Thèse de Doctorat de l'Université Paris 6

Spécialité : Mathématiques

présentée par :

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pour obtenir le grade de

Docteur de l'Université Paris 6

sur le sujet :

Quelques résultats en théorie des déformations en géométrie algébrique

Soutenue le 25 avril 2003 devant le jury composé de:

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Remerciements

Tout d'abord, je voudrais remercier de tout mon coeur ma directrice de thèse, Claire Voisin, qui a fait preuve d'une générosité exceptionelle dans le partage de ses connaissances et de sa créativité mathématique. Cette thèse n'aurait jamais vu le jour sans son aide continuelle, et je voudrais exprimer ma profonde reconnaissance pour tout le temps et l'énérgie qu'elle y a consacré.

Olivier Debarre et Geir Ellingsrud ont accepté d'être rapporteurs de cette thèse, laquelle a été énormement ameliorée par leurs nombreux commentaires détaillés. Je les en remercie très chaleureusement, ainsi de s'être deplacés pour la soutenance.

Je remercie aussi Arnaud Beauville, Daniel Huybrechts et Joseph Le Potier d'avoir accepté de faire partie du jury.

Je tiens à remercier vivement tous ceux à l'Ecole Normale Supérieure qui m'ont accueillie quand je suis arrivée en France il y a cinq ans, et qui m'ont soutenue par la suite. De même, je remercie l'Université Paris 6 et plus particulièrement l'équipe de Géométrie et Topologie Algébrique pour leur accueil et leur soutien. Je remercie chaleureusement Ania Otwinowska pour les échanges très interessants que nous avons eus.

C'est pendant mes études à l'Université de Cambridge que je me suis passionnée pour la première fois pour la géométrie algébrique. Je tiens à remercier tous ceux qui m'y ont enseigné les mathématiques, avec une mention toute particulière pour Alessio Corti, directeur de mon mémoire de maîtrise, qui m'a appris tant de choses sur la géométrie et pourquoi elle mérite d'être aimée.

Et pour terminer avec les remerciements professionels : je remercie Phillip Heyes, mon professeur de mathématiques de collège et de lycée. C'est un homme qui savait être à la fois généreux et rigoureux, et qui pendant des années d'enseignement a lancé la carrière de maints jeunes scientifiques— dont la mienne. Je l'en remercie: je sais que je ne suis pas la seule.

Je remercie tous ceux qui ont été mes camarades ou voisins de bureau pendant ces quatres années de m'avoir empêchée de travailler si souvent, c'était mieux ainsi. Donc merci à Aurélie, Gwendal, Layla, Kristian, Madga, Marcos, Masha, Olivier, Pietro et Samy. Je suis en particulière très reconaissante à mes frères mathématiques, Lorenz et Gianluca, pour quatre ans de soutien professionel et pas-professionel-du-tout, et à Ingo de m'avoir prêté un coin de son bureau, parmi d'autres choses. Et bien sûr, merci a Tanguy.

Les amis qui m'ont accompagnée et qui m'ont supportée sont nombreux et chers: je les remercie tous. Dave en particulier s'est montré plus patient que je n'aurais cru possible.

Ma famille a toujours été là pour moi, et je ne serais jamais parvenue au bout sans eux. Merci donc a mon père, ma mère et ma soeur: ca me fait très plaisir qu'ils puissent être là aujourd'hui. C'est a eux que je dédie ce travail.

Contents

Chapter 1

Introduction

This thesis contains the work on various problems that I have considered over the last three years. It is divided into four independent sections dealing with four independent problems.

1.0.1 Section 1— Deformation theory

This section corresponds to chapter 2 of my thesis. In it, we recall the basic results of deformation theory and then extend these results to a more general problem, namely, the construction of formal neighbourhoods of a given scheme X with specified normal bundle. All our schemes will be of finite type over a base field. More precisely, we define more general deformations in the following way.

Definition 1 *Let X be an l.c.i. scheme and V a vector bundle. An n-th order formal deformation of X with normal bundle V is a scheme X_n together with an embedding $i : X \to X_n$ and an isomorphism*

$$j : I_X/I_X^2 \simeq V^*$$

(I_X is here the ideal sheaf of X in X_n) such that

1. *$I_X^{n+1} = 0$ in $\mathcal{O}_{X_n}$,*
2. *The multiplication map $j^{\otimes n} : \mathrm{Sym}^n V^* \to I_X^n$ is an isomorphism.*

Although the results of abstract deformation theory do not translate directly into this context, the results for embedded deformations carry over. We define an embedded generalised deformation in the following way.

Definition 2 *An embedded generalised n-th order deformation of X with normal bundle V is given by the following data.*

1. *A smooth scheme P and an embedding $X \to P$,*

2. *A vector bundle $\tilde{V} \to P$ and an isomorphism $\tilde{V}|_X = V$,*

3. *A subscheme X_n of P_n (the subscheme of $\tilde{V}$ whose ideal sheaf is $I^{n+1}_{P/\tilde{V}}$) such that $X_n \cap P = X$ and the restriction and multiplication maps*

$$r : V^* \to I_{X/X_n}/I^2_{X/X_n}$$

$$r^{\otimes n} : \mathrm{Sym}^n V^* \to I^n_{X/X_n}/I^{n+1}_{X/X_n}$$

   are isomorphisms.

By I_{X/X_n} we mean the ideal sheaf of X in X_n.

We then prove the following results for these embedded deformations.

Theorem 1 *Let X_n be an n-th order embedded deformation of X. We can assign to any pair (X^1, X^2) of extensions of X_n to $(n+1)$-st order embedded deformations of X an element $d(X^1, X^2) \in \mathrm{Hom}(N^*_X, \mathrm{Sym}^{n+1} V^*)$ such that*

1. *If X^3 is another extension of X_n, then $d(X^1, X^2) + d(X^2, X^3) = d(X^1, X^3)$ and $d(X^1, X^2) = -d(X^2, X^1)$,*

2. *If X is generically smooth and the push-forward of*

$$0 \to N^*_X \to \Omega^1_P \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

   along $d(X^1, X^2)$ is a trivial extension then X^1 and X^2 are isomorphic as abstract infinitesimal neighbourhoods of X_n,

3. *If any extensions of X_n exist they form a principal homogeneous space under $\mathrm{Hom}(N^*_X, \mathrm{Sym}^{n+1} V^*)$.*

Theorem 2 *We can associate to any X_n an element*

$$\omega_{X_n} \in H^1(X, \mathrm{Hom}(N^*_X, \mathrm{Sym}^{n+1} V^*))$$

(or alternatively

$$\omega_{X_n} \in H^1(X, N_X \otimes \mathrm{Sym}^{n+1} V^*))$$

in such a way that X_n can be extended to an $(n+1)$-st order embedded deformation of X if and only if $\omega_{X_n} = 0$.

In all the above, N_X^* denotes the conormal bundle $I_{X/P}/I_{X/P}^2$, and N_X denotes its dual.

These results are fairly straightforward generalisations of the equivalent results for ordinary deformations, but I have not found them in the literature. The results for abstract higher-order deformations cannot be directly translated into this context, since they depend on the existence of a canonical isomorphism between I_{X/X^1} and I_{X/X^2}, (X^1 and X^2 being two separate $(n+1)$-st order formal neighbourhoods of X which have the same n-th part) which does not now exist.

In certain circumstances, all obstruction groups vanish for appropriate choices of normal bundles V. We can then apply these theorems to prove the following results.

Theorem 3 *Let X be a projective local complete intersection scheme. Then there exists a smooth formal neighbourhood X_∞ of X, a vector bundle V on X_∞ and a section $\sigma : X_\infty \to V$ such that*

- *V is a direct sum of line bundles,*
- *The rank of the vector bundle V is equal to the codimension of X in X_∞,*
- *X is schematically the zero locus of σ.*

This last result was the subject of a short article in the C.R.A.S [**?**].

1.0.2 Section 2— The Noether-Lefschetz locus

This section corresponds to chapters 3-5 of my thesis. The material in sections 3-4 form an article submitted for publication, [**?**], and the material in Chapter 5 form another. Whilst writing this thesis I learnt that similar results have been independently obtained by Ania Otwinowska in her articles [**?**] and [**?**]. I am grateful to her for communicating her results to me and for the very interesting discussions we have had.

If X is a generic degree d surface in $\mathbb{P}^3$ and $d > 3$, the Noether-Lefschetz theorem says that

$$H^{1,1}_{\mathrm{prim}}(X, \mathbb{Z}) = 0.$$

Here, $H^{1,1}_{\mathrm{prim}}(X, \mathbb{Z})$ means the group of all primitive Hodge classes on X. A cohomology class on X is said to be primitive if it is orthogonal for the cup product to all cohomology classes inherited from projective space. The space of surfaces which do not satisfy this condition is known as the Noether-Lefschetz locus; it is a union of countably many algebraic subvarieties of U_d, the parameter space of smooth degree d surfaces in $\mathbb{P}^3$.

Let W be a component of the Noether-Lefschetz locus. It can be shown that W is locally the zero locus of a certain section of the bundle $\mathcal{H}^{0,2}_{U_d}$. The latter is the bundle on the parameter space U_d whose fibre over the point $[X]$ is $H^{0,2}(X, \mathbb{C})$; it is therefore a bundle of rank $h^{0,2}(X, \mathbb{C}) = \binom{d-1}{3}$. This gives us a prediction of the codimension of W— namely $\binom{d-1}{3}$— and we say that a component is exceptional if its codimension is strictly smaller than this predicted dimension. In this section we will study the infinitesimal geometry of these exceptional components.

Let X be a point of W and F the defining polynomial of X. Let σ be the section defining W in some sufficiently small neighbourhood of X. In [**?**], Carlson and Griffths gave a complete description of the map

$$d\sigma : TU_d \otimes \mathcal{O}_W \to \mathcal{H}^{0,2}$$

in terms of the multiplication in the Jacobian ring R_F associated to F. In chapter 3 we extend this result by calculating the fundamental quadratic form of σ as a polynomial invariant in the same ring.

In Chapter 4 we use this invariant to study those Noether-Lefschetz components in U_5 whose tangent spaces are of exceptional codimension because they are non-reduced. More precisely, let γ be an element of $H^{1,1}_{\text{prim}}(X, \mathbb{Z})$ and let $NL(\gamma)$ be the component of the Noether-Lefschetz locus associated to γ in some sufficiently small neighbourhood of X. By this we mean the following thing. Fix O, a simply connected neighbourhood of X in U_d, and consider $\overline{\gamma}$, the section of $\mathcal{H}|_O$ which is obtained by flat transport of γ. The scheme $NL(\gamma)$ is then defined to be the set of points of O over which $\overline{\gamma}$ is a $(1, 1)$ Hodge class. We will prove the following theorem.

Theorem 4 *Suppose that $NL(\gamma) \subset O \subset U_5$ is non-reduced and X is a point of $NL(\gamma)$. Then there is a hyperplane $H \subset \mathbb{P}^3$ such that $H \cap X$ contains 2 distinct lines L_1 and L_2 and non-zero distinct integers α and β, such that*

$$\gamma = \alpha[L_1] + \beta[L_2] - \frac{\alpha + \beta}{5} H.$$

In Chapter 5 we then give another application. It was conjectured by Harris and Ciliberto that if X is any point in an exceptional Noether-Lefschetz locus, then there is some surface S of degree $\leq d - 4$ such that $X \cap S$ is reducible. This was shown to be false by Voisin in [**?**]. We will obtain a lower bound which is cubic in d for the codimensions of Noether-Lefschetz components which violate the Ciliberto-Harris conjecture. In the following theorem (and indeed, throughout the rest of this thesis), S^e denotes the vector space of degree e homogeneous polynomials in four variables. (U_e is therefore an open subset of the projectivisation of S^e.)

Theorem 5 *Suppose that* $e \leq \frac{d-1}{2}$. *There then exists an integer* $\phi_e(d)$ *such that if* $NL(\gamma)$ *is reduced and* $\text{codim}(NL(\gamma)) \leq \phi_e(d)$ *then there is a polynomial* $Q \in S^e$ *such that* γ *is supported on* Q. *Further,* $\phi_{\frac{d-1}{2}}(d) = O(d^3)$.

Here, when we say that γ is supported on Q, we mean that γ is a linear combination of classes of curves contained in $\{Q = 0\}$. Note that γ supported on Q implies that $X \cap \{Q = 0\}$ is reducible. Indeed, if the curve $X \cap \{Q = 0\}$ is irreducible, then γ supported on Q implies that γ is a multiple of $X \cap \{Q = 0\}$. However, γ has been assumed primitive, and hence this implies that $\gamma = 0$.

To the best of my knowledge all the bounds previously obtained for the codimension of components violating the Ciliberto-Harris conjecture are linear or quadratic in d (*cf.* [**?**], [**?**], [**?**] and [**?**]).

We will in fact prove a slightly stronger form of this theorem, which may be found on page 72 of this thesis.

In particular, this shows that the number of reduced Noether-Lefschetz components of codimension $\leq \phi_{\frac{d-1}{2},1}(d)$ is finite. This can be seen by Hilbert scheme considerations. If a curve C is contain in the intersection of a smooth degree d surface and a degree e surface, where $e \leq \frac{d-1}{2}$, then there are only a finite number of possibilities for the Hilbert polynomial of C. Indeed, the degree of C is bounded by $d\frac{d-1}{2}$ and its genus is bounded by the genus of smooth complete interesection curves.

If we now denote the set of such possible polynomials by T, we can construct a *finite* number of schemes which parameterise all pairs (C, X) such that

1. The Hilbert polynomial of C is in T.
2. X is a smooth degree d surface containing C.

The fact of being a complete intersection of a degree d and a degree e surface is an open condition in a Hilbert scheme of curves in $\mathbb{P}^3$ (see [**?**]). We can therefore construct a *finite* number of schemes parameterising all pairs (C, X) such that

1. The Hilbert polynomial of C is in T.

2. X is a smooth degree d surface containing C.

3. C is not a complete intersection of X with another surface.

These schemes have a projection to U_d. The result above implies, in particular, that any Noether-Lefschetz component of codimension $\leq \phi_{\frac{d-1}{2},1}(d)$ is the image of a component of one of these schemes.

A special case of the result given on page 72 is the result of Voisin [**?**] and Green [**?**] according to which all Noether-Lefschetz loci have codimension $\geq d-3$,with equality only holding for the space of surfaces containing a line— albeit with the additional hypothesis that $NL(\gamma)$ be reduced. Note that the expected codimension of Noether-Lefschetz loci is itself cubic in d.

1.0.3 Section 3— The Chow groups of K3 surfaces

In this short section which corresponds to chapter 6 of this thesis we study the Chow groups of K3 surfaces. The structure of Chow groups has been known to be intimately linked to the number of sections of K_S ever since Mumford proved in his seminal paper [**?**] that if S is a surface and $H^0(S, K_S) \neq 0$, then the Chow group of S is not representable. Bloch has conjectured that the converse holds— a converse which has been proved, in [**?**], for any surface not of general type.

In this section, we will show that there is a close connection between the fact of having dense Chow group orbits and having $h^0(S, K_S) \leq 1$. We will say that two points x and y are equivalent (and will write $x \equiv y$) if the zero-cycles $[x]$ and $[y]$ are equal in the Chow group $A^0(X)$. We will prove the following theorem.

Theorem 6 *Let S be a general smooth projective K3 surface. Then for general $x \in S$, the set*

$$\{y \in S | y \equiv x\}$$

is dense in S (for the complex topology). Further, if T is a projective complex surface such that the set

$$\{y \in T | y \equiv x\}$$

is Zariski dense in T for a generic point x of T then $h^{2,0}(T) \leq 1$.

1.0.4 Section 4— Gromov-Witten invariants

This section, corresponding to the chapter 7 of the thesis, is devoted to the proof of the following theorem, which was the subject of a note in the C.R.A.S., [**?**]. Let

$$\begin{array}{ccc} F : C & \rightarrow & X \\ \downarrow & & \downarrow \\ f : \Delta & \rightarrow & \Delta \end{array}$$

be a commutative diagram of proper holomorphic maps in which Δ is a complex disc, X is a smooth variety and X_0 is a normal crossing variety. We assume that X_0 is the union of two irreducible smooth varieties X_1 and X_2 which are glued together along isomorphic smooth divisors $Z_1 \subset X_1$ and $Z_2 \subset X_2$. We assume further that C is a flat family of prestable curves and F is a family of stable maps.

The central curve C_0 is the union of two (not necessarily connected) prestable curves C_1 and C_2 such that C_i maps into X_i under F. We denote by F_i the restriction of F to C_i. There exist r-tuples of points, $(x_1^1, \ldots, x_r^1)$ and $(x_1^2, \ldots, x_r^2)$, where for all $j \in \{1, 2 \ldots r\}$, $x_j^1 \in C_1$ and $x_j^2 \in C_2$, such that

$$C_0 = C_1 \cup_{x_j^1 = x_j^2} C_2.$$

In general, given varieties, Y, V and U, a regular embedding $i : U \rightarrow Y$ of codimension d and a map $f : V \rightarrow Y$, Fulton defines in [10], Chapter 6 (see in particular the summary page 92 and §6.1) an element

$$U \cdot_f V \subset A_{k-d}(f^{-1}(U)),$$

where k is the dimension of V. (The subscript f in the notation is not contained in Fulton: it is here included for clarity's sake.) This coincides with the intersection product $U \cap V$ when f is a closed embedding. We will prove the following theorem on the Fulton intersections of C_i and Z_i.

Theorem 7 *There exist integers m_j such that for $i = 1$ or 2 we have*

$$\sum_{j=1}^{j=r} m_j x_j^i = Z_i \cdot_{F_i} C_i$$

as elements of the group

$$A_0(F_i^{-1}(Z_i)).$$

Note that it is immediate by definition of x_j^i that it is a point in $F_i^{-1}(Z_i)$. It follows that $\sum_{j=1}^{j=r} m_j x_j^i$ is a meaningful element of $A_0(F_i^{-1}(Z_i))$.

In the case where C_1 and C_2 intersect Z_1 and Z_2 in points (i.e. there are no components of the central curve contained in the central divisor), then it can be shown that

$$F_i^{-1}(Z_i) = \cup_j x_j^i,$$

and hence

$$A_0(F_i^{-1}(Z_i)) = \oplus_j \mathbb{Z} x_j^i.$$

The result above then says simply the following.

Condition (A): *Let x be a point of C_i which is mapped to Z_i under F_i. Then x is a point of gluing of C_1 and C_2. The multiplicities of x in the intersection products $C_1 \cap Z_1$ and $C_2 \cap Z_2$ are equal.*

The result then says that the condition given in the theorem, which was considered by Gathmann in his definition of α-relative maps in [**?**], is the right generalisation of Condition (A) to the case where C_i has components which are mapped into Z_i under F.

This result provides a geometric justification of the work of Li in [**?**], [**?**] in which, emulating the work of Li and Ruan in the symplectic setting of [**?**], he derives a recurrence formula for the Gromov-Witten invariants of X_t in terms of the relative Gromov-Witten invariants of X_1 and X_2.

Chapter 2

Deformations of l.c.i.s

2.1 Introduction

If $X \subset \mathbb{P}^N(k)$ (k being any field) is a local complete intersection scheme, then X is not necessarily a global complete intersection in projective space — that is, X is not necessarily embedded in $\mathbb{P}^N(k)$ as the vanishing locus of codim X polynomials. It seems natural to ask whether this is true for more general ambient varieties. In particular, given such an X, we may wonder whether it can be embedded in some smooth Y as a globally complete intersection, i.e., as the intersection of codim(X) hypersurfaces. The aim of this chapter is to answer this question in the affirmative, at least formally, by proving the following result.

Theorem 8 *Let $X \subset \mathbb{P}^N(k)$ be a local complete intersection scheme. Then there exists a smooth formal neighbourhood X_∞ of X, a vector bundle V on X_∞ and a section $\sigma : X_\infty \to V$, such that*

1. *V is a direct sum of line bundles,*
2. *The rank of the vector bundle V is equal to the codimension of X in X_∞,*
3. *X is schematically the zero locus of σ.*

We begin this section by recalling the basic objects and results of deformation theory. We will then generalise these ideas to the problem of constructing more general formal neighbourhoods of the given scheme, especially formal neighbourhoods in which the normal bundle of the original variety is not trivial. In certain cases, we will be able to show that the construction of successive formal neighbourhoods is un-obstructed and that hence the formal neighbourhood that we seek exists.

2.2 Preliminaries on deformation theory

The material contained in this section is well-known. Proofs of all results cited below may be found in [?] or in a much more general context in [?]. The presentation of the results in the next section is heavily influenced by [?].

Definition 3 *Suppose that X is a scheme defined over a field k. By a first order deformation of X, we mean a flat pull-back diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_\epsilon \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k[\epsilon]/\epsilon^2). \end{array}$$

More generally, if A is a local Artinian k-algebra, then

Definition 4 *A deformation of X over* $\mathrm{Spec}(A)$ *is a scheme X_A which is flat over A together with a pull-back diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

Two deformations, X_A, X'_A of X over A are said to be isomorphic if there is an isomorphism

$$\begin{array}{ccc} X_A & \simeq & X'_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & = & \mathrm{Spec}(A). \end{array}$$

which induces the identity on the central fibres.

For X a generically smooth locally complete intersection the following result is well-known.

Theorem 9 *The first order deformations of X are classified by elements of* $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$.

This association of an element of $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ to a first order deformation X_ϵ is straightforward— we simply associate to X_ϵ the exact sequence

$$0 \to \mathcal{O}_X \to \Omega_{X_\epsilon} \otimes \mathcal{O}_X \to \Omega_X \to 0. \qquad (2.1)$$

This is exact because X is generically smooth and a locally complete intersection. Similarly, given an extension

$$0 \to \mathcal{O}_X \xrightarrow{i_E} E \xrightarrow{\pi_E} \Omega_X \to 0, \qquad (2.2)$$

we can construct from it a first-order deformation of X in the following way. A first order deformation of X is simply an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \xrightarrow{i_F} F \xrightarrow{\pi_F} \mathcal{O}_X \to 0 \tag{2.3}$$

having the following properties:

1. F is a sheaf of k-algebras,
2. π_F is a morphism of k-algebra sheaves,
3. $\mathrm{Im}(i_F) = I$ is an ideal of F such that $I^2 = 0$.

Given a short exact sequence (**??**), we now construct a sheaf of k-algebras $A(E)$. As a sheaf of abelian groups $A(E)$ is defined to be the kernel of the map

$$\phi : \mathcal{O}_X \oplus E \to \Omega_X$$

which is given by

$$\phi(f, e) = df - \pi_E(e).$$

We impose a multiplication on $A(E)$ given by

$$(f_1, e_1) \times (f_2, e_2) = (f_1 f_2, f_1 e_2 + e_1 f_2).$$

This gives a first-order deformation of X. These two maps, which are inverse to each other, give the required equivalence.

Suppose that X is a locally complete intersection in V, a smooth variety over k. We define first order deformations in the following way.

Definition 5 *A first-order deformation of the embedding $i : X \to V$ is a scheme*

$$X_\epsilon \subset V \times \mathrm{Spec}(k[\epsilon]/\epsilon^2),$$

flat over $k[\epsilon]/\epsilon^2$, such that $X_\epsilon \cap (V \times \mathrm{Spec}(k)) = X$.

We define N_X^*, the co-normal bundle of X in V, to be $I_{X/V}/I_{X/V}^2$. We further define N_X, the normal bundle, to be its dual. We then have the following result which is discussed in [**?**].

Theorem 10 *There is a canonical isomorphism between first-order deformations of the embedding $i : X \to V$ and $H^0(X, N_X)$.*

Note that this result does not require the generic smoothness of X.
When X is generically smooth, there is a natural connection between these two results. Let

$$0 \to N_X^* \to \Omega_V \otimes \mathcal{O}_X \to \Omega_X \to 0 \tag{2.4}$$

be the standard exact sequence of sheaves of Kähler differentials (which is exact because X is l.c.i. and generically smooth). Let $E(X_\epsilon)$ be the extension of Ω_X by $\mathcal{O}_X$ associated to X_ϵ (as an abstract deformation of X) and $d(X_\epsilon)$ the element of $H^0(X, N_X) = \mathrm{Hom}(N_X^*, \mathcal{O}_X)$ associated to X_ϵ (as an embedded deformation of X). Then we have the following result.

Theorem 11 *$E(X_\epsilon)$ is the push-forward of (**??**) along the map $d(X_\epsilon)$.*

Similar results hold for extensions of deformations over Spec(A) to deformations over $\mathrm{Spec}(A')$, when A is a quotient ring of A'. Suppose that we have an exact sequence,

$$0 \to \mathfrak{a} \to A' \to A \to 0,$$

such that

1. A' is a local Artinian k-algebra,

2. $\mathfrak{a}$ is an ideal of A' such that $\mathfrak{a} \cdot \mathfrak{m}_{A'} = 0$.

Let X_A be a deformation of X over $\mathrm{Spec}(A)$. We then define an extension of X_A over $\mathrm{Spec}(A')$ as follows.

Definition 6 *An extension of X_A over $\mathrm{Spec}(A')$ is a flat pull-back diagram*

$$\begin{array}{ccc} X_A & \longrightarrow & X_{A'} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A'). \end{array}$$

The following theorems can be found in [**?**].

Theorem 12 *To any ordered pair (X^1, X^2) of extensions of X_A over $\mathrm{Spec}(A')$, we can assign a difference*

$$\mathcal{D}(X^1, X^2) \in \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X \otimes \mathfrak{a})$$

in such a way that if any extensions of X_A over A' exist, then they form a principal homogeneous space over $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X \otimes \mathfrak{a})$.

Theorem 13 *We can associate to X_A, a deformation of X over* $\mathrm{Spec}(A)$*, an element*

$$\omega_{X_A} \in \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X \otimes \mathfrak{a}),$$

such that extensions of X_A over $\mathrm{Spec}(A')$ *exist if and only if* $\tilde{\omega}_{X_A} = 0$.

The fact that the isomorphism between deformations of X_A and extension classes is no longer canonical corresponds to the fact that there is no canonical inclusion $A \to A'$. It is precisely the inclusion $k \to k[\epsilon]/\epsilon^2$ which allows us to choose, for first order deformations, a canonical base deformation, $\mathcal{O}_{X_\epsilon} = \mathcal{O}_X \oplus \epsilon\mathcal{O}_X$.

There is a concept of extensions of embedded deformations. If V is a smooth variety over k and $X_A \subset V \times \mathrm{Spec}(A)$ is a subscheme which is flat over A such that $X_A \cap (V \times \mathrm{Spec}(k)) = X$, then we define extensions of X_A in the following way.

Definition 7 *An extension of the embedding $i : X_A \to V \times \mathrm{Spec}(A)$ over A' is a variety*

$$X_{A'} \subset V \times \mathrm{Spec}(A'),$$

flat over A', such that $X_{A'} \cap (V \times \mathrm{Spec}(A)) = X_A$.

Once again, there is a theory of the classification of embedded deformations which does not require the generic smoothness of X.

Theorem 14 *To any ordered pair $(X^1, X^2) \hookrightarrow V \times \mathrm{Spec}(A')$ of extensions of X_A over A', we can assign a difference*

$$\mathcal{D}(X^1, X^2) \in H^0(N_X) \otimes \mathfrak{a}$$

in such a way that if any extensions of X_A over A' exist then they form a principal homogeneous space over $H^0(X, N_X) \otimes \mathfrak{a}$.

Here N_X is once again the normal bundle (as defined on page 13) of X with respect to the inclusion $X \to V$.

Theorem 15 *We can associate to $X_A \in V \times \mathrm{Spec}(A)$, an embedded deformation of X over* $\mathrm{Spec}(A)$*, an element*

$$\tilde{\omega}_{X_A} \in H^1(X, N_X) \otimes \mathfrak{a},$$

such that extensions of $X_A \subset V \times \mathrm{Spec}(A)$ over $\mathrm{Spec}(A')$ *exist if and only if*

$$\omega_{X_A} = 0.$$

Compatibility results similar to those given above still hold.

Theorem 16 *$D(X^1, X^2)$ is the push-forward of (**??**) along the map $\mathcal{D}(X^1, X^2)$. ω_{X_A} is the image of $\tilde{\omega}_{X_A}$ under the boundary map*

$$\mathrm{Ext}^1(N_X^*, \mathcal{O}_X \otimes \mathfrak{a}) \to \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X \otimes \mathfrak{a}).$$

Note that $H^1(X, N_X \otimes \mathfrak{a}) = H^1(\mathrm{Hom}(N_X^*, \mathcal{O}_X \otimes a)) \subset \mathrm{Ext}^1(N_X^*, \mathcal{O}_X \otimes \mathfrak{a})$.

2.3 Generalised deformation theory

We will now think of a deformation as a sequence of formal neighbourhoods of the space X, each containing the last, rather than as a family of schemes fibred over an Artinian base. Given this interpretation, it seems natural to ask whether or not the techniques of deformation theory can be applied to the construction of more general formal neighbourhoods— and in particular, to the construction of formal neighbourhoods in which the original space X has non-trivial normal bundle. (It will be noted that X has trivial normal bundle in X_A). More precisely, we ask the following question.

Question 1 *Suppose we have a variety, X, and an n-th formal neighbourhood X_n of X which we want to extend to an $(n+1)$-st formal neighbourhood X_{n+1}. Which of the results of deformation theory are still valid in this context ?*

We need first to define what we mean by a n-th order formal neighbourhood.

Definition 8 *Let X be an l.c.i. variety and V a vector bundle over X. An n-th order formal neighbourhood of X with normal bundle V is a scheme X_n together with an embedding $i : X \to X_n$ such that*

1. *$I_{X/X_n}^{n+1} = 0$ in $\mathcal{O}_{X_n}$,*
2. *There is an isomorphism $j : V^* \simeq I_{X/X_n}/I_{X/X_n}^2$,*
3. *The multiplication map $j^{\otimes n} : \mathrm{Sym}^n V^* \to I_{X/X_n}^n$ is an isomorphism.*

The last condition replaces the fact that X_A should be flat over A. It is justified by the following lemma.

Lemma 1 *Let X_A be a deformation of X over $A = k[x_1, x_2, \ldots, x_l]/\mathfrak{m}^{n+1}$, where $\mathfrak{m}$ is the ideal $[x_1, x_2, \ldots, x_l]$. Define W to be $I_{X/X_A}/I_{X/X_A}^2$. Then X_A is flat over A if and only if*

1. *$W = \mathcal{O}_X^{\oplus l}$,*

2. *The successive multiplication maps*

$$\mathrm{Sym}^k(W) \to I^k_{X/X_A}/I^{k+1}_{X/X_A}$$

are isomorphisms whenever $2 \leq k \leq n$.

Proof of Lemma 1.

We proceed by induction on n.

For $n = 1$, the ideals of A are precisely the subvector-spaces of $\mathfrak{m}_A$, the maximal ideal of A, and the flatness of $\mathcal{O}_{X_A}$ is then equivalent to $W = \mathcal{O}^l_X$.

Assume the result holds for $(n-1)$. We will show that X_A is flat over A if and only if

1. $X_A \otimes_A A/\mathfrak{m}^n_A$ is flat over $A/\mathfrak{m}^n_A$ and
2. The multiplication map $\mathcal{O}_X \otimes \mathrm{Sym}^n(x_1, x_2, \ldots, x_l) \to \mathcal{O}_{X_A}$ is injective.

Here, $\mathrm{Sym}^n(x_1, x_2, \ldots, x_l)$ denotes the k-vector space of all homogeneous polynomials of degree n in the variables $(x_1, \ldots, x_l)$ with coefficients in k.

We prove first the necessity of these two conditions. Consider X_A. We note that in general if M is a flat A module and N is an $A/\mathfrak{m}^n_A$-module, then

$$\mathrm{Tor}^A_i(M, N) = \mathrm{Tor}^{A/\mathfrak{m}^n_A}_i(M \otimes_A (A/\mathfrak{m}^n_A), N)$$

This can be seen from the spectral sequence for a composition of functors, noting that if N is a $A/\mathfrak{m}^n_A$-module then

$$\otimes_A N = (\otimes_{A/\mathfrak{m}^n_A} N) \circ (\otimes_A A/\mathfrak{m}^n_A)$$

and recalling that since M is flat, $\mathrm{Tor}^A_i(M, A/\mathfrak{m}^n_A) = 0$ whenever $i > 0$.

It follows by the Tor characterisation of flat modules that if X_A is flat over A, then $X_A \otimes_A A/\mathfrak{m}^n_A$ is flat over $A/\mathfrak{m}^n_A$. This is the first condition.

Further, if X_A is flat over A, then the multiplication map $\mathcal{O}_{X_A} \otimes_A I \to O_{X_A}$ is injective for any ideal $I \subset A$. But this holds in particular for $I = \mathfrak{m}^n_A$. Writing $\mathfrak{m}^n_A = \mathrm{Sym}^n(x_1, x_2, \ldots, x_l)$, we obtain the second condition.

We now prove sufficience. The scheme X_A is flat over A if and only if

For every ideal $I \subset A$, the multiplication map $I \otimes_A \mathcal{O}_{X_A} \to \mathcal{O}_{X_A}$ is injective.

Suppose that the two conditions hold, and that I is an ideal of A. Then, by the second condition, we know that

$$(\mathcal{O}_{X_A} \otimes A/\mathfrak{m}_A^n) \otimes_{A/\mathfrak{m}_A^n} I/(I \cap \mathfrak{m}_A^n) \to (\mathcal{O}_{X_A} \otimes A/\mathfrak{m}_A^n)$$

is injective. Therefore, given the exact sequence of A-modules

$$I \cap \mathfrak{m}_A^n \to I \to I/(I \cap \mathfrak{m}_A^n) \to 0$$

we see that if $x \in I \otimes_A O_{X_A}$ is contained in the kernel of the multiplication map, then we have

$$x \in \operatorname{Im}(\mathfrak{m}_A^n \cap I) \otimes_k \mathcal{O}_X.$$

The tensor product can be taken over k without loss of information because because the space $\mathfrak{m}_A^n$ is a k-vector space. However, we have that

$$\operatorname{Im}(\mathfrak{m}_A^n \cap I) \otimes_k \mathcal{O}_X \subset \mathcal{O}_X \otimes_k \operatorname{Sym}^n(x_1, x_2, \ldots, x_l)$$

Here we make the natural identification between $\mathfrak{m}_A^n$ and $\operatorname{Sym}^n(x_1, x_2, \ldots, x_l)$. Condition 2 tells us the multiplication map

$$\mathcal{O}_X \otimes \operatorname{Sym}^n(x_1, x_2, \ldots, x_l) \to \mathcal{O}_{X_A}$$

is injective: it follows that the multiplication map on $\operatorname{Im}(\mathfrak{m}_A^n \cap I) \otimes_k \mathcal{O}_X$ is also injective. It follows that for any ideal of A, I, the multiplication map

$$I \otimes_A \mathcal{O}_{X_A} \to O_{X_A}$$

is injective. This is precisely the multiplicative criterion for flatness.

We now complete the proof of the lemma. By the induction hypothesis, the first condition is equivalent to

1. $W = \mathcal{O}_X^l$,
2. $\operatorname{Sym}^k(W) \to I_{X/X_A}^k / I_{X/X_A}^{k+1}$ is an isomorphism for $k < n$.

The second condition is equivalent to $\operatorname{Sym}^n(W) \to I_{X/X_A}^n / I_{X/X_A}^{n+1}$ is an isomorphism. This completes the proof of Lemma 1. □

Our first result is the following.

Proposition 1 *Assume that X is generically smooth. Then first-order formal neighbourhoods of X with normal bundle V are classified up to isomorphism by* $\mathrm{Ext}^1(\Omega_X, V^*)$.

Proof of Proposition 1.

We associate to X_ϵ, a first-order formal neighbourhood of X, the exact sequence

$$0 \to V^* \to \Omega_{X_\epsilon} \otimes \mathcal{O}_X \to \Omega_X \to 0,$$

which we denote by $E(X_\epsilon)$. This sequence is exact because X is generically smooth. We need to show that this sequence is enough to enable us to recover X_ϵ. Given an exact sequence

$$0 \to V^* \xrightarrow{i} E \xrightarrow{\pi} \Omega_X \to 0, \tag{2.5}$$

we define a sheaf of k-algebras on X, $A(E)$, as in the previous section. As a sheaf of abelian groups, $A(E)$ is defined to be the kernel of the map

$$\phi : \mathcal{O}_X \oplus E \to \Omega_X$$

which is given by

$$\phi(e, f) = df - \pi(e).$$

We now put a k-algebra structure on $A(E)$. We define the multiplication on $A(E)$ by

$$(f_1, e_1) \times (f_2, e_2) = (f_1 f_2, f_2 e_1 + f_1 e_2).$$

This gives a generalised first-order deformation of X. We now need to show the following.

Proposition 2 *The algebra sheaf $A(E(X_\epsilon))$ is equal to $\mathcal{O}_{X_\epsilon}$.*

Proof of Proposition 2.

Let E be the $\mathcal{O}_X$-module $\Omega_{X_\epsilon} \otimes \mathcal{O}_X$. Note that there is a map

$$\tau : \mathcal{O}_{X_\epsilon} \to \mathcal{O}_X \oplus E$$

given by $\tau(f) = f|_X + df$. The image of this map lies in $A(E)$. We will show first that τ is injective. We have $\tau(f) = 0$ if and only if

1. $f \in I_{X/X_\epsilon} = V^*$ and

2. $df = 0$.

We know, however, that the map

$$d : V^* \to \Omega_{X_\epsilon} \otimes \mathcal{O}_X$$

is an injection, whence it follows that $f = 0$.

Now we show that τ is surjective. Indeed, defining p to be the projection from $E \oplus \mathcal{O}_X$, we know that

$$p \circ \tau : \mathcal{O}_{X_\epsilon} \to \mathcal{O}_X$$

is a surjection, so it remains only to show that V^* is contained in the image of τ.

If $f \in I_{X/X_\epsilon} = V^*$ then $\tau(f) = (0, df)$. It follows that $i(f) = \tau(f)$, and hence $i(V^*) \in \mathrm{Im}\tau$. This completes the proof of Proposition 2. ☐

We now need the following result.

Proposition 3 *We have $E(A(E)) = E$.*

Proof of Proposition 3.

Since the map $A(E) \to E$ is a derivation, it factors through an $\mathcal{O}_X$-module map

$$d : \Omega_{A(E)} \otimes \mathcal{O}_X \to E.$$

Let π_E and $\pi_{\Omega_{A(E)} \otimes \mathcal{O}_X}$ be respectively the projections from E and $\Omega_{A(E)} \otimes \mathcal{O}_X$ to $\mathcal{O}_X$. Let i_E and $i_{\Omega_{A(E)} \otimes \mathcal{O}_X}$ be the inclusions of V^* into respectively E and $\Omega_{A(E)} \otimes \mathcal{O}_X$. It is immediate that

$$\pi_E \circ d = \pi_{\Omega_{A(E)} \otimes \mathcal{O}_X}$$

and

$$d \circ i_{\Omega_{A(E)} \otimes \mathcal{O}_X} = i_E.$$

It follows that this map is in fact an isomorphism of extensions. This completes the proof of Proposition 3. ☐

Propositions 2 and 3 together establish Proposition 1. ☐

For higher-order deformations, things are a little more complicated. The classification of abstract higher-order deformations does not translate directly into this context. However, the theory of embedded deformations can be translated into this general setting without any problems.

Definition 9 *An embedded generalised n-th order deformation of X with normal bundle V is given by the following data.*

1. *A smooth variety P and an embedding $X \to P$,*
2. *A vector bundle $\tilde{V} \to P$ and an isomorphism $\tilde{V}|_X = V$,*
3. *A subvariety X_n of P_n (the n-th formal neighbourhood of P in $\tilde{V}$), such that the restriction and multiplication maps*

$$r : V^* \to I_{X/X_n}/I^2_{X/X_n},$$

$$r^{\otimes n} : \mathrm{Sym}^n V^* \to I^n_{X/X_n}/I^{n+1}_{X/X_n},$$

are isomorphisms.

Once again, we define N_X^*, the co-normal bundle, by $N_X^* = I_{X/P}/I^2_{X/P}$, and N_X, the normal bundle, to be its dual.

We will now prove the following results.

Proposition 4 *Let X_n be an n-th order embedded deformation of X. We can assign to (X^1, X^2), a pair of extensions of X_n to $(n+1)$-st order deformations of X, an element $d(X^1, X^2) \in \mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1}V^*)$ in such a way that*

1. $d(X^1, X^2) + d(X^2, X^3) = d(X^1, X^3)$ *and* $d(X^1, X^2) = -d(X^2, X^1)$,
2. *If X is geometrically reduced and the push-forward of*

$$0 \to N_X^* \to \Omega_P \otimes \mathcal{O}_X \to \Omega_X \to 0$$

along $d(X^1, X^2)$ is a trivial extension then X^1 and X^2 are isomorphic as abstract deformations of X_n,

3. *If any extensions of $X_n \subset P_n$ exist, then they form a principal homogeneous $\mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1}V^*) = H^0(N_X \otimes Sym^{n+1}V^*)$ space.*

Proposition 5 *We can associate to X_n an element*

$$\omega_{X_n} \in H^1(\mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1}V^*))$$

(or alternatively

$$\omega_{X_n} \in H^1(N_X \otimes \mathrm{Sym}^{n+1}V^*))$$

in such a way that X_n can be extended to an $(n+1)$-st order deformation of X if and only if $\omega_{X_n} = 0$.

Before starting the proof of these two propositions, we will need the following lemma. We choose some open affine neighbourhood U of x in P, such that

1. $\tilde{V}$ is trivial over U.
2. $X \cap U$ is a complete intersection.

We will denote by π_n the projection from P_n to P and by U_n the scheme $\pi_n^{-1}(U)$.

Lemma 2 *Let* $(f_1, f_2, \ldots, f_m)$ *be a regular sequence generating* $I_{X\cap U/U}$. *Let* $\tilde{f}_i$ *be liftings of* f_i *to* I_{X_n/U_n}. *Then* $(\tilde{f}_1, \ldots, \tilde{f}_m)$ *is a regular sequence generating* $I_{X_n\cap U_n/U_n}$.

Proof of Lemma 2.

We proceed by induction on n. The theorem holds for $n = 0$. We assume it holds for $(n-1)$. Consider a lifting $\tilde{f}_i$ of f_i to $I_{X_n\cap U_n/U_n}$.

By the induction hypothesis, the restrictions of $\tilde{f}_i$ to U_{n-1} form a regular sequence for $I_{X_{n-1}\cap U_{n-1}/U_{n-1}}$ in $\mathcal{O}_{U_{n-1}}$. Suppose $\tilde{f}_i$ were not a regular sequence for $I_{X_n\cap U_n/U_n}$. We will use the following criterion for regularity of sequences.

Regularity criterion.

Let R *be a ring. A sequence* $(g_1, \ldots, g_n) \in R^n$ *is a regular sequence in* R *if and only if for all sequences* $(h_1, \ldots, h_n) \in R^n$ *such that* $\sum_i g_i h_i = 0$, *there are* $h_{i,j} \in R$ *such that*

1. $h_{i,j} = -h_{j,i}$ *for all* i *and* j *and*
2. $h_i = \sum_j h_{i,j} g_j$ *for all* i.

Proof of the regularity criterion.

We will first prove that regularity implies the criterion. We proceed by induction on the length of the regular sequence. A regular sequence of length $n = 1$, immediately satifies the criterion, since it then simply says that g_1 is not a zero-divisor in R.

Suppse that regularity implies the criterion for $n-1$. Suppose that $\sum_{i=1}^n h_i g_i = 0$. Then, since g_n is not a zero-divisor in $R/\langle g_1, \ldots, g_{n-1}\rangle$, we know that

$$h_n \in \langle g_1, \ldots, g_{n-1}\rangle;$$

hence we may set

$$h_n = \sum_{j=1}^{n-1} h_{n,j} g_j.$$

Now, upon setting for $i \leq (n-1)$

$$\tilde{g}_i = g_i + h_{n,i} g_i$$

we see that $\sum_{i=0}^{n-1} \tilde{g}_i h_i = 0$. The result now follows by the induction hypothesis.

We will now prove that the criterion implies regularity. Suppose that the sequence is not regular— i.e., that there exists k such that g_k is a zero-divisor in $R/\langle g_1, \dots g_{k-1}\rangle$. In other words, there exists $(h_1, \dots, h_k) \in R$ such that

$$h_k g_k = \sum_{i=1}^{k-1} h_i g_i$$

and $h_k \notin \langle g_1, \dots g_{k-1}\rangle$. This immediately contradicts the statement of the criterion. The criterion is proved. □

If $(\tilde{f}_1, \dots, \tilde{f}_n)$ are not a regular sequence, there are then $\alpha_i \in \mathcal{O}_{U_n}$ such that

1. $\sum_{i=1}^{n} \alpha_i \tilde{f}_i = 0$ in $\mathcal{O}_{U_n}$,
2. There do not exist $a_{i,j} \in \mathcal{O}_{U_n}$ such that $a_{i,j} = -a_{j,i}$ and $\alpha_i = \sum_{j=1}^{n} a_{i,j} \tilde{f}_j$.

Choose such α_i. Since $\tilde{f}_i|_{U_{n-1}}$ are a regular sequence for $I_{X_{n-1} \cap U_{n-1}}$, there exist $\beta_{i,j} \in \mathcal{O}_{U_n}$ such that

1. $\alpha_i|_{U_{n-1}} = \sum_{j=1}^{n} \beta_{i,j}|_{U_{n-1}} \tilde{f}_j|_{U_{n-1}}$,
2. $\beta_{i,j}|_{U_{n-1}} = -\beta_{j,i}|_{U_{n-1}}$.

Set $\mu_i = \alpha_{i=1}^{n} - \sum_j \beta_{i,j} \tilde{f}_j$. It follows that $\sum_i \mu_i \tilde{f}_i = 0$.

We will throughout the rest of the proof of this lemma use the isomorphism $I_{P_{n-1}/P_n} \sim \mathrm{Sym}^n \tilde{V}^*$. Note that, since

$$\alpha_i|_{U_{n-1}} = \sum_{j=1}^{n} \beta_{i,j}|_{U_{n-1}} \tilde{f}_j|_{U_{n-1}},$$

we have $\mu_i \in \mathrm{Sym}^n \tilde{V} j$. By assumption, $\mathrm{Sym}^n \tilde{V} j$ is a free $\mathcal{O}_U$-module. Let $e_1, \ldots, e_l$ be an $\mathcal{O}_U$-module basis for $\mathrm{Sym}^n \tilde{V}^*$. We can choose $g_{i,k} \in \mathcal{O}_U$ such that $\mu_i = \sum_{k=1}^{l} g_{i,k} e_k$. From this it follows that for all k

$$\sum_{i=1}^{n} g_{i,k} f_i = 0.$$

But by assumption f_i is a regular sequence and hence we know that there exist $\nu_{i,j,k}$ such that $\nu_{i,j,k} = -\nu_{j,i,k}$ and for all k

$$g_{i,k} = \sum_{j=1}^{n} \nu_{i,j,k} f_j.$$

From this we see that

$$\alpha_i = \sum_{j=1}^{n} \tilde{f}_j (\beta_{i,j} + \sum_{k=1}^{l} \nu_{i,j,k} e_k).$$

Now, if we set

$$a_{i,j} = (\beta_{i,j} + \sum_{k+1^n} \nu_{i,j,k} e_k),$$

then $a_{i,j} = -a_{i,j}$ and $\alpha_i = \sum_i a_{i,j} \tilde{f}_{i,j}$. This contradicts the assumption above. Hence the $\tilde{f}$ are necessarily a regular sequence.

Finally, if $(\tilde{f}_1, \ldots \tilde{f}_n)$ do not generate $I_{X_n \cap U_n / U_n}$, then consider $g_1 \in I_{X_n \cap U_n / U_n}$ such that

$$g_1 \notin \langle \tilde{f}_1, \ldots \tilde{f}_n \rangle.$$

By the induction hypothesis,

$$g_1|_{U_{n-1}} \in \langle \tilde{f}_1|_{U_{n-1}}, \ldots \tilde{f}_n|_{U_{n-1}} \rangle.$$

Alternatively, there is a

$$g_2 \in \langle \tilde{f}_1|_{U_{n-1}}, \ldots \tilde{f}_n|_{U_{n-1}} \rangle$$

such that $(g_1 - g_2)|_{U_{n-1}} = 0$. Setting $(g_1 - g_2) = g$, we have

$$g \in I_{X_n \cap U_n / U_n} \cap I_{U_{n-1} / U_n}$$

and

$$g \notin \mathrm{Sym}^n \tilde{V}^* \otimes_{\mathcal{O}_U} I_{X \cap U / U}.$$

In other words, the natural surjection

$$\mathrm{Sym}^n(V^*) = I_{U_{n-1}/U_n} \otimes_{\mathcal{O}_U} \mathcal{O}_X \to I_{X_{n-1}\cap U_{n-1}/X_n\cap U_n}$$

has non-trivial kernel. But this is impossible by definition of a generalised deformation, since we have assumed that this map is an isomorphism. This completes the proof of Lemma 2. □

Proof of Proposition 4.

We define a map,

$$\tilde{d}(X^1, X^2) : I_{X_n/P_n} \to \mathrm{Sym}^{n+1}V^*,$$

in the following way. Let f be an element of I_{X_n/P_n}. Choose two liftings of f,

$$f^1 \in I_{X^1/P_{n+1}} \text{ and } f^2 \in I_{X^2/P_{n+1}}.$$

We know that

$$(f^1 - f^2) \in \mathrm{Sym}^{n+1}(\tilde{V}^*).$$

We then define $\tilde{d}(X^1, X^2)$ by

$$\tilde{d}(X^1, X^2)(f) = (f^1 - f^2)|_X.$$

As this map is a map towards an $\mathcal{O}_X$-module, it descends to a map

$$d(X^1, X^2) : N_X^* \to \mathrm{Sym}^{n+1}V^*.$$

We need to prove the following lemma.

Lemma 3 $d(X^1, X^2)$ *is well-defined.*

Proof of Lemma 3.

Suppose that $\tilde{f}'^1$ and $\tilde{f}'^2$ are alternative liftings of f. Then

$$(\tilde{f}'^1 - \tilde{f}^1) \in I_{X^1/P_{n+1}} \cap I_{P_n/P_{n+1}}.$$

Lemma 2 implies that the right-hand side is simply $I_{P_n/P_{n+1}} \otimes_{\mathcal{O}_P} I_{X/P}$. This can be seen locally by taking a regular sequence for $I_{X^1/P_{n+1}}$ which we denote by $(\tilde{f}_1, \ldots, \tilde{f}_n)$. If now we consider

$$(\tilde{f}'^1 - \tilde{f}^1) \in I_{X^1/P_{n+1}} \cap I_{P_n/P_{n+1}},$$

then there are c_i such that

$$\sum_i c_i \tilde{f}_i = (\tilde{f}'^1 - \tilde{f}^1).$$

Since $\tilde{f}_i|_{P_n}$ is a regular sequence, and $\sum_i c_i \tilde{f}_i|_{P_n} = 0$, then there are $d_{i,j} \in \mathcal{O}_{P_{n+1}}$ such that $d_{i,j} = -d_{j,i}$ and $c_i|_{P_n} = \sum_j d_{i,j} \tilde{f}_j|_{P_n}$.

In other words,

$$c_i = \sum_j d_{i,j} \tilde{f}_j + e_i$$

where $e_i \in I_{P_n/P_{n+1}}$. It follows that

$$(\tilde{f}'^1 - \tilde{f}^1) = \sum_i e_i \tilde{f}_i$$

whence it follows that

$$(\tilde{f}'^1 - \tilde{f}^1) \in I_{P_n/P_{n+1}} \otimes_{\mathcal{O}_P} I_{X/P}.$$

The latter space is equal to $\mathrm{Sym}^{n+1}(\tilde{V}^*) \otimes I_{X/P}$. It follows that

$$(\tilde{f}'^1 - \tilde{f}^1 - \tilde{f}'^2 + \tilde{f}^2) \in \mathrm{Sym}^{n+1}\tilde{V}^* \otimes I_{X/P},$$

whence we see that $d(X^1, X^2)$ is independent of the choice of liftings $\tilde{f}^1$ and $\tilde{f}^2$. This completes the proof of Lemma 3. □

To finish the proof of Proposition 4, it will be enough to establish the required properties of the map $d(X^1, X^2)$. It is immediate from the definition that

$$d(X^1, X^2) + d(X^2, X^3) = d(X^1, X^3)$$

and

$$d(X^1, X^2) = -d(X^2, X^1).$$

We will now prove the following lemma.

Lemma 4 *Assume further that X is generically smooth. Let i be the inclusion of N_X^* into $\Omega_P \otimes \mathcal{O}_X$. Assume that there exists a map*

$$d'(X^1, X^2) : \Omega_P \otimes \mathcal{O}_X \to \mathrm{Sym}^{n+1} V^*$$

such that $d(X^1, X^2) = d'(X^1, X^2) \circ i$. The schemes X^1 and X^2 are isomorphic as abstract formal neighbourhoods of X_{n+1}.

Proof of Lemma 4.

Suppose that such a map exists. Then we can define $F : \mathcal{O}_{P_{n+1}} \to \mathcal{O}_{X^2}$ by

$$F(g) = g|_{X^2} + d'(X^1, X^2)d(g|_P).$$

This is a surjective map of sheaves of k-algebras whose kernel is I_{X^1} and which is the identity map on $\mathcal{O}_{X_n}$. Hence X^1 and X^2 are isomorphic. This concludes the proof of Lemma 4. □

It remains only to establish that if there are any extensions of X_n then they form a principal homogeneous $\mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1}V^*)$ space. It follows from the construciton in the proof of Proposition 4 that if $d(X^1, X^2) = 0$, then $X^1 = X^2$. It remains to be shown that given X^1, there exists for every $\phi \in \mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1}V^*)$ some X^2 such that $d(X^1, X^2) = \phi$.

Given X^1, an extension of X_n, and $\phi \in \mathrm{Hom}(N^*, \mathrm{Sym}^{n+1}V^*)$, we define X^2 in the following way. An element $g \in \mathcal{O}_{P_{n+1}}$ is contained in the ideal sheaf of X^2 if and only if there there exists a g' in $I_{X^1/P_{n+1}}$ such that

1. $g|_{P_{n-1}} = g'_{P_{n_1}}$,

2. $(g' - g)|_X = \phi(g')$.

Then X^2 is an extension of X_n to an $(n+1)$-st order generalised formal deformation and $d(X^1, X^2) = \phi$. This completes the proof of Proposition 4. □

Proof of Proposition 5.

Let π_n be the projection from P_n onto P. We choose first of all open affine subsets U^i of P over which $\tilde{V}$ is trivial and on which X is a complete intersection. We denote by U_n^i the scheme $\pi_n^{-1}(U^i)$, and by X_n^i the scheme $U_n^i \cap X_n$.

By Lemma 2, X_n^i is also a complete intersection. (Liftings of a regular sequence for the ideal sheaf of X^i to a regular sequence for the ideal sheaf of X_n^i necessarily exist because U_n^i is an affine scheme).

We then choose $\tilde{X}^i$, extensions of $X_n^i = U_n^i \cap X_n$ to $(n+1)$-st order embedded formal neighbourhoods in U_{n+1}^i. Such things exist because X_n^i is a complete intersection and U_{n+1}^i is an affine scheme. Hence we can simply take a regular sequence defining X_n^i and lift its elements to $\mathcal{O}_{U_{n+1}^i}$.

Consider the family

$$D(\tilde{X}^i, \tilde{X}^j) \in \Gamma(U^{i,j}, \mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1} V^*)).$$

This may be thought of as a Cech cocycle with coefficients in

$$\mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1} V^*).$$

We will prove the following lemma.

Lemma 5 *There exists some extension of the embedding $X_n \to P_n$ if and only if the cohomology class associated to $D(\tilde{X}^i, \tilde{X}^j)$ vanishes.*

Proof of Lemma 5.

We denote by $U^{i,j}$ the intersection $U^i \cap U^j$. By the third property of Proposition 4, the ideals $I_{\tilde{X}^i/U^i_{n+1}}$ and $I_{\tilde{X}^j/U^j_{n+1}}$ are compatible as subsets of $\mathcal{O}_{U^{i,j}}$ if and only if $D(\tilde{X}^i, \tilde{X}^j) = 0$. We prove that if the cohomology class associated to $D(\tilde{X}^i, \tilde{X}^j)$ vanishes then we can choose extensions $\overline{X}^i \in U^i_{n+1}$ of X^i_n which are compatible on the intersections $U^{i,j}_{n+1}$ and which therefore glue together to provide a global extension of X_n.

If this cohomology class vanishes then there exist

$$D^i \in \Gamma(U^i, \mathrm{Hom}(N_X^*, \mathrm{Sym}^{n+1} V^*))$$

such that $D(\tilde{X}^i, \tilde{X}^j) = D^i - D^j$. By the third property of Proposition 4, for all i there exists a unique embedded deformation $\overline{X}^i \subset U^i_{n+1}$ such that

$$D(\overline{X}^i, \tilde{X}^i) = -D^i.$$

Now let us consider the cocycle $D(\overline{X}^i, \overline{X}^j)$. We have

$$D(\overline{X}^i, \tilde{X}^i) + D(\tilde{X}^i, \tilde{X}^j) + D(\tilde{X}^j, \overline{X}^j) = 0.$$

This can be re-expressed as saying (by the second property given in Proposition 4) that

$$D(\overline{X}^i, \overline{X}^j) = 0.$$

It follows that the $\overline{X}^i$ are compatible on the intersections and can be glued together to give a global deformation of X_n.

Likewise, if there exist $\overline{X}^i$ such that the $\overline{X^i}$ and $\overline{X^j}$ are compatible on the intersections, then

$$D(\tilde{X}^i, \tilde{X}^j) = D(\tilde{X}^i, \overline{X}^i) - D(\tilde{X}^j, \overline{X}^j).$$

In other words, $D(\tilde{X}_i, \tilde{X}_j)$ is the boundary of the Cech class

$$D(\tilde{X}^i, \overline{X}^i).$$

We now set

$$\omega_{X_n} = [D(\tilde{X}^i, \tilde{X}^j)].$$

The class ω_{X_n} is independent of the choice of liftings, and satisfies the requirements of the proposition. This completes the proof of the lemma 5. □

This completes the proof of Proposition 5. □

2.4 Proof of Theorem 8

The scheme X is contained in some projective space $\mathbb{P}^N$. We will construct an embedding of $\mathbb{P}^N$ into a smooth variety $\tilde{V}$ with highly negative normal bundle. More precisely, $\mathbb{P}^N$ will be the zero locus of a section of a vector bundle on $\tilde{V}$ which is a direct sum of line bundles.

We will recursively construct an l.c.i. scheme X_{n+1} in $\mathbb{P}^N_{n+1}$ extending X_n. If V is negative enough, the construction of X_n will be unobstructed and we may therefore continue this construction to infinity to obtain X_∞, a formal neighbourhood of X. We will also be able to impose that X_∞ is smooth. The formal scheme X_∞ will then satisfy all the requirements of the theorem.

Consider $I_{X/\mathbb{P}^N}$, the ideal sheaf of X in $\mathbb{P}^N$. We recall Serre's vanishing theorem, which may be found in [**?**].

Proposition 6 *Let F be a coherent sheaf on X, a projective scheme. There exists an m such that, for all $a \geq m$, and for all $j \geq 1$*

1. $H^j(X, F(a)) = 0$,
2. *$F(a)$ is generated by its global sections.*

Define, as in the previous section, N_X^* to be the sheaf $I_{X/\mathbb{P}^N}/I^2_{X/\mathbb{P}^N}$ and N_X to be its dual. In particular, we may choose $m \geq 0$ such that for all $a \geq m$:

1. $H^1(N_X(a))) = 0$,
2. $N_X(a)$ is generated by its global sections.

We make the following definitions.

1. l is the dimension of $H^0(N_X(m))$,
2. V is the vector bundle $\mathcal{O}_{\mathbb{P}^N}(-m)^{\oplus l}$ in which $\mathbb{P}^N$ is naturally embedded as the zero section.

As above, $\mathbb{P}^N_n$ will be the subscheme of V defined by the ideal $I^{n+1}_{\mathbb{P}^N}$. We now have the following result.

Proposition 7 *Let X_n be an extension of X to a generalised embedded deformation in $\mathbb{P}^N_n$. Then there is no obstruction to the extension of X_n to a generalised embedded deformation X_{n+1} in $\mathbb{P}^N_{n+1}$.*

Proof of Proposition 7.

This is immediate, since the obstruction space for this problem is

$$H^1(N_X \otimes \mathrm{Sym}^{n+1}(V^*)).$$

This is 0 by choice of V. Proposition 7 follows. □

We may then recursively choose $X_{n+1} \subset \mathbb{P}^N_{n+1}$ extending X_n for each n. The formal scheme

$$\lim_{n\to\infty} X_n = X_\infty$$

is then a formal neighbourhood of X in which X is embedded as the zero locus of the tautological section of $\pi^*(\mathcal{O}_X(-m))^{\oplus l}$. It remains only to prove the following proposition.

Proposition 8 *For some choice of X_n, the scheme X_∞ is smooth.*

Proof of Proposition 8.

The smoothness of X_∞ depends only on the choice of X_1. All the results we now quote on Kähler differentials may be found in [**?**].
Consider the projection $\pi_1 : \mathbb{P}^N_1 \to \mathbb{P}^N$. The sheaf of Kähler differentials

$$\Omega_{\mathbb{P}^N_1} \otimes \mathcal{O}_{\mathbb{P}^N}$$

is canonically isomorphic to $\pi_1^*\Omega_{\mathbb{P}^N} \oplus V^*$. The universal derivative map

$$d : I_{X_1/\mathbb{P}^N_1} \to \Omega_{\mathbb{P}^N_1} \otimes \mathcal{O}_X$$

is an $\mathcal{O}_{X_1}$ linear map. After tensoring by $\mathcal{O}_X$, we obtain an $\mathcal{O}_X$-linear map

$$d_{X_1} : N_X^* \to \Omega_{\mathbb{P}^N}|_X \oplus V^*|_X.$$

The scheme X_∞ is smooth at x if $d_{X_1}(x)$ is injective.

We now associate X_ϕ to any $\phi : N_X^* \to V^*|_X$. This X_ϕ will be a candidate space for X_1 such that any associated X_∞ will be smooth if $\phi(x)$ is injective for all x.

An element $f \in \mathcal{O}_{\mathbb{P}_1^N}$ will be contained in I_{X_ϕ} if and only if

1. $f|_{\mathbb{P}^N} \in I_X$ and
2. $(f - \pi_1^*(f|_{\mathbb{P}^N}))|_X = \phi(f)$.

For this choice of X_ϕ, we have

$$d_{X_\phi}(f) = \pi_1^* df + \phi(f).$$

From this it follows that $d_{X_\phi}(x)$ is injective for all x if $\phi(x)$ is injective for all x.

It remains to find $\phi \in \mathrm{Hom}(N_X^*, V^*)$ such that $\phi(x)$ is injective for every x. Note that $\mathrm{Hom}(N_X^*, \mathcal{O}_X(m))$ is globally generated. If $(v_1, \ldots, v_l)$ is a basis for $H^0(\mathrm{Hom}(N_X^*, \mathcal{O}_X(m)))$, then the function ϕ defined by

$$\phi = \oplus_{b=1}^l v_b : N_X^* \to V^*$$

is injective on $N_X^*(x)$ for every x. This completes the proof of proposition 8. □

This completes the proof of Theorem 8. □

Question 2 *Is the space X_∞ algebrisable ?*

Note that, according to recent work of Bost and Bogolomov-McQuillan, described in [**?**], we have the following theorem.

Theorem 17 *Let $X \subset V$ be a connected smooth projective subvariety of a quasi-projective variety over an algebraically closed field, and $\hat{X}$ a smooth formal subvariety of $\hat{V}$, the completion of V along X, which contains X. Assume that the normal bundle of X in $\hat{X}$ is ample. Then $\hat{X}$ is algebraic.*

Remark 1 One might wonder whether this work holds for other X. We have used the fact that X is projective only to invoke Serre's vanishing theorem. Suppose that X is a quasi-projective variety. The results of this section will hold for X, provided that we have the following property.

For any coherent sheaf, $\mathcal{F}$ on X, there exists m such that for any $a \geq m$ we have $H^1(\mathcal{F}(a)) = 0$ and $\mathcal{F}(a)$ is generated by its global sections.

Chapter 3

A second order invariant of the Noether-Lefschetz locus

In this part, we will develop a second-order invariant of a component of the Noether-Lefschetz locus which generalises the first-order invariant found by Carlson and Griffiths. In the following chapters, these will be used to prove two new results on components of the Noether-Lefschetz locus of small co-dimension.

3.1 Introduction— the Noether-Lefschetz locus

In this chapter, we will be concerned with smooth surfaces in $\mathbb{P}^3$ and the curves that are contained in them. Suppose that X is a smooth surface of degree d in $\mathbb{P}^3$, cut out by a polynomial $F \in k[X_0, \dots X_3]$. The following three problems are all equivalent.

1. Determine the group $NS(X)$ of 1-cycles of X up to algebraic equivalence,
2. Determine the group $\mathrm{Pic}(X)$ of line bundles on X,
3. Determine the group $H^{1,1}(X, \mathbb{Z})$ of Hodge classes of X.

For a generic X of degree $d \geq 4$ we have the Noether-Lefschetz theorem.

Theorem 18 (Lefschetz) *If X is a generic smooth degree d surface in $\mathbb{P}^3$ and $d \geq 4$ then all curves in X are complete intersections of X with another surface.*

Lefschetz proved this using a monodromy argument and the Hodge decomposition. In the early 80's, Griffiths and Harris reproved the theorem using a degeneration technique, which will be presented in the next chapter, as it will be used to give an alternative proof of one of the two theorems.

Equivalently, the group $H^{1,1}_{\mathrm{prim}}(X, \mathbb{Z})$, consisting of all primitive integral Hodge $(1,1)$-cohomology classes of X, is trivial. We will say that any X such that $H^{1,1}_{\mathrm{prim}}(X, \mathbb{Z}) = 0$ satisfies the Noether-Lefschetz (NL) condition. We will call the set of surfaces containing a non-complete intersection curve the Noether-Lefschetz locus. This locus consists of a countable union of proper algebraic subvarieties of U_d. Ciliberto et al. showed in [**?**] that it is Zariski dense in U_d, and indeed, dense for the usual complex topology.

A stronger version of the Noether-Lefschetz theorem was proved by Carlson, Green, Griffiths and Harris in [**?**], via a study of the first-order infinitesimal variation of the Hodge structure of X. They showed, amongst many other things, that for any X, a general first-order deformation of X contains only complete intersection curves.

If γ is an integral primitive class which is of type $(1,1)$, then $NL(\gamma)$ the locus of surfaces in some suitable simply-connected neighbourhood of X in which γ remains of $(1,1)$ type. More precisely, the locus $NL(\gamma)$ can be locally constructed as the zero locus of a section of a certain vector bundle in the following way.

Let $\mathcal{H}^i$ be the vector bundle whose fibre over the point X is $H^i(X, \mathbb{C})$. This vector bundle is equipped with the Gauss-Manin connection ∇ (a natural flat connection) and has a holomorphic structure. The Hodge filtration on $H^i(X, \mathbb{C})$ gives rise to a descending filtration $F^p(\mathcal{H}^i) \subset \mathcal{H}^i$. We write $F^p/F^{p+1} = \mathcal{H}^{p,q}$. We need the following classical results of Griffiths, which may be found in [**?**].

Theorem 19 (Griffiths)

- $F^p(\mathcal{H}^i)$ *is a holomorphic sub-vector bundle of* $\mathcal{H}^i$,
- *(Transversality)* $\nabla : F^p(\mathcal{H}^i) \to F^{p-1}(\mathcal{H}^i) \otimes \Omega_{U_d}$,
- ∇ *induces an* $\mathcal{O}_{U_d}$*-linear map*

$$\overline{\nabla} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega_{U_d}.$$

Recall that γ is a non-zero element of $H^{1,1}(X, \mathbb{C})$. We now fix O, a simply connected neighbourhood of X in U_d. By $\overline{\gamma}$, we mean the flat section of $\mathcal{H}^2|_O$ which is induced by flat transport of γ. Note that by the definition of the bundles $\mathcal{H}^{p,q}$ there is a projection $\mathcal{H}^2 \to \mathcal{H}^{0,2}$. We denote by $\overline{\gamma}^{0,2}$ the image of $\overline{\gamma}$ under this projection. We are now in a position to define $NL(\gamma)$.

Definition 10 *The space* $NL(\gamma)$ *is the zero locus in* O *of the section* $\overline{\gamma}^{0,2}$.

Being defined locally by $h^{0,2} = \mathrm{rk}(\mathcal{H}^{0,2})$ equations, $NL(\gamma)$ cannot have codimension greater than $h^{0,2}(X) = \binom{d-1}{3}$ in O, and we expect that it should have codimension exactly $h^{0,2}(X)$. However, for $d > 4$, there exist Noether-Lefschetz components of strictly lower codimension. For example, if C is a line in X, then we denote by $[C]_{\mathrm{prim}}$ the primitive part of the cohomology class of C. We can then show by a dimension count that the codimension of $NL([C]_{\mathrm{prim}})$ is $d-3$, since it is simply a component of the set of surfaces containing a line. We will call a component of the NL locus exceptional if it has codimension less than $\binom{d-1}{3}$.

The Zariski tangent space of $NL(\gamma)$ was described by Carlson and Griffiths in [**?**]. The work we will present in this section is heavily based on these results, which are summarised below. If we interpret the defining section $\overline{\gamma}^{0,2}$ as a set of local equations for $NL(\gamma)$, then [**?**] gives a complete description of the first order part of these equations at a point $X \in NL(\gamma)$.

This first-order invariant of $NL(\gamma)$ rendered possible many qualitative results concerning these exceptional components, notably the following.

- (Voisin [**?**], Green [**?**]) Every exceptional NL component has codimension at least $d-3$, and for $d \geq 5$ this bound is obtained only for the component of surfaces containing a line.
- (Voisin, [**?**]) For $d \geq 5$, the second largest NL component of U_d has codimension $2d-7$, and this bound is achieved only by the space of surfaces containing a plane conic.
- (Voisin [**?**]) There are infinitely many exceptional components in U_d, for d sufficiently large.

This last result replies in the negative to a conjecture of Ciliberto and Harris, which will be explained in chapter 5.

We will extend the results of Carlson and Griffiths via a second-order infinitesimal study of $NL(\gamma)$. In this chapter, we will calculate the second order part of the equations for $NL(\gamma)$ at X. This gives new information whenever $NL(\gamma)$ is singular at X or $NL(\gamma)$ is exceptional. We will then use these results in the next chapter to determine the non-reduced components of the Noether-Lefschetz locus in U_5 (the reduced exceptional components all having been found by Voisin in [**?**] and [**?**]) and to get in any degree new bounds on the codimensions of those exceptional loci which violate the conjecture of Ciliberto and Harris. These new bounds may be seen as generalisations of known results in two different ways. Firstly, they may be seen as generalising the work of Voisin in [**?**] to higher degrees than 6 or 7. Secondly, the results given will in fact be conditions for the space of polynomials of degree e vanishing on γ to be dimension at least j. For $e = 1$ and $j = 2$ we will recover the results of Voisin and Green on maximal Noether-Lefschetz loci, albeit with the additional condition that the locus should be reduced.

3.2 Resume of the work of Carlson and Griffiths

In this section, we summarise the results of [**?**] and [**?**]. A summary of this work may also be found in [**?**].
If P is a degree $pd - 4$ polynomial and Ω is the canonical section of the bundle $K_{\mathbb{P}^3}(4)$ then $\frac{P\Omega}{F^p}$ is a holomorphic 3-form on $\mathbb{P}^3 - X$ and has therefore a class in $H^3(\mathbb{P}^3 - X, \mathbb{C})$. The group $H^3(\mathbb{P}^3 - X, \mathbb{C})$ maps via the residue mapping to $H^2_{\text{prim}}(X, \mathbb{C})$: there is therefore in particular a composed mapping

$$\text{res}_X : S^{pd-4} \to H^2_{\text{prim}}(X, \mathbb{C}).$$

This is defined by

$$\text{res}_X(P) = \text{res}_X\left(\left[\frac{P\Omega}{F^p}\right]\right).$$

It is proved in [**?**] (see also [**?**] and [**?**]) that

$$\text{Im}(\text{res}_X) = F^{3-p}H^2_{\text{prim}}(X, \mathbb{C}).$$

It is further proved that $\text{res}_X(Q) \in F^{2-p}H^2(X, \mathbb{C})$ if and only if $Q \in \left\langle \frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_3} \right\rangle$.

We denote by J_F (the Jacobian ideal of F) the homogeneous ideal $\left\langle \frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_3} \right\rangle$.
We further denote by R_F (the Jacobian ring of F) the graded ring $k[X_0 \ldots X_3]/J_F$.
The work above can be summarised in the following way.

Theorem 20 (Carlson, Griffiths) *The residue map induces a natural isomorphism between R_F^{pd-4} and $H^{3-p,p-1}_{\text{prim}}(X, \mathbb{C})$.*

In [**?**], the infinitesimal variation of this Hodge structure with variations of the hypersurface X was also calculated. The map, $\overline{\nabla}_F$, induced by the Gauss Manin connection, can be thought of as a map

$$\overline{\nabla} : \mathcal{H}^{p,q}_{\text{prim}} \to \text{Hom}(T_{U_d}, \mathcal{H}^{p-1,q+1}_{\text{prim}}).$$

Carlson and Griffiths showed that after making the following identifications

1. $T_{U_d}(F) = S^d / \langle F \rangle$,

2. $\mathcal{H}^{p,q}_{\text{prim}}(F) = R_F^{(3-p)d-4}$,

3. $\mathcal{H}^{p-1,q+1}_{\text{prim}}(F) = R_F^{(4-p)d-4}$,

we have the following result.

Theorem 21 (Carlson, Griffiths) *Up to multiplication by a constant, $\overline{\nabla}_F(\text{res}_X P)$ is identified with the multiplication map*

$$\cdot P : R_F^d \to R_F^{(4-p)d-4}.$$

As a first application, we have the following description of the tangent space of the NL locus. $NL(\gamma)$ is the zero locus of $\overline{\gamma}^{0,2}$. If $\gamma \in H^{1,1}_{\text{prim}}(X, \mathbb{C})$ is represented by P then the derivative map:

$$d\gamma^{0,2} : T_{U_d}(X) \to \mathcal{H}^{0,2}(X)$$

is identified with the multiplication map, $\cdot P$. In other words, we have

$$\text{Ker}(d\gamma^{0,2}) = T_{NL(\gamma)}(X).$$

This implies that

$$T_{NL(\gamma)}(X) = \text{Ker}(\cdot P : R_F^d \to R_F^{3d-4}).$$

Thus after identification of $T_{U_d}(X)$ with S^d/F, we have

$$H \in T_{NL(\gamma)}(X) \text{ if and only if there exist } Q_i \in S^{2d-3} \text{ such that } PH = \sum_{i=0}^{3} Q_i \frac{\partial F}{\partial X_i}.$$

We will lean heavily in what follows on the following classical result, due to Macaulay (which may be found in [**?**], for example).

Theorem 22 (Macaulay) *The ring R_F is a Gorenstein graded ring. In other words, $R_F^{4d-8} = \mathbb{C}$ and the multiplication map*

$$R_F^a \otimes R_F^{4d-8-a} \to R_F^{4d-8} = \mathbb{C}$$

is a perfect pairing.

3.3 The second order invariant of IVHS

Throughout the rest of this section, P will be a degree $2d-4$ polynomial whose image under the residue map is γ, and G and H will be degree d polynomials contained in $T_{NL(\gamma)}(X)$. We take $\{Q_i\}_{i=0}^3$, $\{R_i\}_{i=0}^3$ to be degree $2d-3$ polynomials such that

$$PG = \sum_{i=0}^{3} Q_i \frac{\partial F}{\partial X_i}.$$

and

$$PH = \sum_{i=0}^{3} R_i \frac{\partial F}{\partial X_i}$$

and Let $(s_1, \ldots, s_{h^{2,0}(X)})$ be defining equations for $NL(\gamma)$ near X. If

$$d : \langle s_1, \ldots, s_{h^{0,2}(X)} \rangle \to T_{U_d}(X)$$

is the map sending an equation to its first order part at X, then there is a well-defined map

$$r : \mathrm{Ker}(d) \to \mathrm{Sym}^2(T^*_{NL(\gamma)}(X))$$

sending an equation which vanishes to first order at X to its second order at X. This may be interpreted as a set of $\dim(\mathrm{Ker}(d))$ quadratic equations on $T_{NL(\gamma)}(X)$. By Macaulay's theorem, we have

$$\dim(\mathrm{Ker}(d)) = \mathrm{codim}(\mathrm{Im}(\cdot P) \subset R_F^{3d-4}).$$

Indeed, after dualising, there is a canonical identification

$$\mathrm{Ker}(d)^* = R_F^{3d-4}/\mathrm{Im}(\cdot P).$$

The main result of this section is an explicit formula for $q = r^*$.

Theorem 23 *The second quadratic form described above*

$$q : \mathrm{Sym}^2(T_{NL(\gamma)}(X)) \to R_F^{3d-4}/\mathrm{Im}(\cdot P)$$

is given by

$$q(G,H) = \sum_{i=0}^{3} \left(H \frac{\partial Q_i}{\partial X_i} - R_i \frac{\partial G}{\partial X_i} \right).$$

The attentive reader will be surprised to see that this form is apparently not symmetric in G and H. This is, however, only apparent: we have the following lemma.

Lemma 6 *For all H and G in $T_{NL(\gamma)}(X)$,*

$$q(G,H) = q(H,G).$$

Proof of Lemma 6.

We know that

$$\sum_{i=0}^{3} GR_i \frac{\partial F}{\partial X_i} = GHP = \sum_{i=0}^{3} HQ_i \frac{\partial F}{\partial X_i}.$$

Rearranging, we get that

$$\sum_{i=0}^{3} (GR_i - HQ_i) \frac{\partial F}{\partial X_i} = 0.$$

However, F is smooth, and hence the $\frac{\partial F}{\partial X_i}$ form a regular sequence. It follows that there exist $A_{i,j}$, polynomials, such that

1. $A_{i,j} = -A_{j,i}$,
2. $GR_i - HQ_i = \sum_{j=0}^{3} A_{i,j} \frac{\partial F}{\partial X_i}$.

Deriving this second equation and summing over i, we get that

$$\sum_{i=0}^{3} \left(G\frac{\partial R_i}{\partial X_i} + R_i \frac{\partial G}{\partial X_i} \right) - \sum_{i=0}^{3} \left(H\frac{\partial Q_i}{\partial X_i} - Q_i \frac{\partial H}{\partial X_i} \right) = \sum_{i,j} \left(\frac{\partial A_{i,j}}{\partial X_i} \frac{\partial F}{\partial X_i} + A_{i,j} \frac{\partial F}{\partial X_i \partial X_j} \right).$$

From this we deduce that

$$\sum_{i=0}^{3} \left(G\frac{\partial R_i}{\partial X_i} + R_i \frac{\partial G}{\partial X_i} \right) - \sum_{i=0}^{3} \left(H\frac{\partial Q_i}{\partial X_i} + Q_i \frac{\partial H}{\partial X_i} \right) \in \left\langle \frac{\partial F}{\partial X_i} \right\rangle.$$

This completes the proof of Lemma 6. □

3.4 The fundamental quadratic form of a vector bundle section

We will prove Theorem **??** using the fundamental quadratic form of a section of a vector bundle— a generalisation of the Hessian, which we now briefly recall. We consider M, a smooth complex scheme, a vector bundle V on M and a section σ of V. We denote by W the zero scheme of σ and choose a point x of W. We choose holomorphic co-ordinates, $z_1, \ldots, z_m$, in some neighbourhood of x and we

choose a trivialisation of V near x. Of course, having picked such a trivialisation, we can consider σ to be an r-tuple of holomorphic functions $(\sigma_1, \sigma_2 \ldots \sigma_r)$.
We define the map

$$d\sigma_x : T_U(x) \to V_x$$

by

$$d\sigma_x(\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial z_i}) = \sum_{i=1}^{n} \alpha_i \frac{\partial \sigma}{\partial z_i}.$$

This map is independent of the choice of trivialisation and of local co-ordinates. If we think of σ as r holomorphic functions that cut out W then $(d\sigma_x)^*$ is the map sending an equation for W to its first order part at x. Further, Ker$(d\sigma_x)$ is the Zariski tangent space to W at x.

We define the fundamental quadratic form, $q_{\sigma,x}$, of σ at x to be the dual of the map sending an equation for W whose first order part at x vanishes to its second order part at x. It may also be thought of as the second derivative map. More precisely,

$$q_{\sigma,x} : T_W(x) \otimes T_W(x) \to V_x/\mathrm{Im}(d\sigma_x)$$

is defined by

$$q_{\sigma,x}\left(\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial z_i}, \sum_{j=1}^{m} \beta_j \frac{\partial}{\partial z_j}\right) = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial z_i}\left(\sum_{j=1}^{m} \beta_j \frac{\partial}{\partial z_j}(\sigma)\right).$$

This, similarly, is independent of the choice of local trivialisation of V and the choice of local co-ordinates z_j.
This form may also be understood as a first order invariant of the bundle morphism $d\sigma : T_U|_W \to V|_W$ defined by $d\sigma_x$ at the point x. Indeed, given a bundle morphism $\psi : \mathcal{E} \to \mathcal{F}$ on a scheme W we may associate to any $x \in W$ a map

$$d\psi_x : T_W(x) \to \mathrm{Hom}(\mathrm{Ker}(\psi_x), \mathrm{Coker}(\psi_x)).$$

This map $d\psi_x$ is simply the derivative of ψ computed with respect to some local trivialisations of $\mathcal{E}$ and $\mathcal{F}$. The restriction of such a derivation to $\mathrm{Ker}(\psi_x)$ and $\mathrm{Coker}(\psi_x)$ is independent of the choice of local trivialisations. When $\psi = d\sigma$, the map $d\psi_x$ is the fundamental quadratic form of σ at x.
In particular, if x is a smooth point of W_{red} and $\mathrm{rk}(\mathrm{Ker}(d\sigma))$ is constant in a neighbourhood of x, then $q(u, w) = 0$ for any $u \in T_{W_{\mathrm{red}}}$, for the simple reason that we may choose a local trivialisation of T_U such that $\mathrm{Ker}(d\sigma)$ is a trivial subbundle of $\mathcal{E}$ in a neighbourhood of x. This observation will be the starting point for our work in the next section.

3.5 Proof of Theorem ??

In this section, we prove that for $V = \mathcal{H}^{0,2}_{\text{prim}}|_O$ and $\sigma = \overline{\gamma}^{0,2}$ (and hence for $W = NL(\gamma)$), the fundamental quadratic form $q_{\overline{\gamma}^{0,2},X}$ described above is precisely the quadratic form given in Theorem **??**.

Recall that G, H are elements of $T_{NL(\gamma)}(X)$. Whenever f is a section of a vector bundle which vanishes at X, we will denote by $\frac{\partial f}{\partial G}(X)$ the derivative along the tangent vector G of f at the point X.

The fundamental quadratic form, $q_{\overline{\gamma}^{0,2},X}$, is defined as follows

$$q_{\overline{\gamma}^{0,2},X}(G,H) = \frac{\partial(d\overline{\gamma}^{0,2}(H))}{\partial G}(X).$$

This equation is an equality between elements of the space $H^{0,2}(X,\mathbb{C})/\mathrm{Im}(d\overline{\gamma}^{0,2})$.

Carlson and Griffiths' results on the first order IVHS of hypersurfaces tell us that after identification of $\mathcal{H}^{3-p,p-1}_{\text{prim}}$ and R_F^{pd-4} we have

1. $\mathrm{Im}(d\overline{\gamma}^{0,2}(X)) = \mathrm{Im}(\cdot P)$
2. $d\overline{\gamma}^{0,2}(H)(\tilde{X}) = Hs(\tilde{X})$.

Here, s is some section of $S^{2d-4} \otimes \mathcal{O}_{NL(\gamma)}$ such that $\mathrm{res}_{\tilde{X}}(s(\tilde{X})) = \overline{\gamma}(\tilde{X})$, where $\tilde{X}$ is any point in $NL(\gamma)$. Hence $q_{\overline{\gamma}^{0,2}}$ now reduces to

$$q_{\overline{\gamma}^{0,2},X}(G,H) = \frac{\partial(\mathrm{res}_{\tilde{X}}(Hs(\tilde{X})))}{\partial G}(X), \tag{3.1}$$

this last equation being an equality between elements of $R_F^{3p-4}/\mathrm{Im}(\cdot P)$.

Let us explain more precisely what we mean by the formula (**??**). Since $Hs(\tilde{X})$ is a degree $3d-4$ polynomial, it has a residue class in $H^{0,2}(\tilde{X})$ for any $\tilde{X} \in NL(\gamma)$. This residue class disappears at X, and therefore its derivation along the tangent vector $G \in T_{U_d}(X)$ is a well-defined element of $H^{0,2}(X,\mathbb{C})$. This is the entity that we seek to calculate. We note that

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(Hs(\tilde{X})))}{\partial G}(X) = \mathrm{res}_X\left(H\frac{(\tilde{X})}{\partial G}(X)\right) + \frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X).$$

Lemma 7 *We have* $\dfrac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X) = -\mathrm{res}_X(\sum_{i=0}^{3} R_i \dfrac{\partial G}{\partial X_i})$.

Proof of Lemma 7.

If X_ϵ is the variety cut out by the polynomial $F + \epsilon G$, then we have

$$\frac{\partial(\mathrm{res}_{X_\epsilon}(HP))}{\partial \epsilon}(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathrm{res}_{X_\epsilon}(HP).$$

We know that $HP = \sum_{i=0}^{3} R_i \frac{\partial F}{\partial X_i}$, whence we see that

$$HP = \sum_{i=0}^{3} \left(R_i \frac{\partial F + \epsilon G}{\partial X_i} - \epsilon R_i \frac{\partial G}{\partial X_i} \right).$$

Therefore,

$$\mathrm{res}_{X_\epsilon}(HP) = \mathrm{res}_{X_\epsilon} \left(-\epsilon \sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i} \right),$$

and hence

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X) = \frac{\partial(\mathrm{res}_{X_\epsilon}(HP))}{\partial \epsilon}(X) = \lim_{\epsilon \to 0} \mathrm{res}_{X_\epsilon} \left(- \sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i} \right).$$

From this we get that

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(HP))}{\partial G}(X) = \mathrm{res}_{X} \left(- \sum_{i=0}^{3} R_i \frac{\partial G}{\partial X_i} \right).$$

This completes the proof of Lemma 7. □

It remains to calculate $\dfrac{\partial s}{\partial G}(X)$.

Lemma 8 *The section s can be chosen in such a way that* $\dfrac{\partial s}{\partial G}(X) = \displaystyle\sum_{i=0}^{3} \frac{\partial Q_i}{\partial X_i}$.

Proof of Lemma 8.

By definition

$$\mathrm{res}_X(P) = \mathrm{res}_X \left[\frac{P\Omega}{F^2} \right].$$

The polynomial $s(\tilde{X})$ is chosen such that the section $\mathrm{res}_{\tilde{X}}(s(\tilde{X})) = \mathrm{res}_{\tilde{X}} \frac{s(\tilde{X})\Omega}{\tilde{F}^2}$ of $\mathcal{H}^2 \otimes \mathcal{O}_{NL(\gamma)}$ is flat with respect to the Gauss-Manin connection. In particular,

$$\frac{\partial(\mathrm{res}_{\tilde{X}}(s(\tilde{X})))}{\partial G}(X) = 0$$

and hence

$$\operatorname{res}_X\left(\frac{\partial \frac{s\Omega}{F^2}}{\partial G}(X)\right) = 0.$$

On deriving this formula, we obtain that

$$\operatorname{res}_X\left(\frac{(\frac{\partial s}{\partial G}(X))\Omega}{F^2} - 2\frac{GP\Omega}{F^3}\right) = 0.$$

It is proved in [?] that this happens only when there is some $\alpha \in H^0(\Omega^2_{\mathbb{P}^3}(2Y))$ such that

$$\frac{\frac{\partial s}{\partial G}(X)\Omega}{F^2} - 2\frac{GP\Omega}{F^3} = d\alpha.$$

Any $\alpha \in H^0(\Omega^2_{\mathbb{P}^3}(2Y))$ may be written in the form

$$\alpha = \frac{\sum_{i=0}^3 S_i \operatorname{int}(\frac{\partial}{\partial X_i})\Omega}{F^2},$$

where the S_i are degree $2d-3$ polynomials. Here, the operation int is defined for any smooth variety Y as follows. The map

$$\operatorname{int} : T_Y \otimes \Omega^2_Y \to \Omega^1_Y$$

is given by

$$\operatorname{int}(t,\omega)(v) = (\omega(t,v)).$$

It may be verified that (see, for example, [?] or [?]) that

$$d\alpha = \frac{-2}{F^3}\sum_{i=0}^3 S_i \frac{\partial F}{\partial X_i}\Omega + \frac{1}{F^2}\sum_{i=0}^3 \frac{\partial S_i}{\partial X_i}\Omega.$$

Recall that

$$\sum_{i=0}^3 Q_i \frac{\partial F}{\partial X_i} = GP.$$

Therefore, the equation

$$\frac{((\frac{\partial s}{\partial G}(X))\Omega)}{F^2} - 2\frac{HP\Omega}{F^3} = d\alpha$$

is satisfied whenever

$$\frac{\partial s}{\partial G}(X) = \sum_{i=0}^3 \frac{\partial Q_i}{\partial X_i}$$

and

$$\alpha = \frac{\sum_{i=0}^{3} Q_i \mathrm{int}(\frac{\partial}{\partial X_i})\Omega}{F^2}.$$

Since the kernel of the map $S^6 \otimes \mathcal{O}_{U_d} \to \mathcal{H}^2$ is of constant rank, it follows that we may choose s such that $\frac{\partial s}{\partial G}(X) = \sum_{i=0}^{3} \frac{\partial Q_i}{\partial X_i}$.

This completes the proof of Lemma 8. □

It follows that

$$\frac{\partial d_H(\overline{\gamma}^{0,2})}{\partial G}(X) = \mathrm{res}_X \left(\sum_{i=0}^{3} \left(\frac{\partial Q_i}{\partial X_i} H - R_i \frac{\partial G}{\partial X_i} \right) \right).$$

Therefore $q_{\overline{\gamma}^{0,2},X}(H, G)$ is equal to

$$\sum_{i=0}^{3} \left(\frac{\partial Q_i}{\partial X_i} H - R_i \frac{\partial G}{\partial X_i} \right).$$

As always, this is of course an equality of elements of $R_F^{3d-4}/\mathrm{Im}(\cdot \mathrm{P})$.

This completes the proof of Theorem 23. □

Chapter 4

Non-reduced Noether-Lefschetz loci in U_5

In this chapter we will give a first application of Theorem **??** by proving the following theorem.

Theorem 24 *Suppose that $NL(\gamma) \subset O \subset U_5$ be non-reduced and let X be a point of $NL(\gamma)$. There is then a hyperplane $H \subset \mathbb{P}^3$ such that $H \cap X$ contains 2 lines, L_1 and L_2, and non-zero distinct integers α and β such that*

$$\gamma = \alpha[L_1] + \beta[L_2] - \frac{\alpha + \beta}{5} H.$$

This result completes the classification of components of the Noether-Lefschetz locus of U_5 with exceptional codimension of tangent spaces which was started in [**?**]. In this paper, it was shown that the only reduced exceptional Noether Lefschetz components in U_5 are the space of surfaces containing a plane conic and the space of surfaces containing a line. Two proofs of this result will be presented—the first being original, and the second applying the degeneration technique used by Griffiths and Harris in [**?**].

When $d = 5$, any component of the Noether-Lefschetz locus has codimension at most 4. It has been proved in [**?**] and independently in [**?**] that the codimension of $NL(\gamma)$ is ≥ 2 and this bound is achieved only if there is a rational number α such that $\gamma = \alpha(5\Delta - H)$, where H is the class of a hyperplane section and Δ is the class of a line. Further, it has been shown in [**?**] that if $NL(\gamma)$ is of codimension 3, then there is some rational number α such that $\gamma = \alpha(5\Delta - 2H)$ where Δ is the class of a conic.

The only other Noether-Lefschetz loci in U_5 which may have tangent spaces with exceptional codimension are non-reduced components, whose reductions are of codimension 4 or 3.

Proposition 9 *Assume there exists a hyperplane H whose intersection with X has 3 components C_1, C_2, C_3 such that C_1 and C_2 are distinct lines and C_3 is a cubic. If α and β are distinct non-zero integers, then the cohomology class*

$$\gamma = \alpha C_1 + \beta C_2 - \frac{\alpha+\beta}{5} H$$

is such that $NL(\gamma)$ has a non-reduced component.

Proof of Proposition 9.

Since α, β are distinct and non-zero, γ is neither a linear combination of a line and a hyperplane section nor a linear combination of a conic and a hyperplane section. We know by the work of Voisin in [**?**] that codim($T_{NL_\gamma}(X)) > 3$, and hence codim $T_{NL(\gamma)_{\text{red}}}(X)) = 4$. We now show that $NL(\gamma)$ has a non-reduced component.

The space $NL(\gamma)$ contains the space

$$NL(C_{1\text{prim}}) \cap NL(C_{2\text{prim}}).$$

Here, by $C_{1\text{prim}}$, we mean the primitive part of the cohomology class $[C_1]$. Since this set has codimension $\leq 2 + 2 = 4$, it follows that $NL(C_{1\text{prim}}) \cap NL(C_{2\text{prim}})$ is in fact a component of $NL(\gamma)$.

Note further that if $Y \in NL(C_{1\text{prim}})$, then there is a line $C_1^Y \in Y$ such that $\overline{(C_1)}_{\text{prim}}(Y) = [C_1^Y]_{\text{prim}}$. This can be seen by a simple dimension count.

Further, in the surface Y, the intersection number of C_1^Y and C_2^Y is 1— in other words, there is a point

$$p_Y \in C_1^Y \cap C_2^Y$$

The important point is that there is a plane H_Y in $\mathbb{P}^3$ containing $C_1^Y \cap C_2^Y$. Hence, in particular, there is a hyperplane H_Y in $\mathbb{P}^3$ on which γ_Y is supported.

In [**?**] (see also [**?**]) it is shown that if there exists a holomorphic form ω on Y such that γ is supported on the zero locus of ω then codim($T_{NL(\gamma)}(Y)) < 4$. Since $K_Y = \mathcal{O}_Y(1)$, there exists such a holomorphic form, and

$$\text{codim}(T_{NL(\gamma)}(Y)) < 4$$

at every point of $NL(\gamma)$. The space $NL(\gamma)$ is therefore non-reduced. This completes the proof of Proposition 9. □

We will now prove Theorem **??**, which says that this is the only possible type of non-reduced Noether-Lefschetz locus in U_5.

4.1 Proof of Theorem ??

We assume that X is a sufficiently general smooth point of $NL(\gamma)_{\mathrm{red}}$. Recall that F is the polynomial defining X, that R_F is the associated Jacobian ring, and the P is a degree 6 polynomial such that $\mathrm{res}_X(P) = \gamma$. Since codim $T_{NL(\gamma)}(X) < 4$, it follows from the definition of $T_{NL(\gamma)}(X) = \mathrm{Ker}(\cdot P)$ that the map

$$\cdot P : S^5 \to R_F^{11}$$

is not surjective. Alternatively, by Macaulay duality there is an $X_0 \in S^1$ such that

$$X_0 PH = 0 \text{ for all } H \in R_F^5,$$

whence we deduce that $X_0 P = 0$ in R_F. We define H to be the plane $X_0 = 0$.

There exist cubics, $P_i \in S^3$, such that

$$X_0 P = \sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i}.$$

We first use the fundamental quadratic form derived in the last section to obtain relations on the P_i and $\frac{\partial F}{\partial X_i}$ which will imply that $X \cap H$ is reducible.

Proposition 10 *We have*

$$\sum_{i=1}^{3} P_i \frac{\partial F}{\partial X_i}|_H = 0 \tag{4.1}$$

$$\sum_{i=1}^{3} \frac{\partial P_i}{\partial X_i}|_H = 0. \tag{4.2}$$

Note that the first equation implies immediately that $X \cap H$ is a singular curve. We will demonstrate not only that $X \cap H$ is a reducible curve but also that the space of triples P_1, P_2, P_3 satisfying (**??**) and (**??**) has dimension at most $(j-1)$, where j is the number of components of $X \cap H$. We will then deduce that if $D_1, \ldots, D_j$

are the components of $X \cap H$, D_i being of degree d_i, then the subspace V' of $H^{1,1}_{\text{prim}}$ defined by

$$V' = \left\langle \gamma, [D_1] - \frac{d_1 H}{5}, [D_2] - \frac{d_2 H}{5}, \ldots, [D_{j-1}] - \frac{d_{j-1} H}{5} \right\rangle$$

has dimension $(j-1)$— and that therefore γ is supported on H.

Proof of Proposition 10.

We will begin by proving the following lemma.

Lemma 9 *There is a non-zero* L *contained in* S^1 *such that in* R^4_F

$$L\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) = 0. \tag{4.3}$$

Proof of Lemma 9.

We know that $\text{codim} T_{NL(\gamma)}(X) \geq 2$ by the result of Voisin and Green, and codim $T_{NL(\gamma)_{\text{red}}}(X)$ =4, since X is a smooth point of $NL(\gamma)_{\text{red}}$. We treat first the case where the codimension of $T_{NL(\gamma)_{\text{red}}}(X)$ in $T_{NL(\gamma)}(X)$ is 1. We have

$$(X_0 H)P = \sum_{i=0}^{3} P_i H \frac{\partial F}{\partial X_i}$$

and similarly

$$(X_0 G)P = \sum_{i=0}^{3} P_i G \frac{\partial F}{\partial X_i}.$$

Now, suppose that $G \in S^4$ is such that $X_0 G \in T_{NL(\gamma)_{\text{red}}}(X)$. Then for any $H \in S^4$ we have that

$$q_{\overline{\gamma}^{0,2},X}(X_0 H, X_0 G) = 0$$

(by the remark at the end of section 3.4). More precisely, using the equation for the second fundamental form given in **??**, we get that the following equations hold in R_F

$$X_0 G \sum_{i=0}^{3} \frac{\partial (P_i H)}{\partial X_i} - \sum_{i=0}^{3} P_i G \frac{\partial (X_0 H)}{\partial X_i} \in \text{Im}(\cdot P).$$

Rearranging, we get that

$$GH\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) \in \text{Im}(\cdot P).$$

Multiplying by X_0, we get that

$$X_0 GH \left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 \right) = 0,$$

and finally, by Macaulay duality, we have

$$X_0 G \left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 \right) = 0. \tag{4.4}$$

Recall that this last equation holds for any G in the space E which is defined by

$$E = \{G \in S^4 \text{ such that } X_0 G \in T_{NL(\gamma)_{\text{red}}}(X)\}.$$

We have that $\text{codim}(E) \leq 1$ (since we have supposed that the codimension of $T_{NL(\gamma)_{\text{red}}}(X)$ in $T_{NL(\gamma)}(X)$ is 1). Straightforward algebraic manipulations show that the ideal generated by any vector space of codimension 1 in R_F^4 contains R_F^5. It follows that for any $J \in R_F^5$ we have

$$J X_0 \left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 \right) = 0,$$

and hence by Macaulay duality

$$X_0 \left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 \right) = 0.$$

Hence Lemma 9 is proved in this case.

We now treat the case where the codimension of $T_{NL(\gamma)_{\text{red}}}(X)$ in $T_{NL(\gamma)}(X)$ is 2. In this case, there are two distinct elements of S^1, X_0 and X_1 such that $X_0 P = 0$ and $X_1 P = 0$. Once again, we define E by

$$E = \{G \in S^4 \text{ such that } X_0 G \in T_{NL(\gamma)_{\text{red}}}(X)\}.$$

and we then obtain that

$$X_0 G \left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 \right) = 0,$$

and similarly

$$X_1 G \left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 \right) = 0.$$

The codimension of E is at most 2. There are 2 maps,

$$\phi_0 \text{ and } \phi_1 : S^4/E \to \mathrm{Ker}(\cdot E) \subset R_F^8$$

given by multiplication by $X_0(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0)$ and $X_1(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0)$ respectively. Here by $\mathrm{Ker}(\cdot E)$, we mean the set of all elements in R_F^8 which give 0 on multiplying with any element of E. If ϕ_0 is not an isomorphism then the equation (**??**) holds for all G contained in $\phi_0^{-1}(0)$, which is a hyperplane, and the lemma follows as in the case where the codimension of $T_{NL(\gamma)_{\mathrm{red}}}(X)$ in $T_{NL(\gamma)}(X)$ is 1.

Only the case where ϕ_0 is invertible remains. But in this case $\phi_0^{-1} \circ \phi_1$ has an eigenvalue, λ. The multiplication map

$$\cdot(X_0 - \lambda X_1)\left(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0\right) : R_F^4 \to R_F^8$$

has a kernel of codimension at most 1, from which we conclude as before that $(X_0 - \lambda X_1)(X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0) = 0$. This concludes the proof of Lemma 9. □

We will now attempt to prove that this implies that $X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 = 0$. We start with the following technical lemma.

Lemma 10 *If W' is defined to be the space $S^3 \times S^1 \times \{\mathbb{C}^4/0\} \times S^5$, then the map $\phi : W' \to S^4$ given by*

$$\phi(P, L, \alpha_0, \alpha_1, \alpha_2, \alpha_3, F) = PL - \sum_{i=0}^{3} \alpha_i \frac{\partial F}{\partial X_i}$$

is submersive.

Proof of Lemma 10.

Let $(Y_0, \ldots Y_3)$ be a new set of co-ordinates for $\mathbb{P}^3$, such that $\sum_{i=0}^{3} \alpha_i \frac{\partial F}{\partial X_i} = \frac{\partial F}{\partial Y_0}$. We then have

$$\frac{\partial \phi}{\partial F}(G) = \frac{\partial G}{\partial Y_0}.$$

It follows that the map $d\phi : T_{W'} \to T_{S^4}$ is surjective. This completes the proof of Lemma 10. □

From this lemma we will deduce the following:

Lemma 11 *If $U' \subset U_5$ is defined by*

$\{F$ *such that* $\exists L_1 \in R_F^1, L_2 \in R_F^3$ *such that* $L_1 \neq 0, L_2 \neq 0$ *and* $L_1 L_2 = 0$ *in* $R_F^4\}$

then codim $U' \geq 6$.

Proof of Lemma 11.

We now define W to be the subset of W' consisting of all septuples $(P, L, \alpha_0, \alpha_1, \alpha_2, \alpha_3, F)$ such that

$$PL = \sum_{i=0}^{3} \alpha_i \frac{\partial F}{\partial X_i}.$$

It follows that the codimension of W in W' is dim(S^4)= 35, whence we see that

$$\dim(W) = \dim S^5 + 4 + 4 + 20 - 35 = \dim(S^5) - 7.$$

It follows that the codimension of the image of W under projection to U_5 is ≥ 6.

This completes the proof of Lemma 11. □

And finally, this gives us the following.

Lemma 12 *In R_F we have*

$$X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 = 0. \tag{4.5}$$

Proof of Lemma 12.

Indeed, it follows immediately from Lemma **??**, and the fact that

$$\operatorname{codim}(NL(\gamma)_{\text{red}}) = 4,$$

that for a generic point of $NL(\gamma)$ (**??**) implies that

$$X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} - P_0 = 0.$$

So Lemma 12 follows from Lemma 11. This completes the proof of Lemma 12. □

We now complete the proof of the proposition. The equation (**??**) of Proposition **??** follows from the two equations

$$P_0 = X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} \tag{4.6}$$

and

$$\sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i} = X_0 P.$$

We turn now to the equation (**??**), which follows when we differentiate (**??**) with respect to X_0 to obtain

$$\frac{\partial P_0}{\partial X_0} = \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i} + X_0 \frac{\partial (\sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i})}{\partial X_0}.$$

Re-arranging, we get that

$$-X_0 \frac{\partial \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i}}{\partial X_0} = \sum_{i=1}^{3} \frac{\partial P_i}{\partial X_i}.$$

This completes the proof of Proposition 10. □

Now, let us consider the quintic plane curve, $D = X \cap H$. In the next section, we will denote by $\tilde{F}$ the restriction of F to H. We define $D_1, \ldots, D_j$ to be the components of D and d_i to be the degree of D_i.

We now prove the following proposition.

Proposition 11 *The cohomology class γ is a linear combination of $[D_1], \ldots, [D_j]$.*

Proof of Proposition 11.

It will be enough to show that

$$\dim \left(\left\langle \gamma, [D_1] - \frac{d_1 H}{5}, \ldots, [D_{j-1}] - \frac{d_{j-1} H}{5} \right\rangle \right) \leq j - 1. \tag{4.7}$$

Recall that we had denoted this space by V'. We denote by V the space of all triplets of cubics (P_1, P_2, P_3) in variables X_1, X_2, X_3 such that

$$\sum_{i=1}^{3} P_i \frac{\partial \tilde{F}}{\partial X_i} = 0 \tag{4.8}$$

and

$$\sum_{i=1}^{3} \frac{\partial P_i}{\partial X_i} = 0. \tag{4.9}$$

These are of course none other than the equations of Proposition **??**. We will first show that the dimension of V is less than or equal to $(j-1)$ and then construct an injective linear map $V' \to V$, from which (**??**) will follow.

Indeed, we will show that given any $\nu \in V'$ it is represented by $P_\nu \in S^6$ (by which we mean that $\mathrm{res}_X P_\nu = \nu$) such that

$$X_0 P_\nu = \sum_{i=0}^{3} P_\nu^i \frac{\partial F}{\partial X_i}$$

and the triple $(P_\nu^1|_H, \ldots, P_\nu^3|_H,)$ is in V. We will assign this triple to ν.

Lemma 13 *The dimension of* V *is* $\leq j-1$.

Proof of Lemma 13.

For this, we will need to interpret the equations (**??**) and (**??**) geometrically. We consider the maps

$$f : V \to H^0(T_{\mathbb{P}^2}(2))$$

and

$$g : H^0(T_{\mathbb{P}^2}(2)) \to H^0(\Omega_{\mathbb{P}^2}(D))$$

which are given by

$$f(P_1, P_2, P_3) = \sum_{i=1}^{3} P_i \frac{\partial}{\partial X_i}$$

and

$$g(\alpha) = \frac{\mathrm{int}(\alpha)\Omega}{\tilde{F}}.$$

Here once again, for any smooth variety Y, the map

$$\mathrm{int} : T_Y \times \Omega_Y^2 \to \Omega_Y^1$$

is given by

$$\mathrm{int}(t, \omega)(v) = \omega(t, v)$$

In this case, Ω is the canonical section of $K_{\mathbb{P}^2}(3)$. The map g is an isomorphism. We will show the following lemma.

Lemma 14 *The map* f *is injective.*

Proof of Lemma 14.

Suppose that f were not injective, and the triple (P_1, P_2, P_3) were such that $f(P_1, P_2, P_3) = 0$. This would then imply the existence of P such that

$$(P_1, P_2, P_3) = (X_1 P', X_2 P', X_3 P').$$

However we would then have

$$\sum_{i=1}^{3} P_i \frac{\partial \tilde{F}}{\partial X_i} = P' \tilde{F}$$

and hence (**??**) implies that $P' = 0$. This completes the proof of Lemma 14. □

We now consider the image of $g \circ f$ in $H^0(\Omega_{\mathbb{P}^2}(D))$. We will use the following lemma.

Lemma 15 *If* $(P_1, P_2, P_3) \in V$ *then* $g \circ f(P_1, P_2, P_3) \in H^0(\Omega^{1,c}_{\mathbb{P}^2}(\log D))$.

Here, $\Omega^{1,c}_{\mathbb{P}^2}(\log D)$ denotes the sheaf of closed differential forms with logarithmic singularities along D.

Proof of Lemma 15.

It is enough to show that $d(g \circ f(P_1, P_2, P_3)) = 0$. But

$$d\left(\frac{\sum_{i=1}^{3}\left(P_i \mathrm{int}(\frac{\partial}{\partial X_i})(\Omega)\right)}{\tilde{F}}\right) = \sum_{i=1}^{3} \frac{(-P_i \frac{\partial \tilde{F}}{\partial X_i} + \tilde{F}\frac{\partial P_i}{\partial X_i})\Omega}{\tilde{F}^2}.$$

Now, by the equations (**??**) and (**??**), the right hand side of this equation is 0. This completes the proof of Lemma 15. □

We now complete the proof of Lemma 13. By the above, V injects into $H^0(\Omega^{1,c}_{\mathbb{P}^2}(\log D))$. Note that D, being the intersection of a smooth surface and a plane, is reduced.

We define U to be $\mathbb{P}^2 - D_{\mathrm{sing}}$. By the above comment, U is $\mathbb{P}^2$ minus a codimension 2 subset. There is an exact sequence on U,

$$0 \to \Omega^{1,c}_U \to \Omega^{1,c}_U(\log D) \xrightarrow{\mathrm{res}} \mathbb{C}_{D-D_{\mathrm{sing}}} \to 0,$$

from which we get an associated long exact sequence,

$$H^0(\Omega^{1,c}_U) \to H^0(\Omega^{1,c}_U(\log D)) \to H^0(D/D_{\mathrm{sing}}, \mathbb{C}) \xrightarrow{\delta} H^1(\Omega^{1,c}_U)$$

However, since $\Omega^1_{\mathbb{P}^2}$ is free and $\mathbb{P}^2 - U$ is of codimension 2, it follows by Levi's extension theorem that

$$H^0(\Omega^1_U) \simeq H^0(\Omega^1_{\mathbb{P}^2}).$$

In other words, there are no global holomorphic 1-forms, closed or otherwise, on U. Hence,

$$H^0(\Omega^{1,c}_U(\log D)) \simeq \mathrm{Ker}\ \delta.$$

Since $\dim(H^0(D/D_{\mathrm{sing}}, \mathbb{C})) = j$, if we denote the map $H^0(\Omega^{1,c}_U(\log D)) \rightarrow\neq H^0(D - D_{\mathrm{sing}}, \mathbb{C}$ by p, it will be enough to show that

$$\mathrm{Im}(p) \neq H^0(D - D_{\mathrm{sing}}, \mathbb{C}).$$

But if $u \in H^0(\Omega^{1,c}_U(\log D))$ then we have that

$$p(u)(D_i) = \mathrm{res}_{D_i}(u)$$

where $\mathrm{res}_{D_i}(u)$ is the residue of the form u along D_i. But now we know that

$$\sum_{i=1}^{j} d_i \mathrm{res}_{D_i} u = 0.$$

This can be seen by considering the integral of u along a closed path in

$$(\mathbb{P}^2 - D) \cap H,$$

where H is a general line in $\mathbb{P}^2$ and noting that $2\pi i \mathrm{res}_{D_i} u$ is precisely the integral of u along a path looping once around a point of D_i. From this it follows that

$$\dim(H^0(\Omega^{1,c}_{\mathbb{P}^2}(\log D))) \leq j - 1.$$

This completes the proof of Lemma 13. □

We now prove the following lemma.

Lemma 16 *The space V' has dimension $\leq j - 1$.*

Proof of Lemma 16.

The strategy is clear. We will construct a map $L : V' \rightarrow V$ which we will then show to be injective.

We choose a basis $(e_1 \ldots, e_m)$ for V', such that

1. $e_1 = \gamma$

2. $e_2, \ldots, e_m \in \left\langle [D_1] - \frac{d_1 H}{5}, \ldots, [D_{j-1}] - \frac{d_{j-1} H}{5} \right\rangle$.

The point of this choice of basis is the following. The argument presented in the proof of Proposition 10 will also be valid for polynomials representing classes in the space

$$\left\langle [D_1] - \frac{d_1 H}{5}, \ldots, [D_{j-1}] - \frac{d_{j-1} H}{5} \right\rangle,$$

and hence, we will be able to assign to every element of this basis an element of V in the same way as $P^1|_H, P^2|_H, P^3|_H$ was assigned to γ.

For each e_l, we choose Q^l a degree 6 polynomial such that $\mathrm{res}_X(Q^l) = e_l$. By the choice of basis, we have the following.

Lemma 17 *For all l, $X_0 Q^l = 0$ in R_F^7.*

Proof of Lemma 17.

Indeed, this is true for $e_1 = \gamma$ by definition. For $l \geq 2$, it follows from

$$e_l \in \left\langle [D_1] - \frac{d_1 H}{5}, \ldots, [D_{j-1}] - \frac{d_{j-1} H}{5} \right\rangle$$

that

$$X_0 \cdot S^4 \subset T_{NL(e_l)}(X).$$

Indeed, if $G \in S^4$, and Y is the surface defined by the polynomial $F + X_0 G$, then $D_i \subset Y$ for all i. Now assume further that $Y \in O$. There are complex numbers α_i such that

$$e_l = \sum_{i=1}^{j} \alpha_i [D_i],$$

In $H^2(Y, \mathbb{C})$ we then have that

$$\overline{e_l}(Y) = \sum_{i=1}^{j} \alpha_i [D_i]_Y,$$

where in the right hand side $[D_i]_Y$ represents the cohomology class in $H^2(Y, \mathbb{C})$ of $D_i \subset Y$. From the the expression on the left hand side, this is of Hodge type $(1,1)$. Hence the surface Y_ϵ defined by $F + \epsilon X_0 G$ is contained in $NL(e_l)$ for any G and for small ϵ. Hence we have

$$X_0 \cdot S^4 \subset T_{NL(e_l)}(X).$$

This, by Macaulay duality and the results of Carlson and Griffiths, is equivalent to $X_0Q^l = 0$ in R_F. This completes the proof of Lemma 17. □

We now choose polynomials $Q_0^l, Q_1^l, Q_2^l, Q_3^l$ (in four variables) such that

$$X_0Q^l = \sum_{i=0}^{3} Q_i^l \frac{\partial F}{\partial X_i}.$$

We then have the following lemma.

Lemma 18 *The equation (4.5) is valid for* $(Q_0^l, \ldots, Q_3^l)$*. The equations (**??**) and (**??**) are valid for the triple* $(Q_1^l|_H, \ldots, Q_3^l|_H)$*.*

Proof of Lemma 18.

For $l = 1$, this is the statement of Proposition 10. For $l \geq 2$, Lemma 17 implies that for all degree 4 polynomials G_1 and G_2,

$$X_0G_1, X_0G_2 \in T_{NL(e_l)_{\mathrm{red}}}(X).$$

Hence we see that for all G_1 and G_2 in S^4,

$$q_{\overline{e_l}^{0,2},X}(X_0G_1, X_0G_2) = 0$$

Alternatively, as in the proof of Proposition 10

$$G_1G_2\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) \in \mathrm{Im}(\cdot P)$$

and multiplying by X_0 we see that

$$X_0G_1G_2\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) = 0$$

in R_F This time, this relationship is valid for *any* choice of G_1 and G_2, so it follows immediately by Macaulay duality that

$$X_0\left(X_0\sum_{i=0}^{3}\frac{\partial Q_i^l}{\partial X_i} - Q_0^l\right) = 0$$

in R_F. This is precisely the equation (4.5). By Lemma 12, it follows that, since X has been chosen general in $NL(\gamma)$,

$$\left(X_0 \sum_{i=0}^{3} \frac{\partial Q_i^l}{\partial X_i} - Q_0^l \right) = 0$$

in R_F. Indeed, since $\deg(Q_0^l) = 3$, it follows that $(X_0 \sum_{i=0}^{3} \frac{\partial Q_i^l}{\partial X_i} - Q_0^l) = 0$. The two equations (**??**) and (**??**) now follow from the argument given at the bottom of page 51. This completes the proof of Lemma 18. □

We define L by setting $L(e_l) = (Q_1^l|_H, Q_2^l|_H, Q_3^l|_H)$ and extending by linearity.

We will now prove the following lemma.

Lemma 19 *L is injective.*

Proof of Lemma 19.

Let v be any element of V'. By linearity, there are cubic polynomials $Q_0^v, Q_1^v, Q_2^v, Q_3^v$ in variables $X_0, \ldots, X_3$ such that

1. $L(v) = (Q_1^v|_H, Q_2^v|_H, Q_3^v|_H)$,
2. The equation (4.5) is valid for $Q_0^v, \ldots, Q_3^v$,
3. There exists a Q^v such that $\sum_{i=0}^{3} Q_i^v \frac{\partial F}{\partial X_i} = X_0 Q^v$,
4. Q^v represents the cohomology class v.

Lemma 19 now follows from the following lemma.

Lemma 20 *Suppose that $\gamma = \mathrm{res}_X(P)$, and there exist ($P_0, \ldots, P_3$) such that*

$$X_0 P = \sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i}.$$

Suppose further that (4.5) is valid and that

$$P_1|_H = P_2|_H = P_3|_H = 0, i \geq 1.$$

Then $\gamma^{1,1} = 0$.

Proof of Lemma 20.

We have

$$X_0 P = \sum_{i=0}^{3} P_i \frac{\partial F}{\partial X_i}. \tag{4.10}$$

By hypothesis, X_0 divides P_i for $i \geq 1$. It follows from (4.5) that X_0 divides P_0 . Therefore, (**??**) implies that

$$P \in \left\langle \frac{\partial F}{\partial X_i} \right\rangle$$

from which it follows that

$$\mathrm{res}_X P \in F^2(H^2(X, \mathbb{C})).$$

Alternatively, we have that

$$\gamma^{1,1} = 0.$$

This completes the proof of Lemma 20. □

Since all elements of V' are Hodge $(1,1)$ classes, the injectivity of L follows immediately. This completes the proof of Lemma 19. □

This completes the proof of Lemma 16. □

This completes the proof of Proposition 11. □

To complete the theorem, it will be enough to show that D is necesarily the union of two lines and a (possibly reducible) cubic.

Lemma 21 *The curve $X \cap H$ must have at least 3 components.*

Proof of Lemma 21.

We know that γ is a linear combination of classes of curves contained on $X \cap H$. If $X \cap H$ contains only two reducible components, then γ is either the linear combination of

1. a line and a hyperplane section or
2. a conic and a hyperplane section.

This is not possible, since all such cohomology classes have reduced associated Noether-Lefschetz loci. This completes the proof of Lemma 21. □

There are now two possibilities:

1. γ is a linear combination of the cohomology classes of two lines and a hyperplane section,
2. X belongs to S, the space of all quintic hypersurfaces possessing a hyperplane section which is the union of two conics and a line.

The codimension of S is 5 and the codimension of $NL(\gamma)$ is at most 4, so the general element of $NL(\gamma)$ cannot be contained in S.

It remains only to exclude the cases $\gamma = \alpha(L_1 + L_2 - \frac{2H}{5})$ or $\gamma = \alpha(L_1 - \frac{H}{5})$. In the first case, γ is (a multiple of) the primitive part of the cohomology class of a conic, and in the second case γ is (a multiple of) the primitive part of the cohomology class of a line. In either case, γ has a reduced Noether-Lefschetz locus.

This concludes the proof of Theorem **??**. □

4.2 Alternative proof of Theorem ??

This proof will be based on the degeneration argument used by Griffiths and Harris in [**?**]. We consider the space of polynomials

$$S = \{F + X_0 G \text{ where } G \in S^4 \text{ and } F + X_0 G \in U_d\}.$$

We will prove the theorem by considering a degenration of X onto a union of a quartic surface and a line. In order to do this, we need to prove the following proposition.

Proposition 12 *There is a ∇-flat section of $F^1\mathcal{H}(X, \mathbb{C}) \otimes \mathcal{O}_S$ which is equal to γ at the point X.*

Proof of Proposition 12.

Consider the section of $F^1\mathcal{H}(X, \mathbb{C}) \otimes \mathcal{O}_S$ defined in the following way. Let Y be an element of S and G be an element of S^4 such that Y is defined by $F + X_0 G$. Define a function Q on S by

$$Q(Y) = P + \sum_{i=0}^{3} \frac{\partial(GP_i)}{\partial X_i}.$$

We then set

$$\gamma(Y) = \mathrm{res}_Y(Q(Y)).$$

We will prove that the section $\gamma(Y)$ is a ∇-flat section of $\mathcal{H}^2 \otimes \mathcal{O}_S$. We start by proving the

Lemma 22 *At any point Y of S, we have $T_S(Y) \subset T_{NL(\gamma)}(Y)$.*

Proof of Lemma 22.

We know that

$$X_0 \left(P + \sum_{i=0}^{3} \frac{\partial(GP_i)}{\partial X_i} \right) = \sum_{i=0}^{3} \left(P_i \frac{\partial F}{\partial X_i} + X_0 G \frac{\partial P_i}{\partial X_i} + X_0 P_i \frac{\partial G}{\partial X_i}. \right)$$

Now, since $P_0 = X_0 \sum_{i=0}^{3} \frac{\partial P_i}{\partial X_i}$, we have

$$X_0 \left(P + \sum_{i=0}^{3} \frac{\partial(GP_i)}{\partial X_i} \right) = \sum_{i=0}^{3} \left(P_i \frac{\partial F}{\partial X_i} + X_0 P_i \frac{\partial G}{\partial X_i} \right) + GP_0.$$

From this it follows that

$$X_0 \left(P + \sum_{i=0}^{3} \frac{\partial(GP_i)}{\partial X_i} \right) = \sum_{i=0}^{3} P_i \frac{\partial(F + X_0 G)}{\partial X_i}.$$

This last equation implies that in R_{F+X_0G} we have

$$X_0 \in \operatorname{Ker} \left(\cdot (P + \sum_{i=0}^{3} \frac{\partial(GP_i)}{\partial X_i}) \right).$$

Recall that $\operatorname{res}_Y(P + \sum_{i=0}^{3} \frac{\partial(GP_i)}{\partial X_i}) = \gamma(Y)$. Hence, the above statement says that, for all $H \in S^4$,

$$X_0 H \in T_{NL(\gamma)}(Y).$$

In other words, we have

$$T_S(Y) \subset T_{NL(\gamma)}(Y).$$

This completes the proof of Lemma 22. □

The point of this lemma is that over some neighbourhood of Y in $NL(\gamma(Y))$, we know that there exists $\tilde{P}$, a section of $S^6 \otimes \mathcal{O}_{NL(\gamma(Y))}$, such that

$$\operatorname{res}_{\tilde{Y}}(\tilde{P}) = \overline{\gamma(Y)}(\tilde{Y}).$$

Here $\overline{\gamma(Y)}$ denotes the local ∇-flat extension of $\gamma(Y)$ to a section of the vector bundle $\mathcal{H}^2 \otimes \mathcal{O}_{NL(\gamma(Y))}$. Since we know that

$$T_S(Y) \subset T_{NL(\gamma)}(Y)$$

we can compare the derivatives of $\tilde{P}$ and $Q(Y)$ along $X_0 H$ for any $H \in S^4$.

Lemma 23 *We can choose $\tilde{P}$ such that*

$$\frac{\partial \tilde{P}}{\partial (X_0 H)} = \frac{\partial Q(Y)}{\partial (X_0 H)}.$$

Proof of Lemma 23.

From the equation

$$X_0 \tilde{P}(Y) = \sum_{i=0}^{3} P_i \frac{\partial (F + X_0 G)}{\partial X_i},$$

we see that we can choose $\tilde{P}$ in such a way that

$$\frac{\partial \tilde{P}}{\partial (X_0 H)} = \sum_{i=0}^{3} \frac{\partial (H P_i)}{\partial X_i}.$$

It is clear that

$$\frac{\partial Q(Y)}{\partial (X_0 H)} = \sum_{i=0}^{3} \frac{\partial (H P_i)}{\partial X_i}.$$

This completes the proof of Lemma 23. □

This shows that we have indeed constructed a ∇-flat section of primitive Hodge classes over S. Proposition 12 follows. □

We now need to show that, given a ∇-flat section of primitive Hodge classes over S, it actually comes from a curve supported on $X_0 = 0$. The argument that establishes this is given in [**?**] and [**?**]: we summarise it here.

Theorem 25 (Griffiths, Harris) *Under these circumstances, γ is supported on $H \cap X$.*

Proof of Theorem 25.

This is a slightly adapted version of the material contained in [**?**]. There is another, more general version of this in [**?**].

We denote the curve $H \cap X$ by D. Once again, we denote the irreducible components of D by $D_1, \ldots, D_l$, and denote the degree of D_i by d_i. We will need a $G \in S^4$ such that

1. G is smooth,

2. $\mathrm{Pic}(G) = \mathbb{Z}$,

3. G intersects D transversally at smooth points of D,

4. $C = G \cap X_0$ is a smooth curve,

5. Let $p_{i,j}$, with $1 \leq i \leq 4d_j$ be the points of $D_j \cap G$. Then the equation in $\mathrm{Pic}(C)$
$$\mathcal{O}_C(\sum_{i,j} n_{i,j} p_{i,j}) = \mathcal{O}_C(n) \tag{4.11}$$
is only satisfied when there exist integers n_j such that for all i, $n_{i,j} = n_j$.

Lemma 24 *There exists some G satisfying 1-5.*

Proof of Lemma 24.

1), 2) and 4) are satisfied for a generic quartic. 3) holds for generic G because D, being the intersection of a smooth surface with a plane, is reduced. The difficult thing to prove is 5).

Let M be the space of polynomials in S^4 satisfying 3) and 4). The fundamental group of M acts by monodromy as a permutation of the points $p_{i,j}$.

In general, if S is any set, then we denote the group of all permutations of S by $\mathrm{Perm}(S)$.

Let σ be an element of $\mathrm{Perm}(\{p_{i,j}\})$ contained in the image of $\pi_1(M)$. Suppose given integers $n_{i,j}$ such that in $\mathrm{Pic}(C)$
$$\mathcal{O}_C(\sum_{i,j} n_{i,j} p_{i,j}) = \mathcal{O}_C(n)$$
for general G. Then we would also have
$$\mathcal{O}_C(\sum_{i,j} n_{i,j} \sigma(p_{i,j})) = \mathcal{O}_C(n).$$
But it is proved in [**?**] that
$$\mathrm{Im}(\pi_1(M)) = \oplus_j \mathrm{Perm}(\{p_{1,j} \ldots p_{4d_j,j}\}).$$
In other words, the image of $\pi_1(M)$ is precisely the set of all permutations that send points of D_j to points of D_j. It follows that if (**??**) holds for a generic G then for any j, k, l and for any G we have
$$\mathcal{O}_C((n_{k,j} - n_{l,j})(p_{k,j} - p_{l,j})) = \mathcal{O}_C \tag{4.12}$$

in $\mathrm{Pic}(C)$. We want to deduce that $n_{k,j} - n_{l,j} = 0$. We now have to do a little more work than Griffiths and Harris, since we do not have the luxury of starting with a generic F and cannot therefore assume that $p_{k,j}$ and $p_{l,j}$ are general points of C.

Consider a variation of G. This induces a deformation of the triple $(C, p_{k,j}, p_{l,j})$. We choose a variation of G under which the point $p_{k,j}$ approaches the point $p_{l,j}$. If (**??**) holds for generic G then under this deformation $p_{k,j} - p_{l,j}$ can be considered as a section of the Jacobian bundle which is always torsion and which vanishes at the limit point. From this it follows that $p_{k,j} - p_{l,j} = 0$.

This is not possible, since C is a smooth plane curve of degree 4 and is hence non-rational. This completes the proof of Lemma 24. □

We choose such a G, which we will now use to construct a degeneration of X onto $\tilde{Y}_0$, a surface whose Picard group we will be able to describe. Consider Y, the subvariety of $\mathbb{P}^3(\mathbb{C}) \times \mathbb{C}$ given by the equation

$$X_0 G + tF = 0.$$

Here, t parameterises $\mathbb{C}$. The variety Y is smooth over some neighbourhood of the origin except at the points where $t = 0$, $X_0 = 0$, $F = 0$ and $G = 0$. The set of these points will be denoted in what follows by $D \cap G$. At these points Y has ordinary double points of type

$$x_1 x_2 + y_1 y_2 = 0.$$

We denote the central fibre of Y by Y_0. In Proposition 12 we have constructed $\gamma(t)$, a global section of $\mathrm{Pic}(Y_t)$, which we will study via its degeneration to Y_0. In order to be sure that such a degeneration exists, we need to smooth the variety Y by blowing up.

Let us consider the blow-up of Y at $D \cap G$. When we blow up such a point the exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$, and may be contracted along either of its rulings to give a smooth variety.

More concretely, let A be an affine piece of $\mathbb{P}^3$ containing $D \cap G$. Let x_0, f and g be functions on A corresponding to X_0, F and G. Define Y° by

$$Y^\circ = (A \times \mathbb{C}) \cap Y.$$

We will smooth Y° by an explicit blow-up. Consider the space $\tilde{Y}^\circ \subset \mathbb{C}^4 \times \mathbb{P}^1$ which is defined by the equations

$$tf + x_0 g = 0,$$

$$gU = fV,$$
$$x_0 V = -tU,$$

where U and V are the co-ordinate functions on $\mathbb{P}^1$. It is easy to check that this variety is smooth. We now glue this variety to $Y - \{D \cap G\} \times \mathbb{C}$ along $Y^\circ - (D \cap G \times \mathbb{C})$. This forms a smoothing of Y which we denote by $\tilde{Y}$. We will examine the central fibre of $\tilde{Y}$, which we denote by $\tilde{Y}_0$. This fibre is the gluing together of two varieties,

1. the subvariety in $A \times \mathbb{P}^1$ given by the equations $x_0 g = 0$ and $gU = fV$,
2. $Y_0 - D \cap G$,

along the open subset $(Y_0 - D \cap G) \cap A$.

The variety $\tilde{Y}_0$ is the union of two components, $\tilde{H}$ and $\tilde{G}$, where $\tilde{H}$ is the gluing together of

1. the subvariety of $A \times \mathbb{P}^1$ given by the equations $x_0 = 0$ and $gU = fV$,
2. $H - D \cap G$.

along the open subset $(H - D \cap G) \cap A$.

Similarly, $\tilde{G} \subset \mathbb{P}^3 \times \mathbb{P}^1$ is the gluing together of two varieties

1. the subvariety of $A \times \mathbb{P}^1$ given by $g = 0$ and $V = 0$,
2. $G - D \cap G$,

along the open set $(G - D \cap G) \cap A$.

The variety $\tilde{H}$ is the plane H blown up at each of the points $D \cap G$. The variety $\tilde{G}$ is isomorphic to the surface G. They are glued together along the curve $\tilde{C}$ which is the gluing together of

1. the subvariety of $A \times \mathbb{P}^1$ given by the equations $x_0 = 0, g_0 = 0$ and $V = 0$,
2. $C - D \cap G$.

along the open set $(C - D \cap G) \cap A$.

The curve $\tilde{C}$ is simply the proper transform of C in $\tilde{H}$. Since the global variety is smooth, a line bundle L defined on $(\tilde{Y} - \tilde{Y}_0)$ such that

$$L|_{Y_t} = \gamma(t)$$

can be extended to a line bundle on $\tilde{Y}$. We need to understand $\operatorname{Pic}(\tilde{Y}_0)$.
A line bundle on $\tilde{Y}_0$ is given by the following data

1. A line bundle L_G on $\tilde{G}$,
2. A line bundle L_H on $\tilde{H}$,
3. An isomorphism between $L_G|_{\tilde{C}}$ and $L_H|_{\tilde{C}}$.

We know that $\mathrm{Pic}(G) = \mathbb{Z}$ and hence $L_G = \mathcal{O}_{\tilde{G}}(n)$ for some integer n. Letting $E_{i,j}$ be the exceptional divisor in $\tilde{H}$ over the point $D \cap G$, we have $\mathrm{Pic}(\tilde{H})$ is a free group generated by

1. $\mathcal{O}_{\tilde{H}}(1)$ and
2. $\mathcal{O}_{\tilde{H}}(E_{i,j})$.

We can now find a generating set for $\mathrm{Pic}(\tilde{Y}_0)$. Suppose that

$$L = \mathcal{O}_{\tilde{H}}(m + \sum n_{i,j} E_{i,j})$$

is a bundle on $\tilde{H}$. L extends to a bundle on the whole of $\tilde{Y}_0$ if and only if there exists n such that $L|_{\tilde{C}} = \mathcal{O}_{\tilde{C}}(n)$, or equivalently, such that

$$\mathcal{O}_{\tilde{C}}(n) = \mathcal{O}_{\tilde{C}}(m + \sum n_{i,j} p_{i,j}).$$

This is equivalent to saying that in the Picard group of $\tilde{C}$,

$$\mathcal{O}_{\tilde{C}}(\sum n_{i,j} p_{i,j}) = \mathcal{O}_{\tilde{C}}(n - m).$$

By the assumptions on G, we know that this happens only if there exist n_j such that for all i, $n_{i,j} = n_j$. Hence, Pic $\tilde{Y}_0$ is a free abelian group generated by the following bundles:

1. $\mathcal{O}_{\tilde{Y}_0}(1)$,
2. L_j such that $L_j|_{\tilde{H}} = \mathcal{O}_{\tilde{H}}(\sum_{i=1}^{4d_j} E_{i,j})$ and $L_j|_{\tilde{G}} = \mathcal{O}_{\tilde{G}}(d_j)$.

We will now prove the following result.

Proposition 13 *For each L_j, there is a line bundle K_j defined on $\tilde{Y}$ such that*

1. *For small t,*

$$K_{i,t} \in \langle \mathcal{O}_{Y_t}(D_1), \mathcal{O}_{Y_t}(D_2), \ldots, \mathcal{O}_{Y_t}(D_l), \mathcal{O}_{Y_t}(1) \rangle,$$

2. $K_j|_{\tilde{Y}_0 = L_j}$.

Proof of Proposition 13.

Consider the divisor B_j on $\tilde{Y}$ which is the proper transform of $D_j \times \mathbb{C}$ in $\tilde{Y}$. B_j can be explicitly constructed in the following way. B_j is the gluing together of

1. the subvariety of $\mathbb{C} \times A \times \mathbb{P}^1$ defined by the equations
$$U = 0 \text{ and } I_{D_j} = 0,$$
where I_{D_j} is the ideal sheaf of $D_j \cap A$, considered as a sheaf of functions on A,
2. $(D_j - \{D_j \cap G\}) \times \mathbb{C}$,

along the open subset $((D_j - D_j \cap G) \cap A) \times \mathbb{C}$.

We now consider the line bundle $\mathcal{O}_{\tilde{Y}}(B_j)$, which we denote by K'_j.
Note that $K'_j|_{\tilde{Y}_t} = \mathcal{O}_{Y_t}(D_j)$ for $t \neq 0$. The divisor $B_j|_{\tilde{Y}_0}$ is $\overline{D}_j$, the proper transform of D_j in $\tilde{H}$, which does not meet $\tilde{G}$. It follows that

1. $K'_j|_{\tilde{G}} = \mathcal{O}_{\tilde{G}}$,
2. $K'_j|_{\tilde{H}} = \mathcal{O}_{\tilde{H}}(\overline{D_j})$.

We know that
$$\mathcal{O}_{\tilde{H}}(\overline{D_j}) = \mathcal{O}_{\tilde{H}}(p^{-1}(D_j) - \sum_i E_{i,j})$$
whence it follows that
$$\mathcal{O}_{\tilde{H}}(\overline{D_j}) = \mathcal{O}_{\tilde{H}}(d_j H - \sum_i E_{i,j})$$
and finally that
$$K'_j|_{\tilde{Y}_0} = L_j^{-1} \otimes \mathcal{O}_{\tilde{Y}_0}(d_j).$$
We set $K_j = K_j'^{-1}|_{\tilde{Y}_0} \otimes \mathcal{O}_{\tilde{Y}_0}(d_j)$. This completes the proof of Proposition 13. □

Now let $\gamma(0)$ be the restriction of $\gamma(t)$ to $\tilde{Y}_0$. There are integers n_j, m such that
$$\gamma(0) = \mathcal{O}_{\tilde{Y}_0}(m) \otimes^j_{\mathcal{O}_{\tilde{Y}_0}} L_j^{n_j}.$$
From this it follows that in some neighbourhood of 0,
$$\gamma(t) = \mathcal{O}_{\tilde{Y}_t}(m) \otimes^j_{\mathcal{O}_{\tilde{Y}_t}} K_j^{\otimes n_j}(t)$$
and hence we have
$$\gamma(t) \in \langle D_j, H \rangle.$$
This completes the proof of Theorem 25. □

Hence, this gives a different proof of Theorem 24. □

Chapter 5

Exceptional loci of small codimensions

5.1 Ciliberto and Harris' conjecture

Let $NL(\gamma)$ be an exceptional Noether-Lefschetz locus and $X \in NL(\gamma)$ be a surface defined by the polynomial $F \in S^d$. Let $P \in S^{2d-4}$ be a polynomial representing γ. We recall that (see for example page 37) the tangent space $T_{NL(\gamma)}(X)$ is simply the kernel of the map

$$\cdot P : S^d/F \to R_F^{3d-4}$$

which is multiplication by P. If $NL(\gamma)$ is exceptional, then the multiplication map $\cdot P : R_F^d \to R_F^{3d-4}$ is not onto. Since the multiplication map

$$R_F^{d-4} \otimes R_F^{3d-4} \to R_F^{4d-8}$$

is a perfect pairing this is equivalent to saying that there exists $Q \in S^{d-4}$ such that $QP = 0$ in R_F. This is equivalent to saying that

$$Q \cdot S^4 \subset T_{NL(\gamma)}(X).$$

There is one case in which it is clear this will be the case. Denote the surface cut out by Q by Z. Suppose $X \cap Z$ is a reducible curve,

$$X \cap Z = C_1 \cup C_2 \cup \cdots \cup C_l$$

and that γ is a linear combination of the classes $[C_i]$,

$$\gamma = \sum_i a_i [C_i].$$

In this case we say that γ is supported on Q. We have that

Lemma 25 *If γ is supported on Q, then $GQ + F \in NL(\gamma)$ for all $G \in S^4$ such that $\{GQ + F = 0\} \in O$ and hence $QS^4 \subset T_{NL(\gamma)}(X)$.*

Proof of Lemma 25. Indeed, suppose that $Y \in O$ is given by the equation $GQ + F$. We then have that $C_i \in Y$ for all i. Suppose that we have

$$\gamma = \sum_{i=1}^{l} a_i [C_i]$$

then we have that

$$\overline{\gamma}(Y) = \sum_{i=1}^{l} a_i [C_i]_Y$$

where $[C_i]_Y$ is the cohomology of class of C_i in Y. Hence $\overline{\gamma}(Y)$ is of $(1,1)$ type, and hence $Y \in NL(\gamma)$. This completes the proof of Lemma 25. $\square$

These considerations led Ciliberto and Harris to postulate (see for example [**?**] or [**?**]) their

Conjecture 1 (Ciliberto-Harris) *If $NL(\gamma)$ is exceptional, then γ is supported on Q for some $Q \in S^{d-4}$.*

Initial progress on this conjecture was promising. Voisin even showed in [**?**] that the conjecture was true for $d = 6, 7$. Unfortunately, it is now known to be false in general. Voisin proved the following theorem in [**?**].

Theorem 26 (Voisin) *The Ciliberto-Harris conjecture is false for $d = 4s$ and s sufficiently large.*

When γ is supported on Q for some $Q \in S^{d-4}$, we will say that γ satisfies the Ciliberto-Harris conjecture.

The main theorem of this section is as follows.

Theorem 27 *Suppose that $e \leq \frac{d-1}{2}$ and $j \leq \binom{e+3}{3}$. There exists an integer, $\phi_{e,j}(d)$ such that if $NL(\gamma)$ is reduced and codim$(NL(\gamma)) \leq \phi_{e,j}(d)$ then the dimension of the space $\{Q \in S^e$ such that γ is supported on $Q\}$ is $\geq j$.*

Further, $\phi_{\frac{d-1}{2},1}(d) = O(d^3)$.

On setting $j = 1$ in this statement, we obtain the result given in the introduction. Our main theorem implies, in particular, that the Ciliberto-Harris conjecture is true for any reduced locus of codimension $\leq \phi_{\frac{d-1}{2},1}(d)$, which is a function of cubic order in d. To the best of my knowledge, all bounds known for this problem till the present are linear or quadratic in d.

5.2 Proof of Theorem 27

The theorem will follow immediately from the following two propositions.

Proposition 14 *Suppose that $NL(\gamma)$ is reduced and for all Y in some neighbourhood of X, a general element of $NL(\gamma)$, the space*

$$V = \{Q \in S^e | Q \cdot S^{d-e} \subset T_{NL(\gamma)}(Y)\}$$

is of dimension $j > 0$. Suppose further that $e \leq \frac{d-1}{2}$. Then, for all $Q \in V$ we have $F + GQ^2 \in NL(\gamma)$ for all $G \in S^{d-2e}$ such that $F + GQ^2$ defines a smooth variety.

It will then be enough to prove the following proposition.

Proposition 15 *Let X be an element of $NL(\gamma)$. We can construct $\phi_{e,j}(d)$ as above such that if*

$$\textit{codim}(NL(\gamma)) \leq \phi_{e,j}(d)$$

then $\dim\{Q \in S^e | Q \cdot S^{d-e} \subset T_{NL(\gamma)}(X)\} \geq j$.

It follows from Proposition 14, by the argument given in section 2 of [**?**], (pp 56-59), that γ is supported on $Q^2 = 0$— and hence on $Q = 0$. We now summarise this argument.

The aim is, of course, to use the degeneration methods of Griffiths and Harris from [**?**] explained in the alternative proof of Theorem 24, pp 64-69 of the current thesis. The difficulty that arises is that, Q^2 not being smooth (or indeed even reduced), we have no control on the singularities of the central fibre. This is a particular problem since the article of Griffiths and Harris required a detailed description of the family after *base-change* and blow-up, in order the deal with the problem of monodromy. (In the previous section, no such base-change was needed, since we knew by the explicit construction that γ was invariant under monodromy).

In her article, Voisin gets around this problem in the following way. Considering the family of surfaces, Y in $\mathbb{C} \times \mathbb{P}^3$ given by the equation

$$tF + Q^2G = 0$$

for some sufficiently general $G \in S$, she denotes by C the curve $F = G = 0$: she then proves that the derivative $\frac{\partial}{\partial t}$ of $\overline{\gamma}|_C$, considered as an element of $\mathrm{Pic}(C)$, is 0. Since we know (this can be found in [**?**]) that for generic G the restriction map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(C)$$

is injective, it follows that $\overline{\gamma}$ is invariant under monodromy on this family; hence, $\overline{\gamma}$ exists as a *global* section of $\mathcal{H}^{1,1}$ on this space— which is enough to be able to start the degeneration.

Voisin then completes by a more subtle version of the degeneration of Chapter 4, pp.64-69, noting that, although we cannot construct an *explicit* blow-up in this case, we know by resolution of singularities that there exists *some* blow-up, $\tilde{Y}$ of Y, which has the following properties:

The proper transform of $C \times \mathbb{C}$ in $\tilde{Y}$ is smooth and is isomorphic to $C \times \mathbb{C}$ outside of the points p_i given by $F = G = Q = t = 0$.

The proper transform, $\tilde{G}$, of $G = t = 0$ in $\tilde{Y}$ is smooth and is isomorphic to G outside of the points p_i.

If we assume (as we may, since G is generic) that $\mathrm{Pic}(G) = \mathbb{Z}$, then this is enough to be able to apply the argument given in the previous section.

Proof of Proposition 14.

We assume, since the question was dealt with for $d = 6, 7$ in **[?]**, that $d \geq 8$. We choose construct a space W in the following way:

$$W = \{(Y, A) \in NL(\gamma) \times S^e | A \cdot S^{d-e} \subset T_{NL(\gamma)}(Y)\}.$$

If X is a sufficiently general smooth point of $NL(\gamma)$, then the space

$$V_Y = \{A \in S^e | A \cdot S^{d-e} \subset T_{NL(\gamma)}(Y)\}$$

is of constant dimension near X— indeed, given any map of vector bundles over a holomorphic variety, the space over which its kernel is of minimal dimension is an (analytic) open subset. The space W will be a smooth over some neighbourhood of X. We will prove the following lemma.

Lemma 26 *At any point (Y, A) of W we have $(GA^2, 0) \in T_W(Y, A)$ for all G.*

Proof of Lemma 26.

We know that there exists some B such that $(GA^2, B) \in T_W(Y, A)$, since the map $W \to NL(\gamma)$ is locally the projection from a vector bundle onto the base, and hence induces a surjection on the tangent spaces.

Denote this tangent vector by χ. Let us derive the equation

$$AP = \sum_i L_i \frac{\partial F}{\partial X_i}$$

in the direction χ. It was calculated at the end of chapter 3 that

$$\chi(P) = \sum_i \frac{\partial (L_i G A)}{\partial X_i}.$$

By definition of χ we have $\chi(A) = B$ and $\chi(F) = (GA^2)$. Hence we have

$$A \sum_i \left(\frac{\partial (L_i G A)}{\partial X_i} \right) + BP = \sum_i \left(L_i \frac{\partial G A^2}{\partial X_i} + \chi(L_i) \frac{\partial F}{\partial X_i} \right).$$

Rearranging, we get that in R_F

$$GA \sum_i \left(A \frac{\partial L_i}{\partial X_i} - L_i \frac{\partial A}{\partial X_i} \right) = -BP.$$

We will now prove the following result.

Lemma 27 *We have*

$$A \sum_i \left(A \frac{\partial L_i}{\partial X_i} - L_i \frac{\partial A}{\partial X_i} \right) = 0 \textit{ in } R_F.$$

Proof of Lemma 27.

It is in the proof of this key lemma that we will use the fundamental quadratic form. Note that for all $H_1, H_2 \in S^{d-e}$,

$$AH_1 \text{ and } AH_2 \in T_{NL(\gamma)}(X),$$

and further,

$$q_{\overline{\gamma}^{0,2},X}(AH_1, AH_2) = 0.$$

Hence, for all H_1, H_2 the following equality holds in R_F

$$\sum_i \left(AH_1 \frac{\partial (H_2 L_i)}{\partial X_i} - H_1 L_i \frac{\partial (AH_2)}{\partial X_i} \right) \in \operatorname{Im}(\cdot P).$$

Rearranging, we get that

$$H_1 H_2 \sum_i \left(A \frac{\partial L_i}{\partial X_i} - L_i \frac{\partial A}{\partial X_i} \right) \in \operatorname{Im}(\cdot P).$$

From this we see that for all $H \in S^{2d-2e}$,

$$HA \sum_i \left(A \frac{\partial L_i}{\partial X_i} - L_i \frac{\partial A}{\partial X_i} \right) = 0$$

in R_F. We know that

$$\deg HA \sum_i \left(A \frac{\partial L_i}{\partial X_i} - L_i \frac{\partial A}{\partial X_i} \right) = 3d - 4 + e \leq 4d - 8.$$

In the last inequality we have used the fact that $d \geq 8$. It follows that

$$A \sum_i \left(A \frac{\partial L_i}{\partial X_i} - L_i \frac{\partial A}{\partial X_i} \right) = 0$$

in R_F. This completes the proof of Lemma 27. □

Returning to the proof of Lemma 26, we see that $BP = 0$. Hence

$$(0, B) \in T_W(Y, A)$$

and therefore

$$(GA^2, 0) \in T_W(Y, A) \text{ for all } G \in S^{d-2e}.$$

This completes the proof of Lemma 26. □

We now complete the proof of Proposition 14. We have just shown there is a field of tangent vectors on W which we denote by τ_G given by

$$\tau_G(Y, A) = (GA^2, 0).$$

We may now integrate along the tangent field τ_G, at least locally. (Here, we have used the fact that (Y, A) is a smooth point of W). It follows that $F + \epsilon G p^2$ is contained in $NL(\gamma)$ for all sufficiently small ϵ. Hence, since $NL(\gamma)$ is holomorphic, $F + GA^2$ is contained in $NL(\gamma)$, provided that the associated variety is smooth. This completes the proof of Proposition 14. □

It remains only to construct the integer $\phi_{e,j}(d)$ such that if we have

$$\text{codim}(NL(\gamma)) \leq \phi_{e,j}(d)$$

then the dimension of the space

$$V = \{Q \in S^e \text{ such that } Q \cdot S^{d-e} \in T_{NL(\gamma)}(X)\}$$

is at least j.

Proof of Proposition 15.

This theorem is essentially a statement about multiplication in a certain polynomial ring. We will rely on the following theorem, due to Macaulay and Gotzmann which may be found in [**?**] (pp. 64-65).

Theorem 28 (Macaulay, Gotzmann) *Given an integer, d, any other integer c may be written in a unique way as*

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots \binom{k_i}{i},$$

for some integer i. where $k_d > k_{d-1} \cdots > k_i$. We define $c^{<d>}$ by

$$c^{<d>} = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d} + \dots \binom{k_i+1}{i+1}.$$

Let V be a subvector space of S^d of codimension c. Then, the codimension of $\langle V \rangle^{d+1}$ in S^{d+1} is $\leq c^{<d>}$ and if equality holds then for all j we have

$$\text{codim } (\langle V \rangle^{d+j}) = (((c^{<d>})^{<d+1>}) \dots)^{<d+j-1>}.$$

Here, $\langle V \rangle^i$ denotes the degree i part of the ideal generated by V in $\mathbb{C}[X_0, \dots X_3]$. We now define a set of functions, $g_i(n)$. The function $g_i(n)$ should be thought of as the maximal codimension of $\langle V \rangle^{d+i}$ in S^{d+i} if V is a subvector space of S^d of codimension n containing $\langle \frac{\partial F}{\partial X_0}, \dots, \frac{\partial F}{\partial X_3} \rangle$. We define

- $g_0(n) = n$,
- $g_{i+1}(n) = g_i(n)^{<d+i>} - 1$.

Lemma 28 *If $V \subset S^d$ has codimension n and $S^1 \cdot \left\langle \dfrac{\partial F}{\partial X_i} \right\rangle \subset V$, then for any integer j the subspace generated by V in S^{d+j} has codimension $\leq g_j(n)$.*

Proof of Lemma 28.

This follows from Macaulay-Gotzmann by induction once we note that the inclusion

$$S^1 \cdot \left\langle \frac{\partial F}{\partial X_i} \right\rangle \subset V$$

implies that V generates S^{4d-8}, and hence it is not possible to have

$$\mathrm{codim}(\langle V\rangle^{d+j+1}) = (\mathrm{codim}(\langle V\rangle^{d+j}))^{<d+j>}$$

for all j. This completes the proof of Lemma 28. □

We are now in a position to define the integer $\phi_{e,j}(d)$.

Definition 11 *The integer $\phi_{e,j}(d)$ is the smallest integer n having the property that*

$$g_{2d-4-e}(n) \leq \binom{e+3}{3} - j.$$

The above work can be combined to prove the main theorem, with this definition of $\phi_{e,j}$.

Completion of the proof of Theorem 25.

It will be enough to show that if

$$\text{codim } (NL(\gamma)) \leq \phi_{e,j}(d)$$

then dim $\mathrm{Ker}(\cdot P) \geq j$. But the ring

$$S_F = R_F/\mathrm{Ker}(\cdot P)$$

is a Gorenstein graded ring of rang $2d-4$. It follows by duality that

$$\text{dim } (S_F)^e = \text{ dim } (S_F)^{2d-4-e}$$

and hence that

$$\dim(R_F/\mathrm{Ker}(\cdot P))^e \leq \binom{e+3}{3} - j$$

by the definition of $\phi_{e,j}(d)$. Hence we have

$$\text{dim } (\mathrm{Ker}(\cdot P))^e \geq j.$$

Remark 2 When we choose $e = 1$, $j = 2$, we recover the result of [**?**] and [**?**]—albeit with the additional hypothesis that $NL(\gamma)$ should be reduced. Further, for degrees 6 and 7 and using the hypothesis $e \leq d-4$ rather than $e \leq \frac{d-1}{2}$ we recover the work of Voisin in [**?**].

It remains only to prove that $\phi_{\frac{d-1}{2}}(d)$ is indeed a cubic function of d.

Proposition 16 *There exists* $\alpha > 0$ *such that*

$$\phi_{\frac{d-1}{2}}(d) \geq \alpha d^3$$

for d *sufficiently large.*

Proof of Proposition 16.

Since $\binom{\frac{d-1}{2}+3}{3}$ is a cubic in d, there exists $\beta < 1$ such that for d large

$$\binom{\frac{d-1}{2}+3}{3} - 1 \geq (\beta d + 1)\binom{\frac{3d-1}{2}+2}{2}.$$

Hence

$$\binom{\frac{d-1}{2}+3}{3} - 1 \geq \sum_{i=0}^{\lceil \beta d \rceil} \binom{\frac{3d-1}{2} - i + 2}{2},$$

and it follows that

$$g_{\frac{d+1}{2}}(\sum_{i=0}^{\lceil \beta d \rceil} \binom{d-i+2}{2}) \leq \binom{\frac{d-1}{2}+3}{3} - 1.$$

Hence we have

$$\phi_{\frac{d-1}{2},1}(d) \geq \sum_{i=0}^{\lceil \beta d \rceil} \binom{d-i+2}{2}.$$

But we know that

$$\sum_{i=0}^{\lceil \beta d \rceil} \binom{d-i+2}{2} > \frac{\beta(1-\beta)}{2} d^3.$$

and this completes the proof of Proposition 16. □

Theorem 25 follows immediately. □

Chapter 6

The Chow group of K3 surfaces

6.1 Introduction and statement of results

Since Mumford's 1968 paper [**?**], the connection between the Chow group $CH_0(S)$ of 0-cycles on a surface S and sections of the sheaves of 2-forms on S has been an object of study. In this article, Mumford proved the following result.

Theorem 29 *If $CH_0(S)$ is representable, then $h^{2,0}(S) = 0$.*

Bloch [**?**] conjectured that the converse is also true.

Conjecture 2 (Bloch) *If S is a smooth projective surface such that*

$$h^{2,0}(S) = 0$$

then the group $CH_0(S)$ is representable.

This conjecture is a special case of the Bloch-Beilinson conjectures, which predict that the Chow groups of a complex variety possess a filtration which is both compatible with correspondences and in a certain way intimately linked with the Hodge structures on $H^*(S)$. Bloch, Kas and Liebermann proved the Bloch conjecture for surfaces not of general type in [**?**]. This conjecture has also been shown to hold for various surfaces of general type such that $h^{2,0}(S) = 0$— see, for example, [**?**].

Note that, by a result of Roitman's, [**?**], if $CH_0(S)$ is representable, then $CH_0(S)_{hom}$— the subgroup of homologically trivial elements of $CH_0(S)$— is isomorphic to the Albanese variety.

The aim of this section is to show that there is also a close connection between the condition $h^{2,0}(S) = 1$ and the geometry of 0-cycles on S. In particular, we will show the following result.

Theorem 30 *Let S be a general smooth projective K3 surface. Then for general $x \in S$, the set*

$$\{y \in S | y \equiv x\}$$

is dense in S (for the complex topology).

Here $\equiv$ denotes rational equivalence between points. We will also prove a partial converse to this result.

Theorem 31 *Let S be a smooth projective complex surface, such that for a generic point x of S the set*

$$\{y \in S | y \equiv x\}$$

is Zariski dense in S. Then $h^{2,0}(S) \leq 1$.

6.2 Proof of Theorem ??

The strategy for proving Theorem **??** is quite straightforward. Note that there exist many families of elliptic curves in a K3 surface. If E is an elliptic curve and $x \in E$, then the set of points

$$\{y \in E | ny \equiv nx \text{ for some integer } n\}$$

is dense in E. But, by a theorem of Roitman's[**?**], the Chow group of a K3 surface is torsion-free, and hence, for x and y as above we necessarily have that $x \equiv y$ in S, whence the result will follow.

It has been known for some time that there are rational curves and infinitely many elliptic curves on a general K3 surface. Mori and Mukai included in [**?**] a sketch proof of this result which they attributed to Mumford and Bogolomov independently, but for which they did not give a reference. In [**?**], Chen gave a complete proof of the existence of nodal rational curves in all linear systems on a general K3 surfaces.

Theorem 32 (Chen) *For any integers $n \geq 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$ on a general K3 surface S in $\mathbb{P}^n$ contains an irreducible nodal rational curve.*

From this we can deduce the following result, as in [**?**].

Proposition 17 *The linear system $|\mathcal{O}_S(d)|$ contains a 1-dimensional family of curves of geometric genus ≤ 1 whose general element is irreducible and nodal.*

Proof of Proposition 17.

The proof of this is straightforward. The space of nodal curves of geometric genus ≤ 1 is of codimension $g-1$ in the moduli space M_g of stable curves of genus g. There is a map from an open subset (corresponding to nodal curves) of $|\mathcal{O}_S(d)|$ towards M_g. The image of this space meets the subvariety corresponding to nodal curves of genus ≤ 1— since we know that it contains a nodal irreducible rational curve. The space of nodal curves of genus ≤ 1 in $|\mathcal{O}_S(d)|$ therefore has a component of dimension $\leq \dim |\mathcal{O}_S(d)| + 1 - g$. Moreover, the general element of this component is irreducible.

It is therefore enough to show that if C is a generic (smooth, genus g) curve in $|\mathcal{O}_S(d)|$, then $h^0(\mathcal{O}_S(C)) = g+1$ and hence $\dim |\mathcal{O}_S(d)| = g$. By the Kodaira vanishing theorem, we have

$$h^1(\mathcal{O}_S(C)) = h^2(\mathcal{O}_S(C)) = 0.$$

It follows that it will be enough to show that $\chi(\mathcal{O}_S(C)) = g+1$. But by Riemann Roch and the adjunction formula, we have

$$\chi(\mathcal{O}_S(C)) = \chi(\mathcal{O}_S) + \frac{1}{2}C^2 = g+1.$$

This completes the proof of Proposition 17. □

Now, we choose

$$\pi_1 : F_1 \to B_1$$

and similarly

$$\pi_2 : F_2 \to B_2$$

two distinct irreducible 1-dimensional families in the linear systems $|\mathcal{O}_S(1)|$ and $|\mathcal{O}_S(2)|$ respectively whose general elements are integral nodal curves of geometric genus ≤ 1.There are surjective maps

$$\phi_i : F_i \to S.$$

We consider those $x \in S$ such that

x is not contained in the image of any non-integral fibre of π_2.

This is the only condition needed on x to prove the theorem for x.

Choose $y \in F_1$ such that $\phi_1(y) = x$ and denote $\pi_1(y)$ by z. Denote the curve $\pi_1^{-1}(z)$ by D. There is a surjective map from a nodal curve of genus ≤ 1 to D

$$r : \overline{D} \to D.$$

Every component of $\overline{D}$ is of geometric genus ≤ 1. There is some component of $\overline{D}$ which intersects $(\phi_1 \circ r)^{-1}(x)$ and whose image under $\phi_1 \circ r$ is not a single point. Denote the image of this component by E. Then since E has a normalisation of genus ≤ 1, the set

$$\{z \in E \text{ such that } z - x \text{ is torsion in } CH^0(E)\}$$

is dense in E. But by a result of Roitman's[**?**] the torsion part of $CH^0(S)$ is isomorphic to the torsion part of the Jacobian of S which is 0 for a K3 surface. Hence, the set

$$\{z \in E \text{ such that } z \equiv x \text{ in } CH^0(S)\}$$

is dense in E.

Now, the idea of the rest of the proof is clear. There is a curve E containing x of genus at most 1, which is transverse to general elements of a family of nodal elliptic curves in S (namely F_2). Consider the set of those members of this family which meet E in a point rationally equivalent to x. This is a dense set. If E_2 is elliptic or rational and meets E in a point rationally equivalent to x, then the points of E_2 which are rationally equivalent to x are dense in E_2.

More precisely, consider the variety $V = \phi_2^{-1}(E)$, which parameterises points of intersection of the curve E with a curve in the family F_2. The projection of V onto B_2 is surjective because the fibres of π_2 are very ample divisors. We denote by S_E the set

$$\{y \in E \text{ such that } y \equiv x \text{ in } CH^0(S)\}.$$

The set S_E is dense in E for the complex topology. We denote by T the closure of

$$\{y \in S \text{ such that } y \equiv x \text{ in } CH^0(S)\}.$$

We define $\tilde{B_2}$ to be the open set in B_2 parameterising irreducible members of the family F_2. Consider

$$Z = \pi_2 \circ \phi_2^{-1}(S_E)$$

the set parameterising curves in the family F_2 which meet E in at least one point of $S(E)$. We denote by $\tilde{Z}$ the set $Z \cap \tilde{B_2}$. Once again, if $z \in \tilde{Z}$, then the set

$$\{y \in F_{2,z} \text{ such that } y \equiv x \text{ in } CH^0(S)\}$$

is dense in $F_{2,z}$ (the fibre over z in F_2). Hence, T contains $\pi_2^{-1}(\tilde{Z})$. We now need the following lemma.

Lemma 29 *The set Z is dense in B_2.*

Proof of Lemma 29.

It is in the proof of this lemma that we will need the condition that x is not contained in the image of any non-integral fibre of F_2. Note that, since ϕ_2 is surjective, there exists C a component of V which maps surjectively onto E. By the assumption on x, and since E cannot be an element of the linear system $|\mathcal{O}_S(2)|$ for degree reasons, we see that C maps surjectively onto B_2 under π_2.

Since C is irreducible and $\phi_2|_C$ is surjective, $\phi_2|_C^{-1}(S(E))$ is dense in C. It follows that since $\pi_2|_C$ is surjective and continuous Z is dense in B_2. This completes the proof of Lemma 28. □

It immediately follows that T is dense in S. This completes the proof of Theorem 30. □

6.3 Proof of Theorem ??

Now suppose that S satisfies the hypothesis that for general $x \in S$ the set

$$\{y \in S | x \equiv y \in CH^0(S)\}$$

is Zariski dense in S. We want to show that $h^{2,0}(S) \leq 1$. Mumford proved the following result in his paper [**?**].

Theorem 33 (Mumford) *There exists a countable union of maps of reduced algebraic schemes*

$$\phi_i : W_i \to S \times S$$

such that the following hold.

1. *$x \equiv y$ if and only if there exists i such that $(x, y) \in \phi_i(W_i)$,*
2. *Let pr^1 and pr^2 be the two projections from $S \times S$ onto S. Consider the maps*

$$\pi_i^1 \text{ and } \pi_i^2 : W_i \to S$$

given by $\pi_i^j = pr^j \circ \phi_i$. We then have for any 2 form on S, ω,

$$\pi_i^{1*}(\omega) = \pi_i^{2*}(\omega).$$

We may restrict ourselves to the case where the images of all the maps ϕ_i are of dimension ≤ 2, since we have the following lemma, which is also found (in a more general form), in [**?**].

Lemma 30 *If there is an i such that the image of ϕ_i is of dimension ≥ 3 $h^{2,0}(S) = 0$.*

Proof of Lemma 30.

Indeed, assume that $\dim(\mathrm{Im}(\phi_i) \geq 3)$. Note that if π_i^1 is not surjective, then there is some point $q \in S$ such that the space

$$\{r \in S | r \equiv q \in CH^0(S)\}$$

is of dimension 2: in other words, $CH^0(S) = \mathbb{Z}$. From this it follows by the theorem of Mumford's quoted on page 81 that $h^{2,0}(S) = 0$. It follows that we may assume that π_i^1 and π_i^2 are both surjective.

By a similar argument, we may assume $\mathrm{Im}(\phi_i)$ is of dimension exactly 3. Suppose that ω is a non-zero 2-form on S. We may therefore choose a point $p \in W_i$ such that

1. The image of the push-forward map induced on tangent spaces by ϕ_i

$$\phi_{i*}(p) : (T_{W_i})(p) \to T_{S\times S}(\phi(p))$$

   is of dimension 3.

2. π_i^1 and π_i^2 are both submersive at p.

3. $\pi_i^1(p)$ and $\pi_i^2(p)$ are both points at which ω is non-zero.

Now let us consider $\pi_i^{1*}(\omega)(p)$ and $\pi_i^{2*}(\omega)(p)$. By the assumptions 2 and 3, these two forms (which according to Mumford's theorem are equal) are non-zero. We know that in $T_{W_i}(p)$

$$\mathrm{Ker}(\pi^1_{i\,*}(p)) \subset \mathrm{Ker}(\pi_i^{1*}(\omega))(p).$$

Similarly,

$$\mathrm{Ker}(\pi^2_{i\,*}(p)) \subset \mathrm{Ker}(\pi_i^{2*}(\omega))(p).$$

And now since $\pi_i^{1*}(\omega) = \pi_i^{2*}(\omega)$, it follows that

$$\langle \mathrm{Ker}(\pi^1_{i\,*}(p)) \cup \mathrm{Ker}(\pi^2_{i\,*}(p)) \rangle \subset \mathrm{Ker}(\pi_i^{1*}(\omega))(p).$$

Now, by assumption 1

$$\mathrm{codim}(\mathrm{Ker}(\pi^1_{i\,*}(p)) \cap \mathrm{Ker}(\pi^2_{i\,*}(p))) = 3.$$

Further, $\mathrm{codim}(\mathrm{Ker}(\pi^1_{i\,*}(p))) = 2$. Hence,

$$\mathrm{codim}(\mathrm{Ker}(\pi_i^{1*}(\omega)(p))) \leq 1.$$

But now, since $\pi_i^{1*}(\omega)(p)$ is an alternating 2-form on $T_{W_i}(p)$ it follows that

$$\pi_i^{1*}(\omega)(p) = 0.$$

This is contrary to assumptions 2 and 3 which imply that $\pi_i^{1*}(\omega)(p) \neq 0$. This completes the proof of Lemma 30. $\square$

We now choose y such that

1. $y \notin \pi_i^j(W_i)$ for any i such that dim(W_i) ≤ 1,
2. There do not exist x, i, j such that (x, y) is in the image of W_i and π_i^j is not submersive at any point of $\phi_i^{-1}(x, y)$.
3. The set $\{x \in S | y \equiv x\}$ is Zariski dense in S.

Since the varieties described in 1) and 2) are of dimension ≤ 1, and by assumption 3) holds for general y, there exists such a y. The theorem follows from the following proposition.

Proposition 18 *There is no non-zero 2-form ω on S vanishing at y.*

Proof of Proposition 18.

Let ω be such a 2-form, and consider $x \in S$ such that $y \equiv x$. By the assumptions on y we then have the following lemma.

Lemma 31 *The holomorphic 2-form ω vanishes at x.*

Proof of Lemma 31.

There is some W_i such that $(x, y) \in \phi_i(W_i)$. By assumption 2) on y, there exists $p \in W_i$ such that $\phi_i(p) = (x, y)$ and π_i^1, π_i^2 are both submersive at p.

We know that

$$\pi_i^{2*}(\omega)(p) = 0$$

since $\omega(y) = 0$. It follows that $\pi_i^{1*}(\omega)(p) = 0$. But by assumptions 2 and 1, this implies that $\omega(x) = 0$. This completes the proof of Lemma 31. $\square$.

Therefore, since the set of such points is Zariski dense, ω is identically 0. This completes the proof of Proposition 18. □

It follows immediately that

$$h^{2,0}(S) \leq 1.$$

This completes the proof of Theorem 31. □

Chapter 7

Deformations of stable maps of curves

The aim of this chapter will be to prove a geometric property implied by the existence of smoothing morphisms for stable maps of curves. These will give a geometric motivation for the work of [**?**] and [**?**] on relative Gromov-Witten invariants.

7.1 Introduction to Gromov-Witten invariants

The Gromov-Witten invariant of a projective variety is an invariant which, given certain incidence conditions, returns a number which represents the number of curves in the variety satisfying these conditions which " ought to " exist. It bears the same relation to the actual number of such curves as the Fulton intersection bears to the set-theoretic intersection of subschemes of a given variety. In other words, in defining the Gromov-Witten invariant, we may have to ignore certain curves. This counter-balances the fact that obstructions to deformations of curves, which are predicted by deformation theory, turn out to be trivial. Hence, the Gromov-Witten invariant is not generally equal to the actual number of curves that it is supposed to count. The advantage is that the invariant thus defined has good deformation properties— which the number of curves on a variety lacks.

In algebraic geometry, the Gromov-Witten invariant is constructed as an integral over a certain moduli space— namely, the space of pointed stable maps. We define first a stable map and then their moduli space.

Definition 12 *A stable n-pointed map of genus g towards X consists of the following data.*

1. *A projective, connected reduced curve C of genus g with at worst ordinary double points.*

2. *Distinct smooth points, $p_1, \ldots, p_n$, of C.*

3. *A morphism $\mu : C \to X$ such that if C' is a component of C of genus g which is contracted by μ then*

   (a) *if g= 0, then the number of special points (singular points or marked points) on C' is ≥ 3,*

   (b) *if g= 1, then the number of special points (singular points or marked points) on C' is ≥ 1.*

Let β be an element of $H_2(X, \mathbb{Z})$. We can then define as follows the associated moduli space of stable maps.

Definition-Theorem 1 *For any projective algebraic X and any $\beta \in H_2(X)$, there exists a projective coarse moduli space $\overline{M}_{g,n}(X, \beta)$ classifying stable n-pointed maps, μ of genus g towards X such that $\mu_*([C]) = \beta$.*

Further, there are n evaluation maps

$$\sigma_i : \overline{M}_{g,n}(X, \beta) \to X,$$

which send $\{\mu : C \to X; (p_1, \ldots, p_n)\}$ to $\mu(p_i)$.

The idea of Gromov-Witten invariant is now the following: we will pull back cohomology classes on X by σ_i and then integrate their product over the space $\overline{M}_{g,n}(X, \beta)$. If α_i is Poincaré dual to the homology class of a subvariety c_i then $\sigma_i^*(\alpha_i)$ should be Poincaré dual to the subvariety of n-pointed stable maps such that $\mu(p_i)$ lies on c_i.

Hence if $\sigma_1^*(\alpha_1) \smile \cdots \smile \sigma_n^*(\alpha_n)$ is of the appropriate dimension then its integral over $\overline{M}_{g,n}(X, \beta)$ should be the expected number of curves of genus g and cohomology class β in X which meet all the subvarieties c_i.

Unfortunately, this definition does not work except for certain types of X, and in genus 0 (see [**?**]).

Definition-Theorem 2 *If α_i are pure-dimensional elements of $H^*(X, \beta)$ and X is a homogeneous variety G/P, then when $\sigma_1^*(\alpha_1) \smile \cdots \smile \sigma_n^*(\alpha_n)$ is of the appropriate dimension we define*

$$I_\beta(\alpha_1, \ldots, \alpha_n) = \int_{\overline{M}_{0,n}(X,\beta)} \sigma_1^*(\alpha_1) \smile \cdots \smile \sigma_n^*(\alpha_n).$$

If $g_i \in G$ are general elements of G and c_i are subvarieties of X such that $[c_i]^* = \alpha_i$, then the scheme-theoretic intersection

$$\cap \sigma_i^{-1}(g_i(c_i))$$

is a finite number of reduced points on $\overline{M}_{0,n}(X, \beta)$. Further, we have that

$$\#(\cap \sigma_i^{-1}(g_i(c_i))) = I_\beta(\alpha_1, \ldots, \alpha_n).$$

We impose $g = 0$ and X homogeneous to ensure that obstructions to deformations of stable maps should vanish. This is important firstly to be sure that the space $M_{g,n}(X, \beta)$ has nice properties, and secondly to be sure that $I_\beta(\alpha_1, \ldots, \alpha_n)$ behaves well under deformations. For example, if the obstruction space is not zero, the moduli space may suddenly jump dimension in a family of varieties— which naturally makes it impossible to expect that the Gromov-Witten invariants should be deformation-invariant.

These problems were later overcome in the algebraic setting by [**?**] and [**?**], who independently constructed the *fundamental cycle classes* inside $A^*(\overline{M}_{g,n}(X, \beta))$. These are an analogue of Fulton's intersections, in a more general context. Given a moduli space, equipped with a *perfect tangent-obstruction complex*, they construct a cone in an associated vector bundle over the moduli space. They then intersect this cone with the zero section of the vector bundle to construct the fundamental cycle class, $\overline{M}^{vir}_{g,n}(X, \beta)$. More precisely, Li and Tian showed the following.

Definition-Theorem 3 *For any smooth projective variety, X, and any choice of integers g and n and class $\beta \in A_2(X)$, there is a virtual fundamental class $\overline{M}^{vir}_{g,n}(X, \beta)$ in $A^*(\overline{M}_{g,n}(X, \beta))$ such that the associated Gromov-Witten invariants:*

$$I_\beta(\alpha_1, \ldots, \alpha_n) = \int_{\overline{M}^{vir}_{g,n}(X,\beta)} \sigma_1^*(\alpha_1) \smile \cdots \smile \sigma_n^*(\alpha_n)$$

are deformation-invariant.

Gromov-Witten invariants have also been defined for symplectic manifolds using analytic techniques (see [**?**], for example). Although the definitions given in the two cases are different, it has been shown in [**?**] and [**?**] that for complex projective manifolds the symplectic and algebraic Gromov-Witten invariants co-incide.

In [**?**], Li and Ruan related the Gromov-Witten invariants of a symplectic variety X to the relative Gromov-Witten invariants of $X^\pm$, the *symplectic cuttings* of X. We define what we mean by a symplectic cutting. Suppose that X is a symplectic variety, and X^0 is an open domain in X with a Hamiltonian function H

such that the associated vector field χ_H generates a circle action. Assume that $X - H^{-1}(0)$ has two components. Then $\overline{X}^{\pm}$, the component of $X - H^{-1}(0)$ containing $H^{-1}(\mathbb{R}^{\pm})$, is a symplectic manifold with contact boundary. The boundary, $\overline{Z} = H^{-1}(0)$, can be quotiented by the circle action, and the resulting manifold is a symplectic manifold, $X^{\pm}$, containing a distinguished symplectic subvariety, $Z^{\pm}$.

Li and Ruan then obtained a complicated recurrence relation which gives the Gromov-Witten invariants of X in terms of the *log* Gromov-Witten invariants of $X^{\pm}$ relative to $Z^{\pm}$. These are known in the algebro-geometric context as *relative* Gromov- Witten invariants. These log invariants count stable maps of pseudo-holomorphic curves, $F : C \to X^{\pm}$ with prescribed tangency to $Z^{\pm}$ at marked points.

We will now consider comutative diagrams of the following type:

$$\begin{array}{ccc} X_1 \cup_{Z_1=Z_2} X_2 & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Delta \end{array}$$

where $X_1 \cup_{Z_1=Z_2} X_2$, which we will also denote by X_0, is the union of 2 smooth varieties glued along isomorphic smooth codimension 1 subvarieties, Δ is an open complex disc and $X \to \Delta$ has generically smooth fibres. Then the pairs (X_i, Z_i) play the role in algebraic geometry of $(X^{\pm}, Z^{\pm})$ in symplectic geometry, and X_t (for $t \neq 0$) is the algebro-geometric analogue of the symplectic variety X. In this case, Li recently defined in [**?**] both the stack of relative stable maps of a pair (Z, D), and the stack of stable maps to an expanded degeneration of a family of varieties degenrating to a normal crossing variety. He then builds a perfect tangent-obstruction complex on these stacks, allowing him to define a virtual moduli class. In the subsequent paper [**?**] he uses these virtual moduli classes to define relative Gromov-Witten invariant of (Z, D), and re-prove the recursion relation of Li and Ruan.

The result of Li and Ruan has the following intuitive (and incorrect) interpretation:

Given such a commutative diagram, a stable map $f : C_0 \to X_0$ can be smoothed to a flat family of maps,

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \end{array}$$

if and only if any $p \in C_0$ which maps to the singular locus of X_0 is contained in two components C_1 and C_2 of C, such that the image of C_i under f is contained in X_i and the order of tangency of C_i to Z_i at p is independent of i.

This interpretation fails when components of C are sent onto the central divisor $X_1 \cap X_2$: as a result, the space of maps that have the property above will not generally be proper, which prevents us from intersecting cycle classes on it. In the case of genus-0 Gromov-Witten invariants, Gathmann proposed the following definition in [**?**].

Definition 13 *Let X be an algebraic variety, D a codimension-1 subvariety, $\mathfrak{a} = (a_1, \ldots, a_r)$ an r-tuple of non-negative integers, $\beta \in H_2(X)$, and n a non-negative number. Then the moduli space of $\mathfrak{a}$-relative stable genus zero n-pointed maps $F : C \to X$ with homology class β is the closed subspace of the usual Kontsevich space, $M_{0,n+r}(X, \beta)$, of maps $F : C \mapsto X$ such that the Fulton intersection,*

$$D \cdot_F C \in A_0(F^{-1}(D))$$

is $\sum_{i=1}^{r} \alpha_i x_i$.

Here, $D \cdot_F C$ denotes the intersection (with respect to F) constructed by Fulton in [**?**] (see the summary on p.92).

In this chapter we will give a geometric motivation for this definition. Let

$$\begin{array}{rcl} F : C & \to & X \\ \downarrow & & \downarrow \\ f : \Delta & \to & \Delta \end{array}$$

be a commutative diagram of proper holomorphic maps. Assume that X is a smooth variety, and X_0, the central fibre, is a normal crossing variety,

$$X_0 = X_1 \cup_{Z_1 = Z_2} X_2,$$

where X_1 and X_2 are smooth varieties glued together along isomorphic smooth codimension 1 subvarieties, Z_1 and Z_2. The space C is a flat family of stable curves and F is a family of stable maps. Suppose that the central curve C_0 is $C_1 \cup_{x_j^1 = x_j^2} C_2$, where C_i, a not necessarily connected prestable curve (i.e., a union of disjoint nodal curves) maps into X_i under F and $\{x_j^i\}$ is an r-tuple of points of C_i. We denote the restriction of F to C_i by F_i. Then we have the following theorem.

Theorem 34 *There exist integers m_j, such that, for $i = 1, 2$, we have*

$$\sum_j m_j x_j^i = Z_i \cdot_{F_i} C_i$$

in the group

$$A_0(F_i^{-1}(Z_i)).$$

Here once more the symbol $Z_i \cdot_{F_i} C_i$ denotes the intersection class constructed by Fulton in chapter 6 of [**?**] (see in particular page 92 and §6.1)

7.2 Proof of Theorem ??

We prove the theorem first for smooth C. We denote by π_C and π_X the maps from C, X to Δ. As divisors, we have

$$F^*(\pi_X^*(0)) = \pi_C^*(f^{-1}(0)) \text{ and } X_1 + X_2 = \pi_X^*(0)$$

whence it follows that

$$F^*(X_1 + X_2) = nC_0.$$

Here, n is the order of the vanishing of the map f at 0.

The Weil divisors X_1 and X_2 are also Cartier divisors. Since no component of C is sent to X_0 by F, we can pull these back to divisors $F^*(X_1)$ and $F^*(X_2)$ on C.We know that

$$F^*(X_1) + F^*(X_2) = nC_0.$$

We will write L_i for $\mathcal{O}_X(-X_i)$. Notice that

$$L_1|_{X_1} = \mathcal{O}_{X_1}(Z_1),$$

and similarly

$$L_1|_{X_2} = \mathcal{O}_{X_2}(-Z_2).$$

We also have that $L_1 \otimes L_2 = \mathcal{O}_X$.

Further, we have

$$F^*(L_1)|_{C_1} = F_1^*(\mathcal{O}_{X_1}(Z_1)),$$
$$F^*(L_2)|_{C_2} = F_2^*(\mathcal{O}_{X_2}(Z_2)).$$

In a similar way, we see that

$$F^*(L_1)|_{C_2} = F_2^*(\mathcal{O}_{X_2}(-Z_2)),$$

$$F^*(L_2)|_{C_1} = F_1^*(\mathcal{O}_{X_1}(-Z_1)).$$

We will need the following lemma.

Lemma 32 *Consider the set of all ordered triples,* $T = (C', C, p)$ *where*

1. C' *and* C *distinct are components of* C_0*,*
2. $p \in C' \cap C$*.*

To each element T *of the set we can assign an integer,* m_T*, such that*

1. $m_{(C',C'',p)} = -m_{(C'',C',p)}$ *for all elements of* T*,*
2. $m_T = 0$ *if* $F(p) \notin X_1 \cap X_2$*,*
3. *For all components* C'*, in Pic*(C') *we have*

$$F^*(L_1) = \mathcal{O}_{C'}(\sum_{T=(C',C,p)} m_T p) \tag{7.1}$$

where the sum is taken over all triples (C', C, p)*, where* C *is a components of* C_0 *such that* $C \neq C'$ *and* $p \in C' \cap C$*. Equivalently, we could write*

$$F^*(L_2) = \mathcal{O}_{C'}(\sum_{T=(C',C,p)} -m_T p).$$

Proof of Lemma 32.

There exists an n such that $F^*(X_1 + X_2) = nC_0$. For each component of C_1, C', we define an integer $k_{C'}$ by

$$F^*(X_1) = k_{C'}C' + \text{ other components of } C_0.$$

I claim that

$$m(C', C'', p) = k_{C'} - k_{C''}$$

satisfies the conditions of the theorem. Note that

$$F^*(L_1) = F^*(L_1) \otimes \mathcal{O}_C(k_{C'}C_0).$$

The right hand side is the line bundle of the divisor $F^*(-X_1) + k_{C'}C_0$, in which the coefficient of C' is 0.

Let p be a smooth point of C_0, u a local equation for X_1 in X at $F(p)$ and x a local equation at p for C' in C. Near p, $u \circ F = x^{k_{C'}} v$, where v is invertible and holomorphic.

$F^*(-X_1) + k_{C'}C_0$ is represented near p by v. Now, let p be a singular point of C_0, at which the components C' and C'' meet. Let y be a local equation for C'' in C at p. Near p, we have

$$u \circ F = x^{k_{C'}} y^{k_{C''}} w,$$

where w is invertible and holomorphic. It follows that $F^*(-X_1) + k_{C'}C_0$ is represented near p by $y^{k_{C''}-k_{C'}}w$.

Restricting these functions to C' (which we can do, since C' is not a component of $F^*(-X_1) + k_{C'}C_0$) we obtain a Cartier divisor on C' whose associated line bundle is $L_1|C'$. This Cartier divisor is represented at a smooth point of C_0 by an invertible holomorphic function. At an intersection point p of C' and C'' it is represented by $y^{k_{C''}-k_{C'}}$ where y is a local equation for p on C'. This completes the proof of Lemma 32. □

Suppose that x_i^1 and x_i^2 are contained in, respectively, $C_{i_1} \subset C_1$ and $C_{i_2} \subset C_2$, components of C_0.

Lemma 33 *For smooth C, the numbers $m_i = m(C_{i_1}, C_{i_2}, x_i)$ defined in the statement of Lemma 32 satisfy the conditions of the theorem.*

Proof of Lemma 33.

Let q be an isolated point of $F_1^{-1}(Z_1)$. We need to show that $q = x_i$ for some $i \le r$ and that the intersection multiplicity of C_{i_1} and Z_1 at x_i is $m(C_{i_1}, C_{i_2}, x_i)$.

Let u_1, u_2 be local equations for X_1, X_2, respectively. The divisor cut out by $F^*(u_2)$, which we denote by $\sum m_{C_j} C_j$, is non-trivial and contains q in its support. Further, $m_{C_{i_1}} = 0$.

Thus, there is another component of C_0 containing q along which $F^*(u_2)$ vanishes. This component is sent into X_2, and since q was a 0-dimensional connected component of $F_1^{-1}(Z_1)$ it is part of C_2. So q is one of the marked points x_i.

Let $C_{i_1} \subset C_1$ and $C_{i_2} \subset C_2$ be the components of C containing q. Let y_j be a local equation at q for C_{i_j}. Note that at q we have $u_j \circ F = y_j^n v_j$, where v_j is holomorphic. From the definition of $m(T)$ it is clear that

$$m(C_i^1, C_i^2, x_i) = n.$$

Further, the intersection multiplicity of Z_1 and C^1 at q is simply the order of vanishing of $u_2 \circ F_1$ at q, since $u_2|_{X_1}$ is a local equation for Z_1. This is also equal to

n and the lemma follows for a point component.

Now suppose that A is a 1-dimensional topological component of $F_1^{-1}(Z_1)$, with components $A_1, \ldots, A_s$. Let $S_1 \ldots S_j$ be the components of C_1 meeting A which are not contained in A. We note that $F_1(S_j) \not\subset Z_1$. We define p_i to be the point where S_i meets A. From the canonical decomposition described on page 95 of [**?**], the component of $Z_1 \cdot_{F_1} C_1$ supported on $A_0(A)$, which we denote by Z, is given by

$$Z = -c_1(F_1^*(L_1)|_A) + \sum_{S_i} m(S_i)p_i \tag{7.2}$$

where $m(S_i)$ is the intersection multiplicity of S_i and Z_1 at p_i. Now, by (1)

$$c_1(F_1^*(L_1)|_A) = \sum_T m(C', C'', p)p$$

where T is the set of triples, (C', C'', p) such that C' and C'' are components of C_0 which meet at p and C' is contained in A.

Using Lemma 32 and the fact that

$$m(S_i) = m(S_i, A_j, p_i) = -m(A_j, S_i, p_i),$$

we see that in the expression (7.2), any singular point of C_0 which joins two components of C_1, one of which is in A, is counted twice with cancelling coefficients, and so only the $C'' \subset C_2$ will contribute. The expression (**??**) reduces to

$$Z = \sum_{i|x_i^1 \in A} m_i x_i^1.$$

Of course, the same argument applies to a component of $F_2^{-1}(Z_2)$. We have shown that

$$Z_i \cdot_{F_i} C_i = \sum m_j x_j^i$$

in $A_0(F_1^{-1}(Z_1))$. This completes the proof of Lemma 33. □

We now consider a non-smooth family C. The family C is stable and therefore has reduced fibres. After base change and restriction to components of C we may assume that C is smooth, apart from a finite number of A_k singularities in the central fibre. After blow-up of these singularities, we obtain a smooth curve, $\overline{C}$, fibred over Δ, and a map:

$$\overline{F} : \overline{C} \to X$$

factoring through the blow-down, $\tau : \overline{C} \to C$. This blow-down is a contraction of some number of chains of rational components of C_0. We denote $\tau^{-1}(C_1)$ by $\overline{C_1}$ and the closure in $\overline{C_0}$ of $\overline{C} - \overline{C_1}$ by $\overline{C_2}$. We denote by τ_i the restriction of τ to $\overline{C}_i$.

Since $\tau_2^{-1}(x_i^2)$ is a single point, we may denote by $\overline{x_i^2}$ the point $\tau_2^{-1}(x_i^2)$, and by $\overline{x_i^1}$ the point in $\overline{C}_1$ whose image in $\overline{C_0}$ is $\overline{x_i^2}$. We need the following lemma:

Lemma 34 *Let*

$$\tau'_{i*} : A_0(\overline{F_i}^{-1}(Z_i)) \to A_0(F_i^{-1}(Z_i))$$

be the push-forward morphism. Then we have

$$\tau'_{i*}(Z_i \cdot_{\overline{F}_i} \overline{C_i}) = Z_i \cdot_{F_i} C_i.$$

Proof of Lemma 34.

It will be enough to show that, given

1. $c : B \to A$, a map between two nodal curves which is an isomorphism except that there is a unique rational components C which is contracted by c,

2. X a smooth variety containing a smooth divisor, D,

3. F is a map from A to X,

then the image of $D \cdot_{F \circ c} B$ under

$$c_* : A_0((F \circ c)^{-1}(D)) \to A_0(F^{-1}(D))$$

is $A \cdot_F D$.

If $f(C) \notin D$ there is nothing to prove. If B_i are the components of B, and c_i is c restricted to B_i, then

$$c_*(D \cdot_{F \circ c} B) = c_*(\sum_i D \cdot_{F \circ c_i} B_i).$$

Further, $D \cdot_{F \circ c} C$ vanishes, since C is sent under $F \circ c$ to a point in D. This completes the proof of Lemma 34. $\square$.

Since $\overline{C}$ is globally smooth, there exist m_i such that

$$Z_1 \cdot_{\overline{F}_1} \overline{C_1} = \sum_i m_i \overline{x}_i^1$$

and

$$Z_2 \cdot_{\overline{F}_2} \overline{C_2} = \sum_i m_i \overline{x}_i^2.$$

Further, $\tau_{i*}(Z_i \cdot_{\overline{F}_i} \overline{C}_i) = Z_i \cdot_{F_i} C_i$. Since the image under τ_* of $\overline{x}_i^j$ is x_i^j, this completes the proof of Theorem 34. □

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