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Modèles en Arithmétique Bornée

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Introduction

Arithmétique Bornée

On parle d' Arithmétiques Bornées pour évoquer les théories arithmétiques où le schéma d'induction est restreint aux formules dont les quantifications sont bornées: des formules comme $\exists x \leq t \, \forall y \leq s \, \theta$ pour θ ouverte. Elles ont été introduites dans les années 70 par Parikh [17] pour étudier des questions liées à des longueurs des preuves. En particulier le système $I\Delta_0$, induction pour toute formule à quantificateurs bornés dans le langage $\{0, 1, +, \times, <\}$.

Ces théories ne démontrent pas que la fonction exponentielle est totale. En effet, le théorème de Parikh [17] dit que toute fonction prouvablement totale dans un tel système est bornée par un terme du langage, un polynôme dans le cas de $I\Delta_0$. En conséquence la formalisation de certaines notions telles que les fonctions récursives ou la manipulation de la syntaxe, devient une tâche nettement plus compliquée. $I\Delta_0$ peut néanmoins définir les fonctions de la hiérarchie linéaire de Wrathall (voir [30],[23]) ce qui a donné lieu à des travaux pioniers faisant le lien entre complexité d'algorithmes et modèles de l'arithmétique (voir [18]). En revanche, cette théorie est incapable de parler de fonctions calculables en temps polynomial, non plus de substitution dans une formule, ni de preuves de taille polynomiale.

Pour ces raisons Paris et Wilkie [29] ont étudié dans les années 80 l'extension de cette théorie à $I\Delta_0+\Omega_1$, où Ω_1 est un axiome disant que la fonction $x^{|x|}$ existe, et $|x|$ est la longueur de l'expansion binaire de x. Ce n'est pas l'exponentielle mais cette théorie est en fait suffisament forte pour définir les fonctions correspondant à PH, la hiérarchie polynomiale $(\Sigma_i^p)_{i\in\omega}$ de Stockmeyer [24], à laquelle les informaticiens s'intéressaient de plus en plus.

Plus tard Buss [4] introduit un langage avec, notamment, des symboles pour $|x|$ et la fonction $x^{|x|}$. Il définit une théorie S_2 qui est une extension conservative de $I\Delta_0+\Omega_1$ et des fragments qui vont "capturer" chaque niveau de PH. Pour cela Buss introduit une hiérachie de formules $(\Sigma_i^b)_{i\in\omega}$, analogue à la hiérarchie arithmétique, où à chaque niveau correspond un dans la hiérarchie polynomiale. Par exemple les formules Σ_1^b définissent des prédicats NP. Ensuite il considère des théories S_2^i avec un schéma d'induction jusqu'à $|x|$ pour les formules Σ_i^b et caractérise les fonctions Σ_i^b-définissables de S_2^i comme étant celles du niveau correspondant dans la hiérarchie polynomiale, c'est à dire les fonctions calculables en temps polynomial par une machine de Turing qui utilise un oracle pour un prédicat Σ_{i-1}^p. On a en particulier $S_2^i \subset S_2^{i+1}$ et $S_2 = \bigcup_{i\in\omega} S_2^i$.

Montrer que la quantité d'induction disponible dans chaque théorie suffit à définir la classe de fonctions correspondante est une vérification techniquement compliquée mais plus ou moins de routine. C'est le problème inverse qui est plus intéressant, à savoir, montrer que la théorie en question capture exactement la classe $\mathcal{C}$ de fonctions voulue.

Il s'agit de prouver des théorèmes qu'on appelle *witnessing theorems*: si T démontre $\forall x \exists y \phi(x, y)$ alors il existe une fonction f appartenant à $\mathcal{C}$ telle que $\forall x \phi(x, f(x))$, c'est à dire $f(x)$ "témoigne" pour le quantificateur existentiel. Un théorème classique de ce genre est celui de Mints-Parsons's (voir [15],[19],[26]) qui caractérise les fonctions primitives récursives comme étant celles prouvablement totales et récursives dans $I\Sigma_1$. Les techniques utilisées pour le prouver proviennent de la Théorie de la Démonstration.

De même, Buss développa pour ses résultats une technique connue sous le nom de *witness function method*. En analysant la dérivation d'une formule A dans le système donné on constate qu'il est possible de mener au fur et a mesure le calcul d'une suite de valeurs servant a vérifier la véracité de A, et ce par une fonction de $\mathcal{C}$. Il est à noter que cette méthode, bien qu'elle possède un caractère constructif, ne permet pour autant de profiter pour en tirer des algorithmes intéressants car elle se sert, afin de normaliser les preuves, d'un théorème d'élimination des coupures dont le bornes connues à ce jour demandent un temps excessif.

En vue de cette correspondance entre la hiérarchie polynomiale et les théories S_2^i il n'est pas surprenant que l'étude de nombreuses questions concernant la première soit étroitement liée aux secondes. Krajíček, Pudlák et Takeuti [14] ont démontré que si S_2 est finiment axiomatizable alors la hiérarchie polynomiale collapse. Plus tard ceci a été amélioré indépendamment par Buss [5] et Zambella [31] : S_2 est finiment axiomatisable si et seulement si elle est capable de prouver le collapse de PH. Il est relativement simple de voir que chaque S_2^i est finiment axiomatizable (voir [10]), donc la même question pour S_2 est équivalente à savoir si les théories S_2^i forment une hiérarchie stricte, d'où l'importance d'obtenir des résultats de conservation, même partiels, entre ces théories.

Ces dernières années, sous l'impulsion notamment de Pollett [20], d'autres sous-systèmes plus généraux $\hat{T}^{i,\tau}$ ont commencé à être étudiés. Ceux-ci comprennent essentiellement un schéma d'induction de longueur $t(x)$, pour t appartenant à un ensemble de termes τ et des formules $\hat{\Sigma}_i^b$ (une sous-classe de Σ_i^b). Quand τ ne contient que des termes à croissance lente, tels que la fonction $|x|$ itérée plusieurs fois, ces systèmes capturent des classes de complexité plus petites mais où l'on retrouve souvent la même problématique que pour PH, savoir si les inclusions sont strictes. D'où l'intérêt d'obtenir des *witnessing theorems* et des résultats de conservation pour ces théories. Une autre motivation pour l'étude de ces systèmes est de montrer d'éventuels résultats d'indépendance des principales questions concernant PH. Résultats par exemple du type "on ne peut pas prouver le collapse de PH dans le système $\mathcal{S}$", où $\mathcal{S}$ serait quand même capable de formaliser certains arguments connus de complexité servant à séparer d'autres classes. Quelques résultats ont été obtenus dans cette direction par Pollett [21].

La plus uniforme des méthodes utilisées pour obtenir ces résultats continue de se fonder sur la technique de Buss. Néanmoins Wilkie [28] donna une preuve du théorème de Buss par construction d'un modèle non standard (voir une version simplifiée de Pudlák dans [10]). Zambella, qui considère des systèmes du second ordre, en donna une autre dans [31]. Ressayre [22] utilisa une construction de modèle pour démontrer un résultat de conservation. Dans [13] on trouve d'autres constructions de modèles ainsi que dans les travaux de Boughattas (voir [3] par exemple) et Sureson [25].

Problèmes abordés dans la thèse et résultats

Le problème de conservation entre S_2^1 et R_2^2 (voir [7],[2]) a été la motivation pour le travail du chapitre 2 . R_2^2 est la théorie avec induction de longueur $||x||$ pour des formules Σ_2^b, notée aussi Σ_2^b-$LLIND$. En effet, bien que ceci était méconnu par l'auteur au moment d'entreprendre les recherches, il avait été démontré par Buss,Krajíček, Takeuti [7] que $S_3^1 \equiv_{\forall\Sigma_2^b} R_3^2$, l'indice 3 indiquant la présence dans le langage de $\#_3$ (cela revient à se placer dans le contexte des fonctions calculables en temps superpolynomial $2^{|n|^\omega}$). La question se posait si ce résultat était valable pour le langage L_2.

On peut déjà se poser la question de savoir s'il y a conservation entre S_2^1 et R_2^2 pour les formules Σ_1^b. Grâce au théorème de Buss on connaît la classe de fonctions Σ_1^b-définissable dans S_2^1, les fonctions calculables en temps polynomial. Afin de pouvoir démontrer $S_2^1 \equiv_{\forall\Sigma_1^b} R_2^2$ il suffit de construire un sous-modèle Σ_0^b-élémentaire de R_2^2 à l'intérieur d'un ensemble, que nous appelons ressource, de la forme $R(a) = \{M(a)$: M est une machine de Turing de code $\leq r$ et $M(a)$ est calculé en moins de $|a|^r$ étapes $\}$ pour $a, r \in M\backslash\mathbb{N}$, M étant un modèle de S_2^1.

La difficulté de réaliser une telle construction réside dans le fait qu'on veut satisfaire l'induction pour des formules Σ_2^b sans avoir recours à des oracles, comme c'est le cas dans la preuve de Wilkie du théorème de Buss. En effet, dans celle là on prouve, dans le cas $i = 2$, que $T_2^1 \equiv_{\forall\Sigma_2^b} S_2^2$ en construisant un modèle de Σ_2^b-$LIND$ dans une ressource $R(a)$ comme ci-dessus, sauf que les fonctions ont accès à un oracle pour décider sur les prédicats Σ_1^b. Certes, ici on demande seulement d'aller jusqu'à $||x||$, mais le problème de traiter deux alternances de quantificateurs reste.

Les constructions réalisées dans ce chapitre, fondées sur une idée de Ressayre, donnent une solution à ce problème dans le cas où le langage est enrichi avec $\#_3$. On démontre donc qu'il est possible de construire un modèle de $\hat{\Sigma}_2^b$-$LLIND(\#_3)$, qu'on note $\hat{R}_3^2$, à l'intérieur d'une ressource $R(a) = \{M(a)\ :\ M$ est une machine de Turing de code $\leq r$ et $M(a)$ est calculé en moins de $2^{||a||^r}$ étapes $\}$ caractérisant de cette façon les fonctions Σ_1^b-définissables de $\hat{R}_3^2$ et obtenant ainsi une preuve par théorie des modèles d'un résultat de conservation entre S_3^1 et $\hat{R}_3^2$ pour des formules Σ_1^b. Ce travail a fait l'objet d'une publication [8] et une version plus simple contenant des résultats antérieurs se trouve dans [9].

Deux questions se posent ensuite. Premièrement, est-il possible de réaliser une construction similaire si on se restreint au langage L_2? Deuxièmement, y a-t-il une construction permettant d'étendre ce résultat à $\hat{R}_3^2 \equiv_{\forall\hat{\Sigma}_2^b} S_3^1$?

L' intérêt de la première question est évident, une réponse positive donnerait une solution au problème de conservation entre S_2^1 et R_2^2, au niveau des formules Σ_1^b. Nous avons cherché dans cette direction. La ressource $R(a)$ en temps polynomial paraît petite pour abriter un modèle de R_2^2. On peut alors essayer de contruire un modèle d'une théorie entre R_2^2 et S_2^1. Nous pouvons traiter l'induction Σ_2^b jusqu'à $|||x|||$ mais à priori cette théorie ne contient pas S_2^1 et la ressource s'épuise avant de pouvoir inclure des axiomes pour cette partie là. Ceci conduit à se demander dans quelle type de ressource on peut mener à bien la construction d'un modèle de R_2^2 sans avoir recours à des oracles. En trouvant des exemples naturels de telles ressources on peut par comparaison avec $(FNC)^{NP}$, la classe des fonctions Σ_1^b-définissable de R_2^2, conjecturer sur la validité de la conservation entre les deux théories. Cette technique ouvre donc des perspectives intéressantes pour continuer les recherches et rejoint l'esprit du travail mené depuis plusieurs années par Boughattas et Ressayre où l'on cherche des ressources appropiées pouvant servir de terrain pour construire des modèles de l'arithmétique.

Toutefois il faut noter aussi qu'il est naturellement possible qu'une telle construction ne soit pas réalisable et que le résultat de conservation cherché s'avère faux. En effet, Bloch [2] démontre que celui-ci impliquerait l'égalité entre les classes NC^1 et NC, collapse qu'il considère peu probable. Mais l'approfondissement des idées allant dans la direction positive paraît un moyen important d'éclaircir le problème de toute manière.

La deuxième question est intéressante dans le sens où elle conduit à chercher des sous-modèles Σ_1^b-élémentaires et pour ce faire on est amené à considérer des ressources $R(a)$ où les functions ont accès à des oracles NP, des fonctions Σ_2^b-définissables. Mais pour avoir une bonne caractérisation de ces fonctions dans les théories $\hat{T}_2^{i,\tau}$ on admet des multifonctions. Une multifunction f est une relation binaire totale, c.à.d. telle que $\forall x \exists y f(x,y)$. On sait que $\hat{T}_2^{i,\tau}$ peut $\hat{\Sigma}_{i+1}^b$-définir les multifonctions de la classe $FP^{\Sigma_i^p}(wit, |\tau|)$ (voir [20]) . Celle-ci correpond en particulier à un modèle de calcul par machine de Turing avec une borne polynomiale pour le temps et la possibilité de consulter un oracle Σ_i^p au plus $|\tau|$ fois. Le *wit* indique que cet oracle est en mesure de fournir des témoins pour le quantificateur existentiel de la question posée dans le cas où la réponse est affirmative. Une possibilité est alors d'élargir ainsi nos ressources, les propriétés d' élémentarité requises étant satisfaites. Mais la façon de traiter les multifonctions n'est pas claire. Ces classes manquent de certaines propriétés et par conséquent les techniques employées pour les fonctions ne s'appliquent pas. On aborde cette question dans la première partie du chapitre 3, où l'on expose une façon de dépouiller les multifonctions en ne gardant que les "bonnes" images afin d'obtenir des propriétés de clôture adéquates. On obtient comme corollaire une preuve modèle-théorique du *witnessing theorem* pour $\hat{\Sigma}_{i+1}^b$-définissabilité dans $\hat{T}_2^{i,2^{|\tau|^\omega}}$ prouvé par Pollett [20]. En conjonction avec une idée de Visser utilisée par Zambella [31] pour donner une preuve par modèles du théorème de Buss, on emploie dans la deuxième partie notre contruction pour étendre un modèle de $\hat{T}_2^{i,2^{|\tau|^\omega}}$ en un de $\hat{T}_2^{i+1,|\tau|}$. Il découle de ceci les théorèmes de *witnessing* et conservation correspondant pour ces théories (voir [20]) et en particulier $S_3^1 \equiv_{\forall \hat{\Sigma}_2^b} \hat{R}_3^2$ donc une réponse affirmative à la deuxième question posée ci-dessus.

La technique utilisée dans le chapitre 2 exploite les possibilités des modèles non standard en considérant des machines de Turing indexées par des entiers inférieurs à r pour un "petit" $r > \mathbb{N}$ et calculant en un temps polynomial à exposant aussi non standard. Dans le chapitre 3 les multifonctions sont entièrement standard, mis à part le fait qu'elles peuvent utiliser d'autres paramètres. S'ouvre donc ici une voie de recherche, à savoir, combiner les deux méthodes en utilisant le débordement pour une classe de multifonctions. Celle-là pourrait servir, par exemple, pour tenter de construire un modèle de R_2^2 dans une ressource obtenue par débordement à partir de $FP^{\Sigma_1^p}(wit, \log n)(a)$, où $a \in M \models S_2^1$.

Dans le chapitre 4 on considère des schémas dits "de remplacement". Dans [20] l'auteur prouve des *witnessing theorems* pour $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$ caractérisant ainsi la classe des fonctions Σ_{i+1}^b-définissables par cette théorie, et obtient en comparant avec les résultats analogues pour $\hat{T}_2^{i,|\tau|}$ des théorèmes de conservation entre les deux théories. Nous proposons ici une approche purement modèle-théorique de ce problème au moyen d'une technique complètement différente de celles des premiers chapitres. Elle a été employée par Ressayre [22] pour démontrer la Σ_{i+1}^b conservation de Σ_{i+1}^b-$REPL$ sur S_2^i et garde quelques points de contact avec la preuve du résultat analogue de $\forall\Sigma_{n+1}$-conservation entre $B\Sigma_{n+1}$ et $I\Sigma_n$ (voir [10] p.230). Cette méthode est assez propre dans le sens où elle utilise presque directement les axiomes, en plus de quelques outils classiques de Théorie des Modèles comme la saturation récursive. Aucune considération concernant, par exemple, la classe des fonctions définissable par la théorie, n'est nécessaire. Afin d'appliquer cette méthode dans le cas plus général des théories $\hat{T}_2^{i,\tau}$ un travail préliminaire plus attentif est nécessaire. En particulier on a besoin d'un argument de débordement appliqué au schéma d'induction, mais nous restons dans un cadre purement modèle théorique. Nous obtenons ainsi des preuves directes des résultats de conservation pour Σ_{i+1}^b-conséquences entre $\hat{T}_2^{i,|\tau|}$ et $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$.

Chapter 1

Bounded Arithmetic

We introduce basic notions of Bounded Arithmetic and do some preliminary work, proceeding rather swiftly as this is well done in many texts. Good introductory references for these topics are [4], [10], [13]. Here we follow the more general approach of [20].

1.1 Basic notions

Buss introduced in [4] a language which extends that of Peano's Arithmetic by adding $\lfloor \frac{x}{2} \rfloor, |x|, \#$ where $\lfloor \frac{x}{2} \rfloor$ means the integer part of $\frac{x}{2}$, $|x|$ is the length of the binary expansion of x and $x \# y$ (read x smash y) means $2^{|x| \cdot |y|}$. Following [27] and [20] we add two other primitives to allow smooth bootstrapping and to be able to consider theories with very little induction.

Definition 1.1.1 (Language of Bounded Arithmetic)

$$L_{BA} := \{0, +, \cdot, \dot{-}, \leq, \lfloor \frac{x}{2} \rfloor, |x|, \#, MSP\}$$

Here $x \dot{-} y$ means $x - y$ if $y \leq x$ and 0 otherwise. MSP is for "the most significant part": $MSP(x, i)$ corresponds to number x without the last i digits, i.e. $\lfloor \frac{x}{2^i} \rfloor$. As usual we use 1 for $S(0)$, 2 for $S(S(0))$, etc.

In chapter 2 we will consider a language containing a symbol $\#_3$ allowing a bit more exponentiation than $\#$.

Definition 1.1.2 *BASIC is the open theory consisting in the following axioms:*

1. $x \leq y \rightarrow x \leq S(y)$
2. $x \neq S(x)$

3. $0 \leq x$

4. $(x \leq y \wedge x \neq y) \rightarrow S(x) \leq y$

5. $x \neq 0 \rightarrow 2x \neq 0$

6. $x \leq y \vee y \leq x$

7. $(x \leq y \wedge y \leq x) \rightarrow x = y$

8. $(x \leq y \wedge y \leq z) \rightarrow x \leq z$

9. $|0| = 0$

10. $x \neq 0 \rightarrow (|2x| = S(|x|) \ \wedge \ |S(2x)| = S(|x|))$

11. $|1| = 1$

12. $x \leq y \rightarrow |x| \leq |y|$

13. $|x\#y| = S(|x|.|y|)$

14. $0\#x = 1$

15. $x \neq 0 \rightarrow (1\#2x = 2(1\#x) \ \wedge \ 1\#S(2x) = 2(1\#x))$

16. $x\#y = y\#x$

17. $|x| = |y| \rightarrow x\#z = y\#z$

18. $|x| = |y| + |z| \rightarrow x\#t = (y\#t).(z\#t)$

19. $x \leq x + y$

20. $x + 0 = x$

21. $x + (y + 1) = (x + y) + 1$

22. $(x + y) + z = x + (y + z)$

23. $x + y \leq x + z \leftrightarrow y \leq z$

24. $x.0 = 0$

25. $x.S(y) = x.y + x$

26. $x.y = y.x$

27. $x.(y + z) = x.y + x.z$

28. $1 \leq x \rightarrow (x.y \leq x.z \leftrightarrow y \leq z)$

29. $x \neq 0 \rightarrow |x| = S(|\lfloor \frac{x}{2} \rfloor|)$

30. $x = \lfloor \frac{y}{2} \rfloor \leftrightarrow (2x = y \vee S(2x) = y)$

31. $MSP(a, 0) = a$

32. $MSP(a, S(i)) = \lfloor \frac{MSP(a,i)}{2} \rfloor$

33. $x \dot{-} y = z \leftrightarrow (y + z = x \vee (z = 0 \wedge x \leq y))$.

As we have passed to a richer language we can already do some bootstrapping without need of induction. In particular it is possible to define some kind of coding functions as L_{BA}-terms.

Definition 1.1.3 *For $n, k \in \mathbb{N}$ new L_{BA}-terms are defined by:*

$$\begin{aligned}
2^{|y|} &:= 1\#y \\
2^{|y|^n} &:= \lfloor \frac{2^{|y|^{n-1}}}{2} \rfloor \#y \\
2^{k.|y|^n} &:= 2^{|y|^n}.2^{(k-1).|y|^n} \\
cond(x, y, z) &:= (1 \dot{-} x).y + (1 \dot{-} (1 \dot{-} x)).z \\
K_{\leq}(x, y) &:= 1 \dot{-} (x \dot{-} y) \\
max(x, y) &:= cond(K_{\leq}(x, y), x, y) \\
min(x, y) &:= cond(K_{\leq}(x, y), y, x) \\
2^{min(|y|,x)} &:= MSP(2^{|y|}, |y| \dot{-} x) \\
LSP(x, i) &:= x \dot{-} MSP(x, i).2^{min(|x|,i)} \\
\hat{\beta}(x, |y|, w) &:= MSP(LSP(w, Sx.|y|), x.|y|) \\
Bit(x, i) &:= \hat{\beta}(i, 1, x) \\
\dot{\beta}(x, |y|, z, w) &:= cond(K_{\leq}(\hat{\beta}(x, |y|, w), z), \hat{\beta}(x, |y|, w), z).
\end{aligned}$$

Definition 1.1.4 (Bounded formulas)

- *Quantifiers of the form* $Qx \leq t$, *where* t *is a term, are called bounded quantifiers. Those of the form* $Qx \leq |t|$ *are called sharply bounded quantifiers.*

- *Formulas with only sharply bounded quantifiers are called sharply bounded formulas. This class is noted* Δ_0^b, Σ_0^b *or* Π_0^b.

- *For* $i \geq 0$, Σ_{i+1}^b Π_{i+1}^b *are the smallest classes of formulas satisfying*

  1. $\Sigma_i^b \cup \Pi_i^b \subset \Sigma_{i+1}^b \cap \Pi_{i+1}^b$
  2. *Negations of* Π_{i+1}^b *are* Σ_{i+1}^b, *and negations of* Σ_{i+1}^b *are* Π_{i+1}^b
  3. *Both* Σ_{i+1}^b *and* Π_{i+1}^b *are closed by* $\wedge$, $\vee$, *and sharply bounded quantifiers.*
  4. Σ_{i+1}^b *is closed under bounded existential quantification.*
  5. Π_{i+1}^b *is closed under bounded universal quantification.*

- *If* T *is any theory and* $i \geq 1$, *we say that* ψ *is* $\Delta_i^b(T)$ *if*

$$T \vdash (\psi \leftrightarrow \psi_1) \wedge (\psi \leftrightarrow \psi_2)$$

  for some $\psi_1 \in \Sigma_i^b$ *and* $\psi_2 \in \Pi_i^b$.

Another hierarchy of formulas arises if we construct them by alternating existential and universal bounded quantifiers. They are said to be in strict *or* prenex *form.*

- $\hat{\Sigma}_0^b$ *are formulas of the form* $\exists x \leq |t| \psi$ *with* ψ *open.*

- $\hat{\Pi}_0^b$ *are formulas of the form* $\forall x \leq |t| \psi$ *with* ψ *open.*

- $\hat{\Sigma}_{i+1}^b$ *are formulas of the form* $\exists x \leq t\, \psi$ *with* $\psi \in \hat{\Pi}_i^b$.

- $\hat{\Pi}_{i+1}^b$ *are formulas of the form* $\forall x \leq t\, \psi$ *with* $\psi \in \hat{\Sigma}_i^b$.

- *If* T *is any theory and* $i \geq 1$, $\hat{\Delta}_i^b(T)$ *is defined analogously as* $\Delta_i^b(T)$.

For any set of formulas Ψ *we say that* $\alpha \in \mathcal{B}(\Psi)$ *if* α *is a boolean combination of formulas of* Ψ.

Clearly every $\hat{\Sigma}_i^b$-formula is Σ_i^b but the converse need not to be true. In the standard structure Σ_i^b and $\hat{\Sigma}_i^b$-formulas define the same sets. For $i \leq 1$ they are those in the i-th level of the polynomial hierarchy, Σ_i^p. Some theories are strong enough to prove that every Σ_i^b formula is equivalent to a $\hat{\Sigma}_i^b$ one. This has to do with replacement schemes, defined in section 1.2. In this thesis we will be mainly concerned with the prenex hierarchy.

Definition 1.1.5 (Induction) *Given $\alpha(x)$, a formula which might contain parameters, we note by α-IND^a the formula*

$$\alpha(0) \wedge \forall x(\alpha(x) \rightarrow \alpha(x+1)) \;\rightarrow\; \alpha(a).$$

When Ψ is a set of formulas and τ a set of unary terms, we note Ψ-IND^τ the scheme $\{\alpha\text{-}IND^{l(x)} : \alpha \in \Psi, l \in \tau\}$. In particular we write α-IND for α-IND^x, α-$LIND$ for α-$IND^{|x|}$ and α-$LLIND$ for α-$IND^{||x||}$.

Definition 1.1.6 ([20]) *$EBASIC$ is the theory containing $BASIC$ plus the following axioms:*

1. $y < 2^{min(u.|z|,|z|^2)} \rightarrow MSP(x.2^{min(u.|z|,|z|^2)} + y, min(u.|z|, |z|^2)) = x$
2. $y < 2^{|z|} \wedge x < 2^{|z|} \rightarrow (\hat{\beta}(0, |z|, x.2^{|z|} + y) = y \wedge \hat{\beta}(1, |z|, x.2^{|z|} + y) = x)$
3. $S(x).|y| \leq u \rightarrow \hat{\beta}(x, |y|, w) = \hat{\beta}(x, |y|, LSP(w, u))$.

These axioms are necessary to get a form of pairing and coding in our theories. It has to be noted that they can be derived from $BASIC$ using only *Open-LIND* [20], so assuming them costs very little.

Now there are many ways $EBASIC$ can define a pairing function. We will use the following one as it has a clear decoding function. To form a pair $\langle x, y\rangle$ you first add, if necessary, some leading zeros to the binary representation of the shortest number to give them both the same length. Then you add a leading 1 bit to both and finally concatenate them. To recover the coordinates you only have to cut $\langle x, y\rangle$ in half and then delete the first bit of each half.

Definition 1.1.7 ([20]) *For any x, y we set*

$$\langle x, y\rangle := (2^{|max(x,y)|} + y).2^{|max(x,y)|+1} + (2^{|max(x,y)|} + x).$$

The two projection functions are defined by

$$\langle z\rangle_1 := DMSB(LSP(z, \lfloor\frac{|z|}{2}\rfloor))$$

$$\langle z\rangle_2 := DMSB(MSP(z, \lfloor\frac{|z|}{2}\rfloor))$$

where $DMSB$ means "delete most significant bit":

$$DMSB(x) := x \dot{-} \lfloor\frac{2^{|x|}}{2}\rfloor.$$

Contrary to the classical Cantor form of pairing, not every integer codes a pair here. Nevertheless you can define a simple predicate saying when this holds. Moreover you can prove in $EBASIC$ existence and uniqueness of this pairing code.

Definition 1.1.8 ([20]) *We note $ispair(z)$ the following formula*

$$Bit(z, \lfloor \frac{|z|}{2} \rfloor \dot{-} 1) = 1 \ \wedge \ 2.|maz(\langle z \rangle_1, \langle z \rangle_2)| + 2 = |z|.$$

Theorem 1.1.9 *$EBASIC$ proves the following*

1. $\forall x \forall y \exists ! z (ispair(z) \wedge \langle z \rangle_1 = x \wedge \langle z \rangle_2 = y)$
2. $\langle x, y \rangle \leq 16 \cdot max^2(x, y)$, *for* $x, y \neq 0$.

Pairing will allow us to contract variables that are consecutively quantified by the same kind of quantifier. Consider for example a formula $\exists x \exists y \, \alpha(x, y, u)$. Over $EBASIC$ this is equivalent to $\exists z (ispair(z) \wedge \alpha(\langle z \rangle_1, \langle z \rangle_2, u)$. This argument will be used frequently, sometimes implicitly.

Definition 1.1.10 *For $n \geq 3$ we inductively define n-tuples by*

$$\langle x_1, \ldots, x_n \rangle := \langle \langle x_1, \ldots, x_{n-1} \rangle, x_n \rangle$$

and we note by $\langle z \rangle_n$ the n-th projection.

Of course a theorem like 1.1.9 can be proved for any n. In particular we need the notion of triple which can be easily defined as follows. Please note that we use $\equiv$ for syntactical equality.

Definition 1.1.11 $istriple(z) \equiv ispair(z) \wedge ispair(\langle z \rangle_1)$.

Definition 1.1.12 *Sequences w are triples $\langle cd(w), mx(w), lh(w) \rangle$ where $cd(w)$, $mx(w)$, and $lh(w)$ are intended to mean respectively the code, the maximal number coded and the length of the sequence. So we put simply $seq(w) \equiv istriple(w)$ and define functions $cd(w), mx(w), lh(w)$ as the three projections when w is a sequence, and 0 otherwise. Decoding is defined by*

$$(w)_x = \begin{cases} \hat{\beta}(x, |mx(w)|, cd(w)) & \text{if } seq(w) \text{ and } x < lh(w) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1.13 *For $i \geq 0$ and τ any set of unary terms, $\hat{T}_2^{i,\tau}$ is the theory*

$$EBASIC \ \cup \ \hat{\Sigma}_i^b\text{-}IND^\tau.$$

For τ being $\{x\}$, $\{|x|\}$ or $\{||x||\}$, we call it respectively $\hat{T}_2^i$, $\hat{S}_2^i$ and $\hat{R}_2^i$.

The following theorem states a well known property of some induction schemes.

Lemma 1.1.14 *$\hat{T}_2^{i,\tau}$ is equivalent to the theory $EBASIC \ \cup \ \hat{\Pi}_i^b\text{-}IND^\tau$.*

Notation: If t is any L_{BA}-term, $T_2^{i,t(\omega)}$ is the theory $\bigcup_{n\in\omega} T_2^{i,t(n)}$. Similarly, if τ is a set of L_{BA}-terms, τ^ω is the set $\bigcup_{n\in\omega}\{l^n \ : \ l \in \tau\}$.

1.2 Other schemes

In this section we introduce other axiom systems, some of them being equivalent to an induction scheme. We will use them especially in the last two chapters.

Definition 1.2.1 (Replacement) *Given $\alpha(x,y)$, a formula which might contain parameters, we note by α-$REPL_b^a$ the formula*

$$\forall x \leq a \exists y \leq b \ \alpha(x,y) \rightarrow \exists w \ \forall x \leq a((w)_x \leq b \ \wedge \alpha(x,(w)_x)).$$

When Ψ is a set of formulas and τ a set of unary terms, Ψ-$REPL^\tau$ is the scheme

$$\{\forall a \forall b \ \alpha\text{-}REPL_b^{l(a)} \ : \ \alpha \in \Psi, l \in \tau\}.$$

Theorem 1.2.2 *For $i \geq 1$, $\hat{T}_2^{i,|\tau|} \vdash \hat{\Sigma}_i^b$-$REPL^{|\tau|}$.*

Proof This is a straightforward induction argument on the length of w. □

Therefore in the presence of $\hat{T}_2^{i,|\tau|}$ we can push inside $|\tau|$-bounded quantifiers in front of $\hat{\Sigma}_i^b$-formulas enlarging in this way the class of formulas $\hat{T}_2^{i,|\tau|}$-provably equivalent to $\hat{\Sigma}_i^b$-ones.

In particular, if $\tau = \{x\}$ then we are talking about the so-called sharply bounded quantifiers of Definition 1.1.4. If we can push inside those quantifiers then Σ_i^b-formulas become $\hat{\Sigma}_i^b$ after contraction of some variables, thus we have that $\hat{S}_2^i = S_2^i$, and incidentally $\hat{T}_2^i = T_2^i$ too, where S_2^i and T_2^i are the well known theories defined by Buss in [4].

By contracting variables it can be easily seen that $\hat{\Pi}^b_i$-$REPL^\tau$ implies $\hat{\Sigma}^b_{i+1}$-$REPL^\tau$. Thus we have

Lemma 1.2.3 *The schemes $\hat{\Pi}^b_i$-$REPL^\tau$ and $\hat{\Sigma}^b_{i+1}$-$REPL^\tau$ are equivalent over $EBASIC$.*

Definition 1.2.4 (Comprehension) *Given $\alpha(x)$, a formula which might contain parameters, we note by α-$COMP^a$ the formula*

$$\exists w \forall x \leq a \ (\alpha(x) \leftrightarrow Bit(w,x) = 1).$$

When Ψ is a set of formulas and τ a set of unary terms, Ψ-$COMP^\tau$ is the scheme

$$\{\forall a \ \alpha\text{-}COMP^{l(a)} \ : \ \alpha \in \Psi, l \in \tau\}.$$

The following theorem is from [20].

Theorem 1.2.5 *For $i \geq 1$ the theory $\hat{T}^{i,|\tau|}_2$ proves the $\hat{\Sigma}^b_i$-$COMP^{|\tau|}$ axioms.*

As $\hat{\Sigma}^b_i$-$COMP^{|\tau|}$ says that for arguments under $|\tau|$ we can substitute a $\hat{\Sigma}^b_i$-formula for an open one depending on an extra parameter, it follows that this system augmented with $Open$-$IND^{|\tau|}$ is equivalent to $\hat{T}^{i,|\tau|}_2$.

Theorem 1.2.6 *For $i \geq 1$, $Open$-$IND^{|\tau|} \cup \hat{\Sigma}^b_i$-$COMP^{|\tau|}$ implies $\hat{T}^{i,|\tau|}_2$.*

Definition 1.2.7 (Strong replacement) *Given $\alpha(x,y)$, a formula which might contain parameters, we note by α-$STRONG\ REPL^a_b$ the formula*

$$\exists w \forall x \leq a (\exists y \leq b \ \alpha(x,y) \rightarrow \ (w)_x \leq b \ \wedge \alpha(x,(w)_x)).$$

When Ψ is a set of formulas and τ a set of unary terms, Ψ-$STRONG\ REPL^\tau$ is the scheme

$$\{\forall a \forall b \ \alpha\text{-}STRONG\ REPL^{l(a)}_b \ : \ \alpha \in \Psi, l \in \tau\}.$$

Theorem 1.2.8 *For $i \geq 1$ the theory $\hat{T}^{i,|\tau|}_2$ proves the $\hat{\Sigma}^b_i$-$STRONG\ REPL^{|\tau|}$ axioms.*

Proof: Take $\alpha \in \hat{\Pi}^b_{i-1}$ and consider the logically valid formula

$$\forall x \leq |l(a)| \ \exists u \leq b \ (\exists y \leq b \ \alpha(x,y) \rightarrow \ \alpha(x,u)).$$

By Theorem 1.2.5 we can use $\hat{\Sigma}^b_i$-$COMP^{|\tau|}$ to substitute the formula $\exists y \leq b \ \alpha(x,y)$ by an open one $\phi(x,c)$ with one more parameter, the equivalence between both expressions being valid for $x \leq |l(a)|$. Then we can apply $\hat{\Sigma}^b_i$-$REPL^{|\tau|}$ available by Theorem 1.2.2 to the formula $\forall x \leq |l(a)| \ \exists u \leq b \ (\phi(x,c) \rightarrow \ \alpha(x,u))$, the expression in the scope of the $\exists u$ quantifier being $\hat{\Pi}^b_{i-1}$, and the result follows. □

Theorem 1.2.9 *For $i \geq 1$, Open-$IND^{|\tau|} \cup \hat{\Sigma}^b_i$-STRONG $REPL^{|\tau|}$ implies $\hat{T}_2^{i,|\tau|}$.*

Proof This is proved by induction on i using the fact that strong replacement axioms allow us to substitute a $\hat{\Sigma}^b_i$-formula for a $\hat{\Pi}^b_{i-1}$ one with an extra parameter for values under $|\tau|$. □

Following Zambella [31] we introduce another scheme for practical purposes. It will be used in section 3.4.

Definition 1.2.10 (Dependent choices) *Given $\alpha(j, x, y)$, a formula which might contain parameters, we note by α-DC^a_b the formula*

$$\forall j \forall x < b \exists y < b \ \alpha(j, x, y) \rightarrow$$
$$\exists w \forall j < a((w)_j < b \wedge (w)_{j+1} < b \wedge \alpha(j, (w)_j, (w)_{j+1})).$$

When Ψ is a set of formulas and τ a set of unary terms, Ψ-DC^τ is the scheme $\{\forall a \forall b \ \alpha\text{-}DC^{l(a)}_b : \alpha \in \Psi, l \in \tau\}$.

Lemma 1.2.11 *The schemes $\hat{\Sigma}^b_{i+1}$-DC^τ and $\hat{\Pi}^b_i$-DC^τ are equivalent over $EBASIC$.*

Proof By a simple contraction of variables. □

Theorem 1.2.12 *For $i \geq 1$ the theory $\hat{T}_2^{i,|\tau|}$ proves the $\hat{\Sigma}^b_i$-$DC^{|\tau|}$ axioms .*

Proof We derive $\hat{\Pi}^b_{i-1}$-$DC^{|\tau|}$. Let $\alpha(j, x, y) \in \hat{\Pi}^b_{i-1}$, $l \in \tau$, and suppose that $\forall j \forall x < b \, \exists y < b \ \alpha(j, x, y)$. Consider the formula $\theta(z)$ given by

$$\exists w \leq s(a, b) \forall j < |l(a)|(j \leq z \rightarrow (w)_j < b \wedge (w)_{j+1} < b \wedge \alpha(j, (w)_j, (w)_{j+1}))$$

where $s(a, b)$ is a suitable term, a bound for the sequence of length $|l(a)|$ with identical entries b for example. By Theorem 1.2.2 $\theta(z)$ is equivalent to a $\hat{\Sigma}^b_i$-formula. It is easily seen that we have

$$\theta(0) \wedge \forall z(\theta(z) \rightarrow \theta(z + 1)).$$

Then $\theta(|l(a)|)$ by $\hat{\Sigma}^b_i$-$IND^{|\tau|}$. Thus $\hat{T}_2^{i,|\tau|} \vdash \hat{\Pi}^b_{i-1}$-$DC^{|\tau|}$ and we conclude by Lemma 1.2.11. □

Theorem 1.2.13 *For $i \geq 0$, Δ^b_0-$IND^{|\tau|} \cup \hat{\Sigma}^b_i$-$DC^{|\tau|}$ implies $\hat{T}_2^{i,|\tau|}$.*

Proof We use induction on i. The case $i = 0$ is trivial. Let $\psi(j)$ be the $\hat{\Sigma}^b_{i+1}$-formula $\exists x \leq t(j)\varphi(j, x)$ where $\varphi \in \hat{\Pi}^b_i$ with possibly other parameters. Let $l \in \tau$ and suppose we are in a model of Δ^b_0-$IND^{|\tau|} \cup \hat{\Sigma}^b_{i+1}$-$DC^{|\tau|}$ satisfying

$$\psi(0) \wedge \forall j < |l(a)|(\psi(j) \to \psi(j + 1)).$$

Then

$$\exists x_0 \leq t(0)\varphi(0, x_0) \wedge$$

$$\forall j < |l(a)|\, \forall x < b\, \exists y < b(x \leq t(j) \wedge \varphi(j, x) \to y \leq t(j + 1) \wedge \varphi(j + 1, y))$$

where b is an element bounding all the $t(j)$ for $j \leq |l(a)|$. Fix such an x_0. Then we have

$$\forall j < |l(a)|\, \forall x < b\, \exists y < b$$

$$((j = 0 \to x = x_0) \wedge (x \leq t(j) \wedge \varphi(j, x) \to y \leq t(j + 1) \wedge \varphi(j + 1, y))).$$

As the lower part of the formula is equivalent to a Σ^b_{i+1} one, we obtain by $\hat{\Sigma}^b_{i+1}$-$DC^{|\tau|}$

$$\exists w\, \forall j < |l(a)|((w)_0 = x_0 \wedge ((w)_j \leq t(j) \wedge \varphi(j, (w)_j) \to (w)_{j+1} \leq t(j+1) \wedge \varphi(j, (w)_{j+1}))).$$

Putting $\tilde{\varphi}(j, w) \equiv (w)_j \leq t(j) \wedge \varphi(j, (w)_j)$ we have that $\tilde{\varphi} \in \hat{\Pi}^b_i$ and

$$\exists w(\tilde{\varphi}(0, w) \wedge \forall j < |l(a)|(\tilde{\varphi}(j, w) \to \tilde{\varphi}(j + 1, w))).$$

By induction hypothesis and Lemma 1.1.14 our model satisfies $\hat{\Pi}^b_i$-$IND^{|\tau|}$. Thus we get $\tilde{\varphi}(|l(a)|, w)$, hence $\exists x \leq t(|l(a)|)\varphi(|l(a)|, x)$, i.e. $\psi(|l(a)|)$. $\square$

Putting together the above results we get

Theorem 1.2.14 *For $i \geq 1$ and any set of unary terms τ the following schemes are equivalent in the presence of $EBASIC \cup \Delta^b_0$-$IND^{|\tau|}$:*

1. $\hat{\Sigma}^b_i$-$IND^{|\tau|}$
2. $\hat{\Sigma}^b_i$-$COMP^{|\tau|}$
3. $\hat{\Sigma}^b_i$-$STRONG\ REPL^{|\tau|}$
4. $\hat{\Sigma}^b_i$-$DC^{|\tau|}$.

1.3 Some model theory

To end this chapter we mention some classical model theoretic results that we use in this thesis. Apart from the compactness and Löwenheim-Skolem theorems, we use chain constructions and recursive saturation.

Theorem 1.3.1 (Union of $\hat{\Sigma}^b_i$-elementary chains) *Let $(M_n)_{n\in\omega}$ be an increasing chain of L_{BA}-structures such that for every $n \in \omega$*

$$M_n \prec_{\hat{\Sigma}^b_i} M_{n+1}.$$

Let $M := \bigcup_{n\in\omega} M_n$. Then for every $n \in \omega$, $M_n \prec_{\hat{\Sigma}^b_i} M$.

This hold also for unbounded formulas. In particular, $M_n \prec_{\forall\hat{\Sigma}^b_i} M_{n+1}$ implies $M_n \prec_{\forall\hat{\Sigma}^b_i} M$.

Proof By induction on the complexity of formulas. □

Theorem 1.3.2 (Preservation of $\forall\exists\mathcal{B}(\hat{\Sigma}^b_i)$-formulas) *Let $(M_n)_{n\in\omega}$ be an increasing chain of L_{BA}-structures such that for every $n \in \omega$*

$$M_n \prec_{\hat{\Sigma}^b_i} M_{n+1}.$$

Let $M := \bigcup_{n\in\omega} M_n$. Let Θ be a sentence of the form $\forall x\exists y\phi(x,y)$ with $\phi \in \mathcal{B}(\hat{\Sigma}^b_i)$ and such that for every $n \in \omega$, $M_n \models \Theta$. Then $M \models \Theta$.

Proof By induction on the complexity of formulas. □

As a consequence we get that the union of a $\hat{\Sigma}^b_i$-elementary chain of models of $\hat{T}^{i,\tau}_2$ is a itself a model of $\hat{T}^{i,\tau}_2$.

Corollary 1.3.3 *Let $(M_n)_{n\in\omega}$ be an increasing chain of models of $\hat{T}^{i,\tau}_2$ such that for every $n \in \omega$*

$$M_n \prec_{\hat{\Sigma}^b_i} M_{n+1}.$$

Let $M := \bigcup_{n\in\omega} M_n$. Then $M \models \hat{T}^{i,\tau}_2$.

Proof Just note that for $\alpha \in \hat{\Sigma}^b_i$ the formula α-IND^τ is $\forall\exists\hat{\Sigma}^b_i$. □

Theorem 1.3.4 (Tarski-Vaught criterion for BA) *Let $N \subset M$ be an L_{BA}-substructure such that for each formula $\phi(x,\bar{u}) \in \hat{\Pi}^b_i$, $t \in Term(L_{BA})$ and parameters $\bar{b} \in N$ the following holds:*

$$\text{if } M \models \exists x \le t(\bar{b})\phi(x,\bar{b}) \text{ then for some } a \in N, \quad M \models a \le t(\bar{b}) \wedge \phi(a,\bar{b}).$$

Then $N \prec_{\hat{\Sigma}^b_i} M$.

Proof By induction on the complexity of formulas. □

For the notion of recursive saturation and a proof of the following theorem we refer to [11].

Theorem 1.3.5 (Existence of recursively saturated models) *Let M be an L_{BA}-structure. Then there is an elementary extension M' of M of the same cardinality which is recursively saturated.*

Chapter 2

A model of $\hat{R}^2_3$ inside a sub-exponential time resource

This chapter contains essentially the paper published in Notre Dame Journal of Formal Logic [8] with some minor changes. We kept its original notation and it can be read independently of the rest of the dissertation. Using non standard methods we construct a model of $\hat{\Sigma}^b_2$-LLIND inside a "resource" of the form $\{M(a) : M$ is a Turing machine of code $\leq r$, and $M(a)$ is calculated in less than $2^{||a||^r}$ steps$\}$, where a, r are non standard parameters in a model of S^1_3.

2.1 Basic notions and results

We use Buss's notations (see [4]), working in the extended arithmetical language

$$L_3 = \{0, 1, +, \cdot, <, \lfloor x/2 \rfloor, |x|, \#_2, \#_3\}$$

where $|x|$ is the length of the binary expansion of x, $x\#_2y$ means $2^{|x|\cdot|y|}$ and $x\#_3y$ stands for $2^{|x|\#_2|y|}$. Most of Buss's results in [4] were stated for theories in language L_2 without the $\#_3$ symbol (read "smash 3"). But, as he pointed out, they readily generalise to languages L_i including a function symbol $\#_i$ with same rate of growing as function ω_{i-1} of [29] ($x\#_iy = 2^{|x|\#_{i-1}|y|}$), provided we substitute polynomial time by the corresponding S_i-time (also called $\#_i$-time in some texts). In particular, to language L_3 corresponds $2^{|n|^{O(1)}}$-time, to L_4 is $2^{2^{||n||^{O(1)}}}$-time, etc.
Quantifiers of the form $Qx \leq t$, where t is a term, are called bounded quantifiers. Those of the form $Qx \leq |t|$ are called sharply bounded quantifiers. Formulas with only sharply bounded quantifiers are called sharply bounded formulas. This class is noted Δ^b_0, Σ^b_0 or Π^b_0. For $i \geq 0$, Σ^b_{i+1} is the smallest class of formulas containing Σ^b_i, Π^b_i and negations of Π^b_{i+1}, and closed by $\wedge$, $\vee$, sharply bounded quantifiers and $\exists x \leq t$. Classes Π^b_i are defined analogously. A formula is said to be *strict*Σ^b_1 if it

has the form $\exists y \leq t[\Delta_0^b]$. More generally, a formula is $strict\Sigma_i^b$ if it has the form $\exists y \leq t[strict\Pi_{i-1}^b]$. We denote by $\hat{\Sigma}_i^b$ the class of $strict\Sigma_i^b$-formulas. The class $\hat{\Pi}_i^b$ is defined analogously. If T is any theory and $i \geq 1$, we say that Ψ is $\Delta_i^b(T)$ if $T \vdash (\Psi \equiv \Psi_1) \wedge (\Psi \equiv \Psi_2)$ for some $\Psi_1 \in \Sigma_i^b$ and $\Psi_2 \in \Pi_i^b$. By $\alpha(x)$-IND *up to* y we denote the formula

$$[\alpha(0) \wedge \forall x < y(\alpha(x) \rightarrow \alpha(x+1))] \ \rightarrow \ \alpha(y)$$

and if Γ is a class of formulas and $m \in \mathbb{N}$, Γ-L$^{(m)}$IND denote the scheme $\alpha(x)$-IND *up to* $|y|_m$ for α in Γ, where $|y|_m = |(|y|_{m-1})|$ and $|y|_0 = y$. In this chapter we are concerned with $m = 1, 2$ so we write LIND, LLIND and $||y||$ for L$^{(1)}$IND, L$^{(2)}$IND and $|y|_2$. BASIC$_3$ is a finite set of open axioms for the symbols of L_3 and S_3^i is the theory BASIC$_3 + \Sigma_i^b$-LIND (originally it is defined by another induction scheme called PIND, but these two axiomatisations are equivalent (see [6]). R_3^i is the theory

$$\text{BASIC}_3 + \Sigma_i^b\text{-LLIND}$$

By $\hat{S}_3^i$, $\hat{R}_3^i$ we denote the corresponding theories for strict formulas.

We shall suppose included in our language some other useful primitives. These are known to be definable from L_3 with a little amount of induction, and its inclusion does not increase the strength of theories containing S_3^1, for example. In particular we suppose in L_3 the Cantor pairing function $\langle x, y \rangle$ and its projections $\langle z \rangle_1$, $\langle z \rangle_2$, as well as a binary function $y = (c)_x$ for "y is the x-th element in the sequence coded by c". In general we will be able to code sequences of logarithmic length.

By Σ_i^b-replacement we denote the scheme

$$\forall x \leq |a| \exists y \leq b \Psi(x, y) \rightarrow \exists c \forall x \leq |a| \Psi(x, (c)_x)$$

for $\Psi \in \Sigma_i^b$. In fact c can be bounded by a term of L_3, so the conclusion is also Σ_i^b and, moreover, implies trivially the premise. Hence, this scheme allows to push inside sharply bounded quantifiers in Σ_i^b-formulas. This, together with the possibility to merge two consecutive quantifiers of the same type into a single one using coding, permits to put Σ_i^b-formulas in the strict form. As $\hat{S}_3^i \vdash \Sigma_i^b$-replacement, we have that $\hat{S}_3^i \equiv S_3^i$. On the other hand we have that $R_3^i \vdash \Sigma_i^b$-replacement (see [1]), but it is not known if this holds for $\hat{R}_3^i$. Nevertheless, we can derive in $\hat{R}_3^i$ the $\hat{\Sigma}_{i-1}^b$-LIND axioms, thus proving that $\hat{R}_3^i \vdash S_3^{i-1}$.

We note by S_3 the class of total functions which are computable in time $2^{|n|^{O(1)}}$. For an integer a we put $S_3(a) := \{f(a) : f \in S_3\}$ and we say that an L_3-structure K is S_3-closed if $S_3(a) \subset K$ for every $a \in K$.
Let $C(e, T, x, y)$ means

"y is calculated from x in time T by $\{e\}$, the Turing machine coded by e"

Later we will see that this is is definable in S_3^1. The aim of this chapter is to prove:

Theorem 2.1.1 *Let M be a countable non standard model of S^1_3. Let $a, r \in M \backslash \mathbb{N}$ and suppose that $M \models \exists y(y = 2^{2^{||a||^r}})$. Let $R = \{y : M \models \exists e \leq r\ C(e, 2^{||a||^r}, a, y)\}$. There is an L_3-substructure K^* of M such that*

1. *$a \in K^*$*
2. *K^* is S_3-closed, and so $K^* <_{\Delta^b_0} M$.*
3. *$K^* \subset R$*
4. *$K^* \models \hat{R}^2_3$.*

As a consequence we get two known corollaries. Their proofs are classic, we give it for the sake of completeness.

Corollary 2.1.2 *Let $\varphi(x, y)$ a Σ^b_1-formula and suppose that*

$$\hat{R}^2_3 \vdash \ \forall x\, \exists y\, \varphi(x, y) \ .$$

Then for some $f \in S_3$, $S^1_3 \vdash \forall x\, \varphi(x, f(x))$.

Corollary 2.1.3 *The theory $\hat{R}^2_3$ is $\forall\Sigma^b_1$-conservative over S^1_3.*

Proof of corollary 2.1.2 As explained above we can suppose $\varphi \in \hat{\Sigma}^b_1$. Then, using coding to merge two consecutive existential quantifiers into a single one, we can assume that φ is Δ^b_0. Let a be a new constant symbol and let T be the theory

$$S^1_3 \cup \{\forall y(C(e, 2^{||a||^k}, a, y) \rightarrow \neg\varphi(a, y)) : e, k \in \mathbb{N}\}$$

We claim that T is inconsistent.
Suppose the contrary and let

$$T' = T \cup \{\forall y(C(e, , 2^{||a||^k}, a, y) \rightarrow y < d) : e, k \in \mathbb{N}\}$$

where d is another new constant symbol. Clearly T' is also consistent. Let M be a countable model for it. As d is a bound for $S_3(a)$, M must be non standard. We have for every $r_0 \in \mathbb{N}$

$$M \models \forall k \leq r_0 \forall e \leq k \forall y(C(e, 2^{||a||^k}, a, y) \rightarrow \neg\varphi(a, y))$$

In particular

$$M \models \forall k \leq r_0 \forall e \leq k \forall y \leq d(C(e, 2^{||a||^k}, a, y) \rightarrow \neg\varphi(a, y))$$

As we will see later, this last formula is equivalent to a Π_1^b one in S_3^1, and we have $S_3^1 \vdash \Pi_1^b$-LIND. So by overspill it must be valid for some $r_0 \in M\backslash\mathbb{N}$.
If a is interpreted by some standard integer then $S_3(a) = \mathbb{N}$ and thus, as $M \models T$, we would have for every $y \in \mathbb{N}$ $M \models \neg\varphi(a, y)$. By elementarity this formula holds in $\mathbb{N}$, hence $\mathbb{N} \models \forall y \neg\varphi(a, y)$. As $\mathbb{N}$ is obviously a model of $\hat{R}_3^2$, this contradicts the hypothesis of the theorem.
So let suppose $a \in M\backslash\mathbb{N}$ and let $r \leq r_0$ such that $M \models \exists y < d\ (y = 2^{2^{||a||^r}})$ (see lemma 2.4.1). Then we have

$$M \models \forall e \leq r \forall y \leq d (C(e, 2^{||a||^r}, a, y) \rightarrow \neg\varphi(a, y))$$

By definition of R we have $y < 2^{2^{||a||^r}} < d$ for every $y \in R$, and so the last equation reads

$$M \models \forall y \in R\ \neg\varphi(a, y)$$

By theorem 2.1.1 there is a L_3-structure $K^* \subset M$ such that

1. $a \in K^*$
2. K^* is S_3-closed
3. $K^* \subset R$
4. $K^* \models \hat{R}_3^2$.

By (1),(2),(3) we have $K^* \models \forall y \neg\varphi(a, y)$, and by (4) $K^* \models \forall x\, \exists y\, \varphi(x, y)$. Thus we get a contradiction and the claim is proved.
As T is inconsistent, by compactness there is some $n, e_0, \dots, e_n, k_0, \dots, k_n \in \mathbb{N}$ such that

$$S_3^1 \vdash \bigvee_{i=0}^{n} \exists y (C(e_i, 2^{||a||^{k_i}}, a, y) \wedge \varphi(a, y))$$

By theorem on constants

$$S_3^1 \vdash \forall x \bigvee_{i=1}^{n} \exists y (C(e_i, 2^{||x||^{k_i}}, x, y) \wedge \varphi(x, y))$$

Let $f(x)$ be the result of the following search: for $i = 0$ to n we run $\{e_i\}$ on input x with clock $2^{||x||^{k_i}}$ looking for an output y satisfying $\varphi(x, y)$. Clearly $f \in S$ and by the last equation $S_3^1 \vdash \forall x \varphi(x, f(x))$. Hence the corollary is proved. □

Corollary 2.1.3 follows immediately.

Remarks

1. Buss, Krajíček, Takeuti [7] have shown a result stronger than this corollary: the theory R^2_3 is $\forall\Sigma^b_2$-conservative over S^1_3. We prove $S^1_3 \equiv_{\forall\hat{\Sigma}^b_2} \hat{R}^2_3$ in chapter 3 (see 3.4.7).

2. Theorem 2.1.1 can be generalised as follows: if $M \models S^i_3$, $i > 1$, we can consider a larger resource R by giving the Turing machines access to oracles in the i-th level of the S_3-time hierarchy. Then we can construct a Δ^b_{i-1}-elementary L_3-substructure K^* of M which is a model of $\hat{R}^{i+1}_3$. The corresponding witnessing and conservation corollaries follows similarly as 2.1.2 and 2.1.3.

3. To drop the "strict" in theorem 2.1.1 it would suffice to carry out the construction with formulas of the form $\forall x \leq |u| \exists y \leq t \forall z \leq s\ \psi$, $\psi \in \Delta^b_0$, instead of simply $\hat{\Sigma}^b_2$-formulas. The theory obtained in this way would prove Σ^b_2-replacement. But the inclusion of an extra quantifier, even a sharply bounded one, poses some problems. A solution for these could throw some light on how to treat the Σ^b_3 case without use of oracles. Note parenthetically that we cannot use oracles if we want sub-exponential time witnessing theorems, and this makes it non trivial to construct models for Σ^b_i induction axioms inside the corresponding resources.

The rest of the chapter is devoted to prove theorem 2.1.1. In section 2.2 we briefly explains how the proof goes. Section 2.3 presents some tools needed to work with Turing machines. Next we introduce the notions of sparse sequences and resources in 2.4, and finally we present construction of model K^* in section 2.5.

2.2 Sketch of the proof of theorem 2.1.1

Fix an enumeration of axioms θ-IND up to $||d||$ with parameters in M and θ running over $\hat{\Sigma}^b_2$-formulas. We construct K^* as the union of an increasing chain $(K_n)_{n<\omega}$. Let $K_0 = S_3(a) = \{f(a) : f \in S_3\}$ and let θ_1-IND *up to* l_1 be the first axiom in the enumeration having its parameters in K_0. We want $K_1 \supset_{L_3} K_0$, K_1 S_3-closed and satisfying

$$\neg\theta_1(0) \vee \exists j < l_1[\theta_1(j) \wedge \neg\theta_1(j+1)] \vee \theta_1(l_1)$$

where $\theta_1(j) \equiv \exists y \leq t \forall z \leq s\, \psi(j, y, z)$. We can suppose $r < ||a||$ and $r = 2^{|r|-1}$. Let $(T_j)_{j \leq l_1+2}$ be a decreasing sequence such that

$$2^{||a||^r} \gg T_0 \gg T_1 \gg \ldots \gg T_{l_1+2} \gg 1$$

where $A \gg B$ means $A > B.2^{||a||^{O(1)}}$, and such that the T_j's are easy to calculate from a and r (for example $T_j = 2^{||a||^r - (j+1)||a||^{r/2}}$). For $j = 0, \ldots, l_1 + 2$ let

$$R_j(x) = \{ \ y : C(e, T_j, x, y) \ for \ some \ e \leq r\}$$

K_1 will be generated by an element a_1 obtained by running on input a the next program P (which depends on a code for $|r|$):

1: Compute $r = 2^{|r|-1}$.
2: Compute the parameters of θ_1-IND *up to* l_1 and T_0 from the input a.
3: Put $j := 0$, $y_{-1} := 0$.
4: Compute T_{j+1}.
5: Look for $y_j \in R_j(\langle j, a, y_{j-1}\rangle)$, $y_j \leq t$, such that for every $z \in R_{j+1}(\langle j+1, a, y_j\rangle)$ such that $z \leq s$, $M \models \psi(j, y_j, z)$.
6: If there is no such y_j, stop the machine with output $a_1 = \langle j, a, y_{j-1}\rangle$.
7: If y_j is found and $j < l_1$, then put $j := j + 1$ and go to 4.
8: If y_{l_1} is found, stop the machine with output $a_1 = \langle l_1 + 1, a, y_{l_1}\rangle$.

Let $a_1 = \langle J_1 + 1, a, y_{J_1}\rangle$ and suppose for example $0 \leq J_1 < l_1$. Then we have

- for every $z \in R_{J_1+1}(a_1)$ such that $z \leq s$, $M \models \psi(J_1, y_{J_1}, z)$.
- for every $y \in R_{J_1+1}(a_1)$ such that $y \leq t$, there is some $z \in R_{J_1+2}(\langle J_1 + 2, a, y\rangle)$ such that $z \leq s$ and $M \models \neg\psi(J_1 + 1, y, z)$.

So, in order to have $K_1 \models \theta_1(J_1) \wedge \neg\theta_1(J_1 + 1)$, we choose K_1 contained in $R_{J_1+1}(a_1)$ and allowing computations in time T_{J_1+2} :

$$K_1 = \{ \ \{e\}(a_1) < 2^{2^{||a||^{O(1)}}} calculated \ in \ time < O(1).r^2.T_{J_1+2}, \ e < |r|^{O(1)}\} \ .$$

It is easy to see that $K_0 \subset_{L_3} K_1$ and K_1 is S_3-closed. To prove that $K_1 \subset R$ we use the fact that P can be coded by some $p < |r|^{O(1)}$ and calculates a_1 in less than $r^2.T_0$ steps.
Consider now θ_2-IND *up to* l_2, the next axiom in the enumeration having its parameters in K_1. We want $K_2 \supset_{L_3} K_1$ satisfying this axiom while preserving $\theta_1(J_1) \wedge \neg\theta_1(J_1+1)$. The new axiom will be satisfied by letting the construction of K_2 imitate that of K_1, replacing a, θ_1, l_1 by a_1, θ_2, l_2 and the sequence T_i by another sequence T'_i. As explained above, $\theta_1(J_1) \wedge \neg\theta_1(J_1+1)$ will be preserved if $K_2 \subset R_{J_1+1}(a_1)$ and K_2 allows computations in time T_{J_1+2}. In other words, the maximal computation times T'_i are chosen between T_{J_1+1} and T_{J_1+2} (for example $T'_j = T_{J_1+1}/2^{(j+1)||a||^{r/4}}$ if $T_j = 2^{||a||^r - (j+1)||a||^{r/2}}$). In this way

$$T_{J_1+1} \gg T'_0 \gg T'_1 \gg \ldots \gg T'_{l_2+2} \gg T_{J_1+2}$$

Let P' be a program similar to P, running on input a_1, with θ_2-IND *up to* l_2 and T'_i in place of θ_1-IND *up to* l_1 and T_i. Let $a_2 = \langle J_2+1, a_1, y_{J_2}\rangle$ be its output and

$$K_2 = \{\ \{e\}(a_2) < 2^{2^{||a||^{O(1)}}} \text{ calculated in time } < O(1).r^2.T'_{J_2+2},\ e < |r|^{O(1)}\}$$

Then we prove as above that $K_1 \subset_{L_3} K_2$, K_2 is S_3-closed, $K_2 \subset R$ and

$$K_2 \models \theta_1\text{-IND } \textit{up to } l_1 \ \wedge\ \theta_2\text{-IND } \textit{up to } l_2$$

In this way we get $K_3, K_4, \ldots$ and putting $K^* = \bigcup_{n<\omega} K_n$ we have the desired model. □

2.3 Definability of Turing machine computations

We call S_3 the set of total functions which are computable in time $2^{|n|^{O(1)}}$ in the standard structure $\mathbb{N}$. For a predicate X we say that $X \in S_3$ if its characteristic function belongs to S_3. Note that (the intended interpretation in $\mathbb{N}$ of) function symbols of L_3 are in S_3. In particular Δ_0^b predicates are decidable in time $2^{|n|^{O(1)}}$, therefore, S_3-closed substructures are Δ_0^b-elementary. This will be used everywhere. Σ_i^b predicates correspond exactly to predicates in the i-th level of the $2^{|n|^{O(1)}}$-time hierarchy.

We present here some known facts saying roughly that in any model of S_3^1 these functions are definable and have the expected properties, and this will also hold for some non standard functions when $M \neq \mathbb{N}$. Proofs are omitted since they are tedious and contains no new idea. For a reference see [4] and [10].

In order to formalise computations we consider deterministic k-tapes Turing machines, for a fixed $k \in \mathbb{N}$, and a natural coding of its programs and computations. If e is an index for a Turing machine,i.e. a code for its program, we note by $\{e\}$ both the machine itself and the function it computes. By $e \in S_3$ we mean $\{e\} \in S_3$ and $e \in \mathbb{N}$.

Lemma 2.3.1 *For every standard Turing machine M there is a $\Delta_1^b(S_3^1)$-formula $Comp_M(c, x)$ expressing that c is the code of a computation of M on input x.*

In S_3^1 we can code sequences of logarithmic length and there are terms $t_k(x)$ standing for $2^{2^{||x||^k}}$. In consequence we get

Lemma 2.3.2 *Every predicate in S_3 is Δ_1^b definable in S_3^1.*

Lemma 2.3.3 *For every standard Turing machine M*

$$S^1_3 \vdash \forall v \forall x \exists ! c(Comp_M(c,x) \wedge lh(c) = |v|)$$

where $lh(c)$ is the length of the computation coded by c.

If $M \models S^1_3$ and $log(M) := \{|y| : y \in M\}$, this lemma will allow us to define computations in time T provided $T \in log(M)$. In particular, as $2^{||a||^k} \in log(M)$ for every $k \in \mathbb{N}$, we have

Lemma 2.3.4 *Every function in S_3 is provably Δ^b_1 (total) in S^1_3.*

Remark By Buss's theorem (the version for S^1_3) every function provably Σ^b_1 in S^1_3 is in S_3 (see [4]). As a consequence every $\Delta^b_1(S^1_3)$ predicate is decidable in time $2^{|n|^{O(1)}}$.

Now using lemma 2.3.4 we can define a restricted version of an universal Turing machine which will be nevertheless able to simulate all functions in S_3.

Lemma 2.3.5 *There is a $\Delta^b_1(S^1_3)$-formula $U(e,v,x,y)$ expressing that e is the code of a (probably non standard) Turing machine and $\{e\}$ calculates y from x in less than $|v|$ steps.*

We assume that for every term $t(\bar{x})$ in L_3, if $\phi(\bar{x}, y)$ is the Δ^b_1 definition of the corresponding function in S_3, then $S^1_3 \vdash y = t(\bar{x}) \leftrightarrow \phi(\bar{x}, y)$.

Definition 2.3.6 *$C(e,T,x,y)$ is the $\exists\Delta^b_1$-formula $\exists v(|v| = T \ \wedge \ U(e,v,x,y))$. It means that the Turing machine $\{e\}$ running on input x stops with output y before T steps.*

Lemma 2.3.7 *There is $k_0 \in \mathbb{N}$ such that*

1. $S^1_3 \vdash \forall e, e' \ \exists e'' < (e.e')^{k_0} \ \forall x \ (\{e\}(\langle e', x\rangle) = \{e''\}(x))$
2. $S^1_3 \vdash \forall e, e' \ \exists e'' < (e.e')^{k_0} \ \forall T, T', x, y, z, d$
   $(T, T', T+T' < |d| \wedge C(e,T,x,y) \wedge C(e',T',y,z) \rightarrow C(e'', T+T', x, z))$.

Remarks

- Condition (1) will help us to estimate the code of a Turing machine. For example suppose that X is a multiplicative closed cut in a model of S^1_3 and M a Turing machine. If M can be viewed as a standard program with some extra inputs $p_1, \ldots, p_n \in X$, $n \in \mathbb{N}$, then by (1) M can be coded by some $p \in X$.
- By condition (2), if $e, e' \in X$ are Turing machine codes, then the composite function $\{e\} \circ \{e'\}$, if defined, has a code $e'' \in X$.

2.4 Sparse sequences, resources and basic structures.

Notation Let M be a non standard model of S_3^1 and F a function from $\mathbb{N}$ to M. We put

- $A > F(O(1))$ iff $A > F(n)$ for every $n \in \mathbb{N}$
- $F(O(1)) > B$ iff $F(n) > B$ for some $n \in \mathbb{N}$

Even in a nonstandard model we keep $O(1)$ running over standard constants.

Lemma 2.4.1 *Let M be a non standard model of S_3^1 and let $a, d \in M \backslash \mathbb{N}$ such that $S_3(a)$ is bounded by d. There is some $r \in M \backslash \mathbb{N}$ such that following properties hold in M:*

1. $\exists y < d \ (y = 2^{2^{||a||^r}})$.
2. *r is a power of 2, and so* $r = 2^{|r|-1}$.
3. $r < ||a||$.

Moreover, r can be chosen smaller than any given $r_0 \in M \backslash \mathbb{N}$.

Proof We know that for every $k \in \mathbb{N}$, $t_k(a) \in S_3(a)$ and $t_k(a) = 2^{2^{||a||^k}}$ in M. Thus we have for every $r_1 \in \mathbb{N}$,

$$M \models \forall k \leq |r_1| (\exists y < d \ y = 2^{2^{||a||^k}})$$

This formula is Σ_1^b in M and so by overspill it is true for some $r_1 \in M \backslash \mathbb{N}$. Now let $r_2 \in M \backslash \mathbb{N}$ such that $r_2 < |r_1|$ and $r_2 < ||a||$, and put $r = 2^{|r_2|-1}$. Then we have $r \in M \backslash \mathbb{N}$, r is a power of 2, as $|r_2| = |r|$, and finally $r \leq r_2 < ||a||$. □

Remarks

1. In fact we have proved $M \models \forall x \leq r \ \exists y < d \ (y = 2^{2^{||a||^x}})$.
2. By (1) of lemma 2.4.1 we have $[0, 2^{||a||^r}] \subset log(M)$ and then, by lemma 2.3.3, computations in time $T \leq 2^{||a||^r}$ are definable in M.
3. We want r to be computable from some Turing machine of code $< |r|^{O(1)}$. That is why we impose condition 2 (see (3) of lemma 2.4.6).

4. We want also $2^{||a||^r} \in S_3(\langle a, r\rangle)$. For this $r < ||a||^{O(1)}$ would suffice, we put $r < ||a||$ for simplicity. In this way $2^{||a||^r}$ is calculated from $\langle a, r\rangle$ by the function $\langle x, y\rangle \mapsto 2^{||x||^{min(y,||x||)}}$ which is clearly in S_3.

Definition 2.4.2 *Let M be a model of S_3^1, $A, B, l, \alpha \in M$, $(T_j)_{j\leq l}$ a sequence in M and F function from $\mathbb{N}$ to M. Suppose $A > B$.*

1. *The sequence $(T_j)_{j\leq l}$ is* **between** *A and B if $(T_j)_{j\leq l}$ is decreasing and $A > (T_j)_{j\leq l} > B$.*

2. *The sequence $(T_j)_{j\leq l}$ between A and B is* **generated** *by α if for some $e \in S_3$*

   - $T_0 = \{e\}(\langle\alpha, A\rangle)$
   - $T_{j+1} = \{e\}(\langle\alpha, T_j\rangle),\ j < l.$

3. *The sequence $(T_j)_{j\leq l}$ between A and B is $F(O(1))$-***sparse** *if*

   - $A > F(O(1)).T_0$
   - $T_j > F(O(1)).T_{j+1},\ j < l$
   - $T_l > F(O(1)).B$

Lemma 2.4.3 *Let M, a, r be as in lemma 2.4.1. Let $A, B, \alpha \in M$ and suppose that $2^{||a||^r} \geq A > B$, $a \in S_3(\alpha)$, $(T_j)_{j\leq l}$ is a sequence between A and B generated by α, and $l < 2^{||a||^{O(1)}}$. Then for some $e \in S_3$ we have $T_j = \{e\}(\langle j, \alpha, A\rangle),\ j \leq l$.*

Proof Let $e' \in S_3$ such that

$$T_0 = \{e'\}(\langle\alpha, A\rangle) \text{ and } T_{j+1} = \{e'\}(\langle\alpha, T_j\rangle),\ j < l$$

Let $k \in \mathbb{N}$ such that $l < 2^{||a||^k}$ and consider the standard Turing machine which on input $\langle j, \alpha, A\rangle$ calculates a from α, then $2^{||a||^k}$ (k is coded in its program); next it compares j and $2^{||a||^k}$ and if $j < 2^{||a||^k}$ it computes $\{e'\}^{(j+1)}(\langle\alpha, A\rangle)$. It runs in time $2^{|n|^{O(1)}}$ as $e' \in S_3$ and we iterate this function at most $2^{||a||^k}$ times (note that $2^{||a||^k} < 2^{||\alpha||^{O(1)}}$ as $a \in S_3(\alpha)$). Finally, we have that it calculates T_j when $j \leq l$. This can be proved by induction on l as $l \in log(M)$ and the condition considered is Δ_1^b. □

Lemma 2.4.4 *Let M, a, r be as in lemma 2.4.1. Let $A, B, l \in M$ and suppose that*

1. $2^{||a||^r} \geq A > 2^{||a||^{O(1)}}.B$

2. $l < ||a||^{O(1)}$

There is a $2^{||a||^{O(1)}}$*-sparse sequence* $(T_j)_{j\leq l}$ *between* A *and* B *generated by* $\langle a, \rho\rangle$ *for some* $\rho \in M\backslash\mathbb{N}$. *Moreover,* ρ *can be chosen smaller than any given non standard integer in* M*.*

Proof We have for every $k \in \mathbb{N}$

$$M \models \exists y \leq a(y = 2^{||a||^k} \wedge A > y.B)$$

By overspill this formula is true for some $\rho \in M\backslash\mathbb{N}$, and we can choose it as small as we want. Take $\rho < ||a||$ and consider the function

$$f(x, y, z) = msp(x, ||y||^{min(\lfloor z/2\rfloor, ||y||)})$$

where $msp(u, v)$ stands for $\lfloor u/2^v\rfloor$ when $v \leq |u|$ (msp is for "most significant part"; see [4]).

Then clearly $f \in S_3$ and so is g defined by $g(u, x) = f(x, \langle u\rangle_1, \langle u\rangle_2)$. Put

$$T_0 = g(\langle a, \rho\rangle, A) \text{ and } T_{j+1} = g(\langle a, \rho\rangle, T_j), \text{ for } j < l.$$

Then we have

- $T_0 = \lfloor A/2^{||a||^{\lfloor\rho/2\rfloor}}\rfloor$
- For $j < l$, $T_{j+1} = \lfloor T_j/2^{||a||^{\lfloor\rho/2\rfloor}}\rfloor$

It is then clear than $(T_j)_{j\leq l}$ is $2^{||a||^{O(1)}}$-sparse, between A and B and generated by $\langle a, \rho\rangle$. □

Definition 2.4.5 *Let* M *be a model of* S^1_3 *and let* $a, r, T, c \in M$*.*

- *We use* $\boldsymbol{R(r, T, c)}$ *to denote the subset* $\{y \in M : \exists e \leq r\ C(e, T, c, y)\}$*.*
  We call these definable sets **resources***.*
- *The basic* L_3*-structures we will consider are of the form*

  $$\{y \in M : \exists k \in \mathbb{N}\ \exists e < |r|^k\ (y < 2^{2^{||a||^k}} \wedge C(e, k.T, c, y))\}.$$

  We write $\boldsymbol{K(a, r, T, c)}$ *as an abbreviation for the expression above.*

Lemma 2.4.6 *Let M, a, r be as in lemma 2.4.1. Let $c, T \in M$ be such that $2^{||a||^r} > O(1).T$ and let $K = K(a, r, T, c)$. Then K has the following closure property:*

1. *If $y \in K$ and $T' < O(1).T$, then $K(a, r, T', y) \subset K$.*

Moreover, if $T > 2^{||a||^{O(1)}}$ then

2. *K is S_3-closed.*

3. *$[0, |r|^{O(1)}[\cup \{r\} \subset K$*

Proof 1. Let $T' < O(1).T$, $k \in K$, $e < |r|^k$, be such that $C(e, k.T, c, y)$. If $z \in K(a, r, T', y)$ then for some $k' \in \mathbb{N}$, $z < 2^{2^{||a||^{k'}}}$ and $C(e', k'.T', y, z)$ for some $e' < |r|^{k'}$. We have that

$$k.T + k'.T' < O(1).T < 2^{||a||^r}$$

Hence by 2 of lemma 2.3.7 there is some $k'' \in \mathbb{N}$, k'' sufficiently large, and some $e'' < |r|^{k''}$ such that $C(e'', k''.T, c, z)$, i.e. $z \in K$.

2. If $T > 2^{||a||^{O(1)}}$ and $z \in S_3(y)$ for some $y \in K$, then since $y < 2^{2^{||a||^{O(1)}}}$ we have that $z < 2^{2^{||a||^{O(1)}}}$ and $C(e, T', y, z)$ for some $e \in \mathbb{N}$ and $T' < 2^{||a||^{O(1)}} < T$. Hence $z \in K$ and K is S_3-closed.

3. If $p \leq |r|^{O(1)}$ there is some $e \leq |r|^{O(1)}$ such that $\forall x(\{e\}(x) = p)$ and $C(e, |p|, x, p)$ ($\{e\}$ is just a Turing machine that writes p regardless of the input; its program can be coded by some $e < |p|^{O(1)}$). As $|p| < 2^{||a||^{O(1)}} < T$ we have that $p \in K$.
In particular $|r| \in K$. Now, r can be calculated from $|r|$ easily by a standard Turing machine in S_3 because $r = 2^{|r|-1}$. Hence, by (2), $r \in K$. □

Remarks

- We will consider only structures $K(a, r, T, c)$ with $T > 2^{||a||^{O(1)}}$. By (2) we are guaranteed these structures will naturally be L_3-substructures of M and moreover, they will be Δ_0^b-elementary. In particular the $BASIC_3$ axioms will hold.

- In connection with lemma 2.4.4, condition (3) will be useful to generate $2^{||a||^{O(1)}}$-sparse sequences, any "small" non standard integer being available in K.

Lemma 2.4.7 *Let M, a, r be as in lemma 2.4.1. Let $c, c', T_2, T, T_{c'} \in M$ and let $K = K(a, r, T_2, c)$, $K' = K(a, r, T', c')$. Suppose that*

1. $c \in K'$

2. $2^{||a||^r} > O(1).T'$

3. $T' \geq T_2$.

Then $K \subset K'$.

Proof Let $z \in K$. Then $z < 2^{2^{||a||^{O(1)}}}$ and $C(e, k.T_2, c, z)$ for some $k \in \mathbb{N}$ and $e < |r|^k$. But $k.T_2 < O(1).T' < 2^{||a||^r}$ and $c \in K'$, hence, by lemma 2.4.6, $z \in K'$. □

Lemma 2.4.8 *Let* M, a, r *be as in lemma 2.4.1. Let* $c, c', T_1, T', T_{c'} \in M$ *and let* $K' = K(a, r, T', c')$. *Suppose that*

1. $C(p, T_{c'}, c, c')$ *for some* $p < |r|^{O(1)}$

2. $2^{||a||^r} \geq T_1 > T_{c'} + O(1).T'$

Then $K' \subset R(r, T_1, c)$.

Proof Let $y \in K'$ and let $k \in \mathbb{N}$, $e < |r|^k$ such that $C(e, k.T', c', y)$. We have that $C(p, T_{c'}, c, c')$ for some $p < |r|^{O(1)}$ and

$$T_{c'} + k.T' < T_1 \leq 2^{||a||^r}$$

By (2) of lemma 2.3.7 there is some $e' < |r|^{O(1)} < r$ such that $C(e', T_1, c, y)$, hence $y \in R(r, T_1, c)$. □

2.5 Constructing a model of $\hat{R}_3^2$

Let M, a, r be as in lemma 2.4.1. Let R denote the resource $R(r, 2^{||a||^r}, a)$. We call it the main resource. The aim of this section is to construct inside it a model K^* of $\hat{R}_3^2$ containing a. This model will be constructed as the union of an increasing chain $(K_n)_{n \in \mathbb{N}}$, each K_n satisfying a new instance of $\hat{\Sigma}_2^b$-LLIND while preserving those satisfied previously. First we prove the key lemma which will help us to pass from K_n to K_{n+1}.

Lemma 2.5.1 *Let* M, a, r *be as in lemma 2.4.1. Let* $c, T_1, T_2 \in M \backslash \mathbb{N}$ *and let* $K = K(a, r, T_2, c)$. *Let* $b_0, \ldots, b_m \in K$, $l \in log(log(K))$, $\psi(j, y, z, \bar{b})$ *a* Δ_0^b*-formula with parameters* $\bar{b}$ *and let*

$$\theta(j, \bar{b}) \equiv \exists y \leq t \, \forall z \leq s \, \psi(j, y, z, \bar{b})$$

where $t = t(j, \bar{b})$, $s = s(j, y, \bar{b})$ *are* L_3*-terms (parameters* $\bar{b}$ *will frequently be omitted). Suppose that*

a. *$a \in K$ and $c \in K(a, r, T_c, a)$ for some T_c such that $2^{||a||^r} > O(1).T_c$.*

b. *$T_1 \in K$ and $2^{||a||^r} \geq T_1 > T_2 > 2^{||a||^{O(1)}}$.*

c. *$(T_j)_{j \leq l+2}$ is a $||a||^{O(1)}$-sparse sequence between T_1 and T_2 generated by $\langle a, \rho \rangle$ for some $\rho \in K$.*

Then there are integers $p, q, c', Y \in M$, $J \in M \cup \{-1\}$, and an L_3-structure K' satisfying

1. *$p < |r|^{O(1)}$ and $C(p, r^2.T'_0, c, c')$.*
2. *$c' = \langle J+1, c, Y \rangle$, $-1 \leq J \leq l$ and $Y \leq t(J)$.*
3. *If $J \neq -1$ then $\forall z \in R(r, T'_{J+1}, c'), z \leq s(J, Y) \rightarrow \psi(J, Y, z)$*
4. *$q < |r|^{O(1)}$ and $\forall y \exists z \leq s(J+1, y)\ C(q, r^2.T'_{J+2}, \langle c', y \rangle, z)$*
5. *If $J \neq l$ then $\forall y \in R(r, T'_{J+1}, c')$,*
   $y \leq t(J+1) \wedge z = \{q\}(\langle c', y \rangle) \rightarrow z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$
6. *$K' = K(a, r, r^2.T'_{J+2}, c')$*
7. *K' is S_3-closed*
8. *$K \subset K' \subset R$*
9. *$K' \subset R(r, T_1, c)$*
10. *If $x \in K'$, $K(a, r, r^2.T_2, x) \subset K'$*
11. *$K' \models BASIC_3 + \theta(j)$-IND up to l.*

Proof First note that $r \in K$ by lemma 2.4.6 and integers $a, \bar{b}, l, T_1, \rho$ are in K by hypothesis. Hence we can obtain them all from c in time $O(1).T_2$ by means of some (possibly) non standard Turing machine of code $< |r|^{O(1)}$, and these integers are bounded by $2^{2^{||a||^{O(1)}}}$.
The integer p will be the index of the Turing machine P that is working as follows on input c :

1: Compute $r, a, \bar{b}, l, T_1, \rho$ from c.

2: Compute T'_0 from a, ρ, T_1.

3: Put $j := 0$, $y_{-1} := 0$.

4: Compute T'_{j+1} from a, ρ, T'_j.

5: Look for $y_j \in R(r, T'_j, \langle j, c, y_{j-1}\rangle)$ such that

$$y_j \leq t \ \wedge \ \forall z \in R(r, T'_{j+1}, \langle j+1, c, y_j\rangle) \ (z \leq s \rightarrow \psi(j, y_j, z)).$$

(Searching in $R(r, T, x)$ is done by simulating no more than T steps in the computation of $\{e\}(x)$, if e is the code of a Turing machine, and this for all values of e from 0 to r. Verification of a condition for every $z \in R(r, T, x)$ is done in a similar way.)

6: If there is no such y_j, stop the machine with output $P(c) = \langle j, c, y_{j-1}\rangle$.

7: If y_j is found and $j < l$, then put $j := j + 1$ and go to 4.

8: If y_l is found, stop the machine with output $P(c) = \langle l + 1, c, y_l\rangle$.

Let $\langle J + 1, c, Y\rangle$ be the output, i.e. $Y = y_J$, and let us name it c'. Then (2) and (3) follows easily from the definition of P, once the existence of the computation is established.

As explained above, to execute the first line the machine needs a standard number of programs of code $< |r|^{O(1)}$ (namely $6 + m$ programs, as $\bar{b} = b_0, \ldots, b_m$). By (c) a unique standard function in S_3 suffices to obtain T'_0 from a, ρ, T_1 and T'_{j+1} from a, ρ, T'_j. Having r, T'_j, j, c, y_{j-1} we generate the elements of $R(r, T'_j, \langle j, c, y_{j-1}\rangle)$ by means of a standard program. Computation of the values of terms t, s and evaluation of Δ^b_0-formulas is also done by standard programs in S_3. Thus P can be viewed as a standard Turing machine running on c with a standard number of extra inputs bounded by $|r|^{O(1)}$. By (1) of lemma 2.3.7 we conclude that P can be coded by some $p < |r|^{O(1)}$.

For the running time we have that $r, a, b_0, \ldots, b_m, l, T_1, \rho$, are calculated in time $O(1).r^2.T_2$ from c. As $T_1, \rho \in K$ we have $T_1, \rho < 2^{2^{||a||^{O(1)}}}$ and then

$$T'_j < T_1 < 2^{2^{||a||^{O(1)}}}, \ j \leq l + 2$$

By (c) we have $T'_0 \in S_3(\langle a, \rho, T_1\rangle)$ and $T'_{j+1} \in S_3(\langle a, \rho, T'_j\rangle), j \leq l + 1$. Hence T'_j is obtained in time $2^{||a||^{O(1)}}$ for every j. It is known that simulating T'_j steps of the computation of $\{e\}$ can be done in time $O(1).|e|.T'_j$ by an universal program (see Papadimitriou [16], for example). As $e \leq r$ we can bound it by $|r|^2.T'_j$. We calculate the values of terms $t(j, \bar{b}), s(j, y, \bar{b})$ in time $2^{||a||^{O(1)}}$, as they correspond to functions in S_3 and its arguments are all bounded by $2^{2^{||a||^{O(1)}}}$. Deciding if $y_j \leq t$ is done in time $O(1).|t|$, thus less than $2^{||a||^{O(1)}}$ since $t < 2^{2^{||a||^{O(1)}}}$. The same is valid for $z \leq s$.

Evaluation of $\psi(j, y_j, z, \bar{b})$ when $y_j \leq t$ and $z \leq s$ takes time $2^{||a||^{O(1)}}$ because ψ is Δ_0^b and $j, t, s, b_0, \ldots, b_m < 2^{2^{||a||^{O(1)}}}$. Thus, we have that c' is calculated in time T less than

$$O(1).T_2 + 2^{||a||^{O(1)}} + \sum_{j=0}^{l} [2^{||a||^{O(1)}} + r(|r|^2.T'_j + 2^{||a||^{O(1)}} + r(|r|^2.T'_{j+1} + 2^{||a||^{O(1)}}))]$$

Remembering that $T'_j > T_2 > 2^{||a||^{O(1)}}$ we get that

$$T < \sum_{j=0}^{l} r[|r|^2.T'_j + r(|r|^2 + 1)T'_{j+1}]$$

But $r(|r|^2 + 1).T'_{j+1} < T'_j$ since $r < ||a||$ and $(T'_j)_{j \leq l+2}$ is $||a||^{O(1)}$-sparse, thus

$$T < r(|r|^2 + 1).\sum_{j=0}^{l} T'_j < r(|r|^2 + 1)(T'_0 + l.T'_1)$$

Now, $l.T'_1 < T'_0$ because $l < ||a||^{O(1)}$ and $(T'_j)_{j \leq l+2}$ is $||a||^{O(1)}$-sparse. So we conclude that c' is calculated in time

$$T < 2r(|r|^2 + 1).T'_0 < r^2.T'_0$$

Finally note that $r^2.T'_0 \in log(M)$ since $r^2.T'_0 < T_1 \leq 2^{||a||^r}$ and $2^{||a||^r} \in log(M)$ by lemma 2.4.1. Therefore we have

$$\exists w(|w| = r^2.T'_0 \wedge U(p, w, c, c'))$$

i.e. $C(p, r^2.T'_0, c, c')$ and (1) is proved.

The required integer q will be the index of the Turing machine Q working as follows on input $\langle c', y \rangle$:

1: Compute $J + 2, c$ from c'.

2: Compute $r, a, b_0, \ldots, b_m, T_1, \rho$ from c.

3: Compute $t = t(J + 1, \bar{b})$ from $J + 2, b_0, \ldots, b_m$.

4: Compute T'_{J+2} from $J + 2, a, \rho, T_1$.

5: If $y \leq t$, compute $s = s(J + 1, y, \bar{b})$ and look for $z \in R(r, T'_{J+2}, \langle J + 2, c, y \rangle)$ such that $z \leq s \wedge \neg\psi(J + 1, y, z)$.

Else, stop the machine with output 0.

6: If such a z is found, stop the machine with output z.

Else, stop it with output 0.

As $c' = \langle J+1, c, Y\rangle$ we can obtain $J+2$ and c from c' by means of two standard functions in S_3. Integers $r, a, b_0, \ldots, b_m, T_1, l$ can be calculated from c using a standard number of functions of code $< |r|^{O(1)}$ since they belong to K, as we explained above. The values of terms t, s are calculated by standard functions in S_3. By lemma 2.4.3 and hypothesis (c), T'_{J+2} is obtained from $J+2, a, \rho, T_1$ by means of a standard function in S_3. The computations of line 5 requires only a standard program, analogously as for line 5 of program P. In the same way as we did for P, we conclude that Q can be coded by some $q < |r|^{O(1)}$.

For its running time first note that $c < 2^{2^{||a||^{O(1)}}}$ since $c \in K(a, r, T_c, a)$ by hypothesis (a). We have also $t, l < 2^{2^{||a||^{O(1)}}}$, hence

$$Y < t < 2^{2^{||a||^{O(1)}}} \text{ and } J+1 \leq l+1 < 2^{2^{||a||^{O(1)}}}$$

Thus we get that $c' = \langle J+1, c, Y\rangle < 2^{2^{||a||^{O(1)}}}$. As $J+2, c \in S_3(c')$, computations on line 1 are done in time $2^{||a||^{O(1)}}$. Integers in line 2 are in K, hence they are calculated in time $O(1).T_2$ from c. The value of t is calculated in time $2^{||a||^{O(1)}}$ as for program P. We obtain T'_{J+2} in time $2^{||a||^{O(1)}}$ as

$$T'_{J+2} \in S_3(\langle J+2, a, \rho, T_1\rangle) \text{ and } J+2, a, \rho, T_1 < 2^{2^{||a||^{O(1)}}}$$

Deciding if $y \leq t$ takes time $2^{||a||^{O(1)}}$ and when this inequality holds the value of s is calculated in time $2^{||a||^{O(1)}}$ since $y \leq t < 2^{2^{||a||^{O(1)}}}$ and the other arguments of s are also bounded by $2^{2^{||a||^{O(1)}}}$. Searching for z in $R(r, T'_{J+2}, \langle J+2, c, y\rangle)$ verifying the condition in line 5 is done in time less than $r(|r|^2.T'_{J+2} + 2^{||a||^{O(1)}})$. Thus, $Q(\langle c', y\rangle)$ is calculated in time less than

$$2^{||a||^{O(1)}} + O(1).T_2 + r(|r|^2.T'_{J+2} + 2^{||a||^{O(1)}})$$

Since $T'_{J+2} > T_2 > 2^{||a||^{O(1)}}$, we can conclude that $Q(\langle c', y\rangle)$ is calculated in time less than $r^2.T'_{J+2}$. Thus if $z = Q(\langle c', y\rangle)$ then $C(q, r^2.T'_{J+2}, \langle c', y\rangle, z)$ and it is clear that $z \leq s(J+1, y)$ in all cases. This shows (4).

To see (5) suppose $J < l$. As $c' = \langle J+1, c, Y\rangle$ and $Y = y_J$, $J < l$ means that the program P did not find the y_{J+1} it looked for. In other words this says that for every y in $R(r, T'_{J+1}, \langle J+1, c, Y\rangle)$ such that $y \leq t(J+1)$, there is some $z \in R(r, T'_{J+2}, \langle J+2, c, Y\rangle)$ satisfying

$$z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$$

Then, the program Q will eventually find this z and so (5) holds.

Now let $K' = K(a, r, r^2.T'_{J+2}, c')$. We have

$$O(1).r^2.T'_{J+2} > r^2.T_2 > 2^{||a||^{O(1)}}$$

so (7) and (10) follows from lemma 2.4.6. By (2), $c \in S_3(c')$, and by (7) $S_3(c') \subset K'$, so $c' \in K'$. Also

$$2^{||a||^r} > O(1).T_1 > O(1).r^2.T'_{J+2}$$

since $(T_j)_{j\leq l+2}$ is $||a||^{O(1)}$-sparse and $r < ||a||$, and clearly $r^2.T'_{J+2} > T_2$ because $(T_j)_{j\leq l+2}$ is between T_1 and T_2. We can then apply lemma 2.4.7 to conclude that $K \subset K'$.

Now we use lemma 2.4.8 to prove (9) and $K' \subset R$. We have $C(p, r^2.T'_0, c, c')$ and $p < |r|^{O(1)}$ by (1), and

$$2^{||a||^r} \geq T_1 > O(1).r^2.T'_0 > r^2.T'_0 + O(1).r^2.T'_{J+2}$$

thus by lemma 2.4.8 $K' \subset R(r, T_1, c)$ and (9) is proved. By (a) there is some $k \in \mathbb{N}$ and $e < |r|^k$ such that $C(e, k.T_c, a, c)$. By (1) we have $C(p, r^2.T'_0, c, c')$ and $p < |r|^{O(1)}$. Then by (2) of lemma 2.3.7 there is some $e' < |r|^{O(1)}$ such that $C(e', k.T_c + r^2.T'_0, a, c')$. We have

$$2^{||a||^r} > k.T_c + T_1$$

since $2^{||a||^r} > O(1).T_c$ and $2^{||a||^r} > O(1).T_1$ by hypothesis. As indicated above

$$T_1 > r^2.T'_0 + O(1).r^2.T'_{J+2}$$

thus we get that

$$2^{||a||^r} > k.T_c + r^2.T'_0 + O(1).r^2.T'_{J+2}$$

which implies by lemma 2.4.8 that $K' \subset R(r, 2^{||a||^r}, a)$, that is $K' \subset R$ and (8) is proved.

By (7) $K' \prec_{\Delta_0^b} M$ and so $K' \models BASIC_3$. Now we use the previous points to get two easy consequences implying (11). Remember that $-1 \leq J \leq l$.

<u>Fact 1:</u> If $0 \leq J \leq l$ then $K' \models \theta(J)$.

Proof: First note that $J, Y \in S_3(c') \subset K'$ by (2) and (7), and also $K' \subset R(r, T'_{J+1}, c')$, since $K' = K(a, r, r^2.T'_{J+2}, c')$ and $T'_{J+1} > r^2.T'_{J+2}$. Let $z \in K'$, $z \leq s(J, Y)$. Then $z \in R(r, T'_{j+1}, c')$ and by (3) $M \models \psi(J, Y, z)$. We just noted that $K' \prec_{\Delta_0^b} M$, so $K' \models \psi(J, Y, z)$ and thus

$$K' \models \exists y \leq t(J) \forall z \leq s(J, y) \psi(J, y, z)$$

i.e. $K' \models \theta(J)$.

Fact 2: if $-1 \leq J \leq l-1$ then $K' \models \neg\theta(J+1)$.

Proof: Let $y \in K'$, $y \leq t(J+1)$ and let $z = \{q\}(\langle c', y \rangle))$. We have $y \in R(r, T'_{J+1}, c')$, so by (5) we get

$$M \models z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$$

By lemma 2.4.6 and (4), $z \in K'$, so by elementarity,

$$K' \models z \leq s(J+1, y) \wedge \neg\psi(J+1, y, z)$$

Thus we have proved

$$K' \models \forall y \leq t(J+1) \exists z \leq s(J+1, y) \neg\psi(J+1, y, z)$$

that is $K' \models \neg\theta(J+1)$. This proves fact 2.

From Facts 1 and 2 we obtain

$$K' \models \neg\theta(0) \vee \exists j < l[\theta(j) \wedge \neg\theta(j+1)] \vee \theta(l)$$

i.e. $K' \models$ $\theta(j)$-IND *up to l* . □

Now we are ready to construct the chain $(K_n)_{n\in\mathbb{N}}$. Starting from some K_0 (for practical reasons chosen different from the one used in the sketch of the proof), we inductively define K_n for $n \geq 1$, using the procedure of extension exhibited in lemma 2.5.1. This is the content of the next lemma. First we define some useful notation for the rest of the section.

Notation Suppose that M, a, r as in lemma 2.4.1 are fixed and sequences $(T^i_j)_{j \leq l_i}$, $i = 0, 1, \ldots$ are defined. We use the following notation:

- $\boldsymbol{A \gg B}$ means $A > 2^{||a||^{O(1)}}.B$
- $\boldsymbol{R^i_j(x)}$ is the resource $R(r, T^i_j, x)$
- $\boldsymbol{(\bar{b})_i}$ is a set of parameters $b^i_0, \ldots, b^i_{m_i}$

Lemma 2.5.2 *Let M, a, r be as in lemma 2.4.1. Let $T_1^0, T_2^0 \in M$ such that*

$$T_1^0 \in S_3(\langle a, r\rangle) \quad \text{and} \quad 2^{||a||^r} \geq T_1^0 \gg T_2^0 \gg 1$$

Let $K_0 = K(a, r, T_2^0, a)$, $J_0 = 0$, $a_0 = a$.
Let $n \in \mathbb{N}$, $n \geq 1$ and suppose we have n L_3-structures $K_0, \ldots, K_{n-1}$, a $\hat{\Sigma}_2^b$-formula $\theta_n(j) \equiv \exists y \leq t_n \forall z \leq s_n \, \psi_n(j, y, z)$, $\psi_n(j, y, z) \in \Delta_0^b$, with parameters $(\bar{b})_n \in K_{n-1}$, and some integer $l_n \in log(log(K_{n-1}))$. If $n = 1$ we have just K_0, θ_1 and l_1. If $n > 1$ suppose we have also for each $1 \leq i < n$:

- *integers $(\bar{b})_i, \rho_i \in K_{i-1}$, $l_i \in log(log(K_{i-1}))$*
- *a $\hat{\Sigma}_2^b$-formula $\theta_i(j) \equiv \exists y \leq t_i \forall z \leq s_i \; \psi_i(j, y, z)$ with parameters $(\bar{b})_i$, where $\psi_i(j, y, z) \in \Delta_0^b$*
- *integers $p_i, q_i, a_i, Y_i \in M$, $J_i \in M \cup \{-1\}$*
- *a $2^{||a||^{O(1)}}$-sparse sequence $(T_j^i)_{j \leq l_i+2}$ between $T_{J_{i-1}+1}^{i-1}$ and $T_{J_{i-1}+2}^{i-1}$ generated by $\langle a, \rho_i\rangle$*

satisfying (1)-(8) below:

1. *$p_i < |r|^{O(1)}$ and $C(p_i, r^2.T_0^i, a_{i-1}, a_i)$.*
2. *$a_i = \langle J_i + 1, a_{i-1}, Y_i\rangle$, $-1 \leq J_i \leq l_i$ and $Y_i \leq t_i(J_i)$.*
3. *If $J_i \neq -1$ then $\forall z \in R_{J_i+1}^i(a_i)$, $z \leq s_i(J_i, Y_i) \rightarrow \psi_i(J_i, Y_i, z)$*
4. *$q_i < |r|^{O(1)}$ and $\forall y \exists z \leq s_i(J_i + 1, y) C(q_i, r^2.T_{J_i+2}^i, \langle a_i, y\rangle, z)$*
5. *If $J_i \neq l_i$ then $\forall y \in R_{J_i+1}^i(a_i)$,*
   $y \leq t_i(J_i + 1) \wedge z = \{q_i\}(\langle a_i, y\rangle) \rightarrow z \leq s_i(J_i + 1, y) \wedge \neg\psi_i(J_i + 1, y, z)$
6. *$K_i = K(a, r, r^2.T_{J_i+2}^i, a_i)$*
7. *K_i is S_3-closed*
8. *$K_{i-1} \subset K_i \subset R$*

Then there is a $2^{||a||^{O(1)}}$-sparse sequence $(T_j^n)_{j \leq l_n+2}$ between $T_{J_{n-1}+1}^{n-1}$ and $T_{J_{n-1}+2}^{n-1}$ generated by $\langle a, \rho_n\rangle$ for some $\rho_n \in K_{n-1}$, integers $p_n, q_n, a_n, Y_n \in M$, $J_n \in M \cup \{-1\}$, and a L_3-structure K_n such that (1)-(8) holds for $i = n$ and

9. *$K_n \subset R_{J_i+1}^i(a_i)$, for $i = 0, \ldots, n$*

10. If $y \in K_n$ then $\{q_i\}(\langle a_i, y\rangle) \in K_n$, for $i = 1, \ldots, n$

11. $K_n \models BASIC_3 +\ \theta_i(j)$-IND up to l_i, for $i = 1, \ldots, n$.

Proof Let $n \geq 1$. By hypothesis

$$T^{n-1}_{J_{n-1}+1} \gg T^{n-1}_{J_{n-1}+2}$$

and from $l_n \in log(log(K_n))$ it follows that $l_n < ||a||^{O(1)}$. By recurrence on n we have that $2^{||a||^r} \geq T^{n-1}_{J_{n-1}+1}$. Thus by lemma 2.4.4 there is a $2^{||a||^{O(1)}}$-sparse sequence $(T^n_j)_{j\leq l_n+2}$ between $T^{n-1}_{J_{n-1}+1}$ and $T^{n-1}_{J_{n-1}+2}$ generated by $\langle a, \rho_n\rangle$ for some small ρ_n.
As $T^{n-1}_{J_{n-1}+2} \gg 1$ is easily proved by recurrence on n, we can use lemma 2.4.6 to argue that ρ_n can be chosen in K_{n-1}.
We want to apply lemma 2.5.1 for $K = K_{n-1}$. So let us first check its hypothesis (a),(b),(c). If $n = 1$ then

$$K_{n-1} = K_0 = K(a, r, T^0_2, a)$$

and thus hypothesis (a) is trivially verified ($c = a$). We have

$$2^{||a||^r} \geq T^0_1 \gg T^0_2 \gg 1$$

and hence by lemma 2.4.6 K_0 is S_3-closed and $r \in K_0$. Thus

$$T^0_1 \in S_3(\langle a, r\rangle) \subset K_0$$

and so, condition (b) is verified for $T_1 = T^0_1$ and $T_2 = T^0_2$. As the sequence $(T^1_j)_{j\leq l_1+2}$ is obviously $||a||^{O(1)}$-sparse, and $\rho_1 \in K_0$, we have (c) for $T'_j = T^1_j$, $l = l_1$ and $\rho = \rho_1$.

If $n > 1$ we check hypothesis of Lemma 2.5.1 for $c = a_{n-1}$, $T_1 = T^{n-1}_{J_{n-1}+1}$, $T_2 = r^2.T^{n-1}_{J_{n-1}+2}$, $K = K_{n-1}$, $\bar{b} = (\bar{b})_n$, $l = l_n$, $\theta = \theta_n$ and $T'_j = T^n_j$ for $j \leq l_n + 2$. First, we have $(\bar{b})_n \in K_{n-1}$ and $l_n \in log(log(K_{n-1}))$ by hypothesis. Now we check (a),(b),(c):

(a) From $a_i = \langle J_i+1, a_{i-1}, Y_i\rangle$ it follows that $a_{i-1} \in S_3(a_i)$, $i = 1, \ldots, n-1$. It follows also, by recurrence on i, that $a_i, J_i, Y_i < 2^{2^{||a||^{O(1)}}}$. In particular this implies $a_i \in K_i$ for every $i < n$. Composing functions in S_3 we get that that $a = a_0 \in S_3(a_{n-1})$, and by (7) $S_3(a_{n-1}) \subset K_{n-1}$. Therefore $a \in K_{n-1}$.
Now, we have that

$$a_{n-1} = \{p_{n-1}\}(a_{n-2}), \ldots, a_1 = \{p_1\}(a_0) \text{ and } \ p_i < |r|^{O(1)},\ i = 1, \ldots, n-1$$

By (2) of lemma 2.3.7 there is some $e < |r|^{O(1)}$ such that, for $T = \sum_{i=1}^{n-1} r^2.T_0^i$, we have $C(e, T, a_0, a_{n-1})$. But

$$\sum_{i=1}^{n-1} r^2.T_0^i < (n-1).r^2.T_0^1 \ll T_1^0 \ll 2^{||a||^r}$$

Hence $2^{||a||^r} > O(1).T$ and $a_{n-1} \in K(a, r, T, a)$.

(b) We prove by recurrence on n that

$$T_{J_{n-1}+1}^{n-1} \in K_{n-1}.$$

For $n = 1$ it was stated above. Suppose $T_{J_{n-2}+1}^{n-2} \in K_{n-2}$. By (8) $T_{J_{n-2}+1}^{n-2}$ is in K_{n-1} also, as well as ρ_{n-1}, a_{n-1} and, consequently, $J_{n-1}+1$. By lemma 2.4.3

$$T_{J_{n-1}+1}^{n-1} \in S_3(\langle J_{n-1}+1, a, \rho_{n-1}, T_{J_{n-2}+1}^{n-2}\rangle)$$

hence $T_{J_{n-1}+1}^{n-1} \in K_{n-1}$ as K_{n-1} is S_3-closed. The rest follows from

$$2^{||a||^r} \geq T_{J_{n-1}+1}^{n-1} \gg T_{J_{n-1}+2}^{n-1} \gg 1$$

which was remarked at the beginning of the proof.

(c) The sequence $(T_j^n)_{j \leq l_n+2}$ is obviously $||a||^{O(1)}$-sparse and is between $T_{J_{n-1}+1}^{n-1}$ and $r^2.T_{J_{n-1}+2}^{n-1}$ since $r < ||a||$.

Applying now lemma 2.5.1 we get $p_n, q_n, a_n, Y_n \in M$, $J_n \in M \cup \{-1\}$ and a L_3-structure K_n satisfying already (1)-(8). Let us see (9)-(11):

(9) For $i = n$ it is clear by definition (6) of K_n and the fact that

$$T_{J_n+1}^n > O(1).r^2.T_{J_n+2}^n.$$

Consider the case $i < n$. We have that a_n can be calculated from a_i by composing successively $\{p_{i+1}\}, \ldots, \{p_n\}$, and the total computing time is bounded by

$$r^2.(T_0^{i+1} + \ldots + T_0^n) < (n-i).r^2.T_0^{i+1} \ll T_{J_i+1}^i$$

By (2) of lemma 2.3.7 we have $C(e, T, a_i, a_n)$ for some $e < |r|^{O(1)}$ and $T \ll T_{J_i+1}^i$. Since

$$T + O(1).T_{J_n+2}^n < T_{J_i+1}^i < 2^{||a||^r}$$

we can apply lemma 2.4.8 to conclude that $K_n \subset R_{J_i+1}^i(a_i)$.

(10) Let $1 \leq i \leq n$ and $y \in K_n$. Clearly $a_i \in K_n$ and then so is $\langle a_i, y\rangle$ since K_n is S_3-closed. If $z = \{q_i\}(\langle a_i, y\rangle)$ then by (4) we have that

$$z \leq s_i(J_i + 1, y) \text{ and } C(q_i, r^2.T^i_{J_i+2}, \langle a_i, y\rangle, z)$$

If $y \leq t_i(J_i + 1)$ then $s_i(J_i + 1, y) < 2^{2^{||a||^{O(1)}}}$, and when $y > t_i(J_i + 1)$ then $z = 0$ by definition of $\{q_i\}$. In all cases we have $z < 2^{2^{||a||^{O(1)}}}$. But since $T^i_{J_i+2} \leq T^n_{J_n+2}$ when $i \leq n$ we have

$$r^2.T^i_{J_i+2} < O(1).r^2.T^n_{J_n+2}$$

so we can apply lemma 2.4.6 to conclude that $z \in K_n$.

(11) This fact is a direct consequence of (3),(5),(8),(9) and (10). Surprisingly, it will not be used later, and this is because our extensions preserve only Δ^b_0-formulas. We will rather imitate its proof for a bigger model of the form $\bigcup K_n$ in the proof of theorem 2.1.1 below. This is the reason we do not prove it here. □

Proof of theorem 2.1.1 Arguing like in the proof of lemma 2.4.1, there is some $r_0 \in M\backslash\mathbb{N}$, $r_0 \leq r$ (and thus $2^{2^{||a||^{r_0}}}$ exists also), such that $r_0 = 2^{|r_0|-1}$ and $r_0 < ||a||$. Since

$$R(a, r_0, 2^{||a||^{r_0}}) \subset R$$

it suffices to prove the theorem for r_0. So we can assume $r = 2^{|r|-1}$ and $r < ||a||$ without losing generality.

Let $T^0_1 = 2^{||a||^r}$ and let T^0_2 be such that $T^0_1 \gg T^0_2 \gg 1$ (any $2^{||a||^\rho}$ with $r > \rho > O(1)$, for example). As we remarked after lemma 2.4.1, we have $2^{||a||^r} \in S_3(\langle a, r\rangle)$. Set

$$K_0 = K(a, r, T^0_2, a)$$

Fix an enumeration with infinite repetitions of pairs $(\theta(j, \bar{b}), ||d||)$ where θ is a $\hat{\Sigma}^b_2$-formula and $\bar{b}, d$ are parameters in M. Consider the first pair in the enumeration with parameters in K_0 and name it $(\theta_1(j, (\bar{b})_1), l_1)$.

Then $\theta_1(j) \equiv \exists y \leq t_1 \forall z \leq s_1 \psi_1(j, y, z)$, with ψ_1 a Δ^b_0-formula with parameters $(\bar{b})_1$, and we are in the case $n = 1$ of hypothesis of lemma 2.5.2. This gives us K_1. Suppose we have just obtained K_n from K_{n-1} using this lemma, and let

$$(\theta_{n+1}(j, (\bar{b})_{n+1}), l_{n+1})$$

be the first pair in the enumeration after (θ_n, l_n) having its parameters in K_n. Lemma 2.5.2 says that K_n satisfies also (1)-(8), thus we are again verifying its hypothesis and therefore we obtain K_{n+1}.

In this way we get an increasing chain of L_3-structures $(K_n)_{n\in\mathbb{N}}$. At each step a new $\hat{\Sigma}^b_2$-LLIND axiom is satisfied while the precedent ones are preserved. But the chain is only Δ^b_0-elementary and hence preservation of these axioms under the union of the chain is not guaranteed since they are Δ^b_3-formulas. Rather, this preservation is a consequence of the specific way the models are built. In other words, we have not yet proved that

$$K^* := \bigcup_{n\in\mathbb{N}} K_n$$

is a model of $\hat{\Sigma}^b_2$-LLIND . Instead, (a),(b),(c) are promptly verified, and thus we obtain that

$$K^* \prec_{\Delta^b_0} M.$$

Let $\theta(j)$ a $\hat{\Sigma}^b_2$-formula with parameters $\bar{b} \in K^*$ and let $l \in log(log(K^*))$. Suppose that $(\theta(j), l)$ was considered when constructing K_n, i.e. $\theta(j) \equiv \theta_n(j)$ is the formula

$$\exists y \leq t_n \, \forall z \leq s_n \, \psi_n(j, y, z)$$

and $\bar{b} = (\bar{b})_n$, $l = l_n$, with $(\bar{b})_n \in K_{n-1}$, $l_n \in log(log(K_{n-1}))$. Note that $a_n \in K^*$ and hence by (b) J_n and Y_n are also in K^*. Note too that

$$K^* \subset R^n_{J_n+1}(a_n)$$

by (9) of lemma 2.5.2. Remember that $-1 \leq J_n \leq l_n$.

<u>Fact 1:</u> if $0 \leq J_n \leq l_n$ then $K^* \models \theta_n(J_n)$.

Proof: Let $z \in K^*$ such that $z \leq s_n(J_n, Y_n)$. As we just remarked, $z \in R^n_{J_n+1}(a_n)$ so by (5) of Lemma 2.5.2

$$M \models \psi_n(J_n, Y_n, z)$$

and by (2) $Y_n \leq t_n(J_n)$. By Δ^b_0-elementarity $K^* \models \psi_n(J_n, Y_n, z)$. We have proved

$$K^* \models \exists y \leq t_n \, \forall z \leq s_n \, \psi_n(J_n, y, z)$$

that is $K^* \models \theta_n(J_n)$.

<u>Fact 2:</u> if $-1 \leq J_n \leq l_n - 1$ then $K^* \models \neg\theta_n(J_n + 1)$.

Proof: Let $y \in K^*$ such that $y \leq t_n(J_n)$ and let $m \geq n$ such that $y \in K_m$. We have $a_n \in K_n \subset K_m$, so by (10) of lemma 2.5.2 $\{q_n\}(\langle a_n, y\rangle) \in K_m$. By (9)

$$K_m \subseteq R^n_{J_n+1}(a_n)$$

hence $y \in R^n_{J_n+1}(a_n)$ and by (5), if $z = \{q_n\}(\langle a_n, y\rangle)$ then

$$M \models z \leq s_n(J_n, y) \wedge \neg\psi_n(J_n + 1, y, z)$$

Therefore we have that $z \in K^*$ and by Δ^b_0-elementarity

$$K^* \models z \leq s_n(J_n, y) \wedge \neg\psi_n(J_n + 1, y, z)$$

Thus

$$K^* \models \forall y \leq t_n \exists z \leq s_n \neg\psi_n(J_n + 1, y, z)$$

i.e. $K^* \models \neg\theta_n(J_n + 1)$. This proves fact 2.

From Facts 1 and 2,

$$K^* \models \neg\theta_n(0) \vee \exists j < l_n[\theta_n(j) \wedge \neg\theta_n(j + 1)] \vee \theta_n(l_n)$$

i.e. $K^* \models \theta_n(j)$-IND *up to* l_n. Thus we have proved that $K^* \models \hat{\Sigma}^b_2$-LLIND. $\square$

Chapter 3

Multifunction resources

In this chapter we consider models of general theories $\hat{T}_2^{i,2^{|\tau|^\omega}}$ *and* $\hat{T}_2^{i+1,|\tau|}$ *and resources generated by its* $\hat{\Sigma}_{i+1}^b$*-definable multifunctions. In section 3.1 we introduce these classes and do some preliminary work. Given a model* M *of* $\hat{T}_2^{i,2^{|\tau|^\omega}}$ *and a* $\hat{\Sigma}_i^b$*-condition* $\sigma(a,y)$ *depending on one special parameter* a*, we construct in sections 3.2 and 3.3 a* $\hat{\Sigma}_i^b$*-substructure* N *of an elementary extension of* M *whose elements satisfy the condition. In order to have* $\hat{\Sigma}_i^b$*-preservation of formulas we consider structures with some closure properties under multifunctions consulting* $\hat{\Sigma}_i^b$*-oracles. We use this construction in section 3.4 to extend a model of* $\hat{T}_2^{i,2^{|\tau|^\omega}}$ *to one of* $\hat{T}_2^{i+1,|\tau|}$*. We derive also some known witnessing and conservation results for these theories.*

3.1 Definability of (multi)functions in $\hat{T}_2^{i,2^{|\tau|^\omega}}$

In this section we introduce the classes of (multi)functions we will consider. We recall how a definability theorem is proved in $\hat{T}_2^{i,2^{|\tau|^\omega}}$ for the class $F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ and then derive a kind of representation theorem for multifunctions in this theory, a result that proves to be useful further on.

Definition 3.1.1 *Let* $i \geq 0$ *and* τ *a set of unary terms.*

- $F[\tau]^{\Sigma_i^p}$ *is the class of functions which can be computed by a Turing machine equipped with a* Σ_i^p*-oracle, in less than* $O(l(t(x)))$ *steps (if* x *is the input) for some* $l \in \tau$ *and* $t \in Term(L_{BA})$*.*
- $[\tau]^{\Sigma_i^p}$ *is the class of predicates whose characteristic function belongs to* $F[\tau]^{\Sigma_i^p}$*.*
- *We say that a theory* T Ψ-defines *the function* f *if for some* $\varphi(x,y) \in \Psi$ *the following holds:*

1. $T \vdash \forall x \exists y \varphi(x, y)$

2. $T \vdash \forall x \forall y \forall y'(\varphi(x, y) \wedge \varphi(x, y') \rightarrow y = y')$

3. $\mathbb{N} \models \forall m \forall n \;\; f(m) = n \leftrightarrow \varphi(m, n)$.

Theorem 3.1.2 *The theory* $\hat{T}_2^{i,2^{|\tau|^\omega}}$ *can* $\hat{\Sigma}_{i+1}^b$*-define the functions in* $F[|\tau|^\omega]^{\Sigma_i^p}$.

Proof This is classic. The case $\tau = \{x\}$ (i.e. the identity term) corresponds to well known Buss's theories T_2^i and function classes $FP^{\Sigma_i^p}$ of polynomial-time computable functions using oracles from Σ_i^p. In [10] this case is treated using Turing machines. The general case can be handled in the same way. □

Definition 3.1.3 (Multifunctions) *Let* $i \geq 0$ *and* τ *a set of unary terms.*

- *A multifunction* f *is a binary relation such that* $\forall x \exists y f(x, y)$. *We think of a multifunction as a correspondence which to each* x *can associate different values of* y. *To avoid confusion with the single-valued case, we rather note* $y \in Im(f)(x)$ *instead of the usual* $y = f(x)$.
- $F[\tau]^{\Sigma_i^p}(wit)$ *is the same as* $F[\tau]^{\Sigma_i^p}$ *but the oracle gives a witness for the existential query when the answer is YES. So this makes these functions possibly multi-valued.*
- $[\tau]^{\Sigma_i^p}(wit)$ *is the class of predicates whose characteristic function belongs to* $F[\tau]^{\Sigma_i^p}(wit)$.
- *We say that a theory* T Ψ-defines *a multifunction* f *if for some formula* $\varphi(x, y) \in \Psi$ *the following holds*

  1. $T \vdash \forall x \exists y \varphi(x, y)$

  2. $\mathbb{N} \models \forall m \forall n \; f(m, n) \leftrightarrow \varphi(m, n)$.

Let us state more precisely what we mean by a multifunction being computed by a Turing machine. It is possible to adopt two points of view: the first wants the machine to be able to output every $y \in Im(f)(x)$ while the second thinks of f as a search problem where we are looking for some $y \in Im(f)(x)$ and so accepts that for some images there is no computation leading to them. We choose the first definition although this is not relevant for the subsequent work. Choosing the second would make the classes $F[|\tau|]^{\Sigma_i^p}(wit)$ and $FP^{\Sigma_i^p}(wit, |\tau|)$ identical (see [20]) while in our setting we have an only inclusion as $|\tau|$-time is less than or equal to polynomial-time. See also [7] for a discussion of such topics.

Theorem 3.1.4 *The theory $\hat{T}_2^{i,2^{|\tau|^\omega}}$ can $\hat{\Sigma}_{i+1}^b$-define the multifunctions in the class $F[|\tau|^\omega]^{\Sigma_i^p}(wit)$.*

Proof This is a well-known result. We only give some indications of how its proof goes, as this proof has an important application further on. We refer to [12] and [20] for a detailed exposition.

Take $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ and let M be a Turing machine computing f in time $|l(s(x))|^k$ for some $l \in \tau$, $s \in Term(L_{BA})$ and $k \in \mathbb{N}$. There is a $\hat{\Pi}_{i-1}^b$-formula $QComp_M(x, w, v)$ with the meaning that w is like the code of a computation of M on x but requiring only that the positive answers of the oracle are correct (to require the negative answers correct also would make the formula too complex). The variable v keeps track of this having 1 as its j-th *bit* if and only if the answer to the j-th query is YES. Using $\hat{\Sigma}_i^b$-$IND^{2^{|\tau|^\omega}}$ it is proved that there is a maximal v for which $\exists w QComp_M(x, w, v)$ holds. The maximality of this v implies that the negative answers of the oracle must be correct, hence such a w codes in fact a computation of M on x and we can extract the output y by a simple decoding operation, a term $Output \in L_{BA}$ in fact. Hence $\hat{T}_2^{i,2^{|\tau|^\omega}}$ proves

$$\forall x \exists y \exists v \leq 2^{|l(s(x))|^k}[\exists w \leq t(Output(w) = y \wedge QComp_M(x, w, v))$$

$$\wedge \neg \exists v' \leq 2^{|l(s(x))|^k} \exists w' \leq t(v' > v \wedge QComp_M(x, w', v'))]$$

and $y = f(x)$ can be defined by the $\hat{\Sigma}_{i+1}^b$-formula in the scope of $\exists y$. □

As an immediate consequence of this proof we obtain

Theorem 3.1.5 *For every multifunction $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ there is a Π_{i-1}^b-formula A, terms $t, s, h \in L_{BA}$, $l \in \tau$, and $k \in \mathbb{N}$ such that*

1. $\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \exists y \leq t \; B(x, y)$
2. $\mathbb{N} \models \forall m \forall n \; B(m, n) \leftrightarrow n \in Im(f)(m)$.

where $B(x, y)$ is the formula

$$\exists v \leq 2^{|l(s(x))|^k}[\exists w \leq t(h(w) = y \wedge A(x, w, v))$$

$$\wedge \neg \exists v' \leq 2^{|l(s(x))|^k} \exists w' \leq t(v' > v \wedge A(x, w', v'))].$$

□

Classes $F[\tau]^{\Sigma_i^p}$ and $F[\tau]^{\Sigma_i^p}(wit)$ can be alternatively defined as some particular algebras (see [20]). In particular they are closed under the following recursion scheme.

Definition 3.1.6 *We say that f is defined by BPR^{τ} (τ-bounded primitive recursion) from g, h, t and r if*

$$\begin{aligned} F(0,\bar{x}) &= g(\bar{x}) \\ F(n+1,\bar{x}) &= min(h(n,\bar{x},F(n,\bar{x})), r(n,\bar{x})) \\ f(n,\bar{x}) &= F(l(t(n,\bar{x})),\bar{x}) \end{aligned}$$

for some $t, r \in L_{BA}$ and $l \in \tau$.

So we will be able to define new (multi)functions by τ-depth recursion when it will be clear that our functions remain bounded.

Remark 3.1.7 It can also be proved that multifunctions defined as in Theorem 3.1.5 are $\hat{T}_2^{i,2^{|\tau|^{\omega}}}$-provably closed under composition and $BPR^{|\tau|^{\omega}}$ (see [20]). As a consequence, when in a model of $\hat{T}_2^{i,2^{|\tau|^{\omega}}}$, we will be able to define new multifunctions preserving the recursive properties of the definition.

From Theorem 3.1.5 we get the following results.

Theorem 3.1.8 *For every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}(wit)$ there is a $\hat{\Sigma}_i^b$-formula $\theta(x,v,y)$ and a function $g \in F[|\tau|^{\omega}]^{\Sigma_i^p}$ such that $\hat{T}_2^{i,2^{|\tau|^{\omega}}}$ proves*

1. *$\theta(x,v,y) \to y \leq t(x)$, for some term $t \in L_{BA}$*
2. *$\forall x \exists y \theta(x,g(x),y)$*
3. *$\forall x \forall y(\theta(x,g(x),y) \to y \in Im(f)(x))$.*

Proof Let $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}(wit)$ and let A, t, s, h, l be as in Theorem 3.1.5. Consider the function

$$g(x) := max\ v \leq 2^{|l(s(x))|^k}(\exists w \leq t\ A(x,w,v)).$$

By Theorem 3.1.5 such a v exists. Moreover it can be found using binary search in $|l(s(x))|^k$ steps by asking the Σ_i^p-oracle

$$\exists v \leq 2^{|l(s(x))|^k} \exists w \leq t(z_1 < v \leq z_2 \wedge A(x,w,v))$$

for suitably choosen values of z_1, z_2 at each step. This means that $g \in F[|\tau|^{\omega}]^{\Sigma_i^p}$. Defining g appropriately we can have $\hat{T}_2^{i,2^{|\tau|^{\omega}}}$ prove the following properties:

- $\forall x \exists! y(y = g(x))$

- $\forall x(g(x) \leq 2^{|l(s(x))|^k})$
- $\forall x \exists w \leq t\, A(x, w, g(x))$
- $\forall x \forall v \leq 2^{|l(s(x))|^k}(v > g(x) \rightarrow \neg \exists w \leq t\, A(x, w, v))$.

Thus we have

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \exists y \leq t\, \exists w \leq t\, (h(w) = y \wedge A(x, w, g(x))).$$

Taking $\theta(x, v, y)$ to be the $\hat{\Sigma}_i^b$-formula $y \leq t \wedge \exists w \leq t\, (h(w) = y \wedge A(x, w, v))$ gives the desired result. □

Theorem 3.1.9 *For every $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ there is a multifunction $f' \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ such that $\hat{T}_2^{i,2^{|\tau|^\omega}}$ proves:*

1. $\forall x \exists y (y \in Im(f')(x))$
2. $f' \subset f$
3. $y \in Im(f')(x) \leftrightarrow \theta(x, g(x), y)$

for some function $g \in F[|\tau|^\omega]^{\Sigma_i^p}$ and $\hat{\Sigma}_i^b$-formula $\theta(x, v, y)$ satisfying $\theta(x, v, y) \rightarrow y \leq t(x)$ for some $t \in Term(L_{BA})$.

Proof Let $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ and let $\theta(x, v, y)$, $g \in F[|\tau|^\omega]^{\Sigma_i^p}$ be as in Theorem 3.1.8. Define f' as a multifunction that first computes $g(x)$, then asks the oracle for a witness y for $\theta(x, g(x), y)$. □

This theorem says that it is possible to get, maybe not all, but at least some values of the multifunction $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ by asking the oracle just once for a witness. Of course the other YES-NO answers are needed; they are provided by g. Similar results were proved in [12] and [7]. As a corollary of this we have the following equality of classes.

Theorem 3.1.10 $[|\tau|^\omega]^{\Sigma_i^p}(wit) = [|\tau|^\omega]^{\Sigma_i^p}$.

Proof: Let $X \in [|\tau|^\omega]^{\Sigma_i^p}(wit)$ and let f be its characteristic function. By Theorem 3.1.8 there is a $\hat{\Sigma}_i^b$-formula $\theta(x, v, y)$ and a function $g \in F[|\tau|^\omega]^{\Sigma_i^p}$ such that

$$\forall x \forall y (\theta(x, g(x), y) \rightarrow y \in Im(f)(x)).$$

Then we have $\forall x(\theta(x, g(x), 0) \vee \theta(x, g(x), 1))$. Consider the function h that on input x computes $g(x)$, then asks $\theta(x, g(x), 1)$? and outputs 1 if the answer is YES, and 0 otherwise. Clearly $h \in F[|\tau|^\omega]^{\Sigma_i^p}$ and in fact $h = f$. Hence $X \in [|\tau|^\omega]^{\Sigma_i^p}$. We have proved $[|\tau|^\omega]^{\Sigma_i^p}(wit) \subset [|\tau|^\omega]^{\Sigma_i^p}$. The other inclusion is obvious. □

We will have to consider (multi)functions using parameters from a fixed set $\mathcal{S}$. This class can be defined as follows:

Definition 3.1.11 *We fix $\mathcal{S}$ an enumerable set of new constant symbols.*

- $\mathcal{L}_{\mathcal{S}} := L_{BA} \cup \mathcal{S}$.
- *We continue to use the notation $T_2^{i,\tau}$ for our theories when working with $\mathcal{L}_{\mathcal{S}}$.*
- *We say that $\hat{T}_2^{i,2^{|\tau|^\omega}}$ proves that $f \in F[|\tau|^\omega]_{\mathcal{S}}^{\Sigma_i^p}(wit)$ if for some $n \geq 1$ there are symbols $z_1, \dots, z_n \in \{x\} \cup \mathcal{S}$ and $g \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ such that*

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x(Im(f)(x) = Im(g)(\langle z_1, \dots, z_n \rangle))$$

  (recall this is an equality between sets).

- $F[|\tau|^\omega]_{\mathcal{S},c_1,\dots,c_m}^{\Sigma_i^p}(wit)$ *means* $F[|\tau|^\omega]_{\mathcal{S}\cup\{c_1,\dots,c_m\}}^{\Sigma_i^p}(wit)$*, when $c_1, \dots, c_m$ are other constant symbols.*
- $F[|\tau|^\omega]_{\mathcal{S}}^{\Sigma_i^p}$ *is defined similarly.*

We prove some easy facts about these classes that will be used later.

Lemma 3.1.12 *Let $\mathcal{S} \cup \{a, b, c, d\}$ be an enumerable set of constant symbols and suppose that $\hat{T}_2^{i,2^{|\tau|^\omega}}$ proves $d = \langle b, c\rangle$ and $a = t(c)$ for some L_{BA}-term. Then it proves also:*

1. $F[|\tau|^\omega]_{\mathcal{S},d}^{\Sigma_i^p}(wit) = F[|\tau|^\omega]_{\mathcal{S},b,c}^{\Sigma_i^p}(wit)$
2. $F[|\tau|^\omega]_{\mathcal{S},a}^{\Sigma_i^p}(wit) \subset F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$
3. *for every $f \in F[|\tau|^\omega]_{\mathcal{S},b,c}^{\Sigma_i^p}(wit)$ there is some $f' \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$ such that $Im(f)(a) = Im(f')(b)$.*

The same results hold in the single-valued case of $F[|\tau|^\omega]_{\mathcal{S}}^{\Sigma_i^p}$.

Proof: Using $\langle b, c\rangle$ or $\{b, c\}$ as parameters is the same, as you can do some coding or decoding before running your function in $F[|\tau|^\omega]^{\Sigma_i^p}(wit)$, these operations being included in that class. A similar reasoning shows that

$$F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit) = F[|\tau|^\omega]_{\mathcal{S},a,c}^{\Sigma_i^p}(wit)$$

and so condition 2 is clear. Let $g \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ and $s_1, \ldots, s_n \in \mathcal{S}$ such that

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash Im(f)(a) = Im(g)(\langle a, c, b, s_1, \ldots, s_n \rangle)$$

(a previous reordering of variables may be necessary, but this is also allowed in $F[|\tau|^\omega]^{\Sigma_i^p}(wit)$). Let f' be the multifunction defined by

$$f'(x) = g(\langle a, c, x, s_1, \ldots, s_n \rangle).$$

Then $f' \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},a,c}(wit)$ and clearly

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash Im(f)(a) = Im(f')(b).$$

By condition 2 we have in fact $f' \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$. □

Theorem 3.1.4 readily generalises to this setting, as well as the subsequent Theorems 3.1.5, 3.1.8 and 3.1.9. In particular we have the following version of Theorem 3.1.9 which is needed later.

Theorem 3.1.13 *Let T^+ be a theory on $\mathcal{L}_\mathcal{S}$ containing $\hat{T}_2^{i,2^{|\tau|^\omega}}$. Then, for every $f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ there is a multifunction $f' \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ such that T^+ proves*

1. $\forall x \exists y (y \in Im(f')(x))$
2. $f' \subset f$
3. $y \in Im(f')(x) \leftrightarrow \theta(x, g(x), y)$

for some function $g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}$ and $\hat{\Sigma}_i^b$-formula $\theta(x, v, y)$ of $\mathcal{L}_\mathcal{S}$ satisfying $\theta(x, v, y) \rightarrow y \leq t(x)$ for some $t \in Term(\mathcal{L}_\mathcal{S})$. □

This theorem has a useful application in the next lemma.

Lemma 3.1.14 *Let T^+ be a theory on $\mathcal{L}_\mathcal{S}$ containing $\hat{T}_2^{i,2^{|\tau|^\omega}}$. Let $\varphi(x, y, \bar{u})$ be a $\hat{\Pi}_i^b$-formula with parameters $\bar{u}$ in $\mathcal{S}$ and suppose that for every $f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)$*

$$T^+ \nvdash \forall x \forall y (y \in Im(f)(x) \rightarrow \varphi(x, y, \bar{u})).$$

Then the theory $T^+ \cup \{\exists y (y \in Im(f)(a) \wedge \neg\varphi(a, y, \bar{u})) : f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)\}$ is consistent, where a is a new constant symbol not in $\mathcal{L}_\mathcal{S}$.

Proof Suppose that the following theory is inconsistent:

$$T^+ \cup \{\exists y(y \in Im(f)(a) \wedge \neg\varphi(a, y, \bar{u})) : f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)\}.$$

By compactness there are some multifunctions $f_0, \ldots, f_n \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ such that

$$T^+ \vdash \forall x \bigvee_{j \leq n} \forall y(y \in Im(f_j)(x) \to \varphi(x, y, \bar{u})).$$

For every $j \leq n$ let $f'_j \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)$, $\theta_j(x, v, y) \in \hat{\Sigma}^b_i$, $g_j \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}$, and $t_j \in Term(L)$ be as in Theorem 3.1.13, i.e.

1. $f'_j \subset f_j$
2. $y \in Im(f'_j)(x) \leftrightarrow \theta_j(x, g_j(x), y)$
3. $\theta_j(x, v, y) \to y \leq t_j(x)$.

Then we have:

$$T^+ \vdash \forall x \bigvee_{j \leq n} \forall y(\theta_j(x, g_j(x), y) \to \varphi(x, y, \bar{u})). \tag{3.1}$$

Consider the multifunction f operating as follows on input x:

1: For $j = 0$ to n do

2: Calculate $g_j(x)$.

3: Ask $\forall y \leq t_j(x)(\theta_j(x, g_j(x), y) \to \varphi(x, y, \bar{u}))$?

4: If the answer is YES ask for a witness y
for $\theta_j(x, g_j(x), y)$ and STOP with output y.

5: If the answer is NO put $j := j + 1$ and go to 1.

By (1) above there is some $j \leq n$ for which the answer is YES. On the other hand we have that the queries are $\hat{\Pi}^b_i$ and they are, in number, less than

$$(|l_0(s_0(x))|^{k_0} + 1) + \cdots + (|l_n(s_n(x))|^{k_n} + 1)$$

for some terms $l_j \in \tau$, $s_j \in Term(L_{BA})$ and $k_j \in \mathbb{N}$. Note that this is bounded by $|l(s(x))|^k$ for some $l \in \tau$, $s \in Term(L_{BA})$ and $k \in \mathbb{N}$. Also f uses a finite number of parameters from $\mathcal{S}$, thus $f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S}}(wit)$. Now note that f satisfies

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \forall y(y \in Im(f)(x) \to \varphi(x, y, \bar{u}))$$

but this is in contradiction with the hypothesis. □

3.2 Constructing in a multifunction resource

Definition 3.2.1 *For the purposes of this section we fix a set* $\mathcal{S}$ *of constant symbols plus two other constant symbols* a, c*. We use the following notation:*

- $\sigma(x, y)$ *is a* $\hat{\Sigma}_i^b$*-formula of* $\mathcal{L}_{\mathcal{S}}$*.*
- T^+ *is a theory on* $\mathcal{L}_{\mathcal{S}}$ *containing* $\hat{T}_2^{i,2^{|\tau|^\omega}}$*.*
- $\mathcal{L}_0 := \mathcal{L}_{\mathcal{S}} \cup \{a\}$*.*
- T_0 *is the* $\mathcal{L}_0$*- theory*

$$T^+ \cup \{\exists y (y \in Im(f)(a) \wedge \sigma(a, y)) : f \in F[|\tau|^\omega]_{\mathcal{S}}^{\Sigma_i^p}(wit)\}.$$

Moreover, we suppose T_0 *consistent.*

We want a model M of T_0 with a $\hat{\Sigma}_i^b$-substructure $N \models \hat{T}_2^{i,2^{|\tau|^\omega}}$ satisfying $\forall y\, \sigma(a, y)$. This will be done in section 3.3 and will have as a consequence the witnessing Theorem 3.3.4. This will also allow us to get the extension result of section 3.4 and its corresponding conservation and witnessing corollaries 3.4.5 and 3.4.6.

For N to be a $\hat{\Sigma}_i^b$-substructure clearly it suffices to have some kind of closure under multifunctions in $F[|\tau|^\omega]^{\Sigma_i^p}(wit)$. Recall how a similar result is proved for Buss's theory T_2^i : inside a model of it the closure of an element under functions in $FP^{\Sigma_i^p}$ is again a model of T_2^i, which is $\hat{\Sigma}_i^b$-elementary (see [10], [31]). Here we would like for example to take

$$N = \{y \in Im(f)(a) : f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)\}$$

but the problem is that not every image y of a by f is "good" in the sense that it satisfies $\sigma(a, y)$. We can put in N only "good" images y of a but the same problem arises when considering the images of those y.

We will see in this section that it is possible to iterate this process of selecting "good" images to obtain a set N_a that is $F[|\tau|^\omega]^{\Sigma_i^p}(wit)$-*closed* in the sense that

$$\forall x \in N_a\, \exists y \in N_a\, (y \in Im(f)(x)).$$

The problem now is that nothing guarantees that N_a is an L_{BA}-structure. Take for example in N_a some $b_j \in Im(f_j)(a)$ for $j = 1, 2$, and ask if $b_1 + b_2 \in N_a$. If we consider the multifunction $x \mapsto f_1(x) + f_2(x)$, all that we know is that $b_1' + b_2' \in N_a$ for some $b_j' \in Im(f_j)(a)$, but maybe $b_1 + b_2 \neq b_1' + b_2'$. This problem is resolved in section 3.3 by constructing N as an intersection of sets N_c starting with N_a.

Definition 3.2.2 *Let $M \models T_0$ and $c \in M$. We abbreviate by $\mathcal{C}(M, c, a)$ the following conditions*

1. *For every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$, $M \models \exists y(y \in Im(f)(a) \wedge \sigma(a, y))$.*
2. *$a = t(c)$ for some $t \in Term(L_{BA})$.*

So condition 1 means that the fact of including c as a parameter for the multifunctions does not change the property of a of having "good" images under any $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)$. Note for example that we have $\mathcal{C}(M, a, a)$, or $\mathcal{C}(M, s, a)$ for $s \in \mathcal{S}$. We understand condition 2 as "a can be easily extracted from c". To help reading the following lemmas you can think as if $c = a$. In fact we will use this possibility of substituting c by a to construct N_a, but in order to have N closed under L_{BA} we will allow our multifunctions to use parameters $c \in N$, and so we need the more general approach.

Definition 3.2.3 *Let $M \models T_0$ and $c \in M$. We call R_c^M the set defined by*

$$R_c^M := \{x \in M : \text{for every } f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit), M \models \exists y(y \in Im(f)(x) \wedge \sigma(a, y))\}.$$

Elements of R_c^M are those having "good" images under any $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ using c as an additional parameter. For example $a \in R_a^M$. The letter R is for "resource". It is inside this kind of set that we will construct our models.

Lemma 3.2.4 *Let $M \models T_0$ and $c \in M$ satisfy $\mathcal{C}(M, c, a)$. Then*

1. *$a, c \in R_c^M$*
2. *$\forall x \in R_c^M$, $M \models \sigma(a, x)$*

Proof That $a \in R_c^M$ follows from the remark after Definition 3.2.2. To see that $c \in R_c^M$ take $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$. Let f_c be the constant function $x \mapsto c$ and let $h = f \circ f_c$. Then $h \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ and as $a \in R_c^M$ we have that

$$M \models \exists y(y \in Im(h)(a) \wedge \sigma(a, y)).$$

But $Im(h)(a) = Imf(c)$, so $c \in R_c^M$.
For the last point just take f to be the identity function. □

The next lemma is a first tool allowing to keep only "good" images from a multifunction.

Lemma 3.2.5 *Let $M \models T_0$ and let $c \in M$ satisfy $\mathcal{C}(M, c, a)$. Then, for every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ there is a T^+-provably total multifunction $\tilde{f} \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ such that $\tilde{f} \subset f$ and for every $b \in R_c^M$*

$$M \models \forall y(y \in Im(\tilde{f})(b) \to \sigma(a, y)).$$

Proof Let $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$. Using Theorem 3.1.13 (and considering T^+ as an $\mathcal{L}_0 \cup \{c\}$-theory w.l.o.g.) f can be restricted to a T^+-provably total multifunction $f' \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ defined by

$$y \in Im(f')(x) \leftrightarrow \theta(x, g(x), y)$$

for a $\hat{\Sigma}_i^b$-formula $\theta(x, v, y)$ with parameters from $\mathcal{S} \cup \{c\}$ and a function $g \in FP^{\Sigma_i^p}_{\mathcal{S},c}(\dot{\tau})$.

Let $\tilde{f}$ be the multifunction in $F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ which on input x do the following:

1: Compute $g(x)$.

2: Ask $\exists y(\theta(x, g(x), y) \wedge \sigma(a, y))$?

3: If the answer is YES then output a witness y for this.

4: Else, output a witness for $\exists y\, \theta(x, g(x), y)$ (there is always some one).

Clearly $\tilde{f} \subset f$ and T^+ proves $\tilde{f}$ is total. Now let $b \in R_c^M$. By definition of R_c^M we have that $M \models \exists y(y \in Im(f')(b) \wedge \sigma(a, y))$, i.e.

$$M \models \exists y(\theta(x, g(b), y) \wedge \sigma(a, y)).$$

Then for $x = b$ the answer in line 2 is YES, hence $\forall y(y \in Im(\tilde{f})(b) \to \sigma(a, y))$. $\square$

Now we would like to restrict a multifunction f to have its images not only "good" but having themselves "good" images under a fixed g. This is a first kind of closure property we ask for, treating composition of multifunctions. The next lemma shows how to satisfy the condition above but only for the set of images of a fixed element $b \in R_c^M$.

Lemma 3.2.6 *Let $M \models T_0$ and $c \in M$ satisfy $\mathcal{C}(M, c, a)$. Let $b \in R_c^M$. For every $f, g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ there is a T^+-provably total multifunction $f_b^g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ such that*

1. $M \models \ Im(f_b^g)(b) \subset Im(f)(b)$
2. $M \models \forall y \in Im(f_b^g)(b)\ \sigma(a, y)$
3. $M \models \ \forall y \in Im(f_b^g)(b)\ \exists z \in Im(g)(y)\ \sigma(a, z)$.

Proof Consider the multifunction h defined on input x by:

1: Compute $y \in Im(f)(x)$.

2: Compute $z \in Im(g)(y)$, while keeping y in memory.

3: Ask $\sigma(a, z)$?

4: If the answer is YES, output y.

5: Else, output z.

Clearly $h \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ and can be defined in T^+ in such a way as to prove that $w \in Im(h)(x)$ if and only if

$$\exists y \exists z (y \in Im(f)(x) \wedge z \in Im(g)(y) \wedge [(\sigma(a, z) \wedge w = y) \vee (\neg\sigma(a, z) \wedge w = z)]).$$

By Lemma 3.2.5 there is $\tilde{h} \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ such that

$$\tilde{h} \subset h \ \wedge \ \forall w (w \in Im(\tilde{h})(b) \to \sigma(a, w)).$$

Then we have

$$w \in Im(\tilde{h})(b) \to \exists y \exists z (y \in Im(f)(b) \wedge z \in Im(g)(y) \wedge \sigma(a, z) \wedge w = y).$$

From this we get that

$$w \in Im(f)(b) \ \wedge \ \exists z (z \in Im(g)(w) \wedge \sigma(a, z)).$$

Hence $\tilde{h}$ is the multifunction f_b^g we were looking for. □

Thanks to this lemma, for $b \in R_c^M$ and $f, g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ we can speak about the "good" images of b by f which have "good" images under g. This set is defined as follows.

Definition 3.2.7 *For every* $b \in R_c^M$ *and* $f, g \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$ *we put*

$$[f(b)]^g := \{y \in M \ : \ M \models y \in Im(f)(b) \wedge \sigma(a,y) \wedge \exists z(z \in Im(g)(y) \wedge \sigma(a,z))\}.$$

The next lemma contains some easy remarks which are used later.

Lemma 3.2.8 *Let* $M \models T_0$ *and let* $c \in M$ *satisfy* $\mathcal{C}(M,c,a)$. *Then, for every* $b, b' \in R_c^M$ *and* $f, g \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$ *the following holds:*

1. $M \models Im(f_b^g)(b) \subset [f(b)]^g$
2. $M \models [f(b)]^g \neq \emptyset$
3. $M \models \forall y \in [f(b)]^g \ \sigma(a,y)$
4. $M \models \ Im(f)(b) \subset Im(f')(b') \ \rightarrow \ [f(b)]^g \subset [f'(b')]^g$
5. $M \models \ (\forall x \exists ! y \ y \in Im(f)(x)) \ \rightarrow [f(b)]^g = \{f(b)\}$.

Proof From definitions conditions 1,3,4 follows easily. We get condition 2 from condition 1 and the fact that f_b^g is total in M, while condition 5 is implied by condition 2. □

Now we ask for a bit more: we want, for $b \in R_c^M$ and $f \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$, a restriction of the set $Im(f)(b)$ such that every element has "good" images under *every* multifunction $g \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$. For this we naturally want to take intersections of sets $[f(b)]^g$ with g varying in $F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$. The following two lemmas allow us to do this for a finite number of multifunctions g by proving that the intersection is not empty.

Lemma 3.2.9 *Let* $M \models T_0$ *and let* $c \in M$ *satisfy* $\mathcal{C}(M,c,a)$. *Let* $b \in R_c^M$. *Then, for every* $f, g, h \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$ *we have*

$$M \models Im(((f_b^g)_b^h))(b) \subset Im(f_b^g)(b) \cap [f(b)]^h.$$

Proof By Lemma 3.2.6 we have

$$M \models Im((f_b^g)_b^h)(b) \subset Im(f_b^g)(b) \ \wedge \ Im(f_b^g)(b) \subset Im(f)(b).$$

From the second inclusion above we get by Lemma 3.2.8 that

$$M \models Im((f_b^g)_b^h)(b) \subset [f_b^g(b)]^h \subset [f(b)]^h.$$

Hence $M \models Im(((f_b^g)_b^h))(b) \subset Im(f_b^g)(b) \cap [f(b)]^h$. □

Lemma 3.2.10 *Let $M \models T_0$ and $c \in M$ satisfy $\mathcal{C}(M,c,a)$. Let $b \in R_c^M$. Fix $n \in \omega$, $f, g_0, \ldots, g_n \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ and put $f_0 = f_b^{g_0}$ and $f_{k+1} = (f_k)_b^{g_{k+1}}$ for $k < n$. Then*

1. *$M \models Im(f_n)(b) \subset \bigcap_{k \leq n} [f(b)]^{g_k}$*
2. *$M \models \bigcap_{k \leq n} [f(b)]^{g_k} \neq \emptyset$.*

Proof We prove condition 1 by induction on n. The case $n = 0$ is Lemma 3.2.8-1. Suppose condition 1 holds for n. We have

$$Im(f_{n+1})(b) = Im((f_n)_b^{g_{n+1}})(b) = Im(((f_{n-1})_b^{g_n})_b^{g_{n+1}})(b).$$

Then by Lemma 3.2.9 and induction hypothesis

$$\begin{aligned} Im(f_{n+1})(b) &\subset Im((f_{n-1})_b^{g_n})(b) \cap [f_{n-1}(b)]^{g_{n+1}} \\ &\subset Im(f_n)(b) \cap [f_{n-1}(b)]^{g_{n+1}} \\ &\subset \textstyle\bigcap_{k \leq n} [f(b)]^{g_k} \cap [f_{n-1}(b)]^{g_{n+1}} \end{aligned}$$

Now note that by Lemma 3.2.6-1 $Im(f_0)(b) \subset Im(f)(b)$ and for every $k \leq n$

$$Im(f_{k+1})(b) = Im((f_k)_b^{g_{k+1}})(b) \subset Im(f_k)(b).$$

This implies in particular $Im(f_{n-1})(b) \subset Im(f)(b)$ which combined with Lemma 3.2.8-3 gives $[f_{n-1}(b)]^{g_{n+1}} \subset [f(b)]^{g_{n+1}}$. Thus

$$Im(f_{n+1})(b) \subset \bigcap_{k \leq n} [f(b)]^{g_k} \cap [f(b)]^{g_{n+1}} = \bigcap_{k \leq n+1} [f(b)]^{g_k}.$$

From 1 we get 2 as f_n is a total multifunction. $\square$

Now we can prove the main theorem of this section, constructing a set N_c of "good" elements which is $F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$-closed, and setting conditions enabling us to iterate the construction in order to extend this closure property to multifunctions using other parameters. We have to use a compactness argument in order to take infinite intersections, so we move to an elementary extension of the original model.

Theorem 3.2.11 *Let M be a countable model of T_0, and let $c \in M$ satisfy $\mathcal{C}(M,c,a)$. There is a countable elementary extension M_c of M and a subset $N_c \subset M_c$ such that*

1. *$\{a\} \cup \mathcal{S} \subset N_c$*

2. $N_c \subset R_c^{M_c}$

3. $\forall y \in N_c,\ M_c \models \sigma(a,y)$

4. $\forall x \in N_c,\ \forall f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)\ ,\ Im(f)(x) \cap N_c \neq \emptyset$

5. $\forall x \in N_c,\ \langle c,x\rangle \in N_c$ *and* $\mathcal{C}(M_c, \langle c,x\rangle, a)$.

Proof For every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ let e_f be a new constant symbol. As $c \in R_c^M$ we obtain from Lemma 3.2.10 with $b = c$ that the theory

$$Th(M)\ \cup \bigcup_{f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)} \{e_f \in [f(c)]^g :\ g \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)\}$$

is finitely consistent. By compactness and Löwenheim-Skolem theorem we get M_c, a countable model for it, which we can suppose w.l.o.g. extending M. Then for every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$

$$M_c \models \bigcap_{g \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)} [f(b)]^g \neq \emptyset. \tag{3.2}$$

Put

$$N_c := \bigcup_f \bigcap_g [f(c)]^g \quad , \quad f,g \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit).$$

Remark: Note that N_c can alternatively be defined by letting f vary only in $F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ (i.e. both definitions give exactly the same set). This is because for every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ there is some $f' \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ such that $Im(f)(c) = Im(f')(c)$ (by Lemma 3.1.12-3 with $b = c$ and $d \in \mathcal{S}$) and so, by Lemma 3.2.8-4, $[f(c)]^g = [f'(c)]^g$. This will be used in the proof of lemma 3.2.12.

We supposed $\mathcal{C}(M,c,a)$ so there is some term t of L_{BA} such that $a = t(c)$. Consider the function $f(x) := t(x)$. As was remarked in Lemma 3.2.8-5,

$$[f(c)]^g = \{f(c)\} = \{a\}$$

and this for every $g \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$. This proves that $a \in N_c$. A similar argument considering constant functions $f(x) = b$ for every $b \in \mathcal{S}$ shows that $\mathcal{S} \subset N_c$. From Lemma 3.2.8-3 it follows that $\forall y \in N_c,\ M \models \sigma(a,y)$, and of course this is valid in M_c too. From definitions it is clear that $N_c \subset R_c^{M_c}$. So we proved conditions 1,2,3 (condition 3 follows also from condition 2).

To prove condition 4, let $b \in N_c$ and let $h \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ such that

$$b \in \bigcap_{g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)} [h(c)]^g.$$

For every $g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ we have that $[f(b)]^g \subset Im(f)(b)$, and as $b \in Im(h)(c)$ we have $[f(b)]^g \subset [f \circ h(c)]^g$ also. Hence

$$\bigcap_g [f(b)]^g \;\subset\; Im(f)(b) \cap \bigcap_g [f \circ h(c)]^g \;\subset\; Im(f)(b) \cap N_c$$

where g varies over $F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$. By (3.2) we conclude that $Im(f)(b) \cap N_c \neq \emptyset$.

Now we prove condition 5. Let $b \in N_c$. To see that $\langle c, b\rangle \in N_c$ consider the function $h \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ given by $h(x) = \langle c, x\rangle$ and apply condition 4. For $\mathcal{C}(M_c, \langle c, b\rangle, a)$ let $f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},\langle c,b\rangle}(wit)$. As $\mathcal{C}(M, c, a)$ holds, we have $a = t(c)$ for some $t \in Term(L_{BA})$. So by Lemma 3.1.12 there is some $f' \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ such that

$$M_c \models Im(f)(a) = Im(f')(b).$$

By condition 2 we have $b \in R_c^{M_c}$, hence $M_c \models \exists y(y \in Im(f')(b) \wedge \sigma(a, y))$. That gives us the first condition of $\mathcal{C}(M_c, \langle c, b\rangle, a)$. The other one is clear as you can extract first c from $\langle c, b\rangle$, then a from c by using L_{BA}-terms. □

Starting from M and $c \in M$ satisfying some hypotheses, namely $\mathcal{C}(M, c, a)$, Theorem 3.2.11 gives thus an extension M_c of M and a set N_c with some properties. But it says too that those hypotheses are also verified by M_c and any $\langle c, b\rangle$ with $b \in N_c$ (3.2.11-5). So application of the theorem can be iterated to obtain an increasing elementary chain of models $(M_n)_{n\in\omega}$ with corresponding sets N_n. We already know that these sets contain $a \cup \mathcal{S}$. Next we prove that in fact they form a decreasing sequence.

Lemma 3.2.12 *Let M be a countable model of T_0, $c \in M$ satisfying $\mathcal{C}(M, c, a)$, and let M_c, N_c as in the proof of Theorem 3.2.11. Let $b \in N_c$ and $c' = \langle c, b\rangle$, and let $M_{c'}$, $N_{c'}$ be obtained by applying again 3.2.11 to M_c and c'. Then $N_{c'} \subset N_c$.*

Proof By Theorem 3.2.11-5 we have that $c' \in N_c$. So let $h \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$ such that

$$c' \;\in \bigcap_{g \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)} [h(c)]^g.$$

In particular $c' \in Im(h)(c)$. Then for every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ we have

$$Im(f)(c') \subset Im(f \circ h)(c)$$

and then $[f(c')]^{g'} \subset [f \circ h(c)]^{g'}$ by Lemma 3.2.8-4. Thus we have

$$\bigcap_{g' \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c'}(wit)} [f(c')]^{g'} \subset \bigcap_{g' \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c'}(wit)} [f \circ h(c)]^{g'}.$$

From $c' = \langle c, b \rangle$ we get by Lemma 3.1.12 that $F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit) \subset F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c'}(wit)$. Hence

$$\bigcap_{g' \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c'}(wit)} [f(c')]^{g'} \subset \bigcap_{g \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)} [f \circ h(c)]^{g}.$$

Using the remark in the proof of Theorem 3.2.11 we obtain

$$N_{c'} = \bigcup_{f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)} \bigcap_{g' \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c'}(wit)} [f(c')]^{g'}.$$

Hence

$$N_{c'} \subset \bigcup_{f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)} \bigcap_{g \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c}(wit)} [f \circ h(c)]^{g} \subset N_c.$$

□

Remark 3.2.13 Note also that $N_{c'}$ is $F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c'}(wit)$-closed, which is the same as being $F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S},c,b}(wit)$-closed by Lemma 3.1.12. So this says that we have gained a little in terms of closure with respect to N_c as we can use an additional parameter from N_c. By iterating this procedure, as the N_c's are decreasing, we eventually obtain a set N closed under every $f \in F[|\tau|^{\omega}]^{\Sigma_i^p}_{\mathcal{S}}(wit)$ using *any* parameter from N, i.e. N will be closed under two-variables multifunctions. In particular N will be a L_{BA}-substructure of an elementary extension of M preserving $\hat{\Sigma}_i^b$-formulas.

3.3 A model of $\hat{T}_2^{i,2^{|\tau|^{\omega}}}$

Let $M_0 \models T_0$. In this section we use Theorem 3.2.11 to construct an elementary extension of M_0 with a $\hat{\Sigma}_i^b$-substructure $N \models \hat{T}_2^{i,2^{|\tau|^{\omega}}}$ consisting of "good" elements. We start with N_a and proceed as explained in remark 3.2.13.

Theorem 3.3.1 *Let T^+ be a theory on $\mathcal{L}_\mathcal{S}$ containing $\hat{T}_2^{i,2^{|\tau|^\omega}}$, σ a $\hat{\Sigma}_i^b(\mathcal{L}_\mathcal{S})$-formula and M_0 a countable model of*

$$T^+ \cup \{\exists y(y \in Im(f)(a) \wedge \sigma(a,y)) : f \in F[|\tau|^\omega]_\mathcal{S}^{\Sigma_i^p}(wit)\}.$$

There is an elementary extension M of M_0 and an $\mathcal{L}_\mathcal{S}$-substructure N of M such that

1. $a \in N$
2. $N \prec_{\hat{\Sigma}_i^b} M$
3. $N \models \hat{T}_2^{i,2^{|\tau|^\omega}}$
4. $N \models \forall y\ \sigma(a,y)$.

Proof We repeatedly use Theorem 3.2.11 to construct a increasing elementary chain of models $(M_n)_{n\in\omega}$ with corresponding sets $(N_n)_{n\geq 1}$. Each set $N_n \subset M_n$ will be of the form N_{a_n} for some $a_n \in M_n$. Apply Theorem 3.2.11 a first time with $c = a_1 := a$ to obtain $M_1 \succ M_0$ and $N_1 = N_a$ (a is in fact the only element $c \in M_0$ for which we are sure $\mathcal{C}(M,c,a)$ holds). Now fix an enumeration of N_1. Suppose that for $n \geq 1$ we have obtained M_n and N_n from M_{n-1} and a_n as in 3.2.11, and suppose also that $N_n \subset N_1$. Let b_{n+1} be the first element in the enumeration of N_1 lying in N_n and put $a_{n+1} := \langle a_n, b_{n+1}\rangle$. Apply Theorem 3.2.11 to get M_{n+1}, N_{n+1}. By Lemma 3.2.12 $N_{n+1} \subset N_n \subset N_1$, so we can continue in this way. Put

$$M := \bigcup_{n\in\omega} M_n \quad , \quad N := \bigcap_{n\geq 1} N_n.$$

For every $n \geq 1$ we have $\{a\} \cup \mathcal{S} \subset N_n$ and $M_{n-1} \prec M_n$ by Theorem 3.2.11. Hence $M_0 \prec M$ and $\{a\} \cup \mathcal{S} \subset N$. Now we prove the closure property of N we wanted.

Lemma 3.3.2 *Let $c, c' \in N$, $f \in F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit)$. Then $N \cap Im(f)(c') \neq \emptyset$.*

Proof of Lemma 3.3.2 Let $c \in N$. By the construction of $N = \bigcap_{k\in\omega} N_{a_k}$ we have that $a_{n+1} = \langle a_n, c\rangle$ for some $n \in \omega$. Let $k > n$. As $c' \in N_{a_k}$ we have by Theorem 3.2.11-(4) that

$$\forall f \in F[|\tau|^\omega]_{\mathcal{S},a_k}^{\Sigma_i^p}(wit), \quad N_{a_k} \cap Im(f)(c') \neq \emptyset.$$

i.e., $M_k \models b \in Im(f)(c')$ for some $b \in N_{a_k}$. As $M_k \prec M$, this holds in M too. Now note that by the a_n's construction we have $c = t(a_k)$ for some L_{BA}-term t, hence by Lemma 3.1.12 we have

$$F[|\tau|^\omega]_{\mathcal{S},c}^{\Sigma_i^p}(wit) \subset F[|\tau|^\omega]_{\mathcal{S},a_k}^{\Sigma_i^p}(wit)$$

and the result follows. □

Lemma 3.3.3 *Let $\bar{u}$ and $\bar{c}$ means respectively $u_1, \dots, u_n$ and $c_1, \dots, c_m$ and let $\psi(\bar{u}, \bar{c})$ be a $\hat{\Pi}^b_{i-1}$ formula of $\mathcal{L}_\mathcal{S}$ with implicit bounds for $\bar{u}$ and parameters $\bar{c}$ from N. The following hold*

1. *If $c \in N$, then $t(c) \in N$, for every term t of $\mathcal{L}_\mathcal{S}$.*
2. *If $c, c' \in N$, then $\langle c, c' \rangle \in N$.*
3. *If $M \models \exists \bar{u}\ \psi(\bar{u}, \bar{c})$ then $\exists \bar{u} \in N$ such that $M \models \psi(\bar{u}, \bar{c})$.*

Proof of Lemma 3.3.3 By considering respectively the functions $x \mapsto t(x)$ and $x \mapsto \langle c, x \rangle$ we deduce conditions 1 and 2 from Lemma 3.3.2 . By condition 2 we have that $c := \langle c_1, \dots, c_m \rangle \in N$. Consider a $\hat{\Pi}^b_{i-1}$ formula $\tilde{\psi}(u, c)$ such that

$$M \models \tilde{\psi}(u, c) \leftrightarrow \psi(\langle u \rangle_1, \dots, \langle u \rangle_n, \bar{c}).$$

Note that we can suppose w.l.o.g. that u is bounded in $\tilde{\psi}$. As $M \models \exists \bar{u}\ \psi(\bar{u}, \bar{c})$ we have $M \models \exists u \tilde{\psi}(u, c)$. Consider the multifunction f in $F[|\tau|^\omega]^{\Sigma^p_i}_\mathcal{S}(wit)$ which assigns to x a witness u for $\tilde{\psi}(u, x)$ if there is one, and 0 else. Then

$$M \models \forall u (u \in Im(f)(c) \rightarrow \tilde{\psi}(u, c)).$$

By Lemma 3.3.2 there is some $b \in N$ such that $M \models \tilde{\psi}(b, c)$. Hence

$$M \models \psi(\langle b \rangle_1, \dots, \langle b \rangle_n, \bar{c})$$

and by condition 1 we know that $\langle b \rangle_1, \dots, \langle b \rangle_n \in N$. □

As a consequence we get that N is an $\mathcal{L}_\mathcal{S}$-substructure of M (consider formulas like $\exists y (y = t(x_1, \dots, x_n))$ for $t \in \mathcal{L}_\mathcal{S}$) and we can deduce by Theorem 1.3.4 that

$$N \prec_{\hat{\Sigma}^b_i} M. \tag{3.3}$$

Now we prove that $N \models \hat{T}_2^{i, 2^{|\tau|^\omega}}$. Let $\theta(j, c)$ be a $\hat{\Sigma}^b_i$ formula with parameter $c \in N$ (now it suffices to consider formulas with one single parameter as we know N is $\mathcal{L}_\mathcal{S}$-closed). Let $d \in N$ of the form $2^{|l(s(b))|^k}$ for some $l \in \tau$, $s \in Term(\mathcal{L}_\mathcal{S})$, $k \in \mathbb{N}$ and $b \in N$ and suppose that

$$N \models \theta(0, c) \wedge \neg\theta(d, c).$$

By (3.3) above M satisfies this also. Consider the function f with parameter c that on input x uses binary search to find some $j \in M$ such that

$$M \models j < 2^{|l(s(x))|^k} \wedge \theta(j, c) \wedge \neg\theta(j+1, c).$$

As $M \models \hat{T}_2^{i,2^{|\tau|^\omega}}$ we can define f in M. This function queries a $\hat{\Sigma}_i^b$ oracle $|l(s(x))|^k$ times, hence $f \in F[|\tau|^\omega]^{\Sigma_i^p}_{\mathcal{S},c}(wit)$. Moreover f is single-valued, so by Lemma 3.3.2 $j := f(b) \in N$. Finally, as $N \prec_{\hat{\Sigma}_i^b} M$ we have

$$N \models j < d \wedge \theta(j,c) \wedge \neg\theta(j+1,c).$$

Now let $b \in N$. For any $n \geq 1$ we have $b \in N_n$, thus $M_n \models \sigma(a,b)$ by Theorem 3.2.11-3 . But $M_n \prec M$ and $N \prec_{\hat{\Sigma}_i^b} M$, hence $N \models \sigma(a,b)$. So $N \models \forall y\ \sigma(a,y)$. This ends the proof of Theorem 3.3.1. □

As a consequence we obtain a model-theoretic proof of a witnessing theorem for $\hat{\Sigma}_{i+1}^b$-definable multifunctions of $\hat{T}_2^{i,2^{|\tau|^\omega}}$. For another proof using the witness function method of proof-theoretic character see [20].

Theorem 3.3.4 (Witnessing theorem for $\hat{T}_2^{i,2^{|\tau|^\omega}}$) *Suppose that*

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \exists y \varphi(x,y)$$

where $\varphi \in \hat{\Sigma}_{i+1}^b$. Then for some $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$,

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \forall y (y \in Im(f)(x) \rightarrow \varphi(x,y)).$$

Proof First we argue that it suffices to prove the result for $\varphi \in \hat{\Pi}_i^b$. Suppose the theorem established for any $\varphi \in \hat{\Pi}_i^b$ and let us prove it for a $\hat{\Sigma}_{i+1}^b$-formula $\psi(x,y) \equiv \exists u \varphi(x,y,u)$. We suppose that $\varphi(x,y,u)$ is a $\hat{\Pi}_i^b$-formula containing a bound for u. Note that

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \exists y \psi(x,y) \rightarrow \forall x \exists z \varphi(x, \langle z\rangle_1, \langle z\rangle_2).$$

By our assumption there is some $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$ such that

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \forall z (z \in Im(f)(x) \rightarrow \varphi(x, \langle z\rangle_1, \langle z\rangle_2)).$$

Defining $f'(x) := \langle f(x)\rangle_1$ we have $\hat{T}_2^{i,2^{|\tau|^\omega}} \vdash \forall x \forall y (y \in Im(f')(x) \rightarrow \exists u \varphi(x,y,u))$.

We now prove the theorem for $\varphi \in \hat{\Pi}_i^b$. Suppose that for every $f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)$

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \nvdash \forall x \forall y (y \in Im(f)(x) \rightarrow \varphi(x,y)).$$

By Lemma 3.1.14 the theory

$$\hat{T}_2^{i,2^{|\tau|^\omega}} \cup \{\exists y (y \in Im(f)(a) \wedge \neg\varphi(a,y)) : f \in F[|\tau|^\omega]^{\Sigma_i^p}(wit)\}$$

is consistent, where a is a new constant symbol. Take M_0 any countable model of this theory. By Theorem 3.3.1 there is an extension M of M_0 with a substructure N containing a and satisfying $\hat{T}_2^{i,2^{|\tau|^\omega}}$ and $\forall y \neg\varphi(a,y)$. This contradicts the hypothesis. □

3.4 Extending to a model of $\hat{T}_2^{i+1,|\tau|}$

The technique used in this section is inspired from Zambella [31]. In his proof of Buss's theorem he adapts an unpublished model-theoretic argument used by Albert Visser to prove the Mint's-Parsons theorem of Σ_1-conservativity between PRA and $I\Sigma_1$ ([15],[19],[26]). The difficulty of applying this technique is that we need theorem 3.3.1 (see comments at the beginning of section 3.2).

Theorem 3.4.1 *Let $i \geq 1$. Every countable model $M \models \hat{T}_2^{i,2^{|\tau|^\omega}}$ has a $\forall\hat{\Sigma}_i^b$-elementary extension $K \models \hat{T}_2^{i,2^{|\tau|^\omega}}$ such that for every $\hat{\Pi}_i^b$-formula $\varphi(x,y)$, possibly with parameters, there is $f \in F[|\tau|^\omega]_K^{\Sigma_i^p}(wit)$ satisfying*

$$K \models \forall x \exists y \varphi(x,y) \rightarrow \forall x \forall y (y \in Im(f)(x) \rightarrow \varphi(x,y)). \tag{3.4}$$

Proof Model K is constructed as the union of an increasing $\forall\hat{\Sigma}_i^b$-elementary chain

$$M = M_0 \prec_{\forall\hat{\Sigma}_i^b} M_1 \prec_{\forall\hat{\Sigma}_i^b} \dots \prec_{\forall\hat{\Sigma}_i^b} M_n \dots$$

of countable models of $\hat{T}_2^{i,2^{|\tau|^\omega}}$. Let us suppose $M_0, \dots, M_n$ constructed and consider for every $j \leq n$ enumerations $(\varphi_{\langle j,k\rangle})_{k\in\omega}$ of $\hat{\Pi}_i^b$-formulas $\varphi(x,y)$ with parameters in M_j.

To construct M_{n+1} consider the formula $\varphi_{\langle j,k\rangle}$ such that $\langle j,k\rangle = n$ (note that its parameters are in M_j, hence in M_n as $j \leq n$). We want M_{n+1} satisfying $(3.4)_n$, which is formula (3.4) with φ_n in place of φ and parameters in M_n. Note that formula (3.4) is equivalent to a $\exists\forall\hat{\Sigma}_i^b$-formula. As the chain is $\forall\hat{\Sigma}_i^b$-elementary this implies by Theorem 1.3.1 that formulas $(3.4)_n$ will be preserved up to the union K. Note also that at the end all the $\hat{\Pi}_i^b$-formulas with parameters in K will be considered.

If M_n already satisfies $(3.4)_n$ for some $f \in F[|\tau|^\omega]_{M_n}^{\Sigma_i^p}(wit)$ then we let $M_{n+1} = M_n$. Otherwise, we extend M_n in a way to satisfy $\exists x \forall y \neg\varphi_n(x,y)$. If $(3.4)_n$ is false for every $f \in F[|\tau|^\omega]_{M_n}^{\Sigma_i^p}(wit)$ then

$$\forall f \in F[|\tau|^\omega]_{M_n}^{\Sigma_i^p}(wit), \quad M_n \not\models \forall x \forall y (y \in Im(f)(x) \rightarrow \varphi(x,y)).$$

This is the same as saying that for every $f \in F[|\tau|^\omega]_{M_n}^{\Sigma_i^p}(wit)$,

$$Th(M_n) \not\vdash \forall x \forall y (y \in Im(f)(x) \rightarrow \varphi(x,y)).$$

Now $Th(M_n)$ is a theory containing $\hat{T}_2^{i,2^{|\tau|^\omega}}$ in a language $\mathcal{L}_n$ extending L_{BA} by a countable set of constants for M_n. Hence by Lemma 3.1.14 we have that

$$T_n := Th(M_n) \cup \{\exists y (y \in Im(f)(a) \wedge \neg\varphi_n(a,y)) : f \in F[|\tau|^\omega]_{M_n}^{\Sigma_i^p}(wit)\}$$

is consistent, with a a new constant symbol. Using Theorem 3.3.1 we get a countable model M' of T_n, in particular M' is elementarily equivalent to M_n and we can suppose w.l.o.g. that M' is an $\mathcal{L}_n$-extension, and M' contains an $\mathcal{L}_n$-substructure, which is our M_{n+1}, such that

1. $a \in M_{n+1}$
2. $M_{n+1} \prec_{\hat{\Sigma}^b_i} M'$
3. $M_{n+1} \models \hat{T}_2^{i,2^{|\tau|^\omega}}$
4. $M_{n+1} \models \forall y \neg \varphi_n(a, y)$.

As $M_n \subset_{\mathcal{L}_n} M'$ and $M_{n+1} \subset_{\mathcal{L}_n} M'$, and $\mathcal{L}_n$ has constants for the elements of M_n, we have that $M_n \subset_{\mathcal{L}_n} M_{n+1}$. This extension is $\hat{\Sigma}^b_i$-elementary because $M_n \prec M'$ and $M_{n+1} \prec_{\hat{\Sigma}^b_i} M'$. Moreover it preserves also $\forall\hat{\Sigma}^b_i$-formulas: from M_n to M' because $M_n \prec M'$, and from M' to M_{n+1} because $M_{n+1} \prec_{\hat{\Sigma}^b_i} M'$. Now put

$$K := \bigcup_{n \in \omega} M_n.$$

By Theorem 1.3.1 we have $M_n \prec_{\forall\hat{\Sigma}^b_i} K$ for every $n \in \omega$, in particular $M \prec_{\forall\hat{\Sigma}^b_i} K$, and thus $K \models \hat{T}_2^{i,2^{|\tau|^\omega}}$. Finally, as explained above, K satisfies (3.4) for every $\hat{\Pi}^b_i$-formula with parameters in K. □

Lemma 3.4.2 *Let $i \geq 1$. Let $K \models \hat{T}_2^{i,2^{|\tau|^\omega}}$ be such that for every $\hat{\Pi}^b_i$-formula $\varphi(x, y)$ possibly with parameters, there is $f \in F[|\tau|^\omega]_K^{\Sigma^p_i}(wit)$ satisfying*

$$K \models \forall x \exists y \varphi(x, y) \to \forall x \forall y (y \in Im(f)(x) \to \varphi(x, y)).$$

Then $K \models \hat{\Sigma}^b_{i+1}$-$DC^{|\tau|}$.

Proof By Lemma 1.2.11 it suffices to show $K \models \hat{\Pi}^b_i$-$DC^{|\tau|}$. So let $a, b \in K$, $l \in \tau$, and suppose

$$K \models \forall j \forall x < b\ \exists y < b\ \alpha(j, x, y).$$

After the usual transformations that contract variables j and x into a single one using pairing, it follows from hypothesis that there is a multifunction $f \in F[|\tau|^\omega]_K^{\Sigma^p_i}(wit)$ such that

$$K \models \forall j \forall x < b \forall y\ (y \in Im(f)(\langle j, x\rangle) \to y < b \wedge \alpha(j, x, y)).$$

Define $g \in F[|\tau|^\omega]_K^{\Sigma^p_i}(wit)$ by $BPR^{|\tau|^\omega}$ as follows (by (x) we note the one-element sequence w such that $(w)_0 = x$):

$G(0) = (f(\langle 0,0\rangle))$

$G(j+1) = min(G(j)^\frown(f(\langle j,(G(j))_j\rangle)), r(a,b))$

$g(z) = G(|l(z)|)$.

Functions G and g use parameters a, b. Here $r(a,b)$ is a term bounding the sequence v of length $|l(a)|$ such that $\forall j < |l(a)|\ ((v)_j = b)$. As was remarked in 3.1.7 the multifunction g can be defined by $\hat{T}_2^{i,2^{|\tau|^\omega}}$ in such a way that this theory proves g's recursive properties. In particular we have that $lh(g(a)) = |l(a)|$ and

$$\forall j < |l(a)|((g(a))_{j+1} \leq b \wedge (g(a))_{j+1} \in Im(f)(\langle j,(g(a))_j\rangle)).$$

Putting $w = g(a)$ we have

$$K \models \forall j < |l(a)|((w)_j < b \wedge (w)_{j+1} < b \wedge \alpha(j,(w)_j,(w)_{j+1})).$$

□

As a consequence of the preceding results we obtain the following extension theorem for models of $\hat{T}_2^{i,2^{|\tau|^\omega}}$. First we say what we mean by M satisfying an equality between complexity classes.

Definition 3.4.3 *Let $i \geq 1$ and $M \models \hat{T}_2^{i,2^{|\tau|^\omega}}$. We say that*

$$M \models [|\tau|^\omega]^{\Sigma_i^p} = \Sigma_{i+1}^p \cap co\text{-}\Sigma_{i+1}^p$$

if for every formula $\varphi(x)$ satisfying

$$M \models \forall x((\varphi(x) \leftrightarrow \theta(x)) \wedge (\varphi(x) \leftrightarrow \psi(x)))$$

for some $\theta \in \Sigma_{i+1}^b$ and $\psi \in \Pi_{i+1}^b$, there is a 0-1 function $f \in F[|\tau|^\omega]_M^{\Sigma_i^p}$ such that

$$M \models \forall x(\varphi(x) \leftrightarrow f(x) = 1).$$

Theorem 3.4.4 ($i \geq 1$) *Every countable model $M \models \hat{T}_2^{i,2^{|\tau|^\omega}}$ has an extension K such that:*

1. *$M \prec_{\forall\hat{\Sigma}_i^b} K$.*
2. *$K \models \hat{T}_2^{i+1,|\tau|}$.*
3. *$K \models [|\tau|^\omega]^{\Sigma_i^p} = \Sigma_{i+1}^p \cap co\text{-}\Sigma_{i+1}^p$.*

Proof By Theorem 3.4.1 and Lemma 3.4.2 we know that M has a $\forall\hat{\Sigma}^b_i$-elementary extension $K \models \hat{\Sigma}^b_{i+1}\text{-}DC^{|\tau|}$, but clearly K satisfies already $EBASIC$ and $\Delta^b_0\text{-}IND^{|\tau|}$ hence by Lemma 1.2 $K \models \hat{T}^{i+1,|\tau|}_2$. Now let $\varphi(x)$ such that

$$M \models \forall x(\varphi(x) \leftrightarrow \exists y_1\alpha_1(x,y_1) \leftrightarrow \neg\exists y_2\alpha_2(x,y_2)). \tag{3.5}$$

for some $\alpha_j \in \hat{\Pi}^b_i$, $j = 1, 2$. Then we have

$$M \models \forall x\exists y(\alpha_1(x,y) \vee \alpha_1(x,y)).$$

As the formula in the scope of $\exists y$ is $\hat{\Pi}^b_i$, Theorem 3.4.1 applies, so there is some $f \in F[|\tau|^\omega]^{\Sigma^p_i}_K(wit)$ such that

$$M \models \forall x\forall y(y \in Im(f)(x) \to (\alpha_1(x,y) \vee \alpha_1(x,y)). \tag{3.6}$$

Now let g the function that on input x computes some $y \in Im(f)(x)$ and then asks $\alpha_1(x,y)$? If the answer is YES then $g(x) = 1$, else $g(x) = 0$. Note that by (3.5) and (3.6) the value of $g(x)$ does not depend on y. So $g \in F[|\tau|^\omega]^{\Sigma^p_i}_K(wit)$ is single-valued and clearly:

$$K \models \forall x(\varphi(x) \leftrightarrow g(x) = 1).$$

To conclude we argue as in the proof of 3.1.10 to prove that in fact $g \in F[|\tau|^\omega]^{\Sigma^p_i}_K$.
□

In the case $i = 1$ and $\tau = \{x\}$ we get that every model of T^1_2 has a $\forall\hat{\Sigma}^b_i$-elementary extension to a model of S^2_2 satisfying $\Delta^p_2 = \Sigma^p_2 \cap co\text{-}\Sigma^p_2$. See [13] p. 127 for a model of PV satisfying $P = NP \cap co\text{-}NP$.

Theorem 3.4.5 ($i \geq 1$) *The theory* $\hat{T}^{i+1,|\tau|}_2$ *is* $\forall\exists\hat{\Sigma}^b_{i+1}$*-conservative over* $\hat{T}^{i,2^{|\tau|^\omega}}_2$.

Proof Suppose $\hat{T}^{i+1,|\tau|}_2 \vdash \forall x\varphi(x)$, $\varphi \in \exists\hat{\Sigma}^b_{i+1}$. Let $M \models \hat{T}^{i,2^{|\tau|^\omega}}_2$ and let M' be countable and elementarily equivalent to M. Extend M' to $K \models \hat{T}^{i+1,|\tau|}_2$ using Theorem 3.4.4. Then $M' \prec_{\forall\hat{\Sigma}^b_i} K$ and $K \models \forall x\varphi(x)$. But $M' \prec_{\forall\hat{\Sigma}^b_i} K$ is the same as $M' \prec_{\exists\hat{\Sigma}^b_{i+1}} K$, hence $M' \models \forall x\varphi(x)$ and the result follows. □

Theorem 3.4.6 (Witnessing theorem for $\hat{T}^{i+1,|\tau|}_2$) *Suppose that*

$$\hat{T}^{i+1,|\tau|}_2 \vdash \forall x\exists y\varphi(x,y)$$

where $\varphi \in \hat{\Sigma}^b_{i+1}$. *Then for some* $f \in F[|\tau|^\omega]^{\Sigma^p_i}(wit)$,

$$\hat{T}^{i,2^{|\tau|^\omega}}_2 \vdash \forall x\forall y(y \in Im(f)(x) \to \varphi(x,y)).$$

Proof Follows immediately from the witnessing theorem for $\hat{T}_2^{i,2^{|\tau|^\omega}}$ (Theorem 3.3.4) and the conservativity result above. □

Remark 3.4.7 It is not difficult to see that the method used in this chapter applies in the general case of a language containing $\#_k$ for any $k \geq 2$ (see section 2.1 for a definition). In particular we obtain for $k = 3$, $i = 1$ and $\tau = \{||x||\}$:

$$\hat{R}_3^2 \equiv_{\forall \hat{\Sigma}_2^b} \hat{\Sigma}_1^b\text{-}IND^{2^{||x||^\omega}}$$

but the latter theory is S_3^1 as in the presence of $\#_3$ we have that $\forall x\ \exists y\ 2^{||x||^n} = |y|$. So this strengthens corollary 2.1.3.

Chapter 4

Replacement

Inside a recursively saturated model of $T_2^{i,|\tau|}$ we construct a $\hat{\Sigma}_i^b$-elementary substructure satisfying $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$. The technique used is purely model theoretical and comes from Ressayre's [22] conservation theorem between Σ_{i+1}^b-$REPL^{|x|}$ and S_2^i . Thus, we obtain a model theoretic proof of the $\forall\hat{\Sigma}_{i+1}^b$-conservativity of $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$ over $T_2^{i,|\tau|}$ which was proved by Pollett [20] using the witness function method of proof theoretic character. In section 4.1 we do some preliminaries. In section 4.2 we give conditions which suffice to obtain the desired structures, and in section 4.3 we show how these conditions are fulfilled in a recursively saturated model.

4.1 Preliminaries

In this chapter, we will use some equivalent versions of replacement and strong-replacement schemes. They will allow us to keep control of the length of the sequences, particularly when we will concatenate them.

Definition 4.1.1 (Replacement$_+$) *Given $\alpha(x, y)$, a formula which might contain parameters, we note by α-$REPL_{+}{}_b^a$ the formula*

$$\forall x \leq a \exists y \leq b\ \alpha(x, y) \rightarrow$$

$$\exists w\ (seq(w) \wedge lh(w) = a + 1 \wedge mx(w) = b \wedge \forall x \leq a\ \alpha(x, (w)_x)).$$

If τ is a set of terms, Ψ-$REPL_{+}{}^{\tau}$ is the scheme

$$\{\forall a \forall b\ \alpha\text{-}REPL_{+}{}_b^{l(a)}\ :\ \alpha \in \Psi, l \in \tau\}.$$

Lemma 4.1.2 *The schemes $\hat{\Sigma}_i^b$-$REPL^{\tau}$ and $\hat{\Sigma}_i^b$-$REPL_{+}{}^{\tau}$ are equivalent over $EBASIC$.*

Proof: The only important difference with respect to the original replacement scheme is that we impose $lh(w) = a+1$, i.e. $(w)_x = 0$ for $x > a$. However, this does not cause any problem as we want only to conclude $\forall x \leq a\ \alpha(x, (w)_x))$. $\square$

Definition 4.1.3 (Strong replacement$_+$) *Given $\alpha(x, y)$, a formula which might contain parameters, we note by α-STRONG $REPL_{+}{}^{a}_{b}$ the formula*

$$\exists w(seq(w) \wedge lh(w) = a + 1 \wedge mx(w) = b \wedge$$

$$\forall x \leq a(\exists y \leq b\ \alpha(x, y) \rightarrow\ \alpha(x, (w)_x)).$$

If τ is a set of terms, Ψ-STRONG $REPL_+{}^{\tau}$ is the scheme

$$\{\forall a \forall b\ \alpha\text{-}STRONG\ REPL_{+}{}^{l(a)}_{b}\ :\ \alpha \in \Psi, l \in \tau\}.$$

Lemma 4.1.4 *The schemes $\hat{\Sigma}^b_i$-STRONG $REPL^{\tau}$ and $\hat{\Sigma}^b_i$-STRONG $REPL_+{}^{\tau}$ are equivalent over $EBASIC$.*

Proof: Same remarks as for the proof of Lemma 4.1.2. $\square$

So we will continue to refer indistinctly to $REPL$ or $STRONG\ REPL$ when using the $_+$ versions of them.

The following lemmas recall some easy consequences of the theories $T_2^{i,\tau}$. For convenience we state them in a model theoretic setting.

Lemma 4.1.5 *Let M be an L_{BA}-structure, $a \in M$ such that $M \models T_2^{i,a}$. Then $M \models T_2^{i,a^{\omega}}$.*

Proof: This is done by a classical speed-up argument. We only prove the special case $T_2^{i,a} \rightarrow T_2^{i,a^2}$, since the general case can be handled in a similar way. Let $\varphi(x)$ be a $\hat{\Sigma}^b_i$-formula and suppose that

$$M \models \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)).$$

We want to derive $\varphi(a^2)$. Let $\theta(x) \equiv \alpha(ax)$. Then we have $M \models \theta(0)$. Let $b \in M$ and suppose $M \models \theta(b)$. Set $\alpha(b, y) \equiv \varphi(ab + y)$. Then we clearly have

$$M \models \alpha(b, 0) \wedge \forall y\, (\alpha(b, y) \rightarrow \alpha(b, y + 1)).$$

Since $\alpha \in \hat{\Sigma}^b_i$, we get by induction $\alpha(b, a)$, i.e. $\varphi(ab+a)$, i.e. $\theta(a(b+1))$. Therefore, we have proved

$$M \models \forall x\, (\theta(x) \rightarrow \theta(x + 1))$$

so we get $M \models \theta(a)$, i.e. $M \models \varphi(a^2)$. $\square$

Lemma 4.1.6 *Let M be an L_{BA}-structure, $a, b \in M$ such that $a \leq b$ and $M \models T_2^{i,b}$. Then $M \models T_2^{i,a}$.*

Proof: Let $\varphi(x)$ be a $\hat{\Sigma}_i^b$-formula and suppose that

$$M \models \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1)).$$

Apply IND^b to the formula $\theta(x) \equiv (x < a \wedge \varphi(x)) \vee (x > a \wedge \varphi(a))$. □

Lemma 4.1.7 *Let M be an L_{BA}-structure, $a \in M$ such that $M \models T_2^{i,|a|}$. Then $M \models T_2^{i,|a|^\omega} \cup T_2^{i,2^{||a||.\omega}}$.*

Proof: The fact $T_2^{i,|a|} \rightarrow T_2^{i,|a|^\omega}$ follows from Lemma 4.1.5. Let $k \in \mathbb{N}$. Then $2^{||a||.k} < |a|^{k+1}$ and the result follows by Lemma 4.1.6. □

Lemma 4.1.8 *Let M be an L_{BA}-structure, and let $a, b \in M$ such that $a \leq |b|$ and $M \models T_2^{i,a}$. Then*

a. $M \models \hat{\Sigma}_i^b$-$REPL^a$

b. $M \models \hat{\Sigma}_i^b$-$COMP^a$

c. $M \models \hat{\Sigma}_i^b$-$STRONG\ REPL^a$.

Proof: This is a more general statement than the theorems 1.2.2, 1.2.5 and 1.2.8 as a need not to be equal to $|c|$ for some $c \in M$. In other words, the set $log(M) := \{|x| \ : \ x \in M\}$ is not necessarily an initial segment of M (this is true if $M \models S_2^1$ for example). Nevertheless the proof is the same, and it is done by induction on a. The fact that $a < |b|$ allows to bound the corresponding w in each scheme. □

The following lemma says that we can concatenate sequences in a suitable way.

Lemma 4.1.9 *Suppose that w_1 and w_2 are sequences in a model M such that:*

1. *For $j = 1, 2$, $m_j := mx(w_j)$, $l_j := lh(w_j)$*
2. *$M \models \exists z(l_1 + l_2 < |z|)$*
3. *$M \models \hat{\Sigma}_1^b$-$IND^{l_1+l_2}$.*

Then there is a sequence $w \in M$ such that:

- $mx(w) = max(m_1, m_2)$

- $lh(w) = l_1 + l_2$
- $\forall u < max(l_1, l_2)((u < l_1 \to (w)_u = (w_1)_u) \wedge (u < l_2 \to (w)_{l_1+u} = (w_2)_u))$.

Moreover this w is unique. We call it the concatenation *of w_1 and w_2 and note it by $w_1{}^\frown w_2$.*

Proof: Note that we have

$$\forall x \leq l_1 + l_2 \; \exists y \leq max(m_1, m_2) \; ((x < l_1 \wedge y = (w_1)_x) \vee (x \geq l_1 \wedge y = (w_2)_{x-l_1})).$$

By Lemma 4.1.8 we can use $\hat{\Sigma}_1^b$-$REPL^{l_1+l_2}$ in M. We get that

$$\exists w(seq(w) \wedge lh(w) = l_1 + l_2 \wedge mx(w) = max(m_1, m_2) \wedge$$

$$\forall x < l_1 + l_2 \; ((x < l_1 \wedge (w)_x = (w_1)_x) \vee (x \geq l_1 \wedge (w)_x = (w_2)_{x-l_1})).$$

The result follows after some easy transformations. Uniqueness is proved by $\hat{\Sigma}_1^b$-IND on the length of w. $\square$

4.2 Obtaining substructures

The first lemma gives a general way to obtain $\hat{\Sigma}_i^b$-substructures via some simple conditions.

Lemma 4.2.1 *Let M be an L_{BA}-structure, I a cut in M. Suppose for every $k \in I$ we are given some elements $b_k \in M$ and subsets $W_k \subset M$. Let $M_I := \bigcup_{k \in I} W_k$ and suppose that the following holds:*

1. *$W_k \subset W_{k+1}$ for every $k \in I$*
2. *$(b_k)_{k \in I}$ is cofinal in M_I*
3. *for every $\hat{\Pi}_{i-1}^b$-formula φ, $k \in I$ and $x \in W_k$,*

$$\text{if } M \models \exists y \leq b_k \; \varphi(x, y), \quad \text{then } \exists y \in W_{k+1} \text{ s.t. } M \models y \leq b_k \; \varphi(x, y)$$

4. *$M_I \subset_{L_{BA}} M$.*

Then $M_I \prec_{\hat{\Sigma}_i^b} M$.

Proof Let t be a term of L_{BA}, $\alpha(y,c)$ a $\hat{\Pi}^b_{i-1}$-formula with parameter $c \in M_I$, and suppose $M \models \exists y \leq t(c)\alpha(y,c)$ (the case with many parameters can be reduced to this one using the pairing function). As $c \in M_I$ we have by condition 4 that $t(c) \in M_I$, then by conditions 1 and 2 there is some $k \in I$ such that $t(c) \leq b_k$ and $c \in W_k$. Thus, we have

$$M \models \exists y \leq b_k(y \leq t(c) \wedge \alpha(y,c)).$$

By condition 3, there is such a witness y in W_{k+1}, hence in M_I as I is a cut. We conclude applying Theorem 1.3.4. □

The following lemma recalls an easy fact, useful to get elementarity properties when constructing models.

Notation: For an L_{BA}-structure M and $a, b \in M$, we put

$$[a,b]_M := \{x \in M : M \models a \leq x \leq b\}.$$

Lemma 4.2.2 *Let M, M' be L_{BA}-structures such that $M' \prec_{\hat{\Sigma}^b_i} M$. Let $c \in M$ such that $[0,c]_M \subset M'$. If $\theta(x) \in B(\hat{\Sigma}^b_i)$ with parameters in M' then*

$$M \models \exists x \leq c\ \theta(x) \text{ if and only if } M' \models \exists x \leq c\ \theta(x).$$

Proof Obvious. □

By adding such a condition to those of Lemma 4.2.1 we obtain sufficient conditions to get $\hat{\Sigma}^b_i$-elementary substructures of models of $T^{i,\tau}_2$ satisfying also that theory.

Theorem 4.2.3 *Let $i \geq 1$, τ a set of unary terms. Let $M \models T^{i,\tau}_2$, I a cut in M. Suppose for every $k \in I$ we are given some elements $b_k \in M$ and subsets $W_k \subset M$. Let $M_I := \bigcup_{k \in I} W_k$ and suppose that the conditions 1-4 of Lemma 4.2.1 hold, and additionally*

$$\forall\ l \in \tau,\ c \in M_I, \quad [0, l(c)]_M \subset M_I.$$

Then

a. $M_I \prec_{\hat{\Sigma}^b_i} M$

b. $M_I \models T^{i,\tau}_2$.

Proof By Lemma 4.2.1 we get $M_I \prec_{\hat{\Sigma}_i^b} M$. Now let $\alpha(j,d)$ be a $\hat{\Sigma}_i^b$-formula with parameter $d \in M_I$. Let $l \in \tau$, $c \in M_I$ and suppose that

$$M_I \models \alpha(0,d) \wedge \neg\alpha(l(c),d).$$

By (a) this is also satisfied in M. As $M \models T_2^{i,\tau}$ we have that

$$M \models \exists j < l(c)(\alpha(j,d) \wedge \neg\alpha(j+1,d)).$$

By condition 5 we can apply Lemma 4.2.2 to conclude that M_I also satisfies this. □

From now on we suppose τ is recursively given. This is a natural assumption as otherwise the theory $T_2^{i,\tau}$ would not be recursive.

Notation: When $M \models seq(w) \wedge \forall k < r\ seq((w)_k)$ we note:

- $w_k := (w)_k$
- $W_k := \{x \in M : M \models \exists u < lh(w_k)\ (x = (w_k)_u)\}$, the set coded by w_k

Now we give sufficient conditions to get a substructure satisfying a property which is near from our desired replacement scheme. A special cut is used and the model M_J becomes in fact an intersection of sets W_k for $k = r$ to $r - \omega$.

Lemma 4.2.4 *Let M be a recursively saturated L_{BA}-structure, let $r \in M\backslash\mathbb{N}$ and let $w \in M$ such that $M \models seq(w) \wedge \forall k < r\ seq((w)_k)$. Set*

$$J := \{r - n : n < \omega\}$$

and let $M_J := \bigcup_{k\in J} W_k$. Let $c \in M_J$ and suppose the following holds:

1. $W_k \subset W_{k+1}$ *for every* $k \leq r$
2. $(w_k)_{k\in J} \subset M_J$
3. $[0,c]_M \subset M_J$
4. $M_J \prec_{\hat{\Sigma}_i^b} M$.

Then for every $\theta(x,y) \in \hat{\Pi}_i^b$, $d \in M_J$, if

$$M_J \models \forall x \leq c\ \exists y \leq d\ \theta(x,y)$$

there is $k \in J$ such that

$$M \models \forall x \leq c\ \exists y \in W_k\ (y \leq d \wedge \theta(x,y)).$$

Proof Suppose that $M_J \models \forall x \leq c\ \exists y \leq d\ \theta(x,y)$. By condition 3, 4 and Lemma 4.2.2, M also satisfies this formula. By 1 we have $M_J \subset W_{r-n}$ for every $n < \omega$. Thus for every $n < \omega$

$$M \models \forall x \leq c\ \exists y \in W_{r-n}(y \leq d \wedge \theta(x,y)).$$

That means that the following type $t(k)$ is finitely satisfiable in M

$$\{\forall x \leq c\ \exists y \in W_k(y \leq d \wedge \theta(x,y))\} \cup \{k < r-n : n < \omega\}.$$

By saturation there is some k realizing $t(k)$, and this k is clearly in J. □

4.3 A model of $\hat{\Sigma}^b_{i+1}$-$REPL^{|\tau|}$ inside one of $T_2^{i,|\tau|}$

In this section we prove that the conditions given in 4.2 are sufficient to obtain a $\hat{\Sigma}^b_i$-substructure satisfying $\hat{\Sigma}^b_{i+1}$-$REPL^{|\tau|}$ in a recursively saturated model of $T_2^{i,|\tau|}$. Then we show that these conditions can be fulfilled in such a model, by constructing suitable sequences w and b. To this aim we need an overspill argument in the length of the induction available in M. This is done in the following lemma.

Lemma 4.3.1 *Let $i \geq 1$, τ a set of unary terms. Let M be a recursively saturated model of $T_2^{i,|\tau|}$. There is some $q \in M\backslash\mathbb{N}$ such that*

a. $M \models \forall x \exists y\ 2^{||l(x)||\cdot|q|} \leq |y|$ *, for every $l \in \tau$*

b. $M \models T_2^{i,2^{||\tau||\cdot|q|}}$.

Proof Consider the following recursive type $t(q)$

$$\{\forall x \exists y\ 2^{||l(x)||\cdot|q|} \leq |y| : l \in \tau\} \cup \{\psi_n\text{-}IND^{2^{||l(x)||\cdot|q|}} : n < \omega, l \in \tau\} \cup \{q > n : n < \omega\}$$

where $(\psi_n)_{n<\omega}$ is an enumeration of $\hat{\Sigma}^b_i$-formulas. By Lemma 4.1.7 $M \models T_2^{i,2^{||l(x)||.n}}$ for every $l \in \tau, n < \omega$. But

$$T_2^{1,2^{||l(x)||.n}} \vdash \forall x \exists y\ 2^{||l(x)||.n} \leq |y|.$$

Hence $t(q)$ is finitely satisfied and the result follows by saturation. □

The following lemma includes the conditions of section 4.2. These conditions together with the conclusion of Lemma 4.2.4 yield that our substructure also satisfies $\hat{\Sigma}^b_{i+1}$-$REPL^{|\tau|}$.

Lemma 4.3.2 *Let $i \geq 1$, τ a set of unary terms. Let M be a recursively saturated model of $T_2^{i,|\tau|}$ and let $q \in M$ as in Lemma 4.3.1. Let $r \in M \backslash \mathbb{N}$, $r < |||q|||$, and let $w, b \in M$ such that $M \models seq(w) \wedge seq(b) \wedge \forall k \leq r\ seq((w)_k)$. Let J be the cut $\{r - n : n < \omega\}$ and let $M_J := \bigcup_{k \in J} W_k$. Suppose the following holds:*

1. $W_k \subset W_{k+1}$ *for every* $k \in J$
2. $(w_k)_{k \in J}, (b_k)_{k \in J} \subset M_J$, $q \in M_J$
3. $(b_k)_{k \in J}$ *is cofinal in* M_J
4. $[0, 2^{2^{2k}||l(x)||\cdot||q||}]_M \subset M_J$ *for every* $k \in J$, $l \in \tau$, $x \in M_J$
5. $[0, lh(w_k)]_M \subset M_J$ *for every* $k \in J$
6. $M_J \subset_{L_{BA}} M$
7. $M_J \models T_2^{i,lh(w_k)}$ *for every* $k \in J$
8. *for every* $\hat{\Pi}^b_{i-1}$*-formula* φ, $k \in J$ *and* $x \in W_k$,

$$\text{if } \ M \models \exists y \leq b_k\ \varphi(x,y), \quad \text{then } \ \exists y \in W_{k+1} \text{ s.t. } M \models y \leq b_k\ \varphi(x,y).$$

Then

a. $M_J \prec_{\hat{\Sigma}^b_i} M$

b. $M_J \models \hat{\Sigma}^b_{i+1}$-$REPL^{|\tau|^\omega}$.

Proof By Lemma 4.2.1 we have $M_J \prec_{\hat{\Sigma}^b_i} M$. Now note that, as in the proof of Theorem 4.2.3(b), we can deduce from condition 4 and Lemma 4.2.2 that for every $k \in J$

$$M_J \models T_2^{i,2^{2^{2k}||\tau||\cdot||q||}}. \tag{4.1}$$

For $\hat{\Sigma}^b_{i+1}$-$REPL^{|\tau|^\omega}$ it suffices to derive $\hat{\Pi}^b_i$-$REPL^{|\tau|^\omega}$ by Lemma 1.2.3. So let $\theta(x,y) \in \hat{\Pi}^b_i$, $l \in \tau$, $n < \omega$, $c, d \in M_J$ and suppose that

$$M_J \models \forall x \leq |l(c)|^n \exists y \leq d\ \theta(x,y).$$

As $|l(c)|^n < 2^{||l(c)||\cdot||q||}$ we have that $[0, |l(c)|^n]_M \subset M_J$. Then by Lemma 4.2.4 there is some $k \in J$ such that

$$M \models \forall x \leq |l(c)|^n \exists y \in W_k (y \leq d \wedge \theta(x,y)).$$

That is, for some $k \in J$

$$M \models \forall x \leq |l(c)|^n \exists u < lh(w_k)((w_k)_u \leq d \wedge \theta(x, (w_k)_u)). \tag{4.2}$$

By condition 2 this formula has its parameter w_k inside M_J. By condition 5 and Lemma 4.2.2 we have that 4.2 is satisfied also in M_J. Now set

$$\Phi(x, u) \equiv (w_k)_u \leq d \wedge \theta(x, (w_k)_u). \tag{4.3}$$

Using pairing we can code variables x, u into a single one. We set

$$\tilde{\Phi}(z) \equiv ispair(z) \wedge \langle z \rangle_1 \leq |l(c)|^n \wedge \langle z \rangle_2 < lh(w_k) \wedge \Phi(\langle z \rangle_1, \langle z \rangle_2).$$

Then we have

$$M_J \models \forall x \leq |l(c)|^n \, \forall u < lh(w_k)(\Phi(x, u) \leftrightarrow \tilde{\Phi}(\langle x, u \rangle)).$$

By Lemma 1.1.9

$$x \leq |l(c)|^n \wedge u < lh(w_k) \rightarrow \langle x, u \rangle \leq 16 \cdot max^2(|l(c)|^n, lh(w_k)).$$

Let $a := 16 \cdot max^2(|l(c)|^n, lh(w_k))$. As $|l(c)|^n < 2^{||l(c)|| \cdot ||q||}$ we have by Lemma 4.1.6 and (4.1) above that $M_J \models T^{i,|l(c)|^n}$. From this and condition 7 we deduce, by applying Lemmas 4.1.5 and 4.1.6, that $M_J \models T^{i,a}$. Hence, by Lemma 4.1.8 we can code validity of $\tilde{\Phi}(z)$ under a :

$$M_J \models \exists v \forall z \leq a \; (\tilde{\Phi}(z) \leftrightarrow Bit(v, z) = 1).$$

Thus, we have

$$M_J \models \exists v \forall x \leq |l(c)|^n \forall u < lh(w_k)(\Phi(x, u) \leftrightarrow Bit(v, \langle x, u \rangle) = 1). \tag{4.4}$$

From this and (4.2) above with M_J in the place of M we get

$$M_J \models \exists v \forall x \leq |l(c)|^n \exists u < lh(w_k)(Bit(v, \langle x, u \rangle) = 1).$$

Now we can use Δ_0^b-$REPL^{|\tau|^\omega}$ in M_J to obtain

$$M_J \models \exists v \exists s \forall x \leq |l(c)|^n((s)_x < lh(w_k) \wedge (Bit(v, \langle x, (s)_x \rangle) = 1)).$$

Substituting according to (4.4) and (4.3) we obtain successively

$$M_J \models \exists s \forall x \leq |l(c)|^n((s)_x < lh(w_k) \wedge \Phi(x, (s)_x))$$

and

$$M_J \models \exists s(\forall x \leq |l(c)|^n((s)_x \leq lh(w_k) \wedge (w_k)_{(s)_x} \leq d \wedge \theta(x,(w_k)_{(s)_x}))). \qquad (4.5)$$

To conclude we use again Δ_0^b-$REPL^{|\tau|^\omega}$. Clearly we have

$$M_J \models \forall s \forall x \leq |l(c)|^n \exists u \leq d((w_k)_{(s)_x} = u)$$

thus

$$M_J \models \forall s \exists \lambda\ \forall x \leq |l(c)|^n((\lambda)_x \leq d \wedge (w_k)_{(s)_x} = (\lambda)_x).$$

From this and (4.5) we get finally

$$M_J \models \exists \lambda(\forall x \leq |l(c)|^n((\lambda)_x \leq d \wedge \theta(x,(\lambda)_x))).$$

□

In fact nothing is gained by satisfying $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|^\omega}$ instead of $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$ as these two schemes are easily seen to be equivalent over $EBASIC$ (see [20]).

In the following theorem we construct sequences w, b in order to get the conditions of the precedent lemma satisfied.

Lemma 4.3.3 *Let $i \geq 1$, τ a set of unary terms. Let M be a recursively saturated model of $T_2^{i,|\tau|}$ and let $q \in M \backslash \mathbb{N}$ as in Lemma 4.3.1. Let a be an arbitrary element of M.*
Then, there are two sequences w, b of nonstandard length $r + 1 \leq |||q|||$ such that $M \models \forall k \leq r\ seq(w_k)$, and for every $k < r$ the following holds:

a. $q, a \in W_0$

b. $b_{k+1} = max\{w_{k+1}, 2^{|b_k|^2}\}$

c. $w_k, b_k \in W_{k+1}$

d. $W_k \subset W_{k+1}$

e. $W_k \times W_k \subset W_{k+1}$

f. $[0, 2^{2^{2k}||l(x)||.||q||}]_M \subset W_{k+1}$ *for every $l \in \tau$, $x \in W_k$*

g. $[0, lh(w_k)]_M \subset W_{k+1}$

h. $M \models T^{i,lh(w_k)}$

i. *for every $\hat{\Pi}_{i-1}^b$-formula φ,*

$$M \models \forall x \in W_k(\exists y \leq b_k\ \varphi(x,y) \rightarrow \exists y \in W_{k+1}(y \leq b_k\ \wedge \varphi(x,y))).$$

Proof We can express all these conditions by the following type $t(w, b)$ with parameters q, a:

$(t1)$ $seq(w) \wedge seq(b)$

$(t2)$ $\{n < lh(w) = lh(b) < |||q||| \ : \ n < \omega\}$

$(t3)$ $\forall k < lh(w)\ seq(w_k)$

$(t4)$ $q, a \in W_0$

$(t5)$ $\forall k < lh(w) - 1\ (b_{k+1} = max\{w_{k+1}, 2^{|b_k|^2}\})$

$(t6)$ $\forall k < lh(w) - 1\ (\{w_k, b_k\} \cup W_k \cup (W_k \times W_k) \cup [0, lh(w_k)]_M \subset W_{k+1})$

$(t7)$ $\{\forall k < lh(w) - 1\ \forall x \in W_k\ \ [0, 2^{2^{2k}||l(x)||.||q||}]_M \subset W_{k+1} \ : \ l \in \tau\}$

$(t8)$ $\{\psi\text{-}IND^{lh(w_k)} \ : \ \psi \in \hat{\Sigma}^b_i\}$

$(t9)$ $\{\forall k < lh(w) - 1\ \forall x \in W_k(\exists y \leq b_k\ \varphi(x, y) \rightarrow \exists y \in W_{k+1}(y \leq b_k\ \wedge \varphi(x, y))) \ :$
$\varphi \in \hat{\Pi}^b_{i-1}\}$

Now we prove that $t(w, b)$ is finitely satisfied. Let t_{fin} be a finite subset of $t(w, b)$. For suitables $n, m < \omega$ and some finite $\tau_0 \subset \tau$, we have that t_{fin} is included in the type $t^{\tau_0}_{n,m}(w, b)$ obtained from $t(w, b)$ by replacing (t2), (t7) and (t9) respectively by:

$(t2_0)$ $lh(w) = lh(b) = n + 1$

$(t7_0)$ $\{\forall k < n\ \forall x \in W_k\ [0, 2^{2^{2k}||l(x)||.||q||}]_M \subset W_{k+1} \ : \ l \in \tau_0\}$

$(t9_0)$ $\{\forall k < n\ \forall x \in W_k(\exists y \leq b_k\ \varphi_j(x, y) \rightarrow \exists y \in W_{k+1}(y \leq b_k\ \wedge \varphi_j(x, y))) : j \leq m\}$.

Here $(\varphi_j)_{j<\omega}$ is some enumeration of $\hat{\Pi}^b_{i-1}$-formulas. Note parenthetically that $t^{\tau_0}_{n,m}(w, b)$ is infinite, as we have not changed (t8). Consider now the following finite type $s^{\tau_0}_{n,m}(w, b)$:

$(s1)$ $seq(w) \wedge seq(b)$

$(s2)$ $lh(w) = lh(b) = n + 1$

$(s3)$ $\forall k \leq n\ seq(w_k)$

$(s4)$ $q, a \in W_0$

$(s5)$ $\forall k < n\ (b_{k+1} = max\{w_{k+1}, 2^{|b_k|^2}\})$

$(s6)$ $\forall k < n\ (\{w_k, b_k\} \cup W_k \cup (W_k \times W_k) \subset W_{k+1})$

$(s7)$ $\{\forall k < n\ \forall x \in W_k\ [0, 2^{2^{2k}||l(x)||\cdot||q||}]_M \subset W_{k+1}\ :\ l \in \tau_0\}$

$(s8)$ $\forall k \le n\ (lh(w_k) \le 2^{2^{2k}||l_k(e_k)||\cdot||q||})$ for some $l_k \in \tau_0$ and some $e_k \in W_k$

$(s9)$ $\{\forall k < n\, \forall x \in W_k(\exists y \le b_k\ \varphi_j(x,y) \to \exists y \in W_{k+1}(y \le b_k \wedge \varphi_j(x,y)))\ :\ j \le m\}$.

We argue now that in order to show that $t^{\tau_0}_{n,m}(w,b)$ is satisfied, it suffices to check the satisfacion of $s^{\tau_0}_{n,m}(w,b)$. This is trivial for $(t1)$, $(t2_0)$, $(t3)$, $(t4)$, $(t5)$ and $(t9)$. From $(s6)$, $(s7)$ and $(s8)$ we get $(t6)$ and $(t7_0)$. Now note that

$$k < |||q||| \to 2^{2^{2k}||\tau||\cdot||q||} \le 2^{||\tau||\cdot|q|}$$

so by the Lemmas 4.3.1(b), 4.1.6 and 4.1.8 we have

$$\forall c \in M,\ M \models c \le 2^{2^{2k}||\tau||\cdot||q||}\ \Rightarrow\ M \models T_2^{i,c}\ \cup\ \hat{\Sigma}^b_i\text{-}STRONG\ REPL^c. \qquad (4.6)$$

From this and $(s8)$ we get $\hat{\Sigma}^b_i$-$IND^{lh(w_k)}$, i.e. $(t8)$.

Now we prove that $s^{\tau_0}_{n,m}(w,b)$ is satisfied for every $n, m < \omega$. Let us fix an arbitrary $m < \omega$ and finite $\tau_0 \subset \tau$, and prove that $M \vdash s^{\tau_0}_{n,m}(w,b)$ by induction on n.
For $n = 0$ put $b_0 = q$, let $b := (b_0)$, the one element sequence containing b_0, and let w_0 be the two element sequence containing q and a. Conditions $(s1)$-$(s9)$ are trivially satisfied.
Suppose now the type $t^{\tau_0}_{n,m}(w,b)$ is satisfied by some $w, b \in M$, i.e. we have $w_0, \dots, w_n$ and $b_0, \dots, b_n$ satisfying $(s1)$-$(s9)$. We will add elements w_{n+1}, b_{n+1} to each sequence w, b respectively.
Condition $(s5)$ forces $b_{n+1} = max\{w_{n+1}, 2^{|b_n|^2}\}$. Sequence w_{n+1} is defined as the concatenation of some sequences $v_1, \dots, v_{m+5}$ described below:

- v_1 is the two elements sequence containing w_n and b_n
- $v_2 = w_n$
- v_3 codes the $(lh(w_n))^2$ elements of $W_n \times W_n$
- v_4 codes all numbers less than $2^{2^{2n}||\tilde{l}(c_0)||\cdot||q||}$ for some suitable $\tilde{l} \in \tau_0$, $c_0 \in W_n$
- for every $j \le m$, v_{5+j} is such that

$$\forall x \in W_n(\exists y \le b_n\ \varphi_j(x,y) \to \exists u < lh(v_{5+j})((v_{5+j})_u \le b_n\ \wedge \varphi_j(x,(v_{5+j})_u))).$$

Moreover we can ask for

- $lh(v_3) = 2^{2^{2n+1}||l_n(e_n)||\cdot||q||+4}$

- $lh(v_4) = 2^{2^{2n}||\tilde{l}(c_0)||.||q||} + 1$
- $lh(v_{5+j}) = 2^{2^{2n}||l_n(e_n)||.||q||}$, for every $j \leq m$.

The existence of v_1, v_2 is clear. By Lemma 1.1.9 we have that if

$$x, y < lh(w_n) < 2^{2^{2n}||l_n(e_n)||.||q||}$$

then

$$\langle x, y\rangle < 2^{2^{2n+1}||l_n(e_n)||.||q||+4}.$$

By the property (4.6) above we have that

$$\begin{aligned} M \models \;& \exists v_3(seq(v_3) \wedge lh(v_3) = 2^{2^{2n+1}||l_n(e_n)||.||q||+4} \wedge \forall x < 2^{2^{2n+1}||l_n(e_n)||.||q||+4} \\ & (ispair(x) \wedge \langle x\rangle_1 < lh(w_n) \wedge \langle x\rangle_2 < lh(w_n) \rightarrow \langle (w_n)_{\langle x\rangle_1}, (w_n)_{\langle x\rangle_2}\rangle = (v_3)_x)). \end{aligned}$$

In order to satisfy $(s7)$ we have to choose for v_4 some $\tilde{l} \in \tau_0$ and $c_0 \in W_n$ such that $\tilde{l}(c_0) \geq l(x)$ for every $l \in \tau_0$ and $x \in W_n$. To do this consider the formula $\theta(u)$ given by

$$\exists x < lh(w_n)(x \leq u \wedge \bigvee_{j \leq card(\tau_0)} \forall y \leq lh(w_n)(y \leq u \rightarrow \bigwedge_{j' \leq card(\tau_0)} (l_j((w_n)_x) \geq l_{j'}((w_n)_y))))$$

where $(l_j)_{j \leq card(\tau_0)}$ is any enumeration of τ_0. By $(s8)$ and Lemma 4.3.1(a) we have that $\theta(u)$ is in fact a Δ_0^b-formula. It says that a maximal element as the one we are looking for exists if we restrict ourselves to the first u elements coded by W_n. Clearly we have

$$M \models \theta(0) \wedge \forall u(\theta(u) \rightarrow \theta(u+1)).$$

By conditions $(s8)$ and 4.6 we can use induction up to $lh(w_n)$ in M, thus getting $\theta(lh(w_n))$. Once those $\tilde{l} \in \tau_0$ and $c_0 \in W_n$ determined it is easy to get v_4 coding the set $[0, 2^{2^{2n}||\tilde{l}(c_0)||.||q||}]_M$ with the desired length.

The existence of v_{5+j} for $j \leq m$ follows also from $\hat{\Sigma}_i^b$-$STRONG\ REPL^{2^{2^{2n}||l_n(e_n)||.||q||}}$. We have for every $j \leq m$

$$\begin{aligned} M \models \;& \exists v_{5+j}(seq(v_{5+j}) \wedge lh(v_{5+j}) = 2^{2^{2n}||l_n(e_n)||.||q||} \wedge \forall x < 2^{2^{2n}||l_n(e_n)||.||q||} \\ & (x < lh(w_n) \wedge \exists y < b_n \varphi_j((w_n)_x, y) \rightarrow ((v_{5+j})_x \leq b_n \ \wedge \varphi_j((w_n)_x, (v_{5+j})_x))). \end{aligned}$$

By Lemma 4.1.9 there is some w_{n+1} coding the union of the sets coded by $v_1, \ldots, v_{m+5}$ and such that $lh(w_{n+1}) = lh(v_1) + \cdots + lh(v_{5+j})$. Thus,

$$\begin{array}{rcl} lh(w_{n+1}) & = & lh(w_n) + (m+1)2^{2^{2n}||l_n(e_n)||.||q||} + 2^{2^{2n+1}||l_n(e_n)||.||q||+4} + 2^{2^{2n}||\tilde{l}(c_0)||.||q||} + 3 \\ & \leq & 2^{2^{2n+1}||l_n(e_n)||.||q||+5} + 2^{2^{2n}||\tilde{l}(c_0)||.||q||}. \end{array}$$

Now set

$$(l_{n+1}, e_{n+1}) := \begin{cases} (l_n, e_n) & \text{if } l_n(e_n) \geq \tilde{l}(c_0) \\ (\tilde{l}, c_0) & \text{otherwise.} \end{cases}$$

Then $l_{n+1} \in \tau_0$, $e_{n+1} \in W_n \subset W_{n+1}$ and

$$lh(w_{n+1}) \leq 2^{2^{2n+1}||l_{n+1}(e_{n+1})||.||q||+5} \leq 2^{2^{2(n+1)}||l_{n+1}(e_{n+1})||.||q||}.$$

Hence, every type $t^{\tau_0}_{n,m}(w, b)$ is satisfied. We conclude that $t(w, b)$ is finitely satisfied, and the result follows then by the saturation of M. □

Now we can prove the main theorem of this chapter.

Theorem 4.3.4 *Let $i \geq 1$, τ a set of unary terms. Let M be a recursively saturated model of $T_2^{i,|\tau|}$ and let $a \in M$. Then there is a submodel N of M such that*

a. $a \in N$

b. $N \prec_{\hat{\Sigma}^b_i} M$

c. $M \models \hat{\Sigma}^b_{i+1}$*-REPL*$^{|\tau|}$.

Proof Let $q \in M \backslash \mathbb{N}$ as in Lemma 4.3.1 and let $w, b, r \in M$ and $(W_k)_{k \leq r}$ satisfying (a)-(i) of Lemma 4.3.3. Put $J := \{r - n : n < \omega\}$ and $M_J := \bigcup_{k \in J} W_k$. We check that conditions 1-8 of Lemma 4.3.2 are satisfied.

Note that conditions 1,5,8 are respectively consequences of (d),(g),(i) of Lemma 4.3.3; condition 2 follows from (c) and (a), and condition 7 from condition 5 and (h).

Let $x \in M_J$ and $k \in J$ such that $x \in W_k$. By (b) we have $x < w_k \leq b_k$ hence $(b_k)_{k \in J}$ is cofinal in M_J. This is condition 3.

Now let $k \in J$, $l \in \tau$, $x \in M_J$. Let $k' \in J$ such that $k' \geq k$ and $x \in W_{k'}$. Then $2^{2^{2k}||l(x)||.||q||} \leq 2^{2^{2k'}||l(x)||.||q||}$ and by (f) we have

$$[0, 2^{2^{2k'}||l(x)||.||q||}]_M \subset W_{k'+1}.$$

As $W_{k'+1} \subset M_J$ we get condition 4.

Now we prove condition 6, i.e. that M_J is an L_{BA}-structure. Let $c_1, c_2 \in M_J$ and $k \in M_J$ such that $c_1, c_2 < b_k$. Let $t \in Term(L_{BA})$. Clearly we have $t(c_1, c_2) < 2^{|b_k|^{2^n}}$ for some $n < \omega$, and by (b) this is bounded by b_{k+n}. Using (e) and (d) we get that $d := \langle c_1, c_2 \rangle \in W_{k+n}$. Hence

$$M \models \exists y \leq b_{k+n}(y = t(\langle d \rangle_1, \langle d \rangle_2)).$$

By (h) there is such an element y in W_{k+n+1}, hence in M_J. Of course this y is unique and equal to $t(c_1, c_2)$.

So Lemma 4.3.2 can be applied and we get that $M_J \prec_{\hat{\Sigma}_i^b} M$ and $M_J \models \hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$. To conclude note that $a \in M_J$ by (a). □

As a consequence we get a purely model-theoretic proof of the following theorem (see [20] for a proof-theoretic one).

Theorem 4.3.5 *Let $i \geq 1$, τ a set of unary terms. The theory $\hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$ is $\forall\exists\hat{\Sigma}_{i+1}^b$-conservative over $T_2^{i,|\tau|}$.*

Proof Let us suppose that for some $\varphi \in \hat{\Sigma}_{i+1}^b$

$$\hat{\Sigma}_{i+1}^b\text{-}REPL^{|\tau|} \vdash \forall x \exists y \varphi(x, y).$$

Let $M \models T_2^{i,|\tau|}$ and let M' be a recursively saturated model elementarily equivalent to M. Let $a \in M'$ and apply Theorem 4.3.4 to get a $\hat{\Sigma}_i^b$-elementary submodel $N \models \hat{\Sigma}_{i+1}^b$-$REPL^{|\tau|}$ containing a. Then $N \models \exists y \varphi(a, y)$. As $N \prec_{\hat{\Sigma}_i^b} M'$ and $M' \equiv M$ the result follows. □

Bibliography

[1] Bill Allen. Arithmetizing uniform *NC*. *Ann. Pure Appl. Logic*, 53(1):1–50, 1991.

[2] Stephen Bloch. On parallel hierarchies and R^i_k. *Ann. Pure Appl. Logic*, 89(2-3):231–273, 1997.

[3] Sedki Boughattas. Trois théorèmes sur l'induction pour les formules ouvertes munies de l'exponentielle. *J. Symbolic Logic*, 65(1):111–154, 2000.

[4] Samuel R. Buss. *Bounded arithmetic*. Bibliopolis, Naples, 1986.

[5] Samuel R. Buss. Relating the bounded arithmetic and polynomial time hierarchies. *Ann. Pure Appl. Logic*, 75(1-2):67–77, 1995.

[6] Samuel R. Buss and Aleksandar Ignjatovic. Unprovability of consistency statements in fragments of bounded arithmetic. *Ann. Pure Appl. Logic*, 74(3):221–244, 1995.

[7] Samuel R. Buss, Jan Krajíček, and Gaisi Takeuti. Provably total functions in bounded arithmetic theories R^i_3, U^i_2 and V^i_2. In *Arithmetic, proof theory, and computational complexity (Prague, 1991)*, pages 116–161. Oxford Univ. Press, New York, 1993.

[8] Eugenio Chinchilla. A model of $\hat{R}^2_3$ inside a subexponential time resource. *Notre Dame J. Formal Logic*, 39(3):307–324, 1998.

[9] Eugenio Chinchilla. A model-theoretic proof of a subexponential time witnessing theorem. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(8):913–917, 1998.

[10] Petr Hájek and Pavel Pudlák. *Metamathematics of first-order arithmetic*. Springer-Verlag, Berlin, 1993.

[11] Richard Kaye. *Models of Peano arithmetic*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[12] Jan Krajicek. Fragments of bounded arithmetic and bounded query classes. *Trans. Am. Math. Soc.*, 338(2):587–598, 1993.

[13] Jan Krajíček. *Bounded arithmetic, propositional logic, and complexity theory.* Cambridge University Press, Cambridge, 1995.

[14] Jan Krajíček, Pavel Pudlák, and Gaisi Takeuti. Bounded arithmetic and the polynomial hierarchy. *Ann. Pure Appl. Logic*, 52(1-2):143–153, 1991. International Symposium on Mathematical Logic and its Applications (Nagoya, 1988).

[15] G. E. Minc. What can be done in primitive recursive arithmetic. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 60:93–102, 223–224, 1976. Studies in constructive mathematics and mathematical logic, VII.

[16] Christos H. Papadimitriou. *Computational complexity.* Addison-Wesley Publishing Company, Reading, MA, 1994.

[17] Rohit Parikh. Existence and feasibility in arithmetic. *J. Symb. Log.*, 36:494–508, 1971.

[18] Jeff B. Paris and Constantine Dimitracopoulos. Truth definitions for Δ_0 formulae. In *Logic and algorithmic (Zurich, 1980)*, pages 317–329. Univ. Genève, Geneva, 1982.

[19] Charles Parsons. On a number theoretic choice schema and its relation to induction. In *Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968)*, pages 459–473. North-Holland, Amsterdam, 1970.

[20] Chris Pollett. Structure and definability in general bounded arithmetic theories. *Ann. Pure Appl. Logic*, 100(1-3):189–245, 1999.

[21] Chris Pollett. Multifunction algebras and the provability of $PH\downarrow$. In *Proceedings of the Workshop on Proof Theory and Complexity, PTAC'98 (Aarhus)*, volume 104, pages 279–303, 2000.

[22] J. P. Ressayre. A conservation result for systems of bounded arithmetic. Unpublished manuscript, 1986.

[23] J. P. Ressayre and A. J. Wilkie. *Modèles non standard en arithmétique et théorie des ensembles.* Université de Paris VII U.E.R. de Mathématiques, Paris, 1987.

[24] Larry J. Stockmeyer. The polynomial-time hierarchy. *Theoret. Comput. Sci.*, 3(1):1–22 (1977), 1976.

[25] Claude Sureson. P, NP, co-NP and weak systems of arithmetic. *Theoret. Comput. Sci.*, 154(2):145–163, 1996.

[26] Gaisi Takeuti. *Proof theory.* North-Holland Publishing Co., Amsterdam, second edition, 1987. With an appendix containing contributions by Georg Kreisel, Wolfram Pohlers, Stephen G. Simpson and Solomon Feferman.

[27] Gaisi Takeuti. RSUV isomorphisms. In *Arithmetic, proof theory, and computational complexity (Prague, 1991)*, pages 364–386. Oxford Univ. Press, New York, 1993.

[28] A. J. Wilkie. A model theoretic proof of Buss's characterization of the polynomial time computable functions. Unpublished manuscript, 1985.

[29] A.J. Wilkie and J.B. Paris. On the scheme of induction for bounded arithmetic formulas. *Ann. Pure Appl. Logic*, 35:261–302, 1987.

[30] Celia Wrathall. Rudimentary predicates and relative computation. *SIAM J. Comput.*, 7(2):194–209, 1978.

[31] Domenico Zambella. Notes on polynomially bounded arithmetic. *J. Symb. Log.*, 61(3):942–966, 1996.