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Kuperberg invariants for sutured 3-manifolds

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Abstract

In this thesis, we study Kuperberg's Hopf algebra approach to quantum invariants of closed 3-manifolds. We show that, for involutive Hopf superalgebras, Kuperberg invariants extend to the more general class of balanced sutured 3-manifolds, and in particular, to link complements. To achieve this, we bring many aspects of Reidemeister torsion theory into the realm of quantum invariants, such as twisting, Fox calculus and Spin^c structures and we make clear to which aspects of Hopf algebra theory these correspond. When our construction is specialized to an exterior algebra, we show that it recovers the twisted Reidemeister torsion of sutured 3-manifolds.

Keywords: quantum invariants, Hopf algebras, twisted Reidemeister torsion, twisted Alexander polynomials, sutured 3-manifolds, Heegaard diagrams.

Résumé

Dans cette thèse, on étudie les invariants quantiques des 3-variétés de Kuperberg, qui sont basées sur les algèbres de Hopf. On montre que, pour les super-algèbres de Hopf involutives, les invariants de Kuperberg s'étendent à la classe, plus générale, des 3-variétés suturées balancées et en particulier aux complements d'entrelacs. Pour accomplir ceci, on relève plusieurs aspects de la théorie des torsions de Reidemeister au monde des invariants quantiques, tels que la procédure pour tordre des invariants, le calcul de Fox et les structures Spin^c, et on clarifie les aspects de la théorie des algèbres de Hopf auxquels ils correspondent. Quand notre construction est spécialisée au cas d'une algèbre extérieure, on montre qu'elle calcule la torsion de Reidemeister tordue des 3-variétés suturées.

Mots clés: invariants quantiques, algèbres de Hopf, torsion de Reidemeister tordue, polynômes d'Alexander tordus, 3-variétés suturées, diagrammes de Heegaard.

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Contents

Introduction

Low dimensional topology consists of the study of topological objects in dimension less or equal than four. One of the central objects of study are *knots*, that is, strings in three-dimensional space with the ends glued together. The fundamental problem of knot theory is to determine whether, given two knots in space, one can be deformed into the other without breaking the chords.

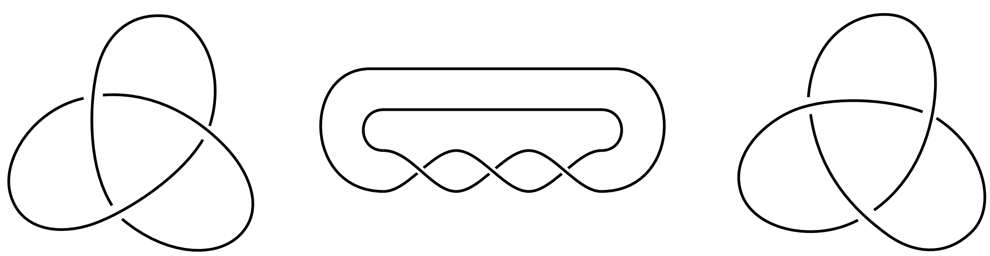


Figure 1: Three knots of which only two are isomorphic.

The other central objects of study are manifolds of dimension three and four, which are tightly linked between them and with knot theory. In order to study knots, 3-manifolds or 4-manifolds one needs *topological invariants*, that is, quantities associated to the given object that are unchanged after a deformation that preserves the topology.

For a long period of time, low dimensional topology was studied as a branch of algebraic topology, that is, through topological invariants such as the fundamental group and homology theory. As there is no efficient algorithm to compare the fundamental groups of distinct knots or 3-manifolds, one usually extracts simpler invariants from the homology (or the chain complex) of appropriate covering spaces, such as the Alexander polynomial of knots and, more generally, the Reidemeister torsion of 3-manifolds [Ale28, Rei35]. Such invariants, usually referred as *classical invariants*, have a very clear topological meaning and capture a good deal of topological information. Moreover, as studied by many people since the 90's, this information becomes sharper if one uses *twisted* versions of these invariants, that is, those associated to non-abelian covering spaces, see e.g. [FV11a].

Quite unexpectedly, low dimensional topology turned out to be not just a branch of algebraic topology. This was evidenced during the 80's through the seminal work of Donaldson in dimension four and that of Jones and Witten in dimension three, in which tight new connections with mathematical physics were found [Don83, Jon85, Wit89]. The theory of Jones and Witten, and its mathematical formulation by Reshetikhin-Turaev [RT90, RT91], grew up into a whole new field now known as *quantum topology*. Here a central role is played by the notion of topological quantum field theory (TQFT), as axiomatized by Atiyah [Ati88]. This notion is radically different from anything existing in algebraic topology, notably because the "morphisms" of the theory are cobordisms instead of the usual continuous functions, and the additive axiom is replaced by a monoidal axiom. The topological invariants of knots and 3-manifolds derived from quantum topology are referred as *quantum invariants*.

There exist several mathematical approaches to define quantum invariants of knots and 3-manifolds, all revolving around the notion of Hopf algebra. Under a condition called quasi-triangularity, the main example being the quantum groups of Drinfeld and Jimbo [Dri89], any representation of the Hopf algebra comes equipped with a solution to the so called quantum Yang-Baxter equation. This easily leads to topological invariants of knots and links in S^3, the Jones polynomial being the most famous example [Tur88, RT90]. However, extending these invariants to links in arbitrary closed 3-manifolds and, in particular, to closed 3-manifolds themselves, is a bit more complicated. The original approach of Reshetikhin and Turaev relied on a deeper study of the representation theory of the given Hopf algebra [RT91], in their case a quantum group, eventually leading to the notion of modular category [Tur94]. This approach has been the subject of intensive research, as modular categories are rich mathematical objects with relations to other fields such as physics and quantum computing. It turns out that there is a simpler procedure to define quantum invariants of closed 3-manifolds, one that relies directly on Hopf algebra theory. This procedure was introduced by Hennings and Kuperberg independently [Hen96, Kup91, Kup96], and both rely on the theory of Hopf algebra integrals, which are analogues of the Haar integral on a Lie group. However, neither of these received as much attention as the approach of Reshetikhin-Turaev, notably because they do not directly extend to a TQFT. This has recently been settled for the Hennings approach in [DRGPM18].

As of today, the differences between classical and quantum invariants are not yet fully understood. Indeed, their difference not only radicates in the mathematics behind their definition, but rather in the information that they contain. This was observed in the early days of quantum topology, when the then new invariants lead to a simple proof of the long-standing Tait conjectures. Later, the $\mathfrak{sl}_2$ quantum knot invariants were observed to be related to hyperbolic geometry, leading to the (still unsolved) volume conjecture of Kashaev and Murakami-Murakami, see e.g. [Mur11]. However, these invariants seem to have no relationship to purely topological properties of knots, such as the Seifert genus, fiberedness or sliceness, as do classical invariants. For closed 3-manifolds, the quantum invariants derived from modular categories have clearer topological limitations [Fun13] and the invariants of Hennings and Kuperberg haven't been studied at all from a topological perspective. These differences are quite puzzling if one takes into account that some classical invariants can be obtained from the methods of quantum topology. This is the case of the Alexander polynomial of links in S^3 [Res92, RS92, Mur92] and the abelian Reidemeister torsion of closed 3-manifolds [BCGPM16], which can be obtained from the representation theory of an appropriate quantum group. However, these results rely on the skein relation characterization of the Alexander polynomial, which completely hides its topological meaning and thus they give no insight on the possible topological content of more general quantum invariants.

The aim of this thesis is twofold. On the one hand, we want to develop further the Hopf algebraic approach to quantum invariants of Kuperberg. Though less developed than its Reshetikhin-Turaev or Hennings counterparts, it has a more topological flavour and we believe that it may help clarify some topological aspects of quantum invariants. On the other hand, we want to better understand Reidemeister torsion as part of quantum topology. We will see that Kuperberg's approach, when appropriately extended, allows to give a Hopf algebra theoretic explanation to many aspects of Reidemeister torsion theory such as twisting, Fox calculus and Spin^c structures. In particular, it allows for a simple way to see classical invariants as *twisted* quantum invariants, in the sense of Turaev's homotopy field theory [Tur00].

Kuperberg invariants

Let H be an arbitrary finite dimensional Hopf algebra over a field $\mathbb{K}$. In [Kup91, Kup96], Kuperberg constructed for any framed closed oriented 3-manifold (Y, f), where f is a trivialization of the tangent bundle TY, a topological invariant

$$Z_H^{Kup}(Y, f) \in \mathbb{K}.$$

To define this invariant, one first encodes Y in a *Heegaard diagram*, consisting of a surface Σ with two sets of pairwise disjoint circles on it. Then the structure maps of the Hopf algebra are combined with the combinatorics of the Heegaard diagram to produce a tensor of the Hopf algebra which is then contracted with the Hopf algebra cointegral and integral. Recall that a *right cointegral* is an element $c \in H$ characterized up to scalar by $c \cdot x = c \cdot \epsilon(x)$ for all $x \in H$, if this relation holds from both sides one says that H is *unimodular*. Kuperberg's invariant is independent of the framing f when H is involutive (that is, the antipode satisfies $S^2 = \mathrm{id}_H$), unimodular and counimodular.

The relation between Kuperberg's invariant and other quantum invariants is now more or less understood. Indeed, as recently shown by Chang-Cui [CC19], $Z_H^{Kup}(Y, f)$

coincides with the Hennings invariants of Y at the Drinfeld double $D(H)$ of H, for any Hopf algebra H. Now, though the invariants of Hennings and WRT are equivalent for links in S^3, the case of arbitrary closed 3-manifolds is less understood. Indeed, there is no general procedure to obtain a modular category from the representations of a non-semisimple ribbon Hopf algebra, and hence there is not a WRT invariant obtained from an arbitrary Drinfeld double. What is known is that the Hennings invariant from quantum $\mathfrak{sl}_2$ at a root of unity coincides with the $\mathfrak{sl}_2$ WRT invariant, at least for homology 3-spheres [CKS09]. Since quantum groups are quotients of the Drinfeld double of its Borel subalgebra, it is reasonable to expect that Kuperberg's invariant at Borel parts of quantum groups at roots of unity should be related to WRT invariants, at least for homology 3-spheres.

Main results

Our first main result, which constitutes Chapter 3, deals with an extension of Kuperberg invariants to a wider class of 3-manifolds, namely, the sutured 3-manifolds introduced by Gabai [Gab83]. By a *sutured 3-manifold* we mean a pair (M,γ) where M is a compact oriented 3-manifold-with-boundary and $\gamma \subset \partial M$ is a collection of pairwise disjoint annuli dividing ∂M into two subsurfaces $R_{\pm}(\gamma)$, we say that (M,γ) is balanced if $\chi(M,R_-(\gamma))=0$ among with some other simple conditions. Balanced sutured 3-manifolds generalize closed 3-manifolds, link complements and Seifert surface complements.

Now consider a finite dimensional Hopf superalgebra H over a field $\mathbb{K}$. Let $\mathrm{Aut}(H)$ be the group of Hopf algebra automorphisms of H. We suppose H is involutive, unimodular and counimodular, this is the context to define unframed Kuperberg invariants as mentioned above.

Theorem 1. *Let (M,γ) be a balanced sutured 3-manifold endowed with a representation $\rho:\pi_1(M)\to Aut(H)$, a relative $Spin^c$ structure $\mathfrak{s}$ and an orientation ω of $H_*(M,R_-(\gamma);\mathbb{R})$. Then Kuperberg's construction can be extended to define a topological invariant*

$$Z_H^{\rho}(M,\gamma,\mathfrak{s},\omega)\in\mathbb{K}$$

of the tuple $(M,\gamma,\rho,\mathfrak{s},\omega)$.

If Y is a closed oriented 3-manifold, then our invariant recovers the original (involutive) Kuperberg invariant $Z_H^{Kup}(Y)$ as follows: let M_0 be the complement of the interior of a closed 3-ball embedded in Y and γ_0 be a single annuli in ∂M_0. Then (M_0,γ_0) is a balanced sutured 3-manifold and if ρ_{triv} denotes the trivial representation of $\pi_1(M_0)$ (that is, $\rho_{\mathrm{triv}}(x)=\mathrm{id}_H$ for all $x\in\pi_1(M_0)$), then

$$Z_H^{\rho_{\mathrm{triv}}}(M_0,\gamma_0,\mathfrak{s},\omega)=\pm Z_H^{Kup}(Y).$$

The idea of the construction of the invariant Z_H^ρ is the following. First, sutured 3-manifolds are represented by sutured Heegaard diagrams, that is, Heegaard diagrams in which the Heegaard surface has boundary [Juh06]. Then the invariant of the above theorem is roughly defined from such a diagram by "twisting" the construction of [Kup91] using the representation ρ. This twisting takes a form of Fox calculus and is justified by considering the semidirect product $\mathbb{K}[\mathrm{Aut}(H)] \ltimes H$. Indeed, the latter Hopf algebra can be considered as a Hopf group-algebra (with group $\mathrm{Aut}(H)$), the dual notion of Turaev's Hopf group-coalgebras [Tur00], and we show that the group-algebra multiplication reduces to "Fox calculus" over H. Note that in [Vir05], Virelizier extended Kuperberg's invariant by using an involutive Hopf G-coalgebra, leading to an invariant of pairs (Y, ρ) where Y is a closed 3-manifold and $\rho : \pi_1(Y) \to G$ is a group homomorphism. Our construction can be seen as an extension of (the dual version of) Virelizier's construction to sutured 3-manifolds, but restricted to a semidirect product. It has to be noted that both [Kup91], [Vir05] assume semisimplicity of the Hopf algebras, hence no extra structure is needed on 3-manifolds.

A relative Spin^c structure is a very simple extra piece of structure on (M, γ) (indeed, they are parametrized by $H_1(M;\mathbb{Z})$) and they appear by the following reason: the semidirect product $\mathbb{K}[\mathrm{Aut}(H)] \ltimes H$ may be non-unimodular, even if H is unimodular. The comodulus of the semidirect product is essentially determined by the homomorphism $r_H : \mathrm{Aut}(H) \to \mathbb{K}^\times$ characterized by $\phi(c) = r_H(\phi)c$ for any $\phi \in \mathrm{Aut}(H)$, hence the invariant does not depend on Spin^c structures when $\mathrm{Im}\,(\rho) \subset \mathrm{Ker}\,(r_H)$ (indeed the Hopf subalgebra $\mathbb{K}[\mathrm{Ker}\,(r_H)] \ltimes H$ is unimodular). This is similar to what happens with Reidemeister torsion (cf. [Tur01] or [FJR11] in the sutured case): the $GL(V)$-torsion is normalized with a Spin^c structure, but the $SL(V)$-torsion is independent of it. An advantage of Spin^c structures is that they are easily represented on a sutured Heegaard diagram via multipoints as in [Juh06], therefore posing no further complexity in computation as opposed to the framings of [Kup96] which would be necessary in an involutive non-unimodular case. On the other hand, homology orientations arise in order to fix some sign indeterminacies when the cointegral has degree one. This is also a phenomenon existing in Reidemeister torsion theory, cf. [Tur01, FJR11].

A special feature of our construction is that, when H is $\mathbb{N}$-graded, the above invariants can be lifted to polynomials by a canonical "degree twist". More precisely, let H_M be H with the coefficients extended to the group ring $\mathbb{K}[H_1(M)]$, that is $H_M \coloneqq H \otimes_{\mathbb{K}} \mathbb{K}[H_1(M)]$. Then there is a canonical "degree" representation $h : H_1(M;\mathbb{Z}) \to \mathrm{Aut}(H_M)$, where $x \in H_1(M)$ acts on the degree n part of H_M by multiplication by x^n. This can be combined with an arbitrary representation $\rho : \pi_1(M) \to \mathrm{Aut}(H)$ to define a *twisted Kuperberg polynomial*

$$Z_{H_M}^{\rho\otimes h}(M, \gamma) \in \mathbb{K}[H_1(M)].$$

This is defined up to unit of $\mathbb{K}[H_1(M)]$, which can be fixed by picking a Spin^c structure and homology orientation.

Our second main result, constituting Chapter 4, specializes the above construction to a particular Hopf algebra and relates it to classical invariants. More precisely, let $\Lambda(V)$ be the exterior algebra over a finite dimensional vector space V. This is a Hopf superalgebra satisfying the hypothesis of our theorem and one has $\mathrm{Aut}(\Lambda(V)) \cong GL(V)$. The homomorphism $r_{\Lambda(V)} : GL(V) \to \mathbb{K}^\times$ is the determinant so $\mathrm{Ker}\,(r_{\Lambda(V)}) = SL(V)$. Since $\Lambda(V)$ is $\mathbb{N}$-graded, we can define twisted Kuperberg polynomials having values in $\mathbb{K}[H_1(M)]$.

Theorem 2. *Let (M, γ) be a balanced sutured 3-manifold and let $\rho : \pi_1(M) \to SL(V)$ be an homomorphism. Then the twisted Kuperberg polynomial at $H = \Lambda(V)$ reduces to twisted Reidemeister torsion:*

$$Z^{\rho\otimes h}_{\Lambda(V)_M}(M, \gamma) = \tau^{(\rho\otimes h)^{-t}}(M, R_-(\gamma))$$

in $\mathbb{K}[H_1(M)]/ \pm H_1(M)$.

In particular, our invariant is equivalent to the twisted Alexander polynomials of Lin and Wada [Lin01, Wad94] if (M, γ) is the sutured manifold associated to a link complement and H is an exterior algebra. In other words, our procedure of "degree twisting" Kuperberg invariants leads to powerful invariants containing a lot of topological information. Indeed, twisted Alexander polynomials can detect mutation, non-invertibility of some knots, the knot genus and fiberedness [Wad94, KL99, FV11b, FV15]. The relative torsion of $(M, R_-(\gamma))$ is also an interesting invariant for Seifert surface complements as shown in [Alt12]. However, if M is the sutured manifold associated to a closed 3-manifold Y (that is $M = Y \setminus B^3$ and $\gamma = S^1 \subset \partial B$ where B is a 3-ball), then $\tau(M, R_-)$ is uninteresting. In particular, our theorem does not recovers the absolute torsion $\tau(Y)$ as in [BCGPM16]. This indicates that Kuperberg's construction admits a further refinement in the case of closed 3-manifolds.

Theorem 2 suggests that Kuperberg invariants may be categorified with the methods of Lagrangian Floer homology. Indeed, it is shown in [FJR11] that the Reidemeister torsion $\tau^{1\otimes h}(M, R_-)$ is the Euler characteristic of the sutured Floer homology of Juhász [Juh06]. That Kuperberg invariants may be categorified was indeed suggested by Crane-Frenkel themselves in their influential paper [CF94] which started the categorification program, though with a more algebraic machinery.

A first version of the above results appeared in [LN19a]. There we introduced relative notions of the Hopf algebra cointegral and integral, where a relative cointegral in a Hopf algebra J is a map $c : A \to J$ satisfying some properties analogue to that of a cointegral, but relative to a Hopf subalgebra $A \subset J$. Similarly, we introduced relative integrals $\mu : J \to B$, where $B \subset J$ is another Hopf subalgebra. Under certain conditions, we showed that an involutive Hopf algebra endowed with these relative structures produced a topological invariant of balanced sutured 3-manifolds endowed with an appropriate representation of $H_1(M)$. We found this structure on the Borel part J of quantum $\mathfrak{gl}(1|1)$ and showed that the resulting invariant was the abelian relative Reidemeister torsion. In our second paper [LN19b], we considerably improved a particular case of this approach. We restricted to a semidirect product $J = \mathbb{K}[\mathrm{Aut}(H)] \ltimes H$ and took the subalgebra $A = \mathbb{K}[\mathrm{Aut}(H)]$. However, instead of using relative cointegrals, we considered J as an $\mathrm{Aut}(H)$-graded Hopf algebra, so we had graded cointegrals for free as shown in [Vir02] (in the dual case). Moreover, we realized that Fox calculus only comes from the semidirect product structure, and is not special to the Borel of quantum $\mathfrak{gl}(1|1)$ as we initially considered.

The manuscript is organized as follows. Chapter 1 contains the necessary preliminaries on Hopf algebras, namely, we discuss the involutivity condition, the theory of Hopf algebra integrals, and Hopf group-algebras from semidirect products. In Chapter 2 we define sutured 3-manifolds, sutured Heegaard diagrams and extended Heegaard diagrams. Chapter 3 is the body of this thesis. Here we construct the invariant of Theorem 1. We treat first the unimodular case, that is, when $\mathrm{Im}\,(\rho) \subset \mathrm{Ker}\,(r_H)$ and then we treat the general case using Spin^c structures. We end this chapter with the definition of the twisted Kuperberg polynomials. Finally, in Chapter 4 we recall some facts from Reidemeister torsion theory and prove Theorem 2.

Chapter 1

Hopf superalgebras

In this chapter, we introduce the notions from Hopf algebra theory that we will use throughout this thesis. In Section 1.1 we define (involutive) Hopf superalgebras, we give several examples and we explore further the involutivity condition through bosonization and its relation to semisimplicity. In Section 1.2 we define integrals and cointegrals of Hopf algebras, which are the main ingredients in the construction of 3-manifold invariants. We end with a short discussion on Hopf G-algebras, the dual notion of Turaev's Hopf G-coalgebras, and we give a good supply of examples.

1.1 Basic notions and examples

In what follows, we let $\mathbb{K}$ be a field. Vector spaces will be assumed to be over $\mathbb{K}$.

1.1.1 Super vector spaces

Definition 1.1.1. A *super vector space* consists of a vector space V endowed with a direct sum decomposition $V = V_0 \oplus V_1$. A vector $v \in V$ is said to be *homogeneous* if $v \in V_0$ or $v \in V_1$. If $v \in V_i$, we say that v has *degree* i, denoted $|v| = i$. A linear map $f : V \to W$ is said to have *degree* k if $f(V_i) \subset W_{i+k}$ for $i = 0, 1$. Both notions of degree are considered mod 2. We denote by $\mathrm{SVect}_{\mathbb{K}}$ the category whose objects are super vector spaces and whose morphisms are degree zero linear maps.

The category $\mathrm{SVect}_{\mathbb{K}}$ has some extra structure, namely, it is a *symmetric monoidal category*. This means that there is a tensor product $\otimes$, a unit object $\mathbb{1}$ and a family of isomorphisms $\tau_{V,W} : V \otimes W \to W \otimes V$ satisfying a bunch of axioms. The symmetry condition refers to the fact that $\tau_{W,V} \circ \tau_{V,W} = \mathrm{id}_{V \otimes W}$. Indeed, if V, W are super vector spaces, then the vector space tensor product $V \otimes_{\mathbb{K}} W$ is a super vector space with

$$(V \otimes W)_i := \bigoplus_j V_j \otimes W_{i-j}$$

where the indices are taken mod 2. The unit object is $\mathbb{K}$, considered as a super vector space concentrated in degree zero. The symmetry

$$\tau_{V,W} : V \otimes W \to W \otimes V$$

is defined on homogeneous elements by

$$\begin{aligned} \tau_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto (-1)^{|v||w|} w \otimes v. \end{aligned}$$

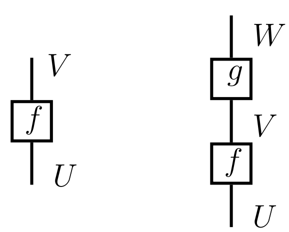
Notation 1.1.2. Let V be a super vector space. For each $n \geq 1$ and $1 \leq i \leq n-1$ we can define $\tau_i : V^{\otimes n} \to V^{\otimes n}$ by $\tau_i := \mathrm{id}_V^{\otimes(i-1)} \otimes \tau_{V,V} \otimes \mathrm{id}_V^{\otimes(n-i-1)}$ where $\tau_{V,V}$ is the above symmetry. It is not difficult to see that the correspondence $\sigma_i \mapsto \tau_i$ defines a representation $S_n \to GL_{\mathrm{SVect}_{\mathbb{K}}}(V^{\otimes n})$, where σ_i is the usual transposition $(i\ i+1)$ (the latter denotes automorphisms in the category $\mathrm{SVect}_{\mathbb{K}}$). We denote by P_τ the image of $\tau \in S_n$ under this map. We also denote by P'_τ the unsigned permutation (i.e. we use the symmetry c of $\mathrm{Vect}_{\mathbb{K}}$ instead). Thus if $v_1, \ldots, v_n \in V$ have degree one, then

$$P_\tau(\overline{v}) = \mathrm{sign}\,(\tau) v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n)} = \mathrm{sign}\,(\tau) P'_\tau(\overline{v})$$

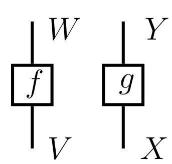
where $\overline{v} = v_1 \otimes \ldots \otimes v_n$.

1.1.2 Graphical notation

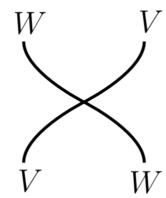
Let $\mathcal{S}$ be a symmetric monoidal category, with tensor product $\otimes$, unit $\mathbb{1}$ and symmetry isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ for any objects V, W of $\mathcal{S}$ (in this thesis we will restrict to $\mathcal{S} = \mathrm{SVect}_{\mathbb{K}}$). We will represent morphisms in $\mathcal{S}$ by diagrams to be read from bottom to top. For example, if $f : U \to V$ and $g : V \to W$ are morphisms in $\mathcal{S}$ we depict f and gf respectively as follows:



If $f : V \to W$ and $g : X \to Y$ are two morphisms, then $f \otimes g$ is represented by juxtaposition:



If V, W are two objects of $\mathcal{S}$, the symmetry isomorphism $c_{V,W}$ is depicted by



There is no over-crossing in the above picture since we work with symmetric monoidal categories (as opposed to braided monoidal categories).

1.1.3 Hopf superalgebras

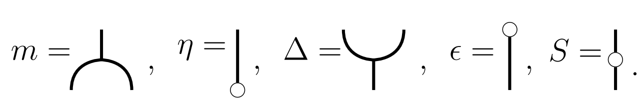
By a *superalgebra* we mean a $\mathbb{K}$-algebra (A, m, η), where $m : A \otimes A \to A$ is the multiplication and $\eta : \mathbb{K} \to A$ is the unit, in which A has a super-vector space structure satisfying $m(A_i \otimes A_j) \subset A_{i+j}$ for any i, j. If A, B are superalgebras, then $A \otimes B$ is a superalgebra with the product defined over homogeneous elements by

$$(a \otimes b)(a' \otimes b') \coloneqq (-1)^{|b||a'|} aa' \otimes bb'.$$

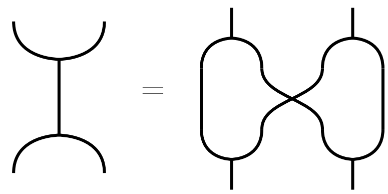
Definition 1.1.3. A *Hopf superalgebra* is a superalgebra (H, m, η) endowed with degree zero linear maps $\Delta : H \to H \otimes H, \epsilon : H \to \mathbb{K}$ and $S : H \to H$ satisfying the following axioms:

1. Δ is a superalgebra morphism for the above superalgebra structure on $H \otimes H$.
2. ϵ is a superalgebra morphism, where $\mathbb{K}$ is concentrated in degree zero.
3. One has
$$(\epsilon \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_H \otimes \epsilon)\Delta = \mathrm{id}_H.$$
4. The map S satisfies
$$m(\mathrm{id}_H \otimes S)\Delta = \eta\epsilon = m(S \otimes \mathrm{id}_H)\Delta.$$

We call Δ the coproduct, ϵ the counit and S the antipode. We will both use Sweedler's notation for the coproduct, that is, we write $\Delta(x) = x_{(1)} \otimes x_{(2)}$ omitting the sumation sign, as well as the following graphical notation for the Hopf algebra tensors:



This way, we can use the graphical notation to write the Hopf superalgebra axioms. For example, the fact that Δ is an algebra morphism is written as



where we used the symmetry of $\mathrm{SVect}_{\mathbb{K}}$ on the right. The Hopf algebra axioms imply that the antipode is an algebra and coalgebra antihomomorphism, that is

$$S(xy) = (-1)^{|x||y|} S(y)S(x), \qquad \Delta(S(x)) = (-1)^{|x_{(1)}||x_{(2)}|} S(x_{(2)}) \otimes S(x_{(1)})$$

for any homogeneous $x, y \in H$.

Remark 1.1.4. Any (ungraded) Hopf algebra can be seen as a Hopf superalgebra concentrated in degree zero. In what follows we reserve the term Hopf algebra exclusively for the ungraded case.

Definition 1.1.5. We say that H is *commutative* if $m = m \circ \tau_{H,H}$ and *cocommutative* if $\Delta = \tau_{H,H} \circ \Delta$ where $\tau_{H,H}$ is the symmetry map. We say that H is *involutive* if $S^2 = \mathrm{id}_H$.

Lemma 1.1.6. *A commutative or cocommutative Hopf superalgebra is involutive.*

Proof. This is standard (see e.g. [Rad12]) but we give a proof for completeness. By definition, the antipode S is characterized by

$$S(x_{(1)})x_{(2)} = \epsilon(x)$$

for any $x \in H$, where we use Sweedler's notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Applying S^{-1} on both sides and assuming H cocommutative, we get

$$S^{-1}(x_{(1)})x_{(2)} = \epsilon(x).$$

Thus, S^{-1} is also an antipode and therefore $S = S^{-1}$ i.e. $S^2 = \mathrm{id}_H$ by uniqueness of the antipode. A similar argument applies if H is commutative. $\square$

Definition 1.1.7. If $(H, m, \eta, \Delta, \epsilon, S)$ is a finite dimensional Hopf superalgebra, then the dual vector space H^* becomes a Hopf superalgebra if we set $H_i^* = (H_i)^*$ for $i = 0, 1$ and we dualize all structure maps, that is, $m_{H^*} := \Delta^*, \eta_{H^*} := \epsilon^*, \Delta_{H^*} := m^*, \epsilon_{H^*} := \eta^*$ and $S_{H^*} := S^*$. This requires an identification $(H \otimes H)^* \cong H^* \otimes H^*$ which we take as

$$(f \otimes g)(x \otimes y) := f(x) \otimes g(y).$$

Note that there is no sign in the latter equation, this ensures that Δ_{H^*} is a superalgebra morphism. Note also that this is the opposite of the convention used in [KV19].

Finally, we give a name to some very special elements of a Hopf superalgebra.

Definition 1.1.8. An element $g \in H$ is said to be *group-like* if $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$. An element $x \in H$ is said to be *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$.

The sets of group-like and primitive elements of H are respectively denoted by $G(H)$ and $P(H)$. It is easy to see that $G(H)$ is a group with the multiplication of H, $P(H)$ is a Lie superalgebra with the bracket $[x, y] := xy-(-1)^{|x||y|}yx$ on homogeneous $x, y \in H$ and that $G(H)$ acts on $P(H)$ by conjugation.

1.1.4 Examples

We now give several examples of involutive Hopf superalgebras. The three examples below are cocommutative and hence involutive.

Example 1.1.9. Let G be a finite group. Let $\mathbb{K}[G]$ be the $\mathbb{K}$-vector space with basis the elements of G. This is an algebra if we extend the multiplication of G by linearity. It is a Hopf algebra if we set

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1},$$

for all $g \in G$.

Example 1.1.10. Let V be a finite dimensional vector space. The *exterior algebra* $\Lambda(V)$ on V is the quotient of the tensor algebra $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ (where $V^{\otimes 0} = \mathbb{K}$) by the ideal generated by the elements of the form $v \otimes w + w \otimes v$ with $v, w \in V$. This becomes a superalgebra by letting $V \subset \Lambda(V)$ be in degree one and it is a Hopf superalgebra if we set

$$\Delta(v) = 1 \otimes v + v \otimes 1, \qquad \epsilon(v) = 0, \qquad S(v) = -\,v$$

for any $v \in V$ and extend Δ, ϵ (resp. S) by letting them be superalgebra homomorphisms (resp. antihomomorphism). More generally, if $\mathfrak{g}$ is a Lie superalgebra, then the universal enveloping algebra $U(\mathfrak{g})$ is a cocommutative Hopf superalgebra constructed in a similar way, only that one takes the quotient of $T(\mathfrak{g})$ by the ideal generated by the $[x, y] - x \otimes y + (-1)^{|x||y|} y \otimes x$ for $x, y \in \mathfrak{g}$. The exterior algebra is obtained when $\mathfrak{g}$ is a trivial Lie algebra concentrated in degree one. However, we will mainly be interested in finite dimensional examples, and exterior algebras are the only ones among enveloping algebras.

Example 1.1.11. Let G be a finite group acting on a finite dimensional vector space V. Then there is a semidirect product Hopf superalgebra $H = \mathbb{K}[G] \ltimes \Lambda(V)$. Here the product is characterized by the fact that $\mathbb{K}[G]$ and $\Lambda(V)$ are Hopf subalgebras, and

$$g \cdot v = g(v) \cdot g$$

for any $g \in G, v \in V$, where $v \to g(v)$ is the action of g on V and $\cdot$ denotes the product of H.

Remark 1.1.12. If $\mathbb{K}$ is algebraically closed of characteristic zero, the above examples exhaust the class of cocommutative Hopf superalgebras. This is a theorem originally due to Milnor-Moore and subsequently refined by Kostant and Cartier-Gabriel, see [Car07, Theorem 3.8.2] or [AEG01, Theorem 2.3.4]. This states that if H is a cocommutative Hopf superalgebra over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}\,(\mathbb{K}) = 0$, then there is an isomorphism of Hopf superalgebras

$$H \cong \mathbb{K}[G(H)] \ltimes U(P(H))$$

where $U(P(H))$ is the universal enveloping algebra of the Lie superalgebra $P(H)$. The above semidirect product is finite dimensional only when $G(H)$ is finite and $P(H)$ is trivial, as in Example 1.1.11.

In positive characteristic there are much more examples of finite dimensional cocommutative Hopf (super)algebras. Indeed, if $\mathfrak{g}$ is an arbitrary finite dimensional *p-restricted* Lie (super)algebra over a field $\mathbb{K}$ with $\mathrm{char}\,(\mathbb{K}) = p > 0$, then the *restricted* enveloping algebra $U^{res}(\mathfrak{g})$ is a finite dimensional cocommutative Hopf (super)algebra. For example, the Hopf algebra $\mathbb{K}[X]/(X^{p^n})$ (in which X is primitive) is among this class for any $n \geq 1$.

Of course, there are (finite-dimensional) involutive Hopf superalgebras which are neither commutative nor cocommutative. The quantum group $U_q(\mathfrak{gl}(1|1))$ at a root of unity (see e.g. [Sar15]) is such an example, for another one see [KV19, Example 5.6].

1.1.5 Bosonization of Hopf superalgebras

We describe a useful procedure to pass from a Hopf superalgebra to a Hopf algebra and conversely. We follow [AEG01].

Let $H = (H_0 \oplus H_1, m, 1, \Delta, \epsilon, S)$ be a Hopf superalgebra. Let $g : H \to H$ be the Hopf automorphism defined by the degree, that is, $g(x) := (-1)^{|x|}x$ for homogeneous $x \in H$. Let H' be the semidirect product Hopf algebra $\mathbb{K}[\mathbb{Z}/2\mathbb{Z}] \ltimes H$, where $\mathbb{Z}/2\mathbb{Z}$ acts over H by g. We use the same notation for the structure maps of H'. Write

$$\Delta(h') = \Delta_0(h') + \Delta_1(h')$$

where $\Delta_i(h') \in H' \otimes H'_i$ for $i = 0, 1$ and homogeneous $h' \in H'$. Now set

$$\Delta'(h') \coloneqq \Delta_0(h') - (-1)^{|h'|}(g \otimes 1)\Delta_1(h'), \qquad S'(h') \coloneqq g^{|h'|}S(h').$$

Then $(H', m, 1, \Delta', S')$ is an ordinary (ungraded) Hopf algebra.

Definition 1.1.13. Given a Hopf superalgebra H, the *bosonization* of H is the Hopf algebra $(H', m, 1, \Delta', \epsilon, S')$ constructed above.

Example 1.1.14. Suppose $H = \mathbb{K}[X]/(X^2)$ is an exterior algebra in one generator (of degree one). It is easy to see that the bosonization of H is the Hopf algebra H' with two generators K, X satisfying $KX = -XK, X^2 = 0, K^2 = 1$ and coproduct defined by

$$\Delta(K) = K \otimes K, \qquad \Delta(X) = X \otimes 1 + K \otimes X.$$

In other words, the bosonization of an exterior algebra is the Borel part of $U_q(\mathfrak{sl}_2)$ at $q = i$. Thus, the bosonization of a commutative and cocommutative Hopf superalgebra can be non-commutative and non-cocommutative.

Proposition 1.1.15. *The representation categories of a Hopf superalgebra and of its bosonization are monoidally equivalent.*

Here, a representation of a Hopf superalgebra is a super vector space $V = V_0 \oplus V_1$ together with an action $H \otimes V \to V$ that preserves the degree. It is easy to see that V extends to a representation (still denoted V) of the semidirect product $\mathbb{K}[\mathbb{Z}/2\mathbb{Z}] \ltimes H$. Now, let V' be the representation of this semidirect product, but where we forget the mod 2 grading. Then one can prove that the correspondence $V \mapsto V'$ defines the desired equivalence of monoidal categories.

Thus, for the purposes of representation theory, there is not an essential difference in working with a Hopf superalgebra H or its bosonization H'. However, in Kuperberg's approach to 3-manifold invariants, involutivity makes things much simpler. Noting that $(S')^2(h') = (-1)^{|h'|}S^2(h')$ for any homogeneous $h' \in H$ we see that if H is involutive and $H_1 \neq 0$, then H' is non-involutive. Thus, for the purpose of Kuperberg's 3-manifold invariants, it is simpler to work with Hopf superalgebras rather than their bosonizations.

1.1.6 Involutivity and semisimplicity

We now discuss some important consequences of the involutivity condition.

Theorem 1.1.16. *([LR88a, LR88b]) Let H be a finite dimensional (ungraded) Hopf algebra over a field of characteristic zero. Then H is involutive if and only if H is semisimple. This extends to positive characteristic as follows: H is involutive and* $\dim(H) \neq 0$ *if and only if H is semisimple and cosemisimple.*

As far as the author knows, all examples of semisimple Hopf algebras (in characteristic zero) are somehow built from group algebras. Therefore, no interesting Hopf algebras (for our purposes) will be found in such case. Fortunately, the aforementioned theorem of Larson-Radford is no longer true for Hopf superalgebras. Indeed, the opposite holds [AEG01, Corollary 3.1.2]:

Proposition 1.1.17. *If $H = H_0 \oplus H_1$ is a finite dimensional involutive Hopf superalgebra over a field of characteristic zero, then H is non-semisimple if and only if $H_1 \neq 0$.*

Proof. We sketch the proof given in [AEG01]. If $H_1 = 0$, then H is semisimple by the theorem above. For the converse, let H' be the bosonization of H. If H is involutive, then $(S')^2(x) = (-1)^{|x|}x$ so H' is non-involutive if $H_1 \neq 0$. Therefore, Larson-Radford's theorem implies that H' is non-semisimple and hence so is H, since their representation categories are equivalent by Proposition 1.1.15. □

1.2 Integrals and cointegrals

We now turn to introduce one of the fundamental notions of Hopf algebra theory, that of cointegrals and integrals. This notion has its origins in the theory of Lie groups, where the property of right invariance of the Haar integral can be appropriately abstracted leading to the notion of Hopf algebra integral. We begin by giving the basic definitions, and then we discuss the unimodularity condition and some properties.

1.2.1 Definitions and examples

Definition 1.2.1. A *right cointegral* in a finite dimensional Hopf superalgebra H is an element $c_r \in H$ such that

$$c_r \cdot x = c_r \cdot \epsilon(x)$$

for all $x \in H$. A left cointegral of H is defined as a right cointegral of H^{op}. A *right integral* is an element $\mu_r \in H^*$ such that

$$(\mu_r \otimes \mathrm{id}_H)\Delta(x) = \mu_r(x)1_H$$

for all $x \in H$. Equivalently, a right integral over H is the same as a right cointegral of H^*.

Theorem 1.2.2 ([LS69])**.** *If H is a finite dimensional Hopf superalgebra over an arbitrary field $\mathbb{K}$, then there is a right cointegral and it is unique up to scalar.*

By applying this theorem to H^{op} or H^* one also obtains existence and uniqueness (up to scalar) of left cointegrals and right or left integrals.

Example 1.2.3. Let G be a finite group. Then $c_G \coloneqq \sum_{g\in G} g$ is a two-sided cointegral in $\mathbb{K}[G]$. The functional defined by $\mu(g) \coloneqq \delta_{g,e}$ for $g \in G$ is a two-sided integral of $\mathbb{K}[G]$.

Example 1.2.4. Let $\Lambda(V)$ be the exterior algebra over a finite-dimensional vector space V. Let $X_1, \ldots, X_n$ be a basis of V. Then the product $c_V \coloneqq X_1 \wedge \cdots \wedge X_n$ is a two-sided cointegral and the functional defined by $\mu(X_1^{i_1} \wedge \cdots \wedge X_n^{i_n}) = \delta_{i_1,1} \ldots \delta_{i_n,1}$ is a two-sided integral over $\Lambda(V)$.

Example 1.2.5. Let G be a finite group acting on a finite dimensional vector space V. Let $H = \mathbb{K}[G] \ltimes \Lambda(V)$ as in Example 1.1.11. Let c_G, and c_V be the cointegrals of $\mathbb{K}[G]$ and $\Lambda(V)$ respectively, as defined above. Then

$$c_l \coloneqq c_G \cdot c_V, \qquad c_r \coloneqq c_V \cdot c_G$$

are respectively a left cointegral and a right cointegral of H. However, noting that $g(c_V) = \det(g)c_V$ for any $g \in G$, one has

$$\begin{aligned} c_l = c_G \cdot c_V &= \sum_{g\in G} g \cdot c_V \\ &= \sum_{g\in G} g(c_V) \cdot g \\ &= \sum_{g\in G} \det(g) c_V \cdot g \end{aligned}$$

which is different from c_r, even up to a scalar, provided $\det(g : V \to V) \neq 1$ for some $g \in G$. Thus, a left cointegral is not a right cointegral in such a case. The integral of H is given by $\mu_{\mathbb{K}[G]} \otimes \mu_{\Lambda(V)}$, which is two-sided (since H is cocommutative).

1.2.2 Unimodularity

As shown in the last section, it is not always true that a left cointegral is also a right cointegral, and similarly for integrals. We now turn to discuss this issue.

Definition 1.2.6. A Hopf superalgebra H is said to be *unimodular* if any right cointegral is also a left cointegral. We say that H is counimodular if H^* is unimodular.

Example 1.2.7. Group-algebras and exterior algebras are always unimodular and counimodular. If a finite group G acts over a finite dimensional vector space V through an homomorphism $\varphi : G \to GL(V)$, then the semidirect product $\mathbb{K}[G] \ltimes \Lambda(V)$ is unimodular if and only if $\varphi(G) \subset SL(V)$ as shown in Example 1.2.5.

Remark 1.2.8. By Theorem 1.1.16, a finite dimensional (ungraded) involutive Hopf algebra H over a characteristic zero field is semisimple and hence unimodular. However, an involutive Hopf superalgebra may be non-unimodular as Example 1.2.5 shows.

We now study the action of the group of Hopf algebra automorphisms over cointegrals.

Definition 1.2.9. Let c_r be a right cointegral in H and let $\alpha \in \mathrm{Aut}(H)$. Clearly, $\alpha(c_r)$ is a right cointegral so by uniqueness, there is a scalar $r_H(\alpha) \in \mathbb{K}^\times$ such that

$$\alpha(c_r) = r_H(\alpha)c_r.$$

This defines a group homomorphism $r_H : \mathrm{Aut}(H) \to \mathbb{K}^\times$. Note that if μ_r is a right integral normalized by $\mu_r(c_r) = 1$, then r_H can also be defined by

$$r_H(\alpha) = \mu_r(\alpha(c_r)).$$

Remark 1.2.10. The homomorphism r_H can be thought as the distinguished group-like of the Hopf G-algebra $\underline{H}$ associated to $\mathbb{K}[G] \ltimes H$ with $G = \mathrm{Aut}(H)$, see Section 1.3 below. Therefore, if $r_H \neq 1$, $\underline{H}$ is non-unimodular even if H is unimodular.

Example 1.2.11. Let $\Lambda(V)$ be the exterior algebra on a finite dimensional vector space V. Any automorphism of $\Lambda(V)$ defines a linear isomorphism over its subspace of primitive elements, which is V. Conversely, any linear isomorphism of V extends to a Hopf automorphism of $\Lambda(V)$ so $\mathrm{Aut}(\Lambda(V)) \cong GL(V)$. Recall from Example 1.2.4 that $c_V := X_1 \wedge \cdots \wedge X_n$ is a cointegral of $\Lambda(V)$, where $X_1, \ldots, X_n$ is any basis of V and that

$$\alpha(c_V) = \alpha(X_1) \wedge \cdots \wedge \alpha(X_n) = \det(\alpha) X_1 \wedge \cdots \wedge X_n = \det(\alpha) c_V$$

for any $\alpha \in \mathrm{Aut}(\Lambda(V))$. Thus, $r_{\Lambda(V)} : GL(V) \to \mathbb{K}^\times$ is just the determinant.

Proposition 1.2.12. *Let H be a finite-dimensional unimodular Hopf superalgebra and c be its two-sided cointegral. Then the following holds:*

1. $S(c) = (-1)^{|c|}c$,
2. $\Delta(c) = \Delta^{op}(c)$,
3. $\mu \circ \alpha = r_H(\alpha)\mu$ *for any* $\alpha \in \mathrm{Aut}(H)$, *where* r_H *is defined as above.*

Proof. The first two properties are standard, cf. [Rad12, Chapter 10]. The third one follows from uniqueness of integrals: if μ_r is a right integral, then $\mu_r \circ \alpha = r'_H(\alpha)\mu_r$ for some scalar $r'_H(\alpha) \in \mathbb{K}^\times$. Evaluating on c_r gives

$$\mu_r(r_H(\alpha)c_r) = \mu_r(\alpha(c_r)) = r'_H(\alpha)\mu_r(c_r)$$

and since $\mu_r(c_r) \neq 0$, the equality $r_H(\alpha) = r'_H(\alpha)$ follows. ☐

Remark 1.2.13. Note that a Hopf algebra is semisimple if and only if $\epsilon(c_r) \neq 0$, see [LS69, Proposition 3]. This implies that $r_H(\alpha) = 1$ for any $\alpha \in \mathrm{Aut}(H)$. Note further that if H is semisimple and the base field has characteristic zero, then $\mathrm{Aut}(H)$ is a finite group by [Rad90].

1.3 Hopf G-algebras

All the concepts introduced so far admit a graded version, where the grading group may be an arbitrary group G. This is the theory of Hopf G-coalgebras introduced by Turaev [Tur00] in order to build *homotopy field theories*, and further developed by Virelizier [Vir02]. This theory can be seen as an “equivariant" Hopf algebra theory. Here we prefer to work with the dual notion of a Hopf G-algebra, since semidirect products provide a general class of examples. All the results of [Vir02] apply to our setting.

1.3.1 Definitions and basic properties

Definition 1.3.1. Let G be a group with neutral element denoted by 1_G. A *Hopf G-algebra* is a family of coalgebras $\underline{H} = \{(H_\alpha, \Delta_\alpha, \epsilon_\alpha)\}_{\alpha\in G}$ indexed by $\alpha \in G$ endowed with coalgebra morphisms

$$m_{\alpha_1,\alpha_2} : H_{\alpha_1} \otimes H_{\alpha_2} \to H_{\alpha_1\alpha_2},$$

a unit $1 \in H_{1_G}$ and maps $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ satisfying graded versions of the associativity, unitality and antipode axioms (see [Vir02] for the dual notion). Note that H_{1_G} is a Hopf algebra in the usual sense. We say that $\underline{H}$ is *involutive* if $S_{\alpha^{-1}}S_\alpha = \mathrm{id}_{H_\alpha}$ for each $\alpha \in G$. We say that $\underline{H}$ is of *finite type* if each H_α is finite dimensional. If $\underline{H}$ is of finite type, a *right cointegral* is a family $c = \{c_\alpha\}_{\alpha\in G}$, where $c_\alpha \in H_\alpha$ for each $\alpha \in G$, satisfying

$$c_{\alpha_1} \cdot x = \epsilon_{\alpha_2}(x) \cdot c_{\alpha_1\alpha_2}$$

for all $\alpha_1, \alpha_2 \in G$ and $x \in H_{\alpha_2}$. Hopf G-superalgebras are defined in a similar way, where each H_α is a super-coalgebra and the multiplication involves a (Koszul) sign when appropriate.

Let $\underline{H} = \{H_\alpha\}_{\alpha\in G}$ be a finite type Hopf G-algebra. Since the dual of a Hopf G-algebra is a Hopf G-coalgebra, the existence and uniqueness theorems of integrals of [Vir02] have analogous statements in the G-algebra case. In particular, there exists a unique family $g^* = \{g^*_\alpha\}_{\alpha\in G}$ where $g^*_\alpha \in H^*_\alpha$ satisfying

$$x \cdot c_{\alpha_2} = c_{\alpha_1\alpha_2} \cdot g^*_{\alpha_1}(x)$$

for all $x \in H_{\alpha_1}$ and $\alpha_1, \alpha_2 \in G$. We call g^* the *comodulus* of $\underline{H}$.

1.3.2 The semidirect product case

Now let H be a finite dimensional Hopf (super)algebra with automorphism group G. Recall that the semidirect product $\mathbb{K}[G] \ltimes H$ is a Hopf algebra with the tensor product coalgebra structure and with the algebra structure given by $\alpha \cdot x = \alpha(x) \cdot \alpha$ for $\alpha \in G, x \in H$.

Proposition 1.3.2. *Let $G = Aut(H)$. If*

$$H_\alpha \coloneqq \{x \cdot \alpha \mid x \in H\} \subset \mathbb{K}[G] \ltimes H$$

then $\underline{H} \coloneqq \{H_\alpha\}_{\alpha \in G}$ is a Hopf G-algebra with the structure morphisms induced from the semidirect product. A right cointegral is given by $c_\alpha \coloneqq c_r \cdot \alpha \in H_\alpha$ where c_r is a right cointegral of H. The comodulus of $\underline{H}$ is given by

$$g^*_\alpha(x \cdot \alpha) \coloneqq r_H(\alpha) g^*_1(x)$$

for $x \in H, \alpha \in G$, where $g^*_1 \in H^*$ is the comodulus of H.

Here, as usual, $r_H : \mathrm{Aut}(H) \to \mathbb{K}^\times$ is the homomorphism characterized by $\alpha(c_r) = r_H(\alpha) c_r$.

Proof. That $\underline{H}$ is a Hopf G-algebra follows from the definitions, for instance, if $x\alpha_1 \in H_{\alpha_1}, x'\alpha_2 \in H_{\alpha_2}$ then

$$(x\alpha_1)(x'\alpha_2) = x(\alpha_1(x'))\alpha_1\alpha_2 \in H_{\alpha_1\alpha_2}.$$

For the second assertion, let $x_{\alpha_2} = x\alpha_2 \in H_{\alpha_2}$ then

$$\begin{aligned} c_{\alpha_1} \cdot x_{\alpha_2} = c_r\alpha_1 x\alpha_2 = c_r\alpha_1(x)\alpha_1\alpha_2 = c_r\epsilon(\alpha_1(x))\alpha_1\alpha_2 &= c_r\epsilon(x)\alpha_1\alpha_2 \\ &= c_{\alpha_1\alpha_2}\epsilon_{\alpha_2}(x_{\alpha_2}) \end{aligned}$$

so that $(c_r \cdot \alpha)$ is a right integral in $\underline{H}$. For the third assertion, let $x_{\alpha_1} = x\alpha_1 \in H_{\alpha_1}$ $(x \in H)$ then one has

$$\begin{aligned} x_{\alpha_1} \cdot c_{\alpha_2} = (x\alpha_1)(c_r\alpha_2) = x\alpha_1(c_r)\alpha_1\alpha_2 &= r_H(\alpha_1) x c_r \alpha_1\alpha_2 \\ &= r_H(\alpha_1) g^*_1(x) c_r \alpha_1\alpha_2 \end{aligned}$$

proving the assertion. ☐

In particular, if H is unimodular, the comodulus of $\underline{H}$ is $g^*_\alpha = r_H(\alpha)\epsilon_\alpha$, see Remark 1.2.10.

Chapter 2

Sutured 3-manifolds

In this chapter we recall some notions of sutured manifold theory. We start by defining sutured manifolds and give some examples in Section 2.1. In Section 2.2 we explain how these manifolds are represented using Heegaard diagrams and we state the strong form of the Reidemeister-Singer theorem, following [JTZ12]. In Section 2.3, we define extended Heegaard diagrams and extend the Reidemeister Singer theorem to this setting, following [LN19a]. We end with a discussion on homology orientations, as in [FJR11].

In what follows, all surfaces and 3-manifolds will be assumed to be compact and oriented, unless explicitly stated. If the boundary of a surface or 3-manifold is considered as an oriented manifold, we will always assume it has the (outward pointing) induced orientation.

2.1 Balanced sutured 3-manifolds

Sutured manifolds were introduced by Gabai in order to study foliations of 3-manifolds [Gab83]. We use a slightly less general definition, as in [Juh06, JTZ12].

Definition 2.1.1. A *sutured manifold* is a pair (M, γ) where M is a 3-manifold-with-boundary and γ is a collection of pairwise disjoint annuli contained in ∂M. Each annuli in γ is supposed to be the tubular neighborhood of an oriented simple closed curve, called a *suture*, the set of which is denoted by $s(\gamma)$. We further suppose that each component of $R(\gamma) \coloneqq \partial M \setminus \text{int}\,(\gamma)$ is oriented and we require that each (oriented) component of $\partial R(\gamma)$ is oriented-parallel to a suture. We denote by $R_+(\gamma)$ (resp. $R_-(\gamma)$) the union of the components of $R(\gamma)$ whose orientation coincides (resp. is opposite) with the induced orientation of ∂M.

Note that the last condition in the above definition implies that for each annuli A in γ, one component of ∂A is contained in $R_-(\gamma)$ while the other is contained in $R_+(\gamma)$.

Definition 2.1.2. A *balanced sutured manifold* is a sutured manifold (M, γ) in which M has no closed components, $\chi(R_-(\gamma)) = \chi(R_+(\gamma))$ and every component of ∂M has at least one suture.

If M has no closed components and every component of ∂M contains a suture (a *proper* sutured manifold as in [JTZ12]), then $s(\gamma)$ determines $R_\pm(\gamma)$. Thus, we only need to specify $s(\gamma)$ in order to define a balanced sutured manifold.

We now give some examples of balanced sutured manifolds.

Example 2.1.3. Pointed closed 3-manifolds: consider a closed 3-manifold Y together with a basepoint $p \in Y$. Then if B is a closed ball neighborhood of p in Y, the complement $Y \setminus \text{int}\,(B)$ becomes a balanced sutured 3-manifold if we let $s(\gamma)$ be a single oriented simple closed curve on ∂B. Both surfaces R_- and R_+ are disks in this case. We denote this sutured manifold by $Y(1)$. Note that a pointed diffeomorphism between pointed closed 3-manifolds is equivalent to a diffeomorphism between the associated sutured 3-manifolds.

Example 2.1.4. Link complements: let L be a link in a closed 3-manifold Y and let $N(L)$ be a closed tubular neighborhood of L. Then $Y \setminus \text{int}\,(N(L))$ becomes a balanced sutured manifold if we put two oppositely oriented meridians on each component of $\partial N(L)$. We denote this sutured 3-manifold by $Y(L)$. Here R_+ consist of one annulus component for each component of L and similarly for R_-.

Example 2.1.5. Seifert surface complements: Let S be a compact oriented surface-with-boundary with no closed components which is embedded in a closed 3-manifold Y and let $N(S) \cong S \times [-1, 1]$ be a closed tubular neighborhood of S in Y. Then $Y \setminus \text{int}\,(N(S))$ is a balanced sutured manifold if we let $s(\gamma) = \partial S \times \{0\}, \gamma = \partial S \times [-1, 1], R_+ = S \times \{1\}$ and $R_- = S \times \{-1\}$.

2.2 Heegaard diagrams

We now describe how to represent sutured manifolds via sutured Heegaard diagrams as in [Juh06, Sect. 2]. We denote by I the interval $[-1, 1]$. By a circle in a surface we will always mean an embedded simple closed curve.

Definition 2.2.1. A *sutured Heegaard diagram* is a tuple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ consisting of the following data:

1. A compact oriented surface-with-boundary Σ,
2. A set $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ of pairwise disjoint embedded circles in $\text{int}\,(\Sigma)$,
3. A set $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_m\}$ of pairwise disjoint embedded circles in $\text{int}\,(\Sigma)$.

A sutured Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defines a sutured 3-manifold, denoted $M_{\mathcal{H}}$, as follows: attach 3-dimensional 2-handles to $\Sigma \times I$ along the curves $\alpha_i \times \{-1\}$ for each $i = 1, \ldots, n$ and along the curves $\beta_j \times \{1\}$ for $j = 1, \ldots, m$. Then let $s(\gamma) := \partial\Sigma \times \{0\}$ and $\gamma := \partial\Sigma \times I$. We orient $M_{\mathcal{H}}$ by extending the product orientation of $\Sigma \times I$ to the 2-handles. Note that the surface R_- (resp. R_+) is obtained by doing surgery on Σ along $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$).

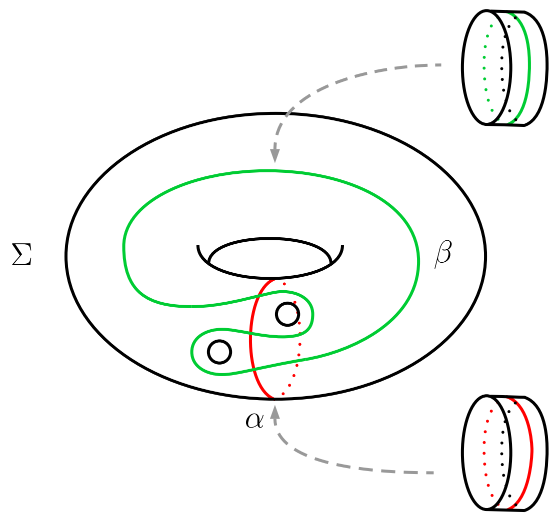


Figure 2.1: A sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$ and the associated sutured manifold $M_{\mathcal{H}}$. The arrows in the picture indicate where the 2-handles are to be glued. For this particular diagram, if one thinks that Σ is embedded in S^3, then it is easy to see that the 2-handles can be attached inside S^3. Thus $M_{\mathcal{H}} \subset S^3$, indeed, $M_{\mathcal{H}}$ is diffeomorphic to the left trefoil complement.

Remark 2.2.2. The manifold $M_{\mathcal{H}}$ could also be constructed in the following way: first, attach n 3-dimensional 1-handles to $R_- \times I$ along $R_- \times \{1\}$ so that the α_i become the belt circles of these 1-handles. The upper boundary of the manifold thus obtained can be identified with Σ. Then, attach m 3-dimensional 2-handles along the curves $\beta_i \subset \Sigma$. The resulting manifold is $M_{\mathcal{H}}$. Thus, a sutured Heegaard diagram specifies a handlebody decomposition of $M_{\mathcal{H}}$ relative to $R_- \times I$, where the handles are attached in increasing order according to their index. In other words, it corresponds to a *self-indexing Morse function* on $M_{\mathcal{H}}$ (see [Mil65]).

Definition 2.2.3. We say that a sutured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is *balanced* if $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and every component of $\Sigma \setminus \boldsymbol{\alpha}$ contains a component of $\partial\Sigma$ (and similarly for $\Sigma \setminus \boldsymbol{\beta}$).

Proposition 2.2.4 ([Juh06, Prop. 2.9])**.** *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a sutured Heegaard diagram. Then the sutured manifold $M_{\mathcal{H}}$ is balanced if and only if $\mathcal{H}$ is balanced.*

Given a sutured manifold (M, γ), we will need Heegaard diagrams to be embedded in M. We follow [JTZ12, Section 2], in which embedded diagrams and the Reidemeister-Singer theorem for these is treated in detail.

Definition 2.2.5. Let (M, γ) be a sutured manifold. An (embedded) *sutured Heegaard diagram* of (M, γ) is a tuple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ consisting of the following data:

1. An embedded oriented surface-with-boundary $\Sigma \subset M$ such that $\partial\Sigma = s(\gamma)$ as oriented 1-manifolds,

2. A set $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ of pairwise disjoint embedded circles in $\mathrm{int}\,(\Sigma)$ bounding disjoint disks to the negative side of Σ,

3. A set $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_m\}$ of pairwise disjoint embedded circles in $\mathrm{int}\,(\Sigma)$ bounding disjoint disks to the positive side of Σ.

We further require that if Σ is surgered along the disks with boundary $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$) inside M, then we get a surface isotopic to R_- (resp. R_+) relative to γ. Thus, M can be written as $M = U_\alpha \cup U_\beta$ with $U_\alpha \cap U_\beta = \Sigma$ where U_α (resp. U_β) is homeomorphic to the sutured manifold obtained from $R_- \times I$ (resp. $R_+ \times I$) by gluing 1-handles to $R_- \times \{1\}$ (resp. $R_+ \times \{0\}$) with belt circles the α curves (resp. β curves). We say that U_α (resp. U_β) is the lower (resp. upper) compression body corresponding to the Heegaard diagram.

If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an embedded diagram of (M, γ), then of course it is also an abstract diagram. Conversely, an abstract diagram $\mathcal{H}$ is an embedded diagram of $M_{\mathcal{H}}$. Thus, from now on we assume all Heegaard diagrams are embedded diagrams (of some sutured manifold). The fact that $\mathcal{H}$ is embedded in M implies that there is a *canonical* homeomorphism (up to isotopy) $d : M \to M_{\mathcal{H}}$. Usually, one says that an abstract diagram $\mathcal{H}$ is a diagram for (M, γ) if *there is* an homeomorphism $M \cong M_{\mathcal{H}}$, but this is not enough to define the homology classes (in $H_1(M)$) of Subsection 2.3.3 below.

The following theorem is proved in [Juh06, Prop. 2.14].

Theorem 2.2.6. *Any balanced sutured 3-manifold admits a balanced sutured Heegaard diagram.*

Proof. We include the proof since some of the ideas behind it will be used later (see Subsection 3.3.1). Fix an orientation-preserving diffeomorphism $h : \gamma \to s(\gamma) \times [-1, 4]$ and let $f : \gamma \to [-1, 4]$ be h composed with projection onto the second factor. Then f can be extended to ∂M by setting $f(R_-) \equiv -1$ and $f(R_+) \equiv 4$. Now, extend f to a Morse function $M \to \mathbb{R}$ which we still denote by f. After an appropriate perturbation, one can assume f is self-indexing, that is $f(p) = i$ for each critical point of index i, and its critical points are of index one or two. We denote these by $p_1, \ldots, p_d$ (index 1) and $q_1, \ldots, q_d$ (index 2). Let $\Sigma \coloneqq f^{-1}(3/2)$ and $U_\alpha \coloneqq f^{-1}[-1, 3/2], U_\beta \coloneqq f^{-1}[3/2, 4]$.

Then U_α is a sutured handlebody built from $R_- \times I$ by attaching one handles with core disks the stable submanifolds $W^s(p_i)$ of f at the index one critical points. Similarly, U_β is a sutured handlebody with core disks the unstable submanifolds $W^u(q_i)$ of the index two critical points. Therefore if we set $\alpha_i := W^s(p_i) \cap \Sigma$ and $\beta_i := W^u(q_i) \cap \Sigma$ for each $i = 1, \dots, d$, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram of (M, γ). $\square$

2.2.1 The Reidemeister-Singer theorem

The Reidemeister-Singer theorem describes how two Heegaard diagrams of a same sutured manifold are related. Since we work with embedded diagrams, we will use a slightly stronger form than the usual theorem. For this, we make a few definitions. If $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1), \mathcal{H}_2 = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ are two Heegaard diagrams, a diffeomorphism $d : \mathcal{H}_1 \to \mathcal{H}_2$ consists of an orientation-preserving diffeomorphism $d : \Sigma_1 \to \Sigma_2$ such that $d(\boldsymbol{\alpha}_1) = \boldsymbol{\alpha}_2$ and $d(\boldsymbol{\beta}_1) = \boldsymbol{\beta}_2$.

Definition 2.2.7 ([JTZ12, Definition 2.34])**.** Let $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1), \mathcal{H}_2 = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ be two Heegaard diagrams of (M, γ) and denote by $j_i : \Sigma_i \to M$ the inclusion map, for $i = 1, 2$. A diffeomorphism $d : \mathcal{H}_1 \to \mathcal{H}_2$ is *isotopic to the identity in* M if $j_2 \circ d : \Sigma_1 \to M$ is isotopic to $j_1 : \Sigma_1 \to M$ relative to $s(\gamma)$.

Definition 2.2.8. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram. Let δ be an arc embedded in $\operatorname{int}(\Sigma)$ connecting a point of a curve α_j to a point of a curve α_i and such that $\operatorname{int}(\delta) \cap \boldsymbol{\alpha} = \emptyset$. There is a neighborhood of $\alpha_j \cup \delta \cup \alpha_i$ which is a pair of pants embedded in Σ and whose boundary consists of the curves α_j, α_i and a curve α_j'. We say that α_j' is obtained by *handlesliding* the curve α_j over α_i along the arc δ. Similarly, we can handleslide a β curve over another along an arc $\delta \subset \operatorname{int}(\Sigma)$ such that $\operatorname{int}(\delta) \cap \boldsymbol{\beta} = \emptyset$. See Figure 2.2 below.

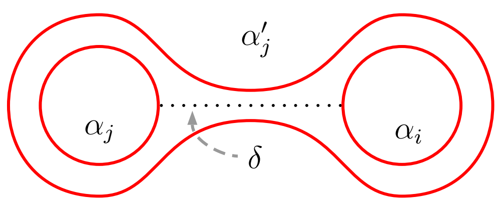


Figure 2.2: Handlesliding a curve α_j over a curve α_i along an arc $\delta \subset \Sigma$.

Definition 2.2.9. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram of a sutured manifold (M, γ). Let $D \subset \operatorname{int}(\Sigma) \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ be a disk. Let Σ' be a connected sum $\Sigma' = \Sigma \# T$ along D, where T is a torus embedded in M. Let α', β' be two curves in T intersecting transversely in one point and suppose that both α', β' bound a disk in M. If $\boldsymbol{\alpha}' = \boldsymbol{\alpha} \cup \{\alpha'\}$, $\boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta'\}$, then $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is a Heegaard diagram of (M, γ) which we say is obtained by a *stabilization* of the diagram $\mathcal{H}$. We also say that $\mathcal{H}$ is obtained by destabilization of $\mathcal{H}'$.

Theorem 2.2.10 ([JTZ12, Prop. 2.36]). *Any two embedded Heegaard diagrams of a sutured 3-manifold (M, γ) are related by a finite sequence of the following moves:*

1. *Isotopy of $\boldsymbol{\alpha}$ (or $\boldsymbol{\beta}$) in $int(\Sigma)$.*
2. *Diffeomorphisms isotopic to the identity in M.*
3. *Handlesliding an α curve (resp. β curve) over another α curve (resp. β curve).*
4. *Stabilization.*

Note that this theorem is stronger than the usual Reidemeister-Singer theorem [Juh06, Prop. 2.15] which states that a *diffeomorphism class* of sutured manifolds is specified by a Heegaard diagram up to isotopy, handlesliding, stabilization and diagram diffeomorphism.

2.3 Extended Heegaard diagrams

In order to extend Kuperberg invariants to sutured 3-manifolds we will need a slight extension of the concept of Heegaard diagram. Most of this section is taken from [LN19a].

2.3.1 Cut systems of surfaces

In what follows, we let R be a compact orientable surface-with-boundary. We suppose R has no closed components.

Definition 2.3.1. A *cut system* of R is a collection $\boldsymbol{a}$ of pairwise disjoint arcs properly embedded in R such that for any component R' of R, $R' \setminus R' \cap N(\boldsymbol{a})$ is homeomorphic to a disk. Here $N(\boldsymbol{a})$ is an open tubular neighborhood of $\boldsymbol{a}$ in R.

A cut system is equivalent to a handlebody decomposition of R with a single 0-handle on each component and no 2-handles. Indeed, the cocores of the 1-handles define a cut system of R, and viceversa.

Definition 2.3.2. Let $\boldsymbol{a} = \{a_1, \dots, a_l\}$ be a cut system of R and suppose that an arc a_j has an endpoint on the same component C of ∂R as another arc a_i. Suppose there is an arc γ in C connecting these two endpoints such that no other endpoint of an arc of $\boldsymbol{a}$ lies on γ. Then there is a neighborhood of $a_j \cup \gamma \cup a_i$ which is a disk D embedded in R and whose boundary $[\partial D] \in H_1(R, \partial R)$, consists of the arcs a_j, a_i and a new arc a_j'. We say that a_j' is obtained by *arc-sliding* (or just *sliding*) a_j over a_i. It is clear that $(\boldsymbol{a} \setminus \{a_j\}) \cup \{a_j'\}$ also defines a cut system of R. See Figure 2.3.

We now prove that these moves suffice to relate any two cut systems over R. In the following lemma, we assume all arcs are properly embedded (see [OS04, Prop. 2.4] for a similar statement).

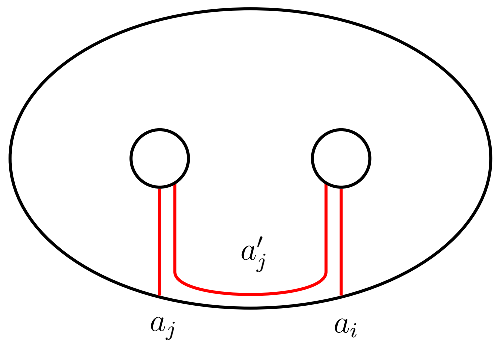


Figure 2.3: An arc a_j is slided over an arc a_i.

Lemma 2.3.3. *Let $\boldsymbol{a} = \{a_1, \dots, a_l\}$ be a cut system of R. If a' is a properly embedded arc in R with $[a'] \neq 0$ in $H_1(R, \partial R)$, then there is an i such that a' is isotopic to an arc obtained by sliding a_i over some of the a_j with $j \neq i$.*

Proof. It suffices to suppose R is connected. We think of R as obtained from a disk D by attaching l one-handles, where each handle is attached along two points $p_i, q_i \in \partial D$ and has cocore a_i for $i = 1, \dots, l$. We isotope each a_i to a small arc contained in D around p_i. Thus, sliding a_i over the other arcs of $\boldsymbol{a}$ corresponds to isotoping the endpoints of a_i through $\partial D \setminus \{p_i, q_i\}$. We suppose a' is transversal to $\boldsymbol{a}$ and proceed by induction on $N = |a' \cap \boldsymbol{a}|$. Suppose first $N = 0$, so a' is disjoint from $\boldsymbol{a}$. Thus, we can suppose that a' is contained in D, so it separates the points of $F \coloneqq \cup_{i=1}^{l} \{p_i, q_i\}$ into two disjoint sets, call them X and Y. Now, since $[a'] \neq 0$ in $H_1(R, \partial R)$, there must exist an i such that $p_i \in X$ and $q_i \in Y$. We can then isotope the endpoints of a_i along ∂D to reach a'. During the isotopy a_i will cross all the points of $X \setminus \{p_i\}$, and as noted above, this corresponds to sliding a_i along the arcs a_j associated to those points. This proves the base of the induction. Now let $N > 0$, let $x_0 \in \partial D \setminus F$ be one of the endpoints of a' and let x_1 be the first intersection point of a' with $\boldsymbol{a}$ starting from x_0, say $x_1 \in a' \cap a_1$. As before, the subarc of a' from x_0 to x_1 (or rather a short extension of it that ends at $p_1 \in \partial D$) separates the points of F into two disjoints sets X, Y. Suppose $q_1 \in Y$, then we can isotope one of the endpoints of a_1 across X to get rid of the intersection point x_1. This corresponds to sliding inside $\boldsymbol{a}$ and decreases N at least by one, so we are done by the induction hypothesis. $\square$

Proposition 2.3.4. *Any two cut systems of a compact orientable surface-with-boundary R (without closed components) are related by isotopy and arc-sliding.*

Proof. It suffices to suppose R connected. Let $\boldsymbol{a}, \boldsymbol{a}'$ be two cut systems over R. Let $a'_1 \in \boldsymbol{a}'$. Since $[a'_1] \neq 0$ in $H_1(R, \partial R)$, by the above lemma we can do isotopy and arc-sliding inside $\boldsymbol{a}$ to get a new cut system (still denoted $\boldsymbol{a}$), with an arc $a_1 \in \boldsymbol{a}$ isotopic to a'_1. We can as well suppose $a'_1 = a_1$, hence we can cut R along a_1 to get two cut systems on a surface R' with $\operatorname{rk} H_1(R') = \operatorname{rk} H_1(R) - 1$. The proof then follows by induction. $\square$

Remark 2.3.5. If the surface R has a basepoint $p \in \partial R$, then the above arguments can be refined to show that any two cut systems are related by isotopy and arc-sliding in the complement of p, see [BVV18, Lemma 2.7].

2.3.2 Extended diagrams and extended moves

Definition 2.3.6. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram of a sutured manifold (M, γ). A *cut system* of $(\Sigma, \boldsymbol{\alpha})$ consists of a set of pairwise disjoint properly embedded arcs $\boldsymbol{a} \subset \Sigma \setminus \boldsymbol{\alpha}$ such that $\boldsymbol{a}$ is a cut system of $\Sigma[\boldsymbol{\alpha}]$, the surface obtained by surgering Σ along the disks (in U_α) bounded by $\boldsymbol{\alpha}$ (note that $\Sigma[\boldsymbol{\alpha}]$ is isotopic to $R_-(\gamma)$). An *extended Heegaard diagram* is a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ endowed with a cut system $\boldsymbol{a}$ of $(\Sigma, \boldsymbol{\alpha})$. We will often denote $\boldsymbol{\alpha}^e = \boldsymbol{\alpha} \cup \boldsymbol{a}$ and if $|\boldsymbol{\alpha}| = d$, $|\boldsymbol{a}| = l$, then we denote $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}$ and $\boldsymbol{a} = \{\alpha_{d+1}, \dots, \alpha_{d+l}\}$.

We will call *extended Heegaard moves* the following moves on an extended Heegaard diagram.

1. Usual Heegaard moves of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, just that we always suppose the α's are isotoped or handleslided in the complement of the arcs in $\boldsymbol{a}$.

2. Isotopies of arcs.

3. Sliding an arc over an arc.

4. Sliding an arc over a curve.

Proposition 2.3.7. *Any two extended Heegaard diagrams of a sutured 3-manifold (M, γ) are related by extended Heegaard moves.*

Proof. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma, \boldsymbol{\alpha}'^e, \boldsymbol{\beta}')$ be two extended Heegaard diagrams of (M, γ). By Proposition 2.3.4 it suffices to prove that whenever the diagrams are related by one of the moves of Theorem 2.2.10, then the latter move can be performed in the complement of the cut systems. This is obvious for all moves except when handlesliding closed curves. So suppose $\mathcal{H}'$ is obtained from $\mathcal{H}$ by handlesliding a curve α_j over a curve α_i along an arc γ. Suppose the arc γ intersects some arcs in $\boldsymbol{a}$. Then we can successively slide these arcs along α_i to get a new cut system $\boldsymbol{a}''$ for $\mathcal{H}$ which is disjoint of γ. Hence, $\boldsymbol{a}''$ is also a cut system for $\mathcal{H}'$. By Proposition 2.3.4, $\boldsymbol{a}'$ and $\boldsymbol{a}''$ are related by isotopy and arc-sliding in $\Sigma'[\boldsymbol{\alpha}']$. But isotoping an arc past the trace of the surgery is the same thing as sliding that arc over the corresponding curve in $\boldsymbol{\alpha}$. Thus we can pass from $\mathcal{H}$ to $\mathcal{H}'$ using extended Heegaard moves, as desired. $\square$

2.3.3 Dual curves

Let (M,γ) be a balanced sutured 3-manifold with a basepoint $p \in s(\gamma)$ and suppose that $R_-(\gamma)$ is *connected*. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an extended Heegaard diagram of (M,γ).

Definition 2.3.8. We say that $\mathcal{H}$ is *oriented* if each curve in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ is oriented.

Suppose $\mathcal{H}$ is oriented and that the arcs in $\boldsymbol{a}$ are also oriented (though the orientation of the arcs will be irrelevant soon). One can construct elements $\alpha^* \in \pi_1(M,p)$ for each $\alpha \in \boldsymbol{\alpha}^e$ as follows. Let $\boldsymbol{\alpha}^e = \boldsymbol{\alpha} \cup \boldsymbol{a}$ where $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}$ are the closed curves and $\boldsymbol{a} = \{\alpha_{d+1}, \dots, \alpha_{d+l}\}$ is a cut system of $(\Sigma, \boldsymbol{\alpha})$. Write $M = U_\alpha \cup U_\beta$ where U_α, U_β denote respectively the lower and upper compression bodies associated to $\mathcal{H}$. We can think of U_α as constructed from $R_- \times I$ by attaching 3-dimensional 1-handles with belt circles the closed α curves. The cocores of these 1-handles are disks $D_1, \dots, D_d$ with $\partial D_i = \alpha_i$ for each $i = 1, \dots, d$. On the other hand, we have disks $D_i = \alpha_i \times I \subset R_- \times I$ for each $i = d+1, \dots, d+l$. Since we assumed that $R_-(\gamma)$ is connected, the complement $U_\alpha \setminus (D_1 \cup \dots \cup D_{d+l})$ is a single 3-ball, we denote it by $B_{\boldsymbol{\alpha}^e}$.

Definition 2.3.9. For each $i = 1, \dots, d+l$ we let $\alpha_i^* \in \pi_1(M,p)$ be the homotopy class of a loop based at p contained in U_α which intersects $D_1 \cup \dots \cup D_{d+l}$ only at a single point in D_i. We orient α_i^* in such a way that $\alpha_i \cdot \alpha_i^* = +1$ when α_i^* is represented as a curve in Σ.

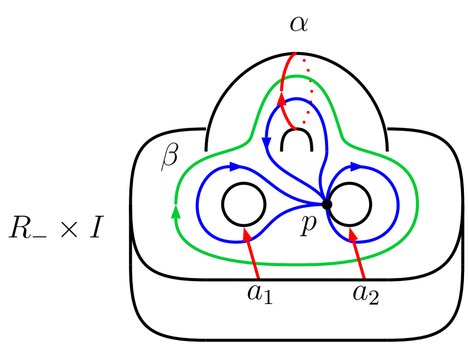


Figure 2.4: An (oriented) extended Heegaard diagram $(\Sigma, \boldsymbol{\alpha}^e, \beta)$ where $\boldsymbol{\alpha} = \{\alpha\}$ and $\boldsymbol{a} = \{a_1, a_2\}$ with its dual curves $\alpha^*, a_1^*, a_2^* \in \pi_1(M,p)$ indicated in blue.

Remark 2.3.10. We will need to understand how the elements $\alpha_i^* \in \pi_1(M,p)$ change when performing extended Heegaard moves:

1. Suppose α_j is handleslided over α_i (with $(\alpha_j, \alpha_i) \notin \boldsymbol{\alpha} \times \boldsymbol{a}$) and let α_j', α_i' denote the curves after handlesliding, so $\alpha_i' = \alpha_i$ and $\alpha_j' = \alpha_i \# \alpha_j$. If P denotes the handlesliding region, we suppose that the basepoint $p \in s(\gamma)$ does not lies in

P (this matters when one of α_i, α_j is an arc). Suppose the curves $\alpha_i, \alpha_j, \alpha'_j$ are oriented so that $\partial P = \alpha_i \cup \alpha_j \cup -\alpha'_j$ in $H_1(\Sigma, \partial\Sigma))$ as oriented 1-manifolds (P has the induced orientation from Σ). It is clear that $(\alpha'_j)^* = \alpha^*_j$. However, the dual α^*_i intersects α'_j positively in one point. To get rid of this intersection point, we slide α^*_i over $(\alpha'_j)^* = \alpha^*_j$, see Figure 2.5. Therefore, the dual of α_i in the Heegaard diagram after handlesliding is

$$(\alpha'_i)^* = \alpha^*_i (\alpha^*_j)^{-1}.$$

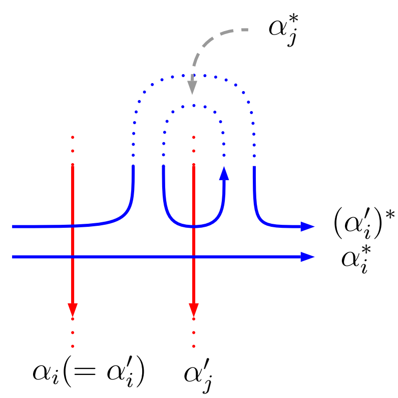


Figure 2.5: A portion of a Heegaard diagram in which a curve or arc α_j has been slided over α_i.

2. Suppose an arc $a \in \boldsymbol{a}$ is isotoped along $\partial\Sigma$ past the basepoint p. We suppose the arc a is oriented so that $a \cdot \delta = +1$ where δ is the oriented boundary component of $\partial\Sigma$ containing p. Suppose further that the arc a is isotoped to the right of p (this has sense since Σ is oriented), and denote by a' the new arc and by $\mathcal{H}'$ the new extended Heegaard diagram. If we denote by α'_i the curves α_i in $\mathcal{H}'$, then

$$(\alpha'_i)^* = a^* \alpha^*_i (a^*)^{-1}$$

for all $i = 1, \dots, d+l$. See Figure 2.6.

Now let $c \subset \Sigma$ be an embedded oriented arc or an embedded oriented circle with a basepoint (which we can consider as an oriented arc by deleting a small neighborhood of the basepoint). Suppose c is transversal to $\boldsymbol{\alpha}^e$ and has endpoints in $\Sigma \setminus \boldsymbol{\alpha}^e$.

Definition 2.3.11. We let $\overline{c} \in \pi_1(M, p)$ be the homotopy class of the loop obtained by joining the basepoint p to the beginning point of c by an arc contained in $B_{\boldsymbol{\alpha}^e} = U_\alpha \setminus (\cup_{i=1}^{d+l} D_i)$, then following c according to its orientation and finally coming back to the basepoint p by an arc contained in $B_{\boldsymbol{\alpha}^e}$. Equivalently, $\overline{c}$ is obtained by taking the product from left to right of the $(\alpha^*)^{m(x)}$'s for each $x \in \boldsymbol{\alpha}^e \cap c$, ordered by their appearance along c and where $m(x) \in \{\pm 1\}$ is the sign of the intersection.

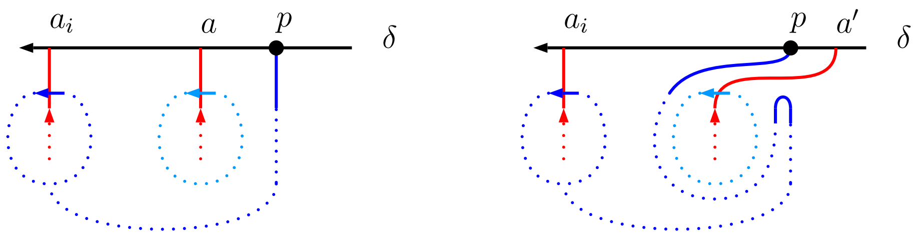


Figure 2.6: An arc a is slided past the basepoint $p \in \delta \subset \partial\Sigma$. On the left we draw a_i^* as $\gamma A_i \gamma^{-1}$, where A_i is a circle in $\operatorname{int}(\Sigma)$ intersecting $\boldsymbol{\alpha}^e$ once at a_i and $\gamma \subset \Sigma \setminus \boldsymbol{\alpha}^e$ is an arc connecting the basepoint p to a point in A_i. On the right the arc γ has been slided over a^* to avoid an intersection with a. This shows that $(a_i')^* = a^* a_i^* (a^*)^{-1}$.

The orientation of the arcs in $\boldsymbol{a}$ is needed to write $\overline{c}$ as a word in the α^*'s, but as an element of $\pi_1(M,p)$, $\overline{c}$ is independent of this. Note that if $c \subset \Sigma$ is an oriented circle with a basepoint, then changing the basepoint along c leaves the conjugacy class of $\overline{c}$ unchanged. As an example, if we put a basepoint on the leftmost part of β in Figure 2.4, then we have $\overline{\beta} = (\alpha^*)^{-1} a_2^* a_1^*$.

Lemma 2.3.12. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an oriented extended Heegaard diagram of (M,γ). The fundamental group of M at p has a presentation*

$$\pi_1(M,p) = \langle \alpha_1^*, \dots, \alpha_{d+l}^* \mid \overline{\beta}_1, \dots, \overline{\beta}_d \rangle.$$

Proof. This is direct from Van Kampen's theorem. □

Definition 2.3.13. Suppose a curve $\beta \in \boldsymbol{\beta}$ is oriented and has a basepoint $q \in \beta \setminus \boldsymbol{\alpha}^e$. For each $x \in \beta \cap \boldsymbol{\alpha}^e$ let $q_x \in \beta$ be a point defined as follows: if the crossing at x is positive (resp. negative), then q_x is right before (resp. after) x along β. Then we denote $\overline{\beta}_x = \overline{c_x}$ where c_x is the subarc of β starting at q and ending at q_x. More precisely, suppose $x \in \alpha_i \cap \beta$ $(i = 1, \dots, d+l)$ and write $\overline{\beta} = w(\alpha_i^*)^{m(x)} w'$ where w (resp. w') is the product of the α^*'s corresponding to the crossings of β that precede (resp. succeed) x. Then

$$\overline{\beta}_x \coloneqq \begin{cases} w & \text{if } m(x) = 1 \\ w(\alpha_i^*)^{-1} & \text{if } m(x) = -1. \end{cases} \tag{2.1}$$

Remark 2.3.14. With this notation the Fox derivatives are computed by

$$\frac{\partial \overline{\beta}}{\partial \alpha_i^*} = \sum_{x \in \alpha_i \cap \beta} m(x) \overline{\beta}_x \in \mathbb{Z}[F], \tag{2.2}$$

for any $\beta \in \boldsymbol{\beta}$, where F is the free group generated by $\alpha_1^*, \dots, \alpha_{d+l}^*$.

2.4 Homology orientations

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a balanced Heegaard diagram of (M, γ) and $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. Denote $R_- = R_-(\gamma)$. Let $A \subset H_1(\Sigma; \mathbb{R})$ (resp. B) be the subspace spanned by $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$). These subspaces have dimension d by [Juh06, Lemma 2.10]. There is a bijection o between orientations of the vector space $H_*(M, R_-; \mathbb{R})$ and orientations of the vector space $\Lambda^d(A) \otimes \Lambda^d(B)$ [FJR11, Sect. 2.4]. This is seen as follows. The Heegaard diagram specifies a handle decomposition of (M, γ) relative to $R_- \times I$ with no handles of index zero or three. There are d handles of index one and two, so the handlebody complex $C_* = C_*(M, R_- \times I; \mathbb{R})$ is just $C_1 \oplus C_2$ where both C_1, C_2 have dimension d. Now let ω be an orientation of $H_*(M, R_-; \mathbb{R})$ and let $h^1_1, \ldots, h^1_m, h^2_1, \ldots, h^2_m$ be an ordered basis of $H_*(M, R_-; \mathbb{R})$ compatible with ω, where $h^i_j \in H_i(M, R_-; \mathbb{R})$. Let $c^1_1, \ldots, c^1_m, c^2_1, \ldots, c^2_m \in C_*$ be chains representing this basis, where $c^i_j \in C_i$. Then, for any $b_1, \ldots, b_{d-m} \in C_2$ such that $c^2_1, \ldots, c^2_m, b_1, \ldots, b_{d-m}$ is a basis of C_2 the collection

$$c^1_1, \ldots, c^1_m, \partial b_1, \ldots, \partial b_{d-m}, c^2_1, \ldots, c^2_m, b_1, \ldots, b_{d-m}$$

is a basis of C_* whose orientation ω' depends only on ω. Now, an orientation of C_* is specified by an ordering and orientation of the handles of index one and two. This is the same as an ordering and orientation of the curves in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ or equivalently, an orientation of $\Lambda^d(A) \otimes \Lambda^d(B)$. This way, the orientation ω induces an orientation of $\Lambda^d(A) \otimes \Lambda^d(B)$ via ω'. We denote this orientation of $\Lambda^d(A) \otimes \Lambda^d(B)$ by $o(\omega)$.

There are a few cases in which there is a canonical orientation of $H_*(M, R_-(\gamma); \mathbb{R})$, and hence there is a canonical sign-ordering of a sutured Heegaard diagram. For example, let Y be a closed oriented 3-manifold. If $M = Y \setminus B$ where B is an open ball in Y and $\gamma = S^1 \subset \partial \overline{B}$, then $H_*(M, R_-(\gamma); \mathbb{R}) = H_2(Y; \mathbb{R}) \oplus H_1(Y; \mathbb{R})$. Now, any basis of $H_1(Y; \mathbb{R})$ determines a Poincaré dual basis of $H_2(Y; \mathbb{R})$ via the (non-degenerate) intersection pairing $H_2(Y) \otimes H_1(Y) \to \mathbb{R}$. The orientation of the basis of $H_*(M, R_-(\gamma))$ thus obtained is independent of the basis of $H_1(Y)$ chosen. However, reversing the orientation of Y, multiplies this orientation by $(-1)^{b_1(Y)}$.

Another situation is when $H_*(M, R_-; \mathbb{R}) = 0$, for example for knots in an homology 3-sphere. Here the orientation ω' of $C_*(M, R_-; \mathbb{R})$ constructed above is canonical and an ordering and orientation of the curves of $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ corresponds to the canonical orientation if and only if

$$\det(\alpha_i \cdot \beta_j) > 0$$

where $\alpha_i \cdot \beta_j \in \mathbb{Z}$ denotes the intersection number.

Chapter 3

Kuperberg invariants for sutured 3-manifolds

In this chapter, we will construct the topological invariant $Z_H^{\rho}(M,\gamma,\mathfrak{s},\omega) \in \mathbb{K}$ of Theorem 1, following mainly [LN19b]. In Section 3.1, we recall the original construction of Kuperberg as well as Virelizier's extension, which are defined for closed 3-manifolds (endowed with a representation of the fundamental group in the second case). In Section 3.2 we define a scalar $Z_H^{\rho}(\mathcal{H})$ from a sutured Heegaard diagram of (M,γ), but we show this is a topological invariant only when $\mathrm{Im}\,(\rho) \subset \mathrm{Ker}\,(r_H)$. The case of arbitrary ρ is treated in Section 3.3 using Spin^c structures.

3.1 Kuperberg invariants of closed 3-manifolds

In this section, we recall the constructions of [Kup91] and [Vir05]. In what follows, we let Y be a compact, oriented, connected closed 3-manifold.

3.1.1 Tensors associated to Heegaard diagrams

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram of Y. We will follow the conventions and notation of [KV19].

Definition 3.1.1. A Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is *ordered* if the sets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are totally ordered. We say that $\mathcal{H}$ is *based* if each curve $\alpha \in \boldsymbol{\alpha}$ (resp. $\beta \in \boldsymbol{\beta}$) has a basepoint $p \in \alpha \setminus \boldsymbol{\beta}$ (resp. $q \in \beta \setminus \boldsymbol{\alpha}$).

Notation 3.1.2. If $\mathcal{H}$ is oriented (i.e. every curve of $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ is oriented), then at each crossing $x \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ of $\mathcal{H}$ we let $m_x \in \{\pm 1\}$ be its intersection sign. Thus, $m_x = 1$ if the tangent vectors to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ at x form a positive basis of $T_x\Sigma$, and $m_x = -1$ otherwise. We define $\epsilon_x \in \{0, 1\}$ by $\epsilon_x = 0$ if $m_x = 1$ and $\epsilon_x = 1$ if $m_x = -1$.

Suppose $\mathcal{H}$ is ordered, oriented and based. We denote by $\mathcal{I}$ the set of crossings of $\mathcal{H}$, that is, $\mathcal{I} = \boldsymbol{\alpha} \cap \boldsymbol{\beta}$. For each $i = 1, \ldots, d$, where $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, the basepoint of α_i together with its orientation determine a total ordering on the set of crossing through α_i. Using the total ordering of $\boldsymbol{\alpha}$, we get a total ordering on the set $\mathcal{I}$. We denote by $\mathcal{I}_{\boldsymbol{\alpha}}$ the set $\mathcal{I}$ with the total ordering coming from $\boldsymbol{\alpha}$. Similarly, the ordering, orientation and basepoints of $\boldsymbol{\beta}$ determine a total ordering on $\mathcal{I}$, we denote by $\mathcal{I}_{\boldsymbol{\beta}}$ the set of crossings with this ordering. These orderings differ by some permutation in S_N, where N is the total number of crossings of $\mathcal{H}$. We let

$$P_{\mathcal{H}} : H^{\otimes N} \to H^{\otimes N}$$

be the map induced by the symmetry of the category $\mathrm{SVect}_{\mathbb{K}}$ and this permutation (see Notation 1.1.2). Here we suppose the domain of $P_{\mathcal{H}}$ is ordered according to $\mathcal{I}_{\boldsymbol{\alpha}}$ and the target is ordered according to $\mathcal{I}_{\boldsymbol{\beta}}$.

Now, for each $\alpha \in \boldsymbol{\alpha}$, let $|\alpha|$ be the number of crossings through α and similarly we define $|\beta|$ for $\beta \in \boldsymbol{\beta}$. We set

$$\Delta_{\boldsymbol{\alpha}} \coloneqq \bigotimes_{i=1}^{d} \Delta^{|\alpha_i|} : H^{\otimes d} \to H^{\otimes N}$$

and

$$m_{\boldsymbol{\beta}} \coloneqq \bigotimes_{i=1}^{d} m^{|\beta_i|} : H^{\otimes N} \to H^{\otimes d}.$$

Here $\Delta^k : H \to H^{\otimes k}$ and $m^k : H^{\otimes k} \to H$ denote iterated coproducts and products for each $k \geq 0$. For $k = 0$ we set $\Delta^0 = \epsilon, m^0 = \eta$ while for $k = 1$, $\Delta^1 = \mathrm{id}_H = m^1$. Finally, we define

$$S_{\boldsymbol{\alpha}} \coloneqq \bigotimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}}} S^{\epsilon_x} : H^{\otimes N} \to H^{\otimes N}.$$

Note that in this tensor product, the order of $\mathcal{I}_{\boldsymbol{\alpha}}$ is relevant. If $S_{\boldsymbol{\beta}}$ denotes the tensor product of the same maps, but using the order of $\mathcal{I}_{\boldsymbol{\beta}}$, then this relates to the previous tensor by $P_{\mathcal{H}} S_{\boldsymbol{\alpha}} = S_{\boldsymbol{\beta}} P_{\mathcal{H}}$.

3.1.2 The original construction of Kuperberg

Let H be a finite dimensional involutive Hopf algebra over a field $\mathbb{K}$. If $\mathrm{char}(\mathbb{K}) = p > 0$, one further supposes $p \nmid \dim(H)$. This guarantees that H is semisimple and hence there is a two-sided cointegral $c \in H$ and a two-sided integral $\mu \in H^*$, we suppose they are normalized by $\mu(c) = 1$.

Definition 3.1.3. Let Y be a (compact, oriented, connected) closed 3-manifold represented by a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ with $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. Suppose $\mathcal{H}$ is arbitrarily ordered, oriented and based. We define

$$K_H^{Kup}(\mathcal{H}) \coloneqq m_{\boldsymbol{\beta}} P_{\mathcal{H}} S_{\boldsymbol{\alpha}} \Delta_{\boldsymbol{\alpha}} : H^{\otimes d} \to H^{\otimes d}$$

and

$$Z_H^{Kup}(\mathcal{H}) \coloneqq \mu^{\otimes d}(K_H^{Kup}(\mathcal{H})(c^{\otimes d})) \in \mathbb{K}.$$

It is shown in [Kup91] that $Z_H^{Kup}(\mathcal{H})$ is invariant under Heegaard moves, the main point being that handlesliding invariance follows directly from the defining equations of the cointegral and integral. By the Reidemeister-Singer theorem, it follows that $Z_H^{Kup}(Y)$ defines a topological invariant of the underlying 3-manifold Y.

Remark 3.1.4. The hypothesis on H may be weakened: one only needs H to be finite dimensional, involutive, unimodular and counimodular for the above formula to define a topological invariant. Thus, H may be a Hopf superalgebra and the characteristic may be positive and divide $\dim(H)$, provided unimodularity holds. However, when H is a Hopf superalgebra with a cointegral of degree one, $Z_H^{Kup}(Y)$ has a sign indeterminacy. This indeterminacy can be removed by picking an homology orientation ω, which can be done in canonical way if Y is oriented (see Section 2.4), and multiplying $Z_H^{Kup}(\mathcal{H})$ by the sign $\delta_\omega(\mathcal{H})$ defined below.

3.1.3 Virelizier's extension

We now sketch Virelizier's extension of Kuperberg's invariant. This relies on the notion of Hopf G-coalgebra, and produces and invariant of a closed 3-manifold Y endowed with a representation $\rho : \pi_1(Y) \to G$. We will actually use the dual notion, that of a Hopf G-algebra as introduced in Section 1.3.

Thus let $\underline{H} = \{H_\alpha\}_{\alpha \in G}$ be a finite type involutive Hopf G-algebra. If $\mathrm{char}\,(\mathbb{K}) = p > 0$, it is further required in [Vir05] that $p \nmid \dim(H_1) \neq 0$. As mentioned above, this implies semisimplicity and hence unimodularity of $\underline{H}$ and counimodularity (of H_1), but only the last two conditions are needed. Let $(c_\alpha)_{\alpha \in G}$ be the two-sided cointegral and $\mu \in H_1^*$ be the two-sided integral, normalized by $\mu(c_1) = 1$.

Let Y be a closed oriented 3-manifold with a basepoint $p \in Y$ and let $\rho : \pi_1(Y, p) \to G$ be a group homomorphism into the grading group of $\underline{H}$. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be an ordered, oriented, based Heegaard diagram of Y with $p \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. More precisely, we let $\mathcal{H}$ be an *embedded* sutured Heegaard diagram of the sutured 3-manifold $(Y \setminus \mathrm{int}\,(B), \gamma)$, where B is a small embedded closed 3-ball with $p \in s(\gamma) \subset \partial B$. This implies that every curve $\alpha \in \boldsymbol{\alpha}$ has a well-defined dual $\alpha^* \in \pi_1(Y, p)$.

We extend the tensors of Subsection 3.1.1 to the Hopf G-algebra setting. For simplicity, we note H_α instead of $H_{\rho(\alpha^*)}$ for each $\alpha \in \boldsymbol{\alpha}$. Suppose a curve $\beta \in \boldsymbol{\beta}$ has associated the word

$$\overline{\beta} = (\alpha_{i_1}^*)^{m_1} \dots (\alpha_{i_k}^*)^{m_k}$$

when starting from its basepoint and following its orientation. Since $\overline{\beta} = 1$ in $\pi_1(M, p)$, we get a multiplication

$$\underline{m}_\beta : H_{\alpha_{i_1}^{m_1}} \otimes \dots \otimes H_{\alpha_{i_k}^{m_k}} \to H_1.$$

We let

$$\underline{m}_{\boldsymbol{\beta}} \coloneqq \bigotimes_{\beta \in \boldsymbol{\beta}} \underline{m}_\beta : \bigotimes_{x \in \mathcal{I}_{\boldsymbol{\beta}}} H_{\alpha_x^{m_x}} \to H_1^{\otimes d}.$$

Now, for each $\alpha \in \boldsymbol{\alpha}$ let $\Delta_\alpha^{|\alpha|} : H_\alpha \to H_\alpha^{\otimes |\alpha|}$ be the $|\alpha|$-iterated coproduct of the coalgebra H_α and let

$$\underline{\Delta}_{\boldsymbol{\alpha}} \coloneqq \bigotimes_{\alpha \in \boldsymbol{\alpha}} \Delta_\alpha^{|\alpha|} : \bigotimes_{\alpha \in \boldsymbol{\alpha}} H_\alpha \to \bigotimes_{\alpha \in \boldsymbol{\alpha}} H_\alpha^{\otimes |\alpha|} = \bigotimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}}} H_{\alpha_x}.$$

We let

$$\underline{S}_{\boldsymbol{\alpha}} \coloneqq \bigotimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}}} \underline{S}_{\alpha_x}^{\epsilon_x} : \bigotimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}}} H_{\alpha_x} \to \bigotimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}}} H_{\alpha_x^{m_x}}.$$

Let $\underline{P}_{\mathcal{H}}$ be the map that takes the $H_{\alpha_x^{m_x}}$'s, ordered according to $\mathcal{I}_{\boldsymbol{\alpha}}$, and reorders them according to $\mathcal{I}_{\boldsymbol{\beta}}$. Then one can define

$$K_{\underline{H}}^\rho(\mathcal{H}) \coloneqq \underline{m}_{\boldsymbol{\beta}} \underline{P}_{\mathcal{H}} \underline{S}_{\boldsymbol{\alpha}} \underline{\Delta}_{\boldsymbol{\alpha}} : \bigotimes_{\alpha \in \boldsymbol{\alpha}} H_{\rho(\alpha^*)} \to H_1^{\otimes d}.$$

If $(c_\alpha)_{\alpha \in G}$ is a (two-sided) cointegral of $\underline{H}$, normalized by $\mu(c_1) = 1$, then let

$$Z_{\underline{H}}^\rho(\mathcal{H}) \coloneqq \mu^{\otimes d} \left(K_{\underline{H}}^\rho(\mathcal{H}) \left(\bigotimes_{\alpha \in \boldsymbol{\alpha}} c_\alpha \right) \right).$$

This is shown to be a topological invariant of (Y, ρ) in [Vir05], denoted $Z_{\underline{H}}^\rho(Y)$. More precisely, if Y' is another closed 3-manifold with a basepoint $p \in Y'$ and $\rho' : \pi_1(Y', p') \to G$ is a group homomorphism for which there is an homeomorphism $f : Y \to Y'$ with $f(p) = p'$ and $\rho' \circ f_* = \rho$ (where $f_* : \pi_1(Y, p) \to \pi_1(Y', p')$ is induced by f), then

$$Z_{\underline{H}}^\rho(Y) = Z_{\underline{H}}^{\rho'}(Y').$$

3.2 Extending to sutured manifolds: the unimodular case

We begin this section by defining a scalar $Z_H^\rho(\mathcal{H})$ out of a sutured Heegaard diagram $\mathcal{H}$ of a balanced sutured 3-manifold (M,γ) with connected $R_-(\gamma)$. We explain the relation of our formula with Virelizier's extension in Subsection 3.2.3. Then in Subsection 3.2.4 we prove some lemmas on $Z_H^\rho(\mathcal{H})$, valid for arbitrary ρ. From these, we deduce in Subsection 3.2.5 that $Z_H^\rho(\mathcal{H})$ is a topological invariant ($=Z_H^\rho(M,\gamma)$) when $\mathrm{Im}\,(\rho)\subset \mathrm{Ker}\,(r_H)$. We say that the condition $\mathrm{Im}\,(\rho)\subset \mathrm{Ker}\,(r_H)$ is the unimodular case, since it corresponds to the unimodular Hopf algebra $\mathbb{K}[\mathrm{Ker}\,(r_H)]\ltimes H$. Finally, in Subsection 3.2.6 we extend everything to the case of disconnected $R_-(\gamma)$ (e.g. link complements), provided $\mathrm{Im}\,(\rho)\subset \mathrm{Ker}\,(r_H)$ as well.

In what follows, we let H be a finite-dimensional Hopf superalgebra over a field $\mathbb{K}$ which we suppose is involutive, unimodular and counimodular. We let $c\in H, \mu\in H^*$ be a two-sided cointegral and integral respectively, normalized by $\mu(c)=1$.

3.2.1 A direct Fox calculus-like formula

Let (M,γ) be a (compact, oriented, connected) balanced sutured 3-manifold with *connected* subsurface $R_-(\gamma)\subset\partial M$. We will suppose that M has a basepoint p in some suture of $s(\gamma)$ (though we show in Corollary 3.2.12 that our formulas do not depend on the basepoint). Let $\rho:\pi_1(M,p)\to \mathrm{Aut}(H)$ be a group homomorphism.

Let $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be a sutured Heegaard diagram of (M,γ). We suppose $\mathcal{H}$ is ordered, oriented and based, that is, both sets $\boldsymbol{\alpha},\boldsymbol{\beta}$ are ordered, oriented and based. The tensors $m_{\boldsymbol{\beta}}, P_{\mathcal{H}}, S_{\boldsymbol{\alpha}}, \Delta_{\boldsymbol{\alpha}}$ defined in Subsection 3.1.1 do not depend on whether the Heegaard surface has boundary or not. Therefore, these tensors are equally defined for $\mathcal{H}$.

Now suppose further that $\mathcal{H}$ is an embedded extended Heegaard diagram, so $\boldsymbol{\alpha}^e=\boldsymbol{\alpha}\cup\boldsymbol{a}$ where the $\boldsymbol{\alpha}$ are closed curves, $\boldsymbol{a}$ consists of arcs and $\Sigma[\boldsymbol{\alpha}]\setminus\boldsymbol{a}$ is homeomorphic to a disk (where $\Sigma[\boldsymbol{\alpha}]$ denotes Σ surgered along $\boldsymbol{\alpha}$). We suppose the basepoints of $\boldsymbol{\beta}$ are disjoint from all of $\boldsymbol{\alpha}^e$. Then for each $\beta\in\boldsymbol{\beta}$ and $x\in\beta\cap\boldsymbol{\alpha}$ there is an element $\overline{\beta}_x\in\pi_1(M,p)$ obtained by joining the basepoint $p\in s(\gamma)$ to the basepoint of β through an arc in $\Sigma\setminus\boldsymbol{\alpha}^e$, traversing β until reaching q_x, and joining q_x back to the basepoint p through an arc in $\Sigma\setminus\boldsymbol{\alpha}^e$, see Definition 2.3.13. Recall that $q_x\in\beta\setminus\boldsymbol{\alpha}^e$ is a point that lies just before x (resp. after x) along β if $m(x)=1$ (resp. $m(x)=-1$). These elements appear when expanding the Fox derivatives $\partial\overline{\beta}/\partial\alpha^*$, see Remark 2.3.14. The arcs in $\boldsymbol{a}$ don't need to be oriented, but sometimes we will assume that they are to have a well-defined dual $a^*\in\pi_1(M,p)$, which together with the α^*, generate $\pi_1(M,p)$. We denote

$$\rho_{\mathcal{H}} \coloneqq \left(\bigotimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}}} \rho(\overline{\beta}_x) \right) : H^{\otimes N} \to H^{\otimes N}$$

where $N = |\boldsymbol{\alpha} \cap \boldsymbol{\beta}|$.

Definition 3.2.1. Let (M,γ) be a balanced sutured 3-manifold with connected $R_-(\gamma)$, endowed with a representation $\rho : \pi_1(M,p) \to \mathrm{Aut}(H)$ where $p \in s(\gamma)$. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an ordered, oriented, based, extended Heegaard diagram of (M,γ) and let $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. We denote by $K_H^\rho(\mathcal{H})$ the tensor

$$K_H^\rho(\mathcal{H}) \coloneqq m_{\boldsymbol{\beta}} P_{\mathcal{H}} \rho_{\mathcal{H}} S_{\boldsymbol{\alpha}} \Delta_{\boldsymbol{\alpha}} : H^{\otimes d} \to H^{\otimes d}.$$

If $c \in H, \mu \in H^*$ denote the two-sided cointegral and integral of H respectively, normalized by $\mu(c) = 1$, then we set

$$Z_H^\rho(\mathcal{H}) \coloneqq \mu^{\otimes d}(K_H^\rho(\mathcal{H})(c^{\otimes d})) \in \mathbb{K}.$$

To treat some sign indeterminacies in the case that the cointegral has degree one, we pick an orientation ω of the vector space $H_*(M, R_-(\gamma); \mathbb{R})$. Since $\mathcal{H}$ is ordered and oriented, we can compare it to the orientation ω via the map o of Section 2.4. If $\delta \in \{\pm 1\}$ is determined by $o(\mathcal{H}) = \delta\omega$, then we define

$$\delta_\omega(\mathcal{H}) \coloneqq \delta^{|c|}. \tag{3.1}$$

In particular, if the cointegral has degree zero then $\delta_\omega(\mathcal{H}) = 1$.

Definition 3.2.2. If ω is an orientation of $H_*(M, R_-(\gamma); \mathbb{R})$, we write

$$Z_H^\rho(\mathcal{H}, \omega) \coloneqq \delta_\omega(\mathcal{H}) Z_H^\rho(\mathcal{H})$$

where $\delta_\omega(\mathcal{H})$ is the sign defined above.

When $\mathcal{H}$ is a Heegaard diagram of a closed 3-manifold and $\rho \equiv 1$, $Z_H^{\rho\equiv 1}(\mathcal{H})$ reduces to $Z_H^{Kup}(\mathcal{H})$ of Definition 3.1.3. Therefore, by Remark 2.3.14, for general ρ one can think of $Z_H^\rho(\mathcal{H})$ as a Fox calculus extension of the construction of [Kup91] to sutured manifolds. We will prove the following in Subsection 3.2.5.

Theorem 3.2.3. *Whenever $r_H \circ \rho \equiv 1$, the scalar $Z_H^\rho(\mathcal{H})$ (resp. $Z_H^\rho(\mathcal{H}, \omega)$ if the cointegral has degree one) defined above is independent of the (ordered, oriented, based) extended sutured Heegaard diagram of (M,γ) chosen. Thus, it defines a topological invariant of the tuple (M,γ,ρ) (resp. (M,γ,ρ,ω)).*

We denote the topological invariant above by $Z_H^\rho(M,\gamma)$ (resp. $Z_H^\rho(M,\gamma,\omega)$).

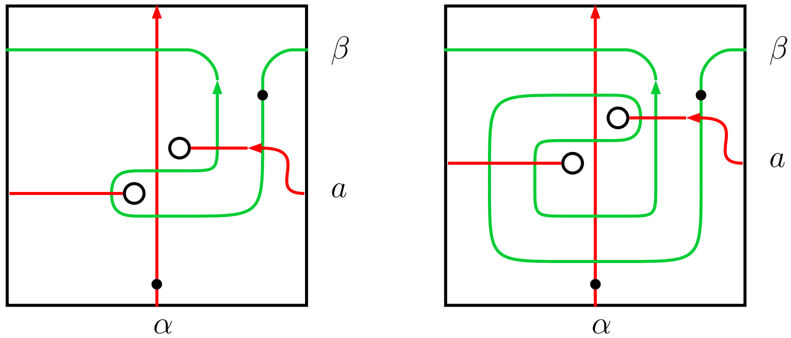


Figure 3.1: From left to right: extended Heegaard diagrams of the left trefoil and figure eight knot respectively. The opposite sides of each square have to be identified, so in both cases Σ is a torus with two punctures. The closed curves of both diagrams are oriented and based as indicated.

3.2.2 Examples of computation

We now compute the tensor $K_H^\rho(\mathcal{H})$ and the scalar $Z_H^\rho(\mathcal{H})$ for some knot complements.

Example 3.2.4. Let $K \subset S^3$ be the left trefoil and $M = S^3 \setminus N(K)$. Recall that M is a sutured 3-manifold if we let γ consist of two meridional sutures in ∂M (see Example 2.1.4). Consider the (oriented, based) extended Heegaard diagram $(\Sigma, \alpha, a, \beta)$ of Figure 3.1. We assume both α, a are oriented, so we have $\alpha^*, a^* \in \pi_1(M)$ and $\rho(\alpha^*), \rho(a^*) \in \mathrm{Aut}(H)$. For simplicity, we will denote these Hopf automorphisms just by α, a. If x_1, x_2, x_3 are the points of $\beta \cap \alpha$, encountered as one follows the orientation of β starting from the given basepoint, then $\overline{\beta}_{x_1} = a, \overline{\beta}_{x_2} = a\alpha a^{-1}\alpha^{-1}$ and $\overline{\beta}_{x_3} = a\alpha a^{-1}\alpha^{-1}a^{-1}$. Hence $K^\rho(\mathcal{H}) : H \to H$ is given by

$$K_H^\rho(\mathcal{H})(h) = a(h_{(1)}) \cdot (a\alpha a^{-1}\alpha^{-1})(S(h_{(2)})) \cdot (a\alpha a^{-1}\alpha^{-1}a^{-1})(h_{(3)}).$$

As a particular example, let $H = \Lambda$ be an exterior algebra on one generator X, so that $\mathrm{Aut}(\Lambda) \cong \mathbb{K}^\times$. Then ρ descends to $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ which is generated by a^*. If $a = \rho(a^*)$ satisfies $a(X) = tX$, then we get

$$\begin{aligned} Z_H^\rho(\mathcal{H}) &= \mu(K^\rho(\mathcal{H})(X)) \\ &= \mu(a(X_{(1)}) \cdot S(X_{(2)}) \cdot a^{-1}(X_{(3)})) \\ &= \mu(a(X)) + \mu(S(X)) + \mu(a^{-1}(X)) \\ &= t - 1 + t^{-1}, \end{aligned}$$

the Alexander polynomial of K.

Example 3.2.5. Consider the (oriented, based) extended Heegaard diagram of the figure eight knot of Figure 3.1. Then we have

$$K^\rho(\mathcal{H})(h) = a(h_{(1)}) \cdot ([a, \alpha] \circ S)(h_{(4)}) \cdot ([a, \alpha]a)(h_{(3)}) \cdot (\alpha^{-1}a \circ S)(h_{(2)}) \cdot \alpha^{-1}(h_{(5)})$$

where we use the commutator notation $[a, \alpha] \coloneqq a\alpha a^{-1}\alpha^{-1}$ (considered as an element of $\mathrm{Aut}(H)$) for simplicity. Note that if H is cocommutative, then this can be rewritten as a convolution product

$$K^\rho(\mathcal{H}) = a * ([a, \alpha] \circ S) * [a, \alpha]a * (\alpha^{-1}a \circ S) * \alpha^{-1}$$

where $f * g \coloneqq m_H(f \otimes g)\Delta_H$ for any $f, g : H \to H$. If $H = \Lambda(\mathbb{C}^2)$ and $\rho : \pi_1(S^3 \backslash K) \to SL(2, \mathbb{C})$ this implies

$$\begin{aligned} Z^\rho_H(\mathcal{H}) &= \mu(K^\rho_{\Lambda(\mathbb{C}^2)}(\mathcal{H})(c)) \\ &= \det(a - [a, \alpha] + [a, \alpha]a - \alpha^{-1}a + \alpha^{-1}) \\ &= \det(\rho(\partial\overline{\beta}/\partial\alpha)). \end{aligned}$$

The second equality follows from Lemma 4.3.1 below together with $r_{\Lambda(\mathbb{C}^2)} = \det$, see Example 1.2.11. This is the $SL(2, \mathbb{C})$ Alexander polynomial of K up to a factor, cf. [DFJ12, Proposition 2.5].

3.2.3 Relation to Virelizier's extension

We now explain how the formula of Definition 3.2.1 can be seen as (a sutured extension of) that of Subsection 3.1.3 specialized to a semidirect product. We also explain why we only consider semidirect products instead of more general Hopf G-algebras.

Let $\underline{H}$ be a Hopf G-algebra and $\rho : \pi_1(M, p) \to G$ a group homomorphism. Let $\mathcal{H}$ be an extended sutured Heegaard diagram which is ordered, oriented, based. Suppose that the set $\boldsymbol{a}$ is also ordered and oriented. We extend the order to all of $\boldsymbol{\alpha}^e$ by declaring the closed curves to come before the arcs. Since all the $\alpha \in \boldsymbol{\alpha}^e$ are oriented, we have a well-defined dual $\alpha^* \in \pi_1(M, p)$ for each $\alpha \in \boldsymbol{\alpha}^e$. For simplicity of notation, for each $\alpha \in \boldsymbol{\alpha}^e$ we denote $H_{\rho(\alpha^*)}$ simply by H_α. Then, the construction of Subsection 3.1.3 extends to a tensor

$$K^\rho_{\underline{H}}(\mathcal{H}) \coloneqq \underline{m}_{\boldsymbol{\beta}} \underline{P}_{\mathcal{H}} \underline{S}_{\boldsymbol{\alpha}^e} \underline{\Delta}_{\boldsymbol{\alpha}^e} : \bigotimes_{\alpha \in \boldsymbol{\alpha}^e} H_\alpha \to H_{1_G}^{\otimes d}$$

where $\underline{\Delta}_{\boldsymbol{\alpha}^e} \coloneqq \otimes_{\alpha \in \boldsymbol{\alpha}^e} \Delta_\alpha^{|\alpha|}$ and $\underline{S}_{\boldsymbol{\alpha}^e} \coloneqq \otimes_{x \in \mathcal{I}_{\boldsymbol{\alpha}^e}} S^{\epsilon_x}_{\alpha_x}$. Here $\mathcal{I}_{\boldsymbol{\alpha}^e}$ denotes the set of crossings $\boldsymbol{\alpha}^e \cap \boldsymbol{\beta}$ ordered according to $\boldsymbol{\alpha}^e$.

Let's see what happens to this tensor when we restrict to the semidirect product Hopf G-algebra associated to $\mathbb{K}[G] \ltimes H$ (where $G = \mathrm{Aut}(H)$) as in Subsection 1.3.2. Recall that this is defined by $H_\alpha = \{x \cdot \alpha \mid x \in H\} \subset \mathbb{K}[G] \ltimes H$ for each $\alpha \in G$ and the structure maps of $\underline{H}$ are induced from those of $\mathbb{K}[G] \ltimes H$. We introduce first some notation. Let $\beta \in \boldsymbol{\beta}$ and suppose that when following the orientaton of $\beta \in \boldsymbol{\beta}$ starting from its basepoint, the crossings through β are $x_1, \dots, x_k$, say $x_i \in \alpha_{x_i} \cap \beta$ for every i, where $\alpha_{x_i} \in \boldsymbol{\alpha}^e$. Let $\epsilon_1, \dots, \epsilon_k \in \{0, 1\}$ be determined by $m_i = (-1)^{\epsilon_i}$

where m_i is the intersection sign at x_i. Recall that, since $\overline{\beta} = \alpha_{x_1}^{m_1} \dots \alpha_{x_k}^{m_k} = 1$ in $\pi_1(M, p)$, there are maps

$$\underline{m}_\beta : H_{\alpha_{x_1}^{m_1}} \otimes \dots \otimes H_{\alpha_{x_k}^{m_k}} \to H_{1_G} = H.$$

We denote $\underline{S}_\beta := \otimes_{i=1}^k S_{\alpha_{x_i}}^{\epsilon_i}$ where $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ denotes the antipode of $\underline{H}$. We denote by m and S the multiplication and antipode of H and let $S_\beta := \otimes_{i=1}^k S^{\epsilon_i}$.

Lemma 3.2.6. *Let $\underline{H}$ be the Hopf G-algebra associated to the semidirect product $\mathbb{K}[G] \ltimes H$, where $G = Aut(H)$. Let $\mathcal{H}$ be an extended Heegaard diagram as above, and let $\beta \in \boldsymbol{\beta}$ and $k = |\boldsymbol{\alpha}^e \cap \boldsymbol{\beta}|$. Then one has*

$$\underline{m}_\beta \circ \underline{S}_\beta \circ \left(\bigotimes_{i=1}^k \kappa_{\alpha_{x_i}} \right) = m^{\otimes k} \circ S_\beta \circ \left(\bigotimes_{i=1}^k \rho(\overline{\beta}_{x_i}) \right) : H^{\otimes k} \to H$$

where $\kappa_\alpha : H \to H_\alpha$ is the canonical coalgebra isomorphism $x \mapsto x \cdot \alpha$.

In other words, the semidirect product structure of $\mathbb{K}[\mathrm{Aut}(H)] \ltimes H$ (i.e. the tensors on the left) reduces to Fox calculus on H (tensors on the right).

Proof. Suppose first that all crossings are positive, that is $\epsilon_i = 0$ for each i (so both $\underline{S}_\beta, S_\beta$ are identity maps). Then $\overline{\beta}_{x_i} = \alpha_{x_1} \alpha_{x_2} \dots \alpha_{x_{i-1}}$ for all $i = 1, \dots, k$ ($\overline{\beta}_{x_1} = 1$) and for any $h_1, \dots, h_k \in H$

$$\begin{aligned}
\underline{m}_\beta \circ \underline{S}_\beta \circ \left(\otimes_{i=1}^k \kappa_{\alpha_{x_i}} \right) &(h_1 \otimes \dots \otimes h_k) \\
&= (h_1 \alpha_1)(h_2 \alpha_2) \dots (h_k \alpha_k) \\
&= h_1 (\alpha_1 (h_2) \alpha_1) \alpha_2 \dots (h_k \alpha_k) \\
&= h_1 \alpha_1 (h_2) (\alpha_1 \alpha_2 (h_3)) \alpha_1 \alpha_2 \alpha_3 \dots \\
&\vdots \\
&= h_1 (\alpha_1 (h_2)) (\alpha_1 \alpha_2 (h_3)) \dots (\alpha_1 \dots \alpha_{k-1})(h_k)(\alpha_1 \dots \alpha_k) \\
&= h_1 (\alpha_1 (h_2)) (\alpha_1 \alpha_2 (h_3)) \dots (\alpha_1 \dots \alpha_{k-1})(h_k) \\
&= \rho(\overline{\beta}_{x_1})(h_1) \rho(\overline{\beta}_{x_2})(h_2) \dots \rho(\overline{\beta}_{x_k})(h_k).
\end{aligned}$$

In the first equalities we successively used the semidirect product relation $\alpha \cdot x = \alpha(x) \cdot \alpha$ for $\alpha \in G, x \in H$ and in the second-to-last equality we used that $\alpha_1 \dots \alpha_k = 1$ in G. This proves what we wanted. If a crossing x_i is negative, then from

$$S_{\alpha_{x_i}}(h_i \alpha_{x_i}) = \alpha_{x_i}^{-1} S(h_i) = \alpha_{x_i}^{-1}(S(h_i)) \alpha_{x_i}^{-1}$$

we get the $\alpha_{x_i}^{-1}$ term at the end of $\overline{\beta}_{x_i}$ (as in the definition of $\overline{\beta}_{x_i}$). This proves the lemma. $\square$

Let $\kappa \coloneqq \otimes_{\alpha\in\boldsymbol{\alpha}^e}\kappa_\alpha : H^{\otimes d+l} \to \otimes_{\alpha\in\boldsymbol{\alpha}^e} H_\alpha$ and $\eta_l \coloneqq \mathrm{id}_H^{\otimes d} \otimes \eta^{\otimes l} : H^{\otimes d} \to H^{\otimes d+l}$ where $\eta : \mathbb{K} \to H$ is the unit of H.

Proposition 3.2.7. *Let $\underline{H}$ be the Hopf G-algebra associated to $\mathbb{K}[G] \ltimes H$. Then*

$$K^\rho_{\underline{H}}(\mathcal{H}) \circ \kappa \circ \eta_l = K^\rho_H(\mathcal{H}).$$

Here the left hand side denotes Kuperberg's tensor at the Hopf G-algebra $\underline{H}$ and the right hand side is the Fox calculus extension of Kuperberg's tensor of Definition 3.2.1.

Proof. We will denote by $m^e_{\boldsymbol{\beta}}, S^e_{\boldsymbol{\beta}}, P^e_{\mathcal{H}}, S_{\boldsymbol{\alpha}^e}, \Delta_{\boldsymbol{\alpha}^e}$ the tensors of Subsection 3.1.1 defined from H using the whole extended diagram $(\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$. Let $m_{\boldsymbol{\beta}}, S_{\boldsymbol{\beta}}, P_{\mathcal{H}}, S_{\boldsymbol{\alpha}}, \Delta_{\boldsymbol{\alpha}}$ be the tensors defined from H using $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, i.e., without using the arcs. (these are the tensors involved in our definition of $K^\rho_H(\mathcal{H})$). Then it is clear that

$$\left(m^e_{\boldsymbol{\beta}} P^e_{\mathcal{H}} \left(\bigotimes_{x\in\mathcal{I}_{\boldsymbol{\alpha}^e}} \rho(\overline{\beta}_x) \right) S_{\boldsymbol{\alpha}^e} \Delta_{\boldsymbol{\alpha}^e} \right) \circ \eta_l = K^\rho_H(\mathcal{H})$$

Now, we have

$$\begin{aligned} K^\rho_{\underline{H}}(\mathcal{H}) \circ \kappa &= \underline{m}_{\boldsymbol{\beta}} \underline{S}_{\boldsymbol{\beta}} \underline{P}_{\mathcal{H}} \underline{\Delta}_{\boldsymbol{\alpha}^e} (\otimes_{\alpha\in\boldsymbol{\alpha}^e} \kappa_\alpha) \\ &= \underline{m}_{\boldsymbol{\beta}} \underline{S}_{\boldsymbol{\beta}} \underline{P}_{\mathcal{H}} (\otimes_{x\in\mathcal{I}_{\boldsymbol{\alpha}^e}} \kappa_{\alpha_x}) \Delta_{\boldsymbol{\alpha}^e} \\ &= \underline{m}_{\boldsymbol{\beta}} \underline{S}_{\boldsymbol{\beta}} (\otimes_{x\in\boldsymbol{\beta}\cap\boldsymbol{\alpha}^e} \kappa_{\alpha_x}) P^e_{\mathcal{H}} \Delta_{\boldsymbol{\alpha}^e} \\ &= m^e_{\boldsymbol{\beta}} S^e_{\boldsymbol{\beta}} (\otimes_{x\in\boldsymbol{\beta}\cap\boldsymbol{\alpha}^e} \rho(\overline{\beta}_x)) P^e_{\mathcal{H}} \Delta_{\boldsymbol{\alpha}^e} \\ &= m^e_{\boldsymbol{\beta}} P^e_{\mathcal{H}} \left(\bigotimes_{x\in\mathcal{I}_{\boldsymbol{\alpha}^e}} \rho(\overline{\beta}_x) \right) S_{\boldsymbol{\alpha}^e} \Delta_{\boldsymbol{\alpha}^e}. \end{aligned}$$

We used that the κ_α's are coalgebra morphisms in the second equality and Lemma 3.2.6 in the fourth one. Composing both sides with η_l gives the desired result. ◻

Remark 3.2.8. In Theorem 3.2.3, we supposed that ρ has image in $\mathrm{Ker}\,(r_H) \subset \mathrm{Aut}(H)$ and obtained an invariant of (M, γ) with no extra structure. Indeed, if $G = \mathrm{Ker}\,(r_H)$, the Hopf G-algebra $\mathbb{K}[G] \ltimes H$ is unimodular. As mentioned in Subsection 3.1.3, this is all that is needed to define invariants of a pair (Y, ρ) (with no extra structure) in the closed case.

Why we do not consider Hopf G-algebras in general? The reason is the following: if $\underline{H}$ is an arbitrary involutive finite type Hopf G-algebra, then to turn the G-graded Kuperberg tensor $K^\rho_{\underline{H}}(\mathcal{H}) : \otimes_{\alpha\in\boldsymbol{\alpha}^e} H_\alpha \to H_{1_G}^{\otimes d}$ into a scalar we need to evaluate it on some special element of H_α for each $\alpha \in \boldsymbol{\alpha}^e$. For closed α it is reasonable to use the cointegral $c_\alpha \in H_\alpha$. However, this is not reasonable to do over the arcs: if we did that, then the scalar obtained would be invariant under handlesliding a curve over an arc, which has no sense. What is reasonable to do is to suppose that each

H_α comes equipped with a group-like element $g_\alpha \in H_\alpha$, satisfying $g_{\alpha_1} g_{\alpha_2} = g_{\alpha_1\alpha_2}$ for each $\alpha_1, \alpha_2 \in G$. Then one can evaluate $K^\rho_{\underline{H}}(\mathcal{H})$ at c_α over each closed $\alpha \in \boldsymbol{\alpha}$ and at g_α for each arc $\alpha \in \boldsymbol{a}$. The resulting scalar would then be invariant exactly under the three extended handlesliding moves of Proposition 2.3.7. However, under these conditions, conjugation by g_α defines a Hopf automorphism $\phi(\alpha)$ of H_{1_G}, defining a group homomorphism $\phi : G \to \mathrm{Aut}(H_{1_G})$. If $\underline{H_{1_G}}$ denotes the Hopf $\mathrm{Aut}(H_{1_G})$-algebra coming from $\mathbb{K}[\mathrm{Aut}(H_{1_G})] \ltimes H_{1_G}$, then it is easy to see that we get a Hopf morphism $\underline{H} \to \underline{H_{1_G}}$ above ϕ, that is, we have coalgebra maps $\Phi_\alpha : H_\alpha \to H_{\phi(\alpha)}$ for each $\alpha \in G$ satisfying obvious algebra properties. It follows that

$$K^\rho_{\underline{H}}(\mathcal{H}) = K^{\phi\circ\rho}_{\underline{H_{1_G}}}(\mathcal{H}) \circ \left(\bigotimes_{\alpha \in \boldsymbol{\alpha}^e} \Phi(\alpha) \right).$$

Since $(\Phi(c_\alpha))$ is a cointegral of the $\phi(G)$-algebra $\{H_{\phi(\alpha)}\}$, it is a multiple of $(c_1 \cdot \phi(\alpha))$, where c_1 is the cointegral of H_{1_G}. It follows that

$$Z^\rho_{\underline{H}}(\mathcal{H}) = Z^{\phi\circ\rho}_{\underline{H_{1_G}}}(\mathcal{H})$$

and so we are reduced to the case of a semidirect product.

3.2.4 Some lemmas

We now establish some properties concerning the tensors Z^ρ_H. In what follows, we let $\mathcal{H}$ be an ordered, oriented, based extended Heegaard diagram of a sutured 3-manifold (M, γ) with connected $R_-(\gamma)$ and let $\rho : \pi_1(M, p) \to \mathrm{Aut}(H)$ be an arbitrary homomorphism, i.e. we do not suppose $\mathrm{Im}\,(\rho) \subset \mathrm{Ker}\,(r_H)$.

Proposition 3.2.9. *Let $\mathcal{H}$ be an ordered, oriented, based Heegaard diagram. Changing the basepoints along the α's has no effect on $Z^\rho_H(\mathcal{H})$. Now, let $\mathcal{H}'$ be the (ordered, oriented, based) Heegaard diagram obtained from $\mathcal{H}$ by moving a single basepoint $q_i \in \beta_i$ of $\mathcal{H}$ to $q'_i \in \beta_i$. Let $b \subset \beta_i$ be the oriented arc from q_i to q'_i. Then*

$$Z^\rho_H(\mathcal{H}) = r_H(\rho(\overline{b})) Z^\rho_H(\mathcal{H}').$$

Proof. Suppose we move the basepoint $p_1 \in \alpha_1$ to p'_1 and let $\mathcal{H}'$ be the new based Heegaard diagram. By definition $K^\rho(\mathcal{H}')$ differs from $K^\rho(\mathcal{H})$ only on the permutations $P(\mathcal{H}')$ and $P(\mathcal{H})$ and these differ as follows: let k denote the number of crossings between p_1 to p'_1 along α_1 and $l = |\alpha_1| - k$. Then one has

$$P(\mathcal{H}) = P(\mathcal{H}') \circ (P_{k,l} \otimes \mathrm{id}_H^{N-k-l})$$

where $P_{k,l} : H^{\otimes(k+l)} \to H^{\otimes(k+l)}$ is induced by the permutation of S_{k+l} that sends the first k letters to the last l and viceversa. But by Proposition 1.2.12, the cyclic order of $\Delta^{(k+l)}(c)$ is irrelevant, that is,

$$\Delta^{(k+l)}(c) = P_{k,l}\Delta^{(k+l)}(c)$$

for any k, l. From this it follows that $Z_H^\rho(\mathcal{H}) = Z_H^\rho(\mathcal{H}')$ as desired. Now suppose we move a basepoint along a curve in $\boldsymbol{\beta}$, say β_1. Then not only the permutation $P(\mathcal{H})$ changes, but the $\overline{\beta}_{1,x}$'s are also affected. For each $\beta \in \boldsymbol{\beta}$ and $x \in \boldsymbol{\alpha} \cap \beta$, let $\overline{\beta}'_x$ be the element corresponding to $\mathcal{H}'$. Then we have $\overline{\beta}_{1,x} = \overline{b} \cdot \overline{\beta}'_{1,x}$ for each $x \in \boldsymbol{\alpha} \cap \beta_1$ while for $\beta \neq \beta_1$, $\overline{\beta}_x = \overline{\beta}'_x$. Thus

$$\begin{aligned}\rho_{\mathcal{H}} &= m_{\boldsymbol{\beta}}(\rho(\overline{b})^{\otimes|\beta_1|} \otimes \mathrm{id}_H^{\otimes N-|\beta_1|})\rho_{\mathcal{H}'} \\ &= (\rho(\overline{b}) \otimes \mathrm{id}_H^{d-1}) m_{\boldsymbol{\beta}} \rho_{\mathcal{H}'}.\end{aligned}$$

Using that $\mu \circ \rho(\overline{b}) = r_H(\rho(\overline{b}))\mu$ (Proposition 1.2.12) and that μ is cyclic as in the argument of the first assertion, we obtain the desired relation.

□

Note that Proposition 3.2.9 requires unimodularity of H and H^* in an essential way. On the other hand Lemma 3.2.10 below requires involutivity of H.

Lemma 3.2.10. *Let $\mathcal{H}$ be an ordered, oriented, based Heegaard diagram. Reversing the orientation of a curve in $\boldsymbol{\beta}$ has no effect in Z_H^ρ. Now let $\mathcal{H}'$ be the ordered, oriented, based Heegaard diagram obtained from $\mathcal{H}$ by reversing the orientation of a single oriented closed curve $\alpha_i \in \boldsymbol{\alpha}$. Then*

$$Z_H^\rho(\mathcal{H}', \omega) = r_H(\rho(\alpha_i^*)) Z_H^\rho(\mathcal{H}, \omega).$$

Proof. Suppose the orientation of $\beta_1 \in \boldsymbol{\beta}$ is reversed. Since $\overline{\beta_1} = 1$ it follows that the $\overline{\beta}_{1,x}$'s are unchanged, hence $\rho_{\mathcal{H}}$ is unchanged. The only changes comes in the order of the multiplication along β_1 and the evaluation of the antipode, since the crossings through β_1 reverse order and sign. But using that S is an algebra anti-automorphism and $S^2 = \mathrm{id}_H$ it is easy to see that

$$K_H^\rho(\mathcal{H}') = (S \otimes \mathrm{id}_H^{d-1}) K_H^\rho(\mathcal{H}).$$

Using that $\mu \circ S = (-1)^{|c|}\mu$ (Proposition 1.2.12), we get $Z_H^\rho(\mathcal{H}) = (-1)^{|c|} Z_H^\rho(\mathcal{H}')$ and hence the equality if we take the orientation ω into account. Now suppose we reverse the orientation of a curve $\alpha_i \in \boldsymbol{\alpha}$, say $\alpha_i = \alpha_1$. The only difference with the preceding argument is that now the $\overline{\beta}_x$'s change. Indeed, for each $\beta \in \boldsymbol{\beta}$ let $m'(x)$ be the sign of intersection, in $\mathcal{H}'$, of $x \in \alpha_j \cap \beta$ so $m'(x) = m(x)$ if $j \neq 1$ and $m'(x) = -m(x)$ if $j = 1$. Recall that if $x \in \alpha_j \cap \beta$, there is a α_j^* at the end of $\overline{\beta}'_x$ if $m'(x)$ is negative, but this α_j^* does not appears if $m'(x)$ is positive. Therefore, we have

$$\overline{\beta}'_x = \overline{\beta}_x \cdot \alpha_1^*$$

for each $x \in \alpha_1 \cap \beta$ and $\overline{\beta}'_x = \overline{\beta}_x$ for $x \in \alpha_j \cap \beta$ with $j \neq 1$. Combining this with the preceding argument (so we use that S is a coalgebra anti-automorphism and $S^2 = \mathrm{id}_H$), it follows that

$$K^\rho(\mathcal{H}') = K^\rho(\mathcal{H})((S\rho(\alpha_1^*)) \otimes \mathrm{id}_H^{\otimes d-1}).$$

Since $(S\rho(\alpha_1^*))(c) = (-1)^{|c|} r_H(\rho(\alpha_1^*))$, it follows that $Z_H^\rho(\mathcal{H}') = (-1)^{|c|} r_H(\rho(\alpha_1^*)) Z_H^\rho(\mathcal{H})$ and hence the desired equality if we consider the orientation ω as before.

□

Now, given a fixed $\phi \in \mathrm{Aut}(H)$, let $\rho_\phi : \pi_1(M, p) \to \mathrm{Aut}(H)$ be the homomorphism defined by $\rho_\phi(x) := \phi \circ \rho(x) \circ \phi^{-1}$ for any $x \in \pi_1(M, p)$.

Lemma 3.2.11. *For any $\phi \in Aut(H)$ we have $Z_H^{\rho_\phi}(\mathcal{H}) = Z_H^\rho(\mathcal{H})$.*

Proof. Since ϕ is a Hopf automorphism and $K_H^\rho(\mathcal{H})$ only involves the structure maps of H, we have

$$K_H^{\rho_\phi}(\mathcal{H}) = \phi^{\otimes d} \circ K_H^\rho(\mathcal{H}) \circ (\phi^{-1})^{\otimes d}.$$

Therefore, by definition of r_H and Proposition 1.2.12, we get

$$\begin{aligned} Z_H^{\rho_\phi}(\mathcal{H}) &= (\mu \circ \phi)^{\otimes d} K_H^\rho(\mathcal{H})((\phi^{-1}(c))^{\otimes d}) \\ &= r_H(\phi)^d \cdot Z_H^\rho(\mathcal{H}) \cdot r_H(\phi^{-1})^d \\ &= Z_H^\rho(\mathcal{H}). \end{aligned}$$

□

Now let $p_1, p_2 \in s(\gamma)$ be basepoints and let $\delta : [0,1] \to M$ be a path from p_1 to p_2. Let $C_{[\delta]} : \pi_1(M, p_1) \to \pi_1(M, p_2)$ be the isomorphism defined by $C_{[\delta]}(\alpha) = [\overline{\delta}]\alpha[\delta]$ for $\alpha \in \pi_1(M, p_1)$.

Corollary 3.2.12. *The scalar $Z_H^\rho(\mathcal{H})$ is independent of the basepoint $p \in s(\gamma)$. More precisely, let $p_1, p_2 \in s(\gamma)$ be two basepoints and let $\rho_i : \pi_1(M, p_i) \to Aut(H)$ for $i = 1, 2$ be group homomorphisms related by $\rho_1 = \rho_2 \circ C_{[\delta]}$ for some path δ from p_1 to p_2. Then $Z_H^{\rho_1}(\mathcal{H}) = Z_H^{\rho_2}(\mathcal{H})$.*

Proof. Note that changing the path δ by another path changes $C_{[\delta]}$ by an inner automorphism of $\pi_1(M, p_2)$. Therefore, by Proposition 3.2.11, it suffices to prove the corollary for a specific path δ. Let $\alpha_{p_i}^* \in \pi_1(M, p_i)$ be the dual curves of the α's coming from the diagram $\mathcal{H}$ with basepoint p_i, $i = 1, 2$. If we just let δ be a path from p_1 to p_2 contained in $\Sigma \setminus \boldsymbol{\alpha}^e$ (which is connected) then $\alpha_{p_2}^* = \overline{\delta}\alpha_{p_1}^*\delta$. Using $\rho_1 = \rho_2 \circ C_{[\delta]}$ we get $\rho_1(\alpha_{p_1}^*) = \rho_2(\overline{\delta}\alpha_{p_1}^*\delta) = \rho_2(\alpha_{p_2}^*)$ for each $\alpha \in \boldsymbol{\alpha}$. It follows that $\rho_1(\mathcal{H}) = \rho_2(\mathcal{H})$ and hence $Z^{\rho_1}(\mathcal{H}) = Z^{\rho_2}(\mathcal{H})$. □

3.2.5 Proof of invariance, special case

We now prove Theorem 3.2.3. Of course, most of the proof is essentially as in [Kup91] or [Vir05], with some extra care because of the appearance of the arcs.

Proof of Theorem 3.2.3. Since $\text{Im}(\rho) \subset \text{Ker}(r_H)$, Proposition 3.2.9 and Lemma 3.2.10 imply that $Z_H^\rho(\mathcal{H})$ is independent of the basepoints and the orientation of the curves in $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$. Changing the ordering of the closed curves introduces a sign (whenever $|c| = 1$) into $Z_H^\rho(\mathcal{H})$, but since $\delta_\omega(\mathcal{H})$ also changes sign, $Z_H^\rho(\mathcal{H}, \omega)$ is unchanged. Therefore, $Z_H^\rho(\mathcal{H})$ is independent of all the extra structure we put on $\mathcal{H}$ (ordering, orientation and basepoints) and so we only need to show it is invariant under extended Heegaard moves. Each time we perform such a move we will denote by $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ the new Heegaard diagram obtained (so $\Sigma' = \Sigma$ except for stabilization), $\overline{\beta}'_x$ the elements in $\pi_1(M, p)$ defined from $\mathcal{H}'$, and so on.

1. Isotopy inside $\boldsymbol{\alpha}^e \cup \boldsymbol{\beta}$: if the isotopy occurs in $\text{int}(\Sigma)$ this follows from the antipode axiom as in [Kup91]. Now, suppose we isotope an arc in $\boldsymbol{a}$ past the basepoint $p \in s(\gamma)$ and let $\mathcal{H}'$ be the extended diagram after the isotopy. Then all the dual curves become conjugated by $(a^*)^{\pm 1}$, see Remark 2.3.10. Therefore, all the $\overline{\beta}'_x$'s get conjugated by $(a^*)^{\pm 1}$. If $\phi = \rho((a^*)^{\pm 1}) \in \text{Aut}(H)$ it follows that $Z^\rho(\mathcal{H}') = Z^{\rho_\phi}(\mathcal{H})$ where recall that ρ_ϕ is defined by $\rho_\phi(x) = \phi \circ \rho(x) \circ \phi^{-1}$ for all $x \in \pi_1(M, p)$. By Proposition 3.2.11, one has $Z^{\rho_\phi}(\mathcal{H}) = Z^\rho(\mathcal{H})$ and so $Z^\rho(\mathcal{H}') = Z^\rho(\mathcal{H})$ as desired.

2. Handlesliding in $\boldsymbol{\alpha}$: suppose we handleslide a closed curve α_2 over α_1 along an arc $\delta \subset \text{int}(\Sigma)$. We denote by α'_2 the curve obtained after handlesliding. We suppose that $\alpha_1, \alpha_2, \alpha'_2$ are oriented so that

   $$\partial P = \alpha_1 \cup \alpha_2 \cup -\alpha'_2$$

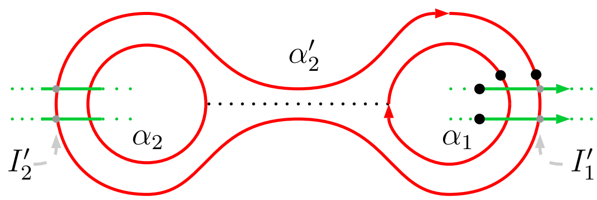
   as oriented 1-manifolds, where P is the handlesliding region. We also suppose that the basepoints of $\boldsymbol{\beta}$ are outside of P. By isotopy invariance, we can further suppose $\delta \cap \boldsymbol{\beta} = \emptyset$, so we can write

   $$\alpha'_2 \cap \boldsymbol{\beta} = I'_1 \cup I'_2$$

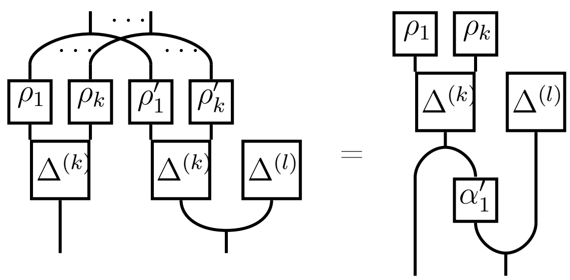
   where I'_i is the set of crossings through α'_2 that sit next to the crossings of α_i for $i = 1, 2$. More precisely, there is an obvious bijection $j : \alpha_i \cap \boldsymbol{\beta} \to I'_i$ for which each point of I'_i is consecutive to the corresponding point in $\alpha_i \cap \boldsymbol{\beta}$ in the order of $\mathcal{I}_{\boldsymbol{\beta}}$ (since the basepoints of $\boldsymbol{\beta}$ are outside P).

   Claim: With these choices, we have $\overline{\beta}'_{j(x)} = \overline{\beta}'_x \cdot (\alpha'_1)^*$ for each $x \in I'_1$ and $\overline{\beta}'_x = \overline{\beta}_x$ for every crossing of $\mathcal{H}'$ not in I'_1 (we identify the points of I'_2 with $\alpha_2 \cap \boldsymbol{\beta}$ for simplicity).

We will further suppose that the basepoint of α_2' is placed right before I_1', so that when following α_2' starting from this basepoint, the crossings in I_1' appear first and then come those of I_2'. Thus, I_1', I_2' inherit the order coming from α_1, α_2 and $x < y$ for all $x \in I_1', y \in I_2'$. The following figure illustrates our choices:



Let $k = |\alpha_1|$ and $l = |\alpha_2|$. For each $i = 1, \ldots, k$, let $\rho_i = \rho(\overline{\beta}'_{x_i})$ and $\rho_i' = \rho(\overline{\beta}'_{j(x_i)}) = \rho_i \circ \alpha_1'$, where $\alpha_1' \in \mathrm{Aut}(H)$ stands for $\rho((\alpha_1')^*)$. With the above choice of orientations and basepoints, the portion of the tensor $K_H^\rho(\mathcal{H}')$ corresponding to α_1, α_2' looks as follows:



On the right hand side we used that each $\rho_i \in \mathrm{Aut}(H)$ is an algebra morphism, that $\alpha_1' \in \mathrm{Aut}(H)$ is a coalgebra morphism and that Δ is an algebra morphism. In other words, we obtained

$$K_H^\rho(\mathcal{H}') = K_H^\rho(\mathcal{H}) \circ (T_{\alpha_1'} \otimes \mathrm{id}_H^{\otimes d-2})$$

where we let $T_\phi \coloneqq (m_H \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \phi \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \Delta_H)$ for any $\phi \in \mathrm{Aut}(H)$. By definition of the cointegral, one can see that

$$T_\phi(c \otimes h) = c \otimes h$$

for any $\phi \in \mathrm{Aut}(H)$ and $h \in H$. It follows that $Z_H^\rho(\mathcal{H}') = Z_H^\rho(\mathcal{H})$ as was to be shown.

3. Sliding an arc over $\boldsymbol{\alpha}^e$: by isotopy invariance, we can suppose that the basepoint $p \in s(\gamma)$ is outside the handlesliding region. Then it is easy to see that sliding an arc over another does not changes the $\overline{\beta}_x$'s and since $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is unaffected, the tensor $K_H^\rho(\mathcal{H})$ is unaffected as well. Now suppose we slide an arc over a closed curve. If we further suppose the basepoints of $\boldsymbol{\beta}$ are outside

the handlesliding region (which we can as a consequence of Proposition 3.2.9), then the $\overline{\beta}_x$'s are unaffected. As before, this implies that $K_H^\rho(\mathcal{H})$, and hence $Z_H^\rho(\mathcal{H})$, is unchanged.

4. Handlesliding in $\boldsymbol{\beta}$: the situation is similar to the case of $\boldsymbol{\alpha}$ handlesliding. If we slide β_2 over β_1 to get a curve β_2' and we put orientations and basepoints as above, then one has $\overline{\beta}'_x = \overline{\beta}_x$ for all $x \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ and also $\overline{\beta}'_{j(x)} = \overline{\beta}_x$ for all $x \in \beta_1 \cap \boldsymbol{\alpha}$. It then follows that

$$K_H^\rho(\mathcal{H}') = (T_{\mathrm{id}_H} \otimes \mathrm{id}_H^{\otimes d-2}) K_H^\rho(\mathcal{H})$$

   and so $Z_H^\rho(\mathcal{H}') = Z_H^\rho(\mathcal{H})$ by the defining property of the integral μ.

5. Stabilization follows directly from the normalization $\mu(c) = 1$.

$\square$

3.2.6 The disconnected case

We now extend the definition of $Z_H^\rho(M, \gamma)$ to the case when $R_-(\gamma)$ is disconnected (e.g. link complements). We also extend Z to disconnected balanced sutured 3-manifolds by declaring it to be multiplicative under disjoint union.

Let (M, γ) be a balanced sutured manifold with possibly disconnected subsurface $R = R_-(\gamma)$ and let $\rho : \pi_1(M, p) \to \mathrm{Ker}\,(r_H) \subset \mathrm{Aut}(H)$. Then we can construct a balanced sutured manifold (M', γ') containing M and with connected $R' = R_-(\gamma')$ as follows (see [FJR11, Section 3.6]). First, attach a 2-dimensional 1-handle h along $s(\gamma)$. This handle can be thickened to a 3-dimensional 1-handle $h \times I$ attached to $\gamma = s(\gamma) \times I$. This produces a new balanced sutured manifold, and after sufficiently many handle attachments, we get (M', γ') with connected R'. Note that $\pi_1(M') = \pi_1(M) * F$ where F is the free group generated by loops piercing the newly attached one-handles in a single point. Thus, a group homomorphism $\rho : \pi_1(M, p) \to \mathrm{Ker}\,(r_H)$ can always be extended to $\rho' : \pi_1(M', p) \to \mathrm{Ker}\,(r_H)$.

Lemma 3.2.13. *Let (M', γ') be a sutured 3-manifold obtained from (M, γ) by attaching 1-handles to γ as above. Let ρ' be an extension of ρ taking values in Ker$(r_H) \subset$ Aut(H). Suppose $R = R_-(\gamma)$ is connected so that both $Z_H^\rho(M, \gamma)$ and $Z_H^{\rho'}(M', \gamma')$ are defined. Then*

$$Z_H^\rho(M, \gamma) = Z_H^{\rho'}(M', \gamma').$$

Proof. It suffices to suppose (M', γ') is obtained by adding a single 3-dimensional 1-handle to $R_- \times I$. Then, an extended Heegaard diagram of M' is obtained from an extended diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ of M by attaching a 2-dimensional 1-handle h to Σ along $\partial\Sigma$ and letting $\boldsymbol{a}' = \boldsymbol{a} \cup \{a\}$, where a is the cocore of the 1-handle attached (as usual we note $\boldsymbol{\alpha}^e = \boldsymbol{\alpha} \cup \boldsymbol{a}$). If Σ' denotes the surface Σ with h attached, then $(\Sigma', \boldsymbol{\alpha}^e \cup \{a\}, \boldsymbol{\beta})$ is an extended diagram of (M', γ'). Thus, the tensors defining $Z(M, \gamma)$ and $Z(M', \gamma')$ are the same except possibly for the $\overline{\beta}_x$'s. Indeed, these also coincide: since the β's are disjoint from a, we have $\overline{\beta}'_x = i_*(\overline{\beta}_x)$ for each $\beta \in \boldsymbol{\beta}$ and $x \in \beta \cap \boldsymbol{\alpha}$, where the $\overline{\beta}'_x \in \pi_1(M', p)$ are defined from $\mathcal{H}'$. From $\rho' i_* = \rho$ we get $\rho(\overline{\beta}_x) = \rho'(\overline{\beta}'_x)$. Thus,

$$Z_H^{\rho}(\mathcal{H}) = Z_H^{\rho'}(\mathcal{H}')$$

proving the lemma.

□

Proposition 3.2.14. *Let (M, γ) be a balanced sutured manifold with possibly disconnected $R = R_-(\gamma)$. Let (M', γ') be any sutured manifold with connected $R' = R_-(\gamma')$ constructed as above and let ρ' be such that $\rho' i_* = \rho$. The scalar $Z_H^{\rho'}(M', \gamma')$ is independent of how the one-handles are attached to R (provided R' is connected) and of the extension ρ' of ρ chosen.*

Proof. Suppose (M', γ') and (M'', γ'') are obtained from (M, γ) by attaching 1-handles to R as above and that R', R'' are connected. One can keep attaching 1-handles to find a sutured manifold (M_0, γ_0) containing both M' and M''. If ρ', ρ'' are extensions of ρ to M', M'' respectively, then there is an extension ρ_0 to M_0 restricting to both ρ', ρ''. This is because $\pi_1(M_0) = \pi_1(M) * F' * F'' * F$ for some free groups F', F'', F, where $\pi_1(M') = \pi_1(M) * F', \pi_1(M'') = \pi_1(M) * F''$. By the lemma above, it follows that $Z_H^{\rho'}(M', \gamma') = Z_h^{\rho_0}(M_0, \gamma_0) = Z_H^{\rho''}(M'', \gamma'')$ as desired. □

Definition 3.2.15. Let (M, γ) be a connected balanced sutured manifold with possibly disconnected $R = R_-(\gamma)$. We define

$$Z_H^{\rho}(M, \gamma) \coloneqq Z_H^{\rho'}(M', \gamma')$$

where (M', γ') is any balanced sutured manifold obtained by adding 1-handles to M as above and ρ' is any extension of ρ to $\pi_1(M)$. If M is disconnected with connected components $M_1, \dots, M_m$ then we let

$$Z_H^{\rho}(M, \gamma) \coloneqq \prod_{i=1}^{m} Z_H^{\rho_i}(M_i, \gamma_i i)$$

where $\gamma_i = \gamma \cap M_i, \rho_i = \rho|_{H_1(M_i)}$ for each $i = 1, \dots, m$.

3.3 Extending to sutured manifolds: non-unimodular case

In this section we explain how to normalize $Z_H^{\rho}(\mathcal{H})$ with Spin^c structures to obtain a topological invariant whenever $\mathrm{Im}\,(\rho)$ is not contained in $\mathrm{Ker}\,(r_H)$. We begin by defining Spin^c structures and explaining how they are represented on a Heegaard diagram. In Subsection 3.3.2, we give a simple trick to turn a Heegaard diagram with a multipoint into a based diagram. Then in Subsection 3.3.3 we show how to turn Z into a well-defined topological invariant $Z^{\rho}_{H}(M,\gamma,\mathfrak{s})$ where $\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)$ using the affine structure of Spin^c.

3.3.1 Spin^c structures and multipoints

Let (M,γ) be a connected sutured manifold. Fix a nowhere vanishing vector field v_0 on ∂M with the following properties:

1. It points into M along $\operatorname{int} R_-$,
2. It points out of M along $\operatorname{int} R_+$,
3. It is given by the gradient of the height function $\gamma = s(\gamma)\times[-1,1]\to[-1,1]$ on γ.

Definition 3.3.1. Let v and w be two non-vanishing vector fields on M such that $v|_{\partial M} = v_0 = w|_{\partial M}$. We say that v and w are *homologous* if they are homotopic rel ∂M in the complement of an open 3-ball embedded in $\operatorname{int}(M)$ where the homotopy is through non-vanishing vector fields. A *Spin^c structure* is an homology class of such non-vanishing vector fields on M. We denote the set of Spin^c structures on M by $\mathrm{Spin}^c(M,\gamma)$.

The space of boundary vector fields v_0 with the above properties is contractible. This implies that there is a canonical identification between the set of Spin^c structures coming from different boundary vector fields. Thus, we make no further reference to v_0.

Proposition 3.3.2 ([Juh10, Prop. 3.6])**.** *Let M be a connected sutured manifold. Then $\mathrm{Spin}^c(M,\gamma)\neq\emptyset$ if and only if M is balanced. In such a case, the group $H^2(M,\partial M)$ acts freely and transitively over $\mathrm{Spin}^c(M,\gamma)$.*

We denote the action of $H^2(M,\partial M)$ over $\mathrm{Spin}^c(M,\gamma)$ by $(h,\mathfrak{s})\mapsto\mathfrak{s}+h$. If $\mathfrak{s}_1,\mathfrak{s}_2$ denote two Spin^c structures on M, we denote by $\mathfrak{s}_1-\mathfrak{s}_2$ the element $h\in H^2(M,\partial M)$ such that $\mathfrak{s}_1=\mathfrak{s}_2+h$.

We now proceed to understand Spin^c structures from a Heegaard diagram. For this, we need the following definition.

Definition 3.3.3. Let $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be a balanced Heegaard diagram, where $\boldsymbol{\alpha}=\{\alpha_1,\dots,\alpha_d\}$ and $\boldsymbol{\beta}=\{\beta_1,\dots,\beta_d\}$. A *multipoint* in $\mathcal{H}$ is an unordered set $\mathbf{x}=\{x_1,\dots,x_d\}$ where $x_i\in\alpha_i\cap\beta_{\sigma(i)}$ for each $i=1,\dots,d$ and σ is some permutation in S_d. The set of multipoints of $\mathcal{H}$ is denoted by $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$.

We use the notation $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ for the set of multipoints by the following reason. If we let $\mathrm{Sym}^d(\Sigma) \coloneqq \Sigma^d/S_d$ where the symmetric group S_d acts on $\Sigma^d = \Sigma \times \ldots \times \Sigma$ by permuting the factors, then the Heegaard diagram induces two tori $\mathbb{T}_\alpha \coloneqq \alpha_1 \times \ldots \times \alpha_d$ and $\mathbb{T}_\beta \coloneqq \beta_1 \times \ldots \times \beta_d$ contained in $\mathrm{Sym}^d(\Sigma)$. A multipoint $\mathbf{x} = \{x_1, \ldots, x_d\}$ corresponds to an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Given a balanced Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of (M, γ) with $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, one can construct a map

$$s : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(M, \gamma)$$

(see [OS04, Section 2.6] for the closed case and [Juh06, Section 4] for the sutured extension). To do this, we first fix a Riemannian metric on M. Now, take a Morse function $f : M \to [-1, 4]$ satisfying the following conditions:

1. $f(R_-) = -1, f(R_+) = 4$ and $f|_\gamma$ is the height function $\gamma = s(\gamma) \times [-1, 4] \to [-1, 4]$. Here we choose a diffeomorphism $\gamma = s(\gamma) \times [-1, 4]$ such that $s(\gamma)$ corresponds to $s(\gamma) \times \{3/2\}$.
2. For $i = 1, 2$, f has d index i critical points and has value i on these points. These lie in $\mathrm{int}\,(M)$ and there are no other critical points.
3. One has $\Sigma = f^{-1}(3/2)$, the α curves coincide with the intersection of the unstable manifolds of the index one critical points with Σ and the β curves coincide with the intersection of the stable manifolds of the index two critical points with Σ.

Such a Morse function always exists (see e.g. [JTZ12, Prop. 6.17]). By the first condition, $\nabla f|_{\partial M}$ satisfies the properties of the vector field v_0 in Definition 3.3.1. Note that the only singularities of ∇f are the index one and index two critical points of f, denote them by $P_1, \ldots, P_d$ and $Q_1, \ldots, Q_d$ respectively. Then the last condition above means

$$W^u(P_i) \cap \Sigma = \alpha_i \quad \text{and} \quad W^s(Q_i) \cap \Sigma = \beta_i$$

for each i. Here W^u and W^s denote respectively the unstable and stable submanifolds of ∇f at the corresponding critical point. Thus, an intersection point $x \in \alpha_i \cap \beta_j$ corresponds to a trajectory of ∇f starting at P_i and ending at Q_j. In particular, a multipoint $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ corresponds to a d-tuple $\gamma_{\mathbf{x}}$ of trajectories of ∇f connecting all the index one critical points to all the index two critical points.

Definition 3.3.4. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a balanced Heegaard diagram of (M, γ) and $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Let f be a Morse function adapted to the Heegaard diagram as above. We define a Spin^c structure $s(\mathbf{x})$ as follows: let N be a tubular neighborhood of $\gamma_{\mathbf{x}}$ in M homeomorphic to a disjoint union of d 3-balls. Then ∇f is a non-vanishing vector field over $M \setminus N$. Since the critical points of f have complementary indices on each component of N, one can extend $\nabla f|_{M \setminus N}$ to a non-vanishing vector field over all of M. We let $s(\mathbf{x})$ be the homology class of this vector field.

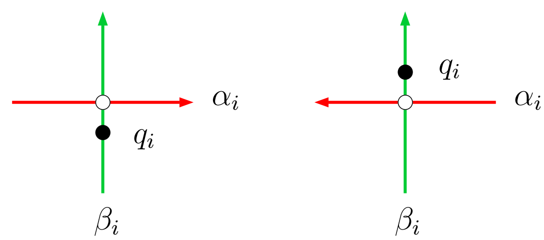


Figure 3.2: Basepoints $q_i = q_i(\mathbf{x})$ on $\boldsymbol{\beta}$ coming from $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We suppose the surface is oriented towards the reader, so that the left crossing is positive and the right one is negative. The white dot represents $x_i \in \alpha_i \cap \beta_i$.

3.3.2 Multipoints and basepoints

In order to fix the indeterminacy of $Z_H^\rho(\mathcal{H})$ coming from the basepoints and orientations of the curves in $\mathcal{H}$ and to obtain a topological invariant defined for an arbitrary ρ we will pick the basepoints in a very special way. The following is [LN19a, Definition 5.22].

Definition 3.3.5. Let $\mathcal{H}$ be an oriented sutured Heegaard diagram and let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a multipoint in $\mathcal{H}$, say $\mathbf{x} = \{x_1, \ldots, x_d\}$ where $x_i \in \alpha_i \cap \beta_i$ for each $i = 1, \ldots, d$. For each i, we let $q_i(\mathbf{x}) \in \beta_i$ be a basepoint defined as follows: if the crossing x_i is positive (resp. negative), then $q_i(\mathbf{x})$ lies just before x_i (resp. after x_i) when following the orientation of β_i, see Figure 3.2. In other words, the basepoint q_i is always to the right side of α_i. If $\mathcal{H}$ is based with these basepoints on $\boldsymbol{\beta}$ and arbitrary basepoints on $\boldsymbol{\alpha}$, we denote the tensor $Z_H^\rho(\mathcal{H})$ (resp. $Z_H^\rho(\mathcal{H}, \omega)$) by $Z_H^\rho(\mathcal{H}, \mathbf{x})$ (resp. $Z_H^\rho(\mathcal{H}, \mathbf{x}, \omega)$).

3.3.3 Normalizing Z via Spin^c

We now treat the case of an arbitrary $\rho : \pi_1(M, p) \to \mathrm{Aut}(H)$. Let (M, γ) be a connected balanced sutured 3-manifold with connected $R_-(\gamma)$ and fix a Spin^c structure $\mathfrak{s}$ on (M, γ).

Let $\mathcal{H}$ be an ordered, oriented, extended Heegaard diagram of (M, γ). Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a multipoint, and suppose $\mathcal{H}$ is based via $\mathbf{x}$ using the convention of Definition 3.3.5. Recall that we note by $Z_H^\rho(\mathcal{H}, \mathbf{x})$ the tensor $Z_H^\rho(\mathcal{H})$ where $\mathcal{H}$ has the basepoints on $\boldsymbol{\beta}$ induced from $\mathbf{x}$ and arbitrary basepoints on $\boldsymbol{\alpha}$. The multipoint $\mathbf{x}$ defines a Spin^c structure $s(\mathbf{x})$ as in Subsection 3.3.1, and comparing this with our chosen $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ we get an homology class

$$h_{\mathfrak{s},\mathbf{x}} := PD[s(\mathbf{x}) - \mathfrak{s}] \in H_1(M).$$

The composition $r_H \circ \rho : \pi_1(M, p) \to \mathbb{K}^\times$ descends to $H_1(M)$ and so can be evaluated over $h_{\mathfrak{s},\mathbf{x}}$.

Theorem 3.3.6. *Let $\mathcal{H}$ be an ordered, oriented, extended Heegaard diagram of (M,γ) and $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a multipoint. Then the scalar*

$$r_H \circ \rho(h_{\mathfrak{s},\boldsymbol{x}}) Z_H^\rho(\mathcal{H}, \boldsymbol{x}, \omega) \in \mathbb{K}$$

is independent of all choices and defines a topological invariant of the tuple $(M,\gamma,\rho,\mathfrak{s})$ (resp. $(M,\gamma,\rho,\mathfrak{s},\omega)$ if the cointegral of H has degree one) where $\mathfrak{s} \in Spin^c(M,\gamma)$.

Definition 3.3.7. We denote the above invariant by $Z_H^\rho(M,\gamma,\mathfrak{s},\omega)$, that is,

$$Z_H^\rho(M,\gamma,\mathfrak{s},\omega) \coloneqq r_H \circ \rho(h_{\mathfrak{s},\mathbf{x}}) Z_H^\rho(\mathcal{H},\mathbf{x},\omega) \in \mathbb{K}$$

where $\mathcal{H}$ is any ordered, oriented, extended Heegaard diagram of (M,γ) with basepoints coming from an arbitrary multipoint $\mathbf{x}$ of $\mathcal{H}$.

Our invariant depends on the Spin^c structure as follows: for any $h \in H^2(M,\partial M)$

$$Z_H^\rho(M,\gamma,\mathfrak{s}+h,\omega) = r \circ \rho(PD[h])^{-1} Z_H^\rho(M,\gamma,\mathfrak{s},\omega) \tag{3.2}$$

where $PD : H^2(M,\partial M) \to H_1(M)$ is Poincaré duality. This follows from the definitions using that $PD[s(\mathbf{x}) - (\mathfrak{s}+h)] = PD[s(\mathbf{x}) - \mathfrak{s}]PD[h]^{-1}$.

3.3.4 Proof of invariance

We now prove Theorem 3.3.6. We need to establish a few easy lemmas beforehand.

To show that the formula of Theorem 3.3.6 is independent of the multipoint chosen, we need to know what happens to $s(\mathbf{x})$ and how do the $\mathbf{x}$-basepoints on $\mathcal{H}$ change when performing Heegaard moves. It turns out that both changes are expressed in terms of the same homology class.

Definition 3.3.8. Let $\mathbf{x},\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be two multipoints say $x_i \in \alpha_i \cap \beta_i$ and $y_i \in \alpha_i \cap \beta_{\sigma(i)}$ for each i. Let c_i be an arc joining x_i to y_i along α_i and d_i be an arc joining $y_{\sigma^{-1}(i)}$ to x_i along β_i. Then

$$\sum_{i=1}^d c_i + \sum_{i=1}^d d_i$$

is a cycle in Σ. We denote by $\epsilon(\mathbf{x},\mathbf{y})$ the element of $H_1(M)$ induced by the cycle above.

Note that for each i there are two choices for an arc c_i joining x_i and y_i along α_i (similarly for d_i) but the class $\epsilon(\mathbf{x},\mathbf{y}) \in H_1(M)$ is independent of which arc is chosen.

Lemma 3.3.9 ([Juh06, Lemma 4.7]). *For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we have*

$$PD[s(\boldsymbol{x}) - s(\boldsymbol{y})] = \epsilon(\boldsymbol{x}, \boldsymbol{y})$$

where $PD : H^2(M, \partial M) \to H_1(M)$ is Poincaré duality.

Now let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, say $x_i \in \alpha_i \cap \beta_i$ and $y_i \in \alpha_i \cap \beta_{\sigma(i)}$ for each $i = 1, \dots, d$. By Definition 3.3.5, we get two basepoints on β_i for each i, which we will denote by $q_i(\mathbf{x})$ and $q_i(\mathbf{y})$. We let $d_i' \subset \beta_i$ be the oriented arc from $q_i(\mathbf{x})$ to $q_i(\mathbf{y})$. Recall that from such an arc we obtain an element $\overline{d_i'} \in \pi_1(M, p)$.

Lemma 3.3.10. *For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ one has*

$$\epsilon(\boldsymbol{y}, \boldsymbol{x}) = \prod_{i=1}^{d} h(\overline{d_i'})$$

in $H_1(M)$, where $\epsilon(\boldsymbol{x}, \boldsymbol{y})$ is the homology class of Subsection 3.3.1 and $h : \pi_1(M) \to H_1(M)$ is the projection.

Proof. For each i let c_i be the oriented subarc of α_i starting at y_i and ending at x_i and let d_i be the oriented subarc of β_i from x_i to $y_{\sigma^{-1}(i)}$, so that $\cup_{i=1}^d (c_i \cup d_i)$ represents $\epsilon(\mathbf{y}, \mathbf{x})$ (see Definition 3.3.8). Let c_i' be an oriented arc parallel to c_i, on its right side, that goes from $q_{\sigma(i)}(\mathbf{y})$ to $q_i(\mathbf{x})$. It is clear that $\cup_{i=1}^d (c_i \cup d_i)$ can be pushed off to the right side of $\boldsymbol{\alpha}$ to match $\cup_{i=1}^d (c_i' \cup d_i')$, so they are isotopic as oriented 1-submanifolds of Σ. Thus, $\epsilon(\mathbf{y}, \mathbf{x})$ is represented by the 1-submanifold $\bigcup_{i=1}^d (c_i' \cup d_i')$, which is transversal to $\boldsymbol{\alpha}^e$ and so

$$\begin{aligned}
\epsilon(\mathbf{y}, \mathbf{x}) &= h(\overline{\cup_{i=1}^d (c_i' \cup d_i')}) \\
&= h(\overline{\cup_{i=1}^d d_i'}) \\
&= \prod_{i=1}^{d} h(\overline{d_i'})
\end{aligned}$$

where we used $\overline{c_i'} = 1$ since the c_i''s are disjoint from $\boldsymbol{\alpha}^e$.

$\square$

We further need to know what happens with the map s when doing Heegaard moves. So let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), \mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ be two balanced Heegaard diagrams of (M, γ). Then we have two maps

$$s : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(M, \gamma) \quad \text{and} \quad s' : \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} \to \mathrm{Spin}^c(M, \gamma).$$

Suppose $\mathcal{H}'$ is obtained from $\mathcal{H}$ by one of the moves of Theorem 2.2.10. In the case of isotopy, we suppose that $\mathcal{H}'$ is obtained by an isotopy of an α or β curve that adds just two new intersection points. For all such moves there is an obvious map

$$j : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}.$$

In the case of isotopies (that increase the number of intersection points) or handlesliding, it is clear that $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ so we let j be the inclusion. For diffeomorphisms isotopic to the identity in M we just let j be the bijection between multipoints induced by the diffeomorphism. For stabilization, if $\alpha_{d+1}, \beta_{d+1}$ are the stabilized curves intersecting in a point $x_{d+1} \in \Sigma'$ then we let j be the bijection $\mathbf{x} \mapsto \mathbf{x} \cup \{x_{d+1}\}$.

Lemma 3.3.11. *If $\mathcal{H}'$ is obtained from $\mathcal{H}$ by a Heegaard move and if the map j is defined as above, then*

$$s'(j(\boldsymbol{x})) = s(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Proof. This is obvious for isotopies and diffeomorphisms isotopic to the identity in M. Suppose $\mathcal{H}'$ is obtained from $\mathcal{H}$ by stabilization. The union of the newly attached one-handle/two-handle pair is a 3-ball and it is clear that in the complement of this ball, the vector fields representing $s(\mathbf{x})$ and $s'(j(\mathbf{x}))$ coincide so $s(\mathbf{x}) = s'(j(\mathbf{x}))$ by definition. Now suppose $\mathcal{H}'$ is obtained from $\mathcal{H}$ by handlesliding a curve α_1 over α_2 and let $\alpha_1' = \alpha_1 \# \alpha_2$. Let U_α be the lower handlebody and let $D_1, D_2, D_1' \subset U_\alpha$ be the compressing disks corresponding to $\alpha_1, \alpha_2, \alpha_1'$ respectively. The complement of these three disks in U_α has two components: one contains the index zero critical point and the other is homeomorphic to a 3-ball B. Note that the boundary of B is the union of D_1, D_2, D_1' and the pair of pants bounded by $\alpha_1, \alpha_2, \alpha_1'$. One can pick Morse functions f, f' adapted to $\mathcal{H}, \mathcal{H}'$ respectively such that in the complement of B in M, the vector fields $-\nabla f$ and $-\nabla f'$ coincide. Moreover, the trajectories of f associated to $\mathbf{x}$ coincide with the trajectories of f' associated to $j(\mathbf{x})$, so the vector fields representing $s(\mathbf{x})$ and $s'(j(\mathbf{x}))$ coincide in the complement of $d+1$ 3-balls, which implies that $s(\mathbf{x}) = s'(j(\mathbf{x}))$. ☐

Proof of Theorem 3.3.6. We have to show that the scalar $r_H \circ \rho(h_{\mathfrak{s},\mathbf{x}}) Z_H^\rho(\mathcal{H}, \mathbf{x}, \omega)$ is independent of the multipoint $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, of the ordering and orientation of $\mathcal{H}$ and of extended Heegaard moves. Most of the proof is exactly as that of Subsection 3.2.5 with some extra care for the multipoint and Spin^c structure. We do the proof step by step.

1. Independence of the multipoint $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$: let $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be another multipoint and for each $i = 1, \dots, d$, let $d_i' \subset \beta_i$ be the arc from $q_i(\mathbf{x})$ to $q_i(\mathbf{y})$. Then

$$\begin{aligned} Z(\mathcal{H}, \mathbf{x}) &= r_H \left(\prod_{i=1}^{d} \rho(\overline{d_i'}) \right) Z(\mathcal{H}, \mathbf{y}) \\ &= r_H \circ \rho(\epsilon(\mathbf{y}, \mathbf{x})) Z(\mathcal{H}, \mathbf{y}). \end{aligned}$$

The first equality follows by using Proposition 3.2.9 over each β curve and the second follows from Lemma 3.3.10 above. Now, we have

$$\begin{aligned} h_{\mathfrak{s},\mathbf{x}} &= PD[s(\mathbf{x}) - \mathfrak{s}] \\ &= PD[s(\mathbf{x}) - s(\mathbf{y}) + s(\mathbf{y}) - \mathfrak{s}] \\ &= \epsilon(\mathbf{x}, \mathbf{y}) + h_{\mathfrak{s},\mathbf{y}} \end{aligned}$$

where we used Lemma 3.3.9 above in the third equality. Hence

$$\begin{aligned} r_H \circ \rho(h_{\mathfrak{s},\mathbf{x}}) Z_H^{\rho}(\mathcal{H}, \mathbf{x}) &= r_H \circ \rho(h_{\mathfrak{s},\mathbf{y}}) r_H \circ \rho(\epsilon(\mathbf{x}, \mathbf{y}))) Z_H^{\rho}(\mathcal{H}, \mathbf{x}) \\ &= r_H \circ \rho(h_{\mathfrak{s},\mathbf{y}}) Z_H^{\rho}(\mathcal{H}, \mathbf{y}) \end{aligned}$$

as was to be shown.

2. Independence of ordering and orientations: independence of ordering follows as before. Now let $\mathcal{H}'$ be the ordered, oriented, based Heegaard diagram obtained from $\mathcal{H}$ only by reversing the orientation of α_i. Let $\overline{\mathcal{H}}$ be the ordered, oriented, based Heegaard diagram obtained from $\mathcal{H}$ by reversing the orientation of α_i and with the basepoints coming from $\mathbf{x}$. Then $\overline{\mathcal{H}}$ differs from $\mathcal{H}'$ only on the basepoint over β_i, so

$$Z_H^{\rho}(\overline{\mathcal{H}}, \mathbf{x}) = r_H(\rho(\alpha_i^*))^{-1} Z_H^{\rho}(\mathcal{H}')$$

by Proposition 3.2.9. By Lemma 3.2.10 we thus get

$$\begin{aligned} Z_H^{\rho}(\overline{\mathcal{H}}, \mathbf{x}) &= r_H(\rho(\alpha_i^*))^{-1} r_H(\rho(\alpha_i^*)) Z_H^{\rho}(\mathcal{H}, \mathbf{x}) \\ &= Z_H^{\rho}(\mathcal{H}, \mathbf{x}) \end{aligned}$$

as we wanted.

3. Independence of extended Heegaard moves: the proof is as before, but we have to be careful on the multipoints and Spin^c structures. Recall that if $\mathcal{H}'$ is obtained from $\mathcal{H}$ by a Heegaard move, then there is a map $j : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. The proof of Subsection 3.2.5 extends to show that $Z(\mathcal{H}, \mathbf{x}) = Z(\mathcal{H}', j(\mathbf{x}))$ and so we only need to show that $h_{\mathfrak{s},\mathbf{x}} = h_{\mathfrak{s},j(\mathbf{x})}$. But this follows from Lemma 3.3.11 above:

$$h_{\mathfrak{s},\mathbf{x}} = PD[s(\mathbf{x}) - \mathfrak{s}] = PD[s'(j(\mathbf{x})) - \mathfrak{s}] = h_{\mathfrak{s},j(\mathbf{x})}.$$

□

3.3.5 The disconnected case with Spin^c

We extend the considerations of Subsection 3.2.6 to the case of arbitrary ρ.

Let (M', γ') be a sutured 3-manifold obtained from (M, γ) by attaching one-handles along γ. Since the newly attached one-handles belong to a neighborhood of R', given $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ there is a unique way to extend $\mathfrak{s}$ to a relative Spin^c structure on M'. We denote it by $i(\mathfrak{s})$. The map $i : \mathrm{Spin}^c(M, \gamma) \to \mathrm{Spin}^c(M', \gamma')$ thus obtained is an affine map: if $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathrm{Spin}^c(M, \gamma)$, then

$$PD[i(\mathfrak{s}_1) - i(\mathfrak{s}_2)] = i_*(PD[\mathfrak{s}_1 - \mathfrak{s}_2])$$

where $i_* : H_1(M) \to H_1(M')$ is induced by inclusion. If $\mathcal{H}$ is a Heegaard diagram of (M, γ) and $\mathcal{H}'$ is the diagram of (M', γ') obtained from $\mathcal{H}$ by attaching handles to $\partial\Sigma$, then there is an obvious bijection $j : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ which clearly satisfies $i(s(\mathbf{x})) = s'(j(\mathbf{x}))$ for any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ (see [FJR11, Section 3.6]). Therefore, one has

$$\begin{aligned} h_{i(\mathfrak{s}),j(\mathbf{x})} &= PD[i(\mathfrak{s}) - s'(j(\mathbf{x}))] \\ &= PD[i(\mathfrak{s}) - i(s(\mathbf{x}))] \\ &= i_* PD[\mathfrak{s} - s(\mathbf{x}))] \\ &= i_*(h_{\mathfrak{s},\mathbf{x}}). \end{aligned}$$

Lemma 3.2.13 shows that $Z_H^\rho(\mathcal{H}, \mathbf{x}) = Z_H^{\rho'}(\mathcal{H}', j(\mathbf{x}))$. Thus, if $R = R_-(\gamma)$ is connected, it follows that

$$Z_H^\rho(M, \gamma, \mathfrak{s}) = Z_H^{\rho'}(M', \gamma', i(\mathfrak{s}))$$

for any $\rho' : \pi_1(M', p) \to \mathrm{Aut}(H)$ such that $\rho' \circ i_* = \rho$, where $i_* : \pi_1(M, p) \to \pi_1(M', p)$ is induced by inclusion. Thus, the argument of Proposition 3.2.14 applies, and we can make the following definition.

Definition 3.3.12. Let (M, γ) be a balanced sutured 3-manifold with possibly disconnected subsurface $R = R_-(\gamma)$ and let $\rho : \pi_1(M, p) \to \mathrm{Aut}(H)$, $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ and ω an orientation of $H_*(M, R; \mathbb{R})$. Let (M', γ') be an arbitrary sutured 3-manifold obtained by adding one-handles to M along γ and let ρ' be an extension of ρ to $\pi_1(M', p)$. Then we define

$$Z_H^\rho(M, \gamma, \mathfrak{s}, \omega) \coloneqq Z_H^{\rho'}(M', \gamma', i(\mathfrak{s}), \omega')$$

where $i : \mathrm{Spin}^c(M, \gamma) \to \mathrm{Spin}^c(M', \gamma')$ is the map defined above. Here ω' is the orientation of $H_*(M', R'; \mathbb{R})$ induced from ω by the isomorphism $H_*(M, R; \mathbb{R}) \cong H_*(M', R'; \mathbb{R})$.

3.4 Twisted Kuperberg polynomials

The Kuperberg invariants for sutured manifolds we defined are scalars in the base field $\mathbb{K}$. We explain a notable feature of our construction: provided H is $\mathbb{N}$-graded, Kuperberg invariants can be upgraded to polynomials.

So suppose that H is a $\mathbb{N}$-*graded* Hopf superalgebra. This means that H has a vector space decomposition $H = \oplus_{n\in\mathbb{N}} H_n$ such that $H_i \cdot H_j \subset H_{i+j}$ and $\Delta(H_n) \subset \sum_{i+j=n} H_i \otimes H_j$. It is required that Δ is a morphism of $\mathbb{N}$-graded algebras, where $H \otimes H$ is $\mathbb{N}$-graded by $(H \otimes H)_n = \oplus_{i+j=n} H_i \otimes H_j$ and the algebra structure is defined by

$$(x \otimes y)(x' \otimes y') = (-1)^{|x'||y|} xx' \otimes yy'$$

where $|\cdot|$ denotes the $\mathbb{N}$-degree of an homogeneous element of H. Thus, H can be considered as a Hopf superalgebra by letting $H_0 = \oplus_{n\geq 0} H_{2n}$ and $H_1 = \oplus_{n\geq 0} H_{2n+1}$. If the product in $H \otimes H$ is defined as above but without the sign, we call H just an $\mathbb{N}$-graded Hopf algebra. In this sense, an $\mathbb{N}$-graded Hopf algebra can be seen as an $\mathbb{N}$-graded superalgebra concentrated in even degree.

Let $\mathrm{Aut}_{gr}(H)$ be the group of Hopf algebra automorphisms of H that preserve the degree. Given a balanced sutured 3-manifold (M, γ), we define

$$H_M \coloneqq H \otimes_{\mathbb{K}} \mathbb{K}[H_1(M)].$$

This is an $\mathbb{N}$-graded $\mathbb{K}[H_1(M)]$-linear Hopf (super)algebra in an obvious way. Now, given an element $\alpha \in \mathrm{Aut}_{gr}(H)$ and $h \in H_1(M)$ we can define a $\mathbb{N}$-graded $\mathbb{K}[H_1(M)]$-linear Hopf automorphism $\alpha \otimes h$ of H_M by

$$\alpha \otimes h(x \otimes f) \coloneqq \alpha(x) \otimes (h^{|x|} \cdot f)$$

where $x \in H$ is homogeneous and $f \in H_1(M)$. Therefore, a group homomorphism $\rho : \pi_1(M, p) \to \mathrm{Aut}_{gr}(H)$ can be combined with the projection $h : \pi_1(M) \to H_1(M; \mathbb{Z})$ to define a representation

$$\begin{aligned} \rho \otimes h : \pi_1(M, p) &\to \mathrm{Aut}_{gr}(H_M) \\ \delta &\mapsto \rho(\delta) \otimes h(\delta). \end{aligned}$$

Note that though $\mathbb{K}[H_1(M)]$ is only a ring, H_M has a two-sided cointegral and integral induced from that of H. More precisely, if $j_H : H \to H_M$ is the inclusion map $x \mapsto x \otimes 1$, then the cointegral of H_M is $j_H(c)$ while the integral is $\mu_{H_M} \coloneqq \mu \otimes \mathrm{id}_{\mathbb{K}[H_1(M)]} : H_M \to \mathbb{K}[H_1(M)]$, where c and μ are the two-sided cointegral and integral of H respectively. Thus, we can apply our construction to the $\mathbb{K}[H_1(M)]$-linear Hopf superalgebra H_M, and this gives a topological invariant

$$Z^{\rho\otimes h}_{H_M}(M, \gamma, \mathfrak{s}, \omega) \in \mathbb{K}[H_1(M)]$$

is one is given $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ and an orientation ω of $H_*(M, R_-(\gamma); \mathbb{R})$

Definition 3.4.1. If $H_M = H \otimes_{\mathbb{K}} \mathbb{K}[H_1(M)]$, we call $Z^{\rho\otimes h}_{H_M}(M,\gamma,\mathfrak{s},\omega) \in \mathbb{K}[H_1(M)]$ the *twisted Kuperberg polynomial* of (M,γ) with respect to ρ. We also refer to $Z^{1\otimes h}_{H_M}$ as the (untwisted) Kuperberg polynomial, which we denote simply by $Z_{H_M}(M,\gamma,\mathfrak{s},\omega) \in \mathbb{K}[H_1(M)]$.

The twisted Kuperberg polynomials specializes to our previous construction as follows. Recall that the augmentation map $\mathrm{aug} : \mathbb{K}[H_1(M)] \to \mathbb{K}$ is the $\mathbb{K}$-linear map defined by $\mathrm{aug}\,(f) = 1$ for all $f \in H_1(M)$.

Proposition 3.4.2. *If $aug\colon \mathbb{K}[H_1(M)] \to \mathbb{K}$ is the augmentation map, then the twisted Kuperberg polynomial satisfies*

$$aug\,(Z^{\rho\otimes h}_{H_M}(M,\gamma,\mathfrak{s},\omega)) = Z^{\rho}_{H}(M,\gamma,\mathfrak{s},\omega).$$

Proof. Set $\mathrm{aug}_H \coloneqq \mathrm{id}_H \otimes \mathrm{aug} : H_M \to H$. It is easy to see that $\mathrm{aug} \circ \mu_{H_M} = \mu \circ \mathrm{aug}_H$ and $\mathrm{aug}_H \circ (\rho \otimes h(\delta)) \circ j_H = \rho(\delta)$ for any $\delta \in \pi_1(M)$, where $j_H : H \to H_M, x \mapsto x \otimes 1$ is the inclusion as above. Since the cointegral of H_M is $j_H(c)$, where c is the cointegral of H, it follows that $\mathrm{aug}\,(Z^{\rho\otimes h}_{H_M}(\mathcal{H})) = Z^{\rho}_{H}(\mathcal{H})$ where $\mathcal{H}$ is a Heegaard diagram of (M,γ). From this the result follows. ∎

Example 3.4.3. Consider the left trefoil complement as in Example 3.2.4 and let $\rho \equiv 1$. Let t be a generator of $H_1(M) \cong \mathbb{Z}$. Then, for the diagram of Figure 3.1, the twisted Kuperberg invariant is given by

$$\begin{aligned} Z^{1\otimes h}_{H}(\mathcal{H}) &= \mu(t^{|c_{(1)}|} c_{(1)} \cdot S(c_{(2)}) \cdot t^{-|c_{(3)}|} c_{(3)}) \\ &= t^{|c_{(1)}| - |c_{(3)}|} \mu(c_{(1)} S(c_{(2)}) c_{(3)}). \end{aligned}$$

As usual, it is understood that this is a sum running through all the terms of $\Delta^{(3)}(c)$. When H is set to be an exterior algebra on one generator of degree one, this formula reduces to the Alexander polynomial of the trefoil knot.

Lemma 3.4.4. *The homomorphism $r_{H_M} : Aut(H_M) \to \mathbb{K}[H_1(M)]^{\times}$ induced from the cointegral of H_M satisfies*

$$r_{H_M}(\alpha \otimes f) = r_H(\alpha) f^{|c|}$$

for any $\alpha \in Aut(H)$ and $f \in H_1(M)$.

Proof. Indeed, the cointegral of $H_M = H \otimes_{\mathbb{K}} \mathbb{K}[H_1(M)]$ is $c \otimes 1$ so that

$$\begin{aligned} \alpha \otimes f(c \otimes 1) &= \alpha(c) \otimes f^{|c|} \\ &= r_H(\alpha) c \otimes f^{|c|} \\ &= r_H(\alpha) f^{|c|} (c \otimes 1) \end{aligned}$$

and so $r_{H_M}(\alpha \otimes f) = r_H(\alpha) f^{|c|}$ by definition of r_{H_M}. ∎

Changing the Spin^c structure has the following effect in the twisted Kuperberg polynomial:

$$Z_{H_M}^{\rho\otimes h}(M,\gamma,\mathfrak{s}+h,\omega) = r_{H_M}\circ(\rho\otimes h)(PD[h])^{-1}Z_{H_M}^{\rho\otimes h}(M,\gamma,\mathfrak{s},\omega).$$

From the above lemma, we find

$$Z_{H_M}^{\rho\otimes h}(M,\gamma,\mathfrak{s}+h,\omega) = r_H\circ\rho(PD[h])^{-1}PD[h]^{-|c|}Z_{H_M}^{\rho\otimes h}(M,\gamma,\mathfrak{s},\omega).$$

Thus if $\mathrm{Im}\,(\rho)\subset \mathrm{Ker}\,(r_H)$, $Z_{H_M}^{\rho\otimes h}$ depends on the Spin^c structure only up to multiplication by $f^{|c|}$, where $f\in H_1(M)$. Hence, we can drop the Spin^c structure and the homology orientation only up to a $\pm H_1(M)$ indeterminacy, and denote

$$Z_{H_M}^{\rho\otimes h}(M,\gamma)\in \mathbb{K}[H_1(M)]/\pm H_1(M).$$

If the cointegral of H has degree zero, we can further drop the sign indeterminacy.

Chapter 4

Recovering Reidemeister torsion

This chapter is devoted to the proof of Theorem 2 and deducing some corollaries. We start by recalling the basics of the theory of Reidemeister torsion in Section 4.1. Then in Section 4.2 we specialize the torsion to sutured 3-manifolds. The proof of Theorem 2 is devoted to Section 4.3.

4.1 Basics of Reidemeister torsion

In this section we recall the definition of the torsion function, the torsion of CW complexes and 3-manifolds, and its relation with (twisted) Alexander polynomials. The material presented here is standard, cf. [Tur01, FV11a].

4.1.1 Algebraic torsion

Let R be a commutative domain with unit. Consider a finitely generated free R-module V. Given two R-basis b and c of V, then we denote by $[b/c] \in R^\times$ the determinant of the change of base matrix from b to c, where $R^\times$ is the group of units of R. Two bases of V are said to be *equivalent* if $[b/c] = 1$. Consider an exact sequence

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

of R-modules C, D, E. Given two bases c, e of C, E, one can lift e to $\widetilde{e} \subset D$ and then $ce := (c, \widetilde{e})$ is a basis of D whose equivalence class is independent of the lifting (indeed, one has $[ce/c'e'] = [c/c'][e/e']$). Now consider an *acyclic* chain complex

$$C : 0 \to C_m \to \cdots \to C_0 \to 0$$

with chosen bases c_i of the C_i. Thus, for each i there is a short exact sequence

$$0 \to B_i \to C_i \to B_{i-1} \to 0$$

where B_i is the image of $\partial : C_{i+1} \to C_i$. For each i, let b_i be a basis of B_i (image at C_i), then $b_i b_{i-1}$ is a basis of C_i.

Definition 4.1.1. The *torsion* of an acyclic based chain complex (C, c) is

$$\tau(C, c) := \prod_{i=0}^{m} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in Q(R)$$

where $Q(R)$ is the quotient field of R.

It is easy to prove that this is well-defined, i.e., independent of the choice of each basis b_i of B_i.

4.1.2 Twisted Reidemeister torsion

We now define the (twisted) Reidemeister torsion of CW complexes and of 3-manifolds. In all that follows, we let X be a finite connected CW complex and Y a (possibly empty) subcomplex such that $\chi(X, Y) = 0$. Let $p : \widetilde{X} \to X$ be the universal covering space of X and $Y' := p^{-1}(Y)$. Let $x_0 \in Y$ be a basepoint (just $x_0 \in X$ if Y is empty) and let $\pi = \pi_1(X, x_0)$.

Consider the cellular chain complex $C_*(\widetilde{X}, Y')$. This becomes a left $\mathbb{Z}[\pi]$-module if we let π act by Deck transformations and it is a free $\mathbb{Z}[\pi]$-module with basis in bijection with the cells of $X \setminus Y$. We will rather consider $C_*(\widetilde{X}, Y')$ as a right $\mathbb{Z}[\pi]$-module by letting $c \cdot g := g^{-1}c$ where $g \in \pi$ and c is a cell of $\widetilde{X}$ (this is the same convention as in [FV11a, DFJ12]).

Now let $\rho : \pi \to GL(V)$ be a representation, where V is a finitely generated free module over some commutative domain with unit R. Then V becomes a $\mathbb{Z}[\pi]$-module on the left via ρ. We thus get a complex of R-modules defined by

$$C_*^{\rho}(X, Y) := C_*(\widetilde{X}, Y') \otimes_{\mathbb{Z}[\pi]} V.$$

We denote the homology of this complex by $H_*^{\rho}(X, Y)$. Suppose the cells of $X \setminus Y$ are ordered and oriented and let e be a choice of lifts to $\widetilde{X}$ of the cells of $X \setminus Y$. Then e defines an (ordered, oriented) $\mathbb{Z}[\pi]$-basis of $C_*(\widetilde{X}, Y')$ and tensoring with an R-basis of V we get a basis $\overline{e}$ of $C_*^{\rho}(X, Y)$. More precisely, if $e = (e_1, \ldots, e_k)$ and $(v_1, \ldots, v_n)$ is an ordered basis of V, then $\overline{e}$ is ordered by $\overline{e} = (e_1 \otimes v_1, \ldots, e_1 \otimes v_n, \ldots, e_k \otimes v_1, \ldots, e_k \otimes v_n)$.

Definition 4.1.2. Let e be a choice of lifts to $\widetilde{X}$ of the cells of $X \setminus Y$. The *Reidemeister torsion* $\tau^{\rho}(X, Y, e)$ is defined as the torsion of the based complex $(C_*^{\rho}(X, Y), \overline{e})$, that is,

$$\tau^{\rho}(X, Y, e) := \tau(C_*(\widetilde{X}, Y') \otimes_{\mathbb{Z}[\pi]} V, \overline{e}) \in Q(R)$$

where the $\mathbb{Z}[\pi]$-module structure of V is defined via ρ. As before, $Q(R)$ denotes the fraction field of R. The torsion $\tau^{\rho}(X, \emptyset, e)$ is denoted just by $\tau^{\rho}(X, e)$.

Changing the basis of V multiplies $\tau^\rho(X, Y, e)$ by $\det(A)^{\chi(X,Y)}$ where $A \in M_{n\times n}(R)$ is the change of basis matrix. Since $\chi(X, Y) = 0$, the torsion only depends on the choice of lifts e. Changing the choice e of lifts of the cells of $X \setminus Y$ or changing the order and orientation of the cells of X multiplies $\tau^\rho(X, Y, e)$ by $\pm\det(\rho(g))$ for some $g \in \pi$.

Now suppose we have a representation $\rho : \pi \to GL(V)$, where V is a finite dimensional vector space over a field $\mathbb{K}$. Let $h : \pi \to F_X$ be the projection onto the free abelian group $F_X := H_1(X)/\operatorname{Tors} H_1(X)$. Note that $\mathbb{K}[F_X]$ is a domain. Then we get a tensor product representation

$$\rho \otimes h : \pi \to GL(V \otimes_{\mathbb{K}} \mathbb{K}[F_X])$$

where for any $\gamma \in \pi$, $\rho \otimes h(\gamma)$ is defined by

$$v \otimes f \mapsto \rho(\gamma)(v) \otimes (h(\gamma) \cdot f)$$

for $v \in V, f \in F_X$.

Definition 4.1.3. Given a representation $\rho : \pi_1(X, x_0) \to GL(V)$ where V is a finite dimensional vector space over a field $\mathbb{K}$, the *twisted Reidemeister torsion* of (X, Y) is the Reidemeister torsion $\tau^{\rho\otimes h}(X, Y, e) \in \mathbb{K}(F_X)$.

By the remarks above, changing the choice of lifts e multiplies the twisted torsion by $\pm\det(\rho(g))f$ for some $g \in \pi$ and $f \in F_X$. We will denote by $\tau^{\rho\otimes h}(X, Y)$ the torsion up to this ambiguity. Note that if ρ takes values in $SL(n, \mathbb{K})$, the indeterminacy is only an element of $\pm F$. This ambiguity can be fixed by picking a *combinatorial Euler structure* of $X \setminus Y$ (cf. [Tur01]).

Now, let M be a compact 3-manifold and $S \subset M$ a compact embedded submanifold such that $\chi(M, S) = 0$. Then M admits a triangulation $\mathcal{T}$ such that $\mathcal{T} \cap S$ is a triangulation of S. This induces a CW structure X on M and a CW structure Y on S such that Y is a subcomplex of X. Given a representation $\rho : \pi_1(M) \to GL(V)$, where V is a finite dimensional vector space over a field $\mathbb{K}$, we get a torsion $\tau^{\rho\otimes h}(X, Y) \in \mathbb{K}(F_M)$ which is defined up to an indeterminacy of the form $\pm\lambda f$ with $\lambda \in \mathbb{K}^\times, f \in F_M$. Since the relative torsion is invariant under cellular subdivision, this torsion depends only on (M, S), cf. [Tur01, Section 14] for more details.

Definition 4.1.4. The twisted Reidemeister torsion of (M, S) is the twisted torsion of any CW pair (X, Y) obtained from a triangulation of M as above.

4.1.3 Twisted Alexander polynomials

Let R be a commutative domain with unit and M a finitely presented R-module. Let $p : R^m \to R^n$ be a presentation of M, so $\operatorname{Coker}(p) \cong M$. If A is the $m \times n$-matrix representing p in the canonical bases of R^m and R^n, then the ideal of R generated by the $(n-k)$-minors of A depends only on M and not on the presentation matrix A chosen. This is the *k-th elementary ideal* of M, denoted $E_k(M)$. If R is a unique factorization domain (UFD), then $\Delta_0(M) \coloneqq \gcd E_0(M)$ is called the *order* of M. Note that this is defined up to multiplication by a unit of R. For more details see [Tur01, Chapter I].

Definition 4.1.5. Let X be a finite CW complex, $\rho : \pi \to GL(V)$ be a representation into a finite dimensional $\mathbb{K}$-vector space V and $h : \pi \to F_X = H_1(X)/\operatorname{Tors} H_1(X)$ be the projection, where $\pi = \pi_1(X, x_0)$ for some $x_0 \in X$. Note that $\mathbb{K}[F_X]$ is a polynomial ring, hence a UFD. We define the *i-th twisted Alexander polynomial* of X, denoted $\Delta^{\rho\otimes h}_{X,i}$, as the order of the $\mathbb{K}[F_X]$-module $H^{\rho\otimes h}_i(X)$. If X is the complement of a link L in a closed 3-manifold Y, then we call $\Delta^{\rho\otimes h}_{X,1}$ the *twisted Alexander polynomial* of L and we denote it by $\Delta^{\rho\otimes h}_L$.

The twisted torsion is related to the twisted Alexander polynomials by the following formula, cf. [Tur01, Theorem 4.7]: if $\Delta^{\rho\otimes h}_{X,i} \neq 0$ for all i then

$$\tau^{\rho\otimes h}(X) \doteq \prod_{i\geq 0} (\Delta^{\rho\otimes h}_{X,i})^{(-1)^{i+1}} \in \mathbb{K}(F_X) \tag{4.1}$$

where $\doteq$ means equality up to multiplication by a unit of $\mathbb{K}[F_X]$.

4.2 Twisted torsion of sutured manifolds

We now specialize the (twisted) torsion to balanced sutured 3-manifolds (M, γ). Note that since $\chi(M, R_-(\gamma)) = 0$, the torsion of $(M, R_-(\gamma))$ is defined (it may be zero, but that would be non-trivial). We begin by giving a handy formula to compute twisted torsion from a sutured Heegaard diagram. In Subsection 4.2.2 we relate the relative torsion to twisted Alexander polynomials in the case of link complements.

4.2.1 Twisted torsion from a Heegaard diagram

Let (M, γ) be a balanced sutured 3-manifold with *connected* $R_- = R_-(\gamma)$. Let $p \in s(\gamma)$ be a basepoint and $\rho : \pi_1(M, p) \to GL(V)$ be a representation, where V is a finite dimensional vector space over a field $\mathbb{K}$. For simplicity, we will denote $\pi = \pi_1(M, p)$, $F_M = H_1(M)/\operatorname{Tors} H_1(M)$ and $h : \pi \to F_M$ the projection.

Now, let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an extended Heegaard diagram of M which is ordered, oriented and based. We write $\boldsymbol{\alpha}^e = \boldsymbol{\alpha} \cup \boldsymbol{a}$ with $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_d\}$ and $\boldsymbol{a} = \{\alpha_{d+1}, \dots, \alpha_{d+l}\}$. We suppose the arcs in $\boldsymbol{a}$ are oriented. Thus, we have a well-defined element $\alpha^* \in \pi$ for every $\alpha \in \boldsymbol{\alpha}^e$ and since the curves in $\boldsymbol{\beta}$ are oriented and have basepoints, we have a presentation of π of the form

$$\pi_1(M, p) = \langle \alpha^*_1, \dots, \alpha^*_{d+l} \mid \overline{\beta}_1, \dots \overline{\beta}_d \rangle \tag{4.2}$$

cf. Subsection 2.3.3. Now, Fox calculus gives elements

$$\frac{\partial \overline{\beta}_j}{\partial \alpha_i^*} \in \mathbb{Z}[\pi]$$

for all i, j, cf. [Tur01]. Let A be the $d \times d$-matrix with $\mathbb{Z}[\pi]$ coefficients whose (i, j)-entry is $\partial\overline{\beta}_j/\partial\alpha_i^*$ for $i, j = 1, \ldots, d$, that is, we only derivate with respect to the closed curves. Let $\sigma : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ be the $\mathbb{Z}$-linear map characterized by $\sigma(g) = g^{-1}$ for each $g \in \pi$ and let $\sigma(A)$ be the matrix obtained by applying this map to each entry of A.

Proposition 4.2.1. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an extended Heegaard diagram of (M, γ) which is ordered, oriented and based and let $A = (\sigma(\partial\overline{\beta}_j/\partial\alpha_i^*))$ as above. The twisted torsion of the pair (M, R_-) at $\rho : \pi_1(M) \to GL(V)$ is computed via Fox calculus by*

$$\tau^{\rho\otimes h}(M, R_-(\gamma)) \doteq \det((\rho \otimes h)(\sigma(A))) \in \mathbb{K}[F_M]$$

where $\doteq$ denotes equality up to a unit in $\mathbb{K}[F_M]$.

Proof. The extended Heegaard diagram specifies a presentation of $\pi_1(M)$ as in (4.2) from which we build a CW complex X as follows: X has a single 0-cell, $(d + l)$ 1-cells corresponding to the generators of the given presentation, and d 2-cells e_j^2 corresponding to the relations. The boundary of each cell e_j^2 is attached to $X^{(1)}$ along the path determined by the word $\overline{\beta}_j$. We let Y be the subcomplex of $X^{(1)}$ determined by the l 1-cells associated to the arcs in $\boldsymbol{a}$. It is easy to see that the pair (M, R_-) is simple homotopy equivalent to (X, Y) and hence

$$\tau^{\rho\otimes h}(M, R_-) = \tau^{\rho\otimes h}(X, Y).$$

Indeed, the pair (X, Y) can be obtained from (M, R_-) by collapsing each handle of M (specified by $\mathcal{H}$) to its core, and this operation preserves the relative torsion by [Tur01, Corollary 8.5]. We now show that $\tau^{\rho\otimes h}(X, Y)$ is given by the above Fox calculus matrix. Thus, let $p : \widetilde{X} \to X$ be the universal covering space of X and let $Y' = p^{-1}(Y)$. The cellular complex $C_* = C_*(\widetilde{X}, Y)$ is thus a complex of free $\mathbb{Z}[\pi_1(X)]$-modules with $C_i = 0$ for $i \neq 1, 2$ and $C_2 \oplus C_1$ has a $\mathbb{Z}[\pi_1(X)]$-basis corresponding to lifts of the cells of X associated to the closed curves of the diagram. Under an appropiate choice of lifts, the boundary map $\partial_2 : C_2 \to C_1$ is represented by the matrix $A \in M_{d\times d}(\mathbb{Z}[\pi])$, cf. [Tur01, Claim 16.6]. Since our convention is that $(g \cdot c) \otimes v = c \otimes [(\rho \otimes h)(\sigma(g)) \cdot v]$ in $C_*^{\rho\otimes h}(X, Y)$, where $g \in \pi, v \in V$ and c is a cell of $\widetilde{X}$, the boundary operator $\partial_2^{\rho\otimes h}$ is represented by the matrix $(\rho \otimes h)(\sigma(A))$. Thus, the torsion of the complex $C_*^{\rho\otimes h}(X, Y)$, and hence the torsion of (M, R_-), is the determinant of this matrix. $\square$

Remark 4.2.2. The right hand side of the above proposition is defined as an element of $\mathbb{K}[H_1(M)]$. Therefore, we will consider the twisted torsion of a balanced sutured

3-manifold as an element of $\mathbb{K}[H_1(M)]$. Note however that if we want to express the torsion in terms of twisted Alexander polynomials (as in Corollary 4.2.4 below), then one has to pass to $\mathbb{K}[F_M]$, which is a Noetherian UFD.

Recall that, if Y is a closed 3-manifold, we get a sutured 3-manifold by $M = Y \setminus \mathrm{int}\,(B)$, $\gamma = S^1 \subset \partial B$ where B is a closed 3-ball embedded in Y (Example 2.1.3). However, as a consequence of the above proposition, the torsion of (M, R_-) is not very interesting.

Corollary 4.2.3. *Let $M = Y \setminus B$ be the sutured manifold associated to a closed 3-manifold Y and let $\rho : \pi_1(M) \to GL(V)$ be a group homomorphism. Then*

$$\tau^\rho(M, R_-) = \begin{cases} 0 & \text{if } \rho \not\equiv 1 \\ \pm|H_1(Y;\mathbb{Z})| & \text{if } \rho \equiv 1. \end{cases}$$

Here $|H_1(Y;\mathbb{Z})|$ is defined to be zero if $b_1(Y) > 0$.

Proof. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a (sutured) Heegaard diagram of M, equivalently, $\mathcal{H}$ is obtained from a Heegaard diagram of Y by removing a disk from the Heegaard surface. This specifies a cell decomposition X of M with one 0-cell and an equal number $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ of 1-cells and 2-cells (see Remark 2.2.2). Let $\partial_i^\rho : C_i^\rho(X) \to C_{i-1}^\rho(X)$ be the boundary operator of the complex $C_*^\rho(X) \coloneqq C_*(\widetilde{X}) \otimes_{Z[\pi]} V$ where $\widetilde{X}$ is the universal cover of X and V is considered as a $\mathbb{Z}[\pi]$-module via ρ. For an appropriate choice of basis in $C_*^\rho(X)$, the boundary operator ∂_1^ρ is represented by $(\rho(\alpha_1^*) - 1, \dots, \rho(\alpha_d^*) - 1)$ and ∂_2^ρ is represented by $A = (\rho(\partial\overline{\beta}_j/\partial\alpha_i^*))$. Suppose first that $\rho \equiv 1$, so $C_*^{\rho\equiv 1}(M) = C_*(M)$ and let $\partial_i \coloneqq \partial_i^{\rho\equiv 1}$. Then $\partial_1 = 0$, and hence $H_1(M) = \mathrm{Coker}(\partial_2)$. It follows that

$$\pm|H_1(Y)| = \pm|H_1(M)| = |\mathrm{Coker}(\partial_2)| = \det(A) = \tau^{\rho\equiv 1}(M, R_-)$$

where the last equality follows from Proposition 4.2.1. Note that $b_1(Y) > 0$ if and only if $\det(A) = 0$, hence $\tau^{\rho\equiv 1}(M, R_-) = 0$. Now, if $\rho \not\equiv 1$, then $\partial_1^\rho \neq 0$ and from $\partial_1^\rho \circ \partial_2^\rho = 0$ it follows that ∂_2^ρ is non-surjective. Therefore $\det(A) = 0$ and hence $\tau^\rho(M, R_-) = \det(A) = 0$. □

4.2.2 Twisted torsion for link complements

Now let L be an ordered oriented link in S^3, with components $L_1, \dots, L_m$. Let (M, γ) be the associated sutured manifold (see Example 2.1.4), that is, $M = S^3 \setminus N(L)$, where $N(L)$ is a tubular neighborhood of L and γ consists of a pair of annuli, one pair for each component of ∂M. The ordering and orientation of L induce an isomorphism $\mathbb{Z}[H_1(M)] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ where t_i corresponds to the (positively oriented) meridian of L_i. Let $\rho : \pi_1(M, p) \to GL(V)$ be an homomorphism, where $p \in s(\gamma)$.

Recall that $R = R_-(\gamma)$ consists of one annuli $R_i \subset N(L_i)$ for each $i = 1, \ldots, m$. For each i, let $y_i \in R_i$ be a basepoint and a_i^* be a generator of $\pi_1(R_i, y_i)$. This has an orientation induced from that of L_i and its image in $H_1(M)$ is t_i. Note that a_i^* is defined in $\pi_1(M, p)$ up to conjugation, so $\det(t_i\rho(a_i^*) - I_n)$ is well-defined (here $n = \dim(V)$).

Corollary 4.2.4. *Let M be the sutured manifold associated to the complement of an m-component oriented link $L \subset S^3$. If $m > 1$ then*

$$\tau^{\rho\otimes h}(M, R_-) \doteq \prod_{i=1}^{m} \det(t_i\rho(a_i^*) - I_n) \cdot \Delta_L^{\rho\otimes h} \in \mathbb{K}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}].$$

The above holds for $m = 1$ provided ρ is irreducible and non-trivial over Ker (h). If $m = 1$ and $\rho \equiv 1$ one has $\tau^h(M, R_-) = \Delta_L$, the multivariable Alexander polynomial of (the knot) L.

Proof. Indeed, since $\chi(R_-) = 0$ we have $\tau^{\rho\otimes h}(M, R_-) = \tau^{\rho\otimes h}(M)\tau^{\rho\otimes h}(R_-)^{-1}$. From the well-known fact that $\tau^{\rho\otimes h}(R_-)^{-1} = \prod_{i=1}^{m} \det(t_i\rho(a_i^*) - I_n)$ together with Formula (4.1) relating torsion to twisted Alexander polynomials we find

$$\tau^{\rho\otimes h}(M, R_-) \doteq \prod_{i=1}^{m} \det(t_i\rho(a_i^*) - I_n) \cdot \frac{\Delta_{L,1}^{\rho\otimes h} \Delta_{L,3}^{\rho\otimes h}}{\Delta_{L,0}^{\rho\otimes h} \Delta_{L,2}^{\rho\otimes h}}.$$

Now we use that $\Delta_{L,3}^{\rho\otimes h} = 1$ always holds, that $\Delta_{L,2}^{\rho\otimes h} = 1$ holds provided $\Delta_{L,1}^{\rho\otimes h} \neq 0$ and that $\Delta_{L,0}^{\rho\otimes h} = 1$ if $m > 1$ or $m = 1$ and ρ is irreducible and non-trivial over Ker (h), see [FV11a, Proposition 3.2]. If $m = 1$ and ρ is trivial over Ker (h), then $\Delta_{L,0}^{\rho\otimes h} = \det(t_i\rho(\alpha_i^*) - I_n)$ and we obtain the second assertion. This completes the proof.

□

4.3 Reidemeister torsion from Hopf algebra theory

In this section we prove Theorem 2. We begin with some easy lemmas on exterior algebras in Subsection 4.3.1. In Subsection 4.3.2 we deduce Theorem 2 from a much stronger statement, namely, that the twisted Kuperberg tensor at an exterior algebra is essentially equivalent to the boundary map $\partial_2^\rho : C_2^\rho(M, R_-) \to C_1(M, R_-)$ after applying the exterior algebra functor. Some corollaries of Theorem 2 are stated in Subsection 4.3.3.

4.3.1 Lemmas on exterior algebras

Recall that for any finite dimensional vector space the exterior algebra $\Lambda(V)$ is the quotient of the tensor algebra $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ by the ideal spanned by the $v \otimes v$

with $v \in V$. It becomes a superalgebra by letting V be in degree one and a Hopf superalgebra if one sets $\Delta(v) = v \otimes 1 + 1 \otimes v$ for any $v \in V$. The exterior algebra construction is *functorial*: if $T : V \to W$ is a linear map between vector spaces V, W, then there is an induced Hopf morphism $\Lambda(T) : \Lambda(V) \to \Lambda(W)$. This is defined to be T over $V \subset \Lambda(V)$ and is extended to all of $\Lambda(V)$ by letting it be an algebra morphism. It is easy to see that $\Lambda(T)$ is indeed a Hopf morphism. We derive a few lemmas concerning the functoriality of the exterior algebra construction.

Lemma 4.3.1. *If $T_1, T_2 : V \to W$ are linear maps, then*

$$\Lambda(T_1 + T_2) = m_{\Lambda(W)} \circ (\Lambda(T_1) \otimes \Lambda(T_2)) \circ \Delta_{\Lambda(V)}.$$

In other words, the exterior algebra functor transforms operator sum into convolution product.

Proof. Indeed, it is easy to see that the right hand side is an algebra morphism by commutativity of $\Lambda(W)$. The left hand side is an algebra morphism by definition. It is easy to see that both coincide over $V \subset \Lambda(V)$ (a generating set of $\Lambda(V)$), hence the lemma follows. $\square$

For the next lemma, note that there is a natural Hopf superalgebra isomorphism

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W)$$

where $(v, w) \in V \oplus W$ corresponds to $v \otimes 1 + 1 \otimes w$. Let T be a linear endomorphism of $V \oplus W$. Denote by $T_{VV}, T_{VW}, T_{WV}, T_{WW}$ its components, so for example $T_{VW} = \pi_W T j_V$ where $j_V : V \to V \oplus W$ is the inclusion and $\pi_W : V \oplus W \to W$ is the projection.

Lemma 4.3.2. *Under the above natural isomorphism, one has*

$$\begin{aligned}\Lambda(T) = (m_{\Lambda(V)} \otimes m_{\Lambda(W)}) \circ (id_{\Lambda(V)} \otimes \tau_{\Lambda(W),\Lambda(V)} \otimes id_{\Lambda(W)}) \\ \circ (\Lambda T_{VV} \otimes \Lambda T_{VW} \otimes \Lambda T_{WV} \otimes \Lambda T_{WW}) \circ (\Delta_{\Lambda(V)} \otimes \Delta_{\Lambda(W)}).\end{aligned}$$

Proof. As above, one can see that both sides are algebra morphisms and coincide over $V \oplus W$, so they are equal. $\square$

Now let T be an endomorphism of a direct sum $V_1 \oplus \dots \oplus V_d$. Then by induction we get

$$\Lambda(T) = \left(\otimes_{j=1}^{d} m^{(d)}_{\Lambda(V_j)}\right) P_d(\Lambda(T_{1*}) \otimes \ldots \otimes \Lambda(T_{d*})) \left(\otimes_{j=1}^{d} \Delta^{(d)}_{\Lambda(V_j)}\right) \tag{4.3}$$

where $\Lambda(T_{i*}) := \Lambda(T_{i1}) \otimes \ldots \otimes \Lambda(T_{id})$ for each $i = 1, \ldots, d$. Here P_d is the isomorphism

$$(\Lambda(V_1) \otimes \ldots \otimes \Lambda(V_d))^{\otimes d} \to \Lambda(V_1)^{\otimes d} \otimes \ldots \otimes \Lambda(V_d)^{\otimes d}$$

induced from the symmetry of the category of super vector spaces by the permutation defined by $P_d((k-1)d + i) = (i-1)d + k$.

4.3.2 Proof of Theorem 2

Let (M,γ) be a balanced sutured 3-manifold, $p \in s(\gamma)$ and let $\rho : \pi_1(M,p) \to GL(V)$ be a representation. We also denote by ρ the map into $\mathrm{Aut}(\Lambda(V)) \cong GL(V)$. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an extended Heegaard diagram of (M,γ) which is ordered, oriented and based.

For the following proposition, suppose that $R_-(\gamma)$ is connected, so that the $\alpha^* \in \pi_1(M,p)$ are defined for all $\alpha \in \boldsymbol{\alpha}^e$. We thus get a presentation of $\pi_1(M,p)$ as in (4.2).

Proposition 4.3.3. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}^e, \boldsymbol{\beta})$ be an extended Heegaard diagram of (M,γ) as above. If $R_-(\gamma)$ is connected, then*

$$Z^{\rho}_{\Lambda(V)}(\mathcal{H}) = \det\left(\rho\left(\frac{\partial \overline{\beta}_i}{\partial \alpha_j^*}\right)\right)_{i,j=1,\dots,d} \in \mathbb{K}.$$

Note that if we had used the convention that $(g\cdot c)\otimes v = c \otimes (\rho(g)^t(v))$ for the tensor product $C_*(\widetilde{M}) \otimes_{\mathbb{Z}[\pi]} V$ (as in [Por18]), then $Z^{\rho\otimes h}_{\Lambda(V)}$ would be exactly $\tau^{\rho\otimes h}$.

Proposition 4.3.3 in turn follows from the following stronger proposition.

Proposition 4.3.4. *Under the hypothesis and notation of Proposition 4.3.3 we have*

$$K^{\rho}_{\Lambda(V)}(\mathcal{H}) = \Lambda(T)$$

where $T : V^{\oplus d} \to V^{\oplus d}$ is the map given in components by $T_{ij} = \rho(\partial \overline{\beta}_j / \partial \alpha_i^*)$, that is, $T_{ij} = \pi_j T \iota_i$ where ι_i (resp. π_j) is the inclusion of V (resp. projection) into the i-th factor of $V^{\oplus d}$ (resp. j-th factor).

In other words, T is dual to the boundary map $\partial_2^{\rho} : C_2^{\rho}(M,R) \to C_1^{\rho}(M,R)$ of the universal covering of M twisted by ρ.

Proof. We will expand the tensor $\Lambda(T)$ using the lemmas of Subsection 4.3.1 and use commutativity and cocommutativity of $\Lambda(V)$ to show it equals Kuperberg's tensor. For simplicity, we will suppose $d = 2$, the general case is proved similarly. For simplicity, denote $\rho_x = \rho(\overline{\beta}_x) \in GL(V)$ for any $\beta \in \boldsymbol{\beta}$ and $x \in \boldsymbol{\alpha} \cap \beta$, so that

$$T_{ij} = \sum_{x \in \alpha_i \cap \beta_j} m(x)\rho_x$$

for any i,j (Remark 2.3.14). By Lemma 4.3.2 we have

$$\Lambda(T) = (m \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Lambda T_{11} \otimes \Lambda T_{12} \otimes \Lambda T_{21} \otimes \Lambda T_{22})(\Delta \otimes \Delta)$$

where m, Δ are the structure maps of the Hopf algebra $\Lambda(V)$. Now, by Lemma 4.3.1 each ΛT_{ij} can be expressed as a convolution

$$\begin{aligned}\Lambda T_{ij} &= \Lambda\left(\sum_{x\in\alpha_i\cap\beta_j} m(x)\rho_x\right)\\ &= m^{(k_{ij})}\left(\bigotimes_{x\in\alpha_i\cap\beta_j}\Lambda(m(x)\rho_x)\right)\Delta^{(k_{ij})}\\ &= m^{(k_{ij})}\circ A_{ij}\circ\Delta^{(k_{ij})}\end{aligned}$$

where $k_{ij} \coloneqq |\alpha_i \cap \beta_j|$ and

$$A_{ij} = \bigotimes_{x\in\alpha_i\cap\beta_j}\Lambda(\rho_x)\circ S^{\epsilon_x}.$$

Recall that $m^{(n)} : H^{\otimes n} \to H$ denotes iterated multiplication and similarly for $\Delta^{(n)}$. Hence, by coassociativity of Δ we can write

$$(\Lambda T_{11}\otimes\Lambda T_{12})\Delta = m^{(k_{11})}\otimes m^{(k_{12})}\,(A_{11}\otimes A_{12})\,\Delta^{|\alpha_1|}$$

and similarly for $(\Lambda T_{21}\otimes\Lambda T_{22})\Delta$. Thus, we obtain

$$\begin{aligned}\Lambda(T) &= m^{\otimes 2}(\mathrm{id}\otimes\tau\otimes\mathrm{id})(m^{(k_{11})}\otimes m^{(k_{12})}\otimes m^{(k_{21})}\otimes m^{(k_{22})})(A_{11}\otimes\ldots\otimes A_{22})\Delta_{\boldsymbol{\alpha}}\\ &= m^{\otimes 2}(m^{(k_{11})}\otimes m^{(k_{21})}\otimes m^{(k_{12})}\otimes m^{(k_{22})})(\mathrm{id}\otimes\tau\otimes\mathrm{id})(A_{11}\otimes\ldots\otimes A_{22})\Delta_{\boldsymbol{\alpha}}\\ &= m_{\boldsymbol{\beta}}(\mathrm{id}\otimes\tau\otimes\mathrm{id})(A_{11}\otimes\ldots\otimes A_{22})\Delta_{\boldsymbol{\alpha}}.\end{aligned}$$

By commutativity and cocommutativity of $\Lambda(V)$, the terms inside the last tensor can be reordered. More precisely, let P_{α_i} $(i = 1, 2)$ be the permutation of the set of crossings through α_i that puts the crossings of $\alpha_i \cap \beta_1$ first, and then those of $\alpha_i \cap \beta_2$. Similarly, let P_{β_j} be the inverse of the permutation of the crossings through β_j that puts the crossings of $\alpha_1 \cap \beta_j$ first, and then those of $\alpha_2 \cap \beta_j$. Then it is clear that

$$P_{\mathcal{H}} = (P_{\beta_1}\otimes P_{\beta_2})(\mathrm{id}\otimes\tau\otimes\mathrm{id})(P_{\alpha_1}\otimes P_{\alpha_2}).$$

By commutativity and cocommutativity of $\Lambda(V)$ we have

$$m_{\boldsymbol{\beta}} = m_{\boldsymbol{\beta}}(P_{\beta_1}\otimes P_{\beta_2}) \quad\text{and}\quad \Delta_{\boldsymbol{\alpha}} = (P_{\alpha_1}\otimes P_{\alpha_2})\Delta_{\boldsymbol{\alpha}},$$

therefore

$$\begin{aligned}\Lambda(T) &= m_{\boldsymbol{\beta}}(P_{\beta_1}\otimes P_{\beta_2})(\mathrm{id}\otimes\tau\otimes\mathrm{id})(A_{11}\otimes\ldots\otimes A_{22})(P_{\alpha_1}\otimes P_{\alpha_2})\Delta_{\boldsymbol{\alpha}}\\ &= m_{\boldsymbol{\beta}}P_{\mathcal{H}}\left(\bigotimes_{x\in\mathcal{I}_{\boldsymbol{\alpha}}}\Lambda(\rho_x)\right)S_{\boldsymbol{\alpha}}\Delta_{\boldsymbol{\alpha}}.\end{aligned}$$

This is exactly $K^{\rho}_{\Lambda(V)}(\mathcal{H})$ as desired. $\square$

Proof of Proposition 4.3.3. By the preceding proposition we get

$$\begin{aligned}
\det(\rho(\partial\overline{\beta}_i/\partial\alpha_j^*)) &= \det(T) \\
&= \mu_{\Lambda(V)^{\otimes d}}(\Lambda(T)(c_{\Lambda(V)^{\otimes d}})) \\
&= \mu_{\Lambda(V)^{\otimes d}}(K^{\rho}_{\Lambda(V)}(\mathcal{H})(c_{\Lambda(V)^{\otimes d}})) \\
&= Z^{\rho}_{\Lambda(V)}(\mathcal{H})
\end{aligned}$$

as was to be shown. $\square$

Proof of Theorem 2. Suppose first that $R_-(\gamma)$ is connected. Note that the inverse-transpose satisfies $(\rho\otimes h)^{-t}(\sigma(x)) = (\rho\otimes h)(x)^t$ for any $x\in\mathbb{Z}[\pi]$, where recall that $\sigma:\mathbb{Z}[\pi]\to\mathbb{Z}[\pi]$ is the map defined by $\sigma(g)=g^{-1}$ for $g\in\pi$. Then we get

$$\begin{aligned}
Z^{\rho\otimes h}_{\Lambda(V)_M}(M,\gamma)\doteq Z^{\rho\otimes h}_{\Lambda(V)_M}(\mathcal{H}) &= \det((\rho\otimes h)(\partial\overline{\beta}_i/\partial\alpha_j^*)) \\
&= \det((\rho\otimes h)(\partial\overline{\beta}_j/\partial\alpha_i^*)^t) \\
&= \det((\rho\otimes h)^{-t}(\sigma(\partial\overline{\beta}_j/\partial\alpha_i^*))) \\
&\doteq \tau^{(\rho\otimes h)^{-t}}(M,R_-)
\end{aligned}$$

where we use Proposition 4.3.3 in the second equality and Proposition 4.2.1 in the last equality. When $R_-(\gamma)$ is disconnected, then by definition $Z^{\rho\otimes h}_{\Lambda(V)_M}(M,\gamma) \coloneqq Z^{\rho'\otimes h'}_{\Lambda(V)_{M'}}(M',\gamma')$ where (M',γ') is obtained by adding one handles to $R=R_-(\gamma)$ in such a way that $R'=R_-(\gamma')$ is connected. But then

$$Z^{\rho\otimes h}(M,\gamma)\coloneqq Z^{\rho'\otimes h'}(M',\gamma') = \tau^{(\rho'\otimes h')^{-t}}(M',R') = \tau^{(\rho\otimes h)^{-t}}(M,R)$$

where the last equality follows from [FJR11, Lemma 3.20]. $\square$

4.3.3 Particular cases of Theorem 2

Combining Theorem 2 to Corollary 4.2.3 we get the following.

Corollary 4.3.5. *Let Y be a closed oriented 3-manifold and $M = Y\setminus int(B)$, $\gamma = S^1\subset\partial B$ where B is some closed embedded 3-ball. Then*

$$Z^{\rho}_{\Lambda(V)}(M,\gamma) = \begin{cases} 0 & \text{if } \rho\neq 1, \\ \pm|H_1(Y;\mathbb{Z})| & \text{if } \rho\equiv 1. \end{cases}$$

Now let L be an ordered oriented m-component link in S^3. As in Subsection 4.2.2 we identify $\mathbb{Z}[H_1(M)]\cong\mathbb{Z}[t_1^{\pm1},\dots,t_m^{\pm1}]$ where $M=S^3\setminus L$. Combining Theorem 2 with Corollary 4.2.4 and using the same notation as in that corollary we get:

Corollary 4.3.6. *Let $L \subset S^3$ be an ordered oriented m-component link and let (M, γ) be the sutured manifold complementary to L. If $m > 1$ or if $m = 1$ and ρ is irreducible with $\rho|_{Ker(h)} \neq 1$, the twisted Alexander polynomial of L is recovered as*

$$\Delta_L^{\rho \otimes h}(t_1, \ldots, t_m) = Z^{\rho \otimes h}_{\Lambda(V)_M}(M, \gamma) \cdot \prod_{i=1}^{m} \det(t_i \rho(a_i^*) - I_n)^{-1}.$$

If $m = 1$, $\rho \equiv 1$ and $\dim(V) = 1$, then $\Delta_L(t) = Z^{1 \otimes h}_{\Lambda(\mathbb{K})_M}(M, \gamma)$.

Remark 4.3.7. The torsion of (M, R_-), and hence the invariant $Z^{\rho}_{\Lambda(V)}$, is also interesting when M is a Seifert surface complement (as in Example 2.1.5). Indeed, it can distinguish minimal genus Seifert surfaces of some knots up to isotopy [Alt12].

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