriptstyle \pi^E} \\ S^K & \longrightarrow & S^E. \end{array} \tag{C.12}$$

   We call the induced morphism $N_{K/E} : S^K \to S^E$.

2. *There is a commutative diagram*

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{h^K} & S^K \\ & \searrow{\scriptstyle h^E} & \downarrow{\scriptstyle N_{K/E}} \\ & & S^E \end{array} \tag{C.13}$$

3. *Let E denote the maximal CM field contained in K, if there is no such subfield we set $E = \mathbb{Q}$. Then $N_{K/E} : S^K \to S^E$ is an isomorphism.*

4. *Let (E, Φ) be a CM type and $K_1 \subset K_2$ two number fields, such that $E^* \subset K_1$, and let $\rho_{\Phi,i} : S^{K_i} \to T^E$ be the corresponding maps from the universal property of the Serre group. Then we have*

$$\rho_{\Phi,1} \circ N_{K_2/K_1} = \rho_{\Phi,2}.$$

Appendix D

Shimura Varieties

Our references are Deligne's foundational [Del79], Milne's [Mil04] and Hida's [Hid04].

Let G be an algebraic group over $\mathbb{Q}$. Then the adjoint group G^{ad} of G is defined to be the quotient of G by its center C. The derived group G^{der} of G is defined to be the intersection of the normal algebraic subgroups of G such that G/N is commutative. By $G(\mathbb{R})^+$ we denote the identity component of $G(\mathbb{R})$ relative to its real topology and set $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$. If G is reductive, we denote by $G(\mathbb{R})_+$ the group of elements of $G(\mathbb{R})$ whose image in $G^{ad}(\mathbb{R})$ lies in its identity component and set $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

D.1 Shimura datum

A **Shimura datum** is a pair (G, X) consisting of a reductive group G (over $\mathbb{Q}$) and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$, such that the following (three) axioms are satisfied

$(SV1)$: For each $h \in X$, the representation $Lie(G_{\mathbb{R}})$ defined by h is of type $\{(-1,1),(0,0),(1,-1)\}$.
$(SV2)$: For each $h \in X$, $ad(h(i))$ is a Cartan involution on $G^{ad}_{\mathbb{R}}$.
$(SV3)$: G^{ad} has no $\mathbb{Q}$-factors on which the projection of h is trivial.

Because $G(\mathbb{R})$ is acting transitively on X it is enough to give a morphism $h_0 : \mathbb{S} \to G_{\mathbb{R}}$ to specify a Shimura datum. Therefore a **Shimura datum** is sometimes written as triple (G, X, h_0) or simply by (G, h_0).
Further in our case of interest the following axioms are satisfied and (simplify the situation enormously).

$(SV4)$: The weight homomorphism $\omega_X : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$.
$(SV5)$: The group $C(\mathbb{Q})$ is discrete in $C(\mathbb{A}_f)$.
$(SV6)$: The identity component of the center C^o splits over a CM field.
(SC): The derived group G^{der} is simply connected.
(CT): The center C is a cohomologically trivial torus.

Remark D.1. 1) Axioms $(SV1-6)$ are taken from [Mil04], the other two axioms are taken from [Hid04].
2) The axioms of a Shimura variety (SV1-3) imply, for example, that X is a finite union of hermitian symmetric domains. When viewed as an analytic space we sometimes write x instead of h for points in X and h_x for the associated morphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$.

3) In 3.1 [HP05] a more general definition of a Shimura datum is given. For our purpose Deligne's original definition, as given above, and so called 0-dimensional Shimura varieties are sufficient.

A **morphism of Shimura data** $(G, X) \to (G', X')$ is a morphism $G \to G'$ of algebraic groups which induces a map $X \to X'$.

D.2 Shimura varieties

Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Set $Sh_K = Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, where $G(\mathbb{Q})$ is acting on X and $G(\mathbb{A}_f)$ on the left, and K is acting on $G(\mathbb{A}_f)$ on the right. On can show (see 5.13 [Mil04]) that there is a homeomorphism $Sh_K \cong \bigsqcup \Gamma_g\backslash X^+$. Here X^+ is a connected component of X and Γ_g is the subgroup $gKg^{-1} \cap G(\mathbb{Q})_+$ where g runs runs over a set of representatives of $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$. When K is chosen sufficiently small, then $\Gamma_g\backslash X^+$ is an arithmetic locally symmetric variety. For an inclusion $K' \subset K$ we obtain a natural map $Sh_{K'} \to Sh_K$ and in this way an inverse system $(Sh_K)_K$. There is a natural right action of $G(\mathbb{A}_f)$ on this system (cf., p 55 [Mil04]).
The **Shimura variety** $Sh(G, X)$ associated with the Shimura datum (G, X) is defined to be the inverse limit of varieties $\varprojlim_K Sh_K(G, X)$ together with the natural action of $G(\mathbb{A}_f)$. Here K runs through sufficiently small compact open subgroups of $G(\mathbb{A}_f)$. $Sh(G, X)$ can be regarded as a scheme over $\mathbb{C}$.
Let (G, X) be a Shimura datum such that $(SV5)$ holds, then one has

$$Sh(G, X) = \varprojlim_K Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f). \tag{D.1}$$

In this case we write $[x, l]$ for an element in $Sh(G, X)$ and the (right) action of an element $g \in G(\mathbb{A}_f)$ is given by

$$g[x, l] = [x, lg]. \tag{D.2}$$

In the general case, wenn $(SV5)$ is not holding we use the same notation, understanding that $[x, l]$ stands for a family $(x_K, l_K)_K$ indexed by compact open subgroups K of $G(\mathbb{A}_f)$.
A **morphism of Shimura varieties** $Sh(G, X) \to Sh(G', X')$ is an inverse system of regular maps of algebraic varieties compatible with the action of $G(\mathbb{A}_f)$. We have the following functorial property:
A morphism $\varphi : (G, X) \to (G', X')$ of Shimura data defines an equivariant morphism $Sh(\varphi) : Sh(G, X) \to Sh(G', X')$ of Shimura varieties, which is a closed immersion if $G \to G'$ is injective (Thm 5.16 [Mil04]).

D.3 Example

We want to give some details about the Shimura varieties attached to the data $\mathcal{S}_{Sh}$ constructed in 1.2.2. For the identification of the $GSp(\mathbb{R})$-conjugacy class of h_{cm} with the higher Siegel upper lower half space

$$\mathbb{H}_g^{\pm} = \{M = A + iB \in M_g(\mathbb{C}) \mid A = A^t, B \text{ positive or negative definitive } \}$$

we refer further to exercise 6.2 [Mil04].
In addition the data $\mathcal{S}_{Sh}$ fulfill all the axioms stated in D.1. The validity of $(SV1\text{-}6)$ is

shown on p. 67 [Mil04] and the validity of (SC) and (CT) in [Hid04]. The latter two axioms are important for making the arguments in [MS81] in this case. From $(SV5)$ follows in particular that we don't have to bother about the limits in the definition of $Sh(GSp, \mathbb{H}_g^{\pm})$ because we have

$$Sh(GSp, \mathbb{H}_g^{\pm}) = GSp(\mathbb{Q})\backslash(\mathbb{H}_g^{\pm} \times GSp(\mathbb{A}_f)).$$

D.4 Connected Shimura varieties

A **connected Shimura datum** is a pair (G, X^+) consisting of a semisimple algebraic group G over $\mathbb{Q}$ and a $G^{ad}(\mathbb{R})^+$-conjugacy class of homomorphisms $h : \mathbb{S} \to G^{ad}_{\mathbb{R}}$ satisfying axioms $(SV1$-$3)$.
The **connected Shimura variety** $Sh^o(G, X^+)$ associated with a connected Shimura datum (G, X^+) is defined by the inverse limit

$$Sh^o(G, X^+) = \varprojlim_{\Gamma} \Gamma\backslash X^+ \tag{D.3}$$

where Γ runs over the torsion-free arithmetic subgroups of $G^{ad}(\mathbb{Q})^+$ whose inverse image in $G(\mathbb{Q})^+$ is a congruence subgroup.

If we start with a Shimura datum (G, X) and choose a connected component X^+ of X, we can view X^+ as a $G^{ad}(\mathbb{R})^+$-conjugacy class of morphisms $h : \mathbb{S} \to G^{ad}_{\mathbb{R}}$ by projecting elements in X^+ to $G^{ad}_{\mathbb{R}}$. One can show that (G^{der}, X^+) is a connected Shimura datum. Further if we choose the connected component $Sh(G, X)^o$ of $Sh(G, X)$ containing $X^+ \times 1$, one has the following compatibility relation

$$Sh(G, X)^o = Sh^o(G^{der}, X^+). \tag{D.4}$$

D.5 0-dimensional Shimura varieties

In section 1.2.1 we defined a "Shimura datum" $\mathcal{S}_K = (T^K, X_K)$ which is not a Shimura datum in the above sense because X_K has more than one conjugacy class (recall that T^K is commutative). Rather $\mathcal{S}_K$ is a Shimura datum in the generalized sense of Pink [Pin90] which we don't want to recall here. Instead we define the notion of a 0-dimensional Shimura varieties following [Mil04] which covers all exceptional Shimura data we consider.
We define a **0-dimensional Shimura datum** to be a triple (T, Y, h), where T is a torus over $\mathbb{Q}$, Y a finite set on which $T(\mathbb{R})/T(\mathbb{R})^+$ acts transitively and $h : \mathbb{S} \to T_{\mathbb{R}}$ a morphism of algebraic groups. We view Y as a finite cover of $\{h\}$. We remark that the axioms $(SV1-3)$ are automatically satisfied in this setup.
The associated **0-dimensional Shimura variety** $Sh(T, Y, h)$ is defined to be the inverse system of finite sets $T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/K$ with K running over the compact open subgroups of $T(\mathbb{A}_f)$.
A **morphism** $(T, Y, h) \to (H, h_0)$ from a 0-dimensional Shimura datum to a Shimura datum, with H an algebraic torus, is given by a morphism of algebraic groups $\varphi : T \to H$ such that $h = \varphi_{\mathbb{R}} \circ h_0$.
If φ is such a morphism it defines a morphism $Sh(\varphi) : Sh(T, Y, h) \to Sh(H, h_0)$ of Shimura varieties.

Remark D.2. We have that $\mathcal{S}_K$ fulfills axiom $(SV5)$ if and only if $K = \mathbb{Q}$ or K an imaginary quadratic field (see 3.2 [HP05]).

D.6 Canonical model of Shimura varieties

Let (G,X) be a Shimura datum. A point $x \in X$ is called a **special point** if there exists a torus $T \subset G$ sucht that h_x factors through $T_{\mathbb{R}}$. The pair (T,x) or (T,h_x) is called special pair. If (G,X) satisfies the axioms $(SV4)$ and $(SV6)$, then a special point is called **CM point** and a special pair is called CM pair.

Now given a special pair (x,T) we can consider the cocharacter μ_x of $G_{\mathbb{C}}$ defined by $\mu_x(z) = h_{x,\mathbb{C}}(z,1)$. Denote by $E(x)$ the field of definition of μ_x, i.e. the smallest subfield k of $\mathbb{C}$ such that $\mu_x : \mathbb{G}_{m,k} \to G_k$ is defined.
Let R_x denote the composition

$$T^{E(x)} \xrightarrow{Res_{E(x)/\mathbb{Q}}(\mu_x)} Res_{E(x)/\mathbb{Q}}(T_E(x)) \xrightarrow{Nm_{E(x)/\mathbb{Q}}} T \tag{D.5}$$

and define the **reciprocity morphism**

$$r_x = R_x(\mathbb{A}_f) : \mathbb{A}^{\times}_{E(x),f} \to T(\mathbb{A}_f). \tag{D.6}$$

Moreover every datum (G,X) defines an algebraic number field $E(G,X)$, the reflex field of (G,X). For the definition we refer the reader to 12.2 [Mil04].

Remark D.3. 1) For the Shimura datum $\mathcal{S}_{Sh} = (GSp, \mathbb{H}_g^{\pm})$ (see 1.2.2) we have $E(\mathcal{S}_K) = \mathbb{Q}$ (cf., p. 112 [Mil04]).
2) For explanations to relations with the reflex field of a CM field (cf., B.3), see example 12.4 b) of pp. 105 [Mil04].

A **model** $M^o(G,X)$ of $Sh(G,X)$ over the reflex field $E(G,X)$ is called **canonical** if
1)$M^o(G,X)$ is equipped with a right action of $G(\mathbb{A}_f)$ that induces an equivariant isomorphism $M^o(G,X)_{\mathbb{C}} \cong Sh(G,X)$, and
2) for every special pair $(T,x) \subset (G,X)$ and $g \in G(\mathbb{A}_f)$ the point $[x,g] \in M^o(G,X)$ is rational over $E(x)^{ab}$ and the action of $\sigma \in Gal(E(x)^{ab}/E(x))$ is given by

$$\sigma[x,g] = [x, r_x(\nu)g] \tag{D.7}$$

where $\nu \in \mathbb{A}^{\times}_{E(x),f}$ is such that $[\nu] = \sigma^{-1}$ under Artin's reciprocity map.
In particular, for every compact open subgroup $K \subset G(\mathbb{A}_f)$, it follows that $M^o_K(G,X) = M^o(G,X)/K$ is a model of $Sh_K(G,X)$ over $E(G,X)$.

Remark D.4. Canonical models are known to exist for all Shimura varieties (see [Mil04]).

D.7 Canonical model of connected Shimura varieties

We refer to 2.7.10 [Del79] for the precise definition of the canonical model $M^o(G,X^+)$ of a connected Shimura variety $Sh(G,X^+)$. Here we just want to mention the compatibility

$$M^o(G^{der}, X^+) = M^o(G,X)^o \tag{D.8}$$

where the latter denotes a correctly chosen connected component of the canonical model $M^o(G,X)$.

Appendix E

Compatibility of symmetries with other constructions

We would like to clarify the relation between the different definitions of symmetries of Bost-Connes systems occurring in the literature.
In [LLN09] or in the framework of endomotives, as in our work, symmetries are always given by automorphisms, on the other hand e.g. in [CMR05] symmetries occur also in form of endomorphisms.
Apart from the two natural actions used to define the Bost-Connes system $\mathcal{A}_K$ in form of the action of $I_K = \widehat{\mathcal{O}}_K^\natural / \widehat{\mathcal{O}}_K^\times$ on $Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} \mathrm{Gal}(K^{ab}/K)$ by

$$s \cdot [\rho, \alpha] = [\rho s, [s]^{-1}\alpha]$$

and the action of $\mathrm{Gal}(K^{ab}/K)$ on Y_K given by

$$\gamma \cdot [\rho, \alpha] = [\rho, \gamma\alpha],$$

there is a third natural action of $\widehat{\mathcal{O}}_K^\natural$ on Y_K given by

$$s \star [\rho, \alpha] = [\rho s, \alpha]$$

In this way we get an action of $\mathrm{Gal}(K^{ab}/K)$ as automorphisms on $C(Y_K)$ by

$$^{\gamma}f([\rho, \alpha]) = f([\rho, \gamma^{-1}\alpha])$$

and an action of $\widehat{\mathcal{O}}_K^\natural$ on $C(Y_K)$ as endomorphisms by

$$^{s\star}f([\rho, \alpha]) = \begin{cases} f([\rho s^{-1}, \alpha]) & \text{, if } \rho s^{-1} \in \widehat{\mathcal{O}}_K \\ 0 & \textit{otherwise} \end{cases}$$

The latter action is used for example in [CMR05] to define the symmetries of the corresponding Bost-Connes systems. The two notions of symmetries are related as follows. If we take $s \in \widehat{\mathcal{O}}_K^\natural$, denote by $\gamma = [s] \in \mathrm{Gal}(K^{ab}/K)$ its image under Artin's reciprocity map and by $\overline{s} \in I_K$ the associated integral ideal, we see that for every function $f \in C(Y_K)$ the following relation holds

$$^{s\star}f(\overline{s} \cdot [\rho, \alpha]) = {}^{\gamma}f([\rho, \alpha]) \tag{E.1}$$

This explains why both definitions of symmetries induce the same action on extremal KMS_β-states, for $\beta > 1$, and on extremal KMS_∞-states evaluated on the arithmetic subalgebra.

Remark E.1. One does immediately see that the strict ray class group $Cl_K^+ = \mathrm{Gal}(K^{ab}/K)/[\widehat{\mathcal{O}}_K^\times]$ of K is responsible for the fact that $\widehat{\mathcal{O}}_K^\natural$ acts by endomorphisms on $C(Y_K)$. If the strict ray class group of K is trivial then $\widehat{\mathcal{O}}_K^\natural$ acts by automorphisms as well and the actions of $\mathrm{Gal}(K^{ab}/K)$ and $\widehat{\mathcal{O}}_K^\natural$ agree, in fact.

Appendix F

On Euler's formula

In the following we show that the classical Euler totient function can be naturally generalized to arbitrary number fields. This is surely a well-known result.

Lemma F.1. *For K a number field define the function $\varphi_K : I_K \to \mathbb{N}$ by setting*

$$\varphi_K(\mathfrak{f}) = |(\mathcal{O}_K/\mathfrak{f})^\times| \tag{F.1}$$

Then the following equality holds

$$N(\mathfrak{f}) = |\mathcal{O}_K/\mathfrak{f}| = \sum_{\mathfrak{d}|\mathfrak{f}} \varphi_K(\mathfrak{d}). \tag{F.2}$$

Proof. Thanks to the Chinese reminder theorem, it is enough to show $\varphi_K(\mathfrak{p}^k) = N(\mathfrak{p}^k) - N(\mathfrak{p}^{k-1})$ for all $k \geq 1$. Using the fact that $\mathcal{O}_K/\mathfrak{p}^k$ is a local ring with maximal ideal $\mathfrak{p}/\mathfrak{p}^k$ we obtain $\varphi_K(\mathfrak{p}^k) = |\mathcal{O}_K/\mathfrak{p}^k| - |\mathfrak{p}/\mathfrak{p}^k| = N(\mathfrak{p}^k) - |\mathfrak{p}/\mathfrak{p}^k|$. The isomorphism $(\mathcal{O}_K/\mathfrak{p}^k)/(\mathfrak{p}/\mathfrak{p}^k) \cong \mathcal{O}_K/\mathfrak{p}$ and the multiplicativity of the norm imply $|\mathfrak{p}/\mathfrak{p}^k| = N(\mathfrak{p}^{k-1})$ which finishes the proof. □

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