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Sur la classification de certaines algèbres de von Neumann

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CHAPITRE 1

Introduction

Durant ces années de thèse, mon travail de recherche a porté sur le problème de la classification (à isomorphisme près) de certaines algèbres de von Neumann, ou encore *facteurs*, provenant de deux contextes très différents : la *théorie ergodique* et les *probabilités libres*. Ce premier chapitre a deux objectifs : à la fois, je veux présenter mon travail de recherche, c'est-à-dire introduire les notions, outils et concepts qui seront utilisés dans les autres chapitres, et je veux donner aussi, pour le lecteur non spécialiste de ces questions, une introduction générale aux algèbres de von Neumann et aux probabilités libres. Cette introduction est donc le reflet de mes années d'apprentissage dans ce domaine tout à fait passionnant.

Outre ce premier chapitre introductif, ma thèse se compose de quatre autres chapitres. Le deuxième chapitre est un article paru au *Journal of Functional Analysis* dans lequel je construis une nouvelle famille de facteurs de type III_1. Je classifie complètement cette famille dans un cas assez général. Le troisième chapitre est un article soumis aux *International Mathematics Research Notices* dans lequel je prouve des résultats de classification pour certains *produits libres* d'algèbres de von Neumann. Le quatrième chapitre est une note dans laquelle je montre que certains produits libres d'algèbres de von Neumann sont *premiers*, au sens où ils ne peuvent se décomposer comme le produit tensoriel de deux algèbres de von Neumann *diffuses*. Enfin, le cinquième chapitre est constitué de quelques problèmes sur lesquels je compte poursuivre mes recherches.

1. Une Introduction aux Algèbres de von Neumann

1.1. Généralités. Fixons H un espace de Hilbert complexe séparable. On note $\langle \cdot, \cdot \rangle$ son produit scalaire que l'on suppose linéaire en la première variable. Notons $B(H)$ l'algèbre de tous les opérateurs bornés sur H. Cette algèbre est munie d'une involution notée $*$ et appelée l'*adjonction*. L'adjonction est définie par

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle, \forall \xi, \eta \in H, \forall T \in B(H).$$

L'algèbre $B(H)$ est de plus normée par la norme d'opérateurs ou norme du *sup*, définie par

$$\|T\| = \sup_{\xi \in H, \|\xi\| \leq 1} \|T\xi\|, \forall T \in B(H).$$

L'algèbre $B(H)$ est complète pour cette norme, et l'on a $\|ST\| \leq \|S\|\|T\|$, pour tous $S, T \in B(H)$: ainsi, $B(H)$ est une *algèbre de Banach*. Enfin, cette norme vérifie la relation fondamentale suivante :

$$\|T^*T\| = \|TT^*\| = \|T\|^2, \forall T \in B(H). \tag{1}$$

La propriété donnée par l'Équation (1) fait de $B(H)$ une C^*-*algèbre*. Plus généralement, on appelle C^*-algèbre toute sous-algèbre involutive de $B(H)$ fermée pour la topologie normique.

On peut en fait définir d'autres topologies sur $B(H)$ plus *faibles* que la topologie donnée par la norme. Ces topologies sont données dans le tableau suivant (on les définit pour des *suites généralisées*) :

Topologie	$T_i \to 0$
normique	$\|T_i\| \to 0$
ultra-$*$-forte	$\sum_n (\|T_i\xi_i\|^2 + \|T_i^*\xi_n\|^2) \to 0,\ \forall (\xi_n) \in \ell^2(\mathbf{N}) \otimes H$
ultraforte	$\sum_n \|T_i\xi_n\|^2 \to 0,\ \forall (\xi_n) \in \ell^2(\mathbf{N}) \otimes H$
ultrafaible	$\sum_n \langle T_i\xi_n, \eta_n\rangle \to 0,\ \forall (\xi_n), (\eta_n) \in \ell^2(\mathbf{N}) \otimes H$
$*$-forte	$\|T_i\xi\|^2 + \|T_i^*\xi\|^2 \to 0,\ \forall \xi \in H$
forte	$\|T_i\xi\| \to 0,\ \forall \xi \in H$
faible	$\langle T_i\xi, \eta\rangle \to 0,\ \forall \xi, \eta \in H$

La relation entre ces différentes topologies est la suivante :

$$\begin{array}{ccccccc}
\text{normique} & \prec & \text{ultra-}*\text{-forte} & \prec & \text{ultraforte} & \prec & \text{ultrafaible} \\
 & & \curlywedge & & \curlywedge & & \curlywedge \\
 & & *\text{-forte} & \prec & \text{forte} & \prec & \text{faible,}
\end{array}$$

où "$\prec$" signifie que le côté gauche est plus fin que le côté droit. Si H est de dimension finie, ces topologies sont évidemment toutes les mêmes. Par contre, si H est de dimension infinie, ces topologies sont toutes distinctes deux à deux. Pourquoi définir des topologies faibles ? L'idée est très simple : *moins* d'ouverts pour *plus* de compacts. Ainsi, on peut facilement montrer que la boule unité de $B(H)$ est compacte pour la topologie faible. Quelle est la relation entre ces différentes topologies sur les parties bornées de $B(H)$? Notons B_1 la boule unité de $B(H)$. On a les relations suivantes :

- topologie ultrafaible sur B_1 = topologie faible sur B_1 ;
- topologie ultraforte sur B_1 = topologie forte sur B_1 ;
- topologie ultra-$*$-forte sur B_1 = topologie $*$-forte sur B_1.

Pour une partie $A \subset B(H)$ non vide, on définit le *commutant* de A noté A' par

$$A' = \{x \in B(H); xy = yx, \forall y \in A\}.$$

On voit facilement que le commutant A' est une sous-$*$-algèbre de $B(H)$. On définit de la même manière A'', $A^{(3)}$, ... La relation entre ces différents *commutants* est la suivante :

$$\begin{array}{rcl}
A & \subset & A''; \\
A^{(2n-1)} & = & A^{(2n+1)}, \forall n \geq 1; \\
A^{(2n)} & = & A^{(2n+2)}, \forall n \geq 1.
\end{array}$$

La définition suivante devient alors naturelle.

DÉFINITION 1.1. *Une* algèbre de von Neumann M *sur* H *est une sous-$*$-algèbre de* $B(H)$ *telle que*

$$M = M''.$$

Un *facteur* est une algèbre de von Neumann M avec centre trivial, i.e., $M \cap M' = \mathbf{C}1$. Pour toute partie non vide $A \subset B(H)$, le commutant A' est une algèbre de von Neumann. Le lien profond entre la notion de commutant et les différentes topologies définies sur l'algèbre $B(H)$ est donné dans le théorème suivant :

THÉORÈME 1.2 (Théorème du Bicommutant de von Neumann). *Soit $M \subset B(H)$ une sous-$*$-algèbre contenant* 1. *Alors, il y a équivalence entre les propriétés suivantes :*

(1) *M est fermée pour la topologie ultra-$*$-forte.*

(2) *$M = M''$.*

On voit donc qu'il y a pour une algèbre de von Neumann une caractérisation *algébrique* et une caractérisation *topologique*. Parmi toutes les topologies que nous avons définies plus haut, l'une d'entre elles joue un rôle particulier : c'est la topologie *ultrafaible*. Étant donnée $M \subset B(H)$ une algèbre de von Neumann, on note M_* l'espace vectoriel des formes ultrafaiblement continues sur M. C'est en fait un espace de Banach normé par $\|\varphi\| = \sup_{x \in M, \|x\| \leq 1} |\varphi(x)|$. On montre l'égalité suivante :

$$M = (M_*)^*. \tag{2}$$

Pour cette raison, M_* est appelé le *prédual* de M. Toute C^*-algèbre A qui est le dual d'un espace de Banach comme dans l'Équation (2) est appelée une *W^*-algèbre*. On montre qu'il y a en fait équivalence entre la notion de W^*-algèbre et celle d'algèbre de von Neumann. Il existe donc une notion *intrinsèque* d'algèbre de von Neumann indépendante de l'espace de Hilbert sur lequel elle est représentée. La topologie ultrafaible est donc intrinsèque. Une forme φ sur M est dite normale si pour toute suite généralisée $(x_i)_{i \in I}$ bornée, croissante, constituée d'éléments positifs de M, on a

$$\varphi(\sup_{i \in I} x_i) = \sup_{i \in I} \varphi(x_i).$$

Une forme sur M est normale si et seulement si elle est ultrafaiblement continue. Un *état* φ sur une algèbre de von Neumann M est une forme qui vérifie

- $\varphi(x^*x) \geq 0$, $\forall x \in M$. C'est la condition de *positivité*.
- $\varphi(1) = 1$.

Un état est dit *fidèle* si de plus $\varphi(x^*x) = 0 \Longrightarrow x = 0$.

1.2. La Classification des Facteurs en Types. Dans toute cette partie, on suppose que l'algèbre de von Neumann $M \subset B(H)$ est un facteur, i.e., $M \cap M' = \mathbf{C}1$. On appelle *projection* de M tout élément $p \in M$ vérifiant $p^2 = p$ et $p^* = p$. L'ensemble des projections de M est noté $\mathcal{P}(M)$. On appelle *isométrie partielle* de M tout élément $u \in M$ tel qu'il existe un sous espace fermé $K \subset H$ avec u isométrique sur K et nulle sur $K^\perp$. On définit sur $\mathcal{P}(M)$ une relation d'équivalence. Soient $e, f \in \mathcal{P}(M)$. On dit que e est *équivalent* à f et on écrit $e \sim f$, s'il existe une isométrie partielle $u \in M$ telle que $e = u^*u$ et $f = uu^*$. On vérifie aisément que c'est bien une relation d'équivalence. Si $p, q \in \mathcal{M}$, on écrit $p \leq q$ pour dire que $pH \subset qH$ ou encore l'image de p est incluse dans celle de q. Si une projection $p \in M$ est équivalente à une projection $q_1 \in M$ avec $q_1 \leq q \in M$, alors on écrit $p \precsim q$ ou $q \succsim p$. Si $p \precsim q$ mais $p \nsim q$, alors on écrit $p \prec q$. Soit $p \in \mathcal{P}(M) \backslash \{0\}$.

- On dit que p est *finie* si
$$\forall q \in \mathcal{P}(M) \text{ telle que } q \leq p,\ q \sim p \ \Rightarrow\ q = p.$$
- On dit que p est *minimale* si
$$\forall q \in \mathcal{P}(M),\ q \leq p \ \Rightarrow\ q = p \text{ ou } q = 0.$$

Il est facile de voir que minimale $\Longrightarrow$ finie. Si p n'est pas finie, on dit qu'elle est *infinie*.

Nous sommes à présent en mesure de pouvoir définir le *type* du facteur M. On dit que M est de type I, s'il existe dans M une projection minimale. On dit que M est de type II, si M possède une projection finie mais aucune projection minimale. On dit alors que M est de type II_1, si l'identité est finie et de type II_∞, si l'identité est infinie. On dit enfin que M est de type III, si M ne possède aucune projection finie. On remarque immédiatement que l'on est bien dans l'un des trois cas de figures cités ci-dessus. Une étude plus approfondie de la géométrie des projections conduit au théorème suivant :

Théorème 1.3 (Murray & von Neumann, [**6**]). *Soit M un facteur. Il existe une application $D : \mathcal{P}(M) \mapsto [0, +\infty]$ unique à un scalaire $\lambda > 0$ près telle que :*

(1) $p \sim q \Longleftrightarrow D(p) = D(q)$.
(2) $p \precsim q \Longleftrightarrow D(p) \leq D(q)$.
(3) $pq = 0 \Longrightarrow D(p + q) = D(p) + D(q)$.
(4) *p finie* $\Longleftrightarrow D(p) < +\infty$.

De plus, à normalisation près, l'image de D est l'un des sous-ensembles suivants de $[0, +\infty]$:

$\{0, \dots, n\}$	*dans ce cas M est un facteur de type*	I_n
$\{0, \dots, +\infty\}$		I_∞
$[0, 1]$		II_1
$[0, +\infty]$		II_∞
$\{0, +\infty\}$		III

Donnons tout de même quelques précisions sur ce théorème. Pour $n \in \mathbf{N}^*$, un facteur de type I_n n'est rien d'autre que $M_n(\mathbf{C})$. Quant au facteur de type I_∞, il est isomorphe à $B(\ell^2)$. Un facteur M de type II_1 est un facteur de dimension infinie munie d'une trace τ normale et fidèle, i.e., un état normal et fidèle qui vérifie de plus $\tau(xy) = \tau(yx), \forall x, y \in M$. On montre qu'une telle trace est nécessairement unique sur M. De plus, on a la notion de *dimension continue*, à savoir

$$\tau : \mathcal{P}(M) \to [0, 1]$$

est une surjection. À chaque projection p dans M, on associe sa *dimension* $\tau(p)$ qui est un nombre réel compris entre 0 et 1. Tous les réels du segment $[0, 1]$ sont atteints de cette manière. Le problème de la classification des facteurs de type II_1 à isomorphisme près est un problème extrêmement difficile. Un facteur de type II_∞ n'est en fait rien d'autre que le produit tensoriel d'un facteur de type II_1 avec $B(\ell^2)$. Enfin, un facteur de type III est un facteur qui n'est ni de type I ni de type II ! Nous donnerons plus loin des théorèmes de structure pour de tels facteurs. Mais notons ici que la géométrie des projections n'est plus d'aucun ressort car toutes les projections non nulles sont équivalentes dans un facteur de type III.

1.3. Quelques Constructions Classiques de Facteurs.

1.3.1. *L'Algèbre de von Neumann d'un Groupe Dénombrable.* Soit Γ un groupe infini dénombrable muni de sa topologie discrète. L'action de Γ par translations à gauche sur lui-même induit une *représentation unitaire* de Γ sur l'espace de Hilbert $\ell^2(\Gamma)$ notée λ_Γ et appelée la *représentation régulière gauche* :

$$\lambda_\Gamma : \begin{array}{ccc} \Gamma & \to & \ell^2(\Gamma) \\ \gamma & \mapsto & (\delta_{\gamma'} \mapsto \delta_{\gamma\gamma'}). \end{array}$$

L'algèbre de von Neumann du groupe Γ est par définition $L(\Gamma) := \lambda_\Gamma(\Gamma)''$. Cette algèbre de von Neumann est *finie*, i.e., elle possède une trace τ fidèle et normale donnée par :

$$\tau(x) = \langle x\delta_e, \delta_e \rangle, \forall x \in L(\Gamma).$$

On dit qu'un groupe Γ est *à classes de conjugaison infinies* ou encore CCI, si toutes les classes de conjugaison hormis celle de l'élément neutre 1 sont infinies. Il est facile de voir que Γ est CCI si et seulement si $L(\Gamma)$ est un facteur. Dans ce cas, $L(\Gamma)$ est un facteur de type II_1.

EXEMPLES 1.4. Nous donnons quelques exemples de groupes CCI.

- Soit $S_\infty = \bigcup_{n\geq 1} S_n$ le groupe des permutations à support fini sur $\mathbf{N}^*$. Le groupe S_∞ est CCI et *moyennable*, i.e., il existe une suite de vecteurs unitaires (ξ_n) dans $\ell^2(S_\infty)$, telle que pour tout $\sigma \in S_\infty$, $\|\lambda_\sigma(\xi_n) - \xi_n\|_2 \to 0$ lorsque $n \to \infty$. On note $\mathcal{R} = L(S_\infty)$ et on l'appelle le facteur *hyperfini* de type II_1. Le célèbre théorème de Connes [**4**] sur l'unicité du facteur *injectif* implique en particulier que tous les groupes moyennables CCI donne exactement le même facteur de type II_1, à savoir $\mathcal{R}$.
- Pour $n \in \mathbf{N}^* \cup \{+\infty\}$, notons $\mathbf{F}_n$ le groupe libre à n générateurs. Pour tout $n \geq 1$, le groupe $\mathbf{F}_n$ est CCI et a la *Propriété de Haagerup*, i.e., il existe sur $\mathbf{F}_n$ une fonction conditionnellement de type négatif propre [**2**]. On ne sait toujours pas à l'heure actuelle si $L(\mathbf{F}_m) \cong L(\mathbf{F}_n)$ pour $m \neq n$. L'alternative dûe à Rădulescu [**27**] et à Dykema [**10**] est la suivante : soit les $L(\mathbf{F}_n)$ sont tous isomorphes, soit ils sont deux à deux non isomorphes.
- Pour $n \in \mathbf{N}^*$, notons Γ_n le groupe $\mathrm{SL}(2n+1, \mathbf{Z})$. Pour tout $n \geq 1$, le groupe Γ_n est CCI et a la *Propriété (T) de Kazhdan*, i.e., toute fonction conditionnellement de type négatif sur Γ_n est bornée [**14**]. Connes a conjecturé que pour $m \neq n$, $L(\Gamma_m)$ et $L(\Gamma_n)$ ne sont pas isomorphes.

1.3.2. *Group Measure Space Construction de Murray & von Neumann.* Soit $(X, \mathcal{A}, \mu)$ un espace mesuré, soit G un groupe dénombrable et soit une action mesurable notée $\cdot$, de G sur $(X, \mathcal{A}, \mu)$. On suppose que l'action

- *préserve la classe de la mesure*, i.e., pour tout $A \in \mathcal{A}$, $\mu(A) = 0 \Longrightarrow \mu(g \cdot A) = 0$, pour tout $g \in G$;
- est *libre*, i.e., pour tout $g \neq 1$, $\mu(\{x \in X, g \cdot x = x\}) = 0$;
- est *ergodique*, i.e., pour tout $A \in \mathcal{A}$, si pour tout $g \in G$ on a $\mu(A \,\Delta\, g \cdot A) = 0$, alors $\mu(A) = 0$ ou $\mu(X \backslash A) = 0$.

Soit ν une autre mesure sur l'espace $(X, \mathcal{A})$. On dit que ν est *équivalente* à μ si pour tout $A \in \mathcal{A}$, $\nu(A) = 0 \Longleftrightarrow \mu(A) = 0$. On dit que ν est G-invariante si pour tout $A \in \mathcal{A}$ et tout $g \in G$, $\nu(g \cdot A) = \nu(A)$.

L'action de G sur $(X, \mathcal{A}, \mu)$ s'interprète facilement en termes d'action sur l'algèbre de von Neumann $L^\infty(X, \mathcal{A}, \mu) \subset B(L^2(X, \mathcal{A}, \mu))$. Pour tout $g \in G$, pour tout $F \in L^\infty(X, \mathcal{A}, \mu)$, on définit

$$\sigma_g(F)(x) = F(g^{-1} \cdot x), \forall x \in X.$$

Notons H l'espace de Hilbert $L^2(X, \mathcal{A}, \mu) \otimes \ell^2(G)$. Soit $\pi_\sigma : L^\infty(X, \mathcal{A}, \mu) \to B(H)$ la représentation définie par :

$$(\pi_\sigma(F)\xi)(x, g) = \xi(g^{-1} \cdot x, g), \forall F \in L^\infty(X, \mathcal{A}, \mu), \forall \xi \in H.$$

Il est clair que $1 \otimes \lambda_G$ définit une représentation unitaire de G sur H. L'action σ et les représentations π_σ et $1 \otimes \lambda_G$ vérifient la relation de *covariance* suivante :

$$(1 \otimes \lambda_G(g))\pi_\sigma(F)(1 \otimes \lambda_G(g))^* = \pi_\sigma(\sigma_g(F)), \forall F \in L^\infty(X, \mathcal{A}, \mu), \forall g \in G.$$

On dit aussi que le système $\{L^\infty(X), G, \sigma\}$ est covariant. On appelle *produit croisé* de $L^\infty(X, \mathcal{A}, \mu)$ par G et on note $L^\infty(X, \mathcal{A}, \mu) \rtimes_\sigma G$, l'algèbre de von Neumann agissant sur

H engendrée par $\pi_\sigma(L^\infty(X, \mathcal{A}, \mu))$ et $(1 \otimes \lambda_G)(G)$. On a alors le théorème de classification suivant :

THÉORÈME 1.5 (Murray & von Neumann, [**41**]). *En reprenant les hypothèses mentionnées plus haut, notons $M = L^\infty(X, \mathcal{A}, \mu) \rtimes_\sigma G$. Alors M est un facteur et*

(1) *M est de type* I *si et seulement si l'action est transitive.*
(2) *M est de type* II_1 *si et seulement s'il existe une mesure finie ν, G-invariante et équivalente à μ.*
(3) *M est de type* II_∞ *si et seulement s'il existe une mesure infinie ν, G-invariante et équivalente à μ.*
(4) *M est de type* III *si et seulement s'il n'existe aucune mesure non triviale ν, G-invariante et équivalente à μ.*

EXEMPLE 1.6 (Décalage de Bernoulli). Prenons $(X_0, \mu_0) = ([0, 1], \lambda)$ l'espace borélien standard avec la mesure de Lebesgue. Soit G un groupe dénombrable et infini. Soit

$$(X, \mu) = \prod_{g \in G} (X_0, \mu_0)_g.$$

On regarde l'action σ de G sur (X, μ) par *décalage*, i.e., $\sigma_g((x_h)) = (x_{g^{-1}h})$, pour tout $g \in G$. Cette action préserve trivialement la mesure μ. On vérifie aisément que cette action est libre et ergodique. Elle vérifie même une propriété d'ergodicité plus forte appelée le *mélange*. Ces dernières années, Popa a montré des résultats de *rigidité* tout à fait spectaculaires pour de telles actions (voir par exemple [**22, 23, 43**]).

Soit N une algèbre de von Neumann quelconque. On peut munir le groupe $\mathrm{Aut}(N)$ d'une structure de groupe topologique. En effet, soient $\theta, (\theta_n)_{n \in \mathbf{N}}$ des éléments de $\mathrm{Aut}(N)$, on définit

$$\theta_n \to \theta \Longleftrightarrow \|\psi \circ \theta_n - \psi \circ \theta\| \to 0, \forall \psi \in N_*.$$

Cette topologie fait de $\mathrm{Aut}(N)$ un groupe *polonais*. Soit maintenant G un groupe localement compact quelconque et $\sigma : G \to \mathrm{Aut}(N)$ une action continue. On peut alors définir de la même manière que précédemment le *produit croisé* $N \rtimes_\sigma G$. Par la suite, cette notion sera utilisée pour établir les théorèmes de structure des facteurs de type III.

1.4. La Classification des Facteurs de Type III.

1.4.1. *La Théorie de Tomita-Takesaki pour un état.* Soit (M, φ) une algèbre de von Neumann munie d'un état fidèle et normal. La construction GNS associée à cet état donne un triplet $(H_\varphi, \pi_\varphi, \xi_\varphi)$ avec $H_\varphi = L^2(M, \varphi)$. Si on note $\eta_\varphi : M \to H_\varphi$ le plongement canonique, on a $\xi_\varphi = \eta_\varphi(1)$ et la représentation π_φ est donnée par :

$$\pi_\varphi(a)\eta_\varphi(b) = \eta_\varphi(ab), \forall a, b \in M.$$

L'état φ est donné par la relation $\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi \rangle$, pour tout $x \in M$. De plus, comme φ est fidèle, la représentation π_φ est elle-même fidèle. Dorénavant, nous regarderons $M \subset B(H_\varphi)$, autrement dit nous supposons $\pi_\varphi = \mathrm{id}$. Le vecteur ξ_φ est *cyclique*, i.e., $\overline{M\xi_\varphi} = H_\varphi$ et *séparant*, i.e., pour tout $x \in M$, $x\xi_\varphi = 0 \Longrightarrow x = 0$.

On définit à présent

$$S_\varphi : \begin{array}{ccc} H_\varphi & \to & H_\varphi \\ x\xi_\varphi & \mapsto & x^*\xi_\varphi. \end{array}$$

L'opérateur S_φ est anti-linéaire, densément défini et non borné *a priori*. Le premier résultat-clé de la Théorie de Tomita-Takesaki affirme que l'opérateur S_φ est fermable.

Nous continuerons à noter S_φ sa fermeture. Écrivons alors sa décomposition polaire :

$$S_\varphi = J_\varphi \Delta_\varphi^{1/2},$$

avec J_φ anti-unitaire, i.e., $J_\varphi^* = J_\varphi$ et $J_\varphi^2 = 1$, et $\Delta_\varphi = S_\varphi^* S_\varphi$ auto-adjoint, positif, densément défini et non borné *a priori*. Le résultat majeur de cette théorie est le suivant :

THÉORÈME 1.7 (Tomita & Takesaki, [**40**]). *Les notations sont celles adoptées plus haut. Soit M' le commutant de M sur H_φ.*

(1) $\forall x \in M,\ J_\varphi x^* J_\varphi \in M'$.

(2) $\sigma_t^\varphi : x \mapsto \Delta_\varphi^{it} x \Delta_\varphi^{-it}$ *définit un flot d'automorphismes sur M.*

1.4.2. *La Classification des Facteurs de Type* III. On suppose que M est un facteur. Notons $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)$ le groupe des *automorphismes extérieurs* de M. Notons $\varepsilon : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ la projection canonique. Connes a montré dans sa thèse [**5**] qu'à l'instant t fixé, $\varepsilon(\sigma_t^\varphi)$ ne dépend pas du choix de l'état φ. Ceci permet alors de définir l'homomorphisme suivant :

$$\delta : \begin{array}{ccc} \mathbf{R} & \to & \mathrm{Out}(M) \\ t & \mapsto & \varepsilon(\sigma_t^\varphi). \end{array}$$

Selon Connes, cette application est l'*évolution naturelle dans le temps* de l'algèbre de von Neumann M. On écrit $T(M) = \ker\delta$: c'est un invariant algébrique pour M. Il montre l'équivalence suivante :

$$T(M) \neq \{0\} \Longleftrightarrow M \text{ est de type III.}$$

Dorénavant, on supposera que M est un facteur de type III. Malheureusement, l'invariant T est insuffisant pour pouvoir définir le sous-type de M. Connes définit alors un autre invariant, l'invariant S. Cette construction est difficile, et nous renvoyons à [**5**]. Nous allons brièvement exposer l'approche de Takesaki de ce problème [**42**].

On appelle *core* de M le produit croisé $M \rtimes_{\sigma^\varphi} \mathbf{R}$. On sait d'après les résultats de Connes [**5**] que ce produit croisé ne dépend pas du choix de l'état φ ; Takesaki montre que ce core est une algèbre de von Neumann semi-finie, i.e., il existe sur $M \rtimes_{\sigma^\varphi} \mathbf{R}$ une trace fidèle, normale et semi-finie. Il définit dans [**42**] l'*action duale* θ^φ de $\mathbf{R}_+^*$ sur $M \rtimes_{\sigma^\varphi} \mathbf{R}$, de la façon suivante (les notations pour les générateurs du produit croisé sont celles utilisées précédemment) :

$$\begin{array}{rcl} \theta_s^\varphi(\pi_{\sigma^\varphi}(x)) & = & \pi_{\sigma^\varphi}(x), \forall x \in M, \forall s \in \mathbf{R}_+^* \\ \theta_s^\varphi(1 \otimes \lambda_{\mathbf{R}}^t) & = & s^{-it}(1 \otimes \lambda_{\mathbf{R}}^t), \forall s \in \mathbf{R}_+^*, \forall t \in \mathbf{R}. \end{array}$$

L'action duale θ^φ *échelle la trace* au sens où il existe une trace fidèle, normale, semi-finie sur $M \rtimes_{\sigma^\varphi} \mathbf{R}$, notée Tr, telle que

$$\mathrm{Tr}(\theta_s^\varphi(y)) = s\,\mathrm{Tr}(y), \forall s \in \mathbf{R}_+^*, \forall y \in (M \rtimes_{\sigma^\varphi} \mathbf{R})^+.$$

Le système covariant $\{M \rtimes_{\sigma^\varphi} \mathbf{R}, \mathbf{R}_+^*, \theta^\varphi\}$ ne dépend pas du choix de l'état φ, et le théorème de dualité de Takesaki montre l'isomorphisme suivant :

$$(M \rtimes_{\sigma^\varphi} \mathbf{R}) \rtimes_{\theta^\varphi} \mathbf{R}_+^* \simeq M \otimes B(L^2(\mathbf{R})).$$

De plus, il montre que l'action duale θ^φ restreinte au centre de $M \rtimes_{\sigma^\varphi} \mathbf{R}$ est ergodique : cette action est donc un invariant de M et est appelée le *flot des poids*. Soit cette action est *proprement ergodique*, et alors M est de type III_0, soit cette action est *transitive*, et

- si θ^φ est l'action $\mathbf{R}_+^* \curvearrowright \mathbf{R}_+^*$, alors M est un facteur de type I ou II ;
- si θ^φ est l'action $\mathbf{R}_+^* \curvearrowright \mathbf{R}_+^*/\lambda^{\mathbf{Z}}$ pour $\lambda \in]0,1[$, alors M est de type III_λ ;

- si θ^φ est l'action $\mathbf{R}_+^* \curvearrowright \{1\}$, alors M est de type III_1.

On obtient alors le théorème de structure pour les facteurs de type III_1 encore appelé *décomposition continue* des facteurs de type III_1.

Théorème 1.8 (Takesaki, [**42**]). *Soit M un facteur de type* III_1*. Alors, il existe N un facteur de type* II_∞ *et une action* $\theta : \mathbf{R}_+^* \to \mathrm{Aut}(N)$ *qui échelle la trace tels que*

$$M = N \rtimes_\theta \mathbf{R}_+^*.$$

De plus, cette décomposition est unique *à conjugaison près.*

Le facteur N est isomorphe à $M \rtimes_{\sigma^\varphi} \mathbf{R}$, le core de M. L'action θ n'est alors rien d'autre que l'action duale de σ^φ. Précisons quelque peu ce que signifie *à conjugaison près.* Pour $i \in \{1, 2\}$, soient $\theta^i : \mathbf{R} \curvearrowright N_i$ une action qui échelle la trace. On dit que θ^1 et θ^2 sont *conjuguées* s'il existe un isomorphisme $\pi : N_1 \to N_2$, tel que

$$\pi^{-1}\theta_t^2\pi(x) = \theta_t^1(x), \forall x \in N_1, \forall t \in \mathbf{R}.$$

Dans sa thèse [**5**], en utilisant d'autres méthodes, Connes avait obtenu un théorème de structure analogue pour les facteurs de type III_λ. C'est la *décomposition discrète* des facteurs de type III_λ.

Théorème 1.9 (Connes, [**5**]). *Soit M un facteur de type* III_λ*, pour* $\lambda \in]0, 1[$*. Alors, il existe N un facteur de type* II_∞ *et un élément* $\theta \in \mathrm{Aut}(N)$ *tels que* $\mathrm{Tr} \circ \theta = \lambda \, \mathrm{Tr}$ *et*

$$M = N \rtimes_\theta \mathbf{Z}.$$

De plus, cette décomposition est unique *à conjugaison près.*

1.5. Un Raffinement de la Classification pour les Facteurs de Type III_1 : le Cas Presque-Périodique. Au vu du Théorème 1.8, les facteurs de type III_1 admettent-ils tous une décomposition discrète ? Connes répond négativement à cette question dans son article [**3**]. Il introduit alors la notion fondamentale suivante :

Définition 1.10 (Connes, [**3**]). *Soit M un facteur. On suppose que le groupe* $\mathrm{Aut}(M)$ *est munie de la topologie introduite précédemment. On dit que M est un facteur* plein *si le sous-groupe* $\mathrm{Int}(M)$ *est fermé dans* $\mathrm{Aut}(M)$*.*

Un facteur de type I, $B(K)$, est trivialement plein car $\mathrm{Aut}(B(K)) = \mathrm{Int}(B(K))$. Connes montre qu'un facteur plein ne peut pas être de type III_0. Pour un facteur de type II_1, il montre l'équivalence entre les trois propriétés suivantes ([**4**]) :

- M est un facteur plein ;
- M n'a pas la propriété Γ de Murray & von Neumann, i.e., toutes les suites *centrales* son triviales ;
- L'algèbre des opérateurs compacts $K(L^2(M))$ est incluse dans la C^*-algèbre $C^*(M, M')$.

Dans toute la suite, on supposera que M est un facteur de type III_1 plein. Soit φ un état fidèle, normal sur M. On dit que φ est *presque périodique* si l'opérateur modulaire Δ_φ dans la théorie de Tomita-Takesaki est diagonalisable. Notons $\mathrm{Sp}(\Delta_\varphi)$ le spectre ponctuel de l'opérateur Δ_φ. Pour $\gamma \in \mathrm{Sp}(\Delta_\varphi)$, notons M^γ l'ensemble des γ-vecteurs propres pour l'état φ dans M, i.e.,

$$\begin{aligned} M^\gamma &= \{x \in M ; \varphi(xy) = \gamma\varphi(yx), \forall y \in M\} \\ &= \{x \in M ; \sigma_t^\varphi(x) = \gamma^{it}x\}. \end{aligned}$$

On remarque immédiatement que $M^{\gamma_1}M^{\gamma_2} \subset M^{\gamma_1\gamma_2}$. Notons que M^1 est le *centralisateur* de l'état φ : on le note en général M^φ. Ainsi φ est presque périodique si et seulement si la sous-algèbre $\bigoplus_{\gamma\in\mathrm{Sp}(\Delta_\varphi)} M^\gamma$ est ultrafaiblement dense dans M. Notons $\Gamma \subset \mathbf{R}_+^*$ le sous-groupe engendré par le spectre ponctuel $\mathrm{Sp}(\Delta_\varphi)$. En fait, le groupe modulaire peut s'étendre par continuité au groupe compact $G = \widehat{\Gamma}$ dual du groupe Γ. Connes prouve le théorème suivant :

THÉORÈME 1.11 (Connes, [**3**], Dykema, [**9**]). *Soit M un facteur de type III_1 plein. Soit φ un état fidèle, normal et presque périodique sur M. On suppose que le centralisateur M^φ est un facteur. Alors $\mathrm{Sp}(\Delta_\varphi) \subset \mathbf{R}_+^*$ est un sous-groupe dénombrable et dense noté Γ. Il existe N un facteur de type II_∞, et une action $\theta : \Gamma \to \mathrm{Aut}(N)$ qui échelle la trace tels que*

$$M = N \rtimes_\theta \Gamma.$$

Le facteur N est isomorphe à $M \rtimes_{\sigma^\varphi} G$ où $G = \widehat{\Gamma}$. Le produit croisé $M \rtimes_{\sigma^\varphi} G$ est parfois appelé le *core discret* de M. L'action θ n'est alors rien d'autre que l'action duale de σ^φ. Bien sûr, les produits croisés $M \rtimes_{\sigma^\varphi} \mathbf{R}$ et $M \rtimes_{\sigma^\varphi} G$ ne sont absolument pas isomorphes. Enfin, Connes montre qu'en fait le groupe Γ est un invariant algébrique de M, il ne dépend pas du choix de l'état φ. On note dans ce cas $\Gamma = \mathrm{Sd}(M)$. Nous verrons dans la deuxième partie que certains *facteurs d'Araki-Woods libres* fournissent des exemples de facteurs de type III_1 pleins munis d'un état presque périodique préféré, à savoir l'*état quasi-libre libre*.

2. Une Introduction aux Probabilités Libres

2.1. Liberté. Produits Libres d'Algèbres de von Neumann. Soit (M, φ) une algèbre de von Neumann munie d'un état normal, fidèle. Dans la suite, une telle algèbre sera appelée *W^*-espace de probabilité non-commutatif.* Ses éléments seront appelés des *variables aléatoires libres.*

DÉFINITION 1.12. *Soit (M, φ) un W^*-espace de probabilité non-commutatif. Soit $x \in M$. La* distribution *de x est la fonctionelle linéaire μ_x sur $\mathbf{C}[X]$ (l'algèbre des polynômes complexes en une variable) définie par $\mu_x(P) = \varphi(P(x))$, pour tout $P \in \mathbf{C}[X]$.*

REMARQUE 1.13 (Théorème Spectral). Si de plus $x = x^*$, μ_x s'étend naturellement en une mesure à support compact sur $\mathbf{R}$. Plus précisément, il existe une unique mesure $d\mu_x$ dont le support est $\mathrm{Sp}(x)$, le spectre de x, et telle que pour toute fonction f bornée, borélienne sur $\mathrm{Sp}(x)$, on a

$$\int f(t)d\mu_x(t) = \varphi(f(x)).$$

DÉFINITION 1.14. *Soit (M, φ) un W^*-espace de probabilité non-commutatif. Soit une famille $(A_\iota)_{\iota\in I}$ de sous-algèbres de von Neumann. On dit que cette famille est* libre *dans M pour l'état φ si pour tout $n \in \mathbf{N}^*$, pour tous $a_j \in A_{\iota_j}$ avec $\iota_1 \neq \cdots \neq \iota_n$, on a*

$$\forall j \in \{1, \ldots, n\}, \varphi(a_j) = 0 \Longrightarrow \varphi(a_1 \cdots a_n) = 0. \tag{3}$$

Supposons que la famille $(A_\iota)_{\iota\in I}$ soit libre dans (M, φ) et que $\bigcup_{\iota\in I} A_\iota$ engendre M en tant qu'algèbre de von Neumann. Alors, grâce à l'Équation (3), l'état φ est totalement déterminé par ses restrictions $(\varphi_{|A_\iota})_{\iota\in I}$. De plus, si chacune de ses restrictions est une trace, φ est lui-même un état tracial. Nous présentons un moyen très naturel d'obtenir des sous-algèbres libres dans une algèbre de von Neumann donnée.

Soit $(M_\iota, \varphi_\iota)_{\iota\in I}$ une famille d'algèbres de von Neumann, chacune d'entre elles étant munie d'un état fidèle, normal φ_ι. On fait la construction GNS pour chacune des (M_ι, φ_ι).

On obtient alors un triplet $(H_\iota, \pi_\iota, \xi_\iota)$ où comme précédemment $H_\iota = L^2(M_\iota, \varphi_\iota)$. Si $\eta_\iota : M_\iota \to H_\iota$ est le plongement canonique, on a $\xi_\iota = \eta_\iota(1)$. On supposera comme avant $M_\iota \subset B(H_\iota)$, autrement dit $\pi_\iota = \mathrm{id}$. On définit tout d'abord (H, ξ) l'espace de Hilbert *produit libre* des espaces (H_ι, ξ_ι). Notons $H_\iota^\circ = H_\iota \ominus \mathbf{C}\xi_\iota$. On pose

$$H = \mathbf{C}\xi \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{\iota_1\neq\cdots\neq\iota_n\in I} H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ\right).$$

On note aussi $(H, \xi) = *_{\iota\in I}(H_\iota, \xi_\iota)$. Afin de pouvoir définir le produit libre des algèbres (M_ι, φ_ι), il faut faire agir chacune des algèbres M_ι sur l'espace de Hilbert H. Pour cela, nous définissons à présent un plongement $\lambda_\iota : B(H_\iota) \to B(H)$. Pour $\iota \in I$, notons

$$H(\iota) = \mathbf{C}\xi \oplus \bigoplus_{n\geq 1}\left(\bigoplus_{\iota_1\neq\cdots\neq\iota_n\in I, \iota_1\neq\iota} H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ\right),$$

et définissons l'unitaire $V_\iota : H_\iota \otimes H(\iota) \to H$ de la façon suivante :

$$\begin{array}{rcl} \xi_\iota \otimes \xi & \mapsto & \xi \\ H_\iota^\circ \otimes \xi & \to & H_\iota^\circ \\ \xi_\iota \otimes (H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ) & \to & H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ \\ H_\iota^\circ \otimes (H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ) & \to & H_\iota^\circ \otimes H_{\iota_1}^\circ \otimes \cdots \otimes H_{\iota_n}^\circ. \end{array}$$

La représentation λ_ι de $B(H_\iota)$ sur H est donnée par

$$\lambda_\iota(a) = V_\iota(a \otimes 1_{H(\iota)})V_\iota^*, \forall a \in B(H_\iota).$$

Définition 1.15. *L'algèbre de von Neumann* produit libre *des (M_ι, φ_ι), notée $M = *_{\iota\in I} M_\iota$, et agissant sur $(H, \xi) = *_{\iota\in I}(H_\iota, \xi_\iota)$ est définie par*

$$M = W^*\left(\bigcup_{\iota\in I} \lambda_\iota(M_\iota)\right).$$

Comme pour chaque $\iota \in I$, la représentation λ_ι est fidèle, on identifiera par la suite M_ι avec $\lambda_\iota(M_\iota)$, son image dans $B(H)$. On définit alors un état sur M par $\varphi(x) = \langle x\xi, \xi\rangle$, pour tout $x \in M$. Il est appelé le *produit libre* des états φ_ι et noté $\varphi = *_{\iota\in I} \varphi_\iota$. On montre aisément que φ est normal, fidèle et la restriction de φ à chacune des sous-algèbres M_ι n'est rien d'autre que φ_ι. Enfin, la famille $(M_\iota)_{\iota\in I}$ est *libre* dans M pour l'état φ. Alors, on peut synthétiser ce résultat dans la proposition suivante :

Proposition 1.16. *Soit $(M_i, \varphi_\iota)_{\iota\in I}$ une famille d'algèbres de von Neumann, chacune d'entre elles étant munie d'un état φ_ι fidèle et normal. Alors, il existe à isomorphisme près qui préserve l'état, une unique algèbre de von Neumann M munie d'un état φ normal et fidèle, telle que*

- *(M_i, φ_i) se plonge dans (M, φ) d'une façon qui préserve l'état,*
- *M est engendrée par la famille $(M_\iota)_{\iota\in I}$ qui est libre dans (M, φ).*

Notation 1.17. Par la suite, nous emploierons librement la notation suivante. Soient (M, φ) et (N, ψ) deux algèbres de von Neumann, chacune munie d'un état normal et fidèle. S'il existe un isomorphisme $\theta : M \to N$ qui *préserve l'état*, i.e., $\psi \circ \theta = \varphi$, on écrit simplement $(M, \varphi) \cong (N, \psi)$.

EXEMPLE 1.18. Soit $(G_\iota)_{\iota\in I}$ une famille de groupes discrets, dénombrables. On note $(L(G_\iota),\tau_\iota)$ leurs algèbres de von Neumann respectives munies de leur trace canonique. Notons $G = *_{\iota\in I} G_\iota$ le *produit libre* des groupes G_ι et $(L(G),\tau_G)$ son algèbre de von Neumann munie de sa trace canonique. On a alors très naturellement l'isomorphisme suivant :

$$(L(G),\tau_G) \cong \mathop{*}_{\iota\in I}(L(G_\iota),\tau_\iota).$$

Ainsi pour tout $n \in \mathbf{N}^*$, si on note $\mathbf{F}_n$ le groupe libre à n générateurs, on a

$$L(\mathbf{F}_n) = \underbrace{L(\mathbf{Z}) * \cdots * L(\mathbf{Z})}_{n \text{ fois}}.$$

2.2. Le Foncteur CAR Libre. Les Facteurs d'Araki-Woods Libres. Nous présentons enfin un analogue *libre* des facteurs hyperfinis. Les facteurs hyperfinis peuvent être construits de bien des manières. Une construction qui retient notre attention est celle du foncteur CAR ou encore foncteur des *relations d'anticommutation canoniques*. Elle a une généralisation très naturelle dans le cadre des probabilités libres. Cette construction est dûe à Voiculescu dans le cas tracial [**48**]. Elle a été étendue dans un cadre plus général par Shlyakhtenko [**39**]. C'est cette dernière que nous allons exposer.

Soit $H_{\mathbf{R}}$ un espace de Hilbert réel. Soit $(U_t)_{t\in\mathbf{R}}$ une représentation orthogonale de $\mathbf{R}$ sur l'espace $H_{\mathbf{R}}$. Notons $H_{\mathbf{C}} = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ l'espace de Hilbert complexifié de $H_{\mathbf{R}}$. La représentation complexifiée de $(U_t)_{t\in\mathbf{R}}$ devient unitaire : elle sera toujours notée $(U_t)_{t\in\mathbf{R}}$. Le théorème de Stone affirme qu'il existe alors un opérateur A de $H_{\mathbf{C}}$, autoadjoint, positif, densément défini, non borné *a priori*, tel que $U_t = A^{it}$, pour tout $t \in \mathbf{R}$. Quant à l'opérateur $\frac{2}{1+A^{-1}}$, il est borné, autoadjoint, positif et injectif. Il permet alors de définir un nouveau produit scalaire sur $H_{\mathbf{C}}$,

$$\langle \xi, \eta\rangle_U = \langle \frac{2}{1+A^{-1}}\xi, \eta\rangle, \forall \xi, \eta \in H_{\mathbf{C}}.$$

Nous noterons H la fermeture de $H_{\mathbf{C}}$ pour le produit scalaire $\langle\cdot,\cdot\rangle_U$. On définit alors l'*espace de Fock plein* de H :

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n\geq 1} H^{\otimes n}.$$

Le vecteur Ω est appelé *vecteur du vide.* On note φ_U l'état vectoriel associé, $\varphi_U(y) = \langle y\Omega, \Omega\rangle_U$, pour tout $y \in B(\mathcal{F}(H))$. Pour tout $\xi \in H$, on définit $l(\xi)$, l'*opérateur de création à gauche* par

$$l(\xi) : \mathcal{F}(H) \to \mathcal{F}(H) : \begin{cases} l(\xi)\Omega = \xi, \\ l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

Sa partie réelle notée $s(\xi)$ est définie par

$$s(\xi) = \frac{l(\xi) + l(\xi)^*}{2}.$$

Le résultat crucial de Voiculescu [**48**] affirme que la distribution de l'opérateur $s(\xi)$ pour l'état φ_U est la loi *semi-circulaire* portée sur le segment $[-\|\xi\|, \|\xi\|]$ et donnée par :

$$d\gamma_{\|\xi\|}(t) = \frac{2}{\pi\|\xi\|^2}\sqrt{\|\xi\|^2 - t^2}dt.$$

Cette loi semi-circulaire est en fait l'analogue libre de la loi *Gaussienne* pour les probabilités classiques. Plus généralement, une variable aléatoire libre dont la loi est semi-circulaire est tout simplement appelée une *variable semi-circulaire*.

Définition 1.19 (Shlyakhtenko, [**39**]). *Soit $H_{\mathbf{R}}$ un espace de Hilbert réel de dimension au moins 2. Soit $(U_t)_{t\in\mathbf{R}}$ une représentation orthogonale de $\mathbf{R}$ sur $H_{\mathbf{R}}$. L'algèbre de von Neumann incluse dans $B(\mathcal{F}(H))$ et engendrée par $\{s(\xi), \xi \in H_{\mathbf{R}}\}$ est un facteur. On l'appelle le* facteur d'Araki-Woods libre *associé à $H_{\mathbf{R}}$ et $(U_t)_{t\in\mathbf{R}}$. L'état φ_U sur $\Gamma(H_{\mathbf{R}}, U_t)''$ est fidèle et normal. Il est appelé l'*état quasi-libre libre.

Regardons plus en détail le cas où la représentation $(U_t)_{t\in\mathbf{R}}$ est triviale. Dans ce cas, l'état du vide φ est tracial et le facteur $\Gamma(H_{\mathbf{R}})''$ est isomorphe à $L(\mathbf{F}_{\dim H_{\mathbf{R}}})$ le facteur du groupe libre à $\dim H_{\mathbf{R}}$ générateurs. De plus, l'application

$$\begin{array}{rcl} H_{\mathbf{R}} & \to & \Gamma(H_{\mathbf{R}})'' \\ \xi & \mapsto & s(\xi) \end{array}$$

vérifie la propriété suivante : pour toute famille orthogonale $(\xi_\iota)_{\iota\in I}$ dans $H_{\mathbf{R}}$, la famille $(s(\xi_\iota))_{\iota\in I}$ est libre dans $\Gamma(H_{\mathbf{R}})''$ pour la trace φ. Pour cette raison, cette application est aussi appelée *foncteur Gaussien libre*. C'est une construction très efficace qui permet d'obtenir un *mouvement Brownien libre*. Prenons, en effet, $H_{\mathbf{R}} = L^2([0, +\infty[, d\lambda)$. Notons pour $t \geq 0$, $\chi_{[0,t[}$ la fonction caractéristique de l'intervalle $[0, t[$ et $B_t = s(\chi_{[0,t[})$. Alors $t \mapsto B_t$ est un analogue libre du mouvement Brownien classique, i.e., il satisfait :

(1) Pour tout $t \geq 0$, B_t est semi-circulaire.

(2) Pour tous $t > s \geq 0$, $B_t - B_s$ est libre avec la famille $\{B_u, 0 \leq u < s\}$.

Dans le cas général d'une représentation $(U_t)_{t\in\mathbf{R}}$ quelconque, la propriété fonctorielle de liberté est toujours valable. Plus précisément, nous avons canoniquement l'isomorphisme suivant :

$$\Gamma\left(\bigoplus_{\iota\in I} H_{\mathbf{R}}^\iota, \bigoplus_{\iota\in I} U_t^\iota\right)'' \cong \mathop{*}_{\iota\in I} \Gamma(H_{\mathbf{R}}^\iota, U_t^\iota)''. \tag{4}$$

Shlyakhtenko obtient le résultat suivant :

Théorème 1.20 (Shlyakhtenko [**34, 37, 38, 39**]). *Soit $H_{\mathbf{R}}$ un espace de Hilbert réel de dimension au moins 2. Soit $(U_t)_{t\in\mathbf{R}}$ une représentation orthogonale de $\mathbf{R}$ sur $H_{\mathbf{R}}$. Notons $N = \Gamma(H_{\mathbf{R}}, U_t)''$. Alors, on a*

(1) *N est un facteur plein ; donc N n'est jamais de type* III_0.

(2) *N est de type* II_1 *si et seulement si $(U_t)_{t\in\mathbf{R}}$ est triviale.*

(3) *N est de type* III_λ $(0 < \lambda < 1)$ *si et seulement si $(U_t)_{t\in\mathbf{R}}$ est périodique de période* $\frac{2\pi}{|\log \lambda|}$.

(4) *N est de type* III_1 *dans tous les autres cas.*

(5) *L'état du vide φ_U est presque périodique si et seulement si $(U_t)_{t\in\mathbf{R}}$ l'est.*

On suppose dorénavant $H_{\mathbf{R}}$ séparable. Nous allons regarder plus en détail le cas où la représentation $(U_t)_{t\in\mathbf{R}}$ est presque périodique, i.e., l'opérateur A est diagonalisable. Notons $\mathrm{Sp}(A)$ le spectre ponctuel (i.e. l'ensemble des valeurs propres) de l'opérateur A et $\Gamma \subset \mathbf{R}_+^*$ le sous-groupe multiplicatif qu'il engendre. Après réduction de la représentation orthogonale $(U_t)_{t\in\mathbf{R}}$ et en utilisant l'Équation (4), on se rend compte qu'il nous suffit de comprendre uniquement les *blocs* de taille 1 et 2, i.e., les cas où $\dim H_{\mathbf{R}} = 1$ et $\dim H_{\mathbf{R}} = 2$. Le cas où $\dim H_{\mathbf{R}} = 1$ est facile car on sait déjà que l'algèbre correspondante n'est autre que $L(\mathbf{Z})$. Pour $\dim H_{\mathbf{R}} = 2$, on adopte la notation suivante :

NOTATION 1.21. Soit $H_{\mathbf{R}} = \mathbf{R}^2$. Pour $0 < \lambda < 1$, soit $(U_t^\lambda)_{t\in\mathbf{R}}$ la représentation orthogonale définie par

$$U_t^\lambda = \begin{pmatrix} \cos(t\log\lambda) & -\sin(t\log\lambda) \\ \sin(t\log\lambda) & \cos(t\log\lambda) \end{pmatrix}.$$

La représentation $(U_t^\lambda)_{t\in\mathbf{R}}$ est $\frac{2\pi}{|\log(\lambda)|}$-périodique, on sait alors que le facteur d'Araki-Woods libre correspondant $\Gamma(\mathbf{R}^2, U_t^\lambda)''$ est de type III_λ. Il sera simplement noté $(T_\lambda, \varphi_\lambda)$, avec φ_λ son état quasi-libre libre.

Ainsi, grâce à la Formule (4), $\Gamma(H_{\mathbf{R}}, U_t)''$ est obtenu comme un produit libre où figurent des $L(\mathbf{Z})$ et des $(T_\lambda, \varphi_\lambda)$ pour $\lambda \in \mathrm{Sp}(A)\backslash\{1\}$. En fait, Shlyakhtenko obtient un résultat remarquable de classification beaucoup plus précis. Le voici :

THÉORÈME 1.22 (Shlyakhtenko, [**34, 39**]). *Soit $H_{\mathbf{R}}$ un espace de Hilbert réel séparable. Soit $(U_t)_{t\in\mathbf{R}}$ une représentation orthogonale de $\mathbf{R}$ sur $H_{\mathbf{R}}$ presque périodique. Notons A son générateur infinitésimal et $N = \Gamma(H_{\mathbf{R}}, U_t)''$ le facteur d'Araki-Woods libre correspondant. Alors, à isomorphisme près qui préserve l'état, N ne dépend que du groupe Γ engendré par* $\mathrm{Sp}(A)$*, le spectre ponctuel de A. Réciproquement,* $\mathrm{Sd}(N) = \Gamma$*. En conclusion, pour tout sous-groupe dénombrable $\Lambda \subset \mathbf{R}_+^*$, il existe (à isomorphisme près qui préserve l'état) un unique facteur d'Araki-Woods libre dont l'invariant* Sd *est exactement Λ. De plus, le centralisateur N^{φ_U} est toujours isomorphe à $L(\mathbf{F}_\infty)$.*

L'outil essentiel pour montrer ce théorème est un *modèle matriciel*, très algébrique en nature, qui généralise dans le cas non tracial celui de Voiculescu [**49**]. Rappelons très sommairement que le modèle de Voiculescu reposait sur le fait que l'on peut approcher en *distribution* une famille libre de variables semi-circulaires par des matrices aléatoires gaussiennes et indépendantes de grande taille.

Ainsi, $(T_\lambda, \varphi_\lambda)$ est l'unique facteur d'Araki-Woods libre de type III_λ. Shlyakhtenko montre de nombreux résultats d'isomorphisme pour ces facteurs. En particulier, il prouve

$$(T_\lambda, \varphi_\lambda) \cong (M_2(\mathbf{C}), \omega_\lambda) * L(\mathbf{Z}),$$

avec $\omega_\lambda(e_{ij}) = \delta_{ij}\lambda^j/(1+\lambda)$, pour $i, j \in \{0, 1\}$. Rădulescu avait montré dans [**28**] que $(M_2(\mathbf{C}), \omega_\lambda) * L(\mathbf{Z})$ est un facteur de type III_λ en identifiant son core et l'action duale correspondante [**29**]. Plus généralement, adoptons la notation suivante :

NOTATION 1.23. Soit $\Gamma \subset \mathbf{R}_+^*$ un sous-groupe dénombrable. On note $(T_\Gamma, \varphi_\Gamma)$ l'unique facteur d'Araki-Woods libre dont l'invariant Sd est exactement Γ. Si $\Gamma = \lambda^{\mathbf{Z}}$, le facteur d'Araki-Woods libre correspondant est noté $(T_\lambda, \varphi_\lambda)$ comme dans la Notation 1.21. Shlyakhtenko obtient l'isomorphisme suivant :

$$(T_\Gamma, \varphi_\Gamma) \cong \mathop{*}_{\gamma\in\Gamma} (T_\gamma, \varphi_\gamma).$$

De plus, soit K un espace de Hilbert complexe de dimension finie ou séparable et ψ un état fidèle et normal sur $B(K)$. Soit Γ le groupe engendré par les valeurs propres de ψ. Shlyakhtenko montre aussi l'isomorphisme suivant :

$$(T_\Gamma, \varphi_\Gamma) \cong (B(K), \psi) * L(\mathbf{Z}). \tag{5}$$

3. Aperçu des Principaux Résultats

3.1. Une Nouvelle Construction de Facteurs de Type III_1. Dans [**15**], j'ai construit une nouvelle famille de facteurs de type III_1. J'ai pu classifier complètement cette famille dans un cas assez général en utilisant un résultat de Popa. Voici tout d'abord l'idée de la construction. On se donne P un facteur de type III_1 et ω un état fidèle, normal sur P. On prend $\gamma : \mathbf{R} \curvearrowright (X, \mu)$ un flot libre, ergodique qui préserve la mesure μ. La mesure μ peut être finie ou infinie. Ainsi, le produit croisé $L^\infty(X) \rtimes_\gamma \mathbf{R}$ est le facteur hyperfini de type II_∞. L'objet d'étude est essentiellement l'algèbre de von Neumann

$$M = (L^\infty(X) \otimes P) \rtimes_{\gamma \otimes \sigma^\omega} \mathbf{R}.$$

Bien entendu, M ne dépend pas du choix de l'état ω sur P. Il s'avère que cette algèbre de von Neumann est en fait un facteur de type III_1 dont la plupart des invariants sont calculables : j'ai pu identifier le core de ce facteur ainsi que son action duale. Ces résultats sont donnés dans le théorème suivant :

THÉORÈME 1.24 ([**15**]). *L'algèbre de von Neumann M est un facteur de type* III_1 *dont le core est isomorphe à*

$$(L^\infty(X) \rtimes_\gamma \mathbf{R}) \otimes (P \rtimes_{\sigma^\omega} \mathbf{R}).$$

Sous cette identification, l'action duale θ est donnée par

$$\theta_s = \widehat{\gamma}_{-s} \otimes \theta^\omega_s, \forall s \in \mathbf{R}^*_+.$$

Au vu de ce résultat, il semble raisonnable de penser qu'une classification est possible pour cette famille. Pour ce faire, j'ai utilisé un très beau résultat de Popa dont j'ai appris l'existence durant son cours au Collège de France à l'Automne 2004. Ce résultat est déroutant car il est à la fois très simple dans son énoncé et puissant dans sa conclusion. J'en donne ici la version pour les facteurs de type II_∞ car c'est celle-ci que je vais utiliser par la suite.

THÉORÈME 1.25 (Popa, [**21**]). *Soient R_1, R_2 deux copies du facteur hyperfini de type* II_∞. *Soient N_1, N_2 deux facteurs pleins de type* II_∞. *Soit un isomorphisme*

$$\rho : R_1 \otimes N_1 \to R_2 \otimes N_2.$$

Alors il existe deux isomorphismes

$$\begin{aligned} \alpha : R_1 &\to R_2 \\ \beta : N_1 &\to N_2 \end{aligned}$$

et un unitaire $u \in \mathcal{U}(R_2 \otimes N_2)$ tels que $\mathrm{Ad}(u) \circ \rho = \alpha \otimes \beta$.

J'ai donné une preuve de ce résultat dans [**15**] ; Popa a démontré son théorème dans [**21**]. Grâce à ce théorème, j'ai pu obtenir la classification suivante.

THÉORÈME 1.26 ([**15**]). *Pour $i \in \{1, 2\}$, soit $\gamma^i : \mathbf{R} \curvearrowright (X_i, \mu_i)$ un flot libre, ergodique qui préserve la mesure μ_i et soit P_i un facteur de type* III_1 *dont le core $P_i \rtimes_{\sigma^{\omega_i}} \mathbf{R}$ est plein. On note pour $i \in \{1, 2\}$,*

$$M_i = (L^\infty(X_i) \otimes P_i) \rtimes_{\gamma^i \otimes \sigma^{\omega_i}} \mathbf{R}.$$

On a alors l'équivalence suivante :

$$M_1 \simeq M_2 \Longleftrightarrow \begin{cases} P_1 \simeq P_2 \\ \gamma^1 \text{ et } \gamma^2 \text{ sont conjugués.} \end{cases}$$

Il est utile de faire quelques commentaires sur ce théorème. Le facteur d'Araki-Woods libre $P = \Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$ associé à la représentation régulière gauche de $\mathbf{R}$ a un core isomorphe à $L(\mathbf{F}_\infty) \otimes B(H)$ (voir [**34**]) : il fournit donc un exemple très naturel de facteur de type III_1 avec core plein. Ainsi, si l'on prend $P_1 = P_2 = P$, on obtient une famille de facteurs de type III_1 complètement classifiée par un flot libre et ergodique qui préserve une mesure.

J'esquisse à présent l'idée de la preuve du Théorème 1.26 afin de voir comment on peut utiliser le théorème de Popa. La construction étant assez naturelle, on voit immédiatement que si $P_1 \simeq P_2$ et si les flots γ^1 et γ^2 sont conjugués, alors $M_1 \simeq M_2$. Réciproquement, supposons $M_1 \simeq M_2$. Sachant que le core de M_i est isomorphe à

$$(L^\infty(X_i) \rtimes_{\gamma^i} \mathbf{R}) \otimes (P_i \rtimes_{\sigma^{\omega_i}} \mathbf{R}),$$

le théorème de structure de Takesaki affirme alors qu'il existe un isomorphisme

$$\rho : (L^\infty(X_1) \rtimes_{\gamma^1} \mathbf{R}) \otimes (P_1 \rtimes_{\sigma^{\omega_1}} \mathbf{R}) \to (L^\infty(X_2) \rtimes_{\gamma^2} \mathbf{R}) \otimes (P_2 \rtimes_{\sigma^{\omega_2}} \mathbf{R})$$

qui *entrelace* les action duales θ^1 et θ^2. Comme $L^\infty(X_i) \rtimes_{\gamma^i} \mathbf{R}$ est isomorphe au facteur hyperfini de type II_∞ et par hypothèse $P_i \rtimes_{\sigma^{\omega_i}} \mathbf{R}$ est plein, le théorème de Popa affirme qu'il existe deux isomorphismes

$$\begin{aligned} \alpha : L^\infty(X_1) \rtimes_{\gamma^1} \mathbf{R} &\to L^\infty(X_2) \rtimes_{\gamma^2} \mathbf{R} \\ \beta : P_1 \rtimes_{\sigma^{\omega_1}} \mathbf{R} &\to P_2 \rtimes_{\sigma^{\omega_2}} \mathbf{R} \end{aligned}$$

tels que, à un unitaire près, $\rho = \alpha \otimes \beta$. Puisque les actions duales θ^i sont données par $\theta^i_s = \widehat{\gamma}^i_{-s} \otimes \theta^{\omega_i}_s$, je montre que, quitte à conjuguer α et β par des unitaires, l'isomorphisme α entrelace les actions duales $\widehat{\gamma}^1$ et $\widehat{\gamma}^2$, et de même, l'isomorphisme β entrelace les actions duales θ^{ω_1} et θ^{ω_2}. En travaillant encore un peu, ceci permet de montrer que $P_1 \simeq P_2$ et que les flots γ^1 et γ^2 sont conjugués.

3.2. Résultats de Classification pour Certains Produits Libres d'Algèbres de von Neumann. Le problème de la classification des produits libres d'algèbres de von Neumann hyperfinies a été étudié par Dykema il y a une quinzaine d'années. Il a complètement résolu la question dans le cas *tracial* [**12**]. Pour ce faire, il a utilisé comme outil essentiel les *facteurs interpolés des groupes libres* [**10, 27**] et a défini la notion de *dimension libre*. Ses techniques reposent principalement sur des généralisations de certains modèles matriciels de Voiculescu. Dans le cas *non tracial* [**8**], il n'obtient plus de résultats de classification comme précédemment. Il montre cependant un résultat très intéressant : un produit libre d'algèbres de von Neumann hyperfinies (selon des états presque périodiques dont l'un au moins est non tracial) contient nécessairement un facteur de type III dont le centralisateur est isomorphe à $L(\mathbf{F}_\infty)$. Voici un énoncé précis dans le cas d'un produit libre de facteurs de type I :

THÉORÈME 1.27 (Dykema, [**8**]). *Soient H_1, H_2 deux espaces de Hilbert séparables de dimension finie ou infinie. Soient ψ_1, ψ_2 deux états fidèles et normaux sur $B(H_1)$ et $B(H_2)$. Alors le produit libre*

$$(\mathcal{M}, \varphi) = (B(H_1), \psi_1) * (B(H_2), \psi_2)$$

est un facteur de type III *plein. L'état φ est presque périodique et le centralisateur $\mathcal{M}^\varphi$ est isomorphe au facteur de type* II_1, $L(\mathbf{F}_\infty)$. *Donc, $\mathcal{M}$ est nécessairement un facteur de type* III_λ *avec $\lambda \in]0,1]$.*

Dykema pose alors la question suivante (Question 9.1 et 9.3 dans [**8**]) : les facteurs de type III obtenus en prenant des produits libres d'algèbres de von Neumann de dimension finie et plus généralement hyperfinis ayant le même invariant Sd sont-ils isomorphes ? Au vu des résultats d'*absorption libre* des facteurs d'Araki-Woods libres, Shlyakhtenko pose à son tour la question suivante : les produits libres d'algèbres de matrices sont-ils isomorphes aux facteurs d'Araki-Woods libres presque périodiques ? On voit immédiatement que répondre à la question de Shlyakhtenko, c'est aussi répondre à la question de Dykema, car les facteurs d'Araki-Woods libres sont complètement classifiés dans le cas presque périodique.

Soit $\lambda \in]0,1[$. Sur $M_2(\mathbf{C})$, l'état ω_λ est défini par : $\omega_\lambda(e_{ij}) = \delta_{ij}\lambda^j/(\lambda+1)$, pour $i,j \in \{0,1\}$. Soit $\beta \in]0,1[$. Si l'on regarde $\mathbf{C}^2$ engendré par une projection p, on définit la trace τ_β sur $\mathbf{C}^2$ par $\tau_\beta(p) = \beta$. Le produit libre $(M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$ est en quelque sorte le plus *petit* produit libre dans lequel peut apparaître un facteur de type III_λ. Les résultats de Dykema (voir [**8**]) assurent que ce produit libre est un facteur de type III_λ dès que $\lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. La question soulevée par Dykema, dans ce cas très particulier, est de savoir si, finalement, ce produit libre dépend du paramètre β. J'ai répondu négativement à cette question et plus précisément, j'ai montré le théorème suivant :

THÉORÈME 1.28 ([**16**]). *Soient $\beta, \lambda \in]0,1[$ tels que $\lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. Alors,*

$$(M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta) \cong (T_\lambda, \varphi_\lambda).$$

Ce cas particulier a été en fait le *socle* pour montrer d'autres résultats de classification du même type. En utilisant, en effet, une technique de *produits libres avec amalgamation* au-dessus de *petites* algèbres comme $(M_2(\mathbf{C}), \omega_\lambda)$ ou $(\mathbf{C}^2, \tau_\beta)$, j'ai pu classifier d'autres produits libres. J'introduis la définition suivante afin de pouvoir énoncer ensuite un résultat de classification plus général.

DÉFINITION 1.29 ([**16**]). *Soient (A, φ_A) et (B, φ_B) deux algèbres de von Neumann munies d'états fidèles et normaux. Soit $\rho : (A, \varphi_A) \hookrightarrow (B, \varphi_B)$ un plongement d'algèbres de von Neumann qui préserve l'état. On dit que ρ est* modulaire *si $\rho(A)$ est globalement invariante sous le groupe modulaire $(\sigma_t^{\varphi_B})$.*

J'ai montré le théorème général suivant :

THÉORÈME 1.30 ([**16**]). *Soient (M_1, φ_1) et (M_2, φ_2) deux algèbres de von Neumann munies d'états fidèles, normaux, presque périodiques. Soient $\beta, \lambda \in]0,1[$ tels que $\lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. On suppose que les plongements*

$$\begin{cases} (M_2(\mathbf{C}), \omega_\lambda) & \hookrightarrow & (M_1, \varphi_1) \\ (\mathbf{C}^2, \tau_\beta) & \hookrightarrow & (M_2, \varphi_2) \end{cases}$$

sont modulaires. Alors,

$$(M_1, \varphi_1) * (M_2, \varphi_2) \cong (M_1, \varphi_1) * (M_2, \varphi_2) * (T_\lambda, \varphi_\lambda).$$

En particulier, ce résultat classifie complètement les produits libres d'algèbres de matrices de dimension paire $M_{2n}(\mathbf{C}) = M_2(\mathbf{C}) \otimes M_n(\mathbf{C})$, selon des états du type $\omega_\lambda \otimes \omega$. Soient $\lambda, \mu \in]0,1]$ tels que $\lambda\mu \neq 1$. Soient d'autre part $(M_{n_1}(\mathbf{C}), \omega_1)$, $(M_{n_2}(\mathbf{C}), \omega_2)$ deux algèbres de matrices munies d'états fidèles. Soit $\Gamma \subset \mathbf{R}_+^*$ le sous-groupe engendré par λ, μ et les spectres ponctuels de ω_1 et ω_2. Alors,

$$(M_{2n_1}(\mathbf{C}), \omega_\lambda \otimes \omega_1) * (M_{2n_2}(\mathbf{C}), \omega_\mu \otimes \omega_2) \cong (T_\Gamma, \varphi_\Gamma).$$

3.3. Les Facteurs d'Araki-Woods Libres sont Premiers. Une algèbre de von Neumann M est dite *diffuse* si elle ne contient pas de projection minimale. Dans une courte note (cf. Chapitre 5), je montre le résultat suivant :

THÉORÈME 1.31. *Tout facteur d'Araki-Woods libre est premier, i.e., il ne peut s'écrire comme le produit tensoriel de deux algèbres de von Neumann diffuses. De plus, tout produit libre de facteurs de type* I *est lui-même premier.*

En fait, je montre un résultat plus fort : tout facteur d'Araki-Woods libre est *solide* dans un sens généralisé. Cette notion de solidité pour une algèbre de von Neumann avait été introduite par Ozawa dans son remarquable article [**18**]. Il avait montré que l'algèbre de von Neumann d'un groupe *hyperbolique* est solide, i.e., le commutant relatif de n'importe quelle sous-algèbre de von Neumann diffuse est injectif.

CHAPITRE 2

A New Construction of Factors of Type III_1

This chapter is the text of an article [**15**] appeared in *Journal of Functional Analysis*, **242** (2007), 375–399.

We give in this chapter a new construction of factors of type III_1. Under certain assumptions, we can, thanks to a result by Popa, give a complete classification for this family of factors. Although these factors are never full, we can nevertheless, in many cases, compute Connes' τ invariant. We obtain a new example of an uncountable family of pairwise non-isomorphic factors of type III_1 with the same τ invariant.

1. Introduction

Let N be a type II_∞ factor endowed with a trace-scaling one-parameter automorphism group denoted by $(\alpha_s)_{s\in\mathbf{R}_+^*}$. Let Γ be a countable, dense subgroup of $\mathbf{R}_+^*$. The crossed product of N by Γ under the action α, denoted by $N \rtimes_\alpha \Gamma$, is a factor of type III_1 ; this factor has almost-periodic states and its invariant Sd is included in Γ (see [**3**] for further details). We want to generalize this construction to the case of a virtual subgroup of $\mathbf{R}$: for (X,μ) a measure space with μ a finite or infinite measure, and $(\gamma_t)_{t\in\mathbf{R}}$ a measure-preserving, free, ergodic action of $\mathbf{R}$ on (X,μ) by automorphisms, we shall give the right definition of the crossed product of N by the virtual subgroup $(L^\infty(X),\gamma)$.

We present the main result of this paper. First, we start with a factor P of type III_1. Its *core* $P \rtimes_{\sigma^\omega} \mathbf{R}$ is a factor of type II_∞ and does not depend on the choice of the state (or weight) ω. Consider also a free, ergodic, measure-preserving flow $\gamma : \mathbf{R} \curvearrowright (X,\mu)$. The object of study is essentially the von Neumann algebra $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$. This von Neumann algebra turns out to be a factor of type III_1 whose core is canonically isomorphic to $(L^\infty(X) \rtimes_\gamma \mathbf{R}) \otimes (P \rtimes_{\sigma^\omega} \mathbf{R})$ (see Theorem 2.11). Under this identification, the *dual action* is given by $\theta_s = \widehat{\gamma}_{-s} \otimes \theta_s^\omega$, with $\widehat{\gamma}$ the dual action of γ and θ^ω the dual action of σ^ω. Notice that since γ is free, ergodic and measure-preserving, and $\mathbf{R}$ is amenable, $L^\infty(X) \rtimes_\gamma \mathbf{R}$ is the hyperfinite type II_∞ factor.

We shall prove a result of complete classification for this family of factors in the case the core $P \rtimes_{\sigma^\omega} \mathbf{R}$ is a *full* factor. For this, we shall use a very nice result by Popa on unique tensor product decomposition of McDuff II_1 factors. The following theorem is at the heart of this paper :

Theorem 2.1. [**21**] *Let R_1 and R_2 be two copies of the hyperfinite type II_1 factor ; let N_1 and N_2 be two full factors of type II_1. Let us assume that the factor N has both of the following decompositions : $N = R_1 \otimes N_1 = R_2 \otimes N_2$. Then there exist $t > 0$ and a unitary $u \in N$ such that $R_2 = u^*(R_1)^{1/t}u$ and $N_2 = u^*(N_1)^t u$.*

We can give an equivalent version of this theorem in the type II_∞ case :

Theorem 2.2 (Mc Duff II_∞ factors). *Let R_1^∞ and R_2^∞ be two copies of the hyperfinite type II_∞ factor ; let N_1^∞ and N_2^∞ be two full factors of type II_∞. Let $\alpha : R_1^\infty \otimes N_1^\infty \to R_2^\infty \otimes N_2^\infty$ be an isomorphism. Then, there exist a unitary $u \in R_2^\infty \otimes N_2^\infty$ and two isomorphisms $\beta : R_1^\infty \to R_2^\infty$, $\gamma : N_1^\infty \to N_2^\infty$ such that for any $x \in R_1^\infty \otimes N_1^\infty$, $u^*\alpha(x)u = (\beta \otimes \gamma)(x)$.*

We can now give our main result concerning the complete classification of the construction in the case $P \rtimes_{\sigma^\omega} \mathbf{R}$ is a full factor.

THEOREM 2.3. *Let P be the factor of type* III$_1$ *whose core is isomorphic to $L(\mathbf{F}_\infty) \otimes B(H)$ and the dual action is given by the trace-scaling one-parameter automorphism group $(\alpha_t)_{t\in\mathbf{R}}$ from Rădulescu* [**29**]*. Let (X_i, μ_i, γ^i) $(i = 1, 2)$ be two measure spaces endowed with measure-preserving, free, ergodic flows. Denote $M_i = (L^\infty(X_i) \otimes P) \rtimes_{\gamma^i \otimes \sigma^\omega} \mathbf{R}$ $(i = 1, 2)$. Then M_1 and M_2 are isomorphic if and only if γ^1 and γ^2 are conjugate.*

Since the construction is canonical, it is obvious that if γ^1 and γ^2 are conjugate then M_1 and M_2 are isomorphic. Let us give a few ideas about the proof of the converse. Assume that M_1 and M_2 are isomorphic. Then, we know that the cores of M_1 and M_2 are isomorphic and the dual actions are cocycle conjugate (see [**40, 42**] for further details). But for $i = 1, 2$, the core of M_i, denoted by Core(M_i), is isomorphic to the McDuff type II$_\infty$ factor $(L^\infty(X_i) \rtimes_{\gamma^i} \mathbf{R}) \otimes (P \rtimes_{\sigma^\omega} \mathbf{R})$. If we apply the result by Popa (Theorem 2.2), we get on the "first leg" of the tensor product decomposition of Core(M_1) and Core(M_2) an isomorphism $\pi : L^\infty(X_1) \rtimes_{\gamma^1} \mathbf{R} \to L^\infty(X_2) \rtimes_{\gamma^2} \mathbf{R}$ and a family $(v_s)_{s\in\mathbf{R}}$ of unitaries in Core(M_1) such that for any $z \in L^\infty(X_1) \rtimes_{\gamma^1} \mathbf{R}$

$$\pi^{-1}\left(\widehat{\gamma^2}_s(\pi(z))\right) \otimes 1 = v_s(\widehat{\gamma^1}_s(z) \otimes 1)v_s^*.$$

Using classical techniques, we show that there exists a family of unitaries $(w_s)_{s\in\mathbf{R}}$ which is a one-cocycle for γ_1 and such that

$$\pi^{-1}\widehat{\gamma^2}_s\pi = w_s\widehat{\gamma^1}_s w_s^*.$$

At last, using classical results by Takesaki [**40, 42**], we prove that γ^1 and γ^2 are necessarily conjugate. Consequently, if the factor P satisfies the assumptions of Theorem 2.3 (and more generally if the core of P is full), the factor of type III$_1$, $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$, entirely remembers the flow γ.

Finally, when P is a free Araki-Woods factor [**39**], we will compute Connes' τ invariant for the factor $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$. We will exhibit a new example of an uncountable family of pairwise non-isomorphic factors of type III$_1$ with the same τ invariant.

2. The General Construction

Notation. We want to introduce here all the notation we will use in this paper. Let $(A, (\alpha_t)_{t\in\mathbf{R}})$ and $(B, (\beta_t)_{t\in\mathbf{R}})$ be two von Neumann algebras endowed with a one-parameter automorphism group. Let $\mathcal{F} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be the Fourier transform on $\mathbf{R}$: for $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, $\mathcal{F}(f) = \xi \mapsto \int_{\mathbf{R}} f(t)\exp(-it\xi)dt$. So, if $\rho = \mathrm{Ad}(\mathcal{F}) : L^\infty(\mathbf{R}) \to B(L^2(\mathbf{R}))$, we get for any $t \in \mathbf{R}$, $\rho(e^{it\cdot}) = \rho_t$, with ρ_t the translation operators on $L^2(\mathbf{R})$. Let $\alpha : A \to A \otimes L^\infty(\mathbf{R})$ defined by : for any $x \in N$ and any $t \in \mathbf{R}$, $\alpha(x)(t) = \alpha_{-t}(x)$. Let $\beta : B \to L^\infty(\mathbf{R}) \otimes B$ defined by : for any $y \in B$ and any $t \in \mathbf{R}$, $\beta(y)(t) = \beta_{-t}(y)$.

DEFINITION 2.4. *We define the* crossed product *of A and B under α and β in the following way : it is the von Neumann subalgebra of $A \otimes B(L^2(\mathbf{R})) \otimes B$ generated by $(\alpha(A) \otimes 1 \cup 1 \otimes \tilde{\beta}(B))$, with $\tilde{\beta} = (\rho \otimes 1)\beta$. We shall denote it by $A_\alpha\bowtie_\beta B$.*

It is clear that $A_\alpha\bowtie_\beta B$ is canonically isomorphic to the von Neumann subalgebra of $A \otimes B(L^2(\mathbf{R})) \otimes B$ generated by $(\tilde{\alpha}(A) \otimes 1 \cup 1 \otimes \beta(B))$, with $\tilde{\alpha} = (1 \otimes \rho)\alpha$. Let $(\gamma_t)_{t\in\mathbf{R}}$ be a free, ergodic flow on (X, μ) and let $(N, (\alpha_t)_{t\in\mathbf{R}})$ be a factor of type II$_\infty$ endowed with a trace-scaling one-parameter automorphism group. Let P be the factor of type III$_1$, $N \rtimes_\alpha \mathbf{R}$. Let τ_N be a faithful, semi-finite, normal trace on N (this trace is unique up to a scalar $\lambda > 0$) and let ω be the dual weight on P of τ_N under the action α ; let

σ^ω be the modular automorphism group of the weight ω and β the action $\gamma \otimes \sigma^\omega$ of $\mathbf{R}$ on $L^\infty(X) \otimes P$. We shall denote by C the crossed product $N_\alpha \bowtie_\gamma L^\infty(X)$ of N and $L^\infty(X)$ under α and γ. We will prove in Proposition 2.5 that $C \otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$. We shall denote by ϕ the n.f.s. weight $\tau \otimes \omega$ on $L^\infty(X) \otimes P$, and $\tilde{\phi}$ the dual weight of ϕ on $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$ under the action β. Moreover, most of the time $L^\infty(X) \otimes P \otimes B(L^2(\mathbf{R}))$ will be denoted by M ; the canonical embedding of $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$ into M will be denoted by π. At last we know, according to Proposition 8.4 in [**42**], that $(N \rtimes_\alpha \mathbf{R}) \rtimes_{\sigma^\omega} \mathbf{R}$ is canonically isomorphic to $N \otimes B(L^2(\mathbf{R}))$, i.e. $P \rtimes_{\sigma^\omega} \mathbf{R} = N \otimes B(L^2(\mathbf{R}))$. We shall always identify both of these factors.

The following proposition gives a justification to Definition 2.4.

Proposition 2.5. *Let $(A, (\alpha_t)_{t\in\mathbf{R}})$ and $(B, (\beta_t)_{t\in\mathbf{R}})$ be two von Neumann algebras endowed with a one-parameter automorphism group. Let $C = A_\alpha \bowtie_\beta B$.*

(1) *If $B = L^\infty\left(\widehat{\Gamma}\right)$, with Γ subgroup of $\mathbf{R}$ and $(\beta_t)_{t\in\mathbf{R}}$ is the action of $\mathbf{R}$ by translations, then C is canonically isomorphic to $A \rtimes_\alpha \Gamma$.*
(2) *If $A = A_0 \rtimes_{\alpha^0} \mathbf{R}$ and α is the dual action of α^0, then C is canonically isomorphic to $(A_0 \otimes B) \rtimes_{\alpha^0\otimes\beta} \mathbf{R}$.*

Proof. (1) Let $B = L^\infty\left(\widehat{\Gamma}\right)$, with $(\beta_t)_{t\in\mathbf{R}}$ the action of $\mathbf{R}$ by translations. Let $G = \widehat{\Gamma}$. For $\gamma \in \Gamma$, let $u_\gamma \in L^\infty(G)$ defined by $u_\gamma(g) = \langle g, \gamma\rangle$. Then for any $t \in \mathbf{R}$,

$$\begin{aligned} \beta_t(u_\gamma)(g) &= \langle g - t, \gamma\rangle \\ &= \exp(-it\gamma)\langle g, \gamma\rangle \\ &= \exp(-it\gamma) u_\gamma(g). \end{aligned}$$

Thus, for any $\gamma \in \Gamma$, u_γ is an eigenvector associated with β_t for the eigenvalue $\exp(-it\gamma)$. Therefore, $\tilde{\beta}(u_\gamma) = \rho_{-\gamma} \otimes u_\gamma$ and consequently

$$\begin{aligned} (1 \otimes \tilde{\beta}(u_\gamma))(\alpha(a) \otimes 1)(1 \otimes \tilde{\beta}(u_\gamma))^* &= (1 \otimes \rho_{-\gamma})\alpha(a)(1 \otimes \rho_{-\gamma})^* \otimes 1 \\ &= \alpha(\alpha_{-\gamma}(a)) \otimes 1. \end{aligned}$$

As $L^\infty(G)$ is spanned by the operators u_γ for $\gamma \in \Gamma$, it is clear that $A_\alpha \bowtie_\beta B$ is canonically isomorphic to $A \rtimes_\alpha \Gamma$.

(2) Let $A = A_0 \rtimes_{\alpha^0} \mathbf{R}$ and let α be the dual action of α^0. We still denote by $\alpha^0 : A_0 \to A$, the mapping defined by : for any $x \in A_0$ and any $t \in \mathbf{R}$, $\alpha^0(x)(t) = \alpha^0_{-t}(x)$. We shall denote by $\lambda_t \in A$ the unitaries which implement the action α^0 of $\mathbf{R}$ on A_0. By definition of the dual actions [**42**], for any $a \in A_0$ and any $t \in \mathbf{R}$, we get :

$$\begin{aligned} \tilde{\alpha}(\alpha^0(a)) &= \alpha^0(a) \otimes 1 \\ \tilde{\alpha}(\lambda_t) &= \lambda_t \otimes \rho_t. \end{aligned}$$

The von Neumann algebra $A_\alpha \bowtie_\beta B$ is generated by $\tilde{\alpha}(\alpha^0(A_0)) \otimes 1$, $\tilde{\alpha}(\lambda_t) \otimes 1$ for $t \in \mathbf{R}$ and $1 \otimes \beta(B)$. But $\tilde{\alpha}(A_0) \otimes 1$ is isomorphic to A_0, $1 \otimes \beta(B)$ is isomorphic to B, and $\tilde{\alpha}(A_0) \otimes 1$ and $1 \otimes \beta(B)$ commute in $A_\alpha \bowtie_\beta B$. Moreover, for any $a \in A_0$ and any $b \in B$, we have

$$\begin{aligned} \tilde{\alpha}(\lambda_t)\tilde{\alpha}(\alpha^0(a))\tilde{\alpha}(\lambda_t)^* &= \alpha^0(\alpha^0_t(a)) \\ (\tilde{\alpha}(\lambda_t) \otimes 1)(1 \otimes \beta(b))(\tilde{\alpha}(\lambda_t) \otimes 1)^* &= 1 \otimes \beta(\beta_t(b)). \end{aligned}$$

Finally, after trivial identifications, we get that $A_\alpha \bowtie_\beta B$ is canonically isomorphic to $(A_0 \otimes B) \rtimes_{\alpha^0\otimes\beta} \mathbf{R}$. $\square$

From now on, let (N, α) be a factor of type II_∞ endowed with a trace-scaling one-parameter automorphism group and let $(\gamma_t)_{t\in\mathbf{R}}$ be a finite or infinite measure-preserving, free, ergodic flow on the measure space (X, μ). Let $P = N \rtimes_\alpha \mathbf{R}$ and $\beta = \gamma \otimes \sigma^\omega$. According to the theorem of duality by Takesaki, we know that $P \rtimes_{\sigma^\omega} \mathbf{R}$ is canonically isomorphic to $N \otimes B(L^2(\mathbf{R}))$. Thus, we get thanks to Proposition 2.5 that $C \otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$, with C the crossed product $N_\alpha \bowtie_\gamma L^\infty(X)$. From now on, we shall identify $C \otimes B(L^2(\mathbf{R}))$ with $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$. For any weight ψ, we shall denote by Δ_ψ the modular operator associated with ψ. Let τ be the faithful, normal, semifinite trace on $L^\infty(X)$ given by the measure μ (preserved by the flow $(\gamma_t)_{t\in\mathbf{R}}$). It is clear that $\phi = \tau \otimes \omega$ is a faithful, normal, semifinite weight on $L^\infty(X) \otimes P$. As τ is preserved by the action γ_t and ω by σ_t^ω, it is obvious that ϕ is preserved by the action $\beta_t = \gamma_t \otimes \sigma_t^\omega$. Let $\tilde{\phi}$ the dual weight of ϕ on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ under the action β and λ_s the unitaries which implement the action β of $\mathbf{R}$ on $L^\infty(X) \otimes P$. According to Proposition 5.15 in [**42**], we know that $\sigma^{\tilde{\phi}}$ acts on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ in the following way : for any $x \in L^\infty(X) \otimes P$ and $s \in \mathbf{R}$,

$$\begin{aligned} \sigma_t^{\tilde{\phi}}(\pi_\beta(x)) &= \pi_\beta(\sigma_t^\phi(x)) \\ \sigma_t^{\tilde{\phi}}(\lambda_s) &= \lambda_s. \end{aligned}$$

Let us denote by $L^2(X, \mu)$ and $L^2(P, \omega)$ the Hilbert spaces on which $L^\infty(X)$ and P act canonically. Hence, the Hilbert space on which $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ acts is nothing but $L^2(X) \otimes L^2(P, \omega) \otimes L^2(\mathbf{R})$; by definition of the dual weight $\tilde{\phi}$ on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ (see [**42**] for further details), we have

$$\begin{aligned} \forall t \in \mathbf{R},\ \Delta_{\tilde{\phi}}^{it} &= \Delta_\phi^{it} \otimes 1 \\ &= 1 \otimes \Delta_\omega^{it} \otimes 1. \end{aligned}$$

On the other hand, as the family (λ_t) is in the centralizer of the weight $\tilde{\phi}$, it is a one-cocycle for the action $\sigma^{\tilde{\phi}}$. We know according to Theorem 1.2.4 of [**5**] that there exists a faithful, normal, semifinite weight ψ on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ such that :

$$\forall t \in \mathbf{R},\ \sigma_t^\psi = \lambda_t^* \sigma_t^{\tilde{\phi}} \lambda_t.$$

We denote by U the unitary representation of $\mathbf{R}$ on $L^2(X, \mu)$ which implement the action γ of $\mathbf{R}$ on $L^\infty(X)$. For any $x \in L^\infty(X) \otimes P$, we have

$$\begin{aligned} \sigma_t^\psi(\pi_\beta(x)) &= \lambda_t^* \sigma_t^{\tilde{\phi}}(\pi_\beta(x)) \lambda_t \\ &= \lambda_t^* \pi_\beta(\sigma_t^\phi(x)) \lambda_t \\ &= \pi_\beta(\beta_{-t}(\sigma_t^\phi(x))) \\ &= \pi_\beta((\gamma_{-t} \otimes \mathrm{id})(x)). \end{aligned}$$

Hence, by definition of the modular operator,

$$\forall t \in \mathbf{R},\ \Delta_\psi^{it} = U_{-t} \otimes 1 \otimes 1. \tag{6}$$

Thus, from the Equation (6), it is not difficult to see that, since the action γ is ergodic, the centralizer of ψ (denoted by $\mathrm{Cent}(\psi)$) is the von Neumann algebra $(\pi_{\sigma^\omega}(P) \cup \lambda(\mathbf{R}))''$ spanned by $\pi_{\sigma^\omega}(P)$ and $\lambda(\mathbf{R})$; after trivial identifications, it is nothing but $N \otimes B(L^2(\mathbf{R}))$ and consequently $\mathrm{Cent}(\psi)$ is a factor of type II_∞. We are now able to prove the following theorem :

THEOREM 2.6. *Let (N, α) be a factor of type* II_∞ *endowed with a trace-scaling one-parameter automorphism group and $(\gamma_t)_{t\in\mathbf{R}}$ be a finite or infinite measure-preserving, free, ergodic flow on (X, μ). Then the crossed product $C = N_\alpha \bowtie_\gamma L^\infty(X)$ is a factor of type* III_1.

PROOF. Let us prove first that C is a factor. We remind that $C \otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$. We denote by π the natural embedding of $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ into $M = L^\infty(X) \otimes P \otimes B(L^2(\mathbf{R}))$. As P is a factor of type III_1, we know according to Theorem XII 1.7 of [**40**] that

$$\{P \otimes B(L^2(\mathbf{R}))\} \cap (P \rtimes_{\sigma^\omega} \mathbf{R})' = \mathbf{C}.$$

So, it is not very difficult to see that

$$\{(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}\} \cap (1 \otimes \pi(P \rtimes_{\sigma^\omega} \mathbf{R}))' \subset L^\infty(X)^\gamma \otimes 1 \otimes 1,$$

with $L^\infty(X)^\gamma$ the fixed points subalgebra of $L^\infty(X)$ under the action γ. But, as γ is ergodic, we know that $L^\infty(X)^\gamma = \mathbf{C}$. Consequently, we obtain that

$$\{(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}\} \cap (1 \otimes \pi(P \rtimes_{\sigma^\omega} \mathbf{R}))' = \mathbf{C} \tag{7}$$

and C is a factor.

Furthermore, we want to prove that C is a factor of type III. Indeed, suppose that C were semifinite. Then $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ would be semifinite and the modular automorphism group σ_t^ψ would be inner. Hence, there would exist unitaries $u_t \in (L^\infty(X)\otimes P) \rtimes_\beta \mathbf{R}$ such that $\sigma_t^\psi = \mathrm{Ad}(u_t)$ for any $t \in \mathbf{R}$. But, this implies that $u_t \in (L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R} \cap \mathrm{Cent}(\psi)'$, and thanks to Equation (7), we get $\sigma_t^\psi = \mathrm{Id}$ for any $t \in \mathbf{R}$. That means exactly that ψ would be a trace and consequently $\Delta_\psi = \mathrm{Id}$. Thus, according to Equation (6), $U_t = 1$ for any $t \in \mathbf{R}$ which contradicts the fact that (γ_t) is free. Therefore, C is a factor of type III. Moreover, as $\mathrm{Cent}(\psi)$ is a factor, we know according to Corollary 3.2.7 of [**5**], that C cannot be a factor of type III_0. So now, if we want to find out the under-type of C, we can compute T invariant of Connes [**5**]. We remind that we have showed that $\sigma_t^\psi(\pi_\beta(x)) = \pi_\beta((\gamma_{-t} \otimes \mathrm{id})(x))$ for any $x \in L^\infty(X) \otimes P$. As the flow (γ_t) is supposed to be free, it cannot be periodic ; moreover, for the same reasons as before, σ_t^ψ cannot be inner. Therefore $T(C) = \{0\}$ and C is a factor of type III_1. □

Before proceeding, we are going to remind the definition of a strongly ergodic action of a locally compact group, the definition of a full factor and state a well-known theorem.

DEFINITION 2.7. [**30, 31**] *Let G be a locally compact group which acts on a probability space X by (γ_g) in a finite measure preserving way. This action is said to be* strongly ergodic *when for any sequence of projections (p_n) in $L^\infty(X)$, if the sequence $(\gamma_g(p_n) - p_n)$ tends to 0 $*-$strongly uniformly on compacts sets, then the sequence $(p_n - \tau(p_n)1)$ tends to 0 $*-$strongly.*

DEFINITION 2.8. [**3**] *Let C be a factor. Let $(x_n)_{n\in\mathbf{N}}$ be a bounded sequence in C. The sequence (x_n) is said to be* centralising *if for any normal state ϕ on C, $\|[x_n, \phi]\| \to 0$ when $n \to +\infty$; it is said to be* trivial *if there exists a sequence of complex numbers $(\lambda_n)_{n\in\mathbf{N}}$ such that $x_n - \lambda_n \to 0$ $*-$strongly. At last, the factor C is said to be a* full factor *if any centralising sequence in C is trivial.*

THEOREM 2.9. [**30, 31**] *Let G be a locally compact group. If G has Kazhdan property (T), then any ergodic action of G on a probability space is strongly ergodic. If G is amenable, then it cannot act strongly ergodically on a probability space.*

We are now able to state the following general proposition :

PROPOSITION 2.10. *Let (N,α) be a factor of type II_∞ endowed with a trace-scaling one-parameter automorphism group and $(\gamma_t)_{t\in\mathbf{R}}$ be a finite measure-preserving, free, ergodic flow on (X,μ). Then the crossed product $C = N_\alpha \bowtie_\gamma L^\infty(X)$ is a non-full factor.*

PROOF. We still denote by M the von Neumann algebra $L^\infty(X)\otimes P\otimes B(L^2(\mathbf{R}))$ and we remind that $C\otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to the factor $(L^\infty(X)\otimes P)\rtimes_{\gamma\otimes\sigma^\omega}\mathbf{R}$. We remind that π is the embedding of $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$ into M. In order to prove that C is not a full factor, we have thus to find a centralising sequence which is not trivial. The action (γ_t) of $\mathbf{R}$ on $L^\infty(X)$ is not strongly ergodic because $\mathbf{R}$ is abelian and thus amenable. Therefore, there exists a sequence of projections (p_n) in $L^\infty(X)$ such that $(\gamma_t(p_n)-p_n)$ tends to 0 $*-$strongly, uniformly on compact sets, but $(p_n-\tau(p_n)1)$ does not tend to 0 $*-$strongly. We have now to study the behaviour of (p_n) in $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$. The fact that $(\gamma_t(p_n)-p_n)$ tends to 0 $*-$strongly, uniformly on compact sets implies that in M, $(\pi(p_n)-p_n\otimes 1\otimes 1)$ tends to 0 $*-$strongly. Furthermore, every single element $p_n\otimes 1\otimes 1$ is central in M and therefore it commutes trivially with any normal form on M : hence $(p_n\otimes 1\otimes 1)$ is a centralising sequence in M. As $(\pi(p_n)-p_n\otimes 1\otimes 1)$ tends to 0 $*-$strongly, $(\pi(p_n))$ is also a centralising sequence in M. Consequently, (p_n) is also a centralising sequence in $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$. Moreover, one can easily prove that (p_n) is not a trivial sequence in $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$ because $(p_n-\tau(p_n)1)$ does not tend to 0 $*-$strongly. Consequently, C is not full. $\square$

We can notice that C may be full when the preserved measure μ is infinite. For example, if $\mathbf{R}$ acts by translations on $(\mathbf{R},\lambda)$, then $N_\alpha\bowtie_\gamma L^\infty(\mathbf{R})$ is isomorphic to $P = N\rtimes_\alpha\mathbf{R}$ according to Proposition 2.5, which is full if N is full [**34**]. We are now going to give a precise description of the core of the factor C.

THEOREM 2.11. *Let (N,α) be a factor of type II_∞ endowed with a trace-scaling one-parameter automorphism group and $(\gamma_t)_{t\in\mathbf{R}}$ be a finite or infinite measure-preserving, free, ergodic flow on (X,μ). Then the crossed product $C = N_\alpha\bowtie_\gamma L^\infty(X)$ is a factor of type III_1 whose core is isomorphic to*

$$(L^\infty(X)\rtimes_\gamma\mathbf{R})\otimes N.$$

Under this identification, the dual action of $\mathbf{R}$ on the core of C, denoted by (θ_s), is given by $\theta_s = \widehat{\gamma}_{-s}\otimes\alpha_s$, with $\widehat{\gamma}$ the dual action of γ.

PROOF. For the convenience of the proof, we shall denote by N_1 the crossed product $P\rtimes_{\sigma^\omega}\mathbf{R}$; we know that N_1 is nothing but $N\otimes B(L^2(\mathbf{R}))$, and according to Theorem 4.6 in [**42**], the dual action of σ^ω on N_1 is nothing but $\alpha_t\otimes\mathrm{Ad}(\rho_t^*)$ with ρ_t the left regular representation of $\mathbf{R}$ on $B(L^2(\mathbf{R}))$. We remind that τ is the faithful, normal, semifinite trace on $L^\infty(X)$ given by the measure μ (preserved by the flow $(\gamma_t)_{t\in\mathbf{R}}$). It is clear that $\phi=\tau\otimes\omega$ is a faithful, normal, semifinite weight on $L^\infty(X)\otimes P$. As τ is preserved by the action γ_t and ω by σ_t^ω, it is obvious that ϕ is preserved by the action $\beta_t=\gamma_t\otimes\sigma_t^\omega$. Let $\tilde{\phi}$ be the dual weight of ϕ on $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$ under the action β. Let us denote by $\mathcal{H}=L^2(X,\mu)$ et $\mathcal{K}=L^2(P,\omega)$ the Hilbert spaces on which $L^\infty(X)$ and P act canonically. Hence, $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$ acts on $L^2(\mathbf{R},\mathcal{H}\otimes\mathcal{K})$. Let us denote by $\mathrm{Core}((L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R})$ the core of the factor $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$. By definition, we have $\mathrm{Core}((L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}) = ((L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R})\rtimes_{\sigma^{\tilde{\phi}}}\mathbf{R}$. The core of $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$ acts on the Hilbert space $L^2(\mathbf{R}\times\mathbf{R},\mathcal{H}\otimes\mathcal{K})$, and it is spanned by $\pi_{\sigma^{\tilde{\phi}}}(N)$ and $\rho(\mathbf{R})$, if we denote by ρ_s the unitaries which implement the action $\sigma^{\tilde{\phi}}$ of $\mathbf{R}$ on $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$. Therefore, $\mathrm{Core}((L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R})$ is spanned by three types of operators :

(1) $\widetilde{a\otimes b} = \pi_{\sigma^{\tilde{\phi}}} \circ \pi_\beta(a\otimes b)$, for $a \in L^\infty(X)$ and $b \in P$; for any $\zeta \in L^2(\mathbf{R}\times\mathbf{R}, \mathcal{H}\otimes\mathcal{K})$, we have $((\widetilde{a\otimes b})\zeta)(s',t') = \left(\gamma_{-t'}(a) \otimes \sigma^\omega_{-(s'+t')}(b)\right)\zeta(s',t')$.

(2) $u_t = \pi_{\sigma^{\tilde{\phi}}}(\lambda_t)$, for $t \in \mathbf{R}$; for any $\zeta \in L^2(\mathbf{R}\times\mathbf{R}, \mathcal{H}\otimes\mathcal{K})$, we have $(u_t\zeta)(s',t') = \zeta(s', t'-t)$.

(3) $v_s = \rho_s$, for $s \in \mathbf{R}$; for any $\zeta \in L^2(\mathbf{R}\times\mathbf{R}, \mathcal{H}\otimes\mathcal{K})$, we have $(v_s\zeta)(s',t') = \zeta(s'-s, t')$.

Let U be the following unitary :

$$U : \begin{array}{ccc} L^2(\mathbf{R},\mathcal{H}) \otimes L^2(\mathbf{R},\mathcal{K}) & \rightarrow & L^2(\mathbf{R}\times\mathbf{R}, \mathcal{H}\otimes\mathcal{K}) \\ \xi\otimes\eta & \mapsto & \{(s',t') \mapsto \xi(t')\otimes\eta(s'+t')\} \end{array}.$$

Let us denote by $\pi_\gamma(L^\infty(X))$ and $\lambda^g(\mathbf{R})$ the images of $L^\infty(X)$ and $\mathbf{R}$ in the crossed product $L^\infty(X,\mu) \rtimes_\gamma \mathbf{R}$; in the same way, we denote by $\pi_{\sigma^\omega}(P)$ and $\lambda^d(\mathbf{R})$ the images of P and $\mathbf{R}$ in $P \rtimes_{\sigma^\omega} \mathbf{R} = N_1$. But now the question is : what is the behaviour of our three types of operators when we intertwine them by the unitary U ? Let $\xi \in L^2(\mathbf{R},\mathcal{H})$, $\eta \in L^2(\mathbf{R},\mathcal{K})$, s, s', t and $t' \in \mathbf{R}$. We have

$$\begin{aligned} \left(U^*\widetilde{a\otimes b}U(\xi\otimes\eta)\right)(s',t') &= \gamma_{-t'}(a)\xi(t') \otimes \sigma^\omega_{-s'}(b)\eta(s') \\ \left(U^*u_tU(\xi\otimes\eta)\right)(s',t') &= \xi(t'-t)\otimes\eta(s'-t) \\ \left(U^*v_sU(\xi\otimes\eta)\right)(s',t') &= \xi(t')\otimes\eta(s'-s). \end{aligned}$$

Using notations we have just introduced, we easily obtain :

$$U^*\widetilde{a\otimes b}U = \pi_\gamma(a)\otimes\pi_{\sigma^\omega}(b) \tag{8}$$

$$U^*u_tU = \lambda^g_t \otimes \lambda^d_t \tag{9}$$

$$U^*v_sU = 1\otimes\lambda^d_s. \tag{10}$$

Therefore at this stage, we have proved that the core of $(L^\infty(X)\otimes P) \rtimes_\beta \mathbf{R}$, intertwined by U, is equal to

$$(L^\infty(X) \rtimes_\gamma \mathbf{R}) \otimes N_1.$$

It remains us to understand what the dual action of $\sigma^{\tilde{\phi}}$ (denoted by θ_s) is. According to [**42**], for any $g \in \mathbf{R}$ and s, $t \in \mathbf{R}$, we have :

$$\begin{aligned} \theta_g(\widetilde{a\otimes b}) &= \widetilde{a\otimes b} \\ \theta_g(u_t) &= u_t \\ \theta_g(v_s) &= e^{-isg}v_s. \end{aligned}$$

For any $g \in \mathbf{R}$, let $\theta'_g = U^*\theta_gU$; we notice, thanks to Equations (8), (9) and (10) that :

$$\begin{aligned} \theta'_g\left(\pi_\gamma(a)\otimes\pi_{\sigma^{\omega_0}}(b)\right) &= \pi_\gamma(a)\otimes\pi_{\sigma^{\omega_0}}(b) \\ \theta'_g\left(\lambda^g_t\otimes 1\right) &= e^{itg}\left(\lambda^g_t\otimes 1\right) \\ \theta'_g\left(1\otimes\lambda^d_s\right) &= e^{-isg}\left(1\otimes\lambda^d_s\right). \end{aligned}$$

Finally, as $\alpha_t \otimes \mathrm{Ad}(\rho^*_t)$ is the dual action of σ^ω, we know according to [**42**] that the dual action θ'_g is given by :

$$\forall g \in \mathbf{R},\ \theta'_g = \widehat{\gamma}_{-g}\otimes\alpha_g\otimes\mathrm{Ad}(\rho^*_g).$$

As $C\otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to $(L^\infty(X)\otimes P)\rtimes_\beta\mathbf{R}$, after trivial identifications, we obtain that the core of C is nothing but $(L^\infty(X)\rtimes_\gamma\mathbf{R})\otimes N$ and the dual action is given by $\widehat{\gamma}_{-s}\otimes\alpha_s$. □

3. Classification in the Case N is a Full Factor

In this section, we shall assume that N is a full factor of type II_∞ endowed with a trace-scaling one-parameter automorphism group denoted by $(\alpha_t)_{t\in\mathbf{R}}$ and $(\gamma_t)_{t\in\mathbf{R}}$ is a free, ergodic, measure-preserving flow on (X,μ). The first example of such a full factor N has been given by Rădulescu in [**29**] : it is $L(\mathbf{F}_\infty)\otimes B(\mathcal{H})$. We can notice that other examples exist thanks to the theory of free Araki-Woods factors developed by Shlyakhtenko in [**32, 34, 39**].

Before stating the main theorem of this section, let us introduce a few notations. Let (X,μ) and (X',μ') be two measure spaces on which $\mathbf{R}$ acts by γ and γ' in a free, ergodic and measure-preserving way. Let (N,α) and (N',α') be two full factors of type II_∞, both of them endowed with a trace-scaling one-parameter automorphism group denoted by α and α'. According to [**42**], the covariant systems (N,α) and (N',α') are said to be *weakly equivalent*, if there exist an isomorphism $\pi : N\to N'$ and a strongly continuous family of unitaries (u_s) of N such that (u_s) is a one-cocycle for the action α and for any $x\in N$ and any $s\in\mathbf{R}$,

$$\pi^{-1}\alpha'_s\pi(x)=u_s\alpha_s(x)u_s^*.$$

Takesaki proved in [**42**], Corollary 8.4, that it is equivalent to say that the factors $N\rtimes_\alpha\mathbf{R}$ and $N'\rtimes_{\alpha'}\mathbf{R}$ are isomorphic. Let us denote by C the crossed product $N_\alpha\bowtie_\gamma L^\infty(X)$ and C' the crossed product $N'_{\alpha'}\bowtie_{\gamma'}L^\infty(X')$. As our construction is canonical in a obvious way, it is clear that if the flows (X,γ) and (X',γ') are conjugate and (N,α) and (N',α') are weakly equivalent, then C and C' are isomorphic (this claim is true even if N and N' are not full). We want to prove the converse here, when N and N' are full :

THEOREM 2.12. *If C and C' are isomorphic, then (N,α) and (N',α') are weakly equivalent, and the flows (X,γ) and (X',γ') are conjugate, i.e. there exists an isomorphism $\pi : L^\infty(X)\to L^\infty(X')$ such that for any $t\in\mathbf{R}$, $\pi\gamma_t=\gamma'_t\pi$.*

First of all, we want to prove an important result about full factors of type II_1.

LEMMA 2.13. *Let M be a full factor of type II_1 and N be any type II_1 factor. Then, for any net of unitaries (u_i) in $M\otimes N$ such that $\|xu_i-u_ix\|_2\to 0$ for all $x\in M\otimes N$,*

$$\sup_{x\in M,\|x\|\le 1}\|u_i(x\otimes 1)-(x\otimes 1)u_i\|_2\to 0.$$

PROOF. Let J_M be the canonical anti-unitary on $L^2(M)$ associated with the trace τ_M. Let (u_i) be a net of unitaires in $M\otimes N$ such that $\|xu_i-u_ix\|_2\to 0$ for all $x\in M\otimes N$. We are going to prove that $\|u_i-1\otimes E_N(u_i)\|_2\to 0$ with $E_N : M\otimes N\to N$ the conditional expectation defined by $E_N=\tau_M\otimes\mathrm{Id}$. It will prove our result ; indeed, for any $x\in M$, as $x\otimes 1$ commutes with $1\otimes E_N(u_i)$, we get $\|(x\otimes 1)u_i-u_i(x\otimes 1)\|_2=\|(x\otimes 1)v_i-v_i(x\otimes 1)\|_2$, with $v_i=u_i-1\otimes E_N(u_i)$. Moreover, for any $x\in M$, $\|x\|\le 1$, we have

$$\begin{aligned}\|(x\otimes 1)v_i-v_i(x\otimes 1)\|_2 &\le \|(x\otimes 1)v_i\|_2+\|v_i(x\otimes 1)\|_2\\ &\le \|(x\otimes 1)v_i\|_2+\|(J_Mx^*J_M\otimes 1)v_i\|_2\\ &\le \|x\|\,\|v_i\|_2+\|J_Mx^*J_M\|\,\|v_i\|_2\\ &\le 2\|v_i\|_2.\end{aligned}$$

Consequently, we get

$$\sup_{x\in M,\,\|x\|\le 1}\|(x\otimes 1)u_i-u_i(x\otimes 1)\|_2\le 2\|u_i-1\otimes E_N(u_i)\|_2\to 0.$$

We shall denote by $\Omega_M\in L^2(M)$ and $\Omega_N\in L^2(N)$ the images of $1_M\in M$ and $1_N\in N$ associated with the GNS constructions respectively for M and N. For any $x\in M$ and $x'\in$

M', $[u_i, xx'\otimes 1] = (x'\otimes 1)[u_i, x\otimes 1]$. Therefore, as (u_i) is a bounded net in $B(L^2(M))\otimes N$, we get that for any $y \in C^*(M, M')$, $[u_i, y\otimes 1] \to 0$ $*-$strongly. According to Theorem 2.1 in [**4**], as M is a full factor of type II_1, we know that $K(L^2(M)) \subset C^*(M, M')$. Thus, if we denote by $P_{\Omega_M} \in B(L^2(M))$ the rank-one projection onto $\mathbf{C}\Omega_M$, we have proved that $[u_i, P_{\Omega_M}\otimes 1] \to 0$ $*-$strongly.

We still denote by $\Omega_M : \mathbf{C} \to L^2(M)$ the map which sends λ onto $\lambda\Omega_M$. It is easy to notice that $\Omega_M\Omega_M^* = P_{\Omega_M} : L^2(M) \to L^2(M)$. We can notice that $E_N(u_i) = (\Omega_M^*\otimes 1)u_i(\Omega_M\otimes 1)$, thus

$$\begin{aligned}
E_N(u_i)^*E_N(u_i) &= (\Omega_M^*\otimes 1)u_i^*(P_{\Omega_M}\otimes 1)u_i(\Omega_M\otimes 1)\\
&= (\Omega_M^*\otimes 1)(P_{\Omega_M}\otimes 1)(\Omega_M\otimes 1) - (\Omega_M^*\otimes 1)u_i^*[u_i, P_{\Omega_M}\otimes 1](\Omega_M\otimes 1).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\|u_i - 1\otimes E_N(u_i)\|_2^2 &= \langle (u_i - 1\otimes E_N(u_i))(\Omega_M\otimes\Omega_N), (u_i - 1\otimes E_N(u_i))(\Omega_M\otimes\Omega_N)\rangle\\
&= 1 - \tau_N(E_N(u_i)^*E_N(u_i))\\
&= \langle u_i^*[u_i, P_{\Omega_M}\otimes 1](\Omega_M\otimes\Omega_N), (\Omega_M\otimes\Omega_N)\rangle\\
&\leq \|[u_i, P_{\Omega_M}\otimes 1](\Omega_M\otimes\Omega_N)\|.
\end{aligned}$$

As we know that $\|[u_i, P_{\Omega_M}\otimes 1](\Omega_M\otimes\Omega_N)\| \to 0$, the proof is complete. $\square$

Before proceeding, we want to remind a few things on basic constructions. Let $B \subset N$ be an inclusion of type II_1 factors; we denote by $e_B : L^2(N) \to L^2(B)$ the orthogonal projection, and $E_B : N \to B$ the unique trace-preserving conditional expectation from N onto B. If we denote by J_N the canonical anti-unitary on $L^2(N)$ associated with the trace τ_N, it is well known that

$$\begin{aligned}
\langle N, e_B\rangle &= (N\cup e_B)''\\
&= J_N\{B'\cap B(L^2(N))\}J_N.
\end{aligned}$$

Moreover, as e_B commutes with B and as for any $x \in N$, $e_Bxe_B = E_B(x)e_B$, Ne_BN turns out to be a $*-$algebra; as the central support of e_B is 1, Ne_BN is weakly dense in $\langle N, e_B\rangle$. Then we can define a semifinite canonical trace Φ on $\langle N, e_B\rangle$ in the following way :

$$\forall x, y \in N\ \Phi(xe_By) = \tau_N(xy).$$

According to [**19**], we remind that if a type II_1 factor M is decomposed as $M = M_1\otimes M_2$ for some type II_1 factors M_1, M_2 and $t > 0$ then, modulo unitary conjugacy, there exists a unique decomposition $M = M_1^t\otimes M_2^{1/t}$, such that $p_1M_1p_1\vee p_2M_2p_2$ and $q_1M_1^tq_1\vee q_2M_2^{1/t}q_2$ are unitary conjugate in M for any projections $p_i \in \mathrm{Proj}(M_i)$, $i = 1, 2$ and $q_1 \in \mathrm{Proj}(M_1^t)$, $q_2 \in \mathrm{Proj}(M_2^{1/t})$, with $\tau(p_1)/\tau(q_1) = \tau(q_2)/\tau(p_2) = t$.

We are now able to state the result by Popa of conjugation of subfactors in type II_1 factors mentioned in the introduction (see [**19**] for other results of the same kind). Our complete classification is based on the following theorem. We wish to gratefully thank Sorin Popa for allowing us to present it here. We should mention that Sorin Popa has recently given a proof of his result in [**21**]. For the sake of completeness, we shall give a proof of this theorem.

THEOREM 2.14. *Let R_1 and R_2 be two copies of the hyperfinite type II_1 factor; let N_1 and N_2 be two full factors of type II_1. Let us assume that the factor N has both of the following decompositions : $N = R_1\otimes N_1 = R_2\otimes N_2$. Then there exist $t > 0$ and a unitary $u \in N$ such that $R_2 = u^*(R_1)^{1/t}u$ and $N_2 = u^*(N_1)^tu$.*

Proof. First of all, we are looking at the inclusion $N_1 \subset R_1 \otimes N_1 = N$; we know now, according to Lemma 2.13, that for any central sequence of unitaries (u_i) in N,

$$\sup_{x \in N_1,\, \|x\| \leq 1} \|(x \otimes 1)u_i - u_i(x \otimes 1)\|_2 \to 0.$$

If we write the hyperfinite type II_1 factor $R_2 = \bigotimes_{n\geq 1}(\mathrm{Mat}_2(\mathbf{C}), \tau)$, and if we denote by $R_2^{(n_0)}$ the product $\bigotimes_{n\geq n_0}(\mathrm{Mat}_2(\mathbf{C}), \tau)$, we get immediatly that for any $n_0 \geq 1$, $R_2 = \mathrm{Mat}_{2^{n_0}}(\mathbf{C}) \otimes R_2^{(n_0)}$. Therefore, as $N = R_2 \otimes N_2$, we obtain that

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbf{N}\ \forall u \in U(N_1)\ \forall v \in U(R_2^{(n_0)})\ \|uv - vu\|_2 < \varepsilon.$$

Let $\varepsilon = \frac{1}{2}$. So, we know there exists $n_0 \in \mathbf{N}$ such that for any $u \in U(N_1)$ and any $v \in U(R_2^{(n_0)})$, $\|uv - vu\|_2 < \frac{1}{2}$. For an inclusion $B \subset N$ of type II_1, we know according to Proposition 1.3.2 in [**26**], that for any $x \in N$

$$E_{B'\cap N}(x) \in \overline{co}^w\{u^*xu, u \in U(B)\}.$$

Here, let $B = N_1$, then $B' \cap N = R_1$. Noticing that for any $u \in U(N_1)$ and any $v \in U(R_2^{(n_0)})$, $\|uv - vu\|_2 = \|v - u^*vu\|_2$, we get that for any $v \in U(R_2^{(n_0)})$, $\|v - E_{R_1}(v)\|_2 \leq \frac{1}{2}$ (the same technique is used in [**24**]). Let now $a \in \overline{co}^w\{v^*e_{R_1}v, v \in U(R_2^{(n_0)})\}$ of minimal $\|\cdot\|_{2,\Phi}-$norm. We are going to prove that a is not 0. From the one hand, for any $u \in N$, the basic construction for $R_1 \subset N$ gives us $e_{R_1}u^*e_{R_1}ue_{R_1} = E_{R_1}(u^*)E_{R_1}(u)e_{R_1}$. From the other hand, as for any $v \in U(R_2^{(n_0)})$, $\|v - E_{R_1}(v)\|_2 \leq \frac{1}{2}$, we get for any $v \in U(R_2^{(n_0)})$, $\|E_{R_1}(v)\|_2 \geq \frac{1}{2}$. Thus, for any $v \in U(R_2^{(n_0)})$,

$$\begin{aligned} \Phi(e_{R_1}v^*e_{R_1}ve_{R_1}) &= \Phi(E_{R_1}(v^*)E_{R_1}(v)e_{R_1}) \\ &= \tau_N(E_{R_1}(v^*)E_{R_1}(v)) \\ &= \|E_{R_1}(v)\|_2^2 \\ &\geq \frac{1}{4}, \end{aligned}$$

therefore $\Phi(e_{R_1}ae_{R_1}) \geq \frac{1}{4}$, and $a \neq 0$. Furthermore, as a is of minimal $\|\cdot\|_{2,\Phi}-$norm, necessarily we have for any $v \in U(R_2^{(n_0)})$, $a = v^*av$. Finally, we have found $a \in \langle N, e_{R_1}\rangle^+ \cap (R_2^{(n_0)})'$ with $a \neq 0$ and $\Phi(a) < \infty$. We can notice that $\langle N, e_{R_1}\rangle = R_1 \otimes B(L^2(N_1))$; so, if we denote by e the spectral projection of a corresponding to the interval $[\|a\|/2, \|a\|]$, then e is a finite non-zero projection in $R_1 \otimes B(L^2(N_1))$, which commutes with $R_2^{(n_0)}$. We can proceed by using the proof of Proposition 12 of [**19**]. The Hilbert space $\mathcal{H} = eL^2(N)$ is an $R_2^{(n_0)} - R_1$ Hilbert bimodule and $\dim \mathcal{H}_{R_1} < \infty$. As $R_1' \cap N = N_1$ is a factor, Proposition 12 of [**19**] claims that there exists $s > 0$ and $u \in U(N)$ such that $uR_2^{(n_0)}u^* \subset R_1^s$. Thus, up to a conjugation by a unitary in N, we get $R_2^{(n_0)} \subset R_1^s$. But by construction, we know that $R_2 = \mathrm{Mat}_{2^{n_0}}(\mathbf{C}) \otimes R_2^{(n_0)}$, so up to a conjugation by a unitary in N, we get $R_2^{1/2^{n_0}} \subset R_1^s$, i.e. up to a conjugation by a unitary in N, we have $R_2 \subset R_1^{s'}$ with $s' = 2^{n_0}s$. So, we get that $N_2 = R_0 \vee N_1^{1/s'}$ with $R_0 = R_2' \cap R_1^{s'}$. Since $N_2 = R_0 \otimes N_1^{1/s'}$, R_0 is a subfactor of $R_1^{s'}$. As N_2 is a full factor, R_0 has to be finite-dimensional. If $d^2 = \dim R_0$, $t = d/s'$, and if we decompose N as $R_1^{1/t} \otimes N_1^t$, we obtain up to a conjugation by a unitary in N that $N_2 = N_1^t$ and $R_2 = R_1^{1/t}$. □

The following lemma is an easy consequence of the previous theorem. It is this result we will use in the proof of Theorem 2.12.

LEMMA 2.15. *Let R_1^∞ and R_2^∞ be two copies of the hyperfinite type II_∞ factor; let N_1^∞ and N_2^∞ be two full factors of type II_∞. Let $\alpha : R_1^\infty \otimes N_1^\infty \to R_2^\infty \otimes N_2^\infty$ be an isomorphism. Then, there exist a unitary $u \in R_2^\infty \otimes N_2^\infty$ and two isomorphisms $\beta : R_1^\infty \to R_2^\infty$, $\gamma : N_1^\infty \to N_2^\infty$ such that for any $x \in R_1^\infty \otimes N_1^\infty$, $u^*\alpha(x)u = (\beta \otimes \gamma)(x)$.*

PROOF. Let α be an isomorphism from $R_1^\infty \otimes N_1^\infty$ onto $R_2^\infty \otimes N_2^\infty$. Let e_1 and f_1 be two finite projections in R_1^∞ et N_1^∞. It is clear that $\alpha(e_1 \otimes f_1)$ is a finite projection in $R_2^\infty \otimes N_2^\infty$; thus there exist two finite projections e_2 and f_2 in R_2^∞ and N_2^∞ such that $\alpha(e_1 \otimes f_1) \sim e_2 \otimes f_2$. If we denote by p_1 the projection $\alpha(e_1 \otimes f_1)$, as $R_2^\infty \otimes N_2^\infty$ is a properly infinite factor, there exists a family (p_n) of pairwise orthogonal and mutually equivalent projections such that $\sum p_n = 1$. If we do again the same thing with $q_1 = e_2 \otimes f_2$, we obtain another family (q_n) of pairwise orthogonal and mutually equivalent projections such that $\sum q_n = 1$. But, as $p_1 \sim q_1$, we obtain for any $n \in \mathbf{N}$, $p_n \sim q_n$. Let $u_n \in R_2^\infty \otimes N_2^\infty$ be the partial isometry such that $u_n^* u_n = p_n$ and $u_n u_n^* = q_n$; then $u = \sum u_n$ is a unitary in $R_2^\infty \otimes N_2^\infty$ and $up_1u^* = q_1$. Consequently, up to a conjugation by a unitary, we can assume that $\alpha(e_1 \otimes f_1) = e_2 \otimes f_2$.

From now on, we denote by R_1 the factor $(R_1^\infty)_{e_1}$ and N_1 the factor $(N_1^\infty)_{f_1}$. We obtain immediately that $\alpha(R_1 \otimes N_1) = (R_2^\infty)_{e_2} \otimes (N_2^\infty)_{f_2}$, because $\alpha(e_1 \otimes f_1) = e_2 \otimes f_2$. According to Theorem 2.14, as $\alpha(R_1 \otimes f_1)$, $(R_2^\infty)_{e_2}$ are two copies of the hyperfinite factor of type II_1, and $a(e_1 \otimes N_1)$, $(N_2^\infty)_{f_2}$ are two full factors of type II_1, we know there exists a unitary $u \in \alpha(R_1 \otimes N_1)$ and $t > 0$ such that :

$$\begin{aligned} u\alpha(R_1 \otimes f_1)u^* &= \{(R_2^\infty)_{e_2}\}^{1/t} \\ \text{and } u\alpha(e_1 \otimes N_1)u^* &= \{(N_2^\infty)_{f_2}\}^{t}. \end{aligned}$$

Moreover, there exist a projection e_1' in R_2^∞ and a projection f_1' in N_2^∞ such that :

$$\{(R_2^\infty)_{e_2}\}^{1/t} = (R_2^\infty)_{e_1'} \text{ and } \{(N_2^\infty)_{f_2}\}^{t} = (N_2^\infty)_{f_1'}.$$

Consequently, we have proved :

$$\begin{aligned} u\alpha(R_1 \otimes f_1)u^* &= (R_2^\infty)_{e_1'} \otimes f_1' \\ \text{and } u\alpha(e_1 \otimes N_1)u^* &= e_1' \otimes (N_2^\infty)_{f_1'}. \end{aligned}$$

Thus, up to a conjugation by a unitary, we can assume that :

$$\begin{aligned} \alpha(R_1 \otimes f_1) &= (R_2^\infty)_{e_1'} \otimes f_1' \\ \text{and } \alpha(e_1 \otimes N_1) &= e_1' \otimes (N_2^\infty)_{f_1'}. \end{aligned}$$

Let $(e_{ij})_{i,j \geq 1}$ be a system of matrix unit in R_1^∞ such that

$$e_{11} = e_1,\ R_1^\infty = V^* \left(R_1 \otimes B(l^2)\right) V,$$

with the unitary V defined in the following way (we shall assume that R_1^∞ acts on the Hilbert space $\mathcal{H}$) :

$$V\ :\ \begin{array}{ccc} \mathcal{H} & \to & e_1\mathcal{H} \otimes l^2 \\ h & \mapsto & (e_{1n}h) \end{array}.$$

In the same way, we denote by $(f_{ij})_{i,j\geq 1}$, $(e'_{ij})_{i,j\geq 1}$, $(f'_{ij})_{i,j\geq 1}$ the systems of matrix unit associated respectively with N_1^∞, R_2^∞, N_2^∞. Let us denote by U the unitary $\sum_{i,j}(e'_{i1} \otimes$

$f'_{j1})\alpha(e_{1i}\otimes f_{1j})$. The computation of $U\alpha(e_{ik}\otimes f_{jl})U^*$ gives us :

$$\begin{aligned} U\alpha(e_{ik}\otimes f_{jl})U^* &= (e'_{i1}\otimes f'_{j1})\alpha(e_{1i}\otimes f_{1j})\alpha(e_{ik}\otimes f_{jl})\alpha(e_{k1}\otimes f_{l1})(e'_{1k}\otimes f'_{1l}) \\ &= (e'_{i1}\otimes f'_{j1})\alpha(e_{11}\otimes f_{11})(e'_{1k}\otimes f'_{1l}) \\ &= (e'_{i1}\otimes f'_{j1})(e'_{11}\otimes f'_{11})(e'_{1k}\otimes f'_{1l}) \\ &= (e'_{ik}\otimes f'_{jl}). \end{aligned}$$

Let $\beta : R_1^\infty \to R_2^\infty$ defined in the following way : $\beta(x)=\alpha(x)$ for all $x\in R_1$ and $\beta(e_{ij})=e'_{ij}$ for all $i,j\geq 1$. It is clear that β is an isomorphism from R_1^∞ onto R_2^∞. In the same way, we define $\gamma : N_1^\infty\to N_2^\infty$ by : $\gamma(x)=\alpha(x)$ for all $x\in N_1$ and $\gamma(f_{ij})=f'_{ij}$ for all $i,j\geq 1$. Once again, γ is an isomorphism from N_1^∞ onto N_2^∞. Finally, with these notations, for any $x\in R_1^\infty\otimes N_1^\infty$, we have :

$$U\alpha(x)U^*=(\beta\otimes\gamma)(x).$$

□

Proof of Theorem 2.12. We keep the notations introduced at the beginning of this section, $C=N{}_\alpha\!\bowtie_\gamma L^\infty(X)$ and $C'=N'{}_{\alpha'}\!\bowtie_{\gamma'}L^\infty(X')$. We have seen, according to Theorem 2.11, that the core of C is isomorphic to $(L^\infty(X)\rtimes_\gamma \mathbf{R})\otimes N$ and the dual action θ_s is given by $\theta_s=\widehat{\gamma}_{-s}\otimes\alpha_s$. The same thing is true for C'. As $C\simeq C'$, $(\mathrm{Core}(C),\theta)$ and $(\mathrm{Core}(C'),\theta')$ are weakly equivalent, according to Corollary 8.4 of [**42**]. That means that there exist an isomorphism $\pi:\mathrm{Core}(C)\to\mathrm{Core}(C')$ and a strongly continuous family of unitaries (u_s) of $\mathrm{Core}(C)$ such that (u_s) is a one-cocycle for θ_s and for any $x\in\mathrm{Core}(C)$ and any $s\in\mathbf{R}$,

$$\pi^{-1}\theta'_s\pi(x)=u_s\theta_s(x)u_s^*.$$

But, according to Lemma 2.15, we know that there exist a unitary $u\in\mathrm{Core}(C')$ and two isomorphisms $\pi_1 : L^\infty(X)\rtimes_\gamma\mathbf{R}\to L^\infty(X')\rtimes_{\gamma'}\mathbf{R}$, $\pi_2:N\to N'$ such that for any $x\in\mathrm{Core}(C)$,

$$u\pi(x)u^*=(\pi_1\otimes\pi_2)(x).$$

Thus we obtain, for any $x\in\mathrm{Core}(C)$ any $s\in\mathbf{R}$:

$$\begin{aligned} \theta'_s(\pi(x)) &= \pi(u_s)\pi(\theta_s(x))\pi(u_s)^* \\ \theta'_s(u)^*\theta'_s((\pi_1\otimes\pi_2)(x))\theta'_s(u) &= \pi(u_s)u^*(\pi_1\otimes\pi_2)(\theta_s(x))u\pi(u_s)^*. \end{aligned}$$

If $v_s=(\pi_1\otimes\pi_2)^{-1}(\theta'_s(u)\pi(u_s)u^*)$, for any $x\in\mathrm{Core}(C)$ and any $s\in\mathbf{R}$, we have

$$(\pi_1\otimes\pi_2)^{-1}\theta'_s((\pi_1\otimes\pi_2)(x))=v_s\theta_s(x)v_s^*.$$

If we take now $x=z\otimes 1$, for $z\in L^\infty(X)\rtimes_\gamma\mathbf{R}$, we have

$$(\pi_1)^{-1}\left(\widehat{\gamma'}_{-s}(\pi_1(z))\right)\otimes 1=v_s(\widehat{\gamma}_{-s}(z)\otimes 1)v_s^*.$$

Let us denote $\beta_s=(\pi_1)^{-1}\widehat{\gamma'}_{-s}\pi_1\widehat{\gamma}_s$; it is an automorphism of $L^\infty(X)\rtimes_\gamma\mathbf{R}$. Moreover, for any $y\in L^\infty(X)\rtimes_\gamma\mathbf{R}$, we have

$$(\beta_s(y)\otimes 1)v_s=v_s(y\otimes 1).$$

Now, we are going to use a classical technique. Let $\phi\in N_*$; we have, for any $y\in L^\infty(X)\rtimes_\gamma\mathbf{R}$, the equality :

$$\beta_s(y)(\mathrm{id}\otimes\phi)(v_s)=(\mathrm{id}\otimes\phi)(v_s)y.$$

As $v_s\neq 0$, there exists necessarily $\phi\in N_*$ such that $(\mathrm{id}\otimes\phi)(v_s)\neq 0$. A classical lemma allows us to claim that β_s is an inner automorphism of $L^\infty(X)\rtimes_\gamma\mathbf{R}$. Therefore, there exists a family of unitaries (w_s) in $L^\infty(X) \rtimes_\gamma \mathbf{R}$ (we can choose it strongly continuous because $s \mapsto \beta_s$ is a continuous map), such that for any $z \in L^\infty(X) \rtimes_\gamma \mathbf{R}$,

$$(\pi_1)^{-1}\widehat{\gamma'}_s\pi_1(z) = w_s\widehat{\gamma}_s(z)w_s^*. \tag{11}$$

But, we have to be careful here because (w_s) is not in general a one-cocycle for $\widehat{\gamma}$; for s and $t \in \mathbf{R}$,

$$\begin{aligned}(\pi_1)^{-1}\widehat{\gamma'}_s\widehat{\gamma'}_t\pi_1 &= \mathrm{Ad}(w_s\widehat{\gamma}_s(w_t))\widehat{\gamma}_{s+t}\\ \text{and } (\pi_1)^{-1}\widehat{\gamma'}_{s+t}\pi_1 &= \mathrm{Ad}(w_{s+t})\widehat{\gamma}_{s+t}.\end{aligned}$$

Consequently, $\mathrm{Ad}(w_s\widehat{\gamma}_s(w_t))\widehat{\gamma}_{s+t} = \mathrm{Ad}(w_{s+t})\widehat{\gamma}_{s+t}$, and we obtain that $w_{s+t}^*w_s\widehat{\gamma}_s(w_t)$ is in the center of $L^\infty(X) \rtimes_\gamma \mathbf{R}$; As $L^\infty(X) \rtimes_\gamma \mathbf{R}$ is a factor, we have finally proved that $w_{s+t}^*w_s\widehat{\gamma}_s(w_t) \in \mathbf{T}$. Hence, there exists $\omega(s,t) \in \mathbf{T}$, such that $w_{s+t} = \omega(s,t)w_s\widehat{\gamma}_s(w_t)$. Moreover, we can easily see that $\omega(s,t)$ satisfies a 2-cocycle relation. But, it is well-known that $Z^2(\mathbf{R},\mathbf{T}) = B^2(\mathbf{R},\mathbf{T})$, therefore $\omega(s,t)$ is a coboundary and there exists a map $\lambda : \mathbf{R} \to \mathbf{T}$, such that for any s, $t \in \mathbf{R}$, $\omega(s,t) = \overline{\lambda(s+t)}\lambda(s)\lambda(t)$. We can notice that if we multiply w_s by $\lambda(s)$, $\lambda(s)w_s$ becomes a one-cocycle for $\widehat{\gamma}$ and we do not change the Equation (11). Finally, up to a multiplication by a scalar $\lambda(s) \in \mathbf{T}$, we can assume that w_s is a one-cocycle for $\widehat{\gamma}$.

We can apply now Proposition 4.2 of [**42**]. We obtain the existence of an isomorphism $\tilde{\pi} : L^\infty(X) \otimes B(L^2(\mathbf{R})) \to L^\infty(X') \otimes B(L^2(\mathbf{R}))$ which intertwines both of the bidual actions $\widehat{\widehat{\gamma}}$ and $\widehat{\widehat{\gamma'}}$, i.e. for any $x \in L^\infty(X) \otimes B(L^2(\mathbf{R}))$,

$$\tilde{\pi}\widehat{\widehat{\gamma}}_s(x) = \widehat{\widehat{\gamma'}}_s\tilde{\pi}(x).$$

Moreover we have both of the equalities $\widehat{\widehat{\gamma}}_s = \gamma_s \otimes \mathrm{Ad}(\lambda_s^*)$ and $\widehat{\widehat{\gamma'}}_s = \gamma'_s \otimes \mathrm{Ad}(\lambda_s^*)$. We can notice that $L^\infty(X) = \mathrm{Z}(L^\infty(X) \otimes B(L^2(\mathbf{R})))$ and $L^\infty(X') = \mathrm{Z}(L^\infty(X') \otimes B(L^2(\mathbf{R})))$. Let us denote by $\rho : L^\infty(X) \to L^\infty(X')$ the isomorphism obtained from $\tilde{\pi}$ by taking restrictions to the centers. Finally, we obtain for any $s \in \mathbf{R}$ and $x \in L^\infty(X)$,

$$\rho\gamma_s(x) = \gamma'_s\rho(x).$$

Therefore, we have proved that the flows (X,γ) and (X',γ'_t) are conjugate; we can prove exactly in the same way that (N,α) and (N',α') are weakly equivalent. ☐

4. Computation of Connes' τ Invariant

For this section, we are going to keep the same notations as before. We shall denote by C the factor of type III_1, $N_\alpha \bowtie_\gamma L^\infty(X)$; we remind that $C \otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to $(L^\infty(X) \otimes P) \rtimes_{\gamma\otimes\sigma^\omega} \mathbf{R}$ with $P = N \rtimes_\alpha \mathbf{R}$. We assume that the flow γ is free and ergodic and finite or infinite measure-preserving; the action $\gamma \otimes \sigma^\omega$ will be still denoted by β and λ_s are the unitaries which implement the action of $\mathbf{R}$ on $L^\infty(X) \otimes P$. Furthermore, we shall assume that P is a free Araki-Woods factor [**39**]; but N is not necessarily full. At last, we shall denote by σ the canonical embedding of $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ into $M = P \otimes L^\infty(X) \otimes B(L^2(\mathbf{R}))$. First of all, let us remind the definition of Connes' τ invariant.

Definition 2.16. *Let C be a factor of type* III_1. *Let $\varepsilon : \mathrm{Aut}(C) \to \mathrm{Out}(C) = \mathrm{Aut}(C)/\mathrm{Inn}(C)$ be the canonical projection. Let ψ be any weight on C. The mapping $\delta : \mathbf{R} \to \mathrm{Out}(C)$, with $\delta(t) = \varepsilon(\sigma_t^\psi)$ is independent of the choice of the weight ψ thanks to Theorem* 1.2.1 *of* [**5**]. *The τ invariant of C, denoted by $\tau(C)$ is the* weakest topology *on $\mathbf{R}$ that makes the map δ continuous.*

Although C is not a full factor, we can nevertheless give an explicit computation of Connes' τ invariant in many cases. We want to remind a few things about free Araki-Woods factors [**39**]. Let $\mathcal{H}_{\mathbf{R}}$ be a real Hilbert space and (U_t) be an orthogonal representation of $\mathbf{R}$ on $\mathcal{H}_{\mathbf{R}}$. Let $\mathcal{H} = \mathcal{H}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified Hilbert space. If A is the infinitesimal generator of (U_t) on $\mathcal{H}$, we remind that $j : \mathcal{H}_{\mathbf{R}} \to \mathcal{H}$ defined by $j(\zeta) = (\frac{2}{A^{-1}+1})^{1/2}\zeta$ is an isometric embedding of $\mathcal{H}_{\mathbf{R}}$ into $\mathcal{H}$. Let $K_{\mathbf{R}} = j(\mathcal{H}_{\mathbf{R}})$. For $\xi \in \mathcal{H}$, we denote by $l(\xi)$ the creation operators on the Fock space $\mathcal{F}(\mathcal{H})$ and $s(\xi)$ their real part. By definition, $\Gamma(\mathcal{H}_{\mathbf{R}}, U_t)'' = \{s(\xi), \xi \in K_{\mathbf{R}}\}''$.

DEFINITION 2.17. *Let $\mathcal{H}_{\mathbf{R}}$ be a real Hilbert space and (U_t) be an orthogonal representation of $\mathbf{R}$ on $\mathcal{H}_{\mathbf{R}}$. Let $P = \Gamma(\mathcal{H}_{\mathbf{R}}, U_t)''$ be the free Araki-Woods factor associated with (U_t) and $\mathcal{H}_{\mathbf{R}}$ [**39**]. We shall say that P satisfies the* condition (M) *of mixing if there exist ξ, $\eta \in \mathcal{H}_{\mathbf{R}}$, such that the continuous function f defined by $f(t) = \langle U_t j\xi, j\eta\rangle$ vanishes at infinity and $f \neq 0$.*

For example, if $\mathcal{H}_{\mathbf{R}} = L^2(\mathbf{R}, \mathbf{R})$ and $U_t = \lambda_t$ for any $t \in \mathbf{R}$, the factor $\Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$ satisfies the condition (M). We remind that $\Gamma(L^2(\mathbf{R}, \mathbf{R}), \lambda_t)''$ is nothing but $N \rtimes_\alpha \mathbf{R}$, with $N = L(\mathbf{F}_\infty) \otimes B(\mathcal{H})$ and α the trace-scaling automorphism group of Rădulescu [**29, 38**]. The aim of this section is to prove the following theorem :

THEOREM 2.18. *Let $P = \Gamma(\mathcal{H}_{\mathbf{R}}, U_t)''$ be a free Araki-Woods factor satisfying the condition (M). For $C = N_\alpha \bowtie_\gamma L^\infty(X)$, $\tau(C)$ = the weakest topology on $\mathbf{R}$ that makes the map $t \mapsto \gamma_t$ continuous for the u-topology on* $\mathrm{Aut}(L^\infty(X))$.

The proof of this result is based on a kind of "14ε lemma", it will be stated in the Appendix (Lemma 2.25).

From now on, we are going to use notations and results of [**3**]. For C our factor of type III$_1$, for any normal faithful state φ on C and for any free ultrafilter ω on $\mathbf{N}$, we shall denote by $A_{\varphi,\omega}$ the norm closed $*-$subalgebra of $l^\infty(\mathbf{N}, C)$ of all sequences $(x_n)_{n\in\mathbf{N}}$ such that $\|[x_n, \varphi]\| \to 0$ when $n \to \omega$. Let $I_\omega = \{(x_n)_{n\in\mathbf{N}}, x_n \to 0 \; *-\text{strongly when } n \to \omega\}$. It is a two-sided ideal of $l^\infty(\mathbf{N}, C)$. Moreover, $A_{\varphi,\omega} \cap I_\omega$ is a two-sided ideal of the C^*-algebra $A_{\varphi,\omega}$, and the canonical quotient map will be denoted by $\rho_{\varphi,\omega}$. We shall denote by C^ω the *ultraproduct* of C along ω, i.e. the quotient of $l^\infty(\mathbf{N}, C)$ by the two-sided ideal I_ω ; we shall denote by $C_{\varphi,\omega}$ the quotient of the C^*-algebra $A_{\varphi,\omega}$ by the two-sided ideal $A_{\varphi,\omega} \cap I_\omega$. We know, according to Proposition 2.2 of [**3**], that $C_{\varphi,\omega}$ is a finite von Neumann algebra. Let φ be a given faithful normal state on C and $\mathcal{D}$ the set of faithful normal states on C with $\alpha\varphi \leq \psi \leq \alpha^{-1}\varphi$ for some $\alpha > 0$. Let

$$C_\omega = \bigcap_{\psi \in \mathcal{D}} \rho_{\varphi,\omega}(A_{\varphi,\omega} \cap A_{\psi,\omega}).$$

According to Theorem 2.9 of [**3**], C_ω is a finite von Neumann called the *asymptotic centralizer* of C at ω. Before proceeding, we want to remind a classical result on compact operators on the Hilbert space $L^2(\mathbf{R})$.

PROPOSITION 2.19. *Let $A \subset B(L^2(\mathbf{R}))$ be a unital C^*-algebra. Assume that for any $t \in \mathbf{R}$, $\lambda_t \in A$ and that there exists $f \in C_o(\mathbf{R})$, $f \neq 0$, such that $M_f \in A$ (M_f is the multiplication operator). Then $K(L^2(\mathbf{R})) \subset A$.*

PROOF. We identify g with M_g for any $g \in C_o(\mathbf{R})$. Since $f \neq 0$ and all its translations belong to $A \cap C_o(\mathbf{R})$, $A \cap C_o(\mathbf{R})$ is a sub$-C^*-$algebra of $C_o(\mathbf{R})$ which separates the points. Consequently, $A \cap C_o(\mathbf{R}) = C_o(\mathbf{R})$ and thus $C_o(\mathbf{R}) \subset A$. Moreover, $C_r^*(\mathbf{R}) \subset A$ because for any $t \in \mathbf{R}$, $\lambda_t \in A$. Therefore, $K(L^2(\mathbf{R})) = [C_r^*(\mathbf{R}) C_o(\mathbf{R})] \subset A$. $\square$

For any subset $E \subset C$ and a sequence (x_k) of elements in C, we shall say that (x_k) *almost commutes* with E if for any $a \in E$, $[x_k, a] \to 0$ $*-$strongly. Let $C = N_\alpha \bowtie_\gamma L^\infty(X)$; we know that $C \otimes B(L^2(\mathbf{R}))$ is canonically isomorphic to $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$. We remind that we have denoted by $\tilde{\phi}$ the dual weight on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ of the weight $\phi = \tau \otimes \omega$ on $L^\infty(X) \otimes P$. Let λ_t be the unitaries in $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ which implement the action $\beta = \gamma \otimes \sigma^\omega$ of $\mathbf{R}$ on $L^\infty(X) \otimes P$. As the family (λ_t) is in the centralizer of the weight $\tilde{\phi}$, it is a one-cocycle for the action $\sigma^{\tilde{\phi}}$. We know according to Theorem 1.2.4 of [**5**] that there exists a faithful, normal, semifinite weight ψ on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ such that :

$$\forall t \in \mathbf{R},\ \sigma_t^\psi = \lambda_t^* \sigma_t^{\tilde{\phi}} \lambda_t.$$

We remind that we have shown in the first section that for any $t \in \mathbf{R}$ and any $x \in L^\infty(X) \otimes P$,

$$\sigma_t^\psi(\pi_\beta(x)) = \pi_\beta((\gamma_{-t} \otimes \mathrm{id})(x)). \tag{12}$$

Thanks to Equation (12), we know that $\mathrm{Cent}(\psi) = (\pi_{\sigma^\omega}(P) \cup \lambda(\mathbf{R}))'' \simeq P \rtimes_{\sigma^\omega} \mathbf{R}$. We are proving now, thanks to Proposition 2.26, the following technical lemma which will turn out to be essential in proof of Theorem 2.18 :

LEMMA 2.20. *Let C be as in Theorem 2.18, and ψ the weight on $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ which satisfies Equation (12). Let ω be any free ultrafilter on $\mathbf{N}$. Any bounded sequence (u_n) of $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ which almost commutes with $\mathrm{Cent}(\psi)$ is centralising. In a shorter way, we have the following equality :*

$$((L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R})^\omega \cap \mathrm{Cent}(\psi)' = ((L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R})_\omega.$$

PROOF. The inclusion $\supset$ is trivial. We have just to prove $\subset$. We assume as in Theorem 2.18, that $P = \Gamma(\mathcal{H}_\mathbf{R}, U_t)''$ is the free Araki-Woods factor associated with the orthogonal representation (U_t). Let Ω be the vacuum vector and let $\varphi_U = \langle \cdot \Omega, \Omega \rangle$ be the free quasi-free state on P, and (σ_t) the modular group associated with φ_U. We remind that we have for any $t \in \mathbf{R}$, and any $\zeta \in K_\mathbf{R}$, $\sigma_t(s(\zeta)) = s(U_t \zeta)$. We remind that we take $M = P \otimes L^\infty(X) \otimes B(L^2(\mathbf{R}))$ and the embedding of $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ into M is denoted by σ. Let (u_n) be a sequence of elements in $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$ which almost commutes with $\mathrm{Cent}(\psi)$ and such that $\|u_n\| \leq 1$ for any $n \in \mathbf{N}$. Thus $(\sigma(u_n))$ almost commutes with $\sigma(\mathrm{Cent}(\psi))$ and in particular $(\sigma(u_n))$ almost commutes with $\sigma(P)$. Hence, according to Proposition 2.26, there exists a bounded sequence (v_n) of elements in $L^\infty(X) \rtimes_\gamma \mathbf{R}$ such that $\sigma(u_n) - 1 \otimes v_n \to 0$ $*-$strongly in $M = P \otimes L^\infty(X) \otimes B(L^2(\mathbf{R}))$ and $\|v_n\| \leq 1$ for any $n \in \mathbf{N}$. Actually, v_n is nothing but $E(u_n)$, for any $n \in \mathbf{N}$, with $E : M \to L^\infty(X) \otimes B(L^2(\mathbf{R}))$ the conditional expectation $\varphi_U \otimes \mathrm{Id}$. We have somewhat reduced the difficulty of the problem : instead of looking at the sequence (u_n) in $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$, we are looking at the sequence (v_n) in $L^\infty(X) \rtimes_\gamma \mathbf{R}$. But we want to go further ; we want to prove that there exists a sequence (w_n) in $L^\infty(X)$ such that $\sigma(u_n) - 1 \otimes w_n \otimes 1 \to 0$ $*-$strongly in M. If we are able to prove such a result, we are done. Indeed, as $1 \otimes w_n \otimes 1 \in \mathrm{Z}(M)$ for any $n \in \mathbf{N}$, the sequence $(1 \otimes w_n \otimes 1)$ turns out to be centralising in M ; but as $\sigma(u_n) - 1 \otimes w_n \otimes 1 \to 0$ $*-$strongly in M, $(\sigma(u_n))$ is also centralising in M. Therefore, (u_n) is centralising in $(L^\infty(X) \otimes P) \rtimes_\beta \mathbf{R}$.

As $\sigma(u_n) - 1 \otimes v_n \to 0$ $*-$strongly in M and (u_n) almost commutes with $\mathrm{Cent}(\psi)$, for any $y \in \mathrm{Cent}(\psi)$, $[1 \otimes v_n, y_{13}] \to 0$ $*-$strongly in M (we are regarding y as a element in $M = P \otimes L^\infty(X) \otimes B(L^2(\mathbf{R}))$, that is why it is denoted by y_{13}). Thus for any normal state ϕ on P, $[v_n, 1 \otimes (\phi \otimes \mathrm{id})(y)] \to 0$ $*-$ strongly in $L^\infty(X) \otimes B(L^2(\mathbf{R}))$. Let $B = C^*((\phi \otimes \mathrm{Id})(y), y \in P \rtimes_{\sigma^\omega} \mathbf{R}, \phi \in P_*)$. We see that $B \subset B(L^2(\mathbf{R}))$ is a unital C^*-algebra, and for any $t \in \mathbf{R}$, $\lambda_t \in B$. Now, since P satisfies condition (M), we know that there exist ξ, $\eta \in \mathcal{H}_{\mathbf{R}}$ such that the continuous function f defined by $f(t) = \langle U_t\xi, \eta\rangle$ vanishes at infinity and $f \neq 0$. Let $\xi' = j(\xi)$ and $\eta' = j(\eta)$. Let $\phi = \varphi_U(s(\eta')\cdot) \in P_*$ and $x = (\sigma_t(s(\xi')))_{t\in\mathbf{R}} \in \mathrm{Core}(P)$, with $\mathrm{Core}(P) = P \rtimes_{\sigma^\omega} \mathbf{R}$. We get immediatly that for any $t \in \mathbf{R}$,

$$\begin{aligned}(\phi \otimes \mathrm{Id})(x)(t) &= \langle s(\eta')\sigma_t(s(\xi'))\Omega, \Omega\rangle \\ &= \langle \sigma_t(s(\xi'))\Omega, s(\eta')\Omega\rangle \\ &= \langle s(U_t\xi')\Omega, s(\eta')\Omega\rangle \\ &= \frac{1}{4}\langle U_t\xi', \eta'\rangle.\end{aligned}$$

Consequently, $(\phi \otimes \mathrm{Id})(x) \in C_o(\mathbf{R})$ and $(\phi \otimes \mathrm{Id})(x) \neq 0$. Then, according to Proposition 2.19, $K(L^2(\mathbf{R})) \subset B$. Hence we have proved that for any $y \in K(L^2(\mathbf{R}))$, $[v_n, 1 \otimes y] \to 0$ $*-$strongly. Let $\xi \in L^2(\mathbf{R})$ such that $\|\xi\| = 1$; for any $\mu \in L^2(\mathbf{R})$ we shall denote again by μ the map from $\mathbf{C}$ into $L^2(\mathbf{R})$ such that for any $\lambda \in \mathbf{C}$, $\mu(\lambda) = \lambda\mu$. Thus for μ, $\mu' \in L^2(\mathbf{R})$, $\theta_{\mu',\mu} = \mu' \circ \mu^*$ is a rank-one operator and it is exactly $\langle\cdot, \mu\rangle\mu'$. In particular, these operators are compact. For any $n \in \mathbf{N}$, let w_n be the element of $L^\infty(X)$ defined by $(1\otimes\xi^*)v_n(1\otimes\xi)$. For any $\eta \in L^2(X)$, we have

$$\begin{aligned}\|w_n\eta\|^2 &= \langle w_n^* w_n\eta, \eta\rangle \\ &= \langle (1\otimes\xi^*)v_n^*(1\otimes\xi)(1\otimes\xi^*)v_n(1\otimes\xi)\eta, \eta\rangle \\ &= \langle v_n^*(1\otimes\theta_{\xi,\xi})v_n(\eta\otimes\xi), \eta\otimes\xi\rangle \\ &= \langle v_n^*((1\otimes\theta_{\xi,\xi})v_n - v_n(1\otimes\theta_{\xi,\xi}))(\eta\otimes\xi), \eta\otimes\xi\rangle + \|v_n(\eta\otimes\xi)\|^2.\end{aligned}$$

Therefore, we get

$$\big|\|w_n\eta\|^2 - \|v_n(\eta\otimes\xi)\|^2\big| \leq \|(v_n(1\otimes\theta_{\xi,\xi}) - (1\otimes\theta_{\xi,\xi})v_n)(\eta\otimes\xi)\|\,\|\eta\|.$$

For any $\eta \in L^2(X)$ and $\mu \in L^2(\mathbf{R})$, we have

$$\begin{aligned}\|(v_n - w_n\otimes 1)(\eta\otimes\mu)\| &= \|(v_n - w_n\otimes 1)(1\otimes\theta_{\mu,\xi})(\eta\otimes\xi)\| \\ &\leq \|(v_n(1\otimes\theta_{\mu,\xi}) - (1\otimes\theta_{\mu,\xi})v_n)(\eta\otimes\xi)\| \\ &\quad + \|(1\otimes\theta_{\mu,\xi})(v_n - w_n\otimes 1)(\eta\otimes\xi)\| \\ &\leq \|(v_n(1\otimes\theta_{\mu,\xi}) - (1\otimes\theta_{\mu,\xi})v_n)(\eta\otimes\xi)\| \\ &\quad + \|(v_n - w_n\otimes 1)(\eta\otimes\xi)\|\,\|\mu\|.\end{aligned}$$

But $\|(v_n - w_n\otimes 1)(\eta\otimes\xi)\|^2 = \|v_n(\eta\otimes\xi)\|^2 - \|w_n\eta\|^2$. Thus, using the several inequalities we obtained above and the fact that for any $y \in K(L^2(\mathbf{R}))$, $[v_n, 1\otimes y] \to 0$ $*-$strongly, for any $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ large enough such that for any $n \geq n_0$,

$$\|(v_n - w_n\otimes 1)(\eta\otimes\mu)\| \leq \varepsilon.$$

That means exactly that $(v_n - w_n\otimes 1) \to 0$ strongly. We can do the same thing with v_n^* and w_n^* instead of v_n and w_n. Hence we have proved that $(v_n - w_n\otimes 1) \to 0$ $*-$strongly. The proof is complete. □

Proof of Theorem 2.18. Here, as usual, $\mathrm{Aut}(C)$ is endowed with the u-topology. Let us denote by τ the weakest topology on $\mathbf{R}$ that makes the map $t \mapsto \gamma_t$ continuous for the u-topology on $\mathrm{Aut}(L^\infty(X))$. According to Equation (6), as (γ_{-t}) and the restriction of (σ_t^ψ) on the von Neumann algebra $L^\infty(X)$ are exactly the same, it is clear that for any sequence of real numbers (t_n), if $t_n \to 0$ w.r.t. the topology τ then $t_n \to 0$ then w.r.t. the topology $\tau(C)$. We are going to prove the converse. Let (t_n) be a sequence of real numbers such that $t_n \to 0$ w.r.t. the topology $\tau(C)$. There exists a sequence of unitaries (u_n) in C

such that $\mathrm{Ad}(u_n) \circ \sigma^{\psi}_{t_n} \to \mathrm{Id}$ in $\mathrm{Aut}(C)$. Then the sequence (u_n) almost commutes with $\mathrm{Cent}\,\psi$. So according to Lemma 2.20, (u_n) is centralising in C. That means exactly that $\mathrm{Ad}(u_n) \to \mathrm{Id}$ in $\mathrm{Aut}(C)$, thus $\sigma^{\psi}_{t_n} \to \mathrm{Id}$ in $\mathrm{Aut}(C)$. Therefore $t_n \to 0$ w.r.t. the topology τ. □

We wish to end this section by giving a consequence of the previous result. First, we are going to remind a few things about mixing actions on a probability space.

DEFINITION 2.21. [**20**] *Let (X, μ) be a probability space, and let γ be a measure-preserving transformation of (X, μ). Let τ be the canonical trace on $L^\infty(X)$ given by the probability measure μ. The transformation γ is said to be* mixing *if for any f, $g \in L^2(X, \mu)$,*

$$\int_X (f \circ \gamma^n) g\, d\mu \to \tau(f)\tau(g) \text{ when } |n| \to +\infty.$$

Example. Let $r \in]0, 1[$. Let $X_0 = \{0, 1\}$ with $\mu_0^r(0) = r$ and $\mu_0^r(1) = 1-r$. Let (X_r, μ_r) be the probability space $\prod_{\mathbf{Z}}(X_0, \mu_0^r)$, and let γ_r be the Bernoulli shift on (X_r, μ_r) defined by :

$$\forall (c_n)_{n\in\mathbf{Z}} \in X,\ \gamma_r \cdot (c_n)_{n\in\mathbf{Z}} = (c_{n+1})_{n\in\mathbf{Z}}.$$

It is well-known that γ_r is a measure-preserving, free, ergodic, mixing transformation. So, we obtain a measure-preserving, free, ergodic, mixing action of $\mathbf{Z}$ on the probability space (X_r, μ_r). Moreover, according to [**20**], the entropy of the transformation γ_r is given by $H(\gamma_r) = -(r\log_2(r) + (1-r)\log_2(1-r))$. Therefore, we obtain a continuum of pairwise non-conjugate measure-preserving, free, ergodic, mixing actions.

From now on, let $(\gamma^n)_{n\in\mathbf{Z}}$ be a measure-preserving, free, ergodic, mixing action of $\mathbf{Z}$ on $L^\infty(X, \mu)$. We are going to induce this action up to $\mathbf{R}$ in the following way. Let

$$\begin{aligned} A &= L^\infty(\mathbf{R} \times X)^{\mathbf{Z}} \\ &= \{F \in L^\infty(\mathbf{R} \times X), n \cdot F = F, \forall n \in \mathbf{Z}\} \\ &= \{F \in L^\infty(\mathbf{R} \times X), F(t, \gamma^n(x)) = F(t+n, x), \forall (n, x, t) \in \mathbf{Z} \times X \times \mathbf{R}\}. \end{aligned}$$

We consider the action (σ_t) of $\mathbf{R}$ on A defined by : for any s, $t \in \mathbf{R}$, $x \in X$ and $F \in A$, $(\sigma_t F)(s, x) = F(s-t, x)$. It is well known according to [**42**] that (σ_t) is a measure-preserving, free, ergodic action of $\mathbf{R}$ on A, and $A \rtimes_\sigma \mathbf{R}$ is isomorphic to $(L^\infty(X) \rtimes_\gamma \mathbf{Z}) \otimes B(L^2(\mathbf{R}/\mathbf{Z}))$. We can identify A with $L^\infty(Y, \nu)$ where (Y, ν) isomorphic to $([0, 1[, \lambda) \times (X, \mu)$ as a probability space. More precisely, for any $t \in \mathbf{R}$, let $t = [t] + \{t\}$, with $[t]$ the entire part of t. The mapping $\theta : L^2([0, 1[, \lambda) \otimes L^2(X, \mu) \to L^2(Y, \nu)$ defined by, $\theta(\xi \otimes \eta)(t, x) = \xi(\{t\})\eta(\gamma^{[t]}(x))$, for $\xi \in L^2([0, 1[, \lambda)$ and $\eta \in L^2(X, \mu)$ is an isomorphism of Hilbert spaces. Through this isomorphism, we can identify $L^2([0, 1[, \lambda) \otimes \mathbf{C}1$ with $L^2(\mathbf{R}/\mathbf{Z}, \lambda)$. We shall denote by $\mathcal{H}_0$ the orthogonal of $L^2(\mathbf{R}/\mathbf{Z})$ in $L^2(Y, \nu)$, i.e. $\mathcal{H}_0$ is nothing but $L^2([0, 1[) \otimes (L^2(X, \mu) \ominus \mathbf{C}1)$. Moreover, the action (σ_t) of $\mathbf{R}$ on $A = L^\infty(Y)$ gives rise to a unitary representation (U_t) of $\mathbf{R}$ on the Hilbert space $L^2(Y, \nu)$: if the canonical embedding of $L^\infty(Y)$ into $L^2(Y, \nu)$ is denoted by η, (U_t) is defined by $U_t\eta(y) = \eta(\sigma_{-t}(y))$ for any $y \in L^\infty(Y)$. We want to go further and prove the following result :

PROPOSITION 2.22. *With the previous notations, we have for any ζ_1, $\zeta_2 \in \mathcal{H}_0$,*

$$\langle \zeta_1, U_t \zeta_2 \rangle \to 0 \text{ when } |t| \to +\infty$$

.

PROOF. It suffices to show that for any ξ_1, $\xi_2 \in L^2([0,1[,\lambda) \cap L^\infty([0,1[)$ and any η_1, $\eta_2 \in (L^2(X,\mu) \ominus \mathbf{C}1) \cap L^\infty(X)$, $\langle \xi_1 \otimes \eta_1, U_t(\xi_2 \otimes \eta_2)\rangle \to 0$ when $|t| \to +\infty$. Let ξ_1, $\xi_2 \in L^2([0,1[,\lambda) \cap L^\infty([0,1[)$, η_1, $\eta_2 \in (L^2(X,\mu) \ominus \mathbf{C}1) \cap L^\infty(X)$ and $t \in \mathbf{R}$. We get

$$\begin{aligned} \langle \xi_1 \otimes \eta_1, U_t(\xi_2 \otimes \eta_2)\rangle &= \iint_{[0,1[\times X} \xi_1(\{s\})\eta_1(\gamma^{[s]}(x))\xi_2(\{s+t\})\eta_2(\gamma^{[s+t]}(x))\mathrm{d}\mu(x)\mathrm{d}\lambda(s) \\ &= \int_{[0,1[} \xi_1(\{s\})\xi_2(\{s+t\}) \left(\int_X \eta_1(\gamma^{[s]}(x))\eta_2(\gamma^{[s+t]}(x))\mathrm{d}\mu(x)\right) \mathrm{d}\lambda(s). \qquad (13) \end{aligned}$$

As the action $(\gamma^n)_{n\in\mathbf{Z}}$ is mixing, it is clear that with $s \in \mathbf{R}$ fixed,

$$\int_X \eta_1(\gamma^{[s]}(x))\eta_2(\gamma^{[s+t]}(x))\mathrm{d}\mu(x) \to 0$$

when $|t| \to +\infty$, because η_1, $\eta_2 \in (L^2(X,\mu) \ominus \mathbf{C}1) \cap L^\infty(X)$. The function $f : s \mapsto \xi_1(\{s\})\xi_2(\{s+t\}) \left(\int_X \eta_1(\gamma^{[s]}(x))\eta_2(\gamma^{[s+t]}(x))\mathrm{d}\mu(x)\right)$ is such that for any $s \in \mathbf{R}$, $|f(s)| \leq \|\xi_1\|\|\xi_2\|\|\eta_1\|\|\eta_2\|$ which is integrable on $[0,1[$ with respect to the Lebesgue measure λ. Finally, thanks to the dominated convergence theorem applied to f with Equation (13), we have proved that $\langle \xi_1 \otimes \eta_1, U_t(\xi_2 \otimes \eta_2)\rangle \to 0$ when $|t| \to +\infty$. □

For the flow $(\sigma_t)_{t\in\mathbf{R}}$ on the probability space (Y,ν), let $\tau(\sigma)$ be the weakest topology on $\mathbf{R}$ that makes the map from $\mathbf{R}$ to $\mathrm{Aut}(L^\infty(Y))$, which sends t onto σ_t, continuous (notice that $\mathrm{Aut}(L^\infty(Y))$ is endowed with the u-topology). We can prove the following result :

PROPOSITION 2.23. *For the flow $(\sigma_t)_{t\in\mathbf{R}}$ on the probability space (Y,ν) as before, the topology $\tau(\sigma)$ is the usual topology on $\mathbf{R}$.*

PROOF. Let $(t_k)_{k\in\mathbf{N}}$ be a sequence of real numbers such that $t_k \to 0$ with respect to the topology $\tau(\sigma)$. Then, $\sigma_{t_k} \to \mathrm{Id}$ in $\mathrm{Aut}(L^\infty(Y))$ with respect to the u-topology. In particular, for any $\zeta \in L^2(Y) \ominus L^2(\mathbf{R}/\mathbf{Z})$, $\zeta \neq 0$, $\langle \zeta, U_{t_k}\zeta\rangle \to \|\zeta\|^2 \neq 0$ when $k \to +\infty$. Consequently, we get, thanks to Proposition 2.22, that (t_k) is necessarily bounded. Moreover, for any cluster point t of the sequence (t_k), we must have $\sigma_t = \mathrm{Id}$. As the flow $(\sigma_s)_{s\in\mathbf{R}}$ is free, $t = 0$ necessarily. Therefore, $t_k \to 0$ w.r.t the usual topology on $\mathbf{R}$. □

At last, it is easy to prove the following claim : let σ_1 and σ_2 be two flows which come from two actions of $\mathbf{Z}$, γ_1 and γ_2 ; if the flows σ_1 and σ_2 are conjugate, then the actions γ_1 and γ_2 are conjugate. Therefore, we have proved the following result :

THEOREM 2.24. *Let N be $L(\mathbf{F}_\infty) \otimes B(\mathcal{H})$ endowed with the one-parameter automorphism group $(\alpha_t)_{t\in\mathbf{R}}$ scaling the trace from Rădulescu [**29**]. For $r \in]0,1/2]$, let γ_r be the Bernoulli shift on (X_r,μ_r) defined as before, and σ_r the flow on $L^\infty(Y_r)$ obtained from γ_r after induction to $\mathbf{R}$. With the family $(N_\alpha \rtimes_{\sigma_r} L^\infty(Y_r))_{r\in]0,1/2]}$, we get an uncountable family of pairwise non-isomorphic factors of type III_1. All these factors are non-full but they have the same τ invariant : it is the usual topology on $\mathbf{R}$. In particular, they have no almost periodic weights.*

5. Appendix

In this last section, we are going to prove a kind of "14ε lemma" which can be viewed as a new version of Lemma 4.1 in [**44**] and Lemma 4.1 in [**45**], which were also a generalization of 14ε lemma due to Murray & von Neumann [**17**] (see [**1**] for another version of this lemma in case of a free product of two von Neumann algebras of type III). We will denote by (P,ω) a free Araki-Woods factor with its free quasi-free state ω : we remind that ω is given by $\langle \cdot\Omega, \Omega\rangle$ with Ω the vacuum vector (see [**39**] for further details). We shall denote by A the von Neumann algebra $L^\infty(X)\otimes B(L^2(\mathbf{R}))$. Let σ be the canonical embedding of $(L^\infty(X)\otimes P)\rtimes_{\gamma\otimes\sigma^\omega}\mathbf{R}$ into $P\otimes A$; in particular, for any $y\in P$, $\sigma(y)=(\sigma^\omega_{-t}(y))_{t\in\mathbf{R}}$ and thus $\sigma(y)\in P\otimes L^\infty(X)$. Let $E: P\otimes A\to A$ be the conditional expectation defined by $E=\omega\otimes\mathrm{id}$. Let $\mathcal{K}=L^2(X)\otimes L^2(\mathbf{R})$ be the Hilbert space on which A acts; let $\xi\in\mathcal{K}$ such that $\|\xi\|=1$, and ω_ξ the vector state on A associated with ξ. We shall denote by $\|\cdot\|_{\omega\otimes\omega_\xi}$ the semi-norm with respect to the normal state $\omega\otimes\omega_\xi$ on $P\otimes A$.

LEMMA 2.25. *Let (P_i,ω_i) be two von Neumann algebras endowed with a faithful normal state ω_i $(i=1,2)$ such that we can write $(P,\omega)=(P_1,\omega_1)*(P_2,\omega_2)$. Let $a\in P_1$ and b, $c\in P_2$. Assume that a, b, c are analytic with respect to the state ω. For any $x\in P\otimes A$ and any $\xi\in\mathcal{K}$, $\|\xi\|=1$,*

$$\begin{aligned}\|x-E(x)\|_{\omega\otimes\omega_\xi} &\leq \mathcal{E}(a,b,c)\max\left\{\|[\sigma(a),x]\|_{\omega\otimes\omega_\xi},\|[\sigma(b),x]\|_{\omega\otimes\omega_\xi},\|[\sigma(c),x]\|_{\omega\otimes\omega_\xi}\right\}\\ &\quad+\mathcal{F}(a,b,c)\,\|x\|\end{aligned}$$

with

$$\begin{aligned}\mathcal{E}(a,b,c) &= 6\|a\|^3+4\|b\|^3+4\|c\|^3,\\ \mathcal{F}(a,b,c) &= 3\mathcal{C}(a)+2\mathcal{C}(b)+2\mathcal{C}(c)+12|\omega(c^*b)|\,\|c^*b\|,\\ \mathcal{C}(a) &= 2\|a\|^3\|a-\sigma^\omega_{i/2}(a)\|+2\|a\|^2\|\sigma(a)(a^*\otimes 1)-1\|^2_{\omega\otimes\omega_\xi}\\ &\quad+3(1+\|a\|^2)\|a^*a-1\|+6|\omega(a)|\,\|a\|.\end{aligned}$$

PROOF. For each P_i let us denote by $\mathcal{H}_i$ the Hilbert space which comes from the GNS representation of ω_i and let ξ_i be the associated cyclic vector. Let us take $(\mathcal{H},\Omega)=(\mathcal{H}_1,\Omega_1)*(\mathcal{H}_2,\Omega_2)$. We remind [**48**] that

$$\mathcal{H}=\mathbf{C}\Omega\oplus(\mathring{\mathcal{H}}_1\otimes\mathcal{H}(2,l))\oplus(\mathring{\mathcal{H}}_2\otimes\mathcal{H}(1,l))$$

with $\mathring{\mathcal{H}}_i=\mathcal{H}_i\ominus\mathbf{C}\Omega_i$,

$$\mathcal{H}(2,l)=\mathbf{C}\Omega\oplus\mathring{\mathcal{H}}_2\oplus(\mathring{\mathcal{H}}_2\otimes\mathring{\mathcal{H}}_1)\oplus(\mathring{\mathcal{H}}_2\otimes\mathring{\mathcal{H}}_1\otimes\mathring{\mathcal{H}}_2)\oplus\cdots,$$

$$\mathcal{H}(1,l)=\mathbf{C}\Omega\oplus\mathring{\mathcal{H}}_1\oplus(\mathring{\mathcal{H}}_1\otimes\mathring{\mathcal{H}}_2)\oplus(\mathring{\mathcal{H}}_1\otimes\mathring{\mathcal{H}}_2\otimes\mathring{\mathcal{H}}_1)\oplus\cdots.$$

Moreover we shall denote by $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_2$ the Hilbert spaces $\mathring{\mathcal{H}}_1\otimes\mathcal{H}(2,l)\otimes\mathcal{K}$ and $\mathring{\mathcal{H}}_2\otimes\mathcal{H}(1,l)\otimes\mathcal{K}$.

Let $x\in P\otimes A$ and let us define $\eta=x\cdot(\Omega\otimes\xi)$. We write $\eta=(1\otimes E(x))\cdot(\Omega\otimes\xi)+\mu+\gamma$ with $\mu\in\widetilde{\mathcal{H}}_1$ and $\gamma\in\widetilde{\mathcal{H}}_2$. We define for $\zeta\in\mathcal{H}$ and $y\in P$, $\zeta\cdot y=Jy^*J\cdot\zeta$, and we notice that $(z\Omega)\cdot\sigma^\omega_{i/2}(y)=zy\Omega$ for $y\in D(\sigma^\omega_{i/2})$. Let $\mathring{x}=x-1\otimes E(x)$, $\eta_0=\mu+\gamma$, $\tilde{\eta}=\sigma(a)\cdot\eta\cdot(a^*\otimes 1)$, $\tilde{\gamma}=\sigma(a)\cdot\gamma\cdot(a^*\otimes 1)$, $\tilde{\mu}=\sigma(a)\cdot\mu\cdot(a^*\otimes 1)$ and $\tilde{\zeta}=\eta_0-\gamma-\tilde{\gamma}$. First of all, we are going to assess the quantity $\|\tilde{\eta}-\eta\|_{\omega\otimes\omega_\xi}$. We have

$$\begin{aligned}\tilde{\eta} &= \sigma(a)\cdot\eta\cdot(a^*\otimes 1)\\ &= \sigma(a)x\cdot(\Omega\otimes\xi)\cdot(a^*\otimes 1)\\ &= [\sigma(a),x]\cdot(\Omega\otimes\xi)\cdot(a^*\otimes 1)+x\sigma(a)\cdot(\Omega\otimes\xi)\cdot(a^*\otimes 1)\\ &= [\sigma(a),x]\cdot(\Omega\otimes\xi)\cdot(a^*\otimes 1)+x\sigma(a)\cdot(\Omega\otimes\xi)\cdot((a^*-\sigma^\omega_{i/2}(a^*))\otimes 1)\\ &\quad+x\sigma(a)\cdot(\Omega\otimes\xi)\cdot(\sigma^\omega_{i/2}(a^*)\otimes 1)\\ &= [\sigma(a),x]\cdot(\Omega\otimes\xi)\cdot(a^*\otimes 1)+x\sigma(a)\cdot(\Omega\otimes\xi)\cdot((a^*-\sigma^\omega_{i/2}(a^*))\otimes 1)\\ &\quad+x\sigma(a)(a^*\otimes 1)\cdot(\Omega\otimes\xi).\end{aligned}$$

Thus, we get immediatly

$$\|\tilde{\eta}-\eta\|_{\omega\otimes\omega_\xi} \leq \|a\|\,\|[\sigma(a),x]\|_{\omega\otimes\omega_\xi}+\|x\|\|a\|\|a-\sigma^{\omega}_{i/2}(a)\| + \|x\|\,\|\sigma(a)(a^*\otimes 1)-1\|_{\omega\otimes\omega_\xi}\,. \tag{14}$$

Moreover, a straightforward computation gives us

$$\begin{aligned}\big|\|\gamma\|^2-\|\tilde{\gamma}\|^2\big| &\leq (1+\|a\|^2)\|a^*a-1\|\,\|\mathring{x}\|^2_{\omega\otimes\omega_\xi}\,,\\ |\langle\tilde{\mu},\tilde{\gamma}\rangle| &\leq (1+\|a\|^2)\|a^*a-1\|\,\|\mathring{x}\|^2_{\omega\otimes\omega_\xi}\,.\end{aligned}$$

Let us denote by Q_2 the projection onto the Hilbert space $\tilde{\mathcal{H}}_2$. As $a\in P_1$, we can easily see that

$$\|Q_2(\sigma(a)(JaJ\otimes 1)\cdot\gamma)\|\leq|\omega(a)|\|(JaJ\otimes 1)\cdot\gamma\|.$$

Thus, we obtain the following inequality

$$|\langle\tilde{\gamma},\gamma\rangle|\leq|\omega(a)|\|a\|\|\mathring{x}\|^2_{\omega\otimes\omega_\xi}.$$

Because $\mu\perp\eta$, we get immediately $|\langle\tilde{\zeta},\gamma\rangle|=|\langle\tilde{\gamma},\gamma\rangle|$. We still have to assess the quantity $|\langle\tilde{\zeta},\tilde{\gamma}\rangle|$. Before doing that, we can notice that as $a\in P_1$, $\sigma(a^*)(1\otimes E(x))(\Omega\otimes\xi)\in\mathbf{C}\Omega\oplus\tilde{\mathcal{H}}_1$. Hence

$$\begin{aligned}\langle\eta-\eta_0,\tilde{\gamma}\rangle &= \langle(1\otimes E(x))\cdot(\Omega\otimes\xi),\sigma(a)\cdot\gamma\cdot(a^*\otimes 1)\rangle\\ &= \langle\sigma(a^*)(1\otimes E(x))\cdot(\Omega\otimes\xi),\gamma\cdot(a^*\otimes 1)\rangle\\ &= 0.\end{aligned}$$

We can prove exactly in the same way that

$$\langle\tilde{\eta}-\tilde{\eta}_0,\tilde{\gamma}\rangle=0.$$

Now

$$\begin{aligned}\langle\tilde{\zeta},\tilde{\gamma}\rangle &= \langle\eta_0-\gamma-\tilde{\gamma},\tilde{\gamma}\rangle\\ &= \langle\eta-\gamma-\tilde{\gamma},\tilde{\gamma}\rangle\\ &= \langle\eta-\tilde{\eta},\tilde{\gamma}\rangle-\langle\gamma,\tilde{\gamma}\rangle+\langle\tilde{\eta}-\tilde{\gamma},\tilde{\gamma}\rangle\\ &= \langle\eta-\tilde{\eta},\tilde{\gamma}\rangle-\langle\gamma,\tilde{\gamma}\rangle+\langle\tilde{\eta}_0-\tilde{\gamma},\tilde{\gamma}\rangle\\ &= \langle\eta-\tilde{\eta},\tilde{\gamma}\rangle-\langle\gamma,\tilde{\gamma}\rangle+\langle\tilde{\mu},\tilde{\gamma}\rangle\end{aligned}$$

because $\langle\eta-\eta_0,\tilde{\gamma}\rangle=\langle\tilde{\eta}-\tilde{\eta}_0,\tilde{\gamma}\rangle=0$. Thus, thanks to Equation (14), we get an inequality for $|\langle\tilde{\zeta},\tilde{\gamma}\rangle|$. Of course

$$\begin{aligned}\|\mu\|^2+\|\gamma\|^2 &= \|\tilde{\zeta}+\gamma+\tilde{\gamma}\|^2\\ &\geq \|\tilde{\zeta}\|^2+\|\gamma\|^2+\|\tilde{\gamma}\|^2-2|\langle\tilde{\zeta},\gamma\rangle|-2|\langle\tilde{\zeta},\tilde{\gamma}\rangle|-2|\langle\gamma,\tilde{\gamma}\rangle|\\ &\geq 2\|\gamma\|^2-\big|\|\gamma\|^2-\|\tilde{\gamma}\|^2\big|-2|\langle\tilde{\zeta},\gamma\rangle|-2|\langle\tilde{\zeta},\tilde{\gamma}\rangle|-2|\langle\gamma,\tilde{\gamma}\rangle|.\end{aligned}$$

Finally, noticing that $\|\mathring{x}\|_{\omega\otimes\omega_\xi}\leq\|x\|_{\omega\otimes\omega_\xi}$ because $E=\omega\otimes\mathrm{id}$, and using all the previous inequalities we obtained above, we have

$$\|\gamma\|^2\leq\|\mu\|^2+2\|a\|^3\|[\sigma(a),x]\|_{\omega\otimes\omega_\xi}\|\mathring{x}\|_{\omega\otimes\omega_\xi}+\mathcal{C}(a)\|x\|\|\mathring{x}\|_{\omega\otimes\omega_\xi}. \tag{15}$$

We can do now exactly the same thing as before with b and c instead of a. Indeed, let $\eta'=\sigma(b)\cdot\eta\cdot(b^*\otimes 1)$, $\eta''=\sigma(c)\cdot\eta\cdot(c^*\otimes 1)$, $\mu'=\sigma(b)\cdot\mu\cdot(b^*\otimes 1)$ and $\mu''=\sigma(c)\cdot\mu\cdot(c^*\otimes 1)$. We define $\zeta'=\eta_0-\mu-\mu'-\mu''$. We find that

$$\begin{aligned}\|\mu\|^2+\|\gamma\|^2 \geq\ & 3\|\mu\|^2-\big|\|\mu\|^2-\|\mu'\|^2\big|-\big|\|\mu\|^2-\|\mu''\|^2\big|-2|\langle\zeta',\mu\rangle|-2|\langle\zeta',\mu'\rangle|\\ &-2|\langle\zeta',\mu''\rangle|-2|\langle\mu,\mu'\rangle|-2|\langle\mu,\mu''\rangle|-2|\langle\mu',\mu''\rangle|.\end{aligned}$$

Once again, we assess the negative terms and we get

$$\begin{aligned}(16)\qquad 2\|\mu\|^2 \quad \leq \quad & \|\gamma\|^2 + 2\|b\|^3\|[\sigma(b), x]\|_{\omega\otimes\omega_\xi} + 2\|c\|^3\|[\sigma(c), x]\|_{\omega\otimes\omega_\xi}\|\mathring{x}\|_{\omega\otimes\omega_\xi} \\ & + (\mathcal{C}(b) + \mathcal{C}(c) + 6|\omega(c^*b)|\|c^*b\|)\,\|x\|\|\mathring{x}\|_{\omega\otimes\omega_\xi}.\end{aligned}$$

As $\|\mu\|^2 + \|\gamma\|^2 = \|\mathring{x}\|^2_{\omega\otimes\omega_\xi}$, a combination of inequalities (15) and (16) gives the inequality of the lemma. □

Let P_i be two free Araki-Woods factors endowed with their free quasi-free state ω_i, $(i = 1, 2)$ such that $(P, \omega) = (P_1, \omega_1) * (P_2, \omega_2)$. We know, thanks to Lemma 4.3 in [**44**], that P_1 contains a bounded sequence of elements (a_n) analytic w.r.t. the state ω and which satisfy $\|\sigma^{\omega}_{i/2}(a_n) - a_n\| \to 0$, $\|a_n^* a_n - 1\| \to 0$, $\omega(a_n) \to 0$ and $\sigma(a_n)(a_n^* \otimes 1) - 1 \to 0$ $*-$strongly. We know besides that P_2 contains bounded sequences (b_n) and (c_n) which satisfy the same condition as (a_n) and the condition $\omega(c_n^* b_n) \to 0$. We can now state the following proposition, used in proof of Lemma 2.20 :

Proposition 2.26. *Let (x_k) be a bounded sequence in $C \overset{\sigma}{\subset} M = P \otimes L^\infty(X) \otimes B(L^2(\mathbf{R}))$ which almost commutes with $\sigma(P)$. Then $E(x_k) - x_k \to 0$ $*-$strongly.*

Proof. The proof of this proposition is a straightforward application of Lemma 2.25. Let $\xi \in \mathcal{K}$, $\|\xi\| = 1$, $\varepsilon > 0$, and $M = \sup\{\|a_n\|, \|b_n\|, \|c_n\|, \|x_k\|\}$. As $\sigma(a_n)(a_n^* \otimes 1) - 1 \to 0$ $*$-strongly, it is clear that $\|\sigma(a_n)(a_n^* \otimes 1) - 1\|_{\omega\otimes\omega_\xi} \to 0$. So, there exists $n \in \mathbf{N}$ such that $\mathcal{F}(a_n, b_n, c_n) < \frac{\varepsilon}{M}$. Then with this particular n, as (x_k) almost commutes with $\sigma(P)$, there exists $k_0 \in \mathbf{N}$ such that for any $k \geq k_0$,

$$\max\left\{\|[\sigma(a_n), x_k]\|_{\omega\otimes\omega_\xi}, \|[\sigma(b_n), x_k]\|_{\omega\otimes\omega_\xi}, \|[\sigma(c_n), x_k]\|_{\omega\otimes\omega_\xi}\right\} < \frac{\varepsilon}{14M^3}.$$

Hence, we have proved that

$$\forall \xi \in \mathcal{K}\ \forall \varepsilon > 0\ \exists\, k_0 \in \mathbf{N}\ \forall k \geq k_0\ \ \|x_k - E(x_k)\|_{\omega\otimes\omega_\xi} \leq \varepsilon.$$

That means exactly that $x_k - E(x_k) \to 0$ strongly in M. We can do the same thing for x_k^*. Thus, we have proved that $x_k - E(x_k) \to 0$ $*-$strongly in M. □

CHAPITRE 3

On some Free Products of von Neumann Algebras which are Free Araki-Woods Factors

This chapter is the text of a paper [**16**] submitted to *International Mathematics Research Notices.*

We prove that certain free products of factors of type I and other von Neumann algebras with respect to nontracial, almost periodic states are almost periodic free Araki-Woods factors. In particular, they have the free absorption property and Connes' Sd invariant completely classifies these free products. For example, for $\lambda, \mu \in]0,1[$, we show that

$$(M_2(\mathbf{C}), \omega_\lambda) * (M_2(\mathbf{C}), \omega_\mu)$$

is isomorphic to the free Araki-Woods factor whose Sd invariant is the subgroup of $\mathbf{R}_+^*$ generated by λ and μ. Our proofs are based on algebraic techniques and amalgamated free products. These results give some answers to questions of Dykema and Shlyakhtenko.

1. Introduction

In [**8**] and [**11**], Dykema investigated free products of finite dimensional and other von Neumann algebras with respect to nontracial faithful states. We are interested in free products of factors of type I. In this respect, we recall Theorem 1 of [**8**] in the particular case of factors of type I (see Proposition 7.3 of [**8**] for a precise statement in full generality).

Theorem 3.1 (Dykema, [**8**]). *Let*

$$(\mathcal{M}, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$$

be the von Neumann algebra free product of factors of type I *with respect to faithful states, at least one of which is nontracial. Then $\mathcal{M}$ is a full factor of type* III *and ϕ is an almost periodic faithful state whose centralizer is isomorphic to the type* II_1 *factor $L(\mathbf{F}_\infty)$. The point spectrum of the modular operator Δ_ϕ of ϕ, is equal to the subgroup of $\mathbf{R}_+^*$ generated by the union of the point spectra of Δ_{ϕ_1} and of Δ_{ϕ_2}. Thus in Connes' classification, $\mathcal{M}$ is always a factor of type* III_λ*, with $0 < \lambda \leq 1$.*

The fact that ϕ is an almost periodic faithful state [**3**] is an easy consequence of basic results on free products (see Section 4 in [**8**] for further details). Of course, the fact that the centralizer of ϕ is isomorphic to the type II_1 factor $L(\mathbf{F}_\infty)$ is the most difficult part of the theorem. To prove this, Dykema uses sophisticated algebraic techniques on free products that he developed also in [**10**] and [**12**]. Finally, the fact that $\mathcal{M}$ is of type III follows from results of Section 4 in [**8**].

However, natural questions were left unanswered. Dykema asked in Question 9.1 [**8**], whether the type III_λ factors that are obtainable by taking various free products of finite dimensional or hyperfinite algebras are isomorphic to each other, and more precisely whether they are isomorphic to the factor of Rădulescu [**28**],

$$(L(\mathbf{Z}), \tau_{\mathbf{Z}}) * (M_2(\mathbf{C}), \omega_\lambda)$$

where $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda+1)$ for $i, j \in \{0,1\}$. Furthermore, he asked in Question 9.3 [**8**], whether the full factors of type III_1 having the same Sd invariant that are obtainable by taking free products of various finite dimensional or hyperfinite algebras are isomorphic to each other. We will see that we partially give positive answers to these questions.

In [**39**], Shlyakhtenko introduced a new class of full factors of type III. His idea is to give a version of the CAR[1] functor and of the associated quasi-free states in the framework of Voiculescu's free probability theory [**48**]. In Section 2, we recall his construction. But roughly speaking, we can say that to each real Hilbert space $H_{\mathbf{R}}$ and to each orthogonal representation (U_t) of $\mathbf{R}$ on $H_{\mathbf{R}}$, he associated a factor $\Gamma(H_{\mathbf{R}}, U_t)''$ called the *free Araki-Woods factor*. He proved that $\Gamma(H_{\mathbf{R}}, U_t)''$ is a type III factor except if $U_t = \mathrm{id}$ for all $t \in \mathbf{R}$. The restriction to $\Gamma(H_{\mathbf{R}}, U_t)''$ of the vacuum state denoted by φ_U and called the *free quasi-free state* is faithful. Moreover, he proved that φ_U is an almost periodic state iff the orthogonal representation (U_t) is almost periodic. Recall in this respect the following definition :

Definition 3.2 (Connes, [**3**]). *Let M be a von Neumann algebra with separable predual which has almost periodic weights. The* Sd invariant *of M is defined as the intersection over all the almost periodic, faithful, normal, semifinite weights φ of the point spectra of the modular operators Δ_φ.*

Connes proved that for a factor of type III, $\mathrm{Sd}(M)$ is a countable subgroup of $\mathbf{R}_+^*$ [**3**]. In the almost periodic case, using a powerful tool called the *matricial model*, Shlyakhtenko obtains this remarkable result :

Theorem 3.3 (Shlyakhtenko, [**34, 39**]). *Let (U_t) be a nontrivial almost periodic orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}} \geq 2$. Let A be the infinitesimal generator of (U_t) on $H_{\mathbf{C}}$, the complexified Hilbert space of $H_{\mathbf{R}}$. Denote $M = \Gamma(H_{\mathbf{R}}, U_t)''$. Let $\Gamma \subset \mathbf{R}_+^*$ be the subgroup generated by the point spectrum of A. Then, M only depends on Γ up to state-preserving isomorphisms.*

Conversely, the group Γ coincides with the Sd *invariant of the factor M. Consequently,* Sd *completely classifies the almost periodic free Araki-Woods factors. Moreover, the centralizer of the free quasi-free state φ_U is isomorphic to the type* II_1 *factor $L(\mathbf{F}_\infty)$.*

He proved also that the (unique) free Araki-Woods factor of type III_λ, denoted by $(T_\lambda, \varphi_\lambda)$, is isomorphic to the factor of Rădulescu [**28**] and "freely absorbs" $L(\mathbf{F}_\infty)$. Since the free Araki-Woods factors satisfy free absorption properties, Shlyakhtenko asked whether the free products of matrix algebras $(A_1, \phi_1) * (A_2, \phi_2)$ are stable by taking free products with $L(\mathbf{Z})$, in other words whether they are free Araki-Woods factors.

We give in this paper a positive answer to the question of Shlyakhtenko for certain free products of matrix algebras and other von Neumann algebras. Thanks to Theorem 6.6 of [**39**], It partially gives positive answers to Questions 9.1 and 9.3 of Dykema [**8**]. Before giving the results, we must introduce a few notation. For an almost periodic state ϕ, we denote by $\mathrm{Sd}(\phi)$ the subgroup of $\mathbf{R}_+^*$ generated by the point spectrum of the modular operator Δ_ϕ. On $B(\ell^2(\mathbf{N}))$, we denote by ψ_λ the state given by $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda)$ for $i, j \in \mathbf{N}$. For $\beta \in]0,1[$, we denote by $(\mathbf{C}^2, \tau_\beta)$ the algebra generated by a projection q with $\tau_\beta(q) = \beta$. The hyperfinite type II_1 factor together with its trace is denoted by $(\mathcal{R}, \tau)$. At last, we denote by $(T_\Gamma, \varphi_\Gamma)$ the unique (up to state-preserving isomorphism) almost periodic free Araki-Woods factor whose Sd invariant is exactly Γ (see Notation 3.13 for further details).

[1]Canonical Anticommutation Relations

DEFINITION 3.4. *Let $\rho : (B, \phi_B) \hookrightarrow (A, \phi_A)$ be an embedding of von Neumann algebras. We shall say that ρ is* modular *if it is state-preserving and if $\rho(B)$ is globally invariant under the modular group $(\sigma_t^{\phi_A})$.*

Our main results are given in the following theorem.

THEOREM 3.5. *Let (A_i, ϕ_i) $i = 1, 2$, be two von Neumann algebras endowed with a faithful, normal, almost periodic state ϕ_i, such that for $i = 1, 2$*

$$(A_i, \phi_i) * (L(\mathbf{Z}), \tau_{\mathbf{Z}}) \cong (T_{\mathrm{Sd}(\phi_i)}, \varphi_{\mathrm{Sd}(\phi_i)}).$$

Let $\Gamma = < \mathrm{Sd}(\phi_1), \mathrm{Sd}(\phi_2) >$ be the subgroup of $\mathbf{R}_+^*$ generated by $\mathrm{Sd}(\phi_1)$ and $\mathrm{Sd}(\phi_2)$. Assume that for some $\lambda, \beta \in]0, 1[$, there exist modular embeddings

$$\begin{aligned} (M_2(\mathbf{C}), \omega_\lambda) &\hookrightarrow (A_1, \phi_1) \\ (\mathbf{C}^2, \tau_\beta) &\hookrightarrow (A_2, \phi_2), \end{aligned}$$

such that $\lambda/(\lambda + 1) \leq \min\{\beta, 1 - \beta\}$. Then

$$(T_\Gamma, \varphi_\Gamma) \cong (A_1, \phi_1) * (A_2, \phi_2).$$

In particular, for any $\lambda, \mu \in]0, 1[$, $(M_2(\mathbf{C}), \omega_\lambda) * (M_2(\mathbf{C}), \omega_\mu)$, $(M_2(\mathbf{C}), \omega_\lambda) * (\mathcal{R}, \tau)$ and $(B(\ell^2(\mathbf{N})), \psi_\lambda) * (\mathcal{R}, \tau)$ are free Araki-Woods factors.

The paper is organized as follows. Section 2 is devoted to a few reminders on free products and free Araki-Woods factors. In Section 3, we show that the free product $(M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$ is isomorphic in a state-preserving way to $(T_\lambda, \varphi_\lambda)$, whenever $\lambda/(\lambda + 1) \leq \min\{\beta, 1 - \beta\}$. In Section 4, we prove Theorem 3.5 using the "machinery" of amalgamated free products. Section 5 is devoted to technical proofs on amalgamated free products. Finally, Section 6 is a remark.

2. Conventions and Preliminary Background

Throughout this paper, we will be working with free products of von Neumann algebras with respect to states. For the convenience of the reader, it is useful to remind the following notation :

NOTATION 3.6. If (M, φ) and (N, ψ) are von Neumann algebras endowed with states φ and ψ, the notation $(M, \varphi) \cong (N, \psi)$ means that there exists a $*-$isomorphism $\alpha : M \to N$ such that $\psi\alpha = \varphi$.

We remind this well-known proposition concerning free products of von Neumann algebras with respect to states.

PROPOSITION 3.7. [**48**] *Let (M_i, φ_i) be a family of von Neumann algebras endowed with faithful normal states. Then, there exists, up to state-preserving isomorphism, a unique von Neumann algebra (M, φ) endowed with a faithful normal state φ such that*

- *(M_i, φ_i) embeds into (M, φ) in a state-preserving way,*
- *M is generated by the family of subalgebras (M_i) which is a free family in (M, φ).*

The free product *of (M_i, φ_i) is denoted by $(M, \varphi) = \underset{i \in I}{*} (M_i, \varphi_i)$.*

NOTATION 3.8. [**12**] For von Neumann algebras A and B, with states φ_A and φ_B, the von Neumann algebra

$$\underset{\alpha}{\overset{p}{A}} \oplus \underset{\beta}{\overset{q}{B}}$$

where α, $\beta \geq 0$ and $\alpha + \beta = 1$, will denote the algebra $A \oplus B$ whose associated state is $\varphi(a, b) = \alpha\varphi_A(a) + \beta\varphi_B(b)$. Moreover, $p \in A$ and $q \in B$ are projections corresponding to the identity elements of A and B.

Now, we want to remind the construction of free Araki-Woods factors [**39**]. Let $H_{\mathbf{R}}$ be a real Hilbert space and let (U_t) be an orthogonal representation of $\mathbf{R}$ on $H_{\mathbf{R}}$. Let $H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexified Hilbert space. If A is the infinitesimal generator of (U_t) on H, we remind that $j : H_{\mathbf{R}} \to H$ defined by $j(\zeta) = (\frac{1}{A^{-1}+1})^{1/2}\zeta$ is an isometric embedding of $H_{\mathbf{R}}$ into H. Let $K_{\mathbf{R}} = j(H_{\mathbf{R}})$. Introduce the *full Fock space* of H :

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector Ω is called *vacuum vector*. For any $\xi \in H$, we have the left creation operator

$$l(\xi) : \mathcal{F}(H) \to \mathcal{F}(H) : \begin{cases} l(\xi)\Omega = \xi, \\ l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

For any $\xi \in H$, we denote by $s(\xi)$ the real part of $l(\xi)$ given by

$$s(\xi) = \frac{l(\xi) + l(\xi)^*}{2}.$$

The crucial result of Voiculescu [**48**] claims that the distribution of the operator $s(\xi)$ with respect to the vacuum vector state $\varphi(x) = \langle x\Omega, \Omega\rangle$ is the semicircular law of Wigner supported on the interval $[-\|\xi\|, \|\xi\|]$.

Definition 3.9. *Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ $(\dim H_{\mathbf{R}} \geq 2)$. The* free Araki-Woods factor *denoted by $\Gamma(H_{\mathbf{R}}, U_t)''$ is defined by*

$$\Gamma(H_{\mathbf{R}}, U_t)'' = \{s(\xi), \xi \in K_{\mathbf{R}}\}''.$$

The vector state $\varphi_U(x) = \langle x\Omega, \Omega\rangle$ is called free quasi-free state.

As we said previously, the free Araki-Woods factors provide many new examples of full factors of type III [**1, 5, 34**]. We can summarize the general properties of free Araki-Woods factors in the following theorem (see also [**44**]) :

Theorem 3.10 (Shlyakhtenko, [**34, 37, 38, 39**]). *Let (U_t) be an orthogonal representation of $\mathbf{R}$ on the real Hilbert space $H_{\mathbf{R}}$ with $\dim H_{\mathbf{R}} \geq 2$. Denote $M = \Gamma(H_{\mathbf{R}}, U_t)''$.*

(1) *M is a full factor.*
(2) *M is of type II_1 iff $U_t = id$ for every $t \in \mathbf{R}$.*
(3) *M is of type III_λ $(0 < \lambda < 1)$ iff (U_t) is periodic of period $\frac{2\pi}{|\log \lambda|}$.*
(4) *M is of type III_1 in the other cases.*
(5) *The factor M has almost periodic states iff (U_t) is almost periodic.*

Let $H_{\mathbf{R}} = \mathbf{R}^2$ and $0 < \lambda < 1$. Let

$$U_t = \begin{pmatrix} \cos(t\log\lambda) & -\sin(t\log\lambda) \\ \sin(t\log\lambda) & \cos(t\log\lambda) \end{pmatrix}. \tag{17}$$

Notation 3.11. [**39**] Denote $(T_\lambda, \varphi_\lambda) := \Gamma(H_{\mathbf{R}}, U_t)''$ where $H_{\mathbf{R}} = \mathbf{R}^2$ and (U_t) is given by (17).

Using a powerful tool called the *matricial model*, Shlyakhtenko was able to prove the following isomorphism

$$(T_\lambda, \varphi_\lambda) \cong (B(\ell^2(\mathbf{N})), \psi_\lambda) * (L^\infty[-1,1], \mu),$$

where $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda)$, $i, j \in \mathbf{N}$, and μ a nonatomic measure on $[-1,1]$. He also proved that $(T_\lambda, \varphi_\lambda)$ is isomorphic to the factor of type III_λ introduced by Rădulescu in [**28**]. Namely,

$$(T_\lambda, \varphi_\lambda) \cong (M_2(\mathbf{C}), \omega_\lambda) * (L^\infty[-1,1], \mu),$$

where $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda+1)$, $i, j \in \{0,1\}$, and μ a nonatomic measure on $[-1,1]$. Moreover, he showed that $(T_\lambda, \varphi_\lambda)$ has a good behaviour when it is compressed by a "right" projection. More precisely, denote $(C, \psi) := (B(\ell^2(\mathbf{N})), \psi_\lambda) * (L^\infty[-1,1], \mu)$ and $(D, \omega) := (M_2(\mathbf{C}), \omega_\lambda) * (L^\infty[-1,1], \mu)$. The following proposition is an easy consequence of proofs of Theorems 5.4 and 6.7 of [**39**]. It will be useful in Section 3.

Proposition 3.12. *Let (C, ψ), (D, ω) defined as above and $e_{00} \in B(\ell^2(\mathbf{N})) \subset C$, p_{00}, $p_{11} \in M_2(\mathbf{C}) \subset D$. Then*

$$\begin{aligned} (T_\lambda, \varphi_\lambda) &\cong (e_{00}Ce_{00}, \frac{1}{\psi(e_{00})}\psi) \\ &\cong (p_{00}Dp_{00}, \frac{1}{\omega(p_{00})}\omega) \\ &\cong (p_{11}Dp_{11}, \frac{1}{\omega(p_{11})}\omega). \end{aligned}$$

When the representation (U_t) is assumed to be almost periodic, we have seen (Theorem 3.3) that $\Gamma \subset \mathbf{R}^*_+$, the subgroup generated by the point spectrum of A completely classified the free Araki-Woods factor $\Gamma(U_t, H_\mathbf{R})''$.

Notation 3.13. For any nontrivial countable subgroup $\Gamma \subset \mathbf{R}^*_+$, we shall denote by $(T_\Gamma, \varphi_\Gamma)$ the unique (up to state-preserving isomorphism) almost periodic free Araki-Woods factor whose Sd invariant is exactly Γ. Of course, φ_Γ is its free quasi-free state. If $\Gamma = \lambda^\mathbf{Z}$ for $\lambda \in]0,1[$, then $(T_\Gamma, \varphi_\Gamma)$ is of type III_λ ; in this case, it will be simply denoted by $(T_\lambda, \varphi_\lambda)$ [**39**], as in Notation 3.11. Theorem 6.4 in [**39**] gives the following formula :

$$(T_\Gamma, \varphi_\Gamma) \cong \mathop{*}_{\gamma \in \Gamma} (T_\gamma, \varphi_\gamma).$$

3. Study of $(M_2(\mathbf{C}),\omega_\lambda) * \left(\underset{\beta}{\mathbf{C}} \oplus \underset{1-\beta}{\mathbf{C}}\right)$ for $\lambda/(\lambda+1) \le \min\{\beta, 1-\beta\}$

For any $\beta \in]0,1[$, the von Neumann algebra $\underset{\beta}{\mathbf{C}} \oplus \underset{1-\beta}{\mathbf{C}}$ is simply denoted by $(\mathbf{C}^2, \tau_\beta)$. Let $\lambda \in]0,1[$ and denote $\alpha = \lambda/(\lambda+1)$. We remind that the faithful state ω_λ on $M_2(\mathbf{C})$ is defined as follows : $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda+1)$, for i, $j \in \{0,1\}$. The aim of this section is to prove the following theorem :

Theorem 3.14. *If $\alpha = \lambda/(\lambda+1) \le \min\{\beta, 1-\beta\}$, then*

$$(M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta) \cong (T_\lambda, \varphi_\lambda). \tag{18}$$

Notation 3.15. The von Neumann algebra of the left-hand side of (18) together with its free product state will be denoted by (M, ω).

To prove Theorem 3.14, we will need the following result due to Voiculescu [**51**] (see also [**10, 12**]) which gives a precise picture of the von Neumann algebra generated by two projections p and q free with respect to a faithful trace. More precisely, THEOREM 3.16 (Voiculescu, [**51**]). *Let* $0 < \alpha \leq \min\{\beta, 1-\beta\} < 1$. *Then*

$$\left(\underset{\alpha}{\overset{p}{\mathbf{C}}} \oplus \underset{1-\alpha}{\overset{1-p}{\mathbf{C}}}\right) * \left(\underset{\beta}{\overset{q}{\mathbf{C}}} \oplus \underset{1-\beta}{\overset{1-q}{\mathbf{C}}}\right) \cong \underset{\beta-\alpha}{\overset{(1-p)\wedge q}{\mathbf{C}}} \oplus \underbrace{\left(L^\infty\left(\left[0, \frac{\pi}{2}\right], \nu\right) \otimes M_2\right)}_{2\alpha} \oplus \underset{1-\alpha-\beta}{\overset{(1-p)\wedge(1-q)}{\mathbf{C}}}, \tag{19}$$

where ν *is a probability measure without atoms on* $[0, \pi/2]$, *and* $L^\infty([0,\pi/2], \nu)$ *has trace given by integration against* ν. *In the picture of the right-hand side of* (19), *we have*

$$p = 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0,$$

$$q = 1 \oplus \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \oplus 0,$$

where $\theta \in [0, \pi/2]$.

REMARK 3.17. The von Neumann algebra of the right-hand side of (19) together with its trace will be denoted by (N, τ). With the previous notations, (N, τ) embeds into (M, ω) in a state-preserving way. We shall assume that $N \subset M$, and $p = p_{11}$. Consequently, p is the "smallest" projection with respect to the state ω, i.e. $\omega(p) = \alpha = \lambda/(\lambda+1)$. As in [**10**], let

$$x = 0 \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus 0 = \mathrm{pol}((1-p)qp),$$

where "pol" means "polar part of". Then x is a partial isometry from p into $1-p$, i.e. $x^*x = p$ and $xx^* \leq 1-p$. Let z be the projection

$$z = 1 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0.$$

Note that $z \leq 1-p$. We see that N is the von Neumann generated by pqp together with x and z. We shall denote $N = W^*(pqp, x, z)$.

DEFINITION 3.18. [**10**] *Let* $(S_\iota)_{\iota\in I}$ *be a family of subsets of a unital algebra* $A \ni 1$. *A nontrivial* traveling product *in* $(S_\iota)_{\iota\in I}$ *is a product* $a_1 \cdots a_n$ *such that* $a_j \in S_{\iota_j}$ $(1 \leq j \leq n)$ *and* $\iota_1 \neq \iota_2 \neq \cdots \neq \iota_{n-1} \neq \iota_n$. *The* trivial traveling product *is the identity element* 1. *The set of all traveling products in* $(S_\iota)_{\iota\in I}$, *including the trivial one is denoted by* $\Lambda((S_\iota)_{\iota\in I})$. *If* $|I| = 2$, *we will call traveling products* alternating products.

We are now ready to prove the following proposition; it gives a precise picture of the compression of the von Neumann algebra (M, ω) by the projection p. The proof is based on algebraic techniques developed in [**8, 10, 12**], and techniques of computation of $*-$distributions developed in [**39**] and [**50**].

PROPOSITION 3.19. *Let* $(M, \omega) = (M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$ *and* $p = p_{11} \in M_2(\mathbf{C})$. *Assume as in Theorem* 3.16, *that* $\alpha = \lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. *Then*

$$\left(pMp, \frac{1}{\omega(p)}\omega\right) \cong L(\mathbf{Z}) * \left((\mathbf{C}^2, \tau_\delta) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)\right),$$

where $(\mathbf{C}^2, \tau_\delta) = \underset{\delta}{\mathbf{C}} \oplus \underset{1-\delta}{\mathbf{C}}$ *with* $\delta = \frac{1-\frac{\beta}{1-\alpha}}{1-\lambda}$ *and* $\psi_\lambda(e_{ij}) = \delta_{ij}\lambda^j(1-\lambda)$, *for* $i, j \in \mathbf{N}$.

PROOF. Let $(M, \omega) = (M_2, \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$. Let p and q be the projections in M such that $N = W^*(p, q)$ as in Theorem 3.16; p and q are free in M with respect to ω and $\omega(p) = \alpha = \lambda/(\lambda+1)$, $\omega(q) = \beta$. Let x and z as in Remark 3.17. We know that $N =$

$W^*(pqp, x, z)$. Denote $u = p_{01} \in M_2(\mathbf{C})$ the partial isometry from p to $1-p$, i.e $u^*u = p$ and $uu^* = 1-p$. Then, thanks to Lemma 5.3 from [**48**]

$$pMp = W^*(pqp, u^*x, u^*zu).$$

Denote $v = u^*x$ and $P = u^*zu$. Since $v^*v = p$ and $vv^* \leq p$, v is an isometry in pMp. Moreover, since $Pv = u^*zuu^*x = u^*zx = u^*x = v$, we get $vv^* \leq P$. Denote $\omega_p = \frac{1}{\omega(p)}\omega$ the canonical state on pMp. First, we are going to compute the $*-$distributions of the elements v and vP in pMp with respect to ω_p.

LEMMA 3.20. *Let $\gamma = \beta(\lambda+1) = \beta/(1-\alpha)$. For any k, $l \in \mathbf{N}$,*

$$\omega_p(v^k(v^*)^l) = \delta_{kl}\lambda^k, \tag{20}$$

$$\omega_p(v^kP(v^*)^l) = \delta_{kl}\lambda^k\gamma. \tag{21}$$

PROOF OF LEMMA 3.20. **Step (0)**. First, we review the "algebraic trick" of Dykema [**10**]. Denote $a = p-\omega(p)$ and $b = q-\omega(q)$; we have $N = \overline{\text{span}}^w\Lambda(\{a\},\{b\})$. Let $w \in N$ such that $\omega(w) = \omega(pw) = 0$. By Kaplansky Density Theorem, w is the s.o.-limit of a bounded sequence in span $\Lambda(\{a\},\{b\})$. Note that since a and b are free and $\omega(a) = \omega(b) = 0$, if $y \in$ span $\Lambda(\{a\},\{b\})$, then $\omega(y)$ is equal to the coefficient of 1 in y. Since $\omega(w) = 0$, we may choose that approximating sequence in span $(\Lambda(\{a\},\{b\})\backslash\{1\})$. Moreover, since $\omega(pw) = 0$, we may also assume that each coefficient of a be zero, i.e. we have a bounded approximating sequence for w of elements of span $(\Lambda(\{a\},\{b\})\backslash\{1,a\})$.

Step (1). We prove now Equation (20). Assume $k \geq 1$ and $l = 0$, then $v^k = (u^*x)^k$ is a nontrivial alternating product in $\{u^*\}$ and $\{x\}$. Since $\omega(x) = \omega(px) = 0$, x is a s.o.-limit of a bounded sequence in span $(\Lambda(\{a\},\{b\})\backslash\{1,a\})$. So to show that $\omega_p(v^k) = 0$, it suffices to show that if s is a nontrivial alternating product in span $(\Lambda(\{a\},\{b\})\backslash\{1,a\})$ and $\{u^*\}$ then $\omega(s) = 0$. But since $u^*a = -\alpha u^*$ and $au^* = (1-\alpha)u^*$, regrouping gives a nontrivial alternating product in $\{a, u^*\}$ and $\{b\}$, hence by freeness $\omega(s) = 0$. We get also immediatly $\omega_p((v^*)^l) = 0$. Assume at last $k \geq 1$ and $l \geq 1$, then $v^k(v^*)^l = (u^*x)^{k-1}u^*xx^*u(x^*u)^{l-1}$. Let $y = xx^* - \alpha 1 + \lambda a$. Since $\omega(xx^*) = \omega(x^*x) = \alpha$ and $py = 0$, $\omega(y) = \omega(py) = 0$, hence y is a s.o.-limit of a bounded sequence in span $(\Lambda(\{a\},\{b\})\backslash\{1,a\})$. Replacing in $(u^*x)^{k-1}u^*xx^*u(x^*u)^{l-1}$ the term xx^* by $y + \alpha 1 - \lambda a$, and since $u^*au = -\alpha p$, we have

$$\omega_p(v^k(v^*)^l) = \omega_p((u^*x)^{k-1}u^*yu(x^*u)^{l-1}) + \lambda\omega_p((u^*x)^{k-1}p(x^*u)^{l-1}). \tag{22}$$

To prove $\omega_p((u^*x)^{k-1}u^*yu(x^*u)^{l-1}) = 0$, it suffices to show that if r is a nontrivial alternating product in span $(\Lambda(\{a\},\{b\})\backslash\{1,a\})$ and $\{u^*,u\}$ then $\omega(s) = 0$. But for the same reasons as above, regrouping gives a nontrivial alternating product in $\{a,u^*,u\}$ and $\{b\}$, hence by freeness $\omega(r) = 0$. If in Equation (20), $k \neq l$, then applying Equation (22) several times we eventually get $\omega_p(u^*x\cdots u^*x)$ or $\omega_p(x^*u\cdots x^*u)$, both of which are zero. If $k = l$, then we eventually get $\lambda^k\omega_p(p) = \lambda^k$. Thus Equation (20) holds.

Step (2). We prove at last Equation (21). Since $\omega(u^*zu) = \lambda\omega(z)$, and $\gamma = \beta(\lambda+1) = \beta/(1-\alpha)$, we get $\omega_p(P) = \gamma$. Assume k, $l \geq 0$, then $v^kP(v^*)^l = (u^*x)^ku^*zu(x^*u)^l$. Since $\omega(z) = \beta$ and $pz = 0$, $y = z - \beta 1 + \gamma a$ satisfies $\omega(y) = \omega(py) = 0$. Consequently, y is a s.o.-limit of a bounded sequence in span$(\Lambda(\{a\},\{b\}\backslash\{1,a\})$. Replacing in the product $(u^*x)^ku^*zu(x^*u)^l$ the term z by $y + \beta 1 - \gamma a$, and since $u^*au = -\alpha p$, we have

$$\omega_p(v^kP(v^*)^l) = \omega_p((u^*x)^ku^*yu(x^*u)^l) + \gamma\omega_p((u^*x)^kp(x^*u)^l).$$

For the same reasons, $\omega_p((u^*x)^ku^*yu(x^*u)^l) = 0$ and $\omega_p(v^kP(v^*)^l) = \gamma\omega_p(v^k(v^*)^l)$. Thus Equation (21) holds. $\square$

LEMMA 3.21. *In pMp, the von Neumann subalgebras $W^*(pqp)$ and $W^*(v,P)$ are $*-$free with respect to ω_p.*

PROOF OF LEMMA 3.21. Lemma 4.2 in [**39**] inspired us to prove Lemma 3.21. It is slightly more complicated because here, in some sense, the assumptions are weaker and we must additionally deal with the projection P. To overcome these difficulties, we will use the "algebraic trick" [**10**] mentioned above.

Let $B = W^*(pqp)$ be the von Neumann subalgebra of pMp generated by pqp and $C = W^*(v,P)$ the von Neumann subalgebra of pMp generated by v and P. Let $g_k = (pqp)^k - \omega_p((pqp)^k)p$ for $k \geq 1$. Let $W_{kl} = v^k(v^*)^l - \delta_{kl}\lambda^k p$, $W'_{rs} = v^r P(v^*)^s - \gamma\delta_{rs}\lambda^r p$ for $k,\ l,\ r,\ s \in \mathbf{N}$, $k+l>0$. Since

$$\begin{aligned} B &= \overline{\operatorname{span}}^w\{p, g_k \mid k \geq 1\}, \\ C &= \overline{\operatorname{span}}^w\{p, W_{kl}, W'_{rs} \mid k,l,r,s \in \mathbf{N}, k+l>0\}, \end{aligned}$$

it follows that to check freeness of B and C, we must show that

$$\omega_p(\underbrace{b_0 w_1 b_1 \cdots w_n b_n}_{W}) = 0 \tag{23}$$

where

$$\begin{aligned} b_j &= g_{m_j}, && (24)\\ w_j &= W_{k_j l_j} && (25)\\ \text{or } w_j &= W'_{r_j s_j}, && (26) \end{aligned}$$

with $k_j, l_j, m_j, r_j, s_j \in \mathbf{N}, k_j + l_j > 0, m_j > 0$ for all j, except possibly b_0 and/or b_n are equal to 1. We shall prove Equation (23) under a weaker assumption, which is that w_j is also allowed to be

$$(u^*x)^{s_j} u^* y u (x^*u)^{t_j}, \tag{27}$$

$s_j, t_j \geq 0$ and $y = xx^* - \alpha 1 + \lambda a$ or $y = z - \beta 1 + \gamma a$ as in proof of Lemma 3.20.

We will denote by $W = b_0 w_1 b_1 \cdots w_n b_n$ such a word with w_j as in Equation (25) or (26). Let w_j be as in Equation (25) with both k_j and l_j nonzero and let $y_1 = xx^* - \alpha 1 + \lambda a$. We will replace this w_j by

$$\begin{aligned} w_j &= ((u^*x)^{k_j-1} u^* y_1 u (x^*u)^{l_j-1}) \\ &\quad + (\lambda (u^*x)^{k_j-1} (x^*u)^{l_j-1} - \delta_{k_j l_j} \lambda^{k_j}) \\ &= A_j + B_j. \end{aligned}$$

Let now w_j be as in Equation (26) with both r_j and s_j nonzero and let $y_2 = z - \beta 1 + \gamma a$. We will replace this w_j by

$$\begin{aligned} w_j &= ((u^*x)^{r_j} u^* y_2 u (x^*u)^{s_j}) \\ &\quad + \gamma \left(\lambda (u^*x)^{r_j} (x^*u)^{s_j} - \delta_{r_j s_j} \lambda^{r_j} \right) \\ &= ((u^*x)^{r_j} u^* y_2 u (x^*u)^{s_j}) \\ &\quad + \gamma \left((u^*x)^{r_j-1} u^* y_1 u (x^*u)^{s_j-1} \right) \\ &\quad + \gamma \left(\lambda (u^*x)^{r_j-1} (x^*u)^{s_j-1} - \delta_{r_j s_j} \lambda^{r_j} \right) \\ &= A'_j + A''_j + C_j. \end{aligned}$$

After such replacements are done, w can be rewritten as a sum of terms, in which some w_j are replaced by A_j's, A'_j's, A''_j's, some by B_j's and some by C_j's. Consider the terms where all replacements are replacements by A_j's, A'_j's, A''_j's. These terms can be written as alternating products in $\Omega = \{x, x^*, y_1, y_2, g_k, xg_k, g_kx^*, xg_kx^*\}$ and $\{u, u^*\}$. But each element $h \in \Omega$ satisfies $\omega(h) = \omega(ph) = 0$, hence h is a s.o.-limit of a bounded sequence in $\operatorname{span}(\Lambda(\{a\},\{b\})\backslash\{1,a\})$. We use the same argument as before. To prove that ω_p is zero on such terms, it suffices to show that ω is zero on a nontrivial alternating product in $\operatorname{span}(\Lambda(\{a\},\{b\})\backslash\{1,a\})$ and $\{u,u^*\}$. But regrouping gives a nontrivial alternating product in $\{a,u,u^*\}$ and $\{b\}$. So, by freeness ω is zero on such a product.

In the rest of the terms at least one w_j is replaced by B_j or C_j. Then, since

$$\begin{aligned} B_j &= \lambda\left((u^*x)^{k_j-1}(x^*u)^{l_j-1} - \delta_{k_j-1,l_j-1}\lambda^{k_j-1}\right) \\ C_j &= \gamma\lambda\left((u^*x)^{r_j-1}(x^*u)^{s_j-1} - \delta_{r_j-1,s_j-1}\lambda^{r_j-1}\right), \end{aligned}$$

we see that such a term is once again

$$b_0 w'_1 b_1 \cdots w'_n b_n,$$

so of the same form as W in Equation (23), but now with the total number of symbols u^* and x strictly smaller than the total number of such symbols in W. Thus applying the replacement procedure to each of these terms repeatedly, we finally get $\omega_p(W) = \omega_p(\sum W_i)$, where each W_i has the same form as W in Equation (23), but for which the substrings w_j are either as in Equation (25) with k_j or l_j equal to zero, or w_j is as Equation (27) (so that no further replacements can be performed). But then each W_i can be rewritten as a nontrivial alternating product in Ω and $\{u,u^*\}$, so as before $\omega_p(W_i) = 0$. Thus $\omega_p(W) = 0$. □

We finish at last the proof of Proposition 3.19. We know that $pMp = W^*(pqp, v, P)$, and thanks to Lemma 3.21, $W^*(pqp)$ and $W^*(v,P)$ are $*-$free in pMp with respect to ω_p. As pqp is with no atoms with respect to ω_p, with the previous notation, we get $W^*(pqp) \cong L(\mathbf{Z})$. Concerning $W^*(v,P)$, let

$$\begin{aligned} e_{ij} &= v^i(p-P)(v^*)^j, \\ f_{kl} &= v^k(P-vv^*)(v^*)^l, \end{aligned}$$

for $i,j,k,l \in \mathbf{N}$. With straightforward computations, we see that $(e_{ij})_{i,j\in\mathbf{N}}$ and $(f_{kl})_{k,l\in\mathbf{N}}$ are systems of matrix units, for all $i,j,k,l \in \mathbf{N}$, $e_{ij}f_{kl} = f_{kl}e_{ij} = 0$, and $W^*(e_{ij}, f_{kl}) = W^*(v,P)$. Moreover, $\omega_p(e_{ii}) = (1-\gamma)\lambda^i$ and $\omega_p(f_{kk}) = (\gamma-\lambda)\lambda^k$, with $\gamma = \beta/(1-\alpha)$. Consequently, with notation of Proposition 3.19, we finally get $(W^*(v,P),\omega_p) \cong (\mathbf{C}^2,\tau_\delta) \otimes (B(\ell^2(\mathbf{N})),\psi_\lambda)$. The proof is complete. □

Notation 3.22. For a von Neumann (A,ϕ_A) endowed with a state ϕ_A, we will denote by A° the kernel of ϕ_A on A.

The next proposition is in some sense a generalization of Theorem 1.2 of [**12**]. For the convenience of the reader, we will write a complete proof.

Proposition 3.23. *Let (A,ϕ_A), (B,ϕ_B) and (C,ϕ_C) be three von Neumann algebras endowed with faithful, normal states such that A is a factor of type* I. *Let*

$$\begin{aligned} (\mathcal{M},\psi) &= ((C,\phi_C)\otimes(A,\phi_A)) * (B,\phi_B) \\ &\cup \\ (\mathcal{N},\psi) &= (A,\phi_A) * (B,\phi_B) \end{aligned}$$

and let e be a minimal projection of A. Then in $e\mathcal{M}e$, we have that $e\mathcal{N}e$ and $C\otimes e$ are free with respect to $\psi_e = \frac{1}{\psi(e)}\psi$ and together they generate $e\mathcal{M}e$, so that

$$(e\mathcal{M}e,\psi_e) \cong (C,\phi_C) * (e\mathcal{N}e,\psi_e).$$

PROOF. We follow step by step the proof of Theorem 1.2 of [**12**]. For notational convenience, we identify C with $C \otimes 1 \subset \mathcal{M}$. To see that $e\mathcal{N}e$ and eC generate $e\mathcal{M}e$, note that $\mathcal{N}$ and eC generate $\mathcal{M}$; so span $\Lambda(\mathcal{N}, eC)$ is dense in $\mathcal{M}$ and $e\Lambda(\mathcal{N}, eC)e = \Lambda(e\mathcal{N}e, eC)$.

We shall show that ψ_e is zero on a nontrivial alternating product in $(e\mathcal{N}e)^\circ$ and eC°. Let $a = e - \psi(e)1$. Then $A^\circ = \mathbf{C}a + S$ where

$$S = \{s \in A \mid \psi(s) = 0,\, ese = 0\}.$$

Let $x \in (e\mathcal{N}e)^\circ$. Then by Kaplansky Density Theorem, x is a s.o.-limit of a bounded sequence $(R_k)_{k\in\mathbf{N}}$ in span $\Lambda(\{a\} \cup S, B^\circ)$. For $Q \in$ span $(\Lambda(\{a\} \cup S, B^\circ)\backslash\{1\})$, we see that ψ on eQe is equal to a fixed constant times the coefficient of a in Q. So since $\psi(R_k) \to 0$ and $\psi(eR_ke) \to 0$, we may assume that the coefficients in each R_k of 1 and a are zero. Since $R_k - eR_ke \to 0$ for the s.o. topology, we may also assume that the coefficient of each element of S in R_k is zero, i.e., that each $R_k \in$ span $(\Lambda(\{a\} \cup S, B^\circ)\backslash(\{1, a\} \cup S))$. To prove the proposition, it suffices to show that ψ is zero on a nontrivial alternating product in $\Lambda(\{a\} \cup S, B^\circ)\backslash(\{1, a\} \cup S)$ and eC°. But regrouping and multiplying some neighboring elements gives (a constant times) a nontrivial alternating product in $\{a\}\cup S\cup(eC^\circ)\cup(SC^\circ)$ and B°. Thus by freeness, ψ is zero on such a product. $\square$

PROOF OF THEOREM 3.14. Apply Proposition 3.23 with $(A, \phi_A) = (B(\ell^2(\mathbf{N})), \psi_\lambda)$, $(B, \phi_B) = (L(\mathbf{Z}), \tau)$, $(C, \phi_C) = (\mathbf{C}^2, \tau_\delta)$. Let $e = e_{00} \in B(\ell^2(\mathbf{N}))$, and denote

$$\begin{array}{ccc} (\mathcal{M}, \psi) & = & \big((\mathbf{C}^2, \tau_\delta) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)\big) * (L(\mathbf{Z}), \tau) \\ \cup & & \\ (\mathcal{N}, \psi) & = & (B(\ell^2(\mathbf{N})), \psi_\lambda) * (L(\mathbf{Z}), \tau). \end{array}$$

We get

$$(e\mathcal{M}e, \psi_e) \cong (\mathbf{C}^2, \tau_\delta) * (e\mathcal{N}e, \psi_e).$$

But with notation of Section 2, $(\mathcal{N}, \psi) \cong (T_\lambda, \varphi_\lambda)$ is the free Araki-Woods factor of type III_λ. Since $e = e_{00}$, applying Proposition 3.12, we get $(e\mathcal{N}e, \psi_e) \cong (T_\lambda, \varphi_\lambda)$. We use now the "free absorption" properties of $(T_\lambda, \varphi_\lambda)$. Denote $L(\mathbf{F}(s))$ the interpolated free factor with s generators. We know that $(T_\lambda, \varphi_\lambda) * (L(\mathbf{F}_\infty), \tau) \cong (T_\lambda, \varphi_\lambda)$ (Corollary 5.5 in [**39**]) and $L(\mathbf{Z}) * (\mathbf{C}^2, \tau_\delta) \cong L(\mathbf{F}(1 + 2\delta(1 - \delta)))$ (Lemma 1.6 in [**12**]). Consequently,

$$(e\mathcal{M}e, \psi_e) \cong (\mathbf{C}^2, \tau_\delta) * (T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda).$$

But, in a canonical way

$$(\mathcal{M}, \psi) \cong (e\mathcal{M}e, \psi_e) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda).$$

Since $(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)$, we get

$$(\mathcal{M}, \psi) \cong (T_\lambda, \psi_\lambda).$$

We remind that we have proved (Proposition 3.19) that

$$(pMp, \omega_p) \cong L(\mathbf{Z}) * \big((\mathbf{C}^2, \tau_\delta) \otimes (B(\ell^2(\mathbf{N})), \psi_\lambda)\big) = (\mathcal{M}, \psi),$$

where $(M, \omega) = (M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta)$ and $p = p_{11} \in M_2(\mathbf{C})$. Thus,

$$(pMp, \omega_p) \cong (T_\lambda, \varphi_\lambda).$$

But once again

$$(M, \omega) \cong (pMp, \omega_p) \otimes (M_2(\mathbf{C}), \omega_\lambda).$$

Since $(T_\lambda, \varphi_\lambda) \cong (T_\lambda, \varphi_\lambda) \otimes (M_2(\mathbf{C}), \omega_\lambda)$, we finally get

$$(M_2(\mathbf{C}), \omega_\lambda) * (\mathbf{C}^2, \tau_\beta) \cong (T_\lambda, \varphi_\lambda).$$

$\square$

4. Proof of the Main Theorem

The aim of this section is to prove Theorem 3.5. We will be using the "machinery" of amalgamated free products of von Neumann algebras. The reader interested in the technical developments on amalgamated free products can find more material background in Section 5.

LEMMA 3.24. *Let (A_1, ϕ_1) and (A_2, ϕ_2) be any von Neumann algebras endowed with faithful, normal states. Assume that for some $\lambda, \beta \in]0,1[$, there exist modular embeddings*

$$\begin{array}{rcl}(M_2(\mathbf{C}), \omega_\lambda) & \hookrightarrow & (A_1, \phi_1) \\ (\mathbf{C}^2, \tau_\beta) & \hookrightarrow & (A_2, \phi_2),\end{array}$$

such that $\lambda/(\lambda+1) \leq \min\{\beta, 1-\beta\}$. Then

$$(A_1, \phi_1) * (A_2, \phi_2) \cong (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) * (A_2, \phi_2).$$

PROOF. We shall simply denote by M_2 the matrix algebra $M_2(\mathbf{C})$. Denote $(\mathcal{M}, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$. Let

$$(N, \omega) = (A_1, \phi_1) \underset{(M_2, \omega_\lambda)}{*} (T_\lambda, \varphi_\lambda).$$

Since $(M_2, \omega_\lambda) * (\mathbf{C}^2, \tau_\beta) \cong (T_\lambda, \varphi_\lambda)$ (Theorem 3.14), applying Proposition 3.28, we get

$$(\mathcal{M}, \phi) \cong (N, \omega) \underset{(\mathbf{C}^2, \tau_\beta)}{*} (A_2, \phi_2).$$

Applying Lemma 3.29, since the modular embedding of (M_2, ω_λ) into $(T_\lambda, \varphi_\lambda)$ is unique (up to a conjugation by a unitary in $T_\lambda^{\varphi_\lambda}$), we conclude that

$$\begin{array}{rcl}(N, \omega) & \cong & (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) \\ & \cong & ((A_1, \phi_1) * (T_\lambda, \varphi_\lambda)) * (T_\lambda, \varphi_\lambda).\end{array}$$

From Theorem 11 of [**1**], we get that the centralizer algebra N^ω is a factor. If $\rho_i : (\mathbf{C}^2, \tau_\beta) \hookrightarrow (N, \omega)$ are two modular embeddings, denote $p_i = \rho_i(p) \in N^\omega$ such that $\omega(p_i) = \beta$. Since p_1 and p_2 are equivalent in N^ω, ρ_1 and ρ_2 are unitarily conjugate. Consequently, using the isomorphism

$$(N, \omega) \cong (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) * (\mathbf{C}^2, \tau_\beta),$$

and applying again Proposition 3.28, we finally get

$$\begin{array}{rcl}(\mathcal{M}, \phi) & \cong & \big((A_1, \phi_1) * (T_\lambda, \varphi_\lambda) * (\mathbf{C}^2, \tau_\beta)\big) \underset{(\mathbf{C}^2, \tau_\beta)}{*} (A_2, \phi_2) \\ & \cong & (A_1, \phi_1) * (T_\lambda, \varphi_\lambda) * (A_2, \phi_2).\end{array}$$

$\square$

Theorem 3.5 is a straightforward corollary of Lemma 3.24. We end this section by giving some examples of von Neumann algebras which satisfy assumptions of Theorem 3.5. We introduce the class $\mathcal{S}$ of all von Neumann algebras (M, ϕ) with separable predual and endowed with a faithful, normal, almost periodic state ϕ such that

$$(M, \phi) * (L(\mathbf{Z}), \tau_{\mathbf{Z}}) \cong (T_{\mathrm{Sd}(\phi)}, \varphi_{\mathrm{Sd}(\phi)}).$$

Note that if (M_1, ϕ_1) and (M_2, ϕ_2) are in $\mathcal{S}$, then $(M_1, \phi_1) * (M_2, \phi_2)$ is also in $\mathcal{S}$.

EXAMPLE 3.25. We give several examples of von Neumann algebras in the class $\mathcal{S}$. This list is not exhaustive and there is nothing really new here : these examples are mere consequences of results in [**8, 10, 12, 27, 39, 48**], and of Proposition 3.23.

- Type I : All factors of type I endowed with a faithful, normal nontracial state ϕ.

- Type III : All the almost periodic free Araki-Woods factors $(T_\Gamma, \varphi_\Gamma)$ endowed with their free quasi-free state.
- Tensor products : All the tensor products $(N, \omega) \otimes (\text{Type I}, \phi)$, where $(\text{Type I}, \phi)$ means a factor of type I endowed with a faithful normal state ϕ, and (N, ω) is :
  - Any finite-dimensional von Neumann algebra of the form $\underset{\alpha_1}{\mathbf{C}} \oplus \cdots \oplus \underset{\alpha_n}{\mathbf{C}}$ with $\alpha_i > 0$ for all i and $\sum \alpha_i = 1$.
  - $(\mathcal{R}, \tau)$ the hyperfinite II_1 factor.
  - Any interpolated free group factor $L(\mathbf{F}(s))$, $s > 1$.
- Free products : All the free products of the previous examples.

5. Appendix on Amalgamated Free-Products

First, we recall this fundamental result by Takesaki on conditional expectations in general von Neumann algebras.

Theorem 3.26. [**40**] *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras and φ a faithful normal state on $\mathcal{M}$. Let (σ_t^φ) be the modular automorphism group of φ on $\mathcal{M}$. There exists a state-preserving conditional expectation $E : \mathcal{M} \to \mathcal{N}$ iff the subalgebra $\mathcal{N}$ is globally invariant under (σ_t^φ). Moreover, whenever E exists, it is unique. We shall call it* the state-preserving conditional expectation *E from $\mathcal{M}$ onto $\mathcal{N}$.*

Let (B, ϕ_B) be a von Neumann algebra together with a faithful normal state. We shall say that (M, ϕ, E, ρ) is a *quadruplet associated to B* if :

- M is a von Neumann algebra with a faithful normal state ϕ.
- $\rho : (B, \phi_B) \to (M, \phi)$ is a modular embedding.
- If E' is the unique state-preserving conditional expectation from M onto $\rho(B)$, then $E = (\rho^{-1})E'$.

Let $(M_i, \phi_i, E_i, \rho_i)$ be two such quadruplets. We shall say that $(M_1, \phi_1, E_1, \rho_1)$ embeds into $(M_2, \phi_2, E_2, \rho_2)$ if there exists a modular embedding $\theta : M_1 \hookrightarrow M_2$ such that the following diagram is commuting

$$\begin{tikzcd}
(M_1, \phi_1) \arrow[r, "\theta"] & (M_2, \phi_2) \\
(B, \phi_B) \arrow[u, "\rho_1"] \arrow[r, equal, "\mathrm{id}"'] & (B, \phi_B) \arrow[u, "\rho_2"']
\end{tikzcd}$$

In particular, thanks to Theorem 3.26, it implies that $E_2\theta = E_1$. We shall denote

$$(M_1, \phi_1, E_1, \rho_1) \underset{B}{\hookrightarrow} (M_2, \phi_2, E_2, \rho_2).$$

If θ is a state-preserving isomorphism, we shall say that the quadruplets $(M_1, \phi_1, E_1, \rho_1)$ and $(M_2, \phi_2, E_2, \rho_2)$ are isomorphic and we will denote $(M_1, \phi_1, E_1, \rho_1) \underset{B}{\cong} (M_2, \phi_2, E_2, \rho_2)$.

We refer to [**52**] for the construction of the reduced amalgamated free product of C^*-algebras. Via a GNS construction, we can define the free product of two von Neumann algebras $(M_i, \phi_i, E_i, \rho_i)$ $i = 1, 2$, with amalgamation over B. The main properties of this amalgamated free product can be summarized in the following proposition (see also [**25**] for the tracial case) :

Proposition 3.27. *Let $(M_i, \phi_i, E_i, \rho_i)$ $i = 1, 2$, be two quadruplets associated to B defined as before. There exists a quadruplet (M, ϕ, E, ρ) such that*

– $(M_i, \phi_i, E_i, \rho_i) \underset{B}{\hookrightarrow} (M, \phi, E, \rho)$.

– *M is generated by M_1 and M_2 which are* free with amalgamation over B, *i.e.,* $E(a_1 \cdots a_n) = 0$ *whenever* $a_j \in M_{\iota_j}$, $\iota_k \neq \iota_{k+1}$ $(1 \leq k \leq n-1)$ *and* $E(a_j) = 0, \forall j$.

We call (M, ϕ, E, ρ) *the* amalgamated free product over B of $(M_i, \phi_i, E_i, \rho_i)$ *and we denote*

$$(M, \phi, E, \rho) = (M_1, \phi_1, E_1, \rho_1) \underset{B}{*} (M_2, \phi_2, E_2, \rho_2).$$

If no confusion is possible, we will simply denote $(M, \phi) = (M_1, \phi_1) \underset{B}{*} (M_2, \phi_2)$.

We describe now a natural way to define a quadruplet. Let (B, ϕ_B) and $(\mathcal{M}, \psi)$ be two von Neumann algebras together with a faithful normal state. Let $(M, \phi) = (B, \phi_B) * (\mathcal{M}, \psi)$ be their free product. We have a canonical state-preserving embedding $\rho : (B, \phi_B) \hookrightarrow (M, \phi)$, such that $\rho(B)$ is globally invariant under (σ_t^ψ) [**11**]. We regard $\mathcal{M} \subset M$. Define as before $E' : M \to \rho(B)$ the (unique) state-preserving conditional expectation and $E = (\rho^{-1})E'$. Let $\mathcal{M}^\circ = \mathcal{M} \cap \ker(\phi)$, $\rho(B)^\circ = \rho(B) \cap \ker(\phi)$ and denote as usual $\Omega = \Lambda(\rho(B)^\circ, \mathcal{M}^\circ)$ the set of alternating products in $\rho(B)^\circ$ and $\mathcal{M}^\circ$. From [**8**], we know that

$$\begin{aligned} \forall b \in B, E(\rho(b)) &= b \\ \forall z \in \Omega \backslash \rho(B)^\circ, E(z) &= 0. \end{aligned}$$

Proposition 3.28. *Let (B, ϕ_B), $(\mathcal{M}_2, \psi_2)$ be two von Neumann algebras together with a faithful normal state. Let $(M_1, \phi_1, E_1, \rho_1)$ be any quadruplet associated to B. Let $(M_2, \phi_2, E_2, \rho_2)$ be the quadruplet associated to B, constructed from $(\mathcal{M}_2, \psi_2)$ as above. Then*

$$(M_1, \phi_1, E_1, \rho_1) \underset{B}{*} (M_2, \phi_2, E_2, \rho_2) \cong (M_1, \phi_1) * (\mathcal{M}_2, \psi_2).$$

Or, more simply

$$(M_1, \phi_1) \underset{B}{*} ((B, \phi_B) * (\mathcal{M}_2, \psi_2)) \cong (M_1, \phi_1) * (\mathcal{M}_2, \psi_2).$$

Proof. Denote $(M, \phi, E, \rho) = (M_1, \phi_1, E_1, \rho_1) \underset{B}{*} (M_2, \phi_2, E_2, \rho_2)$, and

$$\theta_i : (M_i, \phi_i, E_i, \rho_i) \underset{B}{\hookrightarrow} (M, \phi, E, \rho)$$

the embeddings of quadruplets. As $\rho = \theta_1 \rho_1 = \theta_2 \rho_2$, we see immediatly that (M_1, ϕ_1) and $(\mathcal{M}_2, \psi_2)$ embed in (M, ϕ) in a state-preserving way and together they generate (M, ϕ). For notational convenience, we may assume $M_i \subset M$. Denote $M_1^\circ = M_1 \cap \ker(\phi)$, $\mathcal{M}_2^\circ = \mathcal{M}_2 \cap \ker(\phi)$. Let W be a nontrivial alternating product in M_1° and $\mathcal{M}_2^\circ$, so that W can be written

$$W = x_0 w_1 x_1 \cdots w_n x_n,$$

where $x_j \in \mathcal{M}_2^\circ$, $w_j \in M_1^\circ$ for all j, except possibly x_0 and/or x_n are equal to 1. Denote $\Omega = \Lambda(\rho_2(B)^\circ, \mathcal{M}_2^\circ)$. If $W \in M_1^\circ$, there is nothing to do. If not, for each j, replace w_j by

$$w_j = w_j' + b_j,$$

where $w_j' \in M_1 \cap \ker E$ and $b_j \in \rho_1(B)^\circ = \rho_2(B)^\circ$. Applying the replacement procedure and multiplying some neighboring elements, we get $\phi(W) = \phi(\sum W_i)$ where each W_i is a nontrivial alternating product in $M_1 \cap \ker E$ and $\Omega \backslash \rho_2(B)^\circ$. But, we saw that $\Omega \backslash \rho_2(B)^\circ \subset \ker E$. Thus, by freeness over B, we get $E(W_i) = 0$. But $\phi(W_i) = (\phi \circ E)(W_i) = 0$, consequently $\phi(W) = 0$. $\square$

LEMMA 3.29. *Let (N, ω) be a von Neumann algebra endowed with a faithful normal state such that the centralizer N^ω is a factor. Let $\rho_i : (M_2(\mathbf{C}), \omega_\lambda) \hookrightarrow (N, \omega)$, $i = 1, 2$, be two modular embeddings. Then there exists a unitary u in N^ω such that* $\mathrm{Ad}(u)\rho_1 = \rho_2$. *In particular, using our notation*

$$(N, \omega, \rho_1, E_1) \underset{M_2(\mathbf{C})}{\cong} (N, \omega, \rho_2, E_2).$$

PROOF. Let (p_{ij}) $i, j \in \{0, 1\}$, be the matrix unit in $M_2(\mathbf{C})$ such that $\omega_\lambda(p_{ij}) = \delta_{ij}\lambda^j/(\lambda + 1)$. Let $v = \rho_1(p_{10}) \in N$ and $w = \rho_2(p_{10}) \in N$. Let $q_1 = v^*v \in N$ and $q_2 = w^*w \in N$. We see that $\omega(q_1) = \omega(q_2)$ and q_1 and q_2 are in N^ω, the centralizer of the state ω. But N^ω is a factor. Since q_1 and q_2 are equivalent projections in N^ω, there exists a unitary $u \in N^\omega$ such that $uq_1u^* = q_2$. Thus, we may assume that $q_1 = q_2$. We denote by q this projection.

We see that $wv^* \in N^\omega$ is a unitary in $(1 - q)N(1 - q)$. Thus $u = q + wv^* \in N^\omega$ is a unitary in N. Moreover, $uvu^* = wv^*vq = w$. Consequently, $\theta_u = \mathrm{Ad}\, u$ is a state-preserving $*-$automorphism of (N, ω), such that $\theta_u\rho_1 = \rho_2$. □

6. Remark

We still do not know whether all the free products of finite dimensional matrix algebras $(A_1, \phi_1) * (A_2, \phi_2)$ are isomorphic to free Araki-Woods factors. Assume that $A_1 = M_n(\mathbf{C})$ with

$$\phi_1 = \mathrm{Tr}\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot\right), \ \lambda_1 \leq \cdots \leq \lambda_n.$$

Let $\beta \in]0, 1[$ such that $\lambda_1 \leq \min\{\beta, 1 - \beta\}$. With our techniques, it is not difficult to see that if one can prove that $(A_1, \phi_1) * (\mathbf{C}^2, \tau_\beta)$ is a free Araki-Woods factor, then all the free products $(A_1, \phi_1) * (A_2, \phi_2)$ are also free Araki-Woods factors. That is exactly what we did in Section 3 for $n = 2$. But one of the crucial ingredients in the proof was the precise picture of Voiculescu in Theorem 3.16. This precise description no longer exists for $n \geq 3$ (see [**12**] for further details).

CHAPITRE 4

Free Araki-Woods Factors are Generalized Solid

In his remarkable paper [**18**], Ozawa discusses the consequences of the following condition for a von Neumann algebra M. Let $M \subset B(H)$ and denote $\pi : C^*(M, M') \to B(H)/K(H)$ the restriction of the quotient map.

Condition 4.1 (Ozawa, [**18**]). *There are two C^*-subalgebras A of M and B of M', so that*

1. *A is locally reflexive*
2. *A generates M and B generates M'*
3. *The map*
$$\sum a_i \otimes b_i \mapsto \pi(\sum a_i b_i) \in B(H)/K(H)$$
extends to a continuous map on the minimal (spatial) tensor product $A \otimes_{\min} B$.

Ozawa proved that if M is finite and satisfies Condition 4.1, then it is *solid* : the relative commutant $M \cap A'$ of any diffuse von Neumann subalgebra of M is injective. The notion of solidity has been extended by Vaes and Vergnioux [**46**] to arbitrary von Neumann algebras.

Definition 4.2 (Vaes & Vergnioux, [**46**]). *A von Neumann algebra M is said to be* generalized solid *if the relative commutant $M \cap A'$ is injective for any diffuse subalgebra $A \subset M$ for which there exists a* faithful, normal conditional expectation *$E : M \to A$.*

If M is solid and if $N \subset M$ is a von Neumann subalgebra such that there exists a faithful, normal conditional expectation $E : M \to N$, then N is solid. Moreover, a generalized solid factor is *prime*, i.e., it cannot be written as the tensor product of two diffuse von Neumann algebras. They proved the following theorem, which is the exact analogue of Theorem 6 of [**18**] :

Theorem 4.3 (Vaes & Vergnioux, [**46**]). *Any von Neumann algebra $M \subset B(H)$ satisfying Condition* 4.1 *is generalized solid.*

We present now the main theorem of this note :

Theorem 4.4. *Let $H_{\mathbf{R}}$ be a real Hilbert space, and (U_t) be any orthogonal representation. The free Araki-Woods factor $\Gamma(H_{\mathbf{R}}, U_t)'' \subset B(\mathcal{F}(H))$ satisfies Condition* 4.1.

Proof. The proof of this theorem is inspired by the one of Theorem 4.2 in [**33**], where Shlyakhtenko proved that the von Neumann algebra generated by a finite family of q-Gaussian random variables satisfies Condition 4.1 for $q < \sqrt{2} - 1$. First of all, remind the discussion at the beginning of Section 3 in [**39**]. Let $H_{\mathbf{R}}$ be a real Hilbert space, and (U_t) be any orthogonal representation. We recall some notation. Let $H_{\mathbf{C}}$ be the complexified space of $H_{\mathbf{R}}$, and let H be the completion of $H_{\mathbf{C}}$ w.r.t. the following inner product

$$\langle x, y \rangle_U = \langle \frac{2}{1 + A^{-1}} x, y \rangle, \forall x, y \in H_{\mathbf{C}},$$

with A the infinitesimal generator of (U_t) on $H_{\mathbf{C}}$. Let $\mathcal{F}(H)$ be the *full Fock* space of H,

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n \geq 1} H^{\otimes n}.$$

we have the *left* and *right creation* operators :

$$l(\xi) : \mathcal{F}(H) \to \mathcal{F}(H) \quad : \quad \begin{cases} l(\xi)\Omega = \xi, \\ l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n, \end{cases}$$

$$r(\eta) : \mathcal{F}(H) \to \mathcal{F}(H) \quad : \quad \begin{cases} r(\eta)\Omega = \eta, \\ r(\eta)(\eta_1 \otimes \cdots \otimes \eta_m) = \eta_1 \otimes \cdots \otimes \eta_m \otimes \eta. \end{cases}$$

Denote now

$$s(\xi) = \frac{l(\xi) + l(\xi)^*}{2}, d(\eta) = \frac{r(\eta) + r(\eta)^*}{2}$$

the real part of $l(\xi)$ and $r(\eta)$. Consider the real subspace

$$H'_{\mathbf{R}} = iH_{\mathbf{R}}^{\perp_{\Re\langle \cdot,\cdot \rangle_U}} = \{\eta \in H : \langle \eta, \xi \rangle_U \in \mathbf{R}, \forall \xi \in H_{\mathbf{R}}\}.$$

Let $M = \Gamma(H_{\mathbf{R}}, U_t)'' = s(H_{\mathbf{R}})'' \subset B(\mathcal{F}(H))$ be the free Araki-Woods factor associated with the Hilbert space $H_{\mathbf{R}}$ and the orthogonal representation (U_t). Shlyakhtenko proved in Theorem 3.3 of [**39**], that $M' = d(H'_{\mathbf{R}})''$. Set

$$A = C^*(s(H_{\mathbf{R}})),\ B = C^*(d(H'_{\mathbf{R}})),\ \tilde{A} = C^*(l(H_{\mathbf{R}})).$$

From [**7, 48**], it is well known that for $H_{\mathbf{R}}$ of dimension $n < \infty$, $\tilde{A}$ is an extension of the Cuntz algebra $\mathcal{O}_n$ by the compact operators, and for $H_{\mathbf{R}}$ separable infinite dimensional, $\tilde{A}$ is $\mathcal{O}_\infty$. Thus, $\tilde{A}$ is always nuclear. Consequently, A is locally reflexive (see the discussion in [**18**]).

By definition, $[l(\xi), r(\eta)] = 0$ for all ξ, η. From [**39**], we know that if $h \perp \Omega$, then $[l(\xi), r(\eta)^*]h = 0$ for all ξ, η. Moreover,

$$l(\xi)r(\eta)^*\Omega = 0$$

while

$$r(\eta)^*l(\xi)\Omega = \langle \xi, \eta \rangle_U \Omega.$$

Thus, $[l(\xi), d(\eta)] = -\langle \xi, \eta \rangle_U P_{\mathbf{C}\Omega}$, where $P_{\mathbf{C}\Omega}$ is the orthogonal projection on $\mathbf{C}\Omega$. Consequently, we have shown that $[\tilde{A}, B] \subset K(\mathcal{F}(H))$.

Denote by $\pi : B(\mathcal{F}(H)) \to B(\mathcal{F}(H))/K(\mathcal{F}(H))$ the quotient map. It follows that the images $\pi(\tilde{A})$ and $\pi(B)$ commute. Thus, the map

$$\sum a_i \otimes b_i \mapsto \pi(\sum a_i b_i),\ a_i \in \tilde{A},\ b_i \in B$$

is continuous for the max tensor product on $\tilde{A} \otimes B$. Since $\tilde{A}$ is nuclear, we find that the min and max tensor norms on $\tilde{A} \otimes B$ coincide. Thus this map is $\otimes_{\min}$-continuous.

Restricting this map to the image of $A \otimes B \subset A \otimes_{\min} B \subset \tilde{A} \otimes_{\min} B$, we obtain finally that Ozawa's condition is indeed satisfied. $\square$

Corollary 4.5. *All the free Araki-Woods factors are prime. All the free products of factors of type* I *are prime.*

Proof. Let (M_i, φ_i), $i = 1, 2$, be two factors of type I such that at least one of which is non tracial. Let $\Gamma \subset \mathbf{R}^*$ be the subgroup generated by the point spectra of φ_1 and φ_2. From Equation (5), we know that

$$(M_1, \varphi_1) * (M_2, \varphi_2) * L(\mathbf{Z}) \cong (T_\Gamma, \varphi_\Gamma).$$

The inclusion $(M_1, \varphi_1) * (M_2, \varphi_2) \subset (T_\Gamma, \varphi_\Gamma)$ is modular. In particular, there exists a unique state-preserving conditional expectation

$$E : (T_\Gamma, \varphi_\Gamma) \to (M_1, \varphi_1) * (M_2, \varphi_2).$$

Thus, the free product $(M_1, \varphi_1) * (M_2, \varphi_2)$ is solid and prime. $\square$

CHAPITRE 5

Problems

I wish to end this text by giving a few problems which are related to my work.

PROBLEM 5.1. In my article [**15**], from a free, ergodic, measure-preserving flow, I constructed a family of factors of type III_1 which entirely remembers the flow. One can ask whether it is possible to find an invariant for this construction (and more generally for any factor of type III_1) that can give back this flow.

PROBLEM 5.2. The second problem I am thinking about is to prove other isomorphism results in the spirit of Theorem 3.5 ([**16**]). For example I would like to show that the classical Araki-Woods factor of type III_λ, $\bigotimes_{\mathbf{N}}(M_2(\mathbf{C}), \omega_\lambda)$, is in the class $\mathcal{S}$. To do this I think I should further develop the algebraic techniques of Dykema and extend the matricial models of Shlyakhtenko.

PROBLEM 5.3. One the most spectacular applications of the free entropy was the proof of the lack of Cartan subalgebras in the free group factors $L(\mathbf{F}_n)$, $2 \leq n \leq +\infty$ (see [**47**]). Shlyakhtenko showed in [**36**] that the free Araki-Woods factors of type III_λ $(0 < \lambda < 1)$ have no Cartan subalgebras. I think it could be possible to show that any almost periodic free Araki-Woods factor cannot have a Cartan subalgebra (see [**35**]).

PROBLEM 5.4. For M a factor of type II_1, M is full if and only if it does not have the property Γ of Murray & von Neumann, in other words, any central sequence in M is trivial. For a factor of type III_1, this equivalence no longer holds. In general, to prove that a factor of type III_1 is full, one uses a 14ε lemma à la Murray & von Neumann. Thus, one proves a stronger result : one proves that actually any central sequence is trivial. Does there exist a full factor of type III_1 which has non trivial central sequences ?

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Résumé. Ma thèse porte sur le problème de la classification (à $*$-isomorphisme près) de certaines familles d'algèbres de von Neumann. En effet, j'étudie des classes de facteurs de type III provenant de deux contextes très différents : la *théorie ergodique* et la *théorie des probabilités libres*.

Tout d'abord, partant de $\gamma : \mathbf{R} \curvearrowright (X, \mu)$ un flot libre, ergodique qui préserve la mesure, et N un facteur de type II_∞, je construis de manière canonique un facteur de type III_1, noté $C_{\gamma,N}$, pour lequel je peux calculer le core et l'action duale. Cette construction est en quelque sorte le "produit croisé" de N par le sous-groupe virtuel γ. Si je fixe N un facteur plein de type II_∞, en utilisant un résultat de Popa, j'obtiens la classification suivante : si $\gamma_i : \mathbf{R} \curvearrowright (X_i, \mu_i)$, pour $i = 1, 2$, sont deux flots libres, ergodiques qui préservent la mesure, alors

$$\gamma_1 \text{ et } \gamma_2 \text{ sont conjugués } \Longleftrightarrow C_{\gamma_1,N} \text{ et } C_{\gamma_2,N} \text{ sont } *\text{-isomorphes.}$$

C'est un phénomène de rigidité dans la théorie des facteurs de type III analogue à ceux obtenus par Popa ces dernières années pour les facteurs de type II_1.

J'étudie ensuite le problème de la classification des produits libres d'algèbres de von Neumann hyperfinies selon des états non-traciaux. Cette question a été résolue par Dykema dans le cas tracial en utilisant les modèles matriciels de Voiculescu. Dans le cas non-tracial, Dykema a réussi à calculer le centralisateur de ces produits libres mais le problème de la classification restait toujours ouvert. Je classifie complètement certains de ces produits libres en montrant qu'ils sont en fait $*$-isomorphes, en préservant l'état, aux facteurs d'Araki-Woods libres presque périodiques de Shlyakhtenko. Par exemple, pour $\lambda, \mu > 0$, je montre que

$$(M_2(\mathbf{C}), \omega_\lambda) * (M_2(\mathbf{C}), \omega_\mu)$$

est $*$-isomorphe, en préservant l'état, au facteur d'Araki-Woods libre dont l'invariant Sd est engendré par λ et μ. Les preuves reposent sur des techniques algébriques d'étymologie libre et utilisent les produits libres amalgamés.