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Quelques constructions autour de la stabilité de fibrés vectoriels sur les variétés projectives

Some constructions around stability of vector bundles on projective varieties

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Résumé

Cette thèse est un travail autour de la stabilité de fibrés vectoriels. Elle est divisée en quatre parties, dont la première est introductive, et les trois autres sont constituées par des résultats originaux.

Dans la deuxième partie on montre que sur chaque variété projective lisse sur un corps algébriquement clos, les fibrés stables fournissent un ensemble de générateurs pour l'anneau de Chow de la variété. Ce résultat provient de la recherche de formes différentielles particulières fournissant des représentants de classes de Chern des fibrés vectoriels. Pour obtenir des tels représentants on cherche une résolution projective pour chaque fibré vectoriel, faite par des fibrés stables ou polystables. La recherche d'une telle résolution nous a amenés à la construction d'une résolution pour les faisceaux d'idéaux, entraînant le résultat sur les groupes de Chow.

Afin de rechercher si une résolution polystable existe pour tout fibré, un point essentiel est la recherche de transformées stables de fibrés donnés. Par transformée on désigne le noyau de l'application d'évaluation sur un sous-espace de sections globales d'un fibré.

La stabilité de ces noyaux a été étudiée par plusieurs auteurs, avec des motivations différentes. Paranjape et Ramanan, et par Butler, pour étudier la génération normale de certains fibrés vectoriels. Ein and Lazarseld pour montrer la stabilité du fibré de Picard. Beauville pour étudier la réductibilité des diviseur thêta, et Mercat pour étudier la dimension des lieux de Brill-Noether.

Le troisième chapitre donne une réponse à la question de la stabilité des transformées dans des cas particuliers, pour des fibrés en droites sur des courbes projectives lisses de genre plus grand que 1.

Dans la quatrième partie, un autre cas de stabilité est traité, dans le cadre des produits symétriques des courbes.

Les techniques utilisées sont purement algébriques, et valables en toute caractéristique, bien que l'existence des métriques de Hermite-Einstein sur des fibrés vectoriels stables était la motivation pricipale pour le projet.

Bien que la question de la stabilité des transformées soit assez naturelle, il ne s'agit que de résultats partiels qu'on a trouvé, ne permettant pas la construction des résolutions recherchées.

Les résultats obtenus laissent envisager de toute façon, de pouvoir étudier plus profondément la stabilité de certains fibrés sur les produits symétriques de courbes, ou surfaces.

Introduction

This thesis deals with stable vector bundles over projective varieties. Vector bundles are objects used in various areas of mathematics, from differential equations to number theory. In algebraic geometry they are an instrument to study the geometry of the variety over which they are defined. Their simplest numerical invariants are rank and Chern classes.

Stability is a concept arising when we want to construct a moduli space of vector bundles fixing those numerical invariants.

The starting point of our work is the construction of a polystable resolution of ideal sheaves.

The motivations leading to this construction are to be found in the Kobayashi-Hitchin correspondence, relating the stability of a vector bundle on a complex projective variety to the existence of a Hermite-Einstein metric. The existence of a particular metric on a vector bundle implies the possibility of choosing in a "canonical" way differential forms representing Chern classes of the vector bundle.

Hence, looking for some polystable resolution was a first step to find a way of choosing particular differential forms representing Chern classes of *every* vector bundle. And eventually a lift of the cycle map γ_X:

$$\begin{array}{ccc} & & \mathcal{A}^{2i}_{closed}(X) \\ & \nearrow & \downarrow \\ \mathrm{CH}^i(X) & \xrightarrow{\gamma_X} & \mathrm{H}^{2i}_{\mathrm{DR}}(X) \end{array}$$

Unfortunately this turned out to be too optimistic, and we were not able to find such a resolution. In any case, even when the polystable resolutions exist, the associated Chern form might depend on the resolution.

The main reason for this, is the difficulties arising when we want to verify the stability of a given vector bundle coming from some construction.

However, we were able to find such a resolution for ideal sheaves, and this was sufficient to exhibit stable bundles as generators for the Chow ring of any projective smooth variety (and for the K-theory and the derived category as well).

The text is organized as follows.

In the first chapter we find all the basic notions, and the first elementary lemmas. We describe how to construct Chow groups, and Chern classes. We show the origins of stability in the construction of moduli spaces, and the detailed description of slope stability as well as the properties we are going to use. We give also an idea of the meaning of Kobayashi-Hitchin correspondence.

All of the notions in the first chapter are well known, and can be found in many places in the literature. We give the references that we have followed.

The other chapters are our results.

The second part is dedicated to the construction of the polystable resolution of ideal sheaves. This construction is possible by restricting on curves, and using a result of Butler, asserting the stability of kernels of evaluation maps (that we call transforms in the rest of the thesis).

One of the main tool to show stability on a higher dimensional variety is by restriction on curves, where stability can be more easily checked.

A main problem in trying to construct resolutions as above, is to find out whether we have stability of some transforms of vector bundles.

The question about stability of transforms appears in many different studies in the literature. It is observed in particular by Paranjape and Ramanan to prove normal generation of canonical ring of curves, by Butler also to study normal generation of certain vector bundles, by Ein and Lazarsfeld to show the stability of the Picard bundle, by Beauville to study theta divisors, and by Mercat to describe some Brill-Noether loci.

The third chapter is a partial answer to a question of this kind for line bundles on curves.

The fourth chapter is also dedicated to the stability of transforms, on symmetric product of curves. It is shown that tautological bundles and their transforms are stable with respect to a canonical polarization.

Even though stability of transforms is a very natural question, we have only partial results for the time being. We think however that this results can be generalized to symmetric products of surfaces, where tautological sheaves are used on various purposes.

Chapter 1

Notations, and basic lemmas

In algebraic geometry vector bundles are an instrument to study the geometry of the variety over which they are defined. Stability is a concept arising when we want to construct a moduli space of vector bundles having a given rank and Chern classes. Its original definition as well as its numerical characterization are due to Mumford in the case of quotients of varieties by some group (cf. [MFK94]).

In the complex case stability has also a differential description by the existence of a unique metric on the vector bundle satisfying a certain condition on the curvature. This is a Hermite-Einstein metric, and the relation between such metric properties and stability, *i.e.* the Hitchin-Kobayashi correspondence, was described by Kobayashi (cf. [Kob87]). Narasimhan and Seshadri proved it in [NS65] in the case of curves (relating stability and unitary representations of the fundamental group). Donaldson in the case of projective surfaces (cf. [Don85]), and then for all projective variety in [Don87]. Uhlenbeck and Yau, proved it for compact Kähler manifolds (cf. [UY86]).

1.0.1 References of the preliminary notions

In this chapter we will go over the notions just mentioned, even though we will deal in the rest only with their algebraic part. We will try to explain the interest in stability of vector bundles and give some basic lemmas, setting up our context and notations.

All of the theorems and constructions of this chapter are well known, and can be found in various places in the literature, except for theorem 1.2.25 on stability and restrictions, which is widely used, but we were not able to find in this form. We will try to describe how the main objects are constructed, only the theorems we will use the most (existence of a maximal semistable subsheaf and stability through restrictions) will be proved in more details.

For more details on the themes treated in this chapter, here are the references we have followed:

- for the construction of the Chow ring, Chevalley [Che58], and Fulton [Ful98];
- for the Quot and Hilbert's schemes, Grothendieck [Gro95];
- for Chern classes, Grothendieck [Gro58];
- for moduli spaces, Mumford [MFK94] for GIT, Huybrechts and Lehn [HL97] and Le Potier [LP95] for muduli of vector bundles;
- for Hermite Einstein metrics, Kobayashi [Kob87].

1.1 Chow ring and Chern classes

Throughout this thesis by variety we mean a smooth integral projective scheme over an algebraically closed field $\mathbb{k}$.

1.1.1 Algebraic cycles on a projective variety

In this paragraph, we show the classical construction of a ring structure on the set of formal sums of integral closed subschemes modulo rational equivalence in a smooth projective variety, with intersection as product. The functorial construction of such a ring, graded by codimension, is called an "intersection theory for cycles".

The description we follow can be used, or axiomatized, to construct an intersection theory for cycles on nonsingular quasi-projective varieties, with very few chanhgings (as considering only *proper* push-forwards). As we will not use intersection theory on quasi-projective varieties, we stick to our notation and call variety a nonsingular projective scheme over a fixed algebraically closed field $\mathbb{k}$.

We remark that any morphism between projective varieties is proper, and in particular closed.

Definition 1.1.1 *A* cycle *on a variety X is a finite formal sum of closed integral subschemes of X, with coefficients in $\mathbb{Z}$. The set of all cycles of X forms an abelian group, graded by codimension:*

$$C^*(X) = \bigoplus_{p=0}^{n} C^p(X) \ , \ C^p(X) := \bigoplus_{\substack{Y \subset X \\ Y \text{ integral} \\ \operatorname{codim}_X Y = p}} \mathbb{Z}Y \, .$$

Given a morphism of varieties $\varphi \colon X_1 \to X_2$ we define the push-forward morphism $\varphi_* \colon C^*(X_1) \to C^{*+q}(X_2)$ of the groups of cycles (shifting the degrees by $q = \dim X_2 - \dim X_1$) in the following way

$$\varphi_*(Y) = 0 \qquad \text{if } \dim \varphi(Y) < \dim Y$$

$$\varphi_*(Y) = [k(Y) : k(\varphi(Y))] \cdot \varphi(Y) \quad \text{if } \dim \varphi(Y) = \dim Y$$

for all integral subscheme $Y \subset X_1$, then extending to $C^*(X)$ by linearity.

If we have a closed non integral subscheme $Y \subset X$ of codimension p in X, then we associate to Y the cycle

$$\sum_\alpha m_\alpha Y_\alpha \ ,$$

were Y_α are the reduced irreducible components of Y of codimension p in X, with generic points y_α, counted with multiplicities $m_\alpha := \text{length}\mathcal{O}_{Y,y_\alpha}$.

We want to define a graded ring structure in $C^*(X)$, where intersection of subschemes induces the product law. This is not possible as the intersection of two integral subschemes does not necessarily have the good dimension, hence we allow cycles to be deformed in such a way that they can intersect properly.

Definition 1.1.2 *A cycle $Z \in C^*(X \times T)$ is* flat *over T if it is a formal sum of integral subscemes* flat *over T.*

An algebraic family $\{Z_t\}_{t\in T}$ *of p-codimensional cycles on X, parametrized by a connected scheme T, is a p-codimensional cycle $Z \subset X \times T$, flat over T. All the fibers Z_t are said to be* algebraically equivalent*. We note $\sim_{alg}$ the equivalence relation in $C^*(X)$ generated by algebraically equivalent cycles.*

We say that an algebraic family is a rational family*, when it is parametrized by an open subset $T \subseteq \mathbb{P}^1$. We call* rationally equivalent *two cycles in any such family, and we note $\sim_{rat}$ the equivalence relation in $C^*(X)$ generated by rationally equivalent cycles.*

We define Chow group of order q of X the group of cycles of codimension q modulo rational equivalence, noted

$$\text{CH}^q(X) := C^q(X)/ \sim_{rat} =: \text{CH}_{n-q}(X).$$

In the above definition, the cycles Z_t, $t \in T$, are the cycles associated to the schematic intersection $Z \cap (X \times t)$, when Z is a closed integral subscheme, and we extend this in the natural way to a cycle $Z = \sum n_\alpha Y_\alpha$ with every Y_α integral and flat over T.

Example 1.1.3 If we consider divisors, *i.e.* cycles of codimension 1, on a curve C, then $D_1 = \sum n_i x_i$ and $D_2 = \sum m_j y_j$ are algebraically equivalent if and only if they have the same degree, *i.e.* if $\sum m_j = \sum n_i$. To prove this, notice that every two points $x, y \in C$ are algebraically equivalent.

And they are rationally equivalent if and only if they represent the same invertible sheaf, *i.e.* if $\mathcal{O}_C(D_1) = \mathcal{O}_C(D_2)$: in general, for a smooth quasi-projective variety X, the group $C^1(X)$ of 1-codimensional cycles is exactly the group of Weil divisors, and it can be shown that rational equivalence coincides with linear equivalence, hence the group $C^1(X)/ \sim_{rat}$ of divisors modulo rational equivalence, is isomorphic to the group $\text{Pic}(X)$ of line bundles on X.

Definition 1.1.4 *We say that two integral subschemes Z_1 and Z_2* intersect properly *if every component Y_α of $Z_1 \cap Z_2$ verifies* $\text{codim}_X Y_\alpha = \text{codim}_X Z_1 + \text{codim}_X Z_2$.

When two integral subschemes Z_1 and Z_2 intersect properly, and $Z_1 \cap Z_2$ has irreducible components Y_α (with reduced structure), we define their product as

$$Z_1.Z_2 := \sum_\alpha I(Z_1.Z_2, Y_\alpha; X) Y_\alpha \ ,$$

where the coefficients are the intersection multiplicities

$$I(Z_1.Z_2, Y_\alpha; X) := \sum_i (-1)^i \text{lenght } Tor^i_{\mathcal{O}_{X,Y_\alpha}}(\mathcal{O}_{X,Y_\alpha}/\mathcal{I}_{Z_1}, \mathcal{O}_{X,Y_\alpha}/\mathcal{I}_{Z_2}) \ .$$

Thus, intersection product of two integer subschemes intersecting properly is the sum of the intersection components counted with multiplicities. Intersection product $W.Z$ of two cycles $Y = \sum n_\alpha Y_\alpha, Z = \sum m_\beta Z_\beta \in C^*(X)$ is defined any times that every component Y_α of Y intersect properly every component Z_β of Z.

When two subvarieties intersect transversally, then intersection multiplicities are 1 for all components of the intersection.

Proposition 1.1.5 *The intersection product, extended as far as possible to $C^*(X)$, is commutative and associative whenever defined, and has $X \in C^0(X)$ as identity.*

The fact that allows to define intersection products in the Chow groups is Chow's moving lemma, which assures that given two cycles, one can be rationally deformed to intersect properly the other one.

Lemma 1.1.6 (Chow) *Given two cycles Z, W on a variety X, there exist a cycle Y, rationally equivalent to W, such that the intersection cycle $Z.Y$ is defined.*

Then we can define a ring structure on $C^*(X)/\sim_{rat}$ and $C^*(X)/\sim_{alg}$, thanks to he following

Lemma 1.1.7 *Let Z, W, Y be three cycles on a variety X. Suppose that Y is rationally (algebraically) equivalent to W and hat $Z.Y$ and $Z.W$ are defined. Then $Z.W$ is rationally (algebraically) equivalent to $Z.Y$.*

Definition 1.1.8 *We call* Chow ring *of a variey X the commutative graded ring of rationally equivalent cycles on X, and we note it*

$$\mathrm{CH}^*(X) = C^*(X)/\sim_{rat} \ .$$

We call rational Chow ring the ring $\mathrm{CH}^*_{\mathbb{Q}}(X) := \mathrm{CH}^*(X) \otimes \mathbb{Q}$.

Remark 1.1.9 In the following, we will often note in the same way a cycle Z, and its rational class $[Z]_{rat}$. In particular, given two cycles $Z \in C^p(X), Y \in C^q(X)$, we will write $Z.Y \in \mathrm{CH}^{p+q}(X)$ instead of $[Z]_{rat}.[Y]_{rat} \in \mathrm{CH}^{p+q}(X)$.

Given a graded product structure on the Chow groups

$$\mathrm{CH}^q(X) \times \mathrm{CH}^s(X) \to \mathrm{CH}^{q+s}(x)$$

on each variety X, we define the pull-back φ^* of cycles by a morphism $\varphi\colon X_1 \to X_2$ as:

$$\varphi^*(Y) := p_{1*}(\Gamma_\varphi.(X_1 \times Y))$$

where p_1 is the natural projection $X_1 \times X_2 \to X_1$, and Γ_φ is the cycle corresponding to the graph of φ in $X_1 \times X_2$.

Properties 1.1.10 The following are properties of the Chow ring which can be axiomatized for the graded product structures on the Chow groups to give rise to an intersection theory:

i. *Product.* $\mathrm{CH}^*(X)$ is a graded commutative ring with identity, for every variety X.

ii. *Functoriality.* For any morphism $\varphi\colon X_1 \to X_2$ of varieties, $\varphi^*\colon \mathrm{CH}^*(X_2) \to \mathrm{CH}^*(X_1)$ is a graded ring homomorphism. If $\psi\colon X_2 \to X_3$ is another morphism, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

iii. *(Proper) Push-forward.* For any (proper) morphism $\varphi\colon X_1 \to X_2$ of varieties, $\varphi_*\colon \mathrm{CH}^*(X_1) \to \mathrm{CH}^*(X_2)$ is a graded group homomorphism shifting degrees by $\dim X_2 - \dim X_1$. If $\psi\colon X_2 \to X_3$ is another morphism, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

iv. *Projection Formula.* For any (proper) morphism $\varphi\colon X_1 \to X_2$ of varieties, and any classes $W \in \mathrm{CH}^*(X_1)$ and $Z \in \mathrm{CH}^*(X_2)$, we have $\varphi_*(W.\varphi^*Z) = (\varphi_*W).Z \in \mathrm{CH}^*(X_2)$.

v. *Reduction to the diagonal.* If Y and Z are cycles on X, and if $\Delta\colon X \to X \times X$ is the diagonal morphism, then

$$Y.Z = \Delta^*(Y \times Z)\ .$$

vi. *Local Intersection.* If Z_1 and Z_2 are subvarieties of X which intersect properly, then

$$Z_1.Z_2 := \sum_\alpha I(Z_1.Z_2, Y_\alpha; X) Y_\alpha\ ,$$

where Y_α are the components of $Z_1 \cap Z_2$, and $I(Z_1.Z_2, Y_\alpha; X)$ depends only on a neighborhood of the generic point of of Y_α in X.

vii. *Normalization.* Let Z be an effective Cartier divisor given by a section $f\colon X \to L$ of a line bundle L, and let Y be a subvariety of X intersecting properly Z, then $Y.Z$ is the cycle associated to the cartier divisor $Y \cap Z$ on Y, obtained by restricting f to Y.

Proposition 1.1.11 *There exists a unique intersection product on rational equivalence classes of cycles*

$$\mathrm{CH}^q(X) \times \mathrm{CH}^s(X) \to \mathrm{CH}^{q+s}(X)$$

satisfying properties 1.1.10.

The following property of the Chow ring allow us to define Chern classes with values in the Chow groups.

Property 1.1.12 Let E be a rank r vector bundle on the variety X, determining the projective bundle $\pi\colon \mathbb{P}(E) \to X$. And let $\xi \in \mathrm{CH}^1(\mathbb{P}(E)) \cong \mathrm{Pic}(\mathbb{P}(E))$ the class coresponding to the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$. Then the ring homomorphism $\pi^*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(\mathbb{P}(E))$ makes $\mathrm{CH}^*(\mathbb{P}(E))$ a free $\mathrm{CH}^*(X)$-module generated by $1, \xi, \xi^2, \ldots, \xi^{r-1}$.

In this thesis we use intersection theory to compute Chern classes. What we will need in effect, is to cut a given divisor with an ample one many times until we get to some number of points. This number does not depend on the rational or algebraic class of those divisors, as we explain in the following.

Definition 1.1.13 *We call* degree map *the map* $< . >\colon C^*(X) \to \mathbb{Z}$, *which is the zero map on* $C^q(X)$ *when* $q < n = \dim(X)$, *and is the map*

$$\sum_i m_i p_i \mapsto \sum_i m_i$$

couting points on the group of 0*-cycles.*

Remark 1.1.14 Clearly, the map is well defined on algebraic or rational classes of divisors, hence we get degree maps $< . >\colon \mathrm{CH}^*(X) \to \mathbb{Z}$, and $< . >\colon C^*(X)/\sim_{alg} \to \mathbb{Z}$.

When it does not create ambiguities, we will omit the $<>$ signs for the degree map, in particular given a divisor D, and a cycle $Z \in C^q(X)$ we will write $D^{n-q}.Z > 0$ instead of $< D^{n-q}.Z > > 0$.

In this frame, we state the famous ampleness criterion by Nakai and Moishezon. A detailed proof can be found in [Kle66] (in larger generality than here).

Proposition 1.1.15 (Nakai-Moishezon criterion) *Let H be a divisor on a variety X, then H is ample if and only if for every closed integral r-dimensional subscheme $Y \subseteq X$ we have $H^r.Y > 0$.*

Definition 1.1.16 *We say that two cycles Z_1 and Z_2 in a variety X are numerically equivalent, and we note $Z_1 \equiv Z_2$, or $Z_1 \sim_{num} Z_2$, if for every cycle $Y \in C^*(X)$ we have $< Z_1.Y >=< Z_2.Y >$. We call group of Neron-Severi, the finitely generated (by theorem of Neron and Severi) group of Divisors modulo algebraic equivalence, noted*

$$NS(X) := C^1(X)/\sim_{alg},$$

and we note $N^1(X) := C^1(X)/\sim_{num}$.

By Nakai-Moishezon criterion we see that if a divisor $D \in C^1(X)$ verifies $D \equiv H$, H ample, then D itself is ample. So when we are interested only in intersection numbers, *e.g.* in the definition of stability (1.2.9) below, we can consider only divisors up to numerical equivalence, *i.e.* divisors in $N^1(X)$.

Remark 1.1.17 Throughout this thesis, whenever it will not create ambiguities, we will make the abuse of notation of noting in the same way D, a divisor $D \in C^1(X)$, its rational equivalence class $[D]_{rat} \in \mathrm{CH}^1(X)$, its invertible sheaf $\mathcal{O}_X(D) \in \mathrm{Pic}(X)$, and its numerical equivalence class $[D]_{num} \in N^1(X)$.

We will identify as usual any vector bundle $E \to X$ with its locally free sheaf of sections $\mathcal{O}_X(E)$.

We verify immediately the following

Proposition 1.1.18 *Let Y and Z be two cycles in $C^*(X)$, then*

$$Y \sim_{rat} Z \;\Rightarrow\; Y \sim_{alg} Z \;\Rightarrow\; Y \equiv Z\,.$$

1.1.2 Chern classes in the Chow ring

Roughly speaking, the ith Chern class of a vector bundle E is the locus where $r - i + 1$ generic sections don't have maximal rank. We give some example of what this means, and then the general definition.

If a line bundle L on a variety X admits a non zero global section $s\colon X \to L$, then the vanishing locus of this section is a divisor [1] in X, and the rational class of this divisor is exactly L (through the correspondence $\mathrm{CH}^1(X) \cong \mathrm{Pic}(X)$).

In the same way, consider a rank r vector bundle E on the variety X of dimension n, such that $r \leqslant n$. Let us suppose there is a global section $s\colon X \to E$, transversal to the zero section. Then the vanishing locus $Z(s) := \{x \in X \mid s(x) = 0\}$ is a r codimensional subvariety of X. The class of $Z(s)$ in $\mathrm{CH}^r(X)$ does not depend on the section s, and is an invariant of E called the rth Chern class $c_r(E)$ of E.

Furthermore, let us suppose that E is globally generated, and consider r generic global sections $s_1, \ldots, s_r$. Then the locus

$$Z(s_1, \ldots, s_r) := \{x \in X \mid \mathrm{rk}(s_1(x), \ldots, s_r(x)) \leqslant r - 1\}$$

where those sections do not have maximal rank r is a 1-codimensional subvariety of X, *i.e.* a divisor of X. The r sections $s_1, \ldots, s_r$ give us a section $s_1 \wedge \cdots \wedge s_r$ of the line bundle $\det(E)$, that vanishes exactly on $Z(s_1, \ldots, s_r)$. The class of $Z(s_1, \ldots, s_r)$ in $\mathrm{CH}^1(X) \cong \mathrm{Pic}(X)$ does not depend on the chosen sections. It is an invariant of E called the first Chern class $c_1(E)$, and corresponds to the line bundle $\det(E)$.

The properties of Chow rings allow us to formalize such idea and construct Chern classes $c_i(E) \in \mathrm{CH}^i(X)$ for every vector bundle E on X, and then for every sheaf $\mathcal{F} \in \mathrm{Coh}_X$.

[1] the divisor $0 \in C^1(X)$ in the case of the empty set.

If we have a vector bundle E on X, let $\mathbb{P}(E) \to X$ the projective bundle[2] associated to E, and let ξ be the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$ in $CH^1(\mathbb{P}(E))$. Then, by property 1.1.12 of the Chow ring, we can express ξ^r in a unique way as a linear combination of $1, \xi, \xi^2, \ldots, \xi^{r-1}$, with coefficients in $\mathrm{CH}(X)$. Those coefficients define the Chern classes in the following way:

Definition 1.1.19 *Let E be a rank r vector bundle on the variety X, define for all $0 \leqslant i \leqslant r$ the i*th Chern class *$c_i(E) \in \mathrm{CH}^i(X)$ by $c_0(E) = 1$ and*

$$\sum_{i=0}^{r} \pi^* c_i(E).\xi^{r-i} = 0 \,. \tag{1.1}$$

We call total Chern class *$c(E) := c_0(E) + c_1(E) + \cdots + c_r(E)$ and* Chern polynomial

$$c_t(E) := \sum_{i=0}^{r} c_i(E) t^i \in \mathrm{CH}(X)[t] \,.$$

We give a list of main properties of Chern calsses.

Properties 1.1.20 Let E be a vector bundle on a variety X. Then the Chern classes and polynomial satisfy the following properties:

i. If D is a divisor, and $E = \mathcal{O}_X(D)$, then $c_t(E) = 1 + Dt \in \mathrm{CH}(X)[t]$.

ii. If $f\colon Y \to X$ is a morphism, then for all i, $c_i(f^*E) = f^* c_i(E)$.

iii. If E is an extension of vector bundles $0 \to F \to E \to G \to 0$, then $c_t(E) = c_t(F) \cdot c_t(G) \in \mathrm{CH}(X)[t]$.

Thus if E admits a filtration $E = E_0 \supset E_1 \supset \ldots E_r = 0$, whose quotients are invertible sheaves $L_i = \mathcal{O}(D_i) = E_{i-1}/E_i$, then $c_t(E) = \prod_{i=1}^{r}(1 + D_i t)$.

We can always write the Chern polynomial of E as $c_t(E) = \prod_{i=1}^{r}(1 + a_i t)$, where a_i are formal symbols, and the elementary symmetric functions of the a_i are the Chern classes of E. Thanks to this, we can define the Chern character of E as

$$\mathrm{ch}(E) = \sum_{i=1}^{r} e^{a_i} \in \mathrm{CH}^*_{\mathbb{Q}}(X) \,,$$

where $e^{a_i} = 1 + a_i + \frac{1}{2}(a_i)^2 + \ldots$.

This definition is well posed because it is a linear combination of the elementary symmetric functions of a_i with rational coefficients.

We can verify easily that the terms of low order of the Chern character of a rank r vector bundle are

$$\mathrm{ch}(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \ldots \,.$$

[2] we follow the "italian" notations for projective spaces, see examples 1.2.1 and 1.2.3 below.

In the same way the Todd class of E is defined by

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{a_i}} \ .$$

Properties 1.1.21 Let E be a vector bundle on a variety X. Then the Chern character satisfy the following properties:

i. For any variety Y, and any morphism $Y \to X$, $\mathrm{ch}(f^*E) = f^*(\mathrm{ch}(E))$.

ii. If $E = F \oplus G$, then $\mathrm{ch}(E) = \mathrm{ch}(F) + \mathrm{ch}(G)$.

iii. If $E = F \otimes G$, then $\mathrm{ch}(E) = \mathrm{ch}(F) \cdot \mathrm{ch}(G)$.

By the properties above, the Chern character can be extended in a unique natural way to any coherent sheaf on X, considering a locally free resolution. More generally, we have a map $\mathrm{ch}\colon K(X) \to \mathrm{CH}^*_{\mathbb{Q}}(X)$.

For any morphism $f\colon Y \to X$ of projective varieties, we define

$$\begin{array}{rcl} f_!\colon K(Y) & \to & K(X) \\ \mathcal{F} & \mapsto & \sum(-1)^i R^i f_*(\mathcal{F}) \ , \end{array}$$

with these notations, we can state Grothendieck's generalization of Riemann-Roch theorem (cf. Borel and Serre [BS58]):

Theorem 1.1.22 (Grothendieck-Riemann-Roch) *Let $f\colon Y \to X$ be a smooth morphism of nonsingular projective varieties. Then for any $\mathcal{F} \in K(Y)$ we have*

$$\mathrm{ch}(f_!(\mathcal{F})).\mathrm{td}(\mathcal{T}_X) = f_*(\mathrm{ch}(\mathcal{F}).\mathrm{td}(\mathcal{T}_Y))$$

in $\mathrm{CH}^*_{\mathbb{Q}}(X)$.

Considering the application $\varphi\colon X \to \mathrm{Spec}\Bbbk$, the application $\varphi_*\colon \mathrm{CH}^*_{\mathbb{Q}}(X) \to \mathrm{CH}(\mathrm{Spec}\Bbbk)_{\mathbb{Q}} = \mathbb{Q}$ is exactly the degree map. Hence we have Hirzebruch-Riemann-Roch as a particular case Grothendieck-Riemann-Roch theorem:

Theorem 1.1.23 (Hirzebruch-Riemann-Roch) *Let $\mathcal{F}$ be a coherent sheaf on a variety X. Then we have*

$$\chi(X, \mathcal{F}) = < \mathrm{ch}(\mathcal{F}).\mathrm{td}(\mathcal{T}_X) > \ .$$

In particular, this implies that the Hilbert polynomial of a coherent sheaf $\mathcal{F}$, with respect to a polarization H, depends only on the numerical equivalence classes of the Chern classes of $\mathcal{F}$ and H.

1.1.3 Chern classes and cohomology

We briefly review how to construct Chern classes in the cohomology ring of a complex variety, as a different way to view what Chern classes are and how to compute them or their intersections. And we note that given a Hermitian metric on a holomorphic vector bundle, we have a way of choosing differential forms which represent the Chern classes.

There are many ways to consider Chern classes with value in the cohomology ring. Those ways can be found following two main directions.

i. The *cycle map* γ_X is an application associating to any cycle $Z \in C^p(X)$, a cohomology class $\gamma_X(Z) \in H^{2p}(X)$ in a Weil cohomology theory $H^*(X)$. Composing γ with Chern classes or Chern character we obtain Chern classes with values in the cohomology theory $H^*(X)$.

ii. When we have algebraic varieties over the complex numbers $\mathbb{C}$, then we have differential geometric properties other than algebraic ones, and we can use those to obtain Chern forms $c_i(E,h)$ of hermitian vector bundles (E,h), with values in closed differential forms. Passing to the cohomology classes those forms represent, we will get the same Chern classes as those of point *i.* (independent of the hermitian metric).

We explain the second point which is the one of interest to us. Let us consider a differentiable manifold M, and a rank r complex vector bundle $E \to M$. We note $\mathcal{A}^0_M = \mathcal{C}^\infty(M)$ the algebra of $\mathcal{C}^\infty$ functions on M with complex values, and $\mathcal{A}^p(E) = \mathcal{C}^\infty(\bigwedge^p T^* \otimes_{\mathbb{R}} E)$ the p-differential forms with values in E.

A connection on E is a $\mathbb{C}$-linear application

$$\nabla\colon \mathcal{A}^0(E) \to \mathcal{A}^1(E)$$

satisfying Leibniz rule: for all $f \in \mathcal{A}^0_M$, and $s \in \mathcal{A}^0(E)$,

$$\nabla(f \cdot s) = \mathrm{d}f \otimes s + f \cdot \nabla(s)\ .$$

Proposition 1.1.24 *The set of connections on E form an affine space modeled on the vector space $\mathcal{A}^1(\underline{\mathrm{End}}(E))$ of differential 1-forms taking values in the complex vector bundle $\underline{\mathrm{End}}(E)$ of endomorphisms of E.*

A connection can be uniquely extended to a $\mathbb{C}$-linear form,

$$\nabla\colon \mathcal{A}^*(E) \to \mathcal{A}^*(E)$$

of degree 1, *i.e.* such that $\nabla(\mathcal{A}^p(E)) \subseteq \mathcal{A}^{p+1}(E)$. This extension is characterized by the Leibniz rule: for all $\omega \in \mathcal{A}^p_M$ and $s \in \mathcal{A}^0(E)$

$$\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^p \omega \otimes \nabla(s)\ .$$

Given a connection on a complex vector bundle E, the composition $\nabla^2\colon \mathcal{A}^*(E) \to \mathcal{A}^{*+2}(E)$ is a $\mathcal{A}^0_M$-linear operator, *i.e.* $\nabla^2(f\omega) = f\nabla^2(\omega)$, for all $\omega \in \mathcal{A}^q(E)$ and $f \in \mathcal{A}^0_M$.

Such an operator $\nabla^2\colon \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is the coupling with a differential form $\Theta \in \mathcal{A}^2(\underline{\mathrm{End}}(E))$ called the curvature of the connection ∇.

Definition 1.1.25 *We call the* total Chern form *of the connection ∇ the differential form*

$$c(E) := \det(1 + \frac{i}{2\pi}\Theta) = 1 + c_1(E,\nabla) + \cdots + c_r(E,\nabla) \in \mathcal{A}^*_M$$

where $c_i(E,\nabla) \in \mathcal{A}^{2i}_X$ are called the i-th Chern forms *of the connection ∇.*

Properties 1.1.26 The following are the main properties of Chern forms.

i. The differential forms $c_i(E,\nabla)$ are closed.

ii. The cohomology classes $c_i(E) \in \mathrm{H}^{2i}_{\mathrm{DR}}(M)$ of $c_i(E,\nabla)$ do not depend on the connection ∇ on E. They are called *Chern classes* of E. The cohomology class $c(E) = \sum c_i(E)$ is called *total Chern class* of E.

iii. If E and F are 2 vector bundles on M, their total Chern classes verify

$$c(E \oplus F) = c(E) \cdot c(F)$$

in $\mathrm{H}_{\mathrm{DR}}(M) = \oplus \mathrm{H}^i_{\mathrm{DR}}(M)$. In particular $c_1(E \oplus F) = c_1(E) + c_1(F)$

We see from this description that Chern classes of a complex vector bundle E depend only on the topology of E (and M).

Now let X be a complex manifold, and let $E \to X$ be a holomorphic vector bundle on X, carrying a Hermitian metric $h\colon E \times_X E \to \mathbb{C}$. Then there exists a unique connection ∇_h which is holomorphic, *i.e.* in any holomorphic frame field its connection form is of degree $(1,0)$, and which makes h parallel, *i.e.* verifies $\mathrm{d}(h(\xi,\eta)) = h(\nabla\xi,\eta) + h(\xi,\nabla\eta)$, for all $\xi,\eta \in \mathcal{A}^0(E)$.

Thus, we see that when we have a holomorphic and a Hermitian structure, then there are canonical Chern forms $c_i(E,h) = c_i(E,\nabla_h)$ that represent the Chern classes $c_i(E) \in \mathrm{H}^{2i}_{\mathrm{DR}}(X)$.

To see that these Chern classes coincide with those defined in the Chow ring and then in cohomology through the cycle map, one can observe that the De Rham cohomnology $\mathrm{H}_{\mathrm{DR}}(X) \to \mathrm{H}_{\mathrm{DR}}(\mathbb{P}(E))$ of projective bundles verifies properties analogue to property 1.1.12, that Chern classes verify an equation as equation 1.1 in the definition of Chern classes, and that the cycle maps γ_X and $\gamma_{\mathbb{P}(E)}$ are functorial.

1.2 Stable Bundles

1.2.1 Origins of stability

The origins of stability lie in the construction of moduli spaces. A moduli space, is a space classifying objects with some fixed invariants (*e.g.* varieties or vector bundles on a given variety, with fixed Chern classes).

By classify we mean that we want to describe that set of objects, by some other algebraic object. This can be done in two ways: either we find a scheme M whose closed points are in a "natural" bijection with the set of objects we want to describe. Here, natural means that whenever we have a family of objects parametrized by a scheme T, then we have a map from T to M, associating to a closed point $t \in T$ the point of M which correspond to the object in the family over the point t.

Or we find a "universal" family over the scheme M, such that there is a bijection between morphisms from T to M, and families of objects defined over any scheme T. This bijection associating to every morphism $T \to M$ the pull-back to T of the universal family on M.

The natural framework to explain and formalize what this means is the language of categories and representability of functors: we consider the functor from schemes to sets, associating to a scheme the set of families of objects over that scheme. Those two possibilities correspond respectively to a coarse moduli space (or a corepresentable functor), and a fine moduli space (or a representable functor).

We give some example of what we mean by fine moduli space.

Example 1.2.1 The most trivial example of a fine moduli space is the projective space: we describe first the classical definition of the projective space, and then the functorial one, comparing the two.

Let us fix a vector space V of finite dimension over $\mathbb{k}$. By definition a poiint of the projective space [3] is a linear subspace of V of dimension 1:

$$\mathbb{P}(V) := \{\ell \subset V \mid \dim \ell = 1\} .$$

This space is given an algebraic structure in the classical way. On the projective space, consider the subbundle $\mathcal{H} \subset V \times \mathbb{P}(V)$ of the trivial bundle defined in the following way:

$$\mathcal{H} := \{(v, \ell) \in V \times \mathbb{P}(V) \mid v \in \ell\} .$$

The line bundle $\mathcal{H}$ is called the tautological line bundle, and its invertible sheaf of local sections is $\mathcal{O}_{\mathbb{P}(V)}(-1)$. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{Q} \to 0$$

called the Euler sequence, and it can be shown that $\mathcal{T}_{\mathbb{P}(V)} = \mathcal{H}^* \otimes \mathcal{Q}$.

The functorial point of view: given a scheme T over $\mathbb{k}$, let us consider exact sequences

$$0 \to L \to V \otimes_{\mathbb{k}} \mathcal{O}_T \to M \to 0 , \quad (**)$$

where L and M are locally free sheaves on T, respectively of rank 1 and $n-1$.

[3] We follow in this thesis the "italian" notations, dual to Grothendieck's ones, where the projective space associated a vector space V is the space of homogeneous lines in V, rather than hyperplanes.

We say that two such exact sequences are equivalent if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & V\otimes\mathcal{O}_T & \to & M & \to & 0 \\ & & \downarrow\wr & & \| & & \downarrow\wr & & \\ 0 & \to & L' & \to & V\otimes\mathcal{O}_T & \to & M' & \to & 0 \end{array}$$

Then there is a contravariant functor

$$\begin{array}{cccc} \mathcal{P}_V\colon & \mathcal{S}chemes_{/\Bbbk} & \to & \mathcal{S}ets \\ & T & \mapsto & \{\text{sequences } (**)\}/\sim \end{array}$$

where the functor applied to a morphism of schemes $f\colon T \to S$ is the pull-back f^* on exact sequences $(**)$.

Proposition 1.2.2 *The functor $\mathcal{P}_V$ is represented by the projective space $\mathbb{P}(V)$.*

Proof.
To every morphism $f\colon T \to \mathbb{P}(V)$ we can associate the pull-back of the Euler sequence

$$0 \to f^*\mathcal{O}_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O}_T \to f^*\mathcal{Q} \to 0\,.$$

And given an exact sequence

$$0 \to L \to V \otimes_{\Bbbk} \mathcal{O}_T \to M \to 0\,, \quad (**)$$

we can consider the dual map $\varphi\colon V^* \otimes \mathcal{O}_T \twoheadrightarrow L^*$. We define a morphism $T \to \mathbb{P}(V)$ associating to a point $t \in T$ the line $\ell_t \subset V$ defined by the hyperplane $H_t \subset V^*$:

$$H_t := \{s \in V^* \mid (\varphi s)(t) = 0\}\,.$$

□

We see that, the functor $\mathcal{P}_V$ being represented by $\mathbb{P}(V)$, the closed points of $\mathbb{P}(V)$ correspond naturally to $\mathcal{P}_V(\operatorname{Spec}\Bbbk) = \{\ell \subset V\}$. And the universal object on $\mathbb{P}(V)$ is the Euler sequence.

We can say that the classical description of the projective space consists in giving the definition of the space through the properties of its points, while the functorial one looks at properties of families (in the case of projective space, families of linear subspaces parametrized by T), rather than single points.

This has the first advantage of being a "natural" construction, hence it can be generalized easily (*e.g.* when we want to construct the projectivization of a vector bundle).

Also, in some cases it gives a better frame to construct particular spaces. Mainly when when we want to parametrize objects which are naturally given in families (see the example of the Quot scheme below), or when we do not have a representable functor, but only corepresentable, as in the case of coarse moduli spaces of vector bundles.

Example 1.2.3 A natural generalization of the projective space, is the projectivization of a vector bundle E on a variety X. In this case the functor is

$$\begin{array}{rccc} \mathcal{P}_E\colon & \mathcal{S}ch_X & \to & \mathcal{S}ets \\ & t\colon T \to X & \mapsto & \{\text{sequences } 0 \to L \to t^*E \to M \to 0\}/\sim \end{array}$$

where $L \subset t^*E$ is a subbundle of dimension 1. And $\mathcal{P}_E$ is represented by the projectivization $\pi\colon \mathbb{P}(E) \to X$ of E on X. Then giving rise to a tautological sequence $0 \to \mathcal{O}_{\mathbb{P}(E)}(-1) \to \pi^*E \to Q \to 0$ on $\mathbb{P}(E)$.

In the same way the Grassmannian $\mathbb{G}\mathrm{r}(k, E) \to X$ is a scheme representing the functor of exact sequences of sheaves on a X-scheme T

$$0 \to N \to t^*E \to M \to 0\,,$$

with N and M locally free sheaves of ranks k and $n-k$.

One of the schemes constructed by Grothendieck in this framework, and most used in the theory of moduli spaces, is the *Quot* scheme: let us fix a noetherian scheme S, a projective S-scheme X, a coherent sheaf $\mathcal{E}$ on X, and a polynomial $P(d)$.

For an S-scheme $T \to S$, consider the coherent sheaf quotients $(*)\colon \mathcal{E}_T \twoheadrightarrow Q$ on $X \times_S T$, where $\mathcal{E}_T$ is the pull-back to $X \times_S T$ of $\mathcal{E}$, with the equivalence relation $(*) \sim (*)'$ if there is a commuting diagram

$$\begin{array}{ccccc} (*) & \mathcal{E}_T & \twoheadrightarrow & Q & \to & 0 \\ & \| & & \downarrow\wr & & \\ (*)' & \mathcal{E}_T & \twoheadrightarrow & Q' & \to & 0 \end{array}$$

Then there there is a contravariant functor $Q_{\mathcal{E},P(d)}$:

$$\begin{array}{ccc} \mathcal{S}ch_S & \to & \mathcal{S}ets \\ T & & \{\text{equivalence classes of quotients } (*) \\ \downarrow & \mapsto & \text{such that } Q \text{ is flat over } T \text{ and} \\ S & & \text{each } Q_t \text{ has Hilbert polynomial } P(d)\} \end{array}$$

where morphisms $f\colon T' \to T$ induce the map $\tilde{f}^* : Q_{\mathcal{E},P(d)}(T) \to Q_{\mathcal{E},P(d)}(T')$, with $\tilde{f} : X \times_S T' \to X \times_S T$.

Proposition 1.2.4 (Grothendieck) *The functor $Q_{\mathcal{E},P(d)}$ is represented by a projective S-scheme $Quot(\mathcal{E}, P(d))$.*

Proof. [Gro95].

□

If $S = \mathrm{Spec}(\Bbbk)$ then the closed points of $Quot(\mathcal{E}, P(d))$ are exactly the coherent quotient sheaves $\mathcal{E} \twoheadrightarrow Q$ with Hilbert polynomial $P(d)$.

Example 1.2.5 Let us consider the Quot scheme in the following case: $S = \mathrm{Spec}(\Bbbk)$, X is a variety over $\Bbbk$, and $\mathcal{E} = \mathcal{O}_X$ is the structure sheaf. Then the Quot scheme parametrizes subschemes of X with Hilbert polynomial $P(d)$. In fact the kernel of any quotient $\mathcal{O}_X \twoheadrightarrow Q$ is an ideal sheaf determining a subscheme $i\colon Z \hookrightarrow X$, and identifying Q with $i_*\mathcal{O}_Z$. In this case the Quot scheme is often called Hilbert's scheme. If the chosen polynomial is a constant positive integer $n \in \mathbb{N}$, then quotients of $\mathcal{O}_X$ having such Hilbert polynomial are 0-dimensional subschemes of X of lenght n, and the Hilbert's scheme of points is noted $X^{[n]}$. If X is a curve, then $X^{[n]}$ coincides with the symmetric product S^nX; if X is a surface, $X^{[n]}$ is smooth, and is a resolution of singularities of the symmetric product S^nX.

Those are examples of *fine moduli spaces*, *i.e.* schemes $\mathcal{M}$ representing a functor $\mathcal{F}\colon \mathcal{S}ch_S \to \mathcal{S}ets$. Hence any morphism from an object $T \in \mathcal{S}ch_S$ to the fine moduli space $\mathcal{M}$ corresponds to a unique element in $F(T)$. In particular the identity morphism $1_{\mathcal{M}}\colon \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ corresponds to a *universal* object $\mathcal{U} \in F(\mathcal{M})$.

However this is not always the case, and it may happen that a functor is not representable but only corepresentable, and in this case the scheme $\mathcal{M}$ corepresenting the functor is called a *coarse moduli space*. This is what happens in the case of moduli spaces of vector bundles over a variety, or moduli of curves. Even in those cases though, we parametrize only stable objects, and not all sort of vector bundles or curves.

The main ingredient to construct a moduli space of vector bundles (or torsion free sheaves) is Geometric Invariant Theory, which allows to describe quotients of schemes by the action of an algebraic group. This is used because when we want construct a space parametrizing all semistable vector bundles with given Chern classes, we rigidify the problem adding some extra structure. We are able then to construct a (fine) moduli space of vector bundles plus the extra structure. Eventually we want to forget the extra structure by identifying those points giving isomorphic vector bundles.

In particular, instead of considering a semistable (in a sense that we will specify) torsion free sheaf E with Hilbert polynomial P, we consider E and a basis for the vector space $H^0(X, E(m))$ for some fixed large integer m. If m is large enough this basis defines a quotient $\mathcal{O}_X(-m)^{P(m)} \twoheadrightarrow E$, hence a point of the Quot scheme $Quot(\mathcal{O}_X(-m)^{P(m)}, P)$. We have then an open subset of the Quot scheme, consisting of points coming from this construction, *i.e.* points $\mathcal{O}_X(-m)^{P(m)} \twoheadrightarrow F$ such that F is torsion free and semistable, and the quotient induces an isomorphism $\Bbbk^{P(m)} \xrightarrow{\sim} H^0(X, F(m))$. Then we want to divide out the ambiguity in the choice of the basis, by quotienting this open set by the action of $GL(P(m), \Bbbk)$.

In this operation of quotient comes out the notion of GIT-stability. We will not define in detail all the notions in Geometric Invariant Theory, but try to give an idea on the origin of GIT-stability and its relations with stability of vector bundles.

Definition 1.2.6 *Let X be a variety with an action of an algebraic group G. A G-*linearized sheaf *on X, is a coherent sheaf E on X, such that we have an isomorphism $\Phi\colon \sigma^*E \xrightarrow{\sim} p_1^*E$ of sheaves on $X \times G$, where $\sigma\colon X \times G \to X$ is the action, and p_1 is the projection $X \times G \to X$, satisfying the compatibility hypothesis for those morphisms on $X \times G \times G$:*

$$(\mathrm{id}_X \times \mu)^*\Phi = p_{12}^*\Phi \circ (\sigma \times \mathrm{id}_G)^*\Phi$$

where $\mu\colon G \times G \to G$ is the multiplication map, and $p_{12}\colon X \times G \times G \to X \times G$ is the projection on the first two factors.

*A morphism $\psi\colon E \to F$ of G-linearized sheaves is G-*equivariant *if $p_1^*\psi \circ \Phi = \Phi' \circ \sigma^*\psi$, where Φ and Φ' are the given linearizations of E and F respectively.*

Remark 1.2.7 If E is a G-linearized sheaf on X, the linearization $\Phi\colon \sigma^*E \xrightarrow{\sim} p_1^*E$ induces for all $g \in G$, an isomorphism

$$\Phi_g\colon g^*E \xrightarrow{\sim} E\,,$$

and the compatibility hypothesis (also called the cocycle condition) means that, for all g and h in G,

$$\Phi_{gh} = \Phi_{h\circ g} = \Phi_g \circ g^*\Phi_h\colon\ \ g^*h^*E \xrightarrow{\sim} g^*E \xrightarrow{\sim} E\,.$$

In the same way, the G-equivariance of a morphism $\psi\colon E \to F$ of G-linearized sheaves, is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} g^*E & \xrightarrow{g^*\psi} & g^*F \\ {\scriptstyle \Phi_g\,\wr}\big\downarrow & & \big\downarrow{\scriptstyle \wr\,\Phi'_g} \\ E & \xrightarrow{\psi} & F \end{array}$$

for all $g \in G$, where Φ and Φ' are the G-linearizations respectively of E and F.

Let us suppose that X is a projective $\Bbbk$ scheme, with an action of a reductive group G, and that L is a G-linearized ample line bundle. Then $X = Proj(R)$, where $R = \bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ is the ring of homogeneous coordinates. Let Y be the projective scheme associated to the homogeneous ring of G-invariant sections: $Y := Proj(R^G)$. Then $R^G \subset R$ induces a rational map $X \dashrightarrow Y$. This map is defined on points $x \in X$ such that there exist a G-invariant section $s \in H^0(X, L^{\otimes n})^G$ with $s(x) \neq 0$. Those are exactly semistable points:

Definition 1.2.8 *A point $x \in X$ is GIT-semistable with respect to a G-linearized ample divisor L, if there is an integer n, and an invariant global section $s \in H^0(X, L^{\otimes n})^G$, such that $s(x) \neq 0$.*

A point $x \in X$ is GIT-stable, if it is a GIT-semistable point, such that the stabilizer G_x is finite, and the G-orbit is closed in the set of GIT-semistable points of X.

The key point in GIT is the possibility of constructing good quotients Y for the action of G on semistable points of X.

In the construction of moduli spaces one verifies that, in the cases we are interested with, the GIT-(semi)stable points of the Quot scheme $Quot(\mathcal{O}_X(-m)^{P(m)}, P)$ with respect to a well chosen $GL(P(m), \Bbbk)$-invariant polarization, correspond to quotients $\mathcal{O}_X(-m)^{P(m)} \twoheadrightarrow F$ such that F is Gieseker (semi)stable, and the quotient induces an isomorphism $\Bbbk^{P(m)} \xrightarrow{\sim} H^0(X, F(m))$.

We will not define here Gieseker stability, as we will use another notion of stability called slope stability, or μ-stability, that we define in 1.2.9 below. It is related to Gieseker stability through the following implications

$$\mu\text{-stability} \Rightarrow \text{Gieseker stability} \Rightarrow \text{Gieseker semistability} \Rightarrow \mu\text{-semistability} .$$

In the case of vector bundles on projective curves, then μ-(semi)stability and Gieseker (semi)stability coincide.

A detailed description of the construction of moduli spaces of semistable sheaves on a variety can be found in chapter 4 of [HL97], and in chapter 7 of [LP95] in the case of vector bundles on curves.

1.2.2 Definition of stability

The definition of (semi)stability we will use is that of slope stability, or μ-stability, as it is the one that fits well in the context we will treat. As we are interested in vector bundles, we will consider only torsion free sheaves in our definition of stability.

In the following, stability will always mean slope stability with respect to a fixed polarization H. As we see from the definition below, stability with respect to H or to a positive multiple mH are equivalent. Hence we will suppose, when needed, that H is sufficiently positive or very ample.

We recall that a torsion free sheaf E is locally free on an open set $U \subseteq X$, such that $\text{codim}_X(X - U) \geqslant 2$. We say that the rank of E is the rank of the vector bundle $E_{|U}$, *i.e.* the rank of E at the generic point of X.

Definition 1.2.9 *Let E be a torsion free coherent sheaf on a variety X. We say that E is* semistable *(or* slope semistable*, or* μ-semistable*), with respect to a polarization H of X, if for any coherent proper subsheaf $F \hookrightarrow E$,* i.e. *such that $0 \neq F \neq E$, we have*

$$\mu_H(F) = \frac{c_1(F).H^{n-1}}{\text{rk}F} \leqslant \frac{c_1(E).H^{n-1}}{\text{rk}E} = \mu_H(E) . \qquad (1.2)$$

If the equation (1.2) holds with strict inequality for all $F \hookrightarrow E$ with $\text{rk}F < \text{rk}E$, then we say that E is stable*. If E is semistable not stable, we say that E is* strictly semistable*, or* properly semistable*. A torsion free sheaf which is not semistable is called* unstable*.*

Remark 1.2.10 We notice that if L is a line bundle, then a tosion free sheaf E is (semi)stable if and only if $E \otimes L$ is (semi)stable.

Remark 1.2.11 In the definition 1.2.9 of stability, we can suppose that the equation (1.2) holds for all saturated subsheaves $F \hookrightarrow E$, *i.e.* such that E/F is torsion free.

In fact if E/F is not torsion free we can consider its torsion T. We have then a commutative diagram

$$\begin{array}{ccccc} F & \hookrightarrow & E & \twoheadrightarrow & E/F \\ \downarrow & & \| & & \downarrow \\ F' & \hookrightarrow & E & \twoheadrightarrow & \frac{E/F}{T} \end{array}$$

where F' is called the saturation of F in E, and $F'/F \cong T$. As $c_1(T)$ is effective or 0, then $c_1(F).H^{-1} \leqslant c_1(F').H^{n-1}$, therefore $\mu(F) \leqslant \mu(F')$ as they have the same rank. To see that $c_1(T)$ is effective or vanishing, notice that $c_1(T) = c_1(F') \otimes c_1(F)^{-1}$, hence the associated line bundle has a section coming from the injection $F \hookrightarrow F'$.

For the same reason we see that it is sufficient to consider equation (1.2), only for subsheaves $F \hookrightarrow E$ with $\text{rk}F < \text{rk}E$.

Remark 1.2.12 In the definition of stability we can consider torsion free quotient sheaves $E \twoheadrightarrow G$ instead of subsheaves $F \hookrightarrow E$.

Then E is (semi)stable if and only if $\mu(E) < \mu(G)$ for every torsion free quotient $E \twoheadrightarrow G$ ($\leqslant$ for semistability).

In fact from remark 1.2.11 we can consider equation (1.2) for subsheaves $F \hookrightarrow E$ eith torsion free quotient.

And from the exact sequance $0 \to F \to E \to G \to 0$ we deduce that

$$c_1(E).H^{n-1} = c_1(F).H^{n-1} + c_1(G).H^{n-1} \text{ and } \mu(E) = \frac{\text{rk}F\mu(F) + \text{rk}G\mu(G)}{\text{rk}F + \text{rk}G} .$$

Remark 1.2.13 When considering stability on curves, we have

$$c_1(F).1 = c_1(F).[X] = \deg F$$

for every vector bundle, and we can restrict the equation 1.2 to *subbundles* $F \subset E$: we apply remark 1.2.11 and the fact that torsion free sheaves on curves are locally free.

The following are the first properties of stable bundles, allowing us to have some examples

Proposition 1.2.14 *Let E and F be two semistable torsion free sheaves on a variety X. Then if $Hom(E, F) \neq 0$, $\mu(E) \leqslant \mu(F)$. If E is stable, F is semistable, and $\mu(E) = \mu(F)$, then any nontrivial homomorphism $\varphi\colon E \to F$ is injective.*

If E and F are stable vector bundles, and $\mu(E) = \mu(F)$, then any non trivial homomorphism $\varphi\colon E \to F$ is an isomorphism.

Proof.

Let $\varphi\colon E \to F$ be a morphism between the two semistable sheaves E and F, and let $I \subseteq F$ be the image of φ. Then, if $I \neq 0$, by semistability $\mu(E) \leqslant \mu(I) \leqslant \mu(F)$.

If furthermore we have $\mu(E) = \mu(F)$, then $\mu(E) = \mu(I) = \mu(F)$. If E is stable this implies $E = I$, *i.e.* the morphism is injective. In fact I cannot have a smaller rank than E otherwise it would contradict the stability of E. And the map $E \to I$ cannot have a rank 0 kernel because E is torsion free.

And if F is stable, and E and F are vector bundles, this implies that $I = F$, *i.e.* the morphism is surjectve. In fact, E cannot have smaller rank than F, otherwise it would contradict the stability of F. And an injective morphism between vector bundles of the same rank is either an isomorphism or has cokernel supported on an effective divisor, but this last case is not possible because $c_1(E).H^{n-1} = c_1(F).H^{n-1}$.

□

Corollary 1.2.15 *All stable sheaves are simple,* i.e. *their endomorphisms are scalar multiples of the identity.*

Proof.

For every coherent sheaf F, the algebra $End(F)$ has finite dimension over $\mathbb{k}$.

If E is a stable sheaf, then by proposition 1.2.14 $End(E)$ is an extension field of $\mathbb{k}$. As we are supposing that $\mathbb{k}$ is algebraically closed, then $End(E) = \mathbb{k}$.

□

Example 1.2.16 Here are some examples of stable vector bundles:

i. **Line bundles:** every line bundle is trivially stable, as it does not contain any nontrivial subsheaf with torsion free quotient.

ii. **Tangent bundle of a projective space.** Consider a projective space $\mathbb{P}(V)$, with V a $\mathbb{k}$-vector space of dimension $n+1$. We can suppose that $n > 1$ otherwise we have nothing to prove. We have the Euler sequence described in example 1.2.1

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(V)} \to Q \to 0$$

where $Q = T_{\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)$. By remark 1.2.10 it is sufficient to prove the stability of Q.

Let us suppose that Q is not stable, then there is a quotient $Q \twoheadrightarrow F$ with $\mu(F) < \mu(Q) = 1/n$. As $\mathrm{rk}F < \mathrm{rk}Q$, then we have $\mu(F) \leqslant 0$. Without lack of generality we can suppose that F is stable. As we have $V \otimes \mathcal{O}_{\mathbb{P}(V)} \twoheadrightarrow F$, then $\mu(V \otimes \mathcal{O}_{\mathbb{P}(V)}) = 0 \leqslant \mu(F)$, by proposition 1.2.14, so $\mu(F) = 0$. As there must be one of the factors $\mathcal{O}$ of $V \otimes \mathcal{O}_{\mathbb{P}(V)}$ that maps to F, and F is stable of slope 0, then we deduce that $\mathrm{rk}F = 1$, and we have $\mathcal{O} \hookrightarrow F$, with cokernel of codimension at least 2. But dualizing this implies that we have a global section $\mathcal{O} \to Q^*$, and this is impossible because of the dual of the Euler sequence

$$0 \to Q^* \to V^* \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

where $V^* = H^0(\mathbb{P}(V), \mathcal{O}(1))$.

iii. **Extensions of line bundles on curves.** Consider a line bundle L on a genus $g \geqslant 1$ curve. Then $Ext^1(L(p), L) = H^1(C, \mathcal{O}_C(-p)) \cong H^0(C, \omega_C(p)) \neq 0$, and let E the rank 2 vector bundle corresponding to a non trivial extension. Then E is a stable bundle: in fact from the exact sequence

$$0 \to L \to E \to L(p) \to 0$$

we deduce that the slope of E is $\mu(E) = \deg E/2 = \deg L + 1/2$. A subbundle of E is a line bundle F either injecting to L, or having a non zero map to $L(p)$. As a non zero map between two line bundles is always an injection of sheaves, and $F \to L(p)$ cannot be an isomorphism as the sequence does not split, we deduce that $\deg F \leqslant \deg L < \mu(F)$.

iv. **Direct sums.** Let E and F be two semistable shaves of the same slope $\mu(E) = \mu(F) = \mu$. Then $E \oplus F$ is semistable of slope μ

In fact $E \oplus F$ is easily seen to have slope μ. Then assume we have $G \to E \oplus F$ such that $\mu(G) > \mu$. We can suppose that G is semistable, then by proposition 1.2.14, the maps $G \to E$ and $G \to F$ are the zero morphism. Hence $G \to E \oplus F$ is zero.

v. **Tangent bundle of a complex K3 surface.** This is a consequence of the Hitchin-Kobayashi correspondence (cf. section 1.3), and the existence of Kähler-Einstein metrics on a K3 surface.

Definition 1.2.17 *A vector bundle which is a direct sum of* stable *vector bundles of the same slope is called* polystable*.*

Example 1.2.18 A vector bundle which is a direct sum $E_1 \oplus E_2$ of two vector bundles with different slope is unstable: the vector bundle E_1 with higher slope injects into the direct sum, and

$$\mu(E_1) > \mu(E_1 \oplus E_2) = \frac{\mathrm{rk} E_1 \mu(E_1) + \mathrm{rk} E_2 \mu(E_2)}{\mathrm{rk} E_1 + \mathrm{rk} E_2} > \mu(E_2)\,.$$

In the same way any extension $0 \to E \to F \to G \to 0$ is unstable if $\mu(E) > \mu(G)$.

1.2.3 Harder-Narasimhan Filtration

In this section we show how to associate to every vector bundle E on X, a filtration that measures how far it is from being semistable.

The key point is the fact that every vector bundle contains a unique maximal semistable subsheaf:

Definition 1.2.19 *Let E be a torsion free sheaf on a polarized variety (X, H), a subsheaf $F \subseteq E$ is called* maximal semistable subsheaf *of E, if every subsheaf $G \subseteq E$ verifes $\mu(G) \leqslant \mu(F)$, and $G \subseteq F$ whenever $\mu(G) = \mu(F)$.*

Remark 1.2.20 By definition if such a subsheaf exists, it is semistable and unique.

Proposition 1.2.21 *Every torsion free sheaf E admits a* maximal semistable subsheaf *$F \subseteq E$.*

Proof.

Consider the following order relation $\preceq$ on the set of non-trivial subsheaves of E: we say that $F_1 \preceq F_2$ if and only if $F_1 \subseteq F_2$ and $\mu(F_1) \leqslant \mu(F_2)$. As any ascending chain of subsheaves terminates, there is a $\preceq$-maximal element $F \subset E$. Furthermore for every subsheaf $G \subseteq E$ there is a $G \subseteq G' \subseteq E$ such that G' is $\preceq$-maximal. Let F be of minimal rank among $\preceq$-maximal subsheaves of E. Then we can show that F is a maximal semistable subsheaf.

Suppose there exists a $G \subseteq E$ such that $\mu(G) \geqslant \mu(F)$. Then we can assume that $G \subseteq F$, in fact otherwise we have $\mu(G \cap F) > \mu(G) \geqslant \mu(F)$: indeed if $G \nsubseteq F$ then F is a proper subsheaf of $F + G$, and hence $\mu(F) > \mu(F + G)$. Using the exact sequence

$$0 \to F \cap G \to F \oplus G \to F + G \to 0$$

we find that

$$c_1(F).H^{n-1} + c_1(G).H^{n-1} = c_1(F \oplus G).H^{n-1} = c_1(F \cap G).H^{n-1} + c_1(F + G).H^{n-1}$$

and

$$\mathrm{rk}F + \mathrm{rk}G = \mathrm{rk}(F \oplus G) = \mathrm{rk}(F \cap G) + \mathrm{rk}(F + G)\ .$$

Hence we have that

$$\begin{aligned}&\mathrm{rk}(F \cap G)(\mu(G) - \mu(F \cap G)) = \\ &\quad = \mathrm{rk}(F + G)(\mu(F + G) - \mu(F)) + (\mathrm{rk}G - \mathrm{rk}(F \cap G))(\mu(F) - \mu(G))\end{aligned}$$

and using that $\mu(F) < \mu(G)$ and $\mu(F) > \mu(F + G)$, this implies that

$$\mu(F) \leqslant \mu(G) < \mu(F \cap G)\ .$$

So let us suppose there exists $G \subset F$ with $\mu(G) > \mu(F)$, and assume that G is $\preceq$-maximal among subsheaves of F. Then there exists $G' \subseteq E$ which is $\preceq$-maximal such that $G \subseteq G'$. In particular we have $\mu(F) < \mu(G) \leqslant \mu(G')$.

As both F and G' are $\preceq$-maximal, then we cannot have $G' \subseteq F$, otherwise we would either contradict the minimality of rk$F$, or the fact that $\mu(F) < \mu(G')$. Therefore F is a proper subsheaf of $F + G'$, and by maximality of F we have $\mu(F) > \mu(F + G')$. As above, the inequalities $\mu(F) < \mu(G')$ and $\mu(F) \geqslant \mu(F + G')$ imply

$$\mu(F \cap G') > \mu(G') \geqslant \mu(G) .$$

As $G \subset G' \cap F \subset F$, this means that $G \prec G' \cap F$ and they are both subsheaves of F, so we contradict the assumption on G.

□

Definition 1.2.22 *Let E be a torsion free sheaf. A* Harder-Narasimhan filtration *for E is an increasing filtration*

$$0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_l(E) = E ,$$

such that the factors $gr_i^{\mathrm{HN}} = \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ are semistable torsion free sheaves for $i = 1, \ldots, l$, with slopes satisfying

$$\mu_{max}(E) := \mu(gr_1^{\mathrm{HN}}) > \mu(gr_2^{\mathrm{HN}}) > \cdots > \mu(gr_l^{\mathrm{HN}}) =: \mu_{min}(E) .$$

Proposition 1.2.23 *Every torsion free sheaf E has a unique Harder-Narasimhan filtration.*

Proof.

We can proceed by induction on the rank of E: if E has rank 1 we have nothing to prove, assume that E has rank r and is not semistable. Then E has a unique proper maximal semistable subsheaf $E_1 \subset E$. Then E/E_1 has smaller rank than E and so admits a unique Harder-Narasimhan filtration, pulling it back to E we get the desired filtration.

Also for uniqueness part we can use induction, so let us assume that there is a unique Harder-Narasimhan filtration for sheaves of smaller rank than E, and suppose we have two filtrations $E_\bullet$ and $E'_\bullet$. We can suppose that $\mu(E'_1) \geqslant \mu(E_1)$, and let j be minimal such that $E'_1 \subset E_j$. Then we have a non-trivial homomorphism composing

$$E'_1 \to E_j \to E_j/E_{j-1} .$$

By proposition 1.2.14, this implies $\mu(E_j/E_{j-1}) \geqslant \mu(E'_1) \geqslant \mu(E_1) \geqslant \mu(E_j/E_{j-1})$. Then $\mu(E_j/E_{j-1}) = \mu(E'_1) = \mu(E_1) = \mu(E_j/E_{j-1})$, implying that $j = 1$, and so $E'_1 \subset E_1$. But as they are semistable, by proposition 1.2.14 $\mu(E'_1) \leqslant \mu(E_1)$, and we can repeat the argument inverting E'_1 and E_1. So $E'_1 = E_1$, and we can apply the induction on E/E_1.

□

1.2.4 Stability and restrictions

If a vector bundle E on a variety X is stable, it is difficult in general to know whether its restriction to a subvariety is stable or not. Some general results assert stability of the restricion of a stable vector bundle on an ample hypersurface of high degree (notably, theorems of Flenner [Fle84], Mehta and Ramanathan [MR82], Bogomolov [Bog93]).

We will treat one example of stability on a curve in Chapter 3, where we want to know whether the tangent bundle of the projective space restricted to a given curve is stable (see theorem 3.1.8).

On the converse, it is more esasy to show that the stability of the vector bundle $E_{|Y}$ restriction of E on a smooth ample hypersurface Y in $|mH|$ implies stability with respect to H.

We will use the following elementary useful lemma:

Lemma 1.2.24 *Let F be a torsion free sheaf on X, locally free outside of a closed subset $Z \subset X$ of codimension at least 3 in X. Let Y be a smooth hypersurface, and let $H \in \mathrm{CH}^1(X)$ its rational equivalence class. Then*

$$< c_1(F).H^{n-1} >_X = < c_1(F_{|Y}).(H_{|Y})^{n-2} >_Y ,$$

where $<>_X$ and $<>_Y$ are the degree maps on X and Y.

Proof. We remark at first that, if $i\colon Y \hookrightarrow X$ is the immersion of Y in X, then

$$c_1(F_{|Y}) = c_1(i^*F) = i^*c_1(F) .$$

This is clear when F is a vector bundle. Otherwise note that $c_1(F_{|Y})$ and $i^*c_1(F)$ are two line bundles on Y, isomorphic on an open subset $U = Y \setminus Z$ whose complementary $Y \cap Z$ has codimension at least 2. So they are isomorphic. Hence $c_1(F_{|Y}).(H_{|Y})^{n-2} = i^*(c_1(F).H^{n-2})$.

To show the lemma observe that, as i is an immersion, for any cycle $W \in C^*(Y)$, $< W >_Y = < i_*W >_X$. And by the projection formula

$$i_*(c_1(F_{|Y}).(H_{|Y})^{n-2}) = i_*([Y].i^*(c_1(F).H^{n-2})) = i_*([Y]).c_1(F).H^{n-2} ,$$

where $[Y]$ is the identity in $\mathrm{CH}^*(Y)$, and clearly $i_*([Y]) = H$.

□

Theorem 1.2.25 *Let E be a vector bundle on a variety X of dimension $n \geqslant 2$. Assume that $E_{|Y}$ is (semi)stable on a fixed ample smooth hypersurface $Y \in |mH|$, with respect to the polarization $H_{|Y}$. Then E is (semi)stable on X with respect to the polarization H.*

Proof.

We want to prove that given $F \hookrightarrow E$, we have $\mu_H(F) < \mu_H(E)$, knowing that this inequality holds on a given ample hypersurface $Y \in |mH|$.

As we remarked in 1.2.11 we can assume that the quotient $G := E/F$ is torsion free.

Furthermore we can assume that F is a reflexive sheaf, *i.e.* it satisfies $F \xrightarrow{\sim} F^{**}$, where the dual of a sheaf $\mathcal{F}$ is defined as the sheaf of homomorphisms $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Otherwise we have the following diagram

$$\begin{array}{ccccc} F & \hookrightarrow & E & \twoheadrightarrow & G \\ \downarrow & & \| & & \downarrow \\ F^{**} & \hookrightarrow & E & \twoheadrightarrow & G' \end{array}$$

where F^{**} maps in E because E is a vector bundle and the homomorphism sheaf from F to E is $\mathcal{H}om_{\mathcal{O}_X}(F, E) = \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X) \otimes E = F^* \otimes E = (F^{**})^* \otimes E = \mathcal{H}om_{\mathcal{O}_X}(F^{**}, E)$. But as we supposed that G is torsion free it cannot contain a torsion sheaf $T := \ker(G \to G') \cong \operatorname{coker}(F \to F^{**})$.

Reflexive sheaves $\mathcal{F}$ are exactly those torsion free sheaves satisfying Serre's condition S_2: for all schematic points $x \in X$,

$$\operatorname{depth}(\mathcal{F}_x) \geqslant \min\{2, \dim(\mathcal{O}_{X,x})\} .$$

This implies that their singular locus

$$\operatorname{sing}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\} = \{x \in X \mid \operatorname{dh}(\mathcal{F}_x) \neq 0\}$$

has codimension at least 3.

Restricting the sequence $0 \to F \to E \to G \to 0$ to the hypersurface Y we get

$$0 \to \mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y) \to F_{|Y} \to E_{|Y} \to G_{|Y} \to 0 .$$

As $\operatorname{codim}_X \operatorname{sing}(F) \geqslant 3$ in X, then $\operatorname{codim}_Y \operatorname{sing}(F_{|Y}) \geqslant 2$ in Y.

As $\operatorname{supp}(\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y)) \subset \operatorname{sing}(G) \cap Y$, and $\operatorname{codim}_X \operatorname{sing}(G) \geqslant 2$ in X, then $\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y)$ is a torsion sheaf on Y, injecting in $F_{|Y}$. So its support must be contained in $\operatorname{sing}(F_{|Y})$.

Hence $c_1(F_{|Y}) = c_1(F_{|Y}/\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y))$, because sheaves concentrated on high codimension subsets do not affect the first Chern class.

On Y we have the following exact sequence

$$0 \to F_{|Y}/\mathcal{T}or^1_{\mathcal{O}_X}(G, \mathcal{O}_Y) \to E_{|Y} \to G_{|Y} \to 0 .$$

Hence by stability of $E_{|Y}$ we have that

$$\frac{c_1(F_{|Y}).H^{n-2}}{\operatorname{rk} F} < \frac{c_1(E_{|Y}).H^{n-2}}{\operatorname{rk} E} ,$$

where $\dim X = n$. As $\operatorname{codim}_X \operatorname{sing}(F) \geqslant 3$, and E is a vector bundle, then by lemma 1.2.24 $c_1(F_{|Y}).H^{n-2} = c_1(F).H^{n-1}$, and $c_1(E_{|Y}).H^{n-2} = c_1(E).H^{n-1}$, so we get stability of E on X.

Same thing for semistability.

□

1.2.5 Stability and regularity

For vector bundles on curves, stability gives some interesting constraints on global generation and cohomology.

Proposition 1.2.26 *Let E be a semistable vector bundle on a genus g curve. Then the following hold:*

i. if $\mu(E) < 0$, then $H^0(E) = 0$;

ii. if $\mu(E) > 2g - 2$, then $H^1(E) = 0$;

iii. if $\mu(E) > 2g - 1$, then E is globally generated.

Proof. For the first point, observe that if $H^0(E) \neq 0$, then any section gives an injection of sheaves $\mathcal{O} \hookrightarrow E$.

For the second we use Serre's duality, which implies that $H^1(E) \cong H^0(E^* \otimes \omega)^*$. As $\mu(E^* \otimes \omega) = -\mu(E) + 2g - 2$ and $E^* \otimes \omega$ is semistable as well, we can use the first point.

For global generation, let us consider the exact sequence of evaluation on a point p

$$0 \to E(-p) \to E \to E_{|p} \to 0 \, .$$

As $\mu(E(-p)) = \mu(E) - 1 > 2g - 2$ and $E(-p)$ is semistable, we can pass to cohomology and use the second point.

□

1.3 Hermite-Einstein metrics

We define here Hermite-Einstein metrics, and enunciate the Kobayashi-Hitchin correspondence between those metrics and stability.

Let us consider a hermitian holomorphic vector bundle (E, h) on a complex projective variety X.

We have seen in paragraph 1.1.3, that there is a unique holomorphic hermitian connection ∇_h. This connection gives rise to a curvature form $\Theta_h \in \mathcal{A}^2(\underline{\mathrm{End}}(E))$. Contracting this form with the adjoint Λ of the Lefschetz operator (depending on the Kähler metric on X), we obtain the *mean curvature* endomorphism $R_h := i\Lambda\Theta_h \in \mathcal{A}^0(\underline{\mathrm{End}}(E))$.

Definition 1.3.1 *We say that (E, h) is a Hermite-Einstein vector bundle, or that h satisfies the Einstein-Hermite condition, if there exists a constant $\lambda \in \mathbb{R}$ such that $R_h = \lambda \cdot Id$.*

The following theorem is a deep result, relating stability with the existence of Hermite-Einstein metrics. It was described by Kobayashi as a higher dimensional analogue of a characterization, by Narasimhan and Seshadri (cf. [NS65]), of stable vector bundles on curves as those admitting projectively flat unitary connections. It was proved by Donaldson in the case of projective surfaces (cf.

[Don85]), and then for all projective variety in [Don87]. Uhlenbeck and Yau, proved it for compact Kähler manifolds (cf. [UY86]).

We will not use this characterization in the rest of this work, as we will deal only with the algebraic side of stability. As it has been one of the motivations for the constructions that follow, we think it is useful to enunciate this correspondence:

Theorem 1.3.2 (Kobayashi-Hitchin Correspondence) *Let E be a holomorphic vector bundle on a smooth polarized projective variety $(X, \mathcal{O}_X(1))$ over $\mathbb{C}$. Then E is polystable if and only if it admits a ("unique") Hermite-Einstein metric with respect to the Kähler metric of X induced by $\mathcal{O}_X(1)$.*

Thus, we see that a polystable vector bundle E, admits some "canonical" closed differential forms, coming from its unique Hermite-Einstein metric, that represent Chern classes of E.

Chapter 2

Stable bundles and Chow ring

2.1 Introduction

In this chapter, X is a smooth projective variety over an algebraically closed field $\mathbb{k}$, with a fixed polarization H.

The main result shows that the ideal sheaf $\mathcal{I}_Z$ of an effective cycle $Z \subset X$ admits a resolution by polystable vector bundles. In particular, this implies that the rational Chow ring $\mathrm{CH}^*_{\mathbb{Q}}(X)$, the K-theory $K(X)$, and the derived category $\mathcal{D}^b(X)$ are generated (in a sense that we will specify) by stable vector bundles.

Note that, through the Harder-Narashiman and Jordan-Hölder filtrations, it is easy to see that Chern classes of stable not necessary locally free sheaves generate $\mathrm{CH}^*_{\mathbb{Q}}(X)$ or $K(X)$ (cf. Remark 2.3.6). Since polystability for vector bundles on complex varieties is equivalent to the existence of Hermite-Einstein metrics, it seems desirable to work with the more restrictive class of locally free stable sheaves.

In the case of a $K3$-surface S our result can be compared with a recent article of Beauville and Voisin. In [BV04] they show that all points lying on rational curves are rationally equivalent, hence giving rise to the same class $c_X \in \mathrm{CH}^2(S)$, and that $c_2(S)$ and the intersection product of two Picard divisors are multiples of that class. A new formulation of this fact, stating that polynomial cohomological relations involving only $\mathrm{CH}^1(X)$ and the Chern classes of X are satisfied already in $\mathrm{CH}(X)$, can be found in [Voi06], where it is verified on some cases of hyper-Kähler manifolds.

As the tangent bundle T_S and line bundles are stable, one might wonder what happens if we allow arbitrary stable bundles. Our result shows that second Chern classes of stable bundles generate (as a group) $\mathrm{CH}^2(S)$, and that this is true on every surface.

Related results, using the relation between moduli spaces and Hilbert schemes (cf. [GH96]), and between Hilbert schemes and the second Chow group (cf. [Mac03]), had been obtained before.

We will first show the main theorem in the case of surfaces, as it already gives the above description for $\mathrm{CH}^2(X)$. The higher dimensional case is a generalization of this argument.

2.1.1 Notations

Stability will always mean slope stability with respect to the fixed polarization H as in definition 1.2.9. Since stability with respect to H or to a multiple mH are equivalent, we can suppose that H is sufficiently positive, in particular it is globally generated.

2.2 Zero dimensional cycles on a surface

Throughout this section S will be a smooth projective surface, Z a 0-dimensional subscheme of S, and $C \in |H|$ a fixed smooth curve such that $C \cap Z = \emptyset$. As H is very positive, we suppose $g(C) \geqslant 1$.

We will show the following

Proposition 2.2.1 *If $m \gg 0$, and if $V \subset H^0(S, \mathcal{I}_Z(mH))$ is a generic subspace of dimension $h^0(C, \mathcal{O}_C(mH))$, then the sequnce*

$$0 \to \ker(ev) \to V \otimes \mathcal{O}_S \xrightarrow{ev} \mathcal{I}_Z(mH) \to 0 \tag{2.1}$$

is exact and defines a stable vector bundle $M_{Z,m} := \ker(ev)$.

2.2.1 Proof of proposition 2.2.1

We remark that if a subspace $V \subset H^0(S, \mathcal{I}_Z(mH))$ generates $\mathcal{I}_Z(mH)$, the exact sequence

$$0 \to M \to V \otimes \mathcal{O}_S \to \mathcal{I}_Z(mH) \to 0 \tag{2.2}$$

defines a vector bundle M, for S has cohomological dimension 2.

By theorem 1.2.25 a vector bundle on a variety is stable if its restriction to a smooth hypersurface linearly equivalent to a multiple of H is stable.

So it is sufficient to show that the restriction of M to the curve C is a stable vector bundle. As the chosen curve C doesn't intersect Z, the restriction of (2.2) to C yields a short exact sequence:

$$0 \to M|_C \to V \otimes \mathcal{O}_C \to \mathcal{O}_C(mH) \to 0\,. \tag{2.3}$$

We want to choose the space V so that the sequence (2.3) equals:

$$0 \to M_{\mathcal{O}_C(mH)} \to H^0(C, \mathcal{O}_C(mH)) \otimes \mathcal{O}_C \to \mathcal{O}_C(mH) \to 0\,. \tag{2.4}$$

In this case, by general results (cf. [But94], and Theorem 2.3.2 below), the vector bundle $M|_C = M_{\mathcal{O}_C(mH)}$ is stable for $m \gg 0$.

We will use the following lemmas:

Lemma 2.2.2 *For $m \gg 0$, the restriction map*

$$H^0(S, \mathcal{I}_Z(mH)) \to H^0(C, \mathcal{O}_C(mH))$$

induces an isomorphism between a generic subspace $V \subset H^0(S, \mathcal{I}_Z(mH))$ of dimension $h^0(C, \mathcal{O}_C(mH))$, and $H^0(C, \mathcal{O}_C(mH))$.

Proof. This follows immediately from the vanishing of $H^1(S, \mathcal{I}_Z((m-1)H))$ for $m \gg 0$, and from the consideration that, in the grassmanian $Gr(h^0(C, \mathcal{O}_C(mH)), H^0(S, \mathcal{I}_Z(mH)))$, the spaces V avoiding the subspace $H^0(S, \mathcal{I}_Z((m-1)H))$ form an open subset, and project isomorphically on $H^0(C, \mathcal{O}_C(mH))$.

□

So if we show that such a space generates $\mathcal{I}_Z(mH)$, then the sequence (2.2) restricted to the curve will give the sequence (2.4).

Since the dimension $h^0(C, \mathcal{O}_C(mH))$ of such V grows linearly in m, this is a consequence of a general lemma which is true for a variety of any dimension:

Lemma 2.2.3 *Let Y be a variety of dimension n, E a vector bundle of rank r globally generated on Y, $\mathcal{F}$ a coherent sheaf on Y, and H an ample divisor. Then:*

i. If $W \subset H^0(Y, E)$ is a generic subspace of dimension at least $r + n$, then W generates E;

ii. There are two integers $R, m_0 \geqslant 0$, depending on Y, H, and $\mathcal{F}$, such that for any $m \geqslant m_0$, if $V \subset H^0(Y, \mathcal{F}(mH))$ is a generic subspace of dimension at least R, then V generates $\mathcal{F}(mH)$.

Proof.

i. Let $W \subset H^0(Y, E)$ be a generic subspace of dimension v. Then the closed subscheme $Y_s \subset Y$ where the evaluation homomorphism $W \otimes \mathcal{O}_Y \to E$ has rank less than or equal to s is either empty or of codimension $(v-s)(r-s)$ (cf. [HL97] ch.5, p.121)[1]. Hence, taking $v = \dim W \geqslant r + n$, and $s = r - 1$, we see that the evaluation map must be surjective.

ii. By Serre's theorem there exists a $m_1 \geqslant 0$ such that $\mathcal{F}(mH)$ is globally generated and acyclic for any $m \geqslant m_1$. Hence, there exists a (trivial) globally generated vector bundle E of rank $r = h^0(Y, \mathcal{F}(m_1 H))$ and a surjection $E \twoheadrightarrow \mathcal{F}(m_1 H)$; if we call $\mathcal{K}$ its kernel, then $\mathcal{K}(mH)$ is globally generated and acyclic for any $m \geqslant m_2$, and we have for all $m \geq m_1 + m_2$:

$$0 \to H^0(Y, \mathcal{K}((m - m_1)H)) \to H^0(Y, E((m - m_1)H)) \to H^0(Y, \mathcal{F}(mH)) \to 0\,.$$

Let now ν be an integer such that $r + n \leq \nu \leq h^0(Y, \mathcal{F}(mH))$. In $Gr(\nu, H^0(Y, E((m - m_0)H)))$ there is the open subset of the spaces W avoiding $H^0(Y, \mathcal{K}((m - m_0)H))$, and this open set surjects to $Gr(\nu, H^0(Y, \mathcal{F}(mH)))$.

[1] in [HL97] is used a transversal version of Kleiman's theorem which holds only in caracteristic 0, but the dimension count we need is true in any characteristic (see [Kle74]).

So a generic $V \subset H^0(Y, \mathcal{F}(mH))$ of dimension ν lifts to a generic $W \subset H^0(Y, E((m-m_0)H))$ of dimension ν, and since $\nu \geq r+n$, the first part of this lemma gives the result.

□

Lemmas 2.2.2 and 2.2.3 immediately yield Proposition 2.2.1.

2.2.2 Generators for the Chow group of a surface

We have shown that any effective 0-cycle Z admits a resolution

$$0 \to M_{Z,m} \to V \otimes \mathcal{O}_S \to \mathcal{I}_Z(mH) \to 0\,, \tag{2.5}$$

where $M_{Z,m}$ is stable and locally free.

Corollary 2.2.4 *The Chow group* $\mathrm{CH}^2(S)$ *is generated as a group by*

$$\{c_2(M) \mid M \textit{ is a stable vector bundle}\}\,.$$

Proof. The class of Z in $\mathrm{CH}^2(S)$ is given by $[Z] = -c_2(\mathcal{O}_Z)$, hence,

$$c_2(\mathcal{I}_Z) = [Z]\,,$$

furthermore we know that $c_1(\mathcal{O}_Z) = c_1(\mathcal{I}_Z) = 0$.

Using the sequences

$$0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \text{ and,}$$

$$0 \to \mathcal{I}_Z((m-1)H) \to \mathcal{I}_Z(mH) \to \mathcal{O}_C(mH) \to 0\,,$$

we can easily calculate the Chern classes appearing in (2.5):

$$c_1(\mathcal{I}_Z(mH)) = mH \text{ and } c_2(\mathcal{I}_Z(mH)) = c_2(\mathcal{I}_Z) = [Z]\,.$$

So by the sequence (2.5) we obtain

$$c_1(M_{Z,m}) = -c_1(\mathcal{I}_Z(mH)) = -mH\,, \text{ and}$$

$$[Z] = c_2(\mathcal{I}_Z(mH)) = -c_2(M_{Z,m}) + m^2H^2\,,$$

thus second Chern classes of stable vector bundles and the class of H^2 generate the second Chow group of the surface.

Clearly, $H^2 = c_2(H \oplus H)$ is the second Chern class of a polystable vector bundle, but it can also be obtained as a linear combination of $c_2(E_i)$ with E_i stable: since $H^2 = [Z']$ is an effective cycle, we deduce from (2.2) that

$$[Z'] = -c_2(M_{Z',m}) + m^2[Z'] \text{ or, equivalently}$$

$$(m^2-1)H^2 = (m^2-1)[Z'] = c_2(M_{Z',m})$$

for every $m \gg 0$. Choosing m_1 and m_2 such that (m_1^2-1) and (m_2^2-1) are relatively prime, we find that H^2 is contained in the subgroup of $\mathrm{CH}^2(S)$ generated by second Chern classes of stable vector bundles.

□

Remark 2.2.5 This result can also be proven (when $\mathrm{char}(\Bbbk) = 0$) by using the fact that, for every $r > 0$, c_1, and $c_2 \gg 0$, stable locally free sheaves form an open dense subset U in the moduli space $N = N(r, c_1, c_2)$ of semi-stable not necessarily locally free sheaves with fixed rank and homological Chern classes (see [O'G96]).

For any such N, up to desingularizing compactifying and passing to a finite covering, we obtain a homomorphism $\phi_{c_2} : \mathrm{CH}_0(N) \to \mathrm{CH}_0(S)$, which associates the class of a point $E \in N$ to the class $c_2(E) \in \mathrm{CH}_0(S)$. This morphism is given by the correspondance $c_2(F)$, where F is the universal sheaf on $N \times S$.

Next we notice that $\mathrm{CH}_0(U)$ spans $\mathrm{CH}_0(N)$: in fact if we consider a point $x \in N$, we can take a curve passing through x and U. In the normalization of this curve, we see that the class of x is the difference of two very ample divisors, so x is rationally equivalent to a 0-cycle supported on U.

Hence the image of the map $\phi_{c_2} : \mathrm{CH}_0(N) \to \mathrm{CH}_0(S)$ is spanned by the image of $\mathrm{CH}_0(U)$. Letting vary r, c_1, and $c_2 \gg 0$, and observing that $\bigoplus_{r,c_1,c_2} \mathrm{CH}_0(N) \twoheadrightarrow \mathrm{CH}(S)$ we get the result.

(This remark is due to Claire Voisin).

2.2.3 Bounded families of stable vector bundles generating the Chow group of of a surface

Corollary 2.2.4 is interesting in the case of a K3 surface over $\mathbb{C}$, where $\mathrm{CH}(S) = \mathbb{Z} \oplus \mathrm{Pic}(S) \oplus \mathrm{CH}^2(S)$, $\mathrm{Pic}(S)$ is a lattice, and $\mathrm{CH}^2(S)$ is very big (cf. [Mum68]) and torsion free (since $\mathrm{CH}^2(S)_{tor} \subset \mathrm{Alb}(S)_{tor}$ for [Roj80], and $\mathrm{Alb}(S) = 0$).

Beauville and Voisin have shown in [BV04] that every point lying on a rational curve has the same class $c_S \in \mathrm{CH}^2(S)$, that the intersection pairing of divisors maps only to multiples of that class:

$$\mathrm{Pic}(S) \otimes \mathrm{Pic}(S) \to \mathbb{Z} \cdot c_S \subset \mathrm{CH}^2(S) ,$$

and that $c_2(S) = 24 c_S$.

It would be interesting to see whether the fact that $\mathrm{CH}^2(S)$ is generated by second Chern classes of stable vector bundles can be used to get a better understanding of this group.

We have shown that $\{c_2(M) \mid M \textit{ is a stable vector bundle}\}$ is a set of generators for $\mathrm{CH}^2(S)$. This set is "very big" as we are varying arbitrarily the rank and Chern classes of the stable vector bundles. However we can limit this set even in cases where the Chow group is very big.

Proposition 2.2.6 *For every surface S there is a bounded family $\mathcal{V}$ of stable vector bundles on S, such that the second Chern classes of vector bundles in $\mathcal{V}$ generate the Chow group of zero cycles in this surface.*

Proof.

A bounded family of generating stable bundles can be constructed in various ways. We can consider the fact that, as 0-cycles are formal sums of points on the surface, then to generate the Chow group we just need to generate any (rational class of) single point on the surface.

We want to find a bounded family of stable vector bundles, such that their second Chern classes generate every point.

We can apply our construction to find a resolution of the ideal sheaf $\mathcal{I}_Z$, with $Z = \{s\}$ a point on the surface S.

Following the proof of proposition 2.2.1 we see that the numerical invariants chosen in the resolution 2.5

$$0 \to M_{s,V} \to V \otimes \mathcal{O}_S \to \mathcal{I}_s(mH) \to 0 \,,$$

i.e. the twisting factor m and the dimension of the vector space V do not depend on the point $p \in S$, but only on the ample class of the curve C, and can be fixed for all points.

Consider, the diagonal $\Delta \subset S \times S$, and the exact sequence

$$0 \to \mathcal{I}_\Delta(mH_1) \to \mathcal{O}_{S\times S}(mH_1) \to \mathcal{O}_\Delta(mH_1) \to 0 \,,$$

where H_1 is the pull-back of H to $S \times S$ through the projection p_1 of $S \times S$ to the first factor.

Then $I := p_{2*}\mathcal{I}_\Delta(mH_1)$ is a vector bundle on S, whose fiber over $s \in S$ is $H^0(S, \mathcal{I}_s(mH))$.

We can consider the Grassmannian on S,

$$\mathbb{G}\mathrm{r}(k, I) \to S$$

where the number k is $h^0(C, \mathcal{O}_C(mH))$ as in the proof of proposition 2.2.1. Then a point on $\mathbb{G}\mathrm{r}(k, I)$ corresponds to a couple (s, V), where $s \in S$ and $V \subset H^0(S, \mathcal{I}_s(mH))$ is a k-dimensional subsapce.

Hence we have a bounded family $\mathcal{V} = \{M_{s,V}\}_{(s,V)\in\mathbb{G}\mathrm{r}(k,I)}$. And we have shown that second Chern classes of the stable bundles $M_{s,V}$ which are in $\mathcal{V}$ (corresponding to generic V's) generate the Chow group of 0-cycles.

□

2.3 The general case

Let now X be a variety of dimension $n > 2$, with a fixed ample divisor H.

We want to prove the following

Theorem 2.3.1 *For every subscheme $Z \subset X$, its ideal sheaf $\mathcal{I}_Z$ admits a resolution*

$$0 \to E \to P_e \to \cdots \to P_1 \to P_0 \to \mathcal{I}_Z \to 0 \tag{2.6}$$

where E is a stable vector bundle, the P_i are locally free sheaves of the form $V_i \otimes \mathcal{O}_X(-m_iH)$, and $e = \dim X - 2$.

By passing to a multiple of H we may assume that a generic intersection of $n-1$ sections is a smooth curve C such that $g(C) \geqslant 1$. We want to prove Theorem 2.3.1 by the same method as in the surface case, *i.e.* finding vector spaces V_i that can be identified with the spaces of all global sections of a stable vector bundle on a smooth curve.

2.3.1 Proof of the theorem

We recall Butler's theorem for vector bundles on curves [But94]:

Theorem 2.3.2 (Butler) *Let C be a smooth projective curve of genus $g \geqslant 1$ over an algebraically closed field $\Bbbk$, and E a stable vector bundle over C with slope $\mu(E) > 2g$, then the vector bundle $M_E := \ker(H^0(C,E) \otimes \mathcal{O}_C \twoheadrightarrow E)$ is stable.*

Let us now consider a closed sub-scheme Z of codimension at least 2. We want to construct a sequence as in Theorem 2.3.1, which splits into short exact sequences in the following way:

$$\begin{array}{ccccccccccccccc} 0 \to E \to P_e & & \longrightarrow & & P_{e-1} \to \cdots \to P_2 & & \longrightarrow & & P_1 & & \longrightarrow & & P_0 \to \mathcal{I}_Z \to 0 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & K_{e-1} & & & & K_1 & & & & K_0 & & \end{array}$$

where the K_i are stable sheaves on the variety X which restricted to a curve C (an intersection of $n-1$ generic sections of $\mathcal{O}_X(H)$) are stable vector bundles M_i, and the $P_i = V_i \otimes \mathcal{O}_X(-m_i H)$ are obtained by successively lifting the space of global sections $H^0(C, M_{i-1}(m_i H))$ as in the surface case.

In other words the $V_i \subset H^0(X, K_{i-1}(m_i H))$ are spaces isomorphic to $H^0(C, M_i(m_i H))$ by the restriction of global sections to the curve (for the sake of clarity we should pose in the former discussion $K_{-1} := \mathcal{I}_Z$ and $M_{-1} := \mathcal{O}_C$).

We remark that the stability condition is invariant under tensoring by a line bundle.

Proof. (Theorem 2.3.1)

As a first step we want to choose m_0 and $V_0 \subset H^0(X, \mathcal{I}_Z(m_0 H))$.

Choosing $n-1$ generic[2] sections $s_1, ..., s_{n-1} \in |\mathcal{O}_X(H)|$, gives us a filtration of X by smooth sub-varieties:

$$X_0 := X \supset X_1 = V(s_1) \supset X_2 = V(s_1, s_2) \supset ... \supset X_{n-1} = C = V(s_1, \ldots, s_{n-1}).$$

Let $V \subset H^0(X, \mathcal{I}_Z(mH))$ be a subspace generating $\mathcal{I}_Z(mH)$. The restriction of the exact sequence

$$0 \to K \to V \otimes \mathcal{O}_X \to \mathcal{I}_Z(mH) \to 0$$

to the hypersurface X_1 yields an exact sequence

$$0 \to K|_{X_1} \to V \otimes \mathcal{O}_{X_1} \to \mathcal{I}_Z \otimes \mathcal{O}_{X_1}(mH) \to 0\,,$$

[2] By *generic* we mean that the element $(s_1, \ldots, s_{n-1}) \in |\mathcal{O}_X(H)|^{n-1}$ is generic.

due to the generality of the sections.

Restricting further we eventually obtain an exact sequence

$$0 \to K|_C \to V \otimes \mathcal{O}_C \to \mathcal{O}_C(mH) \to 0$$

of vector bundles on the curve C. In other words we are supposing the sequence $(s_1, \dots, s_{n-1})$ to be regular for $\mathcal{I}_Z$, and such that $C \cap Z = \emptyset$, both of which are open conditions. Furthermore, $(s_1, \dots, s_{n-1})$ being generic, we can suppose that all the $\mathcal{T}or^q_{\mathcal{O}_{X_i}}(\mathcal{I}_Z|_{X_i}, \mathcal{O}_{X_{i+1}})$ vanish, for $q > 0$ and $i = 0, \dots, n-2$:

to see this, let us fix an arbitrary locally free resolution

$$0 \to F_s \to \cdots \to F_0 \to \mathcal{I}_Z \to 0$$

of $\mathcal{I}_Z$, which splits into short exact sequences $0 \to P_i \to F_i \to P_{i-1} \to 0$. The sequence $(s_1, \dots, s_{n-1})$ being generic, we can suppose that it is regular for the shaves $\mathcal{I}_Z, P_0, \dots, P_{s-1}$. Hence, from the short exact sequences above, we deduce that $\mathcal{T}or^q_{\mathcal{O}_{X_i}}(\mathcal{I}_Z|_{X_i}, \mathcal{O}_{X_{i+1}}) \cong \mathcal{T}or^1_{\mathcal{O}_{X_i}}(P_{q-2}|_{X_i}, \mathcal{O}_{X_{i+1}}) = 0$.

For $m \gg 0$, we have $H^1(X_i, \mathcal{I}_Z \otimes \mathcal{O}_{X_i}((m-1)H)) = 0$ for every i. As in Lemma 2.2.2, a generic $V \subset H^0(X, \mathcal{I}_Z(mH))$ of dimension $h^0(C, \mathcal{O}_C(m))$ will map injectively to the global sections on the X_i:

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & H^0(X_i, \mathcal{I}_Z \otimes \mathcal{O}_{X_i}((m-1)H)) \\
 & & \downarrow {\scriptstyle \cdot s_{i+1}} \\
V & \hookrightarrow & H^0(X_i, \mathcal{I}_Z \otimes \mathcal{O}_{X_i}(mH)) \\
\wr\downarrow & & \downarrow \\
V & \hookrightarrow & H^0(X_{i+1}, \mathcal{I}_Z \otimes \mathcal{O}_{X_{i+1}}(mH)) \\
 & & \downarrow \\
 & & 0
\end{array}$$

until we have an isomorphism $V \xrightarrow{\sim} H^0(C, \mathcal{O}_C(m))$.

So we can choose $m_0 \gg 0$ and V generating $\mathcal{I}_Z(m_0H)$ such that the kernel K_0 of $V \otimes \mathcal{O}_X(-m_0) \to \mathcal{I}_Z$ is stable (since it's stable on the curve C which is a complete intersection of $n-1$ sections of H), but K_0 is, in general, not locally free (even though it is a vector bundle on the curve C).

As we have chosen $(s_1, \dots, s_{n-1})$ such that $\mathcal{T}or^q_{\mathcal{O}_{X_i}}(\mathcal{I}_Z|_{X_i}, \mathcal{O}_{X_{i+1}}) = 0$ for $q > 0$ and $i = 0, \dots, n-2$, we deduce from the sequence

$$0 \to K_0 \to V \otimes \mathcal{O}_X(-m_0) \to \mathcal{I}_Z \to 0$$

that also the $\mathcal{T}or^q_{\mathcal{O}_{X_i}}(K_0|_{X_i}, \mathcal{O}_{X_{i+1}})$ vanish, for $q > 0$ and $i = 0, \dots, n-2$. In particular, the sequence $(s_1, \dots, s_{n-1})$ is K_0-regular.

Repeating the argument, we obtain, tensoring K_0 by H enough times, exact sequences:

$$o \to K_1(m_1 H)|_{X_i} \to V_1 \otimes \mathcal{O}_{X_i} \to K_0(m_1 H)|_{X_i} \to 0 \, .$$

Again, we can suppose that $H^1(X_i, K_0 \otimes \mathcal{O}_{X_i}(m_1 H)) = 0$ and lift the vector space $H^0(C, K_0(m_1 H)|_C)$ on a generic generating space $V_1 \subset H^0(X, K_0(m_1 H))$. Butler's theorem tells us that the vector bundle $K_1|_C$, satisfying

$$0 \to K_1(m_1 H)|_C \to H^0(C, K_0(m_1 H)|_C) \otimes \mathcal{O}_C \to K_0(m_1 H)|_C \to 0 \, ,$$

is a stable vector bundle (for $m_1 \gg 0$), because $K_0|_C$ is stable and locally free.

So we can continue and find the resolution (2.6), where we remark that if $e \geqslant n-2$, E is a vector bundle because X is smooth and so has cohomological dimension $n = \dim X$, and it is stable because it is so on the curve C.

□

2.3.2 Stable vector bundles as generators

We can apply then this result to calculate the Chern class and character of $\mathcal{I}_Z$; we know that in general for any sheaf $\mathcal{F}$ and any resolution $P^\bullet \to \mathcal{F}$ by vector bundles, its Chern character is $ch(\mathcal{F}) = \sum (-1)^i ch(P^i)$.

Corollary 2.3.3 *A set of generators of* $\mathrm{CH}^*_{\mathbb{Q}}(X)$*, as a group, is*

$$\{ch(E) | E \text{ stable vector bundle}\} \, .$$

Proof. From the resolution (2.6) we have:

$$ch(\mathcal{I}_Z) = (-1)^{e+1} ch(E) + \sum_{i=0}^{e} (-1)^i \dim V_i \cdot ch(\mathcal{O}_X(-m_i H)) \, .$$

From the Grothendieck-Riemann-Roch theorem (cf. [Gro58]) we know that

$$ch(\mathcal{I}_Z) = 1 - ch(\mathcal{O}_Z) = 1 - [Z] + \text{higher order terms}$$

so applying our result to the higher order terms, we see that we can express $[Z]$ as a sum of Chern characters of stable vector bundles.

□

In order to have the same results in the K-theory and the derived category we will use the following

Lemma 2.3.4 *Any coherent sheaf $\mathcal{F}$on X admits a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_\ell = \mathcal{F}$ where each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ admits a polystable resolution.*

Proof. Consider first a torsion sheaf $\mathcal{T}$: it has then a filtration $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_\ell = \mathcal{T}$, where every quotient $\mathcal{T}_i/\mathcal{T}_{i-1}$ is of the form $\mathcal{O}_{Z_i}(-mH)$, for cycles Z_i . Hence $\mathcal{T}$ admits such a filtration.

A torsion free sheaf $\mathcal{F}$ admits an extension

$$0 \to V \otimes \mathcal{O}_X(-m) \to \mathcal{F} \to \frac{\mathcal{F}}{V \otimes \mathcal{O}_X(-m)} \to 0\,,$$

where $m \gg 0$, $V \subseteq H^0(X, \mathcal{F}(m))$ is the subspace generated by R generically independent sections of $\mathcal{F}(m)$, R is the generic rank of $\mathcal{F}$, and $\mathcal{F}/(V \otimes \mathcal{O}_X(-m))$ is a torsion sheaf. Hence taking the pull-back to $\mathcal{F}$ of the torsion sheaf filtration, we get the requested filtration.

Finally, any coherent sheaf fits into an extension with its torsion and torsion free parts:

$$0 \to \mathcal{T}(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}/\mathcal{T}(\mathcal{F}) \to 0\,,$$

so we can take the filtration for $\mathcal{T}(\mathcal{F})$ and the pull-back to $\mathcal{F}$ of the filtration for $\mathcal{F}/\mathcal{T}(\mathcal{F})$.

□

The following result is an immediate consequence:

Corollary 2.3.5 *The Grothendieck ring $K(X)$ is generated, as a group, by the classes of stable vector bundles.*

Remark 2.3.6 Every torsion free sheaf admits a (unique) Harder-Narashiman filtration, whose quotients are semistable sheaves (not necessarely locally free). And every semistable sheaf admits a (non unique) filtration with stable quotients. Mixing those two kinds of filtrations we obtain a filtration with stable quotients of any torsion free sheaf.

Hence, it can be easily proven that the class in $K(X)$ of any coherent sheaf $\mathcal{F}$ is obtained as a sum of classes of stable not necessarily locally free sheaves. In fact we can construct an exact sequence $0 \to K \to V \otimes \mathcal{O}_X(-mH) \to \mathcal{F} \to 0$, and take the filtration of the torsion free sheaf K, whose quotients are stable not necessarily locally free sheaves. (The same argument holds for the Chow group).

For what concerns the derived category, let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X. We will identify, as usual, any coherent sheaf $\mathcal{F}$ to the object $(0 \to \mathcal{F} \to 0) \in \mathcal{D}^b(X)$ concentrated in degree 0.

Definition 2.3.7 *We say that a triangulated subcategory $\mathcal{D} \subseteq \mathcal{D}^b(X)$, is generated by a family of objects $\mathcal{E} \subseteq \mathcal{D}^b(X)$, if it is the smallest full triangulated subcategory of $\mathcal{D}^b(X)$, stable under isomorphisms, which contains $\mathcal{E}$. We will denote it by $< \mathcal{E} >$.*

We can prove the following lemma:

Lemma 2.3.8 *Let $\mathcal{E}$ be a family of objects of $\mathcal{D}^b(X)$. If $< \mathcal{E} >$ contains two coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$, then it contains all their extensions.*

Proof. Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be an extension. Consider the distinguished triangle $\mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to \mathcal{F}_1[1]$ in $\mathcal{D}^b(X)$ associated to this extension. It will induce the distinguished triangle $\mathcal{F}_2 \to \mathcal{F}_1[1] \to \mathcal{F}[1] \to \mathcal{F}_2[1]$. Extending the arrow $\mathcal{F}_2 \to \mathcal{F}_1[1]$ to a distinguished triangle in $< \mathcal{E} >$, we get a distinguished triangle $\mathcal{F}_2 \to \mathcal{F}_1[1] \to \mathcal{G} \to \mathcal{F}_2[1]$. As $\mathcal{G}$ is isomorphic to $\mathcal{F}[1]$ in $\mathcal{D}^b(X)$ and $< \mathcal{E} >$ is stable under isomorphisms, then $\mathcal{F}[1]$ belongs to $< \mathcal{E} >$.

□

Obviously, if a family $\mathcal{E}$ of objects of $\mathcal{D}^b(X)$ is such that $< \mathcal{E} >$ contains every coherent sheaf, then $< \mathcal{E} >= \mathcal{D}^b(X)$. So, using lemma 2.3.4, we get immediately the following

Corollary 2.3.9 *The bounded derived category $\mathcal{D}^b(X)$ is generated by the family of stable vector bundles.*

Chapter 3

Line bundle transforms

3.1 Introduction

In the previous chapter we described how to take stable vector bundles as generators of the Chow ring of a variety. This was done exhibiting for all effective cycles $Z \subset X$ the polystable resolution (2.6):

$$0 \to K \to P_e \to \cdots \to P_1 \to P_0 \to \mathcal{I}_Z \to 0\,. \tag{3.1}$$

A first question concerning resolution (3.1) is how canonical it is, for a given $Z \subset X$.

This resolution is highly non canonical, in fact it depends on e and on the choice of the P_i's, where we have $P_i = V_i \otimes \mathcal{O}_X(-m_i)$. We could fix $e = \dim X - 2$ and the m_i's as the smallest such that our construction works, but as for V_i we have to pick a generic vector space in a Grassmannian, we cannot make a canonical choice.

As we show also that the derived category of the variety is generated by stable vector bundles, some natural questions arise.

Question 3.1.1 Given a variety X,

i. Is any object of the derived category $\mathcal{D}^b(X)$ isomorphic to a complex of polystable vector bundles?

ii. Is any vector bundle on X, as an object in $\mathcal{D}^b(X)$, isomorphic to a complex of polystable vector bundles?

iii. Does any vector bundle on X admits a resolution by polystable vector bundles?

Unfortunately we do not have an answer yet to those questions, and our definition of a derived category being generated by a class of objects is too "naive" to allow us to construct such resolutions. A natural step to try and construct some resolution as in the question above, is the studying of stability of kernels of evaluation maps on subspaces of the space of global sections of a given vector bundle. In fact, the main tool we use for showing stability is the restriction to curves. And when restricting to curve we want to know what happens evaluating on proper subspaces of the space of global sections.

By Butler's result, we know that on a curve the kernel of the evaluation map is not stable if the vector bundle we are starting from is not. So we ask whether we can "improve" stability by considering proper subspaces, and what happens in higher dimensional varieties. This leads to the following natural questions:

Question 3.1.2 Given a globally generated vector bundle E on a curve C, and a subspace $V \subset H^0(E)$, under which hypothesis is $M_{V,E} := \ker(V \otimes \mathcal{O} \twoheadrightarrow E)$ stable?

Question 3.1.3 Let X be a variety of dimension $n \geqslant 2$, and E a globally generated vector bundle on X, is $M_E := \ker(H^0(X, E) \otimes \mathcal{O} \twoheadrightarrow E)$ stable?

There are many cases in the literature where the stability of kernels of evaluation maps on global sections is investigated for various purposes. All results we have found concern vector bundles on curves.

This was used in particular by Paranjape and Ramanan (cf. [PR88]) to prove normal generation of canonical ring of curves, by Butler ([But94]) also to study normal generation of certain vector bundles, by Ein and Lazarsfeld (cf. [EL92]) to show the stability of the Picard bundle, by Beauville (*e.g.* in [Bea03]) to study theta divisors, and by Mercat (cf. [Mer99]) to describe some Brill-Noether loci.

We recall Butler's result that we used to construct the resolution (3.1) we needed in the previous chapter (cf. [But94]):

Theorem 3.1.4 (Butler) *Let C be a smooth projective curve of genus $g \geqslant 1$ over an algebraically closed field $\Bbbk$, and E a semistable vector bundle over C with slope $\mu(E) \geqslant 2g$, then the vector bundle $M_E := \ker(H^0(C, E) \otimes \mathcal{O}_C \twoheadrightarrow E)$ is semistable. Furthermore, if E is stable and $\mu(E) \geqslant 2g$, then M_E is stable, unless $\mu(E) = 2g$, and either C is hyperelliptic or $\omega_C \hookrightarrow E$.*

We will call this vector bundle M_E, a transform of the vector bundle E:

Definition 3.1.5 *Let X be a variety and E a globally generated vector bundle on X. We call $M_{V,E} := \ker(V \otimes \mathcal{O}_C \twoheadrightarrow E)$ the* transform *of the vector bundle E with respect to the generating subspace V, and $M_E := M_{H^0(E),E} = \ker(H^0(C, E) \otimes \mathcal{O}_C \twoheadrightarrow E)$ the* total transform *of E.*

Starting from the result of Butler, we want to investigate here the stability of transforms of line bundles on curves with respect to generic subspaces of certain codimensions.

In the next chapter we will also treat and example concerning question 3.1.3. The results of this chapter will give a partial answer to question 3.1.2, which can be resumed to the following theorem:

Theorem 3.1.6 *Let $\mathcal{L}$ be a line bundle of degree d on a curve C of genus $g \geqslant 2$, such that $d \geqslant 2g + 2c$, with $1 \leqslant c \leqslant g$. Then $M_{V,\mathcal{L}}$ is semistable for a generic subspace $V \subset H^0(\mathcal{L})$ of codimension c. It is stable unless $d = 2g + 2c$ and the curve is hyperelliptic, in which case $M_{V,\mathcal{L}}$ is strictly semistable for a generic $V \subset H^0(\mathcal{L})$ of codimension c.*

Similar results can be deduced by some constructions in Vincent Mercat's work [Mer99] on Brill-Noether's loci, but we think that in our case it is useful to give a more direct proof which applies to all line bundles of degree $d \geqslant 2g + 2c$ and not only generic ones.

Remark 3.1.7 A geometrical interpretation of those kinds of results goes as follows: a generating subspace $V \subset H^0(C, \mathcal{L})$ gives rise to a base point free linear system $|V| \subset |\mathcal{L}|$ on the curve C, and determines a map $\varphi_V : C \to \mathbb{P}(V^*)$, which asociates to a point $x \in C$ the hyperplane of global sections in V vanishing in x. The Euler sequence on $\mathbb{P}(V^*)$ is the dual of the tautological sequence:

$$0 \to \Omega_{\mathbb{P}(V^*)}(1) \to V \otimes \mathcal{O}_{\mathbb{P}(V^*)} \to \mathcal{O}_{\mathbb{P}(V^*)}(1) \to 0$$

which restricted to C gives the evaluation sequence

$$0 \to M_{V,\mathcal{L}} \to V \otimes \mathcal{O}_C \to \mathcal{L} \to 0\,.$$

As stability of a vector bundle is not affected by dualizing and tensorizing by a line bundle, we see that stability of $M_{V,\mathcal{L}} = \Omega(1)_{|C}$ is equivalent to the stability of the restriction of the tangent bundle of the projective space $\mathbb{P}(V^*)$ to the curve C.

So our theorem translates to

Theorem 3.1.8 *Let $C \subset \mathbb{P}^{d-g}$ be a genus $g \geqslant 2$ degree d non-degenerate smooth curve, where $d > 2g + 2c$, and c is a constant such that $0 \leqslant c \leqslant g$. Then for the generic projection $\mathbb{P}^{d-g} \dashrightarrow \mathbb{P}^{d-g-c}$ the restriction $T_{\mathbb{P}^{d-g-c}|C}$ is stable.*

3.2 Stability of transforms

We essentially use the following two lemmas:

Lemma 3.2.1 (Butler) *Let C be a curve of genus $g \geqslant 2$, F a vector bundle on C with no trivial summands, and such that $h^1(F) \neq 0$. Suppose that $V \subset H^0(F)$ generates F. If $N = M_{V,F}$ is stable, then $\mu(N) \leqslant -2$. Furthermore, $\mu(N) = -2$ implies that either C is hyperelliptic F is the hyperelliptic bundle and N its dual, or $F = \omega$ and $N = M_\omega$.*

The proof of this lemma is based on the result by Paranjape, Ramanan asserting the stability of M_ω (see [But94] and [PR88]).

Lemma 3.2.2 *Let $\mathcal{L}$ be a degree $d \geqslant 2g+2c$ line bundle on a curve C of genus $g \geqslant 2$, with $c \leqslant g$ and let $V \subset H^0(\mathcal{L})$ be a generating subspace of codimension c. Supppose there exists a stable subbundle of maximal slope $N \hookrightarrow M_{V,\mathcal{L}}$ such that $0 \neq N \neq M_{V,\mathcal{L}}$ and $\mu(N) \geqslant \mu(M_{V,\mathcal{L}})$.*

Then there exists a line bundle F of degree $f \leqslant d-1$, a generating subspace $W \subset H^0(F)$, and an injection $F \hookrightarrow \mathcal{L}$ such that N fits into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & W \otimes \mathcal{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_{V,\mathcal{L}} & \to & V \otimes \mathcal{O}_C & \to & \mathcal{L} & \to & 0\,, \end{array}$$

i.e. *a destabilization of $M_{V,\mathcal{L}}$ must be the transform of a line bundle injecting into $\mathcal{L}$ such that the global sections we are transforming by are in V.*

The importance of this lemma lies in the fact that we associate a line bundle F to a destabilizing N, and this allows us more easily to parametrize destabilizations and bound their dimension.

Proof. We remark that $\mu(M_{V,\mathcal{L}}) = -d/(d-g-c) \geqslant -2$ for $d \geqslant 2g+2c$. Consider a stable subbundle $N \hookrightarrow M_{V,\mathcal{L}}$ of maximal slope. Then it fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & W \otimes \mathcal{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_{V,\mathcal{L}} & \to & V \otimes \mathcal{O}_C & \to & \mathcal{L} & \to & 0 \end{array}$$

where $W \hookrightarrow V$ is defined by $W^* := \operatorname{Im}(V^* \to H^0(N^*))$, hence W^* generates N^*, and we call $F^* := \ker(W^* \otimes \mathcal{O} \twoheadrightarrow N^*)$.

Then F is a vector bundle with no trivial summands. Moreover the morphism $F \to \mathcal{L}$ is not zero, as $W \otimes \mathcal{O}$ does not map to $M_{V,\mathcal{L}}$. We have to show that $\mathrm{rk}F = 1$ and that $W = H^0(F)$. We distinguish the two cases $h^1(F) = 0$ or $h^1(F) \neq 0$.

• Let us suppose that $h^1(F) = 0$. Then $h^0(F) = \chi(F) = \mathrm{rk}F(\mu(F)+1-g)$. On the other hand, $h^0(F) > \mathrm{rk}F$ as F is globally generated and not trivial.

Together this yields

$$\mu(F) > g\,. \tag{3.2}$$

Furthermore

$$\mu(N) = -\deg F/(\dim W - \mathrm{rk}F) \leqslant -\mu(F)/(\mu(F)-g) = \mu(M_F)\,, \tag{3.3}$$

as $\dim W \leqslant h^0(F) = \mathrm{rk}F(\mu(F)+1-g)$.

Consider the image $I = \operatorname{Im}(F \to \mathcal{L}) \subseteq \mathcal{L}$. The commutative diagram

$$\begin{array}{ccccc} W & \hookrightarrow & H^0(F) & \to & H^0(I) \\ \downarrow & & & & \downarrow \\ V & & \hookrightarrow & & H^0(\mathcal{L}) \end{array}$$

shows that the map $W \to H^0(I)$ is injective and its image $W' \subset H^0(I)$ is contained in $V \subset H^0(\mathcal{L})$, hence $N \hookrightarrow M_{W',I} \hookrightarrow M_{V,\mathcal{L}}$. As N is a subbundle of $M_{V,\mathcal{L}}$ of maximal slope, this yields $\mu(N) \geqslant \mu(M_{W',I})$, *i.e.* $-\deg F/\mathrm{rk}N \geqslant -\deg I/\mathrm{rk}M_{W',I}$. Then

$$\deg F \leqslant \deg I(\mathrm{rk}N/\mathrm{rk}M_{W',I}) \leqslant \deg I \leqslant \deg \mathcal{L} = d\ .$$

If $\mathrm{rk}F \geqslant 2$, then $\mu(F) \leqslant \deg \mathcal{L}/2 = d/2$, so

$$\mu(N) \leqslant \frac{-\mu(F)}{\mu(F)-g} \leqslant \frac{-d/2}{d/2-g} = \frac{-d}{d-2g} \leqslant \frac{-d}{d-g-c} = \mu(M_{V,\mathcal{L}})\ .$$

Here the first inequality is (3.3). For the second one shows that the function $-x/(x-g)$ is strictly increasing for $x > g$. Then use $\mu(F) > g$ due to (3.2). Equality holds only if $\mathrm{rk}F = 2$, $\deg F = d$, $W = H^0(F)$, and $g = c$. But in this case we would find that $\dim W = h^0(F) = d+2-2g > d+1-g-c = \dim V$, wich is impossible as by construction $W \hookrightarrow V$.

Hence $\mathrm{rk}F = 1$. So $F = I$ is a globally generated and acyclic line bundle of degree $f \leqslant d$, and $\mu(N) = -f/(\dim W - 1)$.

It is easy to see that the case $f = d$ cannot hold, as in that case we cannot have $\mu(N) \geqslant \mu(M_{V,\mathcal{L}})$. So $f \leqslant d-1$.

• In the case $h^1(F) \neq 0$, by lemma 3.2.1, $\mu(N) \leqslant -2$. Equality holds only if $F = \omega_C$ and $W = H^0(\omega)$, or if the curve C is hyperelliptic and F is the hyperelliptic bundle. In the latter case the only generating space of global sections is $H^0(F)$. In any case we have $f = \deg F < d-1$. □

Remark 3.2.3 The diagram in the statement of the lemma is a construction from Butler's proof of theorem 3.1.4.

Remark 3.2.4 Loking carefully at the numerical invariants in the above proof, we can deduce some inequalities which will be useful in the following: let us consider again the diagram in the above lemma

$$\begin{array}{ccccccccc} 0 & \to & N & \to & W\otimes\mathcal{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_{V,\mathcal{L}} & \to & V\otimes\mathcal{O}_C & \to & \mathcal{L} & \to & 0 \end{array}$$

and suppose that $h^1(F) = 0$. Let us call $f := \deg F$, $s := d - f$, and $b := \mathrm{codim}_{H^0(F)} W$. Then we can show that

$$0 < c-b < s \leqslant \frac{d}{g+c}(c-b)\ . \tag{3.4}$$

In fact, as $W \hookrightarrow V$, and $W \neq V$, then

$$d-s+1-g-b = h^0(F) - b = \dim W < \dim V = d+1-g-c\ ,$$

hence $c-b < s$. And as

$$-\frac{d-s}{d-s-g-b} = \mu(N) \geqslant \mu(M_{V,\mathcal{L}}) = -\frac{d}{d-g-c}\ ,$$

then $s(g+c) \leqslant d(c-b)$, hence $c-b > 0$ and $s \leqslant \frac{d}{g+c}(c-b)$.

3.2.1 Line bundles of degree $d = 2g + 2$

A first consequence of these lemmas is the following proposition asserting semistability for hyperplane tranforms of line bundles of degree $2g + 2$.

Proposition 3.2.5 *Let $\mathcal{L}$ be a line bundle of degree $d = 2g + 2$ on a curve C of genus $g \geqslant 2$. Then $M_{V,\mathcal{L}}$ is semistable for every generating hyperplane $V \subset H^0(\mathcal{L})$. It is strictly semistable if C is hyperelliptic.*

Proof. Let us prove the semistability of $M_{V,\mathcal{L}}$.

Consider a stable subbundle $N \hookrightarrow M_{V,\mathcal{L}}$ of maximal slope, and suppose that it destabilizes $M_{V,\mathcal{L}}$ in the strict sense, *i.e.* $\mu(N) > -2 = \mu(M_{V,\mathcal{L}})$. By lemma 3.2.2 and remark 3.2.4 (we have $b = 0$ in this case), we know that N fits into a diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & H^0(F) \otimes \mathcal{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_{V,\mathcal{L}} & \to & V \otimes \mathcal{O}_C & \to & \mathcal{L} & \to & 0 \end{array}$$

with F a line bundle of degree $\deg F \leqslant d - 2 = 2g$. Moreover, $h^1(F) = 0$ since otherwise $\mu(N) \leqslant -2$ by lemma 3.2.1. Hence $\mathrm{rk} N = \deg F - g$, and

$$\mu(N) = -\deg F/(\deg F - g) \leqslant -2g/(2g - g) = -2\,,$$

(again, use that the function $-x/(x - g)$ is strictly increasing for $x > g$). So it is not possible to find a strictly destabilizing N.

If the curve is hyperelliptic, then $M_{V,\mathcal{L}}$ is strictly semistable: we can show that there is a line bundle of degree -2 injecting in $M_{V,\mathcal{L}}$. In fact we can consider the line bundle A dual of the only g^1_2 of the curve, *i.e* the dual of the hyperelliptic bundle.

The hyperelliptic bundle A^* has $h^0(A^*) = 2$, and from the exact sequence $0 \to M_{V,\mathcal{L}} \otimes A^* \to V \otimes A^* \to \mathcal{L} \otimes A^* \to 0$, we see that there are destabilizations of $M_{V,\mathcal{L}}$ by the line bundle A if and only if

$$H^0(M_{V,\mathcal{L}} \otimes A^*) = \ker(\varphi\colon V \otimes H^0(A^*) \to H^0(\mathcal{L} \otimes A^*)) \neq 0\,.$$

Counting dimensions we see that the map φ cannot be injective:

$$\dim V \cdot \dim H^0(A^*) = (g + 2)2 > g + 5 = \dim H^0(\mathcal{L} \otimes A^*).$$

□

In order to prove stability for non hyperelliptic curves though, we need to take a generic hyperplane, and not just a generating one.

The following is a special case of a more general result proven in section 3.2.3

Theorem 3.2.6 *Let $\mathcal{L}$ be a line bundle of degree $d = 2g + 2$ on a curve C of genus $g \geqslant 2$. Then $M_{V,\mathcal{L}}$ is stable for a generic hyperplane $V \subset H^0(\mathcal{L})$ if and only if C is non hyperelliptic.*

3.2.2 Line bundles of degree $d > 2g + 2c$

Here we show that for a generic subspace the transform of a line bundle of degree $d > 2g + 2c$ is stable. In contrast to proposition 3.2.5, we have to consider generic hyperplanes, and not just generating ones.

Theorem 3.2.7 *Let $\mathcal{L}$ be a line bundle of degree d on a curve C of genus $g \geqslant 2$, such that $d > 2g + 2c$, with $1 \leqslant c \leqslant g$. Then $M_{V,\mathcal{L}}$ is stable for a generic subspace $V \subset H^0(\mathcal{L})$ of codimension c.*

Proof.

Let us proceed as in proposition 3.2.5. We have that $-2 < \mu(M_{V,\mathcal{L}}) < -1$.

Consider a stable subbundle $N \hookrightarrow M_{V,\mathcal{L}}$ of maximal slope. By lemma 3.2.2 we know it fits into a diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & W \otimes \mathcal{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_{V,\mathcal{L}} & \to & V \otimes \mathcal{O}_C & \to & \mathcal{L} & \to & 0\,. \end{array}$$

We can right away conclude that $h^1(F) = 0$, as by lemma 3.2.1 we would otherwise have $\mu(N) \leqslant -2$.

So F is a globally generated line bundle with $h^1(F) = 0$, $\deg F =: d - s \leqslant d - 2$, and W is a b-codimensional subspace of $H^0(F)$. By remark 3.2.4, we see that for every b with $0 \leqslant b < c$ there is a finite number of s giving rise to a possible destabilization of $M_{V,\mathcal{L}}$.

For any of those b and s we will construct a parameter space allowing F, W, and the subspace $V \subset H^0(\mathcal{L})$ to vary.

For any such b and s we want to consider the parameter space $\mathcal{D}_{b,s}$, parametrizing subspaces $V \subset H^0(\mathcal{L})$ together with a destabilizing bundle of $M_{V,\mathcal{L}}$ of degree $s - d$ originating from a subspace W as in the construction above:

$$\begin{aligned} \mathcal{D}_{b,s} := \{(F\,,\ F \hookrightarrow \mathcal{L}\,,\ W \subset H^0(F))\,,\ V \subset H^0(\mathcal{L})) \mid F \in \mathrm{Pic}^{d-s}(C)\,, \\ (\varphi\colon F \hookrightarrow \mathcal{L}) \in \mathbb{P}(H^0(F^* \otimes \mathcal{L}))\,,\ W \in \mathbb{G}\mathrm{r}(b, H^0(F)) \\ V \in \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))\,,\ \varphi_{|W}\colon W \hookrightarrow V \subset H^0(\mathcal{L})\}\,. \end{aligned}$$

In order to estimate its dimension, we use the natural morphisms

$$\pi_{b,s}\colon \mathcal{D}_{b,s} \to \mathrm{Pic}^{d-s}(C)\,,\ (F, F \hookrightarrow \mathcal{L}, W, V) \mapsto F\,,$$

and $\rho_{b,s}\colon \mathcal{D}_{b,s} \to \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$, $(F, F \hookrightarrow \mathcal{L}, W, V) \mapsto V$.

The image of $\pi_{b,s}$ is formed by all the line bundles $F \in \mathrm{Pic}^{d-s}(C)$ such that $h^0(F^* \otimes \mathcal{L}) \neq 0$. In particular $\dim \pi_s(\mathcal{D}_s) = \min(s, g)$, because the degree of $F^* \otimes \mathcal{L}$ is s. The fiber over $F \in \pi_{b,s}(\mathcal{D}_s)$ has the same dimension as $\mathbb{P}(H^0(F^* \otimes \mathcal{L})) \times \mathbb{G}\mathrm{r}(b, (H^0(F))) \times \mathbb{G}\mathrm{r}(c, (H^0(\mathcal{L})/W))$.

By Clifford's theorem, $h^0(F^* \otimes \mathcal{L}) = s/2 + 1$ if $s \leqslant 2g$, and $h^0(F^* \otimes \mathcal{L}) = s + 1 - g$ otherwise. So,

$$\dim \mathcal{D}_{b,s} \leqslant \min(s, g) + \sup(s/2, s - g) + b(d - s - g + 1 - b) + c(s + b - c) \leqslant$$

$$(3/2)s + b(d - s - g + 1 - b) + c(s + b - c) .$$

Claim: for g, d, c as in the hypothesis and s, b satisfying the inequalities of remark 3.2.4, we have

$$(3/2)s + b(d - s - g + 1 - b) + c(s + b - c) < c(d + 1 - g) - c^2 = \dim \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L})) .$$

Proving the claim, we show that for all s and b giving rise to possible destabilizations, the morphisms $\rho_{b,s} \colon \mathcal{D}_{b,s} \to \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$ have a locally closed image of dimension strictly smaller than $\mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$, hence the generic subspace avoids all possible destabilizations of $M_{V,\mathcal{L}}$.

The claim is equivalent to

$$\frac{3s}{2(c-b)} + s + b < d + 1 - g ,$$

using inequalities (3.4) we get

$$\frac{3s}{2(c-b)} + s + b \leqslant \frac{3/2 + (c-b)}{g+c} d + b ,$$

hence we want to prove

$$\frac{3/2 + (c-b)}{g+c} d + b < d + 1 - g ,$$

which is equivalent to

$$\frac{b + g - 1}{b + g - 3/2} < \frac{d}{g + c} ,$$

and as $b \geqslant 0 \geqslant 2 - g$ then $\frac{b+g-1}{b+g-3/2} \leqslant 2 < \frac{d}{g+c}$. □

3.2.3 Line bundles of degree $d = 2g + 2c$

Theorem 3.2.8 *Let $\mathcal{L}$ be a line bundle of degree $d = 2g + 2c$ on a curve C of genus $g \geqslant 2$. Then $M_{V,\mathcal{L}}$ is semistable for a generic subspace $V \subset H^0(\mathcal{L})$ of codimension c. It is stable if and only if C is non hyperelliptic.*

Proof. As in the proof of theorem 3.2.7 we want to construct parameter spaces for destabilizations, and verify by dimension count that the generic subspace avoids them.

Let us consider a line bundle $\mathcal{L}$ of degree $d = 2g + 2c$ on a curve C of genus $g \geqslant 2$, and the transform $M_{V,\mathcal{L}}$ for a subspace $V \subset H^0(\mathcal{L})$ of codimension c.

To show semistability, let us suppose that there is a destabilizing stable vector bundle $N \hookrightarrow M_{V,\mathcal{L}}$, with $\mu(N) > \mu(M_{V,\mathcal{L}}) = -2$.

By lemma 3.2.2 we know it fits in the diagram

$$\begin{array}{ccccccccc}
0 & \to & N & \to & W \otimes \mathcal{O}_C & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_{V,\mathcal{L}} & \to & V \otimes \mathcal{O}_C & \to & \mathcal{L} & \to & 0 .
\end{array}$$

and we can suppose that $h^1(F) = 0$ by lemma 3.2.1.

In this case we can follow the same computations as in theorem 3.2.7: we have a parameter space for destabilizations

$$\mathcal{D}_{b,s} := \{(F\ ,\ F \hookrightarrow \mathcal{L}\ ,\ W \subset H^0(F))\ ,\ V \subset H^0(\mathcal{L})) \mid F \in \mathrm{Pic}^{d-s}(C)\ ,$$
$$(\varphi\colon F \hookrightarrow \mathcal{L}) \in \mathbb{P}(H^0(F^* \otimes \mathcal{L}))\ ,\ W \in \mathbb{G}\mathrm{r}(b, H^0(F))$$
$$V \in \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))\ ,\ \varphi_{|W}\colon W \hookrightarrow V \subset H^0(\mathcal{L})\}\ ,$$

whose dimension is bounded by

$$\dim \mathcal{D}_{b,s} \leqslant (3/2)s + b(d - s - g + 1 - b) + c(s + b - c)\ ,$$

with b and s satisfying $0 < c - b < s \leqslant \frac{d}{g+c}(c - b)$.

Except in the case $b = 0$ and $g = 2$, we can follow the very same proof of theorem 3.2.7, and we see that this bound shows that the generic subspace avoids the destabilization locus.

In the case $b = 0$ and $g = 2$ as well, it can be easily shown that $\dim \mathcal{D}_{b,s} < \dim \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$, for all s giving rise to destabilizations.

To show that we have strict semistability in the hyperelliptic case, we can proceed as in proposition 3.2.5, and show that dual of the hyperlliptic bundle is a subbundle of $M_{V,\mathcal{L}}$, of slope -2.

To show that we have stability in the non hyperelliptic case, we have to exclude slope -2 subbundles $N \hookrightarrow M_{V,\mathcal{L}}$.

Again we can apply lemma 3.2.2 and consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & W \otimes \mathcal{O}_C & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_{V,\mathcal{L}} & \to & V \otimes \mathcal{O}_C & \to & \mathcal{L} & \to & 0\ , \end{array}$$

where we can distinguish the two cases $H^1(F) = 0$, and $H^1(F) \neq 0$.

In the case $H^1(F) = 0$ we can follow again the same computations as in theorem 3.2.7.

In the case $H^1(F) \neq 0$, lemma 3.2.1 implies $F = \omega$ and $N = M_\omega$, hence the parameter space for destabilizations will be

$$\mathcal{D} := \{(\omega \hookrightarrow \mathcal{L}\ ,\ V \subset H^0(\mathcal{L})) \mid H^0(\omega) \subset V\}\ ,$$

and it can be shown that $\dim \mathcal{D} < \dim \mathbb{G}\mathrm{r}(c, H^0(\mathcal{L}))$.

□

3.3 Conclusions

We have proven stability of transforms of line bundles with respect to subspaces of low codimension. On the converse, it is rather easy to show the stability of transforms with respect to subspaces of low dimension: any stable vector bundle M^* of slope $\mu(M^*) > 2g-1$ is globally generated as it is shown in lemma 1.2.26; hence we can pick any stable vector bundle M^* of determinant $\mathcal{L}$ and rank r, such that $r < d/(2g-1)$, where $\deg \mathcal{L} = d$. Choosing any generating subspace $V^* \subset H^0(M^*)$ of rank $r+1$, we get an exact sequence

$$0 \to \mathcal{L}^* \to V^* \otimes \mathcal{O} \to M^* \to 0 .$$

Dualizing we get an exact sequence

$$0 \to M \to V \otimes \mathcal{O} \to \mathcal{L} \to 0 ,$$

where M is a stable transform of $\mathcal{L}$. Hence, every stable bundle of rank $r < d/(2g-1)$ and determinant $\mathcal{L}^*$, is a stable transform of $\mathcal{L}$. So the rational map $\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})) \dashrightarrow \mathcal{SU}(r, \mathcal{L})$ is dominant.

By the same argument we see that there is only one globally generated vector bundle, among vector bundles of determinant $\mathcal{L}$ and rank $d-g$ with no trivial summands, where $d = \deg \mathcal{L} \geqslant 2g$. Furthermore this is semistable, and even stable if $d > 2g$. In fact having such a globally generated bundle N, we can pick a vector space V of global sections of dimension $\mathrm{rk}N + 1$ generating N. This gives rise to the exact sequence

$$0 \to \mathcal{L}^* \to V \otimes \mathcal{O} \to N \to 0 ,$$

and dualizing

$$0 \to N^* \to V^* \otimes \mathcal{O} \to \mathcal{L} \to 0 .$$

But as N is globally generated and has no trivial summands, then $H^0(N^*) = 0$. And since V^* and $H^0(\mathcal{L})$ have the same dimension, then $V^* \xrightarrow{\sim} H^0(\mathcal{L})$. Hence $N^* = M_{\mathcal{L}}$ is unique.

So when we consider the rational map $\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})) \dashrightarrow \mathcal{SU}(r, \mathcal{L})$, $V \mapsto (M_{V,\mathcal{L}})^*$, we are saying that its image is made by globally generated bundles, and we can sum all this up in the following table, where we suppose that $d > 2g + 2c$, with $1 \leqslant c \leqslant g$:

$\mathrm{rk}(M_{V,\mathcal{L}})$	stability	map
$1 \leqslant r < d/(2g-1)$	stable	$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})) \dashrightarrow \mathcal{SU}(r, \mathcal{L})$ dominant
$\frac{d}{2g-1} \leqslant r < d-g-c$	**??**	**??**
$d-g-c \leqslant r < d-g$	stable	$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})) \dashrightarrow \mathcal{SU}(r, \mathcal{L})$
$r = d-g$	stable	$\{*\} \hookrightarrow \mathcal{SU}(r, \mathcal{L})$

where theorem 3.2.7 corresponds to the existence of the rational map

$$\mathbb{G}\mathrm{r}(r+1, H^0(\mathcal{L})) \dashrightarrow \mathcal{SU}(r, \mathcal{L}) .$$

Chapter 4

Symmetric products

4.1 Introduction

In the previous chapter we have seen a few cases of stability of a line bundle transform with respect to a subspace of global sections. We give here an example of stability of the total transform of a stable vector bundle on a higher dimensional variety.

The examples treated are vector bundles on symmetric products of curves. As stability, or rather poly-stability, is invariant when passing to a finite covering, we use the quotient map from the product to the symmetric product of the curve. We obtain then linearized vector bundles, and we show that having a linearization allows to pose remarkable restrictions on the possible destabilizations.

The examples treated concern tautological sheaves on the symmetric product of curves, and were inspired by the work of Scala [Sca05] and Dănilă [Dăn99], [Dăn01], [Dăn04], about cohomology of tautological sheaves on the Hilbert's schemes of points on surfaces.

4.2 Stability and group actions

Let X be a variety with an action of an algebraic group G. We recall that a G-linearized sheaf on X admits for all $g \in G$, an isomorphism $\Phi_g \colon g^*E \xrightarrow{\sim} E$, satisfying the usual cocycle conditions.

And that a morphism $\psi \colon E \to F$ of G-linearized sheaves is G-equivariant if the following diagram

$$\begin{array}{ccc} g^*E & \xrightarrow{g^*\psi} & g^*F \\ \Phi_g \wr \downarrow & & \downarrow \wr \Phi'_g \\ E & \xrightarrow{\psi} & F \end{array}$$

commutes for all $g \in G$ (cf. definition 1.2.6 and following remark).

Definition 4.2.1 *Let H be a divisor on X, we say that H is* numerically G-invariant, *if for all $g \in G$, $g^*H \sim_{num} H$.*

We have the following property

Proposition 4.2.2 *Let $F \hookrightarrow E$ be the maximal semistable subsheaf with respect to the polarization H, where E is a G-linearized torsion free sheaf, and H is a numerically G-invariant divisor. Then F admits a G-linearization such that $F \hookrightarrow E$ is G-equivariant.*

Proof. Consider the following diagram

$$\begin{array}{ccc} F & \hookrightarrow & E \\ \wr\downarrow & \circlearrowleft & \downarrow\wr \\ \varphi(F) & \hookrightarrow & g^*E \\ g^*F \nearrow & & \end{array}$$

where the isomorphism $\varphi\colon E \tilde{\to} g^*E$ is induced by the G-linearizaion. We want to show that $\varphi(F)$ and g^*F are the same subbundle of g^*E, *i.e.* the linearization of E induces a linearization of F.

We will show that they both are semistable subsheaves of maximal slope of g^*E. First notice that g^*F is a subbundle of g^*E, and its degree is given by

$$c_1(g^*F).H^{n-1} = g^*c_1(F).g^*H^{n-1} = g^*(c_1(F).H^{n-1}) = c_1(F).H^{n-1} .$$

By the same computation we see that the slope of a sheaf is invariant by the action of G, hence also g^*F is semistable, and it is the semistable maximal sheaf of g^*E.

As $\varphi(F) \tilde{\to} F$, then $c_1(\varphi(F)).H^{n-1} = c_1(F).H^{n-1}$ as well. So we deduce that $\varphi(F) \subseteq g^*F$, by maximality of g^*F. We have an exact sequence

$$0 \to \varphi(F) \to g^*F \to T \to 0 ,$$

where T is a torsion sheaf. Suppose that $\operatorname{codim}_X \operatorname{supp}(T) = p$, then $c_p(T)$ is a sum of integer p-codimensional subschemes of X, with positive coefficients and $c_i(T) = 0$ for $i < p$. Then $c_p(\varphi(F)).H^{n-p} = c_p(g^*F).H^{n-p} - c_p(T).H^{n-p}$, and $c_p(T).H^{n-p} > 0$. But this is impossible as $c_p(\varphi(F)).H^{n-p} = c_p(g^*F).H^{n-p}$ by the same argument as above.

Then $\varphi(F) = g^*F$.

Hence, the linearization on E induces a linearization on F, and clearly the morphism is G-equivariant.

□

We see from this proposition that having a group action poses strict conditions on linearized sheaves for being destabilized, and this is what we use to investigate the stability of tautological sheaves.

4.3 Symmetric product of a curve

We define in this section what the tautological sheaves are, and what are their total transforms.

Let C be a smooth projective curve, $X = C^n$ the cartesian product of C, and $G = \mathscr{S}_n$ the symmetric group acting on X permutating the factors.

As C is a smooth curve, the symmetric product $X/G := S^nC$ is a smooth variety. In fact it coincides with the Hilbert scheme of length-n 0-dimensional subschemes of C.

We will denote the elements of S^nC, wich are n-tuples of points of C order free, either $x_1 + \cdots + x_n$, or $[x_1, \ldots, x_n]$.

We can consider the universal family associated to the Hilbert scheme: the universal subscheme $Z \subset S^nC \times C$ consists of all the couples (ξ, x) where ξ is a 0-dimensional subscheme, and x is a point of X lying in ξ. Tere are two natural projections π_1 and π_2 of $S^nC \times C$ to S^nC and C, respectively.

On the product $S^nC \times C$ we have the following exact sequence:

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{S^nC\times C} \to \mathcal{O}_Z \to 0 .$$

As C is a curve, then Z is a divisor in $S^nC \times C$, and $\mathcal{I}_Z = \mathcal{O}_{S^nC\times C}(-Z)$. A "tautological" bundle on S^nC is defined as follows, given a vector bundle E on C:

$$E^{[n]} := \pi_{1*}(\pi_2^* E \otimes \mathcal{O}_Z) .$$

From the exact sequence above we get an exact sequence

$$0 \to \pi_{1*}(\pi_2^* E \otimes \mathcal{O}(-Z)) \to H^0(C,E) \otimes \mathcal{O}_{S^2C} \to E^{[n]} \to R^1\pi_{1*}(\pi_2^* E \otimes \mathcal{O}(-Z)) .$$

By projection formula

$$H^*(S^nC, E^{[n]}) = H^*(C,E) \otimes S^{n-1}H^*(C, \mathcal{O}_C) ,$$

hence, $H^0(S^nC, E^{[n]}) = H^0(C,E)$, and in the sequence above the map $H^0(C,E) \otimes \mathcal{O}_{S^2C} \to E^{[n]}$ is the evaluation map.

When $E^{[n]}$ is globally generated, we call N_E the total transform of $E^{[n]}$, *i.e.* $\pi_{1*}(\pi_2^* E \otimes \mathcal{O}(-Z))$. In the following we want to investigate about the stability of the tautological sheaf $L^{[n]}$ of a line bundle L on the curve C, and of its total transform N_L. We will consider first the case $n = 2$ as it will clarify the strategy used to prove the stability of $L^{[n]}$.

Remark 4.3.1 We recall that, in characteristic 0, if $\varphi \colon X \to Y$ is a finite morphism, and M is a torsion free coherent sheaf on Y, then M is polystable if and only if φ^*M is polystable.

More precisely, if M is (poly)stable, then φ^*M is polystable. And, *in any characteristic*, if φ^*M is (semi)stable, then M is (semi)stable (cf. [HL97], chapter 3).

4.3.1 Tautological sheaf of a line bundle on S^2C

Let us now consider the case where $E = L$ is a degree d line bundle, and $n = 2$, with $d > 2g$. With these hypotheses we have that $H^1(C, L(-x-y)) = 0$ for all x and y in C, hence $R^1\pi_{1*}(\pi_2^* L \otimes \mathcal{O}(-Z)) = 0$.

So we have the exact sequence,

$$0 \to N_L \to H^0(C, L) \otimes \mathcal{O}_{S^2C} \to L^{[2]} \to 0\,. \qquad (4.1)$$

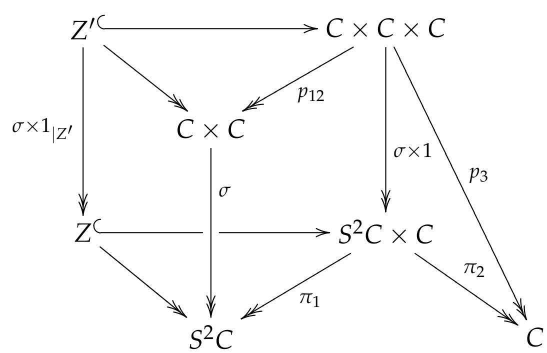
We want to prove that both the right and left side of this exact sequence are stable bundles, with respect to the ample divisor $\tilde{H} = p + C$, whose pull-back to $C \times C$ is the divisor $H = C \times p + p \times C$.

Let us consider first the tautological bundle $L^{[2]}$, it is a rank 2 vector bundle on S^2C. The pull-back of $L^{[2]}$ by the map $\sigma\colon C \times C \to S^2C$ fits in the following sequence:

$$0 \to \sigma^* L^{[2]} \to L \boxplus L \to L_\Delta \to 0\,, \qquad (4.2)$$

where $L \boxplus L = p_1^* L \oplus p_2^* L$, with p_1 and p_2 the 2 projections of $C \times C$, and $L_\Delta = \Delta_* L$, with $\Delta\colon C \hookrightarrow C \times C$ the diagonal inclusion.

To see this, we can use the following diagram



where Z is the universal subscheme defined above,

$$Z := \{(x+y, z) \in S^2C \times C \mid z \in \{x, y\}\} \subset S^2C \times C\,,$$

and Z' is its pull-back to $C \times C \times C$, *i.e.*

$$Z' := \{(x, y, z) \in C \times C \times C \mid z \in \{x, y\}\} = \Delta_{13} \cup \Delta_{23} \subset C \times C \times C\,.$$

We can tensorize by $p_3^* L$ the exact sequence on $C \times C \times C$

$$0 \to \mathcal{O}_{Z'} \to \mathcal{O}_{\Delta_{13}} \oplus \mathcal{O}_{\Delta_{23}} \to \mathcal{O}_{\Delta_{123}} \to 0\,,$$

where Δ_{13} , Δ_{23} , and Δ_{123} are the diagonals defined respectively by $x = z$, $y = z$, and $x = y = z$, on $C \times C \times C$. And the second map is given by $s \oplus t \mapsto s_{|\Delta_{123}} - t_{|\Delta_{123}}$. Then using the projection formula, we obtain the exact sequence (4.2).

As it is a pull-back from S^2C, $\sigma^*L^{[2]}$ is a $\mathcal{S}_2$-equivariant vector bundle. And $L \boxplus L$ is $\mathcal{S}_2$-equivariant as well, by identification of the 2 factors. Furthermore the map $\sigma^*L^{[2]} \to L \boxplus L$ is clearly $\mathcal{S}_2$-equivariant, we will use this map to show the stability of $L^{[2]}$. In fact to show its stability, we need only to suppose that $\deg L > 1$.

Proposition 4.3.2 *Let L be a line bundle on C of degree $d \geqslant 1$, then $\sigma^*L^{[2]}$ is an H-semistable bundle, and $L^{[2]}$ is $\tilde{H}$-semistable.*

Proof. Let us suppose that they are not semistable bundles. There would be a destabilizing line bundle $A \hookrightarrow \sigma^*L^{[2]}$, by proposition 4.2.2 this is an equivariant map.

Hence we have an equivariant map $A \hookrightarrow p_1^*L \oplus p_2^*L$. As A is destabilizing,

$$c_1(A).H = \mu_H(A) > \mu_H(\sigma^*L^{[2]}) = \frac{1}{2}(2d-2) = d-1 \geqslant 0\,.$$

The chosen polarization H is in the numerical equivalence class $[C \times p + p \times C]$, hence $H = f_1 + f_2$, where $f_1 = [C \times p]$ and $f_2 = [p \times C]$. Hence we have either $c_1(A).f_1 > 0$ or $c_1(A).f_2 > 0$.

Let us suppose that $c_1(A).f_1 > 0$. Then $\deg(A_{|C\times p}) > 0$, for all $p \in C$. Hence, restricting to $C \times p$ the map $\psi\colon A \hookrightarrow p_1^*L \oplus p_2^*L$, we have that the second component of $A_{|C\times p} \hookrightarrow L \oplus \mathcal{O}$ vanishes as $A_{|C\times p}$ is a line bundle of positive degree. This being true for all $p \in C$, then the second component of the map ψ vanishes on all $C \times C$.

Now we just need to show that if one component of a $\mathcal{S}_2$-equivariant map to $p_1^*L \oplus p_2^*L$ vanishes, the the whole map vanishes.

In fact, consider any invariant map $\psi = (\psi_1, \psi_2)\colon A \to p_1^*L \oplus p_2^*L$, then if we call $\psi(x,y)\colon A(x,y) \to L(x) \oplus L(y)$, then

$$g^*\psi(x,y) = \psi(y,x) = (\psi_1 \oplus \psi_2)(y,x)\colon g^*A(x,y) = A(y,x) \to L(y) \oplus L(x)\,.$$

Then, using remark 1.2.7, and observing the diagram of the maps ψ and the linearizations on the fibers over $(x,y) \in C \times C$

$$\begin{array}{ccc} & \overset{\psi_1^{(x,y)}}{\nearrow} & L(x) \\ A(x,y) & \hookrightarrow & \oplus \\ & \underset{\psi_2^{(x,y)}}{\searrow} & L(y) \\ \wr\uparrow\Phi_g & & \uparrow\wr \\ & \overset{\psi_1^{(y,x)}}{\nearrow} & L(y) \\ A(y,x) & \hookrightarrow & \oplus \\ & \underset{\psi_2^{(y,x)}}{\searrow} & L(x) \end{array}$$

(the right vertical arrow is the obvious isomorphism) we see that the morphism $\psi\colon A \to p_1^*L \oplus p_2^*L$ of $\mathcal{S}_2$-linearized bundles is equivariant if and only if

$$\psi_1(x,y) \circ \Phi_g(x,y) = \psi_2(y,x)\colon A(y,x) \to L(x)\,,$$

$$\text{and } \psi_2(x,y) \circ \Phi_g(x,y) = \psi_1(y,x) \colon A(y,x) \to L(y) \,.$$

So we see that $\psi_1 = 0$ implies that also $\psi_2 = 0$, hence we cannot have an $\mathcal{S}_2$-equivariant injection $A \hookrightarrow L \boxplus L$, with $c_1(A).H > 0$

Oviously, if we had a destabilization of $L^{[2]}$, we would have a destabilization of $\sigma^* L^{[2]}$ as well, so the proposition is proved.

□

Corollary 4.3.3 *Let L be a line bundle on C of degree $d \geqslant 2$, then $\sigma^* L^{[2]}$ is an H-stable bundle, and $L^{[2]}$ is $\tilde{H}$-stable.*

Proof. By the proposition above we know that the bundles $\sigma^* L^{[2]}$ and $L^{[2]}$ are semistable. Again we need only to show he stability of $\sigma^* L^{[2]}$. By the same argument, a destabilization $A \hookrightarrow \sigma^* L^{[2]}$ is given by an injection of a line bundle A (not necessarily $\mathcal{S}_2$-linearized this time), such that

$$\mu_H(A) = c_1(A).H = \mu_H(\sigma^* L^{[2]}) = \frac{c_1(\sigma^* L^{[2]}).H}{2} = d - 1 > 0 \,.$$

Hence, we would have $c_1(A).f_1 > 0$ or $c_1(A).f_2 > 0$, and therefore $A \hookrightarrow p_i^* L$ with $i = 1$ or 2. But the exact sequence (4.2) implies that $A \hookrightarrow p_i^* L(-\Delta)$ in this case, hence we would have

$$c_1(A).H \leqslant c_1(p_i^* L(-\Delta)) = d - 2 < d - 1 = \mu_H(\sigma^* L^{[2]}) \,.$$

□

Remark 4.3.4 We could use this technique to prove directly proposition 4.3.2, but the method we have shown in the proof will be generalized to higher products.

In fact also N_L is an $\tilde{H}$-stable bundle, we will prove this in the case of $S^n C$ in paragraph 4.3.3.

4.3.2 Tautological sheaf of a line bundle on $S^n C$

We apply in this section the same methods as above to show stability results on $S^n C$.

We will consider the ample divisor $H = \Sigma_{i=1}^n p_i^{-1}(p)$ in C^n, which is the pull-back of the divisor $\tilde{H} = p + S^{n-1} C$ in $S^n C$. In the literature the polarization $\tilde{H}$ is often called **x**. And we call as usual Δ the big diagonal in $S^n C$, *i.e.* the divisor $\{x_1 + \cdots + x_n \in S^n C \mid \exists i \neq j \text{ with } x_i = x_j\}$, which is divisible by 2, as it is the branch locus of the quotient map $\sigma \colon C^n \to S^n C$.

In characteristic 0, if we show that the vector bundle $L^{[n]}$ is stable with respect to the polarization $\tilde{H}$, then the vector bundle $\sigma^* L^{[n]}$ is polystable with respect to H.

Lemma 4.3.5 $c_1(L^{[n]}) \equiv (\deg L)\mathbf{x} - \frac{\Delta}{2}$

Proof.

From Göttsche's appendix in [BS91], we know that

$$c_1(L^{[n]}) = L^{\boxtimes n} - \frac{\Delta}{2}\,,$$

where $L^{\boxtimes n}$ is the unique line bundle on S^nC such that its pull-back via $\sigma\colon C^n \to S^nC$ is equal to $p_1^*L \otimes \dots p_n^*L$. And we can verify that $L^{\boxtimes n} \sim_{alg} (\deg L)\mathbf{x}$, so $L^{\boxtimes n} \equiv_{num} (\deg L)\mathbf{x}$.

□

Proposition 4.3.6 *Let L be a line bundle of degree $d \geqslant n$ on C, then $L^{[n]}$ is an $\tilde{H}$-stable vector bundle on S^nC.*

Proof. Let us suppose that there exists a destabilization of $L^{[n]}$, *i.e.* an injection of sheaves $\tilde{F} \hookrightarrow L^{[n]}$, such that $\mu_{\tilde{H}}(F) \geqslant \mu_{\tilde{H}}(L^{[n]})$, with $\tilde{F}$ torsion free of rank $r < n$. And let us assume that $\tilde{F}$ is locally free. We call $F := \sigma^*\tilde{F}$.

Pulling the injection $\tilde{F} \hookrightarrow L^{[n]}$ back to C^n by the quotient map $\sigma\colon C^n \to S^nC$ we have a $\mathcal{S}_n$-invariant injection $F \hookrightarrow \sigma^*L^{[n]}$. We can compose this injection with the natural $\mathcal{S}_n$-equivariant morphism $\sigma^*L^{[n]} \hookrightarrow \bigoplus_{i=1}^n p_i^*L$, where p_i are the projections $C^n \to C$.

Hence we have a $\mathcal{S}_n$-equivariant injective morphism

$$F \hookrightarrow \bigoplus_{i=1}^n p_i^*L\,.$$

As $A = \bigwedge^r F = \det F$ and $\bigwedge^r \bigoplus_{i=1}^n p_i^*L$ carry a linearization induced by that of F and $\bigoplus_{i=1}^n p_i^*L$ we have a $\mathcal{S}_n$-equivariant morphism

$$\psi\colon A \to \bigwedge^r \bigoplus_{i=1}^n p_i^*L\,.$$

We proceed now as in proposition 4.3.2 to show that this morphism must be zero, hence the maps $F \to \sigma^*L^{[n]}$ and $\tilde{F} \to L^{[n]}$ cannot be injective.

Decomposing $\bigwedge^r \bigoplus_{i=1}^n p_i^*L$ as $\bigoplus_{|J|=r} L_J$, where $L_J = p_{j_1}^*L \otimes \cdots \otimes p_{j_r}^*L$, we decompose also the map $\psi = \bigoplus \psi_J\colon A \to \bigoplus L_J$.

Let us call $f_i := p_i^{-1}(p)$ the hypersurface in C^n. Then $H = \Sigma_{i=1}^n f_i$, $H^{n-1} = \Sigma_{i=1}^n (n-1)! f_1 \dots \hat{f}_j \dots f_n$, and $H^n = n!$, furthermore the class $f_1 \dots \hat{f}_j \dots f_n$ is represented by the curve $x_1 \times \cdots \times x_{j-1} \times C \times x_{j+1} \times \cdots \times x_n$, for any $(x_1, \dots, \hat{x}_j, \dots, x_n) \in C^{n-1}$.

By construction we have that

$$c_1(F).H^{n-1} = c_1(\det F).H^{n-1} \geqslant \frac{r}{n} c_1(\sigma^*L^{[n]}).H^{n-1} = \frac{r}{n}(n!)(d-n+1) > 0\,.$$

We can suppose that $c_1(F).(f_2.f_3 \dots f_n) > 0$. Hence $\deg(A_{|C\times x_2\times\cdots\times x_n}) > 0$, for all $(x_2 \dots, x_n) \in C^{n-1}$.

For all J such that $1 \notin J$, $L_{J|C\times x_2\times\cdots\times x_n} = \mathcal{O}$. Hence, for all such J, $\psi_J \colon A \to L_J$ is the zero map, as we can see by restriction to the curve $C \times x_2 \times \cdots \times x_n$.

Again we can use remark 1.2.7 and proceed as in the proposition 4.3.2: observe that by the following diagram

$$\begin{array}{ccc} A(x_1,\dots,x_n) & \overset{\psi(x_1,\dots,x_n)=\oplus\psi_J}{\hookrightarrow} & \bigoplus L_J \\ \wr\uparrow\Phi_g & & \uparrow\wr \\ A(x_{g_1},\dots,x_{g_n}) & \underset{\psi(x_{g_1},\dots,x_{g_n})}{\hookrightarrow} & \bigoplus g^* L_J \end{array}$$

the map ψ is $\mathcal{S}_n$-equivariant if and only if

$$\psi_J(x_1,\dots,x_n)\circ\Phi_g = \psi_{gJ}(x_{g_1},\dots,x_{g_n}) \colon A(x_{g_1},\dots,x_{g_n}) \to L_J(x_1,\dots,x_n)$$

for all $(x_1,\dots,x_n) \in C^n$ and $g \in \mathcal{S}_n$.

Hence if the map ψ_J vanishes for a J, then the whole map ψ vanishes, and the morphism $F \to \bigoplus_{i=1}^n p_i^* L$ cannot be injective.

The assumption that F is locally free is not limiting, as for any F torsion free of rank r, we have a line bundle A, such that $c_1(A) = c_1(F)$, with $A = \bigwedge^r F$ on the locus where F is locally free, and such that $F \to \bigoplus_{i=1}^n p_i^* L$ induces $A \to \bigwedge^r \bigoplus_{i=1}^n p_i^* L$. This last map being zero, the map $F \to \bigoplus_{i=1}^n p_i^* L$ cannot be injective.

□

Corollary 4.3.7 *In characteristic* 0, $\sigma^* L^{[n]}$ *is poly-stable. In any characteristic, it is semistable.*

Proof.

The first assertion follows from remark 4.3.1. To prove the second assertion, we use the fact that a maximal semistable subsheaf is $\mathcal{S}_n$-linearized, and then follow the proof of proposition 4.3.6.

□

4.3.3 Transform of the tautological sheaf of a line bundle

In this paragraph we want to use the proof of Ein and Lazarsfeld of the stability of M_L for line bundles (cf. [EL92]), to show that when $\deg L > 2g + n$ the total transform

$$N_L = M_{L^{[n]}} = \ker(H^0(C,L) \otimes \mathcal{O}_{S^n C} \twoheadrightarrow L^{[n]})$$

is stable.

In [EL92] the stability of the Picard bundle is shown proving the stability of a total transform, we recall their argument, as it is useful for a better understanding of the stability of N_L.

Consider a pointed smooth curve (C, x_0) of genus $g \geqslant 1$, for all d there is a Picard scheme $\mathrm{Pic}^d(C)$, which is a fine moduli scheme for line bundles of degree d on C. And a universal sheaf $\mathcal{U}$ on $C \times \mathrm{Pic}^d(C)$ such that $\mathcal{U}_{|C\times\xi} = L_\xi$ for all points $\xi \in \mathrm{Pic}^d(C)$, representing a line bundle L_ξ, and $\mathcal{U}_{|x_0\times \mathrm{Pic}^d(C)} = \mathcal{O}_{\mathrm{Pic}^d(C)}$.

The Picard sheaf P_d is the push forward of $\mathcal{U}$ to $\mathrm{Pic}^d(C)$. When $d \geqslant 2g-1$, P_d is a vector bundle, whose fiber upon a point $\xi \in \mathrm{Pic}^d(C)$ is $H^0(C, L_\xi)$.

The Picard schemes are all isomorphic, an isomorphism $\mathrm{Pic}^d(C) \xrightarrow{\sim} \mathrm{Pic}^{d+1}(C)$ is given by the map $L \mapsto L \otimes \mathcal{O}_C(x_0)$. In degree 0 the Picard scheme is the Jacobian of the curve $H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z})$, and is smooth projective of dimension g.

The Picard schemes $\mathrm{Pic}^d(C)$ have a canonical polarization. It is given by the theta divisor $\Theta \subset \mathrm{Pic}^d(C)$, which is the ample divisor represented by the image of the map

$$C^{g-1} \to \mathrm{Pic}^{g-1}(C), (x_1, \dots, x_{g-1}) \mapsto \mathcal{O}_C(x_1 + \dots + x_{g-1})$$

in $\mathrm{Pic}^{g-1}(C)$. By the isomorphisms above it corresponds to ample divisors on all Picard schemes.

Given a line bundle $A \in \mathrm{Pic}^d(C)$, we can consider the applications

$$v_A^r : C^r \to \mathrm{Pic}^{d-r}(C) \ , \ (x_1, \dots, x_r) \mapsto A \otimes \mathcal{O}_C(-x_1 - \dots - x_r) \ .$$

Then by Poincaré's formula, when $r < g$ the class of the image is

$$W_r := [\Theta]^{g-r}/(g-r)!$$

in particular, the class of the image $v_A^1(C)$ is $[\Theta]^{g-1}/(g-1)!$.

Hence to prove stability of P_d with respect to Θ, *i.e.* to show that for all subsheaves $F \hookrightarrow P_d$ we have

$$\frac{c_1(F).[\Theta]^{g-1}}{\mathrm{rk}(F)} < \frac{c_1(P_d).[\Theta]^{g-1}}{\mathrm{rk}(P_d)},$$

it is sufficient to show that $(v_A^1)^* P_d$ is stable for an $A \in \mathrm{Pic}^{d+1}(C)$, and then apply theorem 1.2.25. And this is done, when $d \geqslant 2g$, by showing that $(v_A^1)^* P_d = M_A \otimes \mathcal{O}_C(x_0)$, and that the transform M_A is stable.

Remark 4.3.8 To show that the stability of P_d is implied by the stability of a curve whose class is a multiple of $[\Theta]^{g-1}$, it is not used in [EL92] a restriction result like theorem 1.2.25 but rather the generality of the line bundle $A \in \mathrm{Pic}^{d+1}(C)$.

Remark 4.3.9 Actually Ein and Lazarsfeld show something stronger than stability of M_L, they prove that M_L is cohomologically stable, but we will not use such concept.

We can use the same argument to show the stability of the total transform N_L of a tautological bundle $L^{[n]}$ on S^nC.

Theorem 4.3.10 *Let C be a smooth curve of genus $g \geqslant 2$, and let L be a line bundle on C of degree $d \geqslant 2g+n$. Then N_L is a stable bundle on S^nC with respect to the polarization $\tilde{H}$.*

To prove this we show that for all $(x_1, \dots, x_{n-1}) \in C^{n-1}$ the restriction $N_{L|x_1+\cdots+x_{n-1}+C}$ is stable, and then use theorem 1.2.25

Proof.

We want to show that for all $(x_1, \dots, x_{n-1}) \in C^{n-1}$ the restriction $N_{L|x_1+\cdots+x_{n-1}+C}$ is stable.

To prove it we could just use the restriction to $x_1 + \cdots + x_{n-1} + C$ of the exact sequence

$$0 \to \pi_{1*}(\pi_2^* L \otimes \mathcal{O}(-Z)) \to H^0(C, L) \otimes \mathcal{O}_{S^2C} \to L^{[n]} \to 0 \,,$$

but it seems more clarifying if we look at our total transform as the pull back of the Picard bundle via by the map

$$v_L^n : S^nC \to \mathrm{Pic}^{d-n}(C) \,,\ (x_1, \dots, x_n) \mapsto L \otimes \mathcal{O}_C(-x_1 - \cdots - x_n) \,.$$

Claim: $(v_L^n)^*(\mathcal{P}_{d-n}) = \widetilde{N_L} \otimes \mathcal{O}_{S^nC}(x_0 + S^{n-1}C)$.
From the claim we deduce that $N_{L|x_1+\cdots+x_{n-1}+C}$ is $M_{L(-x_1-\cdots-x_{n-1})}$ tensorized by a line bundle, hence it is stable by Ein and Lazarsfeld results, or by Butler's theorem (3.1.4).

To prove the claim, we can observe that $(v_L^n)^*(\mathcal{P}_{d-n}) = \pi_{1*}((v_L^n \times 1_C)^* \mathcal{U})$ by the theorem on cohomology and base change. Then use see-saw principle to show that

$$(v_L^n \times 1_C)^* \mathcal{U} = \pi_2^* L \otimes \mathcal{O}_{S^nC \times C}(-Z) \otimes \pi_1^* \mathcal{O}_{S^nC}(x_0 + S^{n-1}C) \,,$$

where π_1 and π_2 are the two naturale projections of $S^nC \times C$, p_1 is the projection of $\mathrm{Pic}^{d-n}(C) \times C$ to the first factor. And $\mathcal{U}$ is the universal sheaf on $\mathrm{Pic}^{d-n}(C) \times C$.

□

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