THÈSE de DOCTORAT de l'UNIVERSITÉ PARIS VI

Spécialité :
MATHÉMATIQUES

présentée par

Gianluca PACIENZA

pour obtenir le grade de DOCTEUR de l'UNIVERSITÉ PARIS VI

Sujet :

Courbes rationnelles et diviseurs nef pour certaines variétés génériques

Soutenue le 21 décembre 2001
devant le jury composé de

Ciro CILIBERTO, Rome
Joseph LE POTIER, Paris
Laurent MANIVEL, Grenoble - rapporteur
Christian PESKINE, Paris
Claire VOISIN, Paris - directrice de thèse

Je voudrais d'abord faire part de ma reconnaissance à Claire Voisin. J'ai été très marqué par sa grande générosité scientifique, par le temps et l'énergie qu'elle m'a consacrés. Ses travaux ont été une source d'inspiration majeure et son goût mathématique m'a beaucoup influencé.

Merci à Laurent Manivel pour l'intérêt qu'il a porté à mon travail et pour avoir accepté d'être rapporteur de cette thèse. Ses nombreuses remarques et suggestions m'ont aidé à simplifier mon texte et poussé à améliorer ma rédaction, en particulier dans la deuxième partie.

Je voudrais exprimer ma gratitude à Ciro Ciliberto, pour sa lecture attentive de ma thèse et pour la grande clarté avec laquelle il m'a expliqué son travail sur le produit symétrique second d'une courbe. Je le remercie aussi de s'être déplacé pour faire partie du jury.

Je remercie également Yujiro Kawamata pour avoir accepté d'être rapporteur de ma thèse.

Christian Peskine m'a accueilli avec extrême gentillesse et sympathie en septembre 1997, quand, étudiant Erasmus, j'arrivai pour la première fois à Paris, dépourvu de repères et de vocabulaire ; Joseph Le Potier a été mon enseignant en DEA dans un premier cours de géométrie algébrique où j'ai beaucoup appris : je suis honoré de leur présence aujourd'hui dans le jury de ma thèse.

Je tiens à remercier chaleureusement l'équipe de géométrie de l'Università Roma Tre et, en particulier, Angelo Lopez. Son encouragement constant a été d'une inestimable importance et nos nombreuses discussions mathématiques au cours de ces trois années ont beaucoup contribué à ma formation.

Je remercie Lorenz et Catriona. Pour leur amitié, leur quotidienne aide mathématique, leur soutien psychologique, leur compréhension. Pour avoir passé deux journées entières à chercher avec moi une faute dans une preuve en janvier dernier ...

Merci à Frédéric Han, Guillaume Jamet, Laurent Koelblen et Eric Tjøtta, toujours prêts à répondre avec patience à mes nombreuses questions.

Je profite enfin de l'occasion pour saluer Aurélie, Francesca et Sara, ainsi que toutes les autres personnes qui, à leur façon, ont rendu plus joyeux ces trois années de travail.

J'ai écrit les dernières lignes de ma thèse sous le soleil de l'été calabrais, au milieu de vignes et d'oliviers familiers, entouré par Andrea, ma soeur Carmen, mes parents, nonno Carlo et nonna Franceschina. À eux tous ...

... grazie di cuore.

Résumé en français

Cette thèse, rédigée en anglais, est composée de deux parties distinctes.

Dans la première partie j'étudie la géométrie des sous-variétés de genre géométrique zéro d'une hypersurface projective générique. Je prouve que les seules sous-variétés k-dimensionnelles d'une hypersurface générique $X_d \subset \mathbf{P}^n$, de degré $d = 2n - 2 - k$, où k est un entier tel que $1 \leq k \leq n - 5$, sont les composantes irréductibles du lieu de X couvert par les droites. On en déduit que, pour tout $n \geq 6$, l'hypersurface générique $X_{2n-3} \subset \mathbf{P}^n$ ne contient pas de courbes rationnelles de degré supérieur ou égal à 2.

Dans la deuxième partie, j'étudie le cône des diviseurs numériquement effectifs dans le produit symétrique d'une courbe générique C. Le cône nef de $C^{(k)}$ est facilement déterminé si k est au moins égal à la gonalité de la courbe. Je décris ensuite le cône nef pour $k = gon(C) - 1$, quand le genre de la courbe est pair. Dans ce cas, le bord du cône nef est à pente rationnelle, et est déterminé en utilisant des courbes dans $C^{(k)}$ associées aux pinceaux de degré minimal sur C.

Contents

Part I

Rational curves on general projective hypersurfaces

Introduction to Part I

Let X be a smooth complex projective variety of dimension $dim\ X \geq 3$. The investigation of the existence and of the geometry of "special" subvarieties of X has certainly been one of the major tools in the study of higher dimensional complex geometry. Let Ω^1_X be the holomorphic cotangent bundle of X and $K_X = det\ \Omega^1_X$ its canonical line bundle. The results that have been obtained by several authors in the last two decades or so, or that one might reasonably hope for can be roughly divided into three groups, according to the negativity, triviality or positivity of K_X. The main result of the first part of this thesis is concerned with the geometry of subvarieties with geometric genus zero of general projective hypersurfaces having ample canonical bundle. In this introduction we will briefly recall some of the most significant known results obtained under the hypothesis of the negativity, or triviality of K_X, before shifting our attention to the case when K_X is positive, that will be treated in more detail.

0.1 The case K_X negative

In many explicit examples when K_X is negative it was already classically known that there are "many" rational curves on X. The families of rational curves have then been used in several cases in order to explore the geometry of hypersurfaces of low degree or of certain classes of higher dimensional varieties. The most famous example of such a use is the study of the family of lines on a cubic threefold $X \subset \mathbf{P}^4$, which led independently Mumford, Clemens and Griffiths (see [CG], [C2]) to prove the non-rationality of X.

A systematic way to find rational curves on X when K_X is negative, nowadays referred to as "bend-and-break" technique, was finally settled by Mori in his celebrated proof of Hartshorne's conjecture ([Mo]), where he proves:

Theorem ([Mo]). *If $-K_X$ is ample, then through every point $x \in X$ there is a rational curve $x \in C_x \subset X$ such that $-K_X.C_x \leq dim X + 1$.*

Later Campana and, independently, Kollár, Miyaoka and Mori proved that varieties with ample anticanonical divisor have, moreover, the stronger property of being *rationally connected*:

Theorem ([Ca], [KMM]). *If $-K_X$ is ample, then any two general points $x_1, x_2 \in X$ can be joined by a rational curve $C_{x_1,x_2} \subset X$.*

The details of these developments are described in the monographs [CKM] and [K].

0.2 The case K_X trivial

The situation changes when the canonical bundle is trivial. There are not as many rational curves as in the case K_X negative, but still, rational curves can be used in interesting ways. The best-known case is provided by the general quintic threefold $X_5 \subset \mathbf{P}^4$. Here the word *general* means that the polynomial defining X_5 belongs to the intersection of countably many Zariski open subsets of the space of homogeneous degree 5 polynomials in six variables. In this case, Clemens [C1] constructs a finite number of rational curves of degree δ on X for infinitely many δ, and proves moreover that these rational curves are infinitesimally rigid, *i.e.* for any such a curve $C \subset X$ we have

$$N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

Hence they do not move in X, not even infinitesimally. Finally, he makes use of these infinitely many rational curves to establish that the abelian group quotient $\mathcal{G}_h(X)/\mathcal{G}_a(X)$ is not finitely generated, where $\mathcal{G}_h(X)$ is the group of 1-cycles on X which are homologous to zero modulo those which are rationally equivalent to zero, and $\mathcal{G}_a(X)$ is the subgroup of $\mathcal{G}_h(X)$ generated by cycles algebraically equivalent to zero.

An easy parameter count predicts that X_5 should contain only a finite number of rational curves of each degree. Clemens proposed then the following

Conjecture (Clemens). *The general quintic threefold $X_5 \subset \mathbf{P}^4$ contains only isolated rational curves.*

Mark Green proposed the following stronger conjecture:

Conjecture (Green). *The general quintic threefold $X_5 \subset \mathbf{P}^4$ does not contain a surface S which admits a desingularization $\tilde{S}$ having effective canonical bundle.*

Clemens' conjecture has proved very elusive, and has been checked only for low degrees δ up to 9 (see [JK]). In the direction of Green's conjecture the only general result is provided by a theorem of Chang and Ran:

Theorem ([CR]). *The general quintic threefold $X_5 \subset \mathbf{P}^4$ does not contain a surface S which admits a desingularization $\tilde{S}$ having numerically effective anticanonical bundle.*

We cannot conclude this section without mentioning the recent extraordinary interaction between algebraic geometry and nuclear physics that provided a new interest in the enumerative geometry of rational curves on Calabi-Yau varieties, and then, as a consequence, in Clemens' conjecture (for an introduction to these fascinating developments see for instance [V4]).

0.3 The case K_X positive

Suppose C is a smooth genus g curve contained in X. By standard deformation theory, the *expected dimension* of the Hilbert scheme of subschemes of X having the same Hilbert polynomial as C is given by

$$expdim \ = \ \chi(N_{C/X}). \tag{1}$$

By Riemann-Roch and the adjunction formula we have

$$\chi(N_{C/X}) = -deg(K_{X|C}) + (1-g)(rk(N_{C/X}) - 2), \tag{2}$$

so that one does not expect to find rational curves on X when K_X is positive enough. More explicitly, for a general projective hypersurface $X_d \subset \mathbf{P}^n$, the above predicts that there should be no rational curves at all on X, for $d \geq 2n-2$, $n \geq 4$, and only lines, for $d = 2n-3$, $n \geq 5$.

In this section we will review the various approaches adopted by different authors to detect curves or, more generally, subvarieties on X having low geometric genus, and finally present our results.

0.3.1 Surfaces in $\mathbf{P}^3$

We start with the case of surfaces in $\mathbf{P}^3$. It was known that a degree four surface $S_4 \subset \mathbf{P}^3$ contains a rational curve and a family of elliptic curves. Harris conjectured that this was not the case for higher degrees, *i.e.* a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$ does not contain any rational or elliptic curve. Harris' conjecture has been proved by G. Xu in [X], where he proves the following, more general result:

Theorem ([X]). *On the general surface $S_d \subset \mathbf{P}^3$ of degree $d \geq 5$ there is no curve C with geometric genus $p_g(C) \leq \frac{1}{2}d(d-3)-3$, and the bound is sharp.*

Xu's proof goes as follows. By the Noether-Lefschetz theorem any curve on a generic surface of degree $d \geq 4$ is a complete intersection with another surface of degree k. This allows to compute by the adjunction formula the arithmetic genus of C. Now, let $p_{01}, \ldots, p_{0n_0}$ be the singular points of C. Define the points *infinitely near to p_{0j} to the first order* to be the points in the intersection $E_{0j} \cap C^*$, where E_{0j} is the exceptional divisor of the blow up of S at p_{0j}, and C^* is the strict transform of C. In an analogous way define the points $p_{i1}, \ldots, p_{in_i}$ infinitely near to p_{0j} to the i-th order. Let μ_{ij} be their multiplicities. Xu proves that

$$\begin{aligned} p_g(C) &= p_a(C) - \frac{1}{2}\sum_{i,j} \mu_{ij}(\mu_{ij}-1) \\ &= \frac{1}{2}dk(d+k-4) + 1 - \frac{1}{2}\sum_{i,j} \mu_{ij}(\mu_{ij}-1). \end{aligned}$$

One has then to show that the singularities of C cannot be "too bad". This is done by Xu by means of a careful local study of the deformations of C at its singular points.

Xu's local approach has been interpreted in terms of a global property of the focal locus of a family of curves by Chiantini and Lopez in [CL]. They reprove in this way Xu's result and apply successfully their methods to bound from below the genus of curves on a general projectively Cohen-Macaulay surface in $\mathbf{P}^4$.

0.3.2 Hypersurfaces in $\mathbf{P}^n$, $n \geq 3$

The first result concerning the genus of a curve of a hypersurface of high degree in a projective space of *arbitrary* dimension goes back to Clemens [C3]. He proves

Theorem ([C3]). *There is no rational curve on the general hypersurface $X_d \subset \mathbf{P}^n$ of degree $d \geq 2n-1$.*

Suppose C is a smooth curve on X_d. By adjunction its canonical sheaf is $K_C = \mathcal{O}_C(d-n-1) \otimes det N_{C/X}$, where $N_{C/X}$ is the normal sheaf of C in X. Then it is clear that if one is able to bound from below the negativity of $N_{C/X}$, when the degree d increases, there is no hope to find curves on X with low genus. This is what Clemens does in [C3], via a delicate comparison between the normal bundles $N_{C/X}$ and $N_{X/V} \otimes \mathcal{O}_C$, where V is a general hypersurface of degree d in $\mathbf{P}^{n+d}$ such that X is cut out in V by a generic linear space of dimension n. Clemens' technique, although ingenious, does not seem to apply easily to the study of higher dimensional subvarieties of projective hypersufaces.

A more powerful *variational* approach has been proposed by Ein in his two papers [E1], [E2], where he succeeds in extending Clemens' result to complete intersections in an arbitrary projective ambient variety. In the case of hypersurfaces, this method, as streamlined by Voisin in [V2], is described in detail in §1.2 of Part I, and can be briefly summarized as follows. Consider the family of hypersurfaces of fixed degree

$$\mathcal{X} \subset H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \times \mathbf{P}^n,$$

and, after an étale base change $U \to H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$, a subfamily $\mathcal{Y} \subset \mathcal{X}_U$ of relative dimension k, so that the fiber Y_t identifies to a k-dimensional subvariety of the hypersurface X_t. Let $\tilde{\mathcal{Y}} \to \mathcal{Y}$ be a desingularization inducing a desingularization on the fibers $\tilde{Y}_t \to Y_t$, and N the dimension of the base. To avoid the difficult task of controlling the normal bundle of Y_t in X_t one can apply the adjunction formula to $\tilde{Y}_t \subset \tilde{\mathcal{Y}}$ (notice that the normal bundle of the fiber in the family is trivial!) and use the morphism of restriction of differentials induced by the map from $\tilde{\mathcal{Y}}$ to $\mathcal{X}$:

$$\Omega^{N+k}_{\mathcal{X}_U}{}_{|X_t} \to \Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_t} \cong K_{\tilde{Y}_t}. \tag{3}$$

If d_0 is a degree starting from which the bundle $\Omega^{N+k}_{\mathcal{X}_U}{}_{|X_t}$ is generated by its global sections, by taking the map induced in cohomology by (3)

$$H^0(X_t, \Omega^{N+k}_{\mathcal{X}_U}{}_{|X_t}) \to H^0(\tilde{Y}_t, \Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_t}) \cong H^0(\tilde{Y}_t, K_{\tilde{Y}_t}), \tag{4}$$

we automatically have that the general hypersurface of degree $d \geq d_0$ in $\mathbf{P}^n$ does not contain any k-dimensional subvariety with geometric genus zero. Ein's theorem says exactly that the degree $d_0 = 2n - k$ fits, and thus he re-obtains Clemens' result.

Voisin, in her two papers [V2] and [V3], is then able to sharpen Clemens's result by one, obtaining the following

Theorem ([V2] and [V3]). *Let $X_d \subset \mathbf{P}^n$ be a general hypersurface of degree $d \geq 2n - 1 - k$, where k is an integer such that $1 \leq k \leq n - 3$. Then any k-dimensional subvariety Y of X has desingularization $\tilde{Y}$ with effective canonical bundle.*

We would like to draw the reader's attention to two essential aspects of Voisin's contribution to the problem. First, she points out that, already for $d = 2n - 1 - k$, the sheaf $\Omega^{N+k}_{\mathcal{X}_U}{}_{|X_t}$ is no longer globally generated (for the details, see again §1.2), so that in order to apply the ideas sketched above, one is forced to study the base locus of this sheaf. The second new idea is that one can analyse the cohomological information given by the vanishing of the map (4), concluding that a $\tilde{\mathcal{Y}}$ for which this occurs has to be the solution of a certain distribution on the total space. She then describes these solutions explicitly, showing that they do not give subvarieties $Y \subset X$ with geometric genus zero.

As a concluding remark to this section, we observe that Voisin's technique applies unfortunately only to the study of subvarieties having at least codimension two in the hypersurface. The case of divisors in the general quintic threefold is therefore out of reach: new ideas and techniques are certainly needed in order to attack Clemens' and Green's conjectures.

0.3.3 Results of Part I

Voisin's result is sharp. The general projective hypersurface $X_d \subset \mathbf{P}^n$ of degree $d = 2n - 2 - k$ contains indeed a subvariety of dimension k which is covered by lines, that has, as a consequence, geometric genus zero. The motivation for the first part of this thesis is then provided by the following

Question. *Is the variety covered by the lines the only subvariety of dimension k with geometric genus zero on the general hypersurface $X_{2n-2-k} \subset \mathbf{P}^n$, $1 \leq k \leq n-4$?*

In studying this question, we adopt the variational approach that has been successfully used by Ein and Voisin. As Voisin in [V3], we are able to deduce a geometrical consequence from the vanishing of the map defined in (4), and prove

Proposition A. *Let $X_F \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k$, where k is an integer such that $1 \leq k \leq n-5$. Let $Y_F \subset X_F$ be a subvariety of dimension k such that $H^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$, where $\tilde{Y}_F$ is a desingularization of Y_F. Then Y_F has to be contained in*

$$\Delta_{d,F} = \{x \in X_F : \text{there exists a line } \ell \text{ s.t. } \ell \cap X_F = d{\cdot}x\},$$

a subvariety of X_F of dimension $2n-2-(d-1) = k+1$.

Then, studying the geometry of a desingularization of the locus $\Delta_{d,F}$, we obtain

Proposition B. *Let F be a general polynomial of degree $d = 2n-2-k$ with $1 \leq k \leq n-5$. Let $Y_F \subset \Delta_{d,F}$ be a subvariety of codimension 1 such that $H^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$, where $\tilde{Y}_F$ is a desingularization of Y_F. Then Y_F has to be a component of the k-dimensional subvariety of $\Delta_{d,F}$ covered by the lines lying on X_F.*

Combining these two propositions we obtain our main result which gives a positive answer to the question above for $1 \leq k \leq n-5$:

Theorem I. *Let $X_d \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k$, where k is an integer such that $1 \leq k \leq n-5$. Then any subvariety $Y \subset X$ of dimension k, whose desingularization $\tilde{Y}$ has $h^0(\tilde{Y}, K_{\tilde{Y}}) = 0$, is a component of the (k-dimensional) subvariety covered by the lines lying on X.*

Taking $k = 1$ we get, for $n \geq 6$, a corollary for rational curves on the general projective hypersurface of degree $d = 2n-3$, that, somehow surprisingly, confirms the heuristic prediction:

Corollary. *There is no rational curve of degree $\delta \geq 2$ on the general hypersurface $X_{2n-3} \subset \mathbf{P}^n$, $n \geq 6$.*

Rational curves on general projective hypersurfaces

1.1 Introduction

It was shown by H. Clemens [C] that the general (in the countable Zariski topology) hypersurface of degree d in $\mathbf{P}^n$ does not contain any rational curve, if d is sufficiently large. More precisely, he proved the following:

Theorem (Clemens). *There is no rational curve on the general hypersurface $X_d \subset \mathbf{P}^n$ of degree $d \geq 2n-1$, $n \geq 3$.*

L. Ein studied more generally (see [E1] and [E2]) the geometric genus of subvarieties contained in complete intersections $X_{(d_1,\ldots,d_r)} \subset M$, where M is an arbitrary smooth and projective ambient variety. He proved that if $d_1 + \ldots + d_r \geq 2n - r - k + 1$, any k-dimensional subvariety $Y \subset X$ has desingularization with effective canonical bundle. However, in the case of a hypersurface $X_d \subset \mathbf{P}^n$, he obtained the lower bound $d \geq 2n - k$ on the degree of X, which was the same as Clemens' for $k = 1$, and was not optimal. Indeed, it was classically known that the lines lying on the general hypersurface $X_d \subset \mathbf{P}^n$ of degree $d = 2n - 2 - k$ cover a k-dimensional subvariety, which then has geometric genus zero. Thus nothing was known yet about the canonical bundle of subvarieties of dimension k on the general $X_{2n-1-k} \subset \mathbf{P}^n$. Voisin ([V2], [V3]) showed then that it was possible to sharpen Ein's bound by one, as conjectured by Clemens himself, by proving:

Theorem (Voisin). *Let $X_d \subset \mathbf{P}^n$ be a general hypersurface of degree $d \geq 2n - 1 - k$, where k is an integer such that $1 \leq k \leq n - 3$. Then* *any k-dimensional subvariety Y of X has desingularization $\tilde{Y}$ with effective canonical bundle.*

For $k = 1$ we immediately obtain that the general $X_{2n-2} \subset \mathbf{P}^n$, $n \geq 4$, contains no rational curves. Taking $k = 2$ we get another very interesting corollary whose analogue in the case $n = 4$ would solve Clemens' conjecture on the finiteness of rational curves of fixed degree on the general quintic threefold in $\mathbf{P}^4$:

Corollary (Voisin). *For each integer $\delta \geq 1$, the general hypersurface $X_{2n-3} \subset \mathbf{P}^n$, $n \geq 5$, contains at most a finite number of rational curves of degree δ.*

The goal of our work is to investigate, for the general $X_{2n-2-k} \subset \mathbf{P}^n$, $1 \leq k \leq n-4$, the geometry of its k-dimensional subvarieties having geometric genus zero. Since the locus covered by the lines of X is the only known example of such a subvariety, it seems natural to start with the following:

Question. *Is the variety covered by the lines the only subvariety of dimension k with geometric genus zero on the general hypersurface $X_{2n-2-k} \subset \mathbf{P}^n$, $1 \leq k \leq n-4$?*

Remark that the numerical hypothesis $1 \leq k \leq n-4$ implies the positivity of the canonical bundle of $X_{2n-2-k} \subset \mathbf{P}^n$, and gives meaning to the question in contrast to the case of the Calabi-Yau hypersurface $X_{n+1} \subset \mathbf{P}^n$. The main result of this paper gives actually a positive answer to the previous question for $1 \leq k \leq n-5$:

Theorem. *Let $X_d \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k$, where k is an integer such that $1 \leq k \leq n-5$. Then any subvariety $Y \subset X$ of dimension k, whose desingularization $\tilde{Y}$ has $h^0(\tilde{Y}, K_{\tilde{Y}}) = 0$, is a component of the (k-dimensional) subvariety covered by the lines lying on X.*

Taking $k = 1$ we get, for $n \geq 6$, a corollary on rational curves on the general projective hypersurface of degree $d = 2n-3$. Voisin's corollary already implies that, for each fixed integer $\delta \geq 1$, there are only a finite number of such curves of degree δ. Here we prove that there are only lines - whose number is easily computed as the top Chern class of a certain vector bundle on the Grassmannian of lines in $\mathbf{P}^n$:

Corollary. *There is no rational curve of degree* $\delta \geq 2$ *on the general hypersurface* $X_{2n-3} \subset \mathbf{P}^n, \ n \geq 6$.

Throughout this paper we work on the field of complex number $\mathbb{C}$.

Acknowledgements. This paper, which is a part of my Ph.D. thesis, has greatly benefited from the generous help and guidance of my advisor, Prof. Claire Voisin. I would also like to thank Prof. A. F. Lopez for his encouragement and many useful discussions. I would like to sincerely thank the referee. He or she did a remarkable amount of work, simplifying some of my proofs, checking all details, and suggesting how to put in greater evidence the key underlying ideas driving the result. Finally, with great generosity, he or she shared with us some new ideas of his/hers to sharpen the results presented here.

1.2 Preliminaries and sketch of the proof

To motivate our approach and render our proof more transparent, we will briefly sketch the key ideas contained in [E1], [V2] and [V3]. We start with some

Notation.
$S^d := H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$;
$S^d_x := H^0(\mathbf{P}^n, \mathcal{I}_x \otimes \mathcal{O}_{\mathbf{P}^n}(d))$;
$N := h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) = dim\ S^d$;
$\mathcal{X} \subset \mathbf{P}^n \times S^d$ will denote the universal hypersurface of degree d;
$X_F \subset \mathbf{P}^n$ the fiber of the family $\mathcal{X}$ over $F \in S^d$, i.e. the hypersurface defined by F.

Let $U \to S^d$ be an étale map and $\mathcal{Y} \subset \mathcal{X}_U$ a universal, reduced and irreducible subscheme of relative dimension k (in the following, by abuse of notation, we will often omit the étale base change). We may obviously assume $\mathcal{Y}$ invariant under some lift of the natural action of $GL(n+1)$ on $\mathbf{P}^n \times S^d$: $g(x,F) = (g(x), (g^{-1})^* F)), \ g \in GL(n+1)$. Let $\tilde{\mathcal{Y}} \to \mathcal{Y}$ be a desingularization and $\tilde{\mathcal{Y}} \xrightarrow{j} \mathcal{X}_U$ the natural induced map. Let $\pi : \mathcal{X} \to \mathbf{P}^n$ be the projection on the first component and $T^{vert}_{\mathcal{X}}$ (resp. $T^{vert}_{\mathcal{Y}}$) the vertical part of $T_{\mathcal{X}}$ (resp. $T_{\mathcal{Y}}$) w.r.t. π, i.e. $T^{vert}_{\mathcal{X}}$ (resp. $T^{vert}_{\mathcal{Y}}$) is the sheaf defined by

$$0 \to T^{vert}_{\mathcal{X}} \to T\mathcal{X} \xrightarrow{\pi_*} T\mathbf{P}^n \to 0$$

$$(resp. \quad 0 \to T^{vert}_{\mathcal{Y}} \to T\mathcal{Y} \xrightarrow{\pi_*} T\mathbf{P}^n).$$

The hypothesis of $GL(n+1)$-invariance for $\mathcal{Y}$ has two easy but very important consequences that will be frequently used in the rest of the paper:

Lemma 1.2.1. *Let $\mathcal{Y}$ be a subvariety of $\mathcal{X} \times \mathbf{P}^n$ of relative dimension k and invariant under the action of $GL(n+1)$. Then:*

(i) codim $_{T^{vert}_{\mathcal{X},(y,F)}} T^{vert}_{\mathcal{Y},(y,F)} = codim_{\mathcal{X}}\mathcal{Y} = n-k-1$; in particular, since we are assuming $1 \le k \le n-5$, we have that

$$codim_{\ \mathcal{X}}\ \mathcal{Y}\ \ge 4.$$

(ii) $T^{vert}_{\mathcal{Y},(y,F)} \supset < S^1_y \cdot J^{d-1}_F, F >$, where J^{d-1}_F is the Jacobian ideal of F.

Proof. (i) Use the surjectivity of the map $T\mathcal{Y} \xrightarrow{\pi_*} T\mathbf{P}^n$.
(ii) It follows from the fact that, by $GL(n+1)$-invariance, $T^{vert}_{\mathcal{Y},(y,F)}$ contains the vertical part of the tangent space to the orbit of (y,F) under the action of $GL(n+1)$. □

Assume $d \ge 2n-k$ and let $Y_F \subset X_F$ be a general fiber of the subfamily $\mathcal{Y} \subset \mathcal{X}_U$. Then to prove Clemens' result (which corresponds to the case $k=1$) we have to show that $h^0(\tilde{Y}_F, K_{\tilde{Y}_F}) \neq 0$, where $\tilde{Y}_F \to Y_F$ is the desingularization.

Ein's idea is to produce, by adjunction, a non-zero section in $H^0(\tilde{Y}_F, K_{\tilde{Y}_F})$ by restricting to $\tilde{\mathcal{Y}}$ holomorphic forms on $\mathcal{X}_U$ defined along X_F. The main technical difficulty consists in controlling the positivity of the twisted "vertical" tangent bundle to the universal hypersurfaces. We start then recording, in the first subsection, the needed positivity result, and an equivalent one for a bundle on the Grassmannian of lines in $\mathbf{P}^n$ that will be used later.

1.2.1 Positivity results

Let d be a positive integer. Consider the bundle $M^d_{\mathbf{P}^n}$ defined by the exact sequence

$$0 \to M^d_{\mathbf{P}^n} \to S^d \otimes \mathcal{O}_{\mathbf{P}^n} \xrightarrow{ev} \mathcal{O}_{\mathbf{P}^n}(d) \to 0, \tag{1.5}$$

whose fiber at a point x identifies by definition with S^d_x. From the inclusion $\mathcal{X} \hookrightarrow \mathbf{P}^n \times S^d$ we get the exact sequence

$$0 \to T\mathcal{X}_{|X_F} \to T\mathbf{P}^n{}_{|X_F} \oplus (S^d \otimes \mathcal{O}_{X_F}) \to \mathcal{O}_{X_F}(d) \to 0,$$

which combined with (1.5) gives us

$$0 \to M^d_{\mathbf{P}^n}{}_{|X_F} \to T\mathcal{X}_{|X_F} \to T\mathbf{P}^n{}_{|X_F} \to 0.$$

In other words $M^d_{\mathbf{P}^n |X_F}$ identifies to the vertical part of $T_{\mathcal{X}} \otimes \mathcal{O}_{X_F}$ with respect to the projection to $\mathbf{P}^n$.

Let $G := Grass(1, n)$ be the Grassmannian of lines in $\mathbf{P}^n$, $\mathcal{O}_G(1)$ the line bundle on G giving its Plücker polarization, and $\mathcal{E}_d$ be the d^{th}-symmetric power of the dual of the tautological subbundle on G. Recall that the fibre of $\mathcal{E}_d$ at a point $[\ell]$ is, by definition, given by $H^0(\ell, \mathcal{O}_\ell(d))$.

Let M^d_G be the vector bundle on G defined as the kernel of the evaluation map:

$$0 \to M^d_G \to S^d \otimes \mathcal{O}_G \to \mathcal{E}_d \to 0.$$

Notice that the fiber of M^d_G at a point $[\ell]$ is equal to $I_\ell(d) := H^0(\mathcal{I}_\ell(d))$.

Then we have the following

Proposition 1.2.2. [1] *(i) $M^d_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(1)$ is generated be its global sections; (ii) $M^d_G \otimes \mathcal{O}_G(1)$ is generated be its global sections.*

Proof. (i) Any $P \in S^d_x$ can be written as follows: $P = \sum_{i=1}^n L_i Q_i$, where the L_i's are linear forms vanishing at x. If L_0 is a linear form non zero at x, then $P \otimes L_0 - \sum_{i=1}^n L_0 Q_i \otimes L_i$ is a global section of $M^d_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(1)$ whose value at x is proportional to P.

(ii) First, recall that the space of global sectios of $M^d_G \otimes \mathcal{O}_G(1)$ identifies to the kernel of the contraction $S^d V \otimes \Lambda^2 S^1 \to S^{d+1} \otimes S^1$. Again, remark that any degree d homogenious polynomial P vanishing along a line ℓ can be written as follows: $P = \sum_{i=2}^n L_i Q_i$ where the L_i's are linear forms vanishing along ℓ. If L_0, L_1 are linear forms which are independent after restriction to ℓ, then

$$P \otimes L_0 \wedge L_1 + \sum_{i=2}^n L_0 Q_i \otimes L_1 \wedge L_i - \sum_{i=2}^n L_1 Q_i \otimes L_0 \wedge L_i$$

is a global section of $M^d_G \otimes \mathcal{O}_G(1)$ whose value at x is proportional to P. $\square$

[1] There are many ways to prove this Proposition, beside the one we proposed in a preliminary version of this work. The proof we reproduce here, which is undoubtably the simplest possible, is due to Laurent Manivel.

1.2.2 Proof of Ein's and Voisin's results

Following Ein [E1], [E2], one can use the positivity result (1.2.2), (i), to produce holomorphic forms on the (vertical) tangent space to the family $\mathcal{X}_U$. Then, by pulling them back to $\tilde{Y}_F$ and using the adjunction formula, it will be possible to provide a non zero section of $H^0(\tilde{Y}_F, K_{\tilde{Y}_F})$. To make this more precise, first recall the following elementary facts:

(i) $\Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}$;

(ii) $(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}) \otimes K_{X_F} \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F}$.

Therefore, from the natural morphism $\Omega^1_{\mathcal{X}_U} \to \Omega^1_{\tilde{\mathcal{Y}}}$, we get a map

$$(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}) \otimes K_{X_F} \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F} \to \Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}. \tag{1.6}$$

Since $K_{X_F} = \mathcal{O}_{X_F}(d-n-1) = \mathcal{O}_{X_F}((n-k-1)+(d-2n+k))$ and

$$\wedge^{n-1-k} T\mathcal{X}_{U|X_F}(n-k-1) = \wedge^{n-1-k}(T\mathcal{X}_{U|X_F}(1)),$$

we have

$$(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}) \otimes K_{X_F} = \wedge^{n-1-k}(T\mathcal{X}_{U|X_F}(1)) \otimes \mathcal{O}_{X_F}(d-2n+k). \tag{1.7}$$

Now, since we are supposing $d \geq 2n-k$, Proposition (1.2.2) (i) implies that the vertical part of

$$\wedge^{n-1-k}(T\mathcal{X}_{U|X_F}(1)) \otimes \mathcal{O}_{X_F}(d-2n+k) \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F},$$

namely, the subsheaf

$$\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F} = \wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}(1)) \otimes \mathcal{O}_{X_F}(d-2n+k),$$

is globally generated. Composing the inclusion

$$\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F} \hookrightarrow \wedge^{n-1-k}(T\mathcal{X}_{U|X_F}) \otimes K_{X_F}$$

with the restriction map defined in (1.6), we have a natural morphism

$$\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F} \to K_{\tilde{Y}_F}. \tag{1.8}$$

Ein's result is then given by the following

Lemma 1.2.3. *Let F be a general polynomial of degree $2n-k$. The map*

$$H^0(\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F}) \to H^0(K_{\tilde{Y}_F}),$$

induced in cohomology by (1.72), is non zero.

Proof. By Lemma (1.2.1), (i), we have

$$codim\ _{T^{vert}_{\mathcal{X},(y,F)}} T^{vert}_{\mathcal{Y},(y,F)} = codim_{\mathcal{X}}\mathcal{Y}.$$

Let (y,F) be a smooth point of $\mathcal{Y}$. Since the bundle $\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F}$ is generated by its global sections, there exists a section

$$s \in H^0(\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F})$$

such that

$$< s(y), T^{vert}_{\mathcal{Y},(y,F)} > \neq 0. \tag{1.9}$$

Since $j : \tilde{\mathcal{Y}} \to \mathcal{X}_U$ is generically an immersion, we obtain from the above a non zero element in $H^0(K_{\tilde{Y}_F})$ coming from $H^0(\wedge^{n-1-k}(M^d_{\mathbf{P}^n|X_F}) \otimes K_{X_F})$. □

(For other proofs of Clemens' theorem see, of course, [C] and also [CLR]).

In order to try and improve the bound on the degree by one, we observe that, if $d = 2n-1-k$, then $K_{X_F} = \mathcal{O}_{X_F}(n-2-k)$, so we have, as in (1.6), a map

$$\wedge^{n-1-k} T\mathcal{X}_{U|X_F}(n-k-2) \cong \Omega^{N+k}_{\mathcal{X}_U\ |X_F} \to \Omega^{N+k}_{\tilde{\mathcal{Y}}\ \ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}. \tag{1.10}$$

As we saw in Lemma 2.1, (i), by the hypothesis of $GL(n+1)$-invariance on $\mathcal{Y}$, the relevant part of the tangent space to look at is the vertical one, hence we focus our attention on the map

$$\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-k-2) \to \Omega^{N+k}_{\tilde{\mathcal{Y}}\ \ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}. \tag{1.11}$$

Now, because of the shift between the exterior power and the degree of the canonical bundle we are tensoring with on the lefthand side of (1.11), the global generation of the sheaf $\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-k-2)$ will not follow from the global generation of $M^d_{\mathbf{P}^n}(1)$. Voisin's idea is then to study the positivity of $H^0(\wedge^2 M^d_{\mathbf{P}^n}(1))$, to produce holomorphic forms on the (vertical) tangent space to the universal hypersurface, and use the commutative diagram below to produce sections in $H^0(K_{\tilde{Y}_F})$:

$$\begin{array}{ccc} & H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-2-k)) \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ & \uparrow & \nearrow \\ H^0(\wedge^{n-3-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)) \otimes H^0(\wedge^2 M^d_{\mathbf{P}^n|X_F}(1)) & & \end{array} \tag{1.12}$$

(the vertical map in (1.12) is simply obtained by wedging the sections of the sheaves $\wedge^{n-3-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)$ and $\wedge^2 M^d_{\mathbf{P}^n|X_F}(1)$). Unfortunately the following fact holds:

Fact (Amerik-Voisin). $\wedge^2 M^d_{\mathbf{P}^n}(1)$ *is not generated by its global sections.*

Indeed, in [V3] the following counterexample to the global generation of $\wedge^2 M^d_{\mathbf{P}^n}(1)$ is given. Consider the subvariety

$$\Delta_{d,F} := \{x \in X_F : \textit{there exists a line } \ell \textit{ s.t. } \ell \cap X_F = d \cdot x\}. \tag{1.13}$$

An elementary dimension count shows that, for generic F,

$$dim\ \Delta_{d,F} = 2n-2-(d-1) = 2n-2-(2n-1-k-1) = k$$

(these subvarieties are generically empty for $d \geq 2n-1$, which is the reason they don't come into play in Clemens' and Ein's case). Let Δ_d be the family of the $\Delta_{d,F}$'s, let $\tilde{\Delta}_d \to \Delta_d$ be a desingularization, and $j : \tilde{\Delta}_d \to \mathcal{X}$ the natural morphism. Notice that $\Delta_{d,F}$ parametrizes 0-cycles of X_F which are all rationally equivalent since, by definition, $d \cdot x \equiv H^{n-1}.X_F$, $\forall x \in \Delta_{d,F}$, where H is the hyperplane divisor in $\mathbf{P}^n$. Thus, the variational (and higher dimensional) version of Mumford's fundamental result on 0-cycles on surfaces applies (see [M2], and [V1] for the variational generalization in dimension 2), so we have

$$j^* s = 0 \ \ in \ \ H^0(\Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_F}),\ \forall s \in H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F}),$$

i.e. the map

$$H^0(\wedge^{n-1-k} T\mathcal{X}_{|X_F}(n-2-k)) \cong H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F}) \to H^0(\Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_F}) \cong H^0(K_{\tilde{Y}_F})$$

is identically zero and then so is

$$H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-2-k)) \to H^0(K_{\tilde{Y}_F}).$$

In particular, by (1.12) and Proposition 1.2.2, (i), we have that, at a smooth point $(y, F) \in \mathcal{Y}$, all the global sections of the bundle $\wedge^2 M^d_{\mathbf{P}^n}(1)_{|X_F}$, seen as a line bundle on the Grassmannian of codimension two subspaces of $T\mathcal{X}^{vert}_{|X_F}$, vanish on the codimension two subspaces of $T^{vert}_{\mathcal{X},(y,F)}$ containing $T^{vert}_{\mathcal{Y},(y,F)}$.

Voisin's alternative approach to the problem, as developed in [V3], consists then in studying the base locus of $H^0(\wedge^2 M^d_{\mathbf{P}^n}(1)_{|X_F})$, to investigate the geometry of the subvarieties for which the composite map in (1.12) fails to provide non-zero sections of their canonical bundle. She shows in [V3] that, in the case $d = 2n - 1 - k$, the subvariety $\Delta_{d,F}$ defined in (1.13) is the only one for which this phenomenon occurs. Then, she completes her proof by verifying that each component of $\Delta_{d,F}$ has positive geometric genus.

1.2.3 The strategy of our proof

Our purpose is to study, for $d = 2n - 2 - k$, the geometry of k-dimensional subvarieties of $X_F \subset \mathbf{P}^n$, having geometric genus equal to zero. Recall that, since $d = 2n - 2 - k$, we have $K_{X_F} = \mathcal{O}_{X_F}(n - 3 - k)$, and note that the composite map

$$\begin{array}{ccc} H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)) & \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ \uparrow & \nearrow & \\ H^0(\wedge^{n-5-k} M^d_{\mathbf{P}^n|X_F}(n-5-k)) \otimes H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2)) & & \end{array} \tag{1.14}$$

is obviously zero, since we are supposing $h^0(K_{\tilde{Y}_F}) = 0$. Then the proof of our theorem will naturally be divided into two steps. In section 3, analysing the base locus of $H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2))$, considered as the space of sections of a line bundle on the Grassmannian of codimension four subspaces of $T\mathcal{X}^{vert}_{|X_F}$, we will prove

Proposition A. *Let $X_F \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n - 2 - k$, and $Y_F \subset X_F$ a subvariety of dimension k such that $H^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$, where $\tilde{Y}_F$ is a desingularization of Y_F. Then Y_F has to be contained in*

$$\Delta_{d,F} = \{x \in X_F : \textit{there exists a line } \ell \textit{ s.t. } \ell \cap X_F = d{\cdot}x\},$$

a subvariety of X_F of dimension $2n - 2 - (d - 1) = k + 1$.

In § 4, we will study an explicit desingularization $\tilde{\Delta}_{d,F}$ of $\Delta_{d,F}$, given by the zeroes of a section of a bundle on the incidence variety in $\mathbf{P}^n \times Grass(1, n)$. Denote by $\tilde{\Delta}_d$ the family $(\Delta_{d,F})_{F \in S^d}$, and recall the isomorphism

$$T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}} \cong \Omega^{N+k}_{\tilde{\Delta}_d\ |\tilde{S}_F}.$$

The positivity result (1.2.2), (ii), for the bundle $M^d_G \otimes \mathcal{O}_G(1)$ on $Grass(1, n)$, will allow us to construct a subbundle of $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$ generated by its global sections. Using this fact, together with the vanishing of the natural restriction map

$$H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) = H^0(\Omega^{N+k}_{\tilde{\Delta}_d \ |\tilde{S}_F}) \to H^0(K_{\tilde{Y}_F}), \qquad (1.15)$$

we will prove

Proposition B. *Let F be a general polynomial of degree $d = 2n-2-k$. Let $Y_F \subset \Delta_{d,F}$ be a subvariety of codimension 1 such that $H^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$, where $\tilde{Y}_F$ is a desingularization of Y_F. Then Y_F has to be a component of the k-dimensional subvariety of $\Delta_{d,F}$ covered by the lines lying on X_F.*

These propositions will combine to prove our main theorem.

1.3 Base locus of $\wedge^4 M^d_{\mathbf{P}^n}(2)$ and osculating lines

Let $\mathcal{X} \subset \mathbf{P}^n \times S^d$ be the universal hypersurface of degree $d = 2n-2-k$, $U \to S^d$ an étale map and $\mathcal{Y} \subset \mathcal{X}_U$ a universal, reduced and irreducible subscheme of relative dimension k (to simplify the notation, in what follows we will occasionally omit the étale base change). Assume $\mathcal{Y}$ invariant under some lift of the action of $GL(n+1)$, denote by $\tilde{\mathcal{Y}} \to \mathcal{Y}$ a desingularization, and suppose that the fibres of $\tilde{\mathcal{Y}}$ verify $h^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$.

Consider the bundle $M^d_{\mathbf{P}^n}$ defined by the exact sequence

$$0 \to M^d_{\mathbf{P}^n} \to S^d \otimes \mathcal{O}_{\mathbf{P}^n} \xrightarrow{ev} \mathcal{O}_{\mathbf{P}^n}(d) \to 0,$$

whose fiber at a point x identifies by definition with S^d_x. Recall from §2.1 that

$$M^d_{\mathbf{P}^n|X_F} = T^{vert}_{\mathcal{X}} \otimes \mathcal{O}_{X_F}, \qquad (1.16)$$

where $T^{vert}_{\mathcal{X}}$ is the sheaf defined by

$$0 \to T^{vert}_{\mathcal{X}} \to T\mathcal{X} \xrightarrow{\pi_*} T\mathbf{P}^n \to 0.$$

From the vanishing of the composite map

$$\begin{array}{ccc} H^0(\wedge^{n-1-k} T\mathcal{X}_{U|X_F}(n-3-k)) & \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ \uparrow & \nearrow & \\ H^0(\wedge^{n-5-k} T\mathcal{X}_{U|X_F}(n-5-k)) \otimes H^0(\wedge^4 T\mathcal{X}_{U|X_F}(2)) & & \end{array} \qquad (1.17)$$

and (1.16) we deduce that the composite map

$$\begin{array}{ccc} H^0(\wedge^{n-1-k} M^d_{\mathbf{P}^n|X_F}(n-3-k)) & \longrightarrow & H^0(K_{\tilde{Y}_F}) \\ \uparrow & \nearrow & \\ H^0(\wedge^{n-5-k} M^d_{\mathbf{P}^n|X_F}(n-5-k)) \otimes H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2)) & & \end{array} \quad (1.18)$$

is also zero. Since, by Lemma 1.2.2, (i), $\wedge^{n-5-k} M^d_{\mathbf{P}^n|X_F}(n-5-k)$ is generated by its global sections, the vanishing of the composite map in (1.18) and the $GL(n+1)$ invariance of $\mathcal{Y}$ implies that, at a smooth point $(y, F) \in \mathcal{Y}$, any codimension four subspace of $T^{vert}_{\mathcal{X}_U,(y,F)} = S^d_y$ containing $T^{vert}_{\mathcal{Y},(y,F)}$ is in the base locus of $H^0(\wedge^4 M^d_{\mathbf{P}^n|X_F}(2))$, considered as a space of sections of a line bundle over the Grassmannian of codimension four subspaces of $T^{vert}\mathcal{X}_{|X_F}$. Studying this base locus we will see how, at each point y of a subvariety $Y_F \subset X_F$ with zero geometric genus, the ideal of a line through y naturally comes into play. More precisely, we will prove

Proposition 1.3.1. *Let $\mathcal{Y} \subset \mathcal{X}_U$ be such that the composite map in (1.18) is zero. Then, at a smooth point (y, F), the vertical tangent space $T^{vert}_{\mathcal{Y},(y,F)}$, which is a subspace of $T^{vert}_{\mathcal{X}_U,(y,F)} = S^d_y$, has to contain (at least) a hyperplane $H_{\ell_{(y,F)}} \subset I_{\ell_{(y,F)}}(d)$, where $\ell_{(y,F)}$ is a line passing through y.*

We will then study the distribution $\mathcal{H} \subset T^{vert}_{\mathcal{Y}}$, pointwise defined by $H_{\ell_{(y,F)}}$, and prove its integrability. The description of the corresponding foliation and the $GL(n+1)$-invariance of $\mathcal{Y}$ will allow us to conclude that the line $\ell_{(y,F)}$ is such that $\ell_{(y,F)} \cap X_F = d \cdot y$, thus proving Proposition A.

1.3.1 Proof of Proposition 3.1

We start with the following

Lemma 1.3.2. *Let T be a codimension four subspace of $S^d_x = (T_{\mathcal{X},(x,F)})^{vert}$ which is in the base locus of $H^0(\bigwedge^4 M^d_{\mathbf{P}^n}(2))$. Then T has to contain (at least) a hyperplane of $I_\ell(d)$, where ℓ is a line passing through x.*

Proof. Recall that $H^0(\bigwedge^2 M^d_{\mathbf{P}^n}(1))$ can be naturally interpreted as the kernel of the Koszul map $\bigwedge^2 S^d \otimes S^1 \to S^d \otimes S^{d+1}$. Hence one easily verifies that $Im\ (H^0(\bigwedge^2 M^d_{\mathbf{P}^n}(1)) \to \bigwedge^2 M^d_{\mathbf{P}^n,x})$ contains $PA_1 \wedge PA_2$, for all $P \in S^{d-1}$ and $A_i \in S^1_x$. Then

$$Im\ (H^0(\bigwedge^4 M^d_{\mathbf{P}^n}(2)) \to \bigwedge^4 M^d_{\mathbf{P}^n,x})$$

contains elements of the form

$$PA_1 \wedge PA_2 \wedge QB_1 \wedge QB_2,$$

for all $P, Q \in S^{d-1}$ and $A_i, B_i \in S^1_x$, coming from the wedge product of elements in $Im\ (H^0(\bigwedge^2 M^d_{\mathbf{P}^n}(1)) \to \bigwedge^2 M^d_{\mathbf{P}^n,x})$. Since we are supposing that T is in the base locus of $H^0(\bigwedge^4 M^d_{\mathbf{P}^n}(2))$, the previous fact implies in particular that the dimension of the subspace $\{AP : A \in S^1_x\}$ *modulo* T is at most 3, i.e. the multiplication map

$$\begin{aligned} m_P : \quad S^1_x &\to S^d_x/T \\ A &\mapsto A \cdot P \ mod\ T \end{aligned}$$

cannot be surjective, for any $P \in S^{d-1}$.
Recall that if V and W are vector spaces, and $Z_k := \{\phi \in Hom(V, W) : rank\ \phi \leq k\}$, then

$$T_{Z_k,\phi} = \{\psi \in Hom(V, W) : \psi(ker\phi) \subset Im\phi\}. \tag{1.19}$$

If, for generic P, the map m_P has rank one, from (1.19) we obtain that $Q \cdot Ker\ m_P\ mod\ T \subset Im\ m_P$, for any $Q \in S^{d-1}$, i.e. $I_{\ell_P}(d) \subset P \cdot S^1_x + T$, where ℓ_P is the line determined by $Ker\ m_P$. Then T contains a hyperplane of $I_{\ell_P}(d)$ and the lemma is proved.

Thus, we can assume that, for generic P. the map m_P has rank at least two. Let $A_1, A_2 \in S^1_x$ such that $T' := < A_1P, A_2P, T >$ is of codimension 2 in S^d_x. For generic $Q \in S^{d-1}_x$, consider the map

$$m_Q : S^1_x \to S^d_x/T',$$

whose rank is then equal to 0 or 1. In the former case T' would then contain $S^{d-1} \cdot Ker\ m_Q = S^{d-1} \cdot S^1_x = S^d_x$, which is absurd since T' has codimension 2. Hence we can suppose $rk\ m_Q = 1$. Then by [V2], Lemma 2.3, T' contains the degree d part of the ideal of a line ℓ_Q passing through x, and hence T contains at least a codimension two subspace of $I_{\ell_Q}(d)$. Assume first that the line does not vary with Q, and denote it by ℓ. If

$$codim_{I_\ell(d)}\ T \cap I_\ell(d) = 2,$$

then the image $\overline{T}$ of T in $H^0(\mathcal{O}_\ell(d))$ has codimension 2 in $H^0(\mathcal{O}_\ell(d)(-x))$. On the other hand, since T' contains $I_\ell(d)$, its reduction $\overline{T'}$ modulo $I_\ell(d)$

has also codimension 2 in $H^0(\mathcal{O}_\ell(d)(-x))$. Hence $\overline{T} = \overline{T'}$. By the genericity of the choice P in S^d_x, this fact would imply that

$$\overline{T} = H^0(\mathcal{O}_\ell(d)(-x)),$$

thus leading to a contradiction.

Assume now that $\ell_Q \neq \ell_{Q'}$, for generic $Q, Q' \in S^{d-1}$. Since T contains a codimension two subspace of $I_{\ell_Q}(d)$, from the exact sequence

$$0 \to I_{\ell_Q}(d) \cap I_{\ell_{Q'}}(d) \to I_{\ell_Q}(d) \oplus I_{\ell_{Q'}}(d) \to S^d_x \to 0,$$

and the fact that $T \subset S^d_x$ has codimension 4, it follows that $T \supset I_{\ell_Q}(d) \cap I_{\ell_{Q'}}(d)$. Let $\mathbf{P}^2_{Q,Q'}$ be the span of ℓ_Q and $\ell_{Q'}$: we study the variation of this plane with Q'. If for generic $Q'_1 \neq Q'$ the intersection $\mathbf{P}^2_{Q,Q'} \cap \mathbf{P}^2_{Q,Q'_1}$ is equal to the line ℓ_Q, then $T \supset I_{\ell_Q}(d)$ and we are done. If otherwise $\mathbf{P}^2_{Q,Q'} = \mathbf{P}^2_{Q,Q'_1} = \mathbf{P}^2$, then it is immediate to see that T contains

$$\{F \in S^d_x : F_{|\mathbf{P}^2} \textit{ is singular at the point } x\},$$

because ℓ_Q and $\ell_{Q'}$ will vary in this plane. But this is absurd since $T \subset S^d_x$ is of codimension 4. □

From the previous lemma and the vanishing of the composite map

$$H^0(\wedge^{n-5-k} T\mathcal{X}_{U|X_F}(n-5-k)) \otimes H^0(\wedge^4 T\mathcal{X}_{U|X_F}(2)) \to H^0(K_{\tilde{Y}_F})$$

we have that any codimension four subspace $T \subset T^{vert}_{\mathcal{X}_U,(y,F)}$ containing $T^{vert}_{\mathcal{Y},(y,F)}$ contains a hyperplane $H_{\ell_{(y,F)}}$ of $I_{\ell_{(y,F)}}(d)$, where $\ell_{(y,F)}$ is a line through y. Note that *a priori* the hyperplane $H_{\ell_{(y,F)}}$ could vary with T. We have then to verify that $T_{\mathcal{Y},(y,F)}$ is forced to contain one of those $H_{\ell_{(y,F)}}$.

Proof of Proposition 1.3.1. Remark that a codimension four subspace $T \subset T^{vert}_{\mathcal{X}_U,(y,F)}$ containing $T^{vert}_{\mathcal{Y},(y,F)}$ cannot contain two hyperplanes H and H' in the ideals $I_\ell(d)$ and $I_{\ell'}(d)$ of different lines $\ell \neq \ell'$. Indeed, if this were the case, by the surjectivity of

$$I_\ell(d) \oplus I_{\ell'}(d) \twoheadrightarrow S^d_y,$$

then T would contain at least a codimension 2 subspace of S^d_y, thus violating the hypothesis on its codimension. Set $V := S^d_y$, and $V_0 := T^{vert}_{\mathcal{Y},(y,F)}$. Denote by ℓ_T the unique line such that T contains a hyperplane in its ideal.

Then, by the above, we have a morphism from $G' := Grass(V/V_0, 4)$, the Grassmannian of codimension 4 subspaces of V/V_0, to $G := Grass(1, \mathbf{P}^n)$, the Grassmannian of lines in $\mathbf{P}^n$:

$$\varphi \ : \ G' \to G; \ T \longmapsto \ell_T.$$

Suppose that φ is a not constant map. Let ℓ be in its image, and $T \in \varphi^{-1}(\ell)$. Then it is easy to construct from such a T a positive dimensional family of codimension 4 subspaces containing V_0 and a hyperplane in $I_\ell(d)$. Thus φ has positive dimensional fibers and we are done, since in this case the ample line bundle $\varphi^*\mathcal{O}_G(1) = \mathcal{O}'_G(s), \ s > 0$, would have zero intersection with the curves in the fiber, which is absurd. Now, let ℓ be the unique line in the image of φ. A dimension count shows that if $V_0 \cap I_\ell(d)$ had codimension ≥ 2, then there would exist a $T \in Grass(V/V_0, 4)$ with $codim_{I_\ell(d)} T \cap I_\ell(d) \geq 2$, thus contradicting Lemma 1.3.2.

□

1.3.2 The vertical contact distribution

We now want to use Proposition 3.1 to construct a (well defined) distribution in $T_{\mathcal{Y}}^{vert}$, and show its integrability.

From (1.3.1) we know that $T_{\mathcal{Y},(y,F)}$ contains at least a hyperplane $H := H_{\ell_{(y,F)}}$ in $I := I_{\ell_{(y,F)}}(d)$. Remark that $T_{\mathcal{Y},(y,F)}$ cannot contain two different hyperplanes $H \neq H'$ in different ideals $I \neq I'$, otherwise it would contain a codimension two subspace of S_y^d. But this is absurd, since by Lemma 1.2.1

$$codim \ _{T^{vert}_{\mathcal{X},(y,F)}} \ T^{vert}_{\mathcal{Y},(y,F)} \ = \ codim_{\mathcal{X}} \mathcal{Y} \ \geq 4.$$

Hence the line $\ell_{(y,F)}$ is unique and we have a well defined map

$$\begin{array}{rccc} \phi : & \mathcal{Y} & \longrightarrow & G(1,n) \\ & (y,F) & \longmapsto & \ell_{(y,F)} \end{array} \tag{1.20}$$

If, at a generic point (y, F), $T_{\mathcal{Y},(y,F)}$ contains the hyperplane $H_{\ell_{(y,F)}}$, but not the whole ideal $I_{\ell_{(y,F)}}(d)$, we get a well defined distribution $\mathcal{H} \subset T_{\mathcal{Y}}^{vert}$, whose fiber at a point (y, F) is given by $H_{\ell_{(y,F)}}$. We will call $\mathcal{H}$ the *vertical contact distribution*.

If at a generic point $T_{\mathcal{Y},(y,F)}$ contains $I_{\ell_{(y,F)}}(d)$, then one can consider the distribution $\mathcal{I} \subset T_{\mathcal{Y}}^{vert}$ fiberwise defined by $I_{\ell_{(y,F)}}(d)$. This case is easier and is actually the case considered in [V3]. It will be briefly treated at the end of this section.

In the former case, as in [V3] and by simply adapting the arguments given there to our situation, we now want to show the following natural fact: if we move infinitesimally in the directions parametrized by $H_{\ell_{(y,F)}} \subset I_{\ell_{(y,F)}}(d)$, then the line $\ell_{(y,F)}$ remains fixed. The integrability of $\mathcal{H}$ will then immediately follow.

Lemma 1.3.3. *(i) The differential ϕ_* at the point (y, F) vanishes on $H_{\ell_{(y,F)}}$.*

(ii) The vertical contact distribution $\mathcal{H} \subset T^{vert}_{\mathcal{Y}}$ is integrable.

Proof. Since the distribution $T^{vert}_{\mathcal{Y},(y,F)} = ker\ p_*$ is integrable, the brackets induce a map

$$\Psi : \bigwedge^2 \mathcal{H} \to T^{vert}_{\mathcal{Y}}/\mathcal{H} \subset T^{vert}_{\mathcal{X}}{}_{|\mathcal{Y}}/\mathcal{H},$$

which is given at the point (y, F) by

$$\psi : \wedge^2 H_{\ell_{(y,F)}} \to T^{vert}_{\mathcal{Y},(y,F)}\ mod\ H_{\ell_{(y,F)}} \subset S^d_y\ mod\ H_{\ell_{(y,F)}}.$$

Since we are supposing that $T^{vert}_{\mathcal{Y},(y,F)}$ contains $H_{(y,F)}$ but not the whole ideal $I_{\ell_{(y,F)}}(d)$, there is a canonical isomorphism

$$T^{vert}_{\mathcal{Y},(y,F)}\ mod\ H_{\ell_{(y,F)}} \cong T^{vert}_{\mathcal{Y},(y,F)}\ mod\ I_{\ell_{(y,F)}}(d),$$

and hence ψ identifies with a map

$$\wedge^2 H_{\ell_{(y,F)}} \to H^0(\mathcal{O}_\ell(d)(-y))$$

which we also denote by ψ. To prove the integrability of $\mathcal{H}$, by Frobenius' theorem it will suffice to show that Ψ is zero. In what follows we will denote $\ell_{(y,F)}$ and $H_{(y,F)}$ respectively by ℓ and H. Now, choose coordinates on $\mathbf{P}^n$ such that $\ell = \{X_2 = \ldots = X_n = 0\}$ and $y = [1, 0, \ldots, 0]$. Recall that $H^0(N_{\ell/\mathbf{P}^n}(-1))$ identifies naturally with the set of (-1)-graded homomorphisms from $\bigoplus_d I_\ell(d)/I^2_\ell(d)$ to $\bigoplus_d S^d/I_\ell(d)$. Hence there is a natural bilinear map, denoted by $(a, b) \mapsto a \cdot b$:

$$I_\ell(d) \otimes H^0(N_{\ell/\mathbf{P}^n}(-y)) \to H^0(\mathcal{O}_\ell(d)(-y)),$$

which is explicitly given by

$$P \cdot (X_1 \sum_{i=2}^n b_i \frac{\partial}{\partial X_i}) = \sum_{i=2}^n b_i X_1 (\frac{\partial P}{\partial X_i})_{|\ell} \in H^0(\mathcal{O}_\ell(d)(-y)).$$

Remark that, since $y \in \ell$ and $T^{vert}_{\mathcal{Y},(y,F)} \subset S^d_y$, any deformation of ℓ belonging to $\phi_*(T^{vert}_{\mathcal{Y},(y,F)})$ passes through y, i.e. $\phi_*(T^{vert}_{\mathcal{Y},(y,F)}) \subset H^0(N_{\ell/\mathbf{P}^n}(-y))$. A verification in local coordinates shows that

$$\psi(A \wedge B) = A \cdot \phi_*(B) - B \cdot \phi_*(A), \; A, B \in H. \tag{1.21}$$

Note that

$$(QX_iX_j) \cdot (\sum_2^n b_l \frac{\partial}{\partial X_l}) = \sum_2^n b_l (\frac{\partial Q X_i X_j}{\partial X_l})_{|\ell} = 0,$$

for every $Q \in S^{d-2}$ and $i, j \geq 2$, and therefore $\phi_*(A) \cdot B = 0$, for every $A \in H \cap I^2_\ell(d)$, $B \in H$. We first show that ϕ_* vanishes on $I^2_\ell(d)$: if we had $\phi_*(A) \neq 0$ with $A \in H \cap I^2_\ell(d)$, then $T^{vert}_{\mathcal{Y},(y,F)} mod\; I_\ell(d)$ would contain the elements $B \cdot \phi_*(A)$ for any $B \in H$, hence at least a hyperplane of $H^0(\mathcal{O}_\ell(d)(-y))$. Thus ϕ_* vanishes on $H \cap I^2_\ell(d)$, giving a map $H/H \cap I^2_\ell(d) \to H^0(N_{\ell/\mathbf{P}^n}(-y))$, which we still call ϕ_*.
Identify $H^0(\mathcal{O}_\ell(d)(-y))$ with $H^0(\mathcal{O}_\ell(d-1))$, and recall again the natural isomorphism

$$I_\ell(d)/I^2_\ell(d) \cong H^0(\mathcal{O}_\ell(d-1)) \otimes H^0(N_{\ell/\mathbf{P}^n}(-y))^*.$$

Then $H/H \cap I^2_\ell(d)$ corresponds to a subspace

$$\overline{H} \subset H^0(\mathcal{O}_\ell(d-1)) \otimes H^0(N_{\ell/\mathbf{P}^n}(-y))^*,$$

with *codim* $\overline{H} \leq 1$, and the dot map is simply given by the contraction

$$< \cdot, \cdot > : \overline{H} \otimes H^0(N_{\ell/\mathbf{P}^n}(-y)) \to H^0(\mathcal{O}_\ell(d)(-y))$$

Hence, by (1.21) the map $\overline{\psi} : \bigwedge^2 H/H \cap I^2_\ell(d) \to H^0(\mathcal{O}_\ell(d)(-y))$ identifies with

$$\begin{aligned} &\bigwedge^2 \overline{H} \longrightarrow H^0(\mathcal{O}_\ell(d)(-y)) \\ &A \wedge B \mapsto <A, \phi_*(B)> - <B, \phi_*(A)> . \end{aligned} \tag{1.22}$$

To conclude we need the following linear algebra result:

Lemma 1.3.4. *Let W and K be two vector spaces, $\overline{H}$ a codimension 1 subspace of in $W \otimes K^*$ and $\phi_* : \overline{H} \to K$ a linear map. If $\phi_* \neq 0$, then the image of the map*

$$\begin{aligned} \overline{\psi} \;\; : \;\; &\bigwedge^2 \overline{H} \longrightarrow W \\ &A \wedge B \mapsto < A, \phi_*(B) > - < B, \phi_*(A) > \end{aligned}$$

contains at least a codimension 2 subspace of W.

Proof. Let $J = \overline{\psi}(\wedge^2 \overline{H})$. Pick a complementary subspace $J^\perp$ to J in W and a basis $\{w_j\}$ for W which is compatible with the decomposition

$$W = J \oplus J^\perp.$$

Let $\{k_i\}$ be a basis of K and $\{k_i^*\}$ the dual one. Pick a complementary space $\overline{H}^\perp$ to $\overline{H}$ in $W \otimes K^*$ which will be generated by a monomial $w_{j_0} \otimes k^*_{i_0}$, and extend ϕ_* to the whole $W \otimes K^*$ by setting

$$\phi_*(w_{j_0} \otimes k^*_{i_0}) = 0.$$

The map $\overline{\psi}$ extends naturally to $\wedge^2(W \otimes K^*)$. Since

$$\dim \left(\psi \left(\overline{H}^\perp \otimes (W \otimes K^*) \right) \right) \leq 1,$$

we are reduced to proving that if the extended map $\phi_* : W \to K$ is not zero, then the codimension of the image of

$$\overline{\psi} \ : \ \bigwedge^2 W \otimes K^* \to W$$

is at least 1. This has already been checked in [V3], Lemma 3, and so we are done. □

Take $W := H^0(\mathcal{O}_\ell(d)(-y))$ and $K := H^0(N_{\ell/\mathbf{P}^n}(-y))$, and apply Lemma 1.3.4 to our situation. If we had $\phi_* \neq 0$, then the image of the map $\overline{\psi}$ would contain at least a codimension 2 subspace of $H^0(\mathcal{O}_\ell(d)(-y))$. But the image of $\overline{\psi}$ is contained in $T^{vert}_{\mathcal{Y},(y,F)}$ *mod* $I_\ell(d)$, and $T^{vert}_{\mathcal{Y},(y,F)}$ contains a hyperplane in $I_\ell(d)$. Hence the codimension of $T^{vert}_{\mathcal{Y}}$ in $T^{vert}_{\mathcal{X}}$ would be at most 3, which is in contradiction with Lemma 1.2.2. Thus $\phi_* = 0$, hence ψ is zero and so is Ψ. By Frobenius' theorem the distribution $\mathcal{H}$ is integrable. □

1.3.3 Proof of Proposition A

We can now prove Proposition A, i.e. we show that the line $\ell_{(y,F)}$ defined by the ideal $I_{\ell_{(y,F)}}(d)$ is such that

$$X_F \cap \ell_{(y,F)} = d \cdot y.$$

From (1.3.3) we know that $\mathcal{H}$ is integrable and ϕ is constant along the leaves of the corresponding foliation. Therefore the line $\ell_{(y,F)}$ is fixed along the leaf, and because its tangent space is contained at each point in $I_{\ell_{(y,F)}}(d)$, it follows that the restriction $G_{|\ell_{(y,F)}}$ is constant, for any polynomial G belonging to the leaf through (y, F). This means that the leaf is locally of the form $y \times F + W_{(y,F)}$, where $W_{(y,F)} \subset I_{\ell_{(y,F)}}(d)$ is a germ of a complex hypersurface. Then consider the following diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_{\ell_{(y,F)}} & \longrightarrow & I_{\ell_{(y,F)}}(d) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^{vert}_{\mathcal{Y},(y,F)} & \longrightarrow & T^{vert}_{\mathcal{X},(y,F)} = S^d_y \\ & & \downarrow & & \downarrow \\ & & H^0(\mathcal{O}_\ell(d)(-y)) & = & H^0(\mathcal{O}_\ell(d)(-y)) \\ & & & & \downarrow \\ & & & & 0 \end{array} \tag{1.23}$$

By Lemma 1.2.1,

$$codim\ _{T^{vert}_{\mathcal{X},(y,F)}}\ T^{vert}_{\mathcal{Y},(y,F)} = codim\ _{\mathcal{X}}\ \mathcal{Y} = n - k - 1.$$

Then, by (1.23) the image

$$Im := \ Im\ (T^{vert}_{\mathcal{Y},(y,F)} \to H^0(\mathcal{O}_\ell(d)(-y)))$$

has codimension $(n-k-1)-1 = n-k-2$, and therefore, since $d = 2n-2-k$

$$dim\ Im\ = (2n - 2 - k) - (n - k - 2) = n. \tag{1.24}$$

At the same time, again by Lemma 1.2.1, $T^{vert}_{\mathcal{Y},(y,F)}$ contains $S^1_y \cdot J^{d-1}_F$ and F itself. Take coordinates $X_0, \dots, X_n$ on $\mathbf{P}^n$ such that $y = [1, 0, \dots, 0]$, and $\ell := \ell_{(y,F)} = \{X_2 = \dots = X_n = 0\}$. Since ϕ is constant along the leaves of the foliation, we can generically choose a polynomial G in $F + W_{(y,F)}$, so that the $(n-1)$-elements $X_1 \frac{\partial G}{\partial X_i}$, $i \geq 2$, are generic in a hypersurface. Consider the subspace

$$K :=< G_{|\ell}, X_1(\frac{\partial G}{\partial X_0})_{|\ell}, X_1(\frac{\partial G}{\partial X_1})_{|\ell} > \subset H^0(\ell, \mathcal{O}_\ell(d) \otimes \mathcal{I}_y),$$

which is uniquely determined by $F_{|\ell}$ and hence is constant along the leaf. Its codimension in $H^0(\ell, \mathcal{O}_\ell(d) \otimes \mathcal{I}_y)$ is at least $d - 3 \geq n$ (since, by hypothesis,

$k \leq n-5$ so that $d = 2n-2-k \geq n+3$). Since we know that along the leaf, G moves freely in the complex hypersurface $W_{(y,F)}$, the polynomials $X_1(\frac{\partial G}{\partial X_i})_{|\ell}$ are generic in a codimension 1 subspace of $H^0(\ell, \mathcal{O}_\ell(d) \otimes \mathcal{I}_y)$, and it then follows that they will be generically independent modulo K. From (1.24) we thus get that

$$dim\ K \leq 1 \tag{1.25}$$

and so Proposition A is proved, since by (1.25) $F_{|\ell}$ has to be of the form αX_1^d.

If, for generic (y,F), $T^{vert}_{\mathcal{Y},(y,F)}$ contains the whole ideal $I_{\ell_{(y,F)}}(d)$, then consider the distribution $\mathcal{I} \subset T^{vert}_{\mathcal{X}}$ pointwise defined by $I_{\ell_{(y,F)}}(d)$. Arguing as we did before, we get

$$dim\ Im\ (T^{vert}_{\mathcal{Y},(y,F)} \to H^0(\mathcal{O}_\ell(d)(-y))) = n-1,$$

thus deducing

$$dim\ K = 0.$$

Then the polynomial F belongs to $I_{\ell_{(y,F)}}(d)$, and the theorem is true in this case, i.e. Y_F is a component of the subvariety of X_F covered by lines. □

1.4 The geometry of $\Delta_{d,F}$

Let $X_F \subset \mathbf{P}^n$ be a general hypersurface of degree $d = 2n-2-k,\ 1 \leq k \leq n-5$, and $Y_F \subset X_F$ a k-dimensional subvariety whose desingularization $\tilde{Y}$ has $h^0(\tilde{Y}, K_{\tilde{Y}}) = 0$. Then, by Proposition A, we know that Y_F has to be contained in $\Delta_{d,F} \subset X_F$, the $(k+1)$-dimensional subvariety of points in X_F through which there is a d-osculating line. To prove Proposition B and hence our theorem, we have then to show that the only subvariety of dimension k of $\Delta_{d,F}$ with geometric genus zero is the subvariety covered by the lines in X_F.

1.4.1 A desingularization of $\Delta_{d,F}$

We start by giving an explicit description of a desingularization $\tilde{\Delta}_{d,F}$ of $\Delta_{d,F}$ in terms of the zero locus of a section of a vector bundle. This fact will allow us to calculate, by adjunction, the canonical bundle of $\tilde{\Delta}_{d,F}$ and see that it is very ample.

Let $G := Gr(1,n)$ be the Grassmannian of lines in $\mathbf{P}^n$. Let $\mathcal{O}_G(1)$ be the line bundle on G which gives the Plücker embedding, so that we have $H^0(\mathcal{O}_G(1)) = \bigwedge^2 S^1$. Let $\mathcal{P} \subset \mathbf{P}^n \times G$ be the incidence variety $\{(x,[\ell]) : x \in \ell\}$ with projections

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{p} & \mathbf{P}^n \\ \downarrow q & & \\ G & & \end{array} \tag{1.26}$$

and $H := p^*\mathcal{O}_{\mathbf{P}^n}(1)$, $L := q^*\mathcal{O}_G(1)$ the line bundles generating the Picard group of $\mathcal{P}$.

Define

$$\tilde{\Delta}_r := \{(x,[\ell],F) \ : \ \ell \cdot X_F \geq r \cdot x\} \subset \mathcal{P} \times S^r \subset \mathbf{P}^n \times G \times S^r,$$

(by $\ell \cdot X_F \geq r \cdot x$ we mean that the line has a contact of order at least r with X_F at x) and consider the various projections as illustrated in the following commutative diagram:

$$\begin{array}{ccccc} \tilde{\Delta}_r & \xrightarrow{\rho_r} & \mathcal{X} & \xrightarrow{s} & S^r \\ \downarrow \pi & & \downarrow t & & \\ \mathcal{P} & \xrightarrow{p} & \mathbf{P}^n & & \\ \downarrow q & & & & \\ G & & & & \end{array} \tag{1.27}$$

Since the tangency of order at least r imposes r conditions, the fibres of the projection $\pi : \tilde{\Delta}_r \to \mathcal{P}$ are punctured vector spaces of dimension $N-r$. Hence $\tilde{\Delta}_r$ is smooth and irreducible of dimension

$$N - r + 2(n-1) + 1.$$

Lemma 1.4.1. *(i) The projection $\rho_r : \tilde{\Delta}_r \to \mathcal{X}$ is surjective for $r \leq n$, and generically injective for $r > n$.*

(ii) The composite projection $s \circ \rho_r \ : \tilde{\Delta}_r \to S^r$ is surjective if $r \leq 2(n-1)$. In particular, in that case, its fiber $\tilde{\Delta}_{r,F} := s \circ \rho_r^{-1}(F)$ is smooth for generic $F \in S^r$, and the composite projection $t \circ \rho_r : \tilde{\Delta}_{r,F} \to \mathbf{P}^n$, gives a desingularization of

$$\Delta_{r,F} := \{x \in X_F \ : \ \exists \ell \ s.t. \ \ell \cdot X_F \geq r \cdot x\}$$

Proof. (i) Assume $x = [1, 0, \ldots, 0] \in Proj\,(\mathbb{C}\,[X_0, \ldots, X_n])$. Then the assertion follows from the fact that the contact condition $\ell \cdot X_F \geq r$ for a line ℓ through x with respect to

$$F = \sum\nolimits_{j=1}^{n} X_0^{d-j} F_j\,(X_1, \ldots, X_n)$$

becomes

$$\{F_1 = \ldots = F_{r-1} = 0\} \subseteq Proj\,(\mathbb{C}\,[X_1, \ldots, X_n])\,.$$

(ii) A dimension count shows that all hypersurfaces X_F in $\mathbf{P}^n$ of degree $d \leq 2n-2$ admit a point through which passes a line having contact with X_F of maximal order. □

In what follows, by abuse of notation, we will identify $\tilde{\Delta}_{r,F}$ to its image, $\pi(\tilde{\Delta}_{r,F})$, in $\mathcal{P}$. We will show that $\tilde{\Delta}_{r,F}$ can be seen as the zero locus of a global section of a vector bundle over $\mathcal{P}$. This will enable us to compute its canonical bundle.

Let $\mathcal{E}_d$ be the d^{th}-symmetric power of the dual of the tautological subbundle on G, and recall that, by definition, its fibre at a point $[\ell]$ is then given by $H^0(\ell, \mathcal{O}_\ell(d))$, and its first Chern class is

$$c_1(\mathcal{E}_d) = \mathcal{O}_G(\frac{d(d+1)}{2}).$$

Let $\mathcal{L}_d := dL - dH$ be the rank 1 subbundle of $q^*\mathcal{E}_d$. Note that its fibre $\mathcal{L}_{d,(x,[\ell])}$ is equal to the space of degree d homogeneous polynomials on ℓ vanishing to the order d at x. Finally, let $\mathcal{F}_d$ be the quotient

$$0 \to \mathcal{L}_d \to q^*\mathcal{E}_d \to \mathcal{F}_d \to 0. \tag{1.28}$$

It is possible to associate to every $F \in S^d$ a section $\sigma_F \in H^0(G, \mathcal{E}_d)$, whose value at a point $[\ell]$ is exactly the polynomial $F_{|\ell}$. We will denote by $\overline{\sigma}_F$ the induced section in $H^0(\mathcal{P}, \mathcal{F}_d)$. Then $V(\overline{\sigma}_F)$ is equal to $\tilde{\Delta}_{r,F}$, and, as checked in Lemma 1.4.1, for generic F, $\tilde{\Delta}_{r,F} = V(\overline{\sigma}_F) \stackrel{p}{\longrightarrow} \Delta_{r,F}$ is a desingularization. It is computed in [V3] that

$$K_{\mathcal{P}} = -2H - nL, \tag{1.29}$$

and then

$$det\ \mathcal{F}_d = det\ q^*\mathcal{E}_d - dL + dH = \frac{d(d-1)}{2}L + dH. \tag{1.30}$$

Hence, by adjunction,

$$K_{\tilde{\Delta}_{d,F}} = (K_{\mathcal{P}} \otimes det\ \mathcal{F}_d)_{|\tilde{\Delta}_{d,F}} = [(d-2)H + (\frac{d(d-1)}{2} - n)L]_{|\tilde{\Delta}_{d,F}}, \quad (1.31)$$

so $K_{\tilde{\Delta}_{d,F}}$ is very ample under our numerical hypothesis $d = 2n-2-k,\ 1 \leq k \leq n-5$.

1.4.2 Proof of Proposition B

Let $\Delta_d \subset \mathbf{P}^n \times S^d$ be the family of the $\Delta_{d,F}$'s, and $\tilde{\Delta}_d \subset \mathcal{P} \times S^d$ the family of the desingularizations. Let $\mathcal{Y} \subset \tilde{\Delta}_d$ be a subscheme of relative dimension k, invariant under the action of $GL(n+1)$, and $\tilde{\mathcal{Y}} \to \mathcal{Y}$ a desingularization. Assume $h^0(\tilde{Y}_F, K_{\tilde{Y}_F}) = 0$. Recall the isomorphisms

$$T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}} \cong \Omega^{N+k}_{\tilde{\Delta}_d\ |\tilde{\Delta}_{d,F}} \quad (1.32)$$

$$\Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F} \quad (1.33)$$

and consider the natural map

$$T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}} \cong \Omega^{N+k}_{\tilde{\Delta}_d\ |\tilde{\Delta}_{d,F}} \to \Omega^{N+k}_{\tilde{\mathcal{Y}}\ |\tilde{Y}_F} \cong K_{\tilde{Y}_F}. \quad (1.34)$$

Then, by assumption, the induced map in cohomology

$$H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) \to H^0(K_{\tilde{Y}_F}) \quad (1.35)$$

is zero. Let $T^{vert}_{\tilde{\Delta}_d}$ be the sheaf defined by

$$0 \to T^{vert}_{\tilde{\Delta}_d} \to T\tilde{\Delta}_d \xrightarrow{\pi_*} T\mathcal{P} \to 0.$$

Using the positivity result proved in Lemma 1.2.2, (ii), we will construct a subbundle of $(T^{vert}_{\tilde{\Delta}_d})_{|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$, generated by its global sections. This will allow us to show that any point $(y, [\ell], F) \in \mathcal{Y}$ is such that $y \in \ell \subset X_F$. Comparing the dimension, we will thus obtain that Y_F has to be a component of the subvariety of lines in X_F.

From (1.35) we see that, at a smooth point $(y, [\ell], F) \in \mathcal{Y} \subset \mathcal{P} \times S^d$, the tangent space $T_{\mathcal{Y},(y,[\ell],F)}$ is in the base locus of $H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})$, considered as the space of sections of a line bundle on the Grassmannian of hyperplanes in $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}}$. Consider the vector bundle M^d_G on $G := Gr(1,n)$ defined by the short exact sequence:

$$0 \to M^d_G \to S^d \otimes \mathcal{O}_G \to \mathcal{E}_d \to 0.$$

Notice that the fiber of M^d_G at a point $[\ell]$ is equal to $I_\ell(d)$, and recall that, by Proposition 1.2.2, (ii), $M^d_G \otimes \mathcal{O}_G(1)$ is generated by its global sections.

Then it follows that the vector bundle $q^*M^d_G \otimes det\mathcal{F}_d \otimes K_{\mathcal{P}}$, that, by (1.29), (1.30) and (1.73) can be written as

$$\begin{aligned} q^*M^d_G \otimes det\mathcal{F}_d \otimes K_{\mathcal{P}} &= q^*(M^d_G) \otimes \mathcal{O}_{\mathcal{P}}((d-2)H + (\frac{d(d-1)}{2} - n)L) \\ &= q^*(M^d_G(1)) \otimes \mathcal{O}_{\mathcal{P}}((d-2)H + (\frac{d(d-1)}{2} - n - 1)L), \end{aligned}$$

is generated by its global sections, and so is its restriction to $\tilde{\Delta}_{d,F}$, i.e. the sheaf

$$q^*M^d_{G|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$$

is generated by its global sections.

Let $\mathcal{N}_d$ be the vector bundle over $\mathcal{P}$ defined by the exact sequence

$$0 \to \mathcal{N}_d \to S^d \otimes \mathcal{O}_{\mathcal{P}} \to \mathcal{F}_d \to 0. \tag{1.36}$$

We have

$$0 \to \mathcal{N}_{d|\tilde{\Delta}_{d,F}} \to T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \xrightarrow{\pi_*} T\mathcal{P}_{|\tilde{\Delta}_{d,F}} \to 0,$$

where $\mathcal{S} \subset \mathcal{P} \times S^d \xrightarrow{\pi} \mathcal{P}$ is the projection on the first component, i.e. $\mathcal{N}_{d|\tilde{\Delta}_{d,F}}$ is the vertical component of $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}}$ w.r.t. π. Now consider the vector bundle $\mathcal{M}^d_G$ defined by the exact sequence

$$0 \to \mathcal{M}^d_G \to S^d \otimes \mathcal{O}_{\mathcal{P}} \xrightarrow{ev} q^*\mathcal{E}_d \to 0,$$

whose fiber at a point $(y, F, [\ell])$ is equal to $I_\ell(d)$. From (1.28) and the definition of $\mathcal{N}_d$ we also obtain that

$$0 \to \mathcal{M}^d_G \to \mathcal{N}_d \to \mathcal{L}_d \to 0.$$

So, $\mathcal{M}^d_G \otimes K_{\tilde{\Delta}_{d,F}}$ is a subbundle of $T\tilde{\Delta}^{vert}_{d|\tilde{\Delta}_{d,F}}$. Finally note that $\mathcal{M}^d_G = q^*M^d_G$, hence

$$\mathcal{M}^d_G \otimes K_{\tilde{\Delta}_{d,F}}$$

is generated by its global sections. Using this property of the bundle $\mathcal{M}^d_G \otimes K_{\tilde{\Delta}_{d,F}} \subset T\tilde{\Delta}^{vert}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}$ we are now able to conclude our proof.

Proof of Proposition B. Let $H \subset T_{\tilde{\Delta}_{d,(x,\Delta,F)}}$ be a hyperplane contained in the base locus of $H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})$, considered as the space of sections of a line bundle on the Grassmannian of codimension 1 subspaces of $T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}}$. Then we must have

$$H^{vert} := H \cap \mathcal{N}_{d,(x,[\ell])} = \mathcal{M}^d_{G,(x,\ell)}. \tag{1.37}$$

Indeed, if $\bar{H} := H \cap \mathcal{M}^d_{G,(x,[\ell])}$ were strictly contained in $\mathcal{M}^d_{G,(x,[\ell])}$, then consider the following, well defined, commutative diagram:

$$\begin{array}{ccccc} H^0(\mathcal{M}^d_G \otimes K_{\tilde{\Delta}_{d,F}}) & \xrightarrow{ev} & (\mathcal{M}^d_G \otimes K_{\tilde{\Delta}_{d,F}})_{(x,[\ell])} & \xrightarrow{<\cdot,\bar{H}>} & \mathbb{C} \\ \downarrow & & \downarrow & & \| \\ H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) & \xrightarrow{ev} & (T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})_{(x,[\ell])} & \xrightarrow{<\cdot,H>} & \mathbb{C} \end{array} \tag{1.38}$$

(ev is the evaluation of the sections at the point $(x,[\ell])$, and $< \cdot, H >$ is the contraction defined by the hyperplane H). Since H belongs to the base locus of $H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}})$, then the composite map $< \cdot, H > \circ\, ev$ is zero, and so would be $< \cdot, \bar{H} > \circ\, ev$. But this is absurd, because $\mathcal{M}^d_G \otimes K_{\tilde{\Delta}_{d,F}}$ is generated by its global sections.

Let then $\mathcal{Y} \subset \Delta_d$ be a subvariety, which is stable under the action of $GL(n+1)$ and of relative codimension 1. Assume moreover that the restriction map (1.35)

$$H^0(T\tilde{\Delta}_{d|\tilde{\Delta}_{d,F}} \otimes K_{\tilde{\Delta}_{d,F}}) \to H^0(K_{\tilde{Y}_F})$$

is zero. By (1.37), $T^{vert}_{\mathcal{Y},(y,[\ell],F)}$ is equal to

$$\mathcal{M}_{d,(y,[\ell])} = \{G \in S^d : G_{|\ell} = 0\}. \tag{1.39}$$

On the other hand, by Lemma 1.2.1, (ii), $T^{vert}_{\mathcal{Y},(y,[\ell],F)}$ contains F itself. So by (1.39) we have that $F_{|\ell} = 0$ for every point $(y,[\ell]) \in Y_F$, i.e. Y_F is a component of the subvariety covered by the lines contained in X_F. □

Remark 1.4.2. *If $k > 1$, the k-dimensional subvariety covered by the lines of the general hypersurface of degree $d = 2n-2-k$ is irreducible (see [DM]), so in this case Y_F has to coincide with it.*

Bibliography of Part I

[B] R. Bott, *Homogeneous vector bundles*, Ann. of Math. **66** (1957), 203–248.

[Ca] F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 539–545.

[C1] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, IHES Publ. Math. **58** (1983), 19–38.

[C2] H. Clemens, *The infinitesimal Abel-Jacobi mapping for hypersurfaces*, The Lefschetz centennial conference, Part I (Mexico City, 1984), Contemp. Math. **58**, Amer. Math. Soc., Providence, RI, 1983, 81–89.

[C3] H. Clemens, *Curves in generic hypersurfaces*, Ann. Sci. École Norm. Sup. **19** (1986), 629–636.

[CG] H. Clemens, P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95** (1972), 281–356.

[CKM] H. Clemens, J. Kollár, S. Mori, *Higher dimensional complex geometry*, Astérisque **166** (1988).

[CL] L. Chiantini, A. F. Lopez, *Focal loci of families and the genus of curves on surfaces*, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3451–3459.

[CLR] L. Chiantini, A. F. Lopez and Z. Ran, *Subvarieties of generic hypersurfaces in any variety*, Math. Proc. Cambr. Phil. Soc., to appear.

[CR] M. C. Chang, Z. Ran *Divisors on some generic hypersurfaces*, J. of Diff. Geometry **38** (1993), 671–678.

[DM] O. Debarre and L. Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. **312** (1998), 549–574.

[E1] L. Ein, *Subvarieties of generic complete intersections*, Invent. Math. **94** (1988), 163–169.

[E2] L. Ein, *Subvarieties of generic complete intersections II*, Math. Ann. **289** (1991), 465–471.

[Gre] M. Green, *Koszul cohomology and the geometry of projective varieties II*, J. of Diff. Geometry **20** (1984), 279–289.

[Gri] P. A. Griffiths, *On the periods of certain rational integrals, I and II*, Ann. of Math. **90** (1969), 460–541.

[JK] T. Johnsen, S. Kleiman, *Rational curves of degree at most* 9 *on a general quintic threefold*, Comm. Algebra **24** (1996), no. 8, 2721–2753.

[K] J. Kollár, *Rational curves on algebraic varieties*, Springer (1996).

[KMM] J. Kollár, Y. Miyaoka, S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no. 3, 765–779.

[Mo] S. Mori, *Threefolds whose canonical bundles are not effective*, Ann. of Math. **116** (1979), 593–606.

[Mum1] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J. 1966.

[Mum2] D. Mumford, *Rational equivalence of* 0*-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1968), 195–204.

[R] Z. Ran, *Beyond a conjecture of Clemens*, preprint math.AG/9911161

[V1] C. Voisin, *Variations de structure de Hodge et zéro-cycles sur les surfaces générales*, Math. Ann. **299** (1994), 77–103.

[V2] C. Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. of Diff. Geometry **44** (1996), 200–214.

[V3] C. Voisin, *A correction on "A conjecture of Clemens on rational curves on hypersurfaces"*, J. of Diff. Geometry **49** (1998), 601–611.

[V4] C. Voisin, *Symétrie miroir*, Panoramas et Synthèses, **2**, Société Mathématique de France, Paris, 1996.

[X] G. Xu, *Subvarieties of general hypersurfaces in projective space*, J. of Differential Geom. **39** (1994), no. 1, 139–172.

Part II

On the nef cone of symmetric products of a generic curve

Introduction to Part II

Let C be a smooth irreducible algebraic curve of genus g. For any integer $k \geq 2$, one can consider $C^{(k)}$, the symmetric product of C, defined as the quotient of the k-th cartesian product of C under the natural action of the symmetric group $\mathfrak{S}_k$. This smooth, irreducible, k-dimensional variety, naturally associated to C, parametrizes effective divisors $E = x_1 + x_2 + \ldots + x_k$ of degree k on C, and plays a central role in curve theory. Its geometry has been extensively investigated, although the problem of determining the nef and the effective cone of $C^{(k)}$ has only recently found more attention, thanks to two papers by Kouvidakis [K] and Ciliberto-Kouvidakis [CK]. This interesting question lies on the borderline between higher dimensional algebraic geometry and curve theory. Indeed, on one hand, for $g(C) \geq 2$, one can be motivated by the possibility of providing the description of the nef and effective cones for a new class of higher dimensional varieties of general type. On the other hand, the study of curves on $C^{(k)}$, seen as continuous families of linear systems on C, can be considered as a natural generalization of the classical Brill-Noether theory, and have some import in the theory of algebraic curves. In this introduction, after having recalled some well known facts about $C^{(k)}$, we review Ciliberto and Kouvidakis' work on the effective cone of $C^{(2)}$. Then, we present our result which gives a complete description of the nef and the effective cone of the k-th symmetric product of a generic curve C, for $k = gon(C) - 1$, in the even genus case.

0.1 Classical results on $C^{(k)}$

The generators of the integer cohomology of $C^{(k)}$ and the relations among them are exhibited by Macdonald in [Mac], where he also computes the Chern classes of $C^{(k)}$, while Mattuck [Mat] and Collino [Co] determine the structure of the Chow ring $A(C^{(k)})$. The principle asserting that the geometry of $C^{(k)}$ reflects that of C is well testified by a famous result by Kempf [Ke], subsequently sharpened by Fantechi [F], saying that the deformation functors for C, with $g(C) \geq 3$, and for any symmetric product $C^{(k)}$ are isomorphic if and only if C is not hyperelliptic.

A key role in understanding the relation of the symmetric product with the extrinsic geometry of the curve C is played by Abel's map

$$u_k \ : C^{(k)} \to Pic^k(C), \ E = x_1 + \ldots + x_k \mapsto \mathcal{O}_C(E).$$

By Riemann's theorem, $u_{g-1}(C^{(g-1)})$ identifies to a translate of the theta divisor, while, for $k \geq 2g-1$, the map u_k endows, by Riemann-Roch, the symmetric product $C^{(k)}$ with the structure of a projective bundle. Recall the definition of the subvariety of $C^{(k)}$ (respectively $Pic^k(C)$):

$$C_k^r := \{E \in C^{(k)} : dim|E| \geq r\}$$

$$(resp. \ \ W_k^r := \{L \in Pic^k(C) : h^0(L) \geq r+1\}).$$

Then $u(C_k^r) = W_k^r$, and Brill-Noether theory (see [ACGH]) says that, for a generic curve, these varieties are non-empty if and only if $\rho \geq 0$, and

$$dim \ C_k^r = \rho + r, \ \ dim \ W_k^r = \rho,$$

where $\rho = \rho(g,r,k) := g - (r+1)(g-k+r)$ is the Brill-Noether number. Recall the definition of the gonality of a curve C

$$gon(C) := min\{d : \exists \ \ a \ covering \ C \xrightarrow{d:1} \mathbf{P}^1\}.$$

Hence, if $g(C) = 2k$ then $gon(C) = k+1$.

More recently other properties of the symmetric product have been investigated. Bertram and Thaddeus explore in [BT] the (little) quantum cohomology of $C^{(k)}$ for most k. Shiffmann and Zaidenberg ([SZ]) prove that the k-th symmetric product of a generic curve is hyperbolic for $k \leq gon(C) - 1$. They use this fact, plus Kouvidakis' result on the nef cone of $C^{(2)}$ that will be discussed in the next section, to provide new examples of hyperbolic surfaces of low degree in $\mathbf{P}^3$.

0.2 Known results on the nef and the effective cone of $C^{(2)}$

For any variety X one defines $N_1(X)$ (resp. $N^1(X)$) to be the set of $\mathbb{R}$-linear combinations of 1-cycles (resp. divisors) on X modulo numerical equivalence. $N^1(X)$ is called the Néron-Severi group of X and denoted $NS(X)$. $N_1(X)$ and $N^1(X)$ are finite dimensional vector spaces, and the bilinear form given by the intersection pairing is non-degenerate. The convex cone of $N_1(X)$ given by the classes of effective 1-cycles is denoted by $NE(X)$ and called the *cone of curves*. Dually, one can consider, inside $N^1(X)$, the *nef cone* $Nef(X)$ of X, *i.e.* the closure of the convex cone of classes of ample divisors. By Kleiman's criterion of ampleness ([Kl]), $Nef(X) = \overline{NE(X)^*}$.

When the curve has general moduli, for any k the Néron-Severi group of $C^{(k)}$ has rank two (see [GH], Ch. 2, §5). If x is a point of C, by abuse of notation, we will still denote by x the divisor

$$\{x + Z : Z \in C^{(k-1)}\} \subset C^{(k)}.$$

Let Δ the class of the diagonal $\{2x_1 + x_2 + \ldots + x_{k-1}\}$. Notice that Δ is divisible by 2. To see this, consider $C^k_* \subset C^k$ the open subset of k-tuples with at most two equal entries, and denote by U its image in $C^{(k)}$. Let $\mathcal{A}_k$ be the alternating subgroup of $\mathfrak{S}_k$. Then

$$C^k_*/\mathcal{A}_k \to C^k_*/\mathfrak{S}_k = U$$

is a double cover ramified exactly over $\Delta \cap U$. Since $codim(C^{(k)} \backslash U) = 2$, this proves that Δ is divisible by 2.

The classes of x and $\Delta/2$ provide a basis for the Néron-Severi group, and we will denote by $D_{n,\gamma}$ the divisors $(n + \gamma) \cdot x - \gamma \cdot \Delta/2$.

Coming back to $C^{(2)}$, recall the intersection numbers

$$x^2 = 1,\ x \cdot \Delta = 2,\ \Delta^2 = 2 \cdot degT_C = 4 - 4g.$$

Then $D_{n,\gamma} \cdot D_{n',\gamma'} = nn' - g\gamma\gamma'$ and so

$$D^2_{n,\gamma} = n^2 - g\gamma^2. \tag{40}$$

One can consider in the $x, \frac{\Delta}{2}$-plane the *positive* cone of $C^{(2)}$, *i.e.* the cone of divisors $D \subset C^{(2)}$ with positive self-intersection $D^2 > 0$ (and such that $D.x \geq 0$). By (40) the positive cone is delimited by the rays R_+ and R_-, given respectively by the equations $n = \gamma\sqrt{g}$ and $n = -\gamma\sqrt{g}$. By Riemann-Roch, any rational ray in the positive cone is contained in the interior of the effective cone. Moreover, if D is an irreducible curve in $C^{(2)}$ such that $D^2 \leq 0$, then $NE(X)$ is spanned by the class of D and $NE(X) \cap \{D' : D'.D \geq 0\}$ and D belongs to the boundary of $NE(X)$ (see, for instance, [CKM], Lemma (4.5), p. 23). One side of the effective cone of $C^{(2)}$ is then given by the diagonal, since its self-intersection is equal to $4-4g$, which is non-positive for $g \geq 1$. Hence, in order to complete the description of the effective cone either one provides an irreducible curve D, different from the diagonal, with negative or zero self-intersection, in which case $NE(X)$ is closed on both sides and bounded by the classes of $\frac{\Delta}{2}$ and D, or one proves that such a curve doesn't exist and then the effective cone has a closed boundary generated by $\frac{\Delta}{2}$, and is open on the other side. Then, by the non-existence of a curve with non-positive self-intersection, its open boundary coincides with the boundary R_{-} of the positive cone, that has slope equal to $-\frac{1}{\sqrt{g}+1}$. Dually, the description of the nef cone goes as follows: either there exists an irreducible curve $D \subset C^{(2)}$, with $D^2 \leq 0$, and such that its numerical equivalence class c_D differs from that of the diagonal, in which case

$$Nef(C^{(2)}) = \{\alpha \in NS(C^{(2)}) : \alpha \cdot \frac{\Delta}{2} \geq 0 \textit{ and } \alpha \cdot c_D \geq 0\},$$

or such a curve does not exist and

$$Nef(C^{(2)}) = \{\alpha \in NS(C^{(2)}) : \alpha \cdot \frac{\Delta}{2} \geq 0 \textit{ and } \alpha^2 \geq 0\}.$$

If $g(C) \leq 4$, it is easy to check (see [K]) that there are curves with non-positive self-intersection, but this does not seem to be the case for higher genera. Indeed, Kouvidakis proved in [K] that, if $g(C) = m^2$ and $m \geq 3$, the diagonal is the unique irreducible curve with non-positive self-intersection, *i.e.* the effective cone of $C^{(2)}$ has an open boundary given by R_{-}. This phenomenon has been interpreted by Ciliberto and Kouvidakis ([CK]) in terms of certain linear systems of plane curves which are *expected* to be empty. To see this, they let the curve C degenerate to a rational g-nodal curve C_0. The normalization $\mathbf{P}^1 \to C_0$ induces a birational morphism

$$\rho : \mathbf{P}^2 \cong Sym^2\mathbf{P}^1 \to Sym^2 C_0.$$

A careful analysis of the degeneration and of the map above together with the study of the limits of line bundles under this degeneration enable them to deduce the following: if there exists an irreducible curve $D_{n,\gamma} \subset C^{(2)}$, different from the diagonal, with non-positive self-intersection, then the inverse image $\Gamma_0 := \rho^{-1}(D_0)$ of its limit $D_0 \subset Sym^2 C_0$ is a plane curve of degree n, passing through g points in general positions with multiplicity γ.

Recall now the

Nagata conjecture. *Let g be an integer ≥ 10. There exists no plane curve of degree n passing through g points in general position in $\mathbf{P}^2$ with multiplicity γ at each point, with $n \leq \gamma \cdot \sqrt{g}$.*

Since the Nagata conjecture has been proved for a quadratic number of points (see [N]), they obtain that the effective cone of a curve of quadratic genus at least 16, in contrast to the case of curves of low genus, is closed on one side, and open on the other one. They also verify, using a supplementary argument, that the same holds for the generic curve of genus 9.

Remark that, by the above, the complete description of the effective cone of $C^{(2)}$, for a genus $g(C) \geq 10$ which is not a perfect square, would then furnish either a counterexample to Nagata's conjecture or an example of a cone that is bounded by a ray with irrational slope $-\frac{1}{\sqrt{g}+1}$.

0.3 Results of Part II

The geometry of symmetric products of an elliptic curve E has been studied by Catanese and Ciliberto in [CC]. They prove in particular that the cone of effective divisors in $E^{(r)}$, for any $r \geq 3$, has rational slope and is open on one side and closed on the other. The nef cone of the r-th symmetric product is easy to describe for elliptic curves and curves of genus 2, since they possess a g^1_2, as well as for the generic curve of genus 3 or 4, since it has a g^1_3: in these cases we can in fact apply Lemma 1.2.1 (see §1.2 below). As in the case of the second symmetric product, divisors in $C^{(r)}$ are much more difficult to study for higher genera, and there are only some bounds, due to Kouvidakis [K], for the slope of the effective cone of $C^{(r)}$ for any $r \geq 3$ and $g(C) \geq 2$.

Let δ be the small diagonal in $C^{(k)}$ defined by $\{x_1 = \ldots = x_k\}$, and c_δ its numerical equivalence class. It is not difficult to check that, on one side, the nef cone of $C^{(k)}$ is given by the condition of having non-negative intersection with the class c_δ. Moreover, if $k \geq gon(C)$, one can define a rational curve $\mathbf{P}^1 \subset C^{(k)}$, whose class $c_{\mathbf{P}^1} \in N_1(C^{(k)})$ differs from the class c_δ of the small diagonal. We first establish the following easy

Proposition. *Let C be a curve with general moduli. For any $k \geq gon(C)$*

the nef cone of $C^{(k)}$ *is given by*

$$Nef(C^{(k)}) = \{\alpha \in NS(C^{(k)}) : \alpha \cdot c_\delta \geq 0 \text{ and } \alpha \cdot c_{\mathbf{P}^1} \geq 0\}.$$

Hence, we turn our attention to the case $k = gon(C) - 1$. If the genus $g(C) = 2k$, then, by the Brill-Noether theory, we have $gon(C) = k + 1$. If L' is a pencil on C giving the gonality of the curve, then one can consider the curve

$$D := \{E \in C^{(k)} : \exists\, x \in C \ s.t.\ x + E \in |L'|\},$$

whose class c_D in the Néron-Severi group differs from c_δ. Our main result is the following

Theorem II. *Let* C *be a curve with general moduli of genus* $2k$*. The nef cone of* $C^{(k)}$ *is given by*

$$Nef(C^{(k)}) = \{\alpha \in NS(C^{(k)}) : \alpha \cdot c_\delta \geq 0 \text{ and } \alpha \cdot c_D \geq 0\}.$$

Our approach to the problem is quite different from that adopted by Ciliberto and Kouvidakis. First, we establish a criterion for certain line bundles on $C^{(k)}$, namely the Göttsche line bundles $\mathcal{G}_L$ associated to any $L \in Pic(C)$, to be nef. This criterion can be seen as a finer version of a criterion due to Göttsche himself, and is based essentially on the obvious fact that two line bundles $L_1, L_2 \in Pic(C)$ having k-global sections and the same degree have associated Göttsche line bundles on $C^{(k)}$ that are numerically equivalent (see Lemma 1.3.3). Then, for $k = gon(C) - 1$, we want to check this criterion for a certain numerical equivalence class of line bundles, which have zero intersection with the curve $D \subset C^{(k)}$ defined above. In order to do that, instead of using a degeneration argument, we specialize to the case of a curve on a $K3$ surface, as it is natural to do when dealing with linear systems on a curve. Indeed, such a curve, as shown by Lazarsfeld [L1], behaves generically from the viewpoint of Brill-Noether theory, and, even more strikingly, verifies Green's syzygy conjecture in its canonical embedding, as recently proved by Voisin [V]. We then make use of many techniques and results borrowed from these two remarkable references and conclude the proof of our theorem.

On the nef cone of symmetric products of a generic curve

1.1 Introduction

Let C be a smooth irreducible curve of genus $g \geq 1$. For any $k \geq 2$, denote by $C^{(k)}$ its k-th symmetric product that parametrizes effective degree k divisors $E = x_1 + x_2 + \ldots + x_k$ on C. Let $N_1(C^{(k)})$ (resp. $N^1(C^{(k)})$) be the set of $\mathbb{R}$-linear combinations of 1-cycles (resp. divisors) on $C^{(k)}$ modulo numerical equivalence. $N^1(C^{(k)})$ is called the Néron-Severi group of $C^{(k)}$ and denoted $NS(C^{(k)})$. $N_1(C^{(k)})$ and $N^1(C^{(k)})$ are finite dimensional vector spaces, and the bilinear form given by the intersection pairing is non-degenerate. If C has general moduli, $N_1(C^{(k)})$ and $N^1(C^{(k)})$ have rank two. An interesting problem is to try to determine, inside $N_1(C^{(k)})$, the convex cone of classes of effective 1-cycles, denoted by $NE(C^{(k)})$ and called the *cone of curves*. Dually, one can study, inside $N^1(C^{(k)})$, the *nef cone* $Nef(C^{(k)})$, *i.e.* the closure of the convex cone of classes of ample divisors. By Kleiman's criterion of ampleness ([Kl]), $Nef(C^{(k)}) = \overline{NE(C^{(k)})^*}$.

If k is at least equal to the gonality $gon(C)$ of C (recall that $gon(C) := min\{d : \exists \ \textit{a covering } C \xrightarrow{d:1} \mathbf{P}^1\}$), then $NE(C^{(k)})$ is easily determined (see §2 for the details). The first two interesting cases to look at are then provided by the second symmetric product $C^{(2)}$, and by $C^{(k)}$, with $k = gon(C) - 1$.

The former has been studied by Kouvidakis and Ciliberto ([K],[CK]). In [CK], they degenerate $C^{(2)}$ to the symmetric product of a rational g-nodal curve, and reduce the problem of determining $NE(C^{(2)})$ to the Nagata conjecture, concerning the existence of plane curves of fixed degree passing through given points in general position with prescribed multiplicities (see [N] for the precise statement). In such a way, they succeed in describing

$NE(C^{(2)})$, when the genus of C is a perfect square ≥ 9. In this case, the cone of curves turns out to have rational slope, and to be closed on one side, and open on the other one.

In this work, we focus our attention on $C^{(k)}$, for $k = gon(C)-1$. If C is a curve with general moduli and that has even genus $g = 2k$, by Brill-Noether theory (see [ACGH]) its gonality is equal to $k+1$, *i.e.* $k = gon(C) - 1$. Let δ be the small diagonal in $C^{(k)}$ defined by $\{x_1 = \ldots = x_k\}$. Let $L' \to C$ be a g^1_{k+1} on C, and consider the curve $D \subset C^{(k)}$ defined as follows:

$$D := \{E \in C^{(k)} : \exists\, x \in C \ s.t.\ x + E \in |L'|\}.$$

The class $c_D \in N_1(C^{(k)})$ of the curve D is independent from the g^1_{k+1} chosen, can be explicitly computed and is not proportional to the class $c_\delta \in N_1(C^{(k)})$ of the small diagonal. These two classes will enable us to give a description of the nef cone of $C^{(k)}$. More precisely, we will prove the following

Theorem 1.1.1. *Let C be a curve with general moduli, of genus $g = 2k$. Then the nef cone of $C^{(k)}$*

$$Nef(C^{(k)}) := \{\alpha \in N^1(C^{(k)}) : \alpha \textit{ is numerically effective}\}$$

has rational slope and is determined by the conditions

$$(i)\ \alpha \cdot c_\delta \geq 0, \quad (ii)\ \alpha \cdot c_D \geq 0.$$

The first step in the proof of the theorem will be to refine a criterion, due to Göttsche ([BS]), for certain line bundles on the punctual Hilbert scheme (in our case $C^{(k)}$) to be nef (see Lemma 1.3.3). The second key ingredient will be to specialize C to a curve on a $K3$ surface S. By [L], if $C \subset S$ generates $Pic(S)$, then C is generic in the sense of Brill-Noether theory. In the spirit of Voisin [V], we will then make use of the rich geometrical constructions introduced by Lazarsfeld [L] to check the criterion in this case.

The paper is organized as follows: we start recalling in §2 the basic definitions and proving some preliminary results; in section 3 we recall Göttsche's criterion for nefness, and give a refinement of it. Then, we put the stress on two sufficient conditions (A) and (B) for the class of a set of numerically equivalent divisors on $C^{(k)}$ to be on the boundary of the nef cone. We exhibit in §4 a set of "natural candidates" to satisfy (A) and (B), when $k = gon(C)-1$, and check they verify condition (B). In section 5, after having recalled Lazarsfeld's construction, we see, following [V], how condition

(A) can be expressed, for curves on a K3 surface, in terms of the cohomology of a certain sheaf on a Grassmannian. Finally, we prove the last technical lemma and conclude our proof.

Throughout this paper we work on the field of complex numbers $\mathbb{C}$.

Acknowledgements. I would like to thank my Ph.D. advisor, Prof. Claire Voisin, who brought the problem to my attention, and helped me throughout the preparation of this work.

1.2 Preliminaries

Let π be the projection from the cartesian product C^k to $C^{(k)}$, p_i the projection of C^k to the i-th factor, and $J(C)$ the Jacobian of C. Recall that, fixing a point $p \in C$, there are two maps defined as follows:

$$\nu_k : C^{(k)} \longrightarrow J(C),\ E \longmapsto alb(E - kp),$$

and

$$i_{k-1} : C^{(k-1)} \longrightarrow C^{(k)},\ E \longmapsto E + p.$$

On $C^{(k)}$, associated to these maps, there are three natural divisors:

- the diagonal Δ, that can be thought of as the branch locus of the finite $k! : 1$ map π, and can be shown to be divisible by 2;
- the divisor $\Theta_k := \nu_k^*\Theta$, where $\Theta \subset J(X)$ is the ample theta divisor;
- the image $i_{k-1}(C^{(k-1)})$.

If C has general moduli, its Néron-Severi group is of rank two, and any two of the above divisors provide a basis for $NS(C^{(k)})$.

For any line bundle L on C, consider the line bundle

$$L^{\boxtimes k} := p_1^*L \otimes \ldots \otimes p_k^*L$$

on the k-th cartesian product of C. By abuse of notation, when no confusion is possible, we will still denote by $L^{\boxtimes k}$ the unique line bundle on $C^{(k)}$ such that its pull-back via $\pi : C^k \to C^{(k)}$ is equal to $p_1^*L \otimes \ldots \otimes p_k^*L$. Then $i_{k-1}(C^{(k)})$ is equal to $\mathcal{O}_C(p)^{\boxtimes k}$.

One side of the nef cone can be described as follows. Consider the map from the cartesian product of C to the cartesian product of order $\binom{k}{2}$ of its Jacobian:

$$\begin{aligned} \varphi \quad &: C^k \longrightarrow J(C)^{\binom{k}{2}} \\ &(x_1, \dots, x_k) \mapsto (x_i - x_j)_{i<j}. \end{aligned}$$

This map contracts exactly the small diagonal $\delta := \{x_1 = \ldots = x_k\}$. Let π be the projection $C^k \to C^{(k)}$, and

$$p_i : J(C)^{\binom{k}{2}} \to J(C)$$

the projection on the i-th factor. Then the *unique* divisor E on $C^{(k)}$ such that

$$\pi^* E = \varphi^*(\otimes_i p_i^* \Theta) \tag{1.41}$$

is nef, and $E.\delta = 0$, since φ contracts δ. Hence the class of E belongs to the boundary of the nef cone of C_k. In other words, the existence of the contraction φ implies that one slope of the nef cone of $C^{(k)}$ is determined by the inequality

$$\alpha \cdot c_\delta \geq 0. \tag{1.42}$$

A standard way to produce curves in symmetric products of a curve is to consider, for a given g^1_h on C, $h \geq k$, the curve

$$\Gamma_k(g^1_h) := \{Z \in C^{(k)} : \ E - Z \geq 0 \ \textit{for some} \ E \in g^1_h\}. \tag{1.43}$$

Using these curves we check

Lemma 1.2.1. *If a curve C possesses a g^1_h then Θ_k is a nef but not ample divisor on $C^{(k)}$, for any $k \geq h$.*

Proof. Fix a point $p \in C$. Consider the rational curve

$$\Gamma_h(g^1_h) + (k-h) \cdot p \subset C^{(k)}.$$

This $\mathbf{P^1}$ is contracted by the morphism $\nu_k : C^{(k)} \to J(C)$, since there are no holomorphic non constant maps from $\mathbf{P^1}$ to $J(C)$. Hence $\Theta_k \subset C^{(k)}$ is numerically effective but not ample. □

The rational curve $\Gamma_h(g^1_h) + (k-h) \cdot p \subset C^{(k)}$ does not depend on the choice of the g^1_h and its class $c_{\mathbf{P^1}} \in N_1(C^{(k)})$ is not proportional to the class of the small diagonal. Hence, putting together (1.42) and Lemma 1.2.1 with $h = gon(C)$, we immediately obtain the following

Proposition 1.2.2. *Let C be a curve with general moduli. For any $k \geq gon(C)$ the nef cone of $C^{(k)}$ is determined by the conditions*

$$(i)\ \alpha \cdot c_\delta \geq 0, \quad (ii)\ \alpha \cdot c_{\mathbf{P}^1} \geq 0,$$

and its boundary rays are generated by the class of Θ_k, and the class of the divisor E defined by (1.41).

From the above, the first two interesting cases to look at are then provided by the second symmetric product $C^{(2)}$, and by $C^{(k)}$, with $k = gon(C) - 1$.

In this work we focus our attention on the case of $C^{(k)}$, for $k = gon(C) - 1$. If C has general moduli and $g(C) = 2k$ then, by the classical Brill-Noether theory, the gonality of C is equal to $k+1$, and there are a finite number of g^1_{k+1}'s, say $L'_1, \ldots, L'_s$, where s is the Castelnuovo number

$$s = \frac{g!}{(g-k)!(g-k+1)!},$$

(see [ACGH] for the computation of s). We will consider the curves D_i in $C^{(k)}$ associated to the L'_i:

$$D_i := \Gamma_k(L'_i) = \{Z \in C^{(k)} : L'_i - Z \geq 0\}, \tag{1.44}$$

and the numerically equivalent line bundles on $C^{(k)}$

$$L_i^{\boxtimes k} - \frac{\Delta}{2}, \tag{1.45}$$

where L_i are the line bundles on C defined as $L_i := K_C - L'_i$. Our goal is to prove that

$$L_i^{\boxtimes k} - \frac{\Delta}{2} \textit{ is nef} \tag{1.46}$$

and that, moreover,

$$c_1(L_i^{\boxtimes k} - \frac{\Delta}{2}).D_j = 0 \tag{1.47}$$

for any $i \neq j$. This will automatically provide the description given in Theorem 1.1.1 of the nef cone of $C^{(k)}$, and dually, of the effective cone of curves in $C^{(k)}$.

1.3 The nef cone of $C^{(k)}$: a first reduction

In the beginning of this section, following Göttsche (see his appendix to [BS]), we associate to any $L \in Pic(C)$ a line bundle $\mathcal{G}_{k,L}$ on $C^{(k)}$.

Let X be a complete algebraic variety defined over an algebraically closed field. Let $Z \subset X$ be a zero dimensional subscheme of length $k := dim\ H^0(\mathcal{O}_Z)$, defined by an ideal sheaf $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$.

Let L be an invertible sheaf over X and, for any zero dimensional subscheme $Z \subset X$, consider the restriction map

$$r_Z\ : H^0(X, L) \to H^0(L \otimes \mathcal{O}_Z).$$

Definition 1.3.1. *L is said to be k-very ample if the restriction map r_Z is surjective, for any zero dimensional subscheme $Z \subset X$ of length less than or equal to $k+1$.*

The classical notions of global generation and very ampleness correspond in this way to, respectively, 0-very ampleness and 1-very ampleness.

For any $(k-1)$-very ample line bundle L one can then define a morphism from $X^{[k]}$, the Hilbert scheme of 0-dimensional subschemes of length k of X, to the Grassmannian $G(k, H^0(X, L))$ of codimension k subspaces of $H^0(X, L)$:

$$\varphi_{k,L}\ :\ X^{[k]} \to G(k, H^0(X, L)). \tag{1.48}$$

This morphism associates to $Z \in X^{[k]}$ the kernel $H^0(X, L \otimes \mathcal{I}_Z)$ of the surjective map

$$H^0(X, L) \xrightarrow{r_Z} H^0(X, L \otimes \mathcal{O}_Z) \to 0.$$

In order to simplify the notation, we will now focus our attention on the case when $X = C$ is a curve: this is the one we need in the rest of the paper. Recall that the Hilbert scheme of points of length k on a curve C is given by the symmetric product $C^{(k)}$.

Consider the incidence variety in $C \times C^{(k)}$

$$\begin{array}{ccc} \Sigma_k & \xrightarrow{q} & C \\ \downarrow{\scriptstyle p} & & \\ C^{(k)} & & \end{array} \tag{1.49}$$

and p and q its natural projections. Let L be a line bundle on C. Consider the rank k vector bundle $\mathcal{E}_{k,L}$ over $C^{(k)}$ defined as

$$\mathcal{E}_{k,L} := p_* q^* L. \tag{1.50}$$

We give the following

Definition 1.3.2. *The line bundle*

$$\mathcal{G}_{k,L} := det\ \mathcal{E}_{k,L} \to C^{(k)}, \tag{1.51}$$

will be called the Göttsche line bundle on $C^{(k)}$ associated to L.

Let $\mathcal{O}_G(1)$ be the line bundle on the Grassmannian $G := G(k, H^0(X,L))$ giving its Plücker polarization. Let $\mathcal{S}$ be the universal subbundle over G, so that $det\ \mathcal{S}^* = \mathcal{O}_G(1)$. By definition, whenever L is $(k-1)$-very ample, we have

$$\mathcal{E}_{k,L} = \varphi^*_{k,L}\mathcal{S}^*, \tag{1.52}$$

and then the Göttsche line bundle associated to L is given by

$$\mathcal{G}_{k,L} = \varphi^*_{k,L}\mathcal{O}_G(1). \tag{1.53}$$

Göttsche, in his appendix to [BS], proved the following

Theorem (Göttsche). *For any $L \in Pic(C)$ we have*

$$L^{\boxtimes k} \otimes \mathcal{O}_{C^{(k)}}(-\frac{\Delta}{2}) = \mathcal{G}_{k,L}.$$

This theorem together with (1.53) gives immediately a sufficient condition for $L^{\boxtimes k} \otimes \mathcal{O}_{C^{(k)}}(-\frac{\Delta}{2})$ to be nef:

(Göttsche's criterion for nefness): *For any $L \in Pic(C)$, the line bundle $L^{\boxtimes k} \otimes \mathcal{O}_{C^{(k)}}(-\frac{\Delta}{2})$ over $C^{(k)}$ is nef whenever L is $(k-1)$-very ample.*

Let L be a line bundle over C, and suppose $h^0(C,L) = k$. The hypothesis $h^0(C,L) = k$ implies that $\mathcal{E}_{k,L}$ is generically globally generated. Hence the Göttsche line bundle $\mathcal{G}_{k,L} \to C^{(k)}$ is effective, since k global sections of L generating $H^0(C,L)$ will be independent at the generic set of k points $x_1, \ldots, x_k$ of C and then their wedge product gives a non zero global section of $\mathcal{G}_{k,L}$.

The k-th exterior power of the evaluation map $H^0(C,L) \otimes \mathcal{O}_C \to L$ induces a morphism (which is actually an isomorphism)

$$\bigwedge^k H^0(C,L) \to H^0(C^{(k)}, \mathcal{G}_{k,L}), \tag{1.54}$$

and we will denote by σ_L a non zero global section of $\mathcal{G}_{k,L}$ generating the image of this morphism. Notice that, by construction

$$V(\sigma_L) = \{Z \in C^{(k)} : H^0(L) \xrightarrow{r_Z} H^0(L_{|Z}) \textit{ is not surjective}\}, \qquad (1.55)$$

i.e.

> *the zeroes of σ_L are given by the k-tuples of points of C which fail to be separated by the sections of L.*

Now, we have the following

Lemma 1.3.3. *Let $L_1, \ldots, L_s$ be a finite number of line bundles of fixed degree d on C, with $h^0(C, L_i) = k$. If they verify the property*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$,

then the class in $NS(C^{(k)})$ of the numerically equivalent line bundles $\mathcal{G}_{k,L_i}$ is nef.

Proof. For any curve $X \subset C_k$, by **(A)** we can find an index i such that $\sigma_{L_i|X} \neq 0$. Then $(det\ \mathcal{E}_{L_i})_{|X}$ is effective, i.e.

$$c_1(\mathcal{G}_{k,L_i}).X \geq 0,$$

and the lemma is proved. □

Remark 1.3.4. *From (1.55) it follows immediately that condition* **(A)** *is satisfied if and only if for any (possibly coinciding) k points $x_1, \ldots, x_k \in C$, there is one of the L_i's separating them.*

Notice that if, moreover, there existed an irreducible curve $D \subset C^{(k)}$ such that

(B) $c_1(\mathcal{G}_{k,L_i}).D = 0$,

the class of $\mathcal{G}_{k,L_i}$ in $NS(C^{(k)})$ would belong to the boundary of the nef cone.

We summarize the above in the following

Proposition 1.3.5. *Let $L_1, \ldots, L_s$ be a finite number of line bundles of fixed degree d on C, with $h^0(C, L_i) = k$, and $D \subset C^{(k)}$ an irreducible curve whose class $c_D \in N_1(C^{(k)})$ is not proportional to the class c_δ of the small diagonal. If the following properties are verified*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$,

(B) $c_1(\mathcal{G}_{k,L_i}).c_D = 0$,

then

$$Nef(C^{(k)}) = \{\alpha \in NS(C^{(k)}) : \alpha \cdot c_\delta \geq 0 \text{ and } \alpha \cdot c_D \geq 0\},$$

and the boundary rays of the nef cone are generated by the class of $\mathcal{G}_{k,L_i}$ *and the class of the divisor* E *defined by (1.41).*

1.4 The case of $C^{(k)}$, for $k = gon(C) - 1$

From now on C will be a generic curve of genus $g(C) = 2k$. According to the notation introduced in §2, we denote by L'_i the g^1_{k+1}'s on C, and by L_i the line bundles

$$L_i := K_C - L'_i. \tag{1.56}$$

By Riemann-Roch, we have $h^0(C, L_i) = k$. Thus, as we saw in the previous section, we have an effective line bundle on $C^{(k)}$, given by $\mathcal{G}_{k,L_i} = L_i^{\boxtimes k} - \frac{\Delta}{2}$. Notice that the line bundles $\mathcal{G}_{k,L_i}$ are numerically equivalent since the L_i's all have the same degree.

For any L'_i, consider the irreducible curve

$$D_i := \Gamma_k(L'_i) \subset C^{(k)}$$

defined by (1.44), which is actually a copy of C in its symmetric product $C^{(k)}$. The curves D_i all give the same class in $N_1(C^{(k)})$, which we call c_D. A computation shows that c_D is not proportional to c_δ.

Our purpose is to prove the following

Theorem 1.4.1. *Let C be a curve with general moduli, of genus $g = 2k$. Then the line bundles $L_1, \ldots, L_s$ defined in (1.56), and the class $c_D \in N_1(C^{(k)})$ of the curves $D_i \subset C^{(k)}$ verify the properties:*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$;

(B) $c_1(\mathcal{G}_{k,L_i}).c_D = 0$.

By Proposition 1.3.5, the previous theorem immediately implies Theorem 1.1.1. The proof of **(A)** will occupy the rest of the paper, while in the remainder of this section we will verify **(B)**.

Proposition B. *The section $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$ is everywhere non zero along the curve D_i. In particular this implies*

(B) $c_1(\mathcal{G}_{k,L_i}).c_D = 0$.

Proof. Recall first that, by the base point free pencil trick, the space of sections $H^0(K_C - 2L'_i)$ identifies to the kernel of the multiplication map

$$\mu_0 : H^0(L'_i) \otimes H^0(K_C - L'_i) \to H^0(K_C). \tag{1.57}$$

By Gieseker [G] (or Lazarsfeld [L1]) we know that μ_0 is injective for generic C and hence we have

$$H^0(K_C - 2L'_i) = 0. \tag{1.58}$$

Now, we check

Lemma 1.4.2. *For any $x \in C$ such that $L'_i - x \geq 0$,*

$$H^0(L_i - (L'_i - x)) = H^0(K_C - 2L'_i + x) = H^0(K_C - 2L'_i).$$

Proof. By Riemann-Roch, the equality

$$H^0(K_C - 2L'_i + x) = H^0(K_C - 2L'_i).$$

is equivalent to

$$h^0(2L'_i - x) = h^0(2L'_i) - 1. \tag{1.59}$$

But $|L'_i|$ is base point free, hence $|2L'_i|$ is base point free and (1.59) immediately follows. □

We conclude the proof of the Proposition B. By Lemma 1.4.2, and recalling that

$$Z \in D_i \;\Leftrightarrow\; \exists x : \; Z \sim L'_i - x,$$

we see that, for any $Z \in D_i$,

$$H^0(C, L_i - Z) = 0.$$

By (1.55), this implies that $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$ is everywhere non zero along D_i. □

1.5 Proof of condition (A)

In this section we will state a result (see Proposition 1.5.1 below) and prove that it implies condition (A) holds for the line bundles $L_i = K_C - L'_i$. Then we turn to the case of a curve on a $K3$ surface, recall some constructions due to Lazarsfeld [L1] and Voisin [V], and some results from [V] that will enable us to prove Proposition 1.5.1 for such a curve.

1.5.1 A second reduction

Let L_i' be a g^1_{k+1} on C. The choice of a section $\tau \in H^0(C, L_i')$ furnishes an inclusion

$$H^0(K_C - L_i') = H^0(L_i) \overset{\times\tau}{\hookrightarrow} H^0(K_C), \tag{1.60}$$

which induces a $\mathbb{C}$-linear inclusion

$$H^0(C^{(k)}, \mathcal{G}_{k,L_i}) = \bigwedge^k H^0(C, L_i) \hookrightarrow H^0(C^{(k)}, \mathcal{G}_{k,K_C}) = \bigwedge^k H^0(C, K_C). \tag{1.61}$$

Fix a base $s_1, ..., s_k$ of $H^0(C, L_i)$, so that $s_1 \wedge ... \wedge s_k = \sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$. For any i, consider more generally the map

$$\phi_{k,i}: \ S^k H^0(C, L_i') \to H^0(C^{(k)}, \mathcal{G}_{k,K_C}), \tag{1.62}$$

which sends τ^k to the image of $s_1 \wedge ... \wedge s_k$ via the inclusion (1.61). We will denote this image by $\tau^k \cdot \sigma_{L_i}$. Dually, the map

$$\phi^*_{k,i}: \ \bigwedge^k H^0(\mathcal{O}_C(C))^* \cong \bigwedge^k H^0(\mathcal{O}_C(C)) \to S^k H^0(C, L_i')^*$$

sends $w \in \wedge^k H^0(\mathcal{O}_C(C))^*$ to the polynomial $Q_w \in S^k H^0(C, L_i')^*$ defined as follows

$$Q_w(t) := < t^k \cdot \sigma_{L_i}, w > .$$

We want to prove

Proposition 1.5.1. *The elements $\tau^k \cdot \sigma_{L_i}$, $\forall i$, $\tau \in H^0(C, L_i')$ generate the whole space of sections $H^0(C^{(k)}, \mathcal{G}_{k,K_C}) = \wedge^k H^0(C, K_C)$, i.e. the map*

$$\phi_k \ : \ \bigoplus_i S^k H^0(C, L_i') \longrightarrow \bigwedge^k H^0(C, K_C), \tag{1.63}$$

described above is surjective.

Assuming Proposition 1.5.1 for the moment, we see how it implies

Proposition A. *Let C be a curve with general moduli, of genus $g = 2k$. Consider the line bundles $L_i = K_C - L_i'$. Then the sections $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$ verify the property:*

(A) $\bigcap_i V(\sigma_{L_i}) = \emptyset$.

Proof. The space of global sections $H^0(C^{(k)}, \mathcal{G}_{k,K_C})$ has no base points. Indeed, for any $Z \in C^{(k)}$, we have a surjection

$$H^0(K_C) \twoheadrightarrow H^0(K_{C|Z}).$$

Otherwise, by Riemann-Roch, $h^0(\mathcal{O}_C(Z)) = h^1(\mathcal{O}_C(K_C(-Z))) \geq 2$, contradicting the fact that $gon(C) = k+1$. This, in turn, implies the surjectivity of the evaluation map

$$H^0(C^{(k)}, \mathcal{G}_{k,K_C}) = \bigwedge^k H^0(C, K_C) \twoheadrightarrow \bigwedge^k H^0(K_{C|Z}) = (\mathcal{G}_{k,K_C})_{|Z}. \quad (1.64)$$

If $Z \in C^{(k)}$ belongs to $\bigcap_i V(\sigma_{L_i})$, with $\sigma_{L_i} \in H^0(C^{(k)}, \mathcal{G}_{k,L_i})$, then, *a fortiori*, $Z \in \bigcap_{\tau \in H^0(L_i'),\ i \in I} V(\tau^k \cdot \sigma_{L_i})$. But this is absurd, since by Proposition 1.5.1 the elements $\tau^k \cdot \sigma_{L_i}$ generate the base point-free space of sections $H^0(C^{(k)}, \mathcal{G}_{k,K_C})$. □

1.5.2 Curves on a $K3$ surface

Let C be a curve of genus $g(C) = 2k$ contained in a K3 surface S, such that C generates the Picard group of S. By Lazarsfeld [L1], the curve $C \subset S$ is Brill-Noether-Petri generic, and hence in particular its gonality is equal to $k+1$, and the number of g^1_{k+1}'s on C is finite. Let L' be a g^1_{k+1} on C. In what follows we will recall a classical construction due to Lazarsfeld, which associates to the data (C, L') a vector bundle E on S. This, along the lines of Voisin [V], will allow us to put a structure of complete intersection on the set of g^1_{k+1}'s over C, seen as points of a certain Grassmannian.

The line bundle L' is obviously base point free. Then we can consider the sheaf F defined by the exact sequence

$$0 \to F \to H^0(L') \otimes \mathcal{O}_S \xrightarrow{ev} L' \to 0, \quad (1.65)$$

where, by abuse of notation, we still denote by L' the sheaf on S obtained by extending L' to zero away from C. F is a vector bundle of rank 2 and we define $E := F^*$. The rank 2 vector bundle E on S sits inside the short exact sequence

$$0 \to H^0(L')^* \otimes \mathcal{O}_S \to E \to K_C - L' \to 0, \quad (1.66)$$

obtained by dualizing (1.65). We record now a series of facts about the bundle E:

(i) $c_1(E) = C$, $c_2(E) = deg\ L' = k+1$, $h^0(E) = k+2$;

(ii) E is stable;

(iii) E is independent from the choice of C in its linear system, and from the g^1_{k+1} considered on C.

The numerical conditions in (i) follow easily from (1.66). The stability of E, as remarked in [M], follows from the fact that $detE = L$ generates the Picard group of S and from the vanishing $H^0(S, E(-L)) = H^0(S, F) = 0$ (this second fact follows from (1.65)). Finally, (iii) is a consequence of (i) and (ii). Indeed, if there were another stable rank two bundle F on S satisfying (i), then, as computed in [L1], $\chi(E, F) = 2$, and hence, by Riemann-Roch, either $Hom(E, F) \neq 0$ or $Hom(F, E) \neq 0$. But such a non trivial homomorphism would furnish a destabilizing subbundle of either E or F, or would be an isomorphism.

The exact sequence (1.66) realizes $H^0(C, L')^* \cong H^0(C, L')$ as a rank 2 subspace of $H^0(S, E)$. Let s_1 and s_2 be two generators of $H^0(C, L')$. Then, by (1.66) restricted to C, we have that C is the zero locus of $s_1 \wedge s_2 \in H^0(S, detE)$, and the image of the morphism

$$H^0(C, L') \otimes \mathcal{O}_C \to E_{|C}$$

identifies to L'.

Consider now the Grassmannian $G_2 := Grass(2, H^0(S, E))$ of dimension two subspaces of $H^0(S, E)$, which has dimension equal to 2k, and let $\mathcal{O}_{G_2}(1)$ be the line bundle on G_2 giving its Plücker embedding. Notice that the determinant

$$det\ : H^0(\mathcal{O}_{G_2}(1))^* = \bigwedge^2 H^0(S, E) \to H^0(S, det\ E)$$

is not zero on any rank two element W of $H^0(\mathcal{O}_{G_2}(1))^*$. Indeed, if this were the case, then the subspace $W \subset H^0(S, E)$ would generate a rank one subsheaf of E with at least two sections.

By the above we can then define a map

$$d\ :\ G_2 \to \mathbf{P}H^0(S, \mathcal{O}_S(C)) = \mathbf{P}H^0(S, det(E)),$$

sending a point $w \in G_2$, that corresponds to a subspace $W \subset H^0(S, E)$, to the element in $|C|$ defined by the zeroes of $\bigwedge^2 W$. The target space has dimension 2k (since $g(C) = 2k$) and the map d is finite. Its fibre over C, which we will denote Z_C, is in 1-1 correspondence with the set of the g^1_{k+1}'s over C. Finally, by construction, we have the commutativity of the following diagram

$$\begin{array}{ccc} G_2 & \hookrightarrow & \mathbf{P}^N = \mathbf{P}H^0(\mathcal{O}_{G_2}(1)) \\ & \searrow^{d} & \downarrow \pi \\ & & \mathbf{P}H^0(\mathcal{O}_S(C)) \end{array} \tag{1.67}$$

Hence the set of the g^1_{k+1}'s over C, seen as a finite subset of G_2, is endowed, by this projection, with a scheme-theoretic structure of complete intersection of elements of $|\mathcal{O}_{G_2}(1)|$.

1.5.3 Proof of Proposition 1.5.1

Let C be a curve of genus $2k$ on a K3 surface S, generating the Picard group of S, and $L'_1, \ldots L'_s$ the g^1_{k+1}'s on C, parametrized by the finite complete intersection $Z_C \subset G_2$. Consider the bundle $E \to S$. By the exact sequence (1.66) defining E, there is a natural map from $H^0(C, L'_i)^* \cong H^0(C, L'_i)$ to $H^0(S, E)$. Denote by ψ_l the induced map

$$\psi_l \ : \ \bigoplus_{L'_i \in Z_C} S^l H^0(C, L'_i) \to S^l H^0(S, E). \tag{1.68}$$

The maps ψ_l have been considered by Voisin in [V], where she proves

Proposition [V]. *The maps ψ_l are surjective for any $l \leq k-1$.*

We will prove in the next section the following

Proposition 1.5.2. *The image of*

$$\psi_k \ : \ \bigoplus_{L'_i \in Z_C} S^k H^0(C, L'_i) \to S^k H^0(S, E),$$

has dimension $\geq$ dim $\wedge^k H^0(C, K_C)$.

Now, recall that $h^0(S, E) = k+2$, and $detE = \mathcal{O}_S(C)$. Let γ be a global section of E and $e_1, \ldots, e_{k+1} \in H^0(E)$ such that $\gamma, e_1, \ldots, e_{k+1}$ form a basis of $H^0(E)$. For any $i = 1, \ldots, k+1$, the element $\gamma \wedge e_i$ gives a global section of $\mathcal{O}_S(C)$, and then $(\gamma \wedge e_1) \wedge \ldots \wedge (\gamma \wedge e_{k+1})$ divided by $\gamma \wedge e_1 \wedge e_2 \wedge \ldots \wedge e_{k+1} \in \wedge^{k+2} H^0(S, E) \cong \mathbb{C}$ gives a well defined element in $\wedge^{k+1} H^0(\mathcal{O}_S(C))$. By abuse of notation we will call this element $det(\gamma \wedge H^0(E))$. Notice that, for any $0 \neq \lambda \in \mathbb{C}$, we have

$$det(\lambda \cdot \gamma \wedge H^0(E)) = \frac{\lambda^{k+1}}{\lambda} \cdot \frac{(\gamma \wedge e_1) \wedge \ldots \wedge (\gamma \wedge e_{k+1})}{\gamma \wedge e_1 \wedge e_2 \wedge \ldots \wedge e_{k+1}} = \lambda^k \cdot det(\gamma \wedge H^0(E)).$$

Recall that $h^0(S, \mathcal{O}_S(C)) = 2k+1$, and hence we have the isomorphism:

$$\bigwedge^{k+1} H^0(\mathcal{O}_S(C))^* \cong \bigwedge^{k} H^0(\mathcal{O}_S(C)).$$

We define a morphism

$$\nu_k \ : \ S^k H^0(E) \to \bigwedge^{k+1} H^0(\mathcal{O}_S(C)), \tag{1.69}$$

by describing its dual

$$\nu_k^* : \ \bigwedge^{k+1} H^0(\mathcal{O}_S(C))^* \to S^k H^0(E)^*.$$

The map ν_k^* sends $u \in \bigwedge^{k+1} H^0(\mathcal{O}_S(C))^*$ to the polynomial $P_u \in S^k H^0(E)^*$ defined as follows:

$$P_u(\gamma) := < det(\gamma \wedge H^0(E)), u > .$$

We have

Theorem (Voisin, [V] 3.9). *The map ν_k is an isomorphism.*

This result, together with Proposition 1.5.2 yields the

Proof of Proposition 1.5.1. Consider the short exact sequence defining $C \subset S$

$$0 \to \mathcal{O}_S(-C) \xrightarrow{\cdot \sigma_C} \mathcal{O}_S \to \mathcal{O}_C \to 0,$$

where σ_C is the section defining C. By tensoring it with $\mathcal{O}_S(C)$, recalling that $K_S = \mathcal{O}_S$ and $h^1(\mathcal{O}_S) = 0$, and using the adjunction formula, we obtain

$$0 \to H^0(\mathcal{O}_S) \xrightarrow{\cdot \sigma_C} H^0(\mathcal{O}_S(C)) \to H^0(K_C) \to 0.$$

We then have an injective morphism

$$\iota \ : \ \bigwedge^{k} H^0(K_C) \overset{\wedge \sigma_C}{\hookrightarrow} \bigwedge^{k+1} H^0(\mathcal{O}_S(C)). \tag{1.70}$$

The maps defined in (1.63), (1.68), (1.69) and (1.70) sit inside the following diagram:

$$\begin{array}{ccc} S^k H^0(S,E) & \xrightarrow[\sim]{\nu_k} & \bigwedge^{k+1} H^0(S,\mathcal{O}_S(C)) \\ \uparrow{\scriptstyle \psi_k} & & \uparrow{\scriptstyle \iota} \\ \bigoplus_{L_i'\in Z_C} S^k H^0(C,L_i') & \xrightarrow{\phi_k} & \bigwedge^k H^0(C,K_C). \end{array} \tag{1.71}$$

We check the following

Lemma 1.5.3. *$Im(\nu_k \cdot \psi_k) = Im(\iota \cdot \phi_k)$.*

Proof of Lemma 1.5.3. It is of course sufficient to verify that

$$Im(\nu_k \cdot \psi_{k,i}) = Im(\iota \cdot \phi_{k,i})$$

holds for any i. By duality, we have to prove that, for any $w \in \wedge^k H^0(\mathcal{O}_C(C))^*$ and for any $t \in H^0(C,L_i')$, up to a scalar we have

$$\phi_{k,i}^*(w)(t) := Q_w(t) = P_u(t) =: \nu_k^*(u)(t),$$

with $u = w \wedge \sigma_C^* \in \wedge^{k+1} H^0(\mathcal{O}_S(C))^*$.

Let $\tau \in H^0(C,L_i')$ and $s_1,\ldots,s_k \in H^0(C,L_i)$ be such that $< t,\tau >= H^0(C,L_i')$ and $< s_1,\ldots,s_k >= H^0(C,L_i)$. Then $t\wedge\tau \in H^0(S,detE)$ is such that $V(t\wedge\tau) = C \subset S$ and $t,\tau,s_1,\ldots,s_k$ regarded as elements of $H^0(S,E) = H^0(C,L_i') \oplus H^0(C,L_i)$ give a basis for this space of sections. Moreover we have

$$< t\cdot s_1,\ldots,t\cdot s_k,\tau\cdot s_1,\ldots,\tau\cdot s_k >= H^0(C,\mathcal{O}_C(C)),$$

and

$$< t\wedge s_1,\ldots,t\wedge s_k,\tau\wedge s_1,\ldots,\tau\wedge s_k > \oplus < t\wedge\tau >= H^0(C,\mathcal{O}_S(C)).$$

Then

$$\nu_k^*(u)(t) = P_{w\wedge(t\wedge\tau)^*}(t) = \alpha < (t\wedge s_1)\wedge\ldots\wedge(t\wedge s_k)\wedge(t\wedge\tau), w\wedge(t\wedge\tau)^* >,$$

with $\alpha^{-1} = s_1\wedge\ldots\wedge s_k \in \wedge^k H^0(C,L_i) \cong \mathbb{C}$. But

$$< (t\wedge s_1)\wedge\ldots\wedge(t\wedge s_k)\wedge(t\wedge\tau), w\wedge(t\wedge\tau)^* >=< (t\cdot s_1)\wedge\ldots\wedge(t\cdot s_k), w >,$$

and the term on the right is exactly $\phi_{k,i}^*(w)(t) = Q_w(t)$. □

By the previous lemma, Proposition 1.5.2 immediately implies Proposition 1.5.1. □

What is left to be done in order to complete the proof of Theorem 1.4.1 is to prove Proposition 1.5.2, which we will do in the next subsection.

1.5.4 Proof of Proposition 1.5.2

Let $Z_C \subset G_2$ be the finite complete intersection of hyperplane sections parametrizing the g^1_{k+1}'s over C. Let $\mathcal{E} \to G_2 = Grass(2, H^0(E))$ be the dual of the universal subbundle. In order to prove Proposition 1.5.2 we study the cohomology group $H^0(G_2, \mathcal{I}_{Z_C} \otimes Sym^k\mathcal{E})$ via the Koszul resolution of the ideal sheaf of the complete intersection $Z_C \subset G_2$.

Let $K := H^0(C, \mathcal{O}_C(C))^* = H^0(K_C)^*$. Recall the injection

$$H^0(C, \mathcal{O}_C(C))^* \subset H^0(S, \mathcal{O}_S(C))^* \stackrel{d^*}{\hookrightarrow} \bigwedge^2 H^0(S, E)^* = H^0(G_2, \mathcal{O}_{G_2}(1)).$$

The Koszul resolution of $\mathcal{I}_{Z_C}$ is then given by

$$0 \to \bigwedge^{2k} K \otimes \mathcal{O}_{G_2}(-2k) \to \ldots \to K \otimes \mathcal{O}_{G_2}(-1) \to \mathcal{I}_{Z_C} \to 0. \qquad (1.72)$$

By tensoring it with $S^k\mathcal{E}$ we get the exact complex

$$\mathcal{K}^\bullet : \ 0 \to \bigwedge^{2k} K \otimes \mathcal{O}_{G_2}(-2k) \otimes S^k\mathcal{E} \to \ldots \to \mathcal{I}_{Z_C} \otimes S^k\mathcal{E} \to 0, \qquad (1.73)$$

where the term $S^k\mathcal{E}$ is put in degree zero. We have

Lemma 1.5.4. *Let C be a curve of genus $2k$ on a $K3$ surface S, with $< \mathcal{O}_S(C) >= Pic(S)$. Let $\{L'_i\}_i$ be the finite set of g^1_{k+1}'s on C, which are parametrized by the finite complete intersection $Z_C \subset G_2$. Then the following inequality holds:*

$$h^0(G_2, \mathcal{I}_{Z_C} \otimes S^k\mathcal{E}) \leq \binom{2k}{k+1}.$$

Proof. The proof goes as in [V], Lemma 2. The hypercohomology $\mathbb{H}^0(G_2, \mathcal{K}^\bullet)$ vanishes. Now we have the spectral sequence

$$E_1^{p,q} = H^q(G_2, \mathcal{K}^p) \Rightarrow \mathbb{H}^{p+q}(G_2, \mathcal{K}^\bullet).$$

Using the following facts

- all the differentials d_r starting from $E_r^{0,0}$ vanish;
- $H^q(G_2, \mathcal{O}_{G_2}(-q-1) \otimes S^k\mathcal{E}) = 0$ for all $q \neq k$ (see [V], Proposition 9);
- $H^k(G_2, \mathcal{O}_{G_2}(-k-1) \otimes S^k\mathcal{E}) \cong \mathbb{C}$ (see again [V], Proposition 9);

one constructs a surjective map from a subquotient of $E_1^{-k-1,k} = \wedge^{k+1}K$ to $H^0(G_2, \mathcal{I}_{Z_C} \otimes S^k\mathcal{E})$, and this proves the lemma. □

We can finally give the

Proof of Proposition 1.5.2. Recall that, by (1.66), $H^0(C, L_i')$ identifies to a rank two subspace of $H^0(S, E)$. Consider $\mathcal{E} \to G_2 = Grass(2, H^0(E))$. The fiber of $\mathcal{E}$ at a point $W \in G_2$ is equal to W^*, and the space of its global sections $H^0(G_2, \mathcal{E})$ is given by $H^0(S, E)^*$. Thus, for reduced Z_C (which is always the case for C generic), the dual of the map

$$\psi_k \ : \ \bigoplus_{L_i' \in Z_C} S^k H^0(C, L_i') \to S^k H^0(S, E),$$

identifies to the evaluation map

$$S^k H^0(G_2, \mathcal{E}) = H^0(G_2, S^k\mathcal{E}) \stackrel{ev_{Z_C}}{\longrightarrow} S^k\mathcal{E}_{|Z_C},$$

whose kernel is exactly $H^0(\mathcal{I}_{Z_C} \otimes S^k\mathcal{E})$. Notice that

$$dim\ S^k H^0(S, E) - dim\ \bigwedge^k H^0(C, K_C) = \binom{2k+1}{k} - \binom{2k}{k}. \qquad (1.74)$$

Then from Lemma 1.5.4 and (1.74) it immediately follows that the dimension of the image of ψ_k is greater than or equal to the dimension of $\bigwedge^k H^0(C, K_C)$. □

Bibliography of Part II

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of algebraic curves*, vol. 1, Springer, New York, 1985.

[BS] M. Beltrametti, A. J. Sommese, *Zero cycles and k-th order embeddings of smooth projective surfaces*, with an appendix by L. Göttsche, Sympos. Math., XXXII, Problems in the theory of surfaces and their classification (Cortona, 1988), 33-48, Academic Press, London, 1991.

[BT] A. Bertram, M. Thaddeus, *On the quantum cohomology of a symmetric product of an algebraic curve*, Duke Math. J. **108** (2001), no. 2, 329–362.

[CC] F. Catanese, C. Ciliberto, *Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$*, J. of Algebraic Geom. **2** (1993), 389–411.

[CG] F. Catanese, L. Göttsche, *d-very-ample line bundles and embeddings of Hilbert schemes of 0-cycles*, Manuscripta Math. **68** (1990), no. 3, 337–341.

[CK] C. Ciliberto, A. Kouvidakis, *On the symmetric product of a curve with general moduli*, Geom. Dedicata **78** (1999), no. 3, 327–343.

[Co] A. Collino, *The rational equivalence ring of symmetric products of curves*, Illinois J. Math. **19** (1975), no. 4, 567–583.

[F] B. Fantechi, *Deformations of symmetric products of curves*, Classification of algebraic varieties (L'Aquila, 1992), 135–141, Contemp. Math., 162, Amer. Math. Soc., Providence, RI, 1994.

[G] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982), no. 2, 251–275.

[GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Ke] G. R. Kempf, *Deformations of symmetric products*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 319–341, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.

[Kl] S. L. Kleiman, *Towards a numerical theory of ampleness*, Annals of Math. **(84)** (1966), 293–344.

[K] A. Kouvidakis, *Divisors on symmetric products of curves*, Trans. Amer. Math. Soc. **337**(1) (1993), 117–128.

[L1] R. Lazarsfeld, *Brill-Noether-Petri without degenerations*, J. Differential Geom. **23** (1986), no. 3, 299–307.

[L2] R. Lazarsfeld, *Lectures on linear series*, IAS/Park City Math. Ser., 3, Complex algebraic geometry (Park City, UT, 1993), 161-219, Amer. Math. Soc., Providence, RI, 1997.

[Mac] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.

[Mat] A. Mattuck, *Symmetric products and Jacobians*, Amer. J. Math. **83** (1961), 189–206.

[Mu] S. Mukai, *Biregular classification of Fano* 3*-folds and Fano manifolds of coindex* 3, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), no. 9, 3000–3002.

[N] M. Nagata, *On the fourteenth problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

[SZ] B. Shiffman, M. Zaidenberg, *Two classes of hyperbolic surfaces in* $\mathbf{P}^3$, Internat. J. Math. **11** (2000), no. 1, 65–101.

[V] C. Voisin, *Green's generic syzygy conjecture for curves of even genus lying on a K3 surface*, preprint 2001.