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Sujet de la thèse :

Obstruction au prolongement des formes différentielles régulières et codimension du lieu singulier

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devant le jury composé de

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Cette thèse se compose de deux parties indépendantes. La première «Sections hyperplanes et endomorphismes de l'espace projectif» est consacrée à l'étude des variétés obtenues comme images inverses de sous variétés lisses par un endomorphisme (non-trivial et non-bijectif) d'un espace projectif. Ce travail a fait l'objet d'une publication aux Comptes rendus de l'Académie des Sciences ([Jam97]). Nous renvoyons à l'introduction de cette partie pour une présentation des résultats obtenus.

La seconde partie «Extendability of regular differential forms & codimension of the singular locus» ainsi que les appendices ont été rédigés en langue anglaise en vue d'une publication ultérieure et c'est avec regret que nous n'avons pu, par manque de temps, en donner ici une version écrite en français. Le lecteur trouvera dans l'introduction une présentation et un résumé en français de cette seconde partie ainsi que des appendices B et C.

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Introduction

Obstruction au prolongement des formes différentielles régulières et codimension du lieu singulier

C'est une question posée par J. Fogarty dans ([Fog88]) qui est à l'origine de ce travail (tous les schémas ou variétés considérés ici sont supposés de type fini sur un corps de caractéristique nulle) :

Question *Soit G un groupe fini agissant sur une variété affine lisse X et $\pi : X \longrightarrow Y$ le quotient. La surjectivité de l'application naturelle*

$$\Omega^1_Y \longrightarrow (\Omega^1_X)^G$$

est-elle une condition nécessaire et suffisante pour que Y soit lisse ?

Dans ce même article, J. Fogarty remarque que la condition de surjectivité est effectivement nécessaire et démontre qu'elle est suffisante lorsque le groupe G est abélien.

Remarquons que le module $(\Omega^1_X)^G$ s'identifie à $\Omega^1_Y{}^{\vee\vee}$ et, la variété Y étant normale, le module $\Omega^1_Y{}^{\vee\vee}$ est lui-même isomorphe au module ω^1_Y des 1-formes régulieres (cf. appendice B). Remarquons aussi que l'on peut aussi toujours se ramener au cas où X est une représentation linéaire de G. Il est assez facile de voir que cette condition de surjectivité est suffisante si l'on sait par ailleurs que Y est intersection complète. Ainsi, lorsque $G \subset \mathrm{SL}(\mathbb{C}^2)$ on peut donner une réponse affirmative à la question. En revanche, si l'on suppose seulement que $G \subset \mathrm{GL}(\mathbb{C}^2)$ plus rien ne semble évident.

A défaut de réponse définitive au problème il nous a semblé naturel de le reformuler et de le généraliser dans deux directions :

Cas des quotients par un groupe réductif Considérons l'action d'un groupe réductif G sur une variété affine lisse X et notons encore $\pi : X \longrightarrow Y$ le quotient. Nous rappelons dans (4.1) la définition de l'algèbre graduée $\Omega_{X,G}$ des formes différentielles horizontales et l'existence d'un morphisme naturel :

$$(d\pi)^G \quad : \quad \Omega_Y \longrightarrow (\Omega_{X,G})^G$$

D'après M. Brion, on sait que

Théorème ([Bri98, Theorem 1]) *Sous les hypothèses précédentes, si la variété Y est lisse alors le morphisme $(d\pi)^G$ est un isomorphisme.*

Il semble donc raisonnable de reformuler la question de Fogarty dans ce cadre de la manière suivante :

Question *Quelles relations peut-on envisager entre la surjectivité de $(d\pi)^G$ en un certain degré k (resp. en degrés $\leq k$) et la codimension du lieu singulier de Y ?*

Cas «général» Nous rappelons dans l'appendice B la construction du Ω_Y-module différentiel gradué ω_Y des formes différentielles régulières et l'existence, d'après El Zein, d'un morphisme classe fondamentale pour un schéma Y :

$$\mathrm{c}_Y : \Omega_Y \longrightarrow \omega_Y$$

qui est un morphisme de Ω_Y-modules différentiels gradués. Nous rappelons aussi la construction du module $\tilde{\Omega}_Y$ des formes différentielles absolument régulières. Si $i : Y_{\text{lisse}} \subset Y$ est le lieu lisse d'un schéma réduit on a une suite de morphismes

$$\Omega_Y \longrightarrow \tilde{\Omega}_Y \longrightarrow \omega_Y \longrightarrow i_*\Omega_{Y_{\text{lisse}}}$$

dont la composition est le morphisme de restriction au lieu lisse. Lorsque Y est une variété, il est facile de voir que le noyau du morphisme $\Omega_Y \longrightarrow \tilde{\Omega}_Y$ est exactement le sous-module des éléments de torsion de Ω_Y. Ce morphisme se factorise donc par l'inclusion $\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y$, où $\bar{\Omega}_Y$ désigne le quotient de Ω_Y par sa torsion. Lorsque Y est une variété normale on a un isomorphisme $\omega_Y \simeq i_*\Omega_{Y_{\text{lisse}}}$.

Question *Pour une variété Y ayant des singularités d'un type «raisonnable» quelles relations peut-on envisager entre la surjectivité de c_Y (resp. du morphisme de restriction au lieu lisse) en un certain degré k (resp. en degrés $\leq k$) et la codimension du lieu singulier de Y ?*

Le théorème suivant dû à H. Flenner (voir aussi [vSS85] dans le cas des singularités isolées) peut aider à situer le problème dans le cas général :

Théorème ([Fle88]) *Soit Y une variété algébrique normale (resp. le spectre d'un anneau local normal complet, resp. un espace analytique normal). L'application naturelle $\tilde{\Omega}_Y \longrightarrow i_*\Omega_{Y_{\text{lisse}}}$ est surjective en degrés $< \operatorname{codim}(Y_{\text{sing}}) - 1$.*

Cependant, le cas des singularités quotients par un groupe fini (4.4.5) et des singularités rationnelles isolées montre que la réciproque est fausse : En effet dans ces cas on a des isomorphismes $\tilde{\Omega}_Y \simeq \omega_Y \simeq i_* \Omega_{Y_{\text{lisse}}}$.

Quant au rapport entre le cas des quotients par un groupe réductif et le cas général, on dispose tout d'abord du résultat suivant

Théorème ([Bri98, Theorem 2]) *Soit G un groupe réductif agissant sur une variété affine lisse X et $\pi : X \longrightarrow Y$ le quotient. Si aucun diviseur G-invariant dans X n'est contracté par π en codimension ≥ 2 dans Y alors le morphisme $(d\pi)^G$ induit un isomorphisme*

$$\Omega_Y{}^{\vee\vee} \xrightarrow{\sim} (\Omega_{X,G})^G.$$

Rappelons ici que le quotient d'une variété affine lisse par un groupe réductif est une variété à singularités rationnelles ([Bou87]). Une telle variété est donc en particulier normale et Cohen-Macaulay.

D'autre part nous avons démontré la proposition suivante qu'il nous semble intéressant de mettre en regard des deux théorèmes ci-dessus.

Proposition (4.4.4) *Soit G un groupe réductif agissant sur une variété affine lisse X et $\pi : X \longrightarrow Y$ le quotient. On a des inclusions*

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq \omega_Y$$

qui sont des isomorphismes sur le lieu lisse de Y et qui en degré $n = \dim(Y)$ induisent des isomorphismes :

$$\tilde{\Omega}_Y^n \xrightarrow{\sim} (\Omega_{X,G}^n)^G \xrightarrow{\sim} \omega_Y^n.$$

Ce résultat nous semble aussi intéressant en ce qu'il donne une description du faisceau dualisant de Y. A ce sujet nous renvoyons à l'article de F. Knop ([Kno89]).

Mentionnons enfin un résultat fondamental de E. Kunz et R. Waldi :

Théorème ([KW88, 5.22, p107]) *Si Y est un schéma de type fini, Cohen-Macaulay, réduit et équidimensionnel de dimension n, alors le support de* $\operatorname{Coker}(\mathrm{c}_Y)^n$ *coïncide avec le lieu singulier de Y.*

Ce théorème est un ingrédient essentiel dans la démonstration de nos principaux résultats concernant les quotients par des groupes réductifs :

Théorème (4.5.1) *Soit G un groupe réductif agissant sur une variété affine lisse X et $\pi : X \longrightarrow Y$ le quotient. Soit un entier $k \leq \dim(Y)$. Si le morphisme $(d\pi)^G$ est surjectif en degré k alors Y est lisse en codimension $k+1$.*

Revenons au cas où G est un groupe fini. Alors $\Omega_{X,G} = \Omega_X$ et le théorème 4.5.1 permet de répondre affirmativement à la question de J. Fogarty lorsque $\dim(X) \leq 2$.

Dans l'énoncé suivant et contrairement au précédent, le critère proposé est intrinsèque à Y.

Théorème (4.5.2) *Soit Y une variété quotient d'une variété affine lisse par un groupe algébrique réductif. Soit un entier $k \leq \dim(Y)$. Si le morphisme classe fondamentale pour Y est surjectif en degré k alors Y est lisse en codimension $k+1$.*

Précisons que la réciproque de ces énoncés est fausse, comme le montre l'exemple suivant : Soit ζ une racine non-triviale de l'unité et le groupe fini cyclique $G =< \zeta\mathrm{Id} > \subset \mathrm{GL}(\mathbb{C}^3)$. Alors $Y = \mathbb{C}^3/G$ est une singularité isolée, donc lisse en codimension 2 et l'on sait d'après le théorème de Fogarty dans le cas abélien que le morphisme $\Omega^1_Y \longrightarrow (\Omega^1_{\mathbb{C}^3})^G$ n'est pas surjectif.

Nous avons aussi essayé d'obtenir des résultats du même type pour des singularités autres que les singularités quotients. Le cas des intersections complètes semblait évidemment le plus accessible. Pour plus de clarté, nous énonçons ici les résultats obtenus pour des schémas de type fini sur un corps de caractéristique 0 :

Théorème (5.0.3) *Soit Y un schéma localement intersection complète. Pour un entier $k \geq 0$ les conditions suivantes sont équivalentes :*

(i) Y est lisse en codimension k.

(ii) Pour tout $i = 0 \ldots k$, le module Ω^i_Y est sans torsion en codimension i.

Sous ces conditions, pour tout $i = 1 \ldots k+1$, on a $\mathrm{dim.proj}\ \Omega^i_Y = i$ sauf dans le cas où Y est lisse.

De ce théorème nous déduisons :

Corollaire (5.0.4) *Soit Y un schéma réduit et localement intersection complète. Pour un entier $k \geq 0$ les conditions suivantes sont équivalentes :*

(i) Y est lisse en codimension $k+1$.

(ii) Le morphisme de restriction $\Omega_Y \longrightarrow i_*\Omega_{Y_{\mathrm{lisse}}}$ est surjectif en degrés $\leq k$.

La démonstration du théorème 5.0.3 repose sur les constructions faites dans l'appendice C où nous définissons entre autres les puissances symétriques et extérieures d'un complexe. On dispose alors de complexes qui sont de bons candidats pour être des résolutions projectives des modules Ω^i_Y et à partir de là, la preuve se fait par réccurence en utilisant le lemme d'acyclicité de Peskine et Szpiro.

Précisions Nous devons apporter les précisions suivantes concernant la section 5 et l'appendice C. Nous avons commencé à écrire de manière indépendante la «théorie» des puissances extérieures et symétriques de complexes de (C) en ayant avant tout pour objectif les résultats de la section 5, tout en doutant qu'une construction aussi «naturelle» n'ait pas déjà été développée antérieurement. Après diverses recherches et discussions, il s'est avéré qu'effectivement ceci avait été fait, sous des formes différentes par A. Lascoux, H. A. Nielsen et d'autres auteurs. Cependant nous n'avons pas trouvé dans ces références les énoncés dont nous avions précisément besoin et il nous a donc semblé en définitive utile d'inclure ici cette appendice.

Des remarques du même ordre s'imposent à propos des résultats de la section 5. Un résultat—dont nous ignorions l'existence—assez proche du théorème 5.0.3 a déjà été publié ([Emm95]). Le lecteur bienveillant pourra vérifier que notre démonstration a le mérite d'être indépendante sur un point où Emmanouil fait appel à un théorème de U. Vetter ([Vet70]). En fait, nous retrouvons ce dernier théorème de Vetter dans nos énoncés (5.0.3) et (5.0.4).

Sections hyperplanes et endomorphismes de l'espace projectif

Introduction

G. Ellingsrud et C. Peskine ont démontré ([EP89]) que, excepté pour un nombre fini de composantes du schéma de Hilbert, une surface lisse de $\mathbb{P}_4$ est de type général. D'autre part, les seuls exemples de surfaces lisses irrégulières de $\mathbb{P}_4$ connus sont d'irrégularité égale à 1 ou 2. Ces surfaces sont construites comme images réciproques par des endomorphismes d'un nombre *fini* de surfaces irrégulières connues. Et il est conjecturé de longue date que l'irrégularité des surfaces de $\mathbb{P}_4$ est bornée. Dans [Pes94], Peskine conjecture que, excepté pour un nombre fini de composantes du schéma de Hilbert, la section hyperplane générale d'une surface lisse de $\mathbb{P}_4$ est linéairement complète. Il suggère aussi que la section hyperplane générale d'une surface obtenue comme image réciproque par un endomorphisme de $\mathbb{P}_4$ d'une surface irrégulière, est linéairement complète, excepté peut-être pour un nombre fini de familles de surfaces irrégulières. Nous montrons ici que toute section hyperplane d'une surface image réciproque d'une surface par un endomorphisme (qui n'est pas un automorphisme) est linéairement complète. Ce qui, vue la nature des surfaces irrégulières connues, va dans le sens de la conjecture de Peskine. La démonstration repose sur une décomposition de certaines images directes de fibrés (1.1) et conduit en fait à un énoncé général sur les sections hyperplanes de variétés images réciproques de sous-variétés lisses par un endomorphisme de l'espace projectif (2.2). La dernière section regroupe quelques remarques sur l'adjonction de ces variétés.

1 Décomposition de certaines images directes

Soit n un entier ≥ 1. Soient $\mathbb{P}$ et $\mathbb{P}'$ des espaces projectifs sur $\mathbb{C}$ de dimension n et H (resp. H') une section hyperplane de $\mathbb{P}$ (resp. $\mathbb{P}'$). Nous dirons qu'un sous-schéma X d'un espace projectif $\mathbb{P}$ est linéairement complet (resp. non dégénéré) si le morphisme de restriction : $\mathrm{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(H)) \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X(H))$ est surjectif (resp.

injectif).

Soient $\mathbb{P}' \xrightarrow{\pi} \mathbb{P}$ un morphisme d'image non réduite à un point, donc fini, surjectif et plat, $X \subset \mathbb{P}$ un sous-schéma de degré d et $X' = \pi^{-1}(X) \subset \mathbb{P}'$ son image réciproque. Si $\pi^*(\mathcal{O}_{\mathbb{P}}(H)) = \mathcal{O}_{\mathbb{P}'}(kH')$, X' est de degré $d.k^{\mathrm{codim}(X)}$. En particulier π est de degré k^n.

Lorsque X est localement Cohen-Macaulay, on désignera par ω_X son faisceau dualisant, et, dans ce cas, π étant fini et plat, X' est aussi localement Cohen-Macaulay et son faisceau dualisant est $\omega_{X'} = \pi^!(\omega_X)$ (où $\pi^!(\mathcal{F})$ est défini par $\pi_*\pi^!(\mathcal{F}) = \underline{\mathrm{Hom}}(\pi_*(\mathcal{O}_{X'}), \mathcal{F})$, cf. [Har66] ou [Har77, III.7]). On souhaite étudier les faisceaux $\mathcal{O}_{X'}(lH')$ et, lorsque X est localement Cohen-Macaulay, $\omega_{X'}(lH')$, et en particulier leur cohomologie. La proposition suivante et l'annulation des images directes supérieures de π nous permettent de «travailler sur X».

Proposition 1.1 *Le fibré $\pi_*(\mathcal{O}_{X'}(lH'))$, $l \in \mathbb{Z}$, se décompose naturellement :*

$$\pi_*(\mathcal{O}_{X'}(lH')) = \bigoplus_{d\in\mathbb{Z}} E_{l,d} \otimes \mathcal{O}_X(-dH)$$

où $E_{l,d}$ est le conoyau de la multiplication :

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}}(H)) \otimes \mathrm{H}^0(\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))((d-1)H)) \longrightarrow \mathrm{H}^0(\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))(dH)),$$

non nul exactement lorsque $-[l/k] \leq d \leq \delta(n,k,l)$, avec $\delta(n,k,l) = n+1+\left[-\frac{n+1+l}{k}\right]$.

Démonstration de 1.1 Comme π est affine, $\pi_*(\mathcal{O}_{X'}(lH')) = \pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))_{|X}$, on peut donc supposer que $X = \mathbb{P}$. Or $\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))$ est un fibré sur $\mathbb{P}$ et

$$\mathrm{H}^i(\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))(dH)) = \mathrm{H}^i(\mathcal{O}_{\mathbb{P}'}((l+k.d)H')).$$

Le terme de droite s'annule pour $0 < i < n$ et tout entier d. Donc, par le critère de Horrocks, le fibré $\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))$ est une somme de fibrés en droites. Un tel fibré est engendré en degré $\leq \delta$ si et seulement si

$$\mathrm{H}^n(\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))((\delta-n)H)) = 0,$$

ce qui est équivalent à $l + k(\delta - n) > -n-1$. Notons $V_{l,d} = \mathrm{H}^0(\pi_*(\mathcal{O}_{\mathbb{P}'}(lH'))(dH))$ (la composante homogène de degré $l+kd$ de l'anneau gradué de $\mathbb{P}'$). Le diagramme commutatif suivant (où $l + kd \geq 0$ et les flèches horizontales (resp. verticales) sont induites par la multiplication des sections sur $\mathbb{P}$ (resp. $\mathbb{P}'$)) :

$$\begin{array}{ccccccc}
\mathrm{H}^0(\mathcal{O}_{\mathbb{P}}(H)) \otimes V_{l,d-1} \otimes V_{0,1} & \longrightarrow & V_{l,d} \otimes V_{0,1} & \longrightarrow & E_{l,d} \otimes V_{0,1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}^0(\mathcal{O}_{\mathbb{P}}(H)) \otimes V_{l,d} & \longrightarrow & V_{l,d+1} & \longrightarrow & E_{l,d+1} & \longrightarrow & 0
\end{array}$$

montre que $E_{l,d} = 0$ entraîne $E_{l,d+1} = 0$. D'où la décomposition annoncée. ■

1.2 Lorsque π n'est pas un automorphisme (i.e. $k > 1$), on vérifie que

$$\mathrm{H}^0(\mathcal{I}_{X'}(H')) = 0,$$

c'està dire que X' est non dégénéré.

1.3 D'autre part si $\mathrm{h}_1(\mathcal{I}_X(dH)) = 0$ pour tout $d < s$ alors

$$\mathrm{h}_1(\mathcal{I}_{X'}(lH')) = 0 \text{ pour tout } l < sk.$$

En effet, si $\mathcal{F}$ est un faisceau cohérent sur X, on a

$$\mathrm{H}^i(\pi^*(\mathcal{F})(lH')) \simeq \mathrm{H}^i(\mathcal{F} \otimes \pi_*(\mathcal{O}_{X'}(lH'))) .$$

En particulier si $\mathrm{h}_0(\mathcal{O}_X) = 1$ et si π n'est pas un automorphisme, X' est linéairement complet. Remarquons cependant que $\mathrm{h}_1(\mathcal{I}_X(dH)) = 0$ pour tout d si et seulement si $\mathrm{h}_1(\mathcal{I}_{X'}(lH)) = 0$ pour tout l.

1.4 Si X est lisse on sait d'après le théorème de transversalité de Kleiman ([Kle74], [Har77, III.10.8, p. 273]) que, pour un automorphisme linéaire général σ, $\pi^{-1}(\sigma X)$ est lisse. Donc si X est lisse et en position générale, X' est lisse et $\mathrm{h}_i(\mathcal{O}_{X'}) = \mathrm{h}_i(\mathcal{O}_X)\,,\ i < \dim(X)$.

Dans toute la suite on supposera que π n'est pas un automorphisme (i.e. $k > 1$).

2 La section hyperplane

Soient $X \subset \mathbb{P}$ une variété lisse connexe, s une section non-nulle de $\mathcal{O}_{\mathbb{P}'}(H')$ et μ_s la multiplication par $s\colon \mathcal{O}_{X'} \xrightarrow{\mu_s} \mathcal{O}_{X'}(H')$.

Proposition 2.1 $\mathrm{H}^i(\mu_s)$ *est injective pour* $i < \dim(X)$.

Comme d'après 1.2 et 1.3 l'image réciproque de X est linéairement complète et non dégénérée, on en déduit :

Corollaire 2.2 *Soient $X \subset \mathbb{P}$ une varieté lisse de dimension au moins égale à 2, $\pi : \mathbb{P}' \longrightarrow \mathbb{P}$ un endomorphisme qui n'est pas un automorphisme et $X' = \pi^{-1}(X)$. Alors pour tout hyperplan H' coupant X' proprement, la section $X' \cap H'$ est linéairement complète. En particulier, si X' est irréductible, toute section hyperplane de X est linéairement complète.*

Démonstration de 2.1 D'après 1.1, $\pi_*(\mu_s)$ se décompose sous la forme: $\pi_*(\mu_s) = (\mu_s^{p,q})_{p,q\geq 0}$ avec :

$$\mu_s^{p,q} : E_{0,q} \otimes \mathcal{O}_X(-qH) \longrightarrow E_{1,p} \otimes \mathcal{O}_X(-pH)$$

Remarquons que $\mu_s^{0,0}$ s'identifie à :

$$\mathcal{O}_X \xrightarrow{s\otimes 1} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}'}(H')) \otimes \mathcal{O}_X$$

En appliquant le théorème d'annulation de Kodaira on voit que, pour $i < \dim(X)$, $\mathrm{H}^i(\pi_*(\mu_s))$ se réduit à :

$$\mathrm{H}^i(\mathcal{O}_X) \xrightarrow{s\otimes 1} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}'}(H')) \otimes \mathrm{H}^i(\mathcal{O}_X)$$

qui est évidemment injective. ■

Sans supposer X lisse, on démontre de la même manière :

Proposition 2.3 *Soit j un entier. Si $\mathrm{H}^i(\mathcal{O}_X(-dH)) = 0$ pour tout entier $i < j$ et tout entier d tel que $0 < d \leq \delta(n,k,0)$ alors $\mathrm{H}^i(\mu_s)$ est injective pour tout entier $i < j$.*

3 L'application canonique

On suppose toujours que X est lisse, connexe et en position générale. Si $0 \leq l < k$ et $\delta_l = \delta(n,k,l)$, on a $0 < \delta_l < n+1$. D'après 1.1 :

$$\pi_*(\omega_{X'}(-lH')) = \underline{\mathrm{Hom}}(\pi_*(\mathcal{O}_{X'}(lH')), \omega_X) = (E_{l,0}^\vee \otimes \omega_X) \oplus \ldots \oplus (E_{l,\delta_l}^\vee \otimes \omega_X(\delta_l H))$$

Un élément non nul de $E_{l,\delta_l}^\vee$ définit naturellement un morphisme non nul :

$$\pi^*(\omega_X(\delta_l H)) \longrightarrow \omega_{X'}(-lH')$$

3.1 Si $l = 0$, on obtient une factorisation :

$$\begin{array}{ccc} \mathbb{P}(\mathrm{H}^0(\omega_{X'})) & \dashleftarrow & X' \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathrm{H}^0(\omega_X(\delta_0 H))) & \dashleftarrow & X \end{array}$$

La flèche verticale de gauche est une projection.

3.2 Si $l = 1$ et $\delta_1 \geq \dim(X)$, on sait (cf. [BS95, 8.8.5, p. 239]) que $\omega_X(\dim(X)\,H)$ est engendré par ses sections sauf si $(X, \mathcal{O}_X(H)) = (\mathbb{P}_m, \mathcal{O}_{\mathbb{P}_m}(1))$. Donc, hormis ce cas, $\mathrm{H}^0(\omega_{X'}(-H'))$ est $\neq 0$. Si s est une section non nulle de $\omega_{X'}(-H')$, le système linéaire $s \otimes \mathrm{H}^0(\mathcal{O}_{X'}(H')) \subset \mathrm{H}^0(\omega_{X'})$, définit une application rationnelle qui est un plongement hors du lieu des zéros de s et donc l'application canonique de X' est birationnelle.

En particulier :

Proposition 3.3 *Soient $S \subset \mathbb{P}_4$ une surface lisse et $\pi : \mathbb{P}_4 \longrightarrow \mathbb{P}_4$ un endomorphisme qui n'est pas un automorphisme. Alors, si S est en position générale, $S' = \pi^{-1}(S)$ est lisse et son fibré canonique est très ample, sauf si S est un plan et $k = 2$.*

3.4 La surface S' est donc de type général, sauf dans le cas d'exception de la proposition, S' étant alors une surface de Del Pezzo.

Démonstration de 3.3 Comme $N_{S'}\mathbb{P}' = \pi^*(N_S\mathbb{P})$, alors $\omega_{S'} \otimes \pi^*(\omega_{S'})^\vee = (\omega_{\mathbb{P}'} \otimes \pi^*(\omega_{\mathbb{P}})^\vee)_{|S'} = \mathcal{O}_{S'}(5(k-1)H')$. Donc $\omega_{S'} = \pi^*(\omega_S(2H)) \otimes \mathcal{O}_{S'}((3k-5)H')$. Cela démontre le résultat puisque, sauf si S est un plan, $\omega_S(2H)$ est engendré par ses sections. ■

Extendability of regular differential forms & codimension of the singular locus

The starting point of this work was the following question, raised by J. Fogarty in [Fog88] (here we assume that all the schemes are of finite type over a field of characteristic 0) :

Question *Let G be a finite group acting on a smooth variety X and $\pi : X \longrightarrow Y$ the quotient. Is the surjectivity of the natural morphism*

$$\Omega^1_Y \longrightarrow (\Omega^1_X)^G$$

a necessary and sufficient condition for the quotient Y to be smooth ?

In this article J. Fogarty verifies that the surjectivity condition is indeed necessary. He also proves that, under the additional assumption that the group G is abelian, this condition is sufficient.

Observe that the module $(\Omega^1_X)^G$ is naturally isomorphic to $\Omega^1_Y{}^{\vee\vee}$ and, the variety Y being normal, also isomorphic to the module ω^1_Y of regular 1-forms (cf. appendix B). It is also easily checked that this problem reduces to the case where X is a rational representation of G. In particular when $G \subset \mathrm{SL}(\mathbb{C}^2)$, then $Y = \mathbb{C}^2/G$ is a complete intersection and one can give an affirmative answer to the question above. However, already in dimension 2 (i.e. $G \subset \mathrm{GL}(\mathbb{C}^2)$) this question appears to be quite tricky.

For want of solving this problem, we reformulated and generalized it in two directions :

Quotients by reductive groups Let G be a reductive group acting on a smooth affine variety X and denote by $\pi : X \longrightarrow Y$ the quotient. In section 4.1 we recall the construction of the graded algebra $\Omega_{X,G}$ of horizontal differential forms and the existence of a morphism

$$(d\pi)^G \quad : \quad \Omega_Y \longrightarrow (\Omega_{X,G})^G$$

Following M. Brion, we have :

Theorem ([Bri98, Theorem 1]) *Let G be a reductive algebraic group acting on a smooth affine variety X, and let $\pi : X \longrightarrow Y$ be the quotient. If Y is smooth then $(d\pi)^G$ is an isomorphism.*

In analogy with the case of finite groups, we therefore ask :

Question *Is there any relationship between the surjectivity of the morphism $(d\pi)^G$ in some degree k (resp. in degrees $\leq k$) and the codimension of the singular locus of Y ?*

General case Let Y be a reduced scheme. In appendix B, we recall the construction of the Ω_Y-module ω_Y of regular differential forms and the existence, following El Zein, of the fundamental class morphism :

$$\mathrm{c}_Y : \Omega_Y \longrightarrow \omega_Y$$

We also recall the definition of the module $\tilde{\Omega}_Y$ of absolutely regular differential forms. Let $i : Y_{\mathrm{smth}} \subset Y$ be the smooth locus, we then have a sequence of morphisms

$$\Omega_Y \longrightarrow \tilde{\Omega}_Y \longrightarrow \omega_Y \longrightarrow i_*\Omega_{Y_{\mathrm{smth}}}$$

whose composition is the restriction morphism to Y_{smth}. If Y is normal then there are isomorphisms $\omega_Y \simeq i_*\Omega_{Y_{\mathrm{smth}}} \simeq \Omega_Y{}^{\vee\vee}$.

With these notations fixed, we can now ask :

Question *Is there any relationship between the surjectivity of the fundamental class morphism $(d\pi)^G$ (resp. the restriction morphism) in some degree k (resp. in degrees $\leq k$) and the codimension of the singular locus of Y ?*

Let's give an example of the kind of relationship we are thinking about : H. Flenner (see also Van Straten-Steenbrink [vSS85] in the case of isolated singularities) proved the following

Theorem ([Fle88]) *Let Y be a normal variety (resp. the spectrum of a complete normal local ring, resp. a normal analytic space). The natural morphism $\tilde{\Omega}_Y \longrightarrow i_*\Omega_{Y_{\mathrm{smth}}}$ is surjective in degrees $< \mathrm{codim}\,(Y_{\mathrm{sing}}) - 1$.*

However, the converse of this theorem is false : If Y is a quotient of a smooth variety by a finite group (4.4.5) or an isolated rational singularity, we have that $\tilde{\Omega}_Y \simeq \omega_Y \simeq i_*\Omega_{Y_{\mathrm{smth}}}$. We recall that a n-dimensional variety Y is said to have rational singularities if it is normal, Cohen-Macaulay and satisfies $\tilde{\Omega}_Y^n \simeq \omega_Y^n$.

The following two statements combined with the preceding theorem might help understanding a little more the case of quotients by reductive groups :

Theorem ([Bri98, Theorem 2]) *Let G be a reductive algebraic group acting on a smooth affine variety X, and let $\pi : X \longrightarrow Y$ be the quotient. Assume moreover that no invariant divisor in X is contracted by π in codimension ≥ 2 in Y. Then the morphism $(d\pi)^G$ induces an isomorphism*

$$\Omega_Y{}^{\vee\vee} \xrightarrow{\sim} (\Omega_{X,G})^G.$$

And without assumption on divisors, we prove :

Proposition (4.4.4) *Let G be a reductive algebraic group acting on a smooth affine variety X and let $\pi : X \longrightarrow Y$ be the quotient. Then we have natural inclusions*

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq \omega_Y$$

which are isomorphisms on the smooth locus of Y and which induce isomorphisms in degree $n = \dim(Y)$:

$$\tilde{\Omega}_Y^n \xrightarrow{\sim} (\Omega_{X,G}^n)^G \xrightarrow{\sim} \omega_Y^n.$$

Last but not least, we have the following fundamental theorem of E. Kunz and R. Waldi :

Theorem ([KW88, 5.22, p107]) *Let Y be a reduced scheme, Cohen-Macaulay and equidimensional of dimension n. Then the support of* $\mathrm{Coker}\,(\mathrm{c}_Y)^n$ *is precisely the singular locus of Y.*

Section 4 is devoted to the proof of our main results concerning the case of quotients by reductive groups :

Theorem (4.5.1) *Let G be a reductive algebraic group acting on a smooth affine variety X and let $\pi : X \longrightarrow Y$ the quotient. If the morphism $(d\pi)^G$ is surjective in a fixed degree $k \leq \dim(Y)$, then Y is smooth in codimension $k+1$.*

In the particular case of finite groups, we have that $\Omega_{X,G} = \Omega_X$. Therefore, this theorem provides an affirmative answer to the question of J. Fogarty in dimension 2. The reader will find in (4.6) additional results and remarks about quotients by finite groups.

We have another statement where the criterion is intrinsic to Y :

Theorem (4.5.2) *Let Y be the quotient of a smooth affine variety by a reductive algebraic group. If the fundamental class morphism $c_Y : \Omega_Y \longrightarrow \omega_Y$ is surjective in a fixed degree $k \leq \dim(Y)$, then Y is smooth in codimension $k+1$.*

Note that the converse of these theorems is false (4.6.2).

In section 5 we establish similar results relating local homological properties of Ω_Y and codimension of the singular locus for complete intersections. In the sake of streamliness, we state here these results only in the case of schemes of finite type over a characteristic 0 field :

Theorem (5.0.3) *Let Y be a complete intersection scheme. For an integer $k \geq 0$, the following properties are equivalent :*

(i) Y is smooth in codimension k.

(ii) For all $i = 0 \ldots k$, Ω^i_Y is torsion-free in codimension i.

Moreover, under those conditions, for all $i = 1 \ldots k+1$ we have that $\text{proj.dim}\ \Omega^i = i$ *unless Y is smooth.*

Corollary (5.0.4) *Let Y be a reduced complete intersection scheme. For an integer $k \geq 0$, the following conditions are equivalent :*

(i) Y is smooth in codimension k.

(ii) The restriction morphism $\Omega_Y \longrightarrow i_* \Omega_{Y_{\mathrm{smth}}}$ is surjective in degrees $< k$.

Important remarks We started writing the "theory" of exterior and symetric power of complexes of appendix C in view of the results of section 5. With the sound feeling that, very likely, such a natural construction had already been considered. Indeed, it appeared to us later that this had already been done, in a mildly different way, by A. Lascoux, H. A. Nielsen and other authors. However we did not find an adequate reference for the precise results we needed. Therefore, we finally thought that it might be of some use to include this appendix here.

Similar remarks apply to the results in section 5 : We discovered afterward that a result very close to our theorem 5.0.3 had been published by I. Emmanouil in [Emm95]. The careful reader might observe that our proof is self-contained whereas Emmanouil needs to invoke a theorem of U. Vetter ([Vet70]) at some place. In fact one recovers this last theorem of Vetter as a corollary of our statements (5.0.3) and (5.0.4).

Notations and conventions We work over a fixed field $\mathbf{k}$ of characteristic 0 with algebraic closure $\bar{\mathbf{k}}$. All the schemes we consider are of finite type over $\mathbf{k}$. We denote by Ω_X the differential graded algebra $\oplus_{k\geq 0}\Omega^k_{X/\mathbf{k}}$ of Kähler differentials, and write Ω^k_X for $\Omega^k_{X/\mathbf{k}}$.

For G an algebraic group and a G-scheme X, we denote by G-$\mathcal{O}_X$-mod the category of G-equivariant $\mathcal{O}_X$-modules.

For an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ the category of complexes.

An affine $\mathbb{G}_m$-scheme X is said to be quasi-conical (this is an ugly terminology, but, we believe it is consistent with the algebraic definitions of homogeneous and quasi-homogeneous ideals) if $\mathcal{O}_X$ is generated by homogeneous sections of non-negative weights. We recall that X is said to be conical when $\mathcal{O}_X$ is generated by homogeneous sections of weight 1.

By differential operator, we mean differential operator relative to $\mathbf{k}$ in the sense of [Gro67, 16.8].

We denote by Γ the decreasing filtration by codimension of the support : Let c be an integer. For any $\mathcal{O}_X$-module M and $U \subset X$ an open subset, $\Gamma_c M(U)$ is the subgroup of $M(U)$ consisting of the sections having support of codimension $\geq c$ in X. We write $\Gamma_{(c)}$ for Γ_c/Γ_{c+1} and $\bar{M}$ for $\Gamma_{(0)}M$. In particular, when X is integral, $\Gamma_1 M$ is the submodule of torsion elements and $\bar{M} = \Gamma_{(0)}M$ is M modulo torsion. We recall that this filtration is preserved by differential operators and in particular by $\mathcal{O}_X$-linear morphisms. These definitions extend to categories of complexes in the obvious way.

Let X be a scheme and M a coherent $\mathcal{O}_X$-module. If $x \in X$, we denote by $\mathbf{k}(x)$ the residual field at x. We denote by $\beta(M)$ the upper-semicontinuous function $X \to \mathbb{N}$, $x \mapsto \operatorname{rank}_{\mathbf{k}(x)} M \otimes \mathbf{k}(x)$.

By a desingularisation of X, we always mean a desingularisation of X_{red}. We take ([EV98]) as a general reference for resolution of singularities, in particular for the existence of equivariant resolutions.

4 Quotient singularities

4.1 Horizontal differentials : Construction

Let G be an algebraic group, $\mathfrak{g}$ its Lie algebra considered as a G-module via the adjoint representation, and X a G-scheme. We will also consider G as a G-scheme by the action of G on itself by inner automorphism. We have the following diagram of equivariant maps :

$$\begin{array}{ccccc} G & \xleftarrow{\;p\;} & G \times X & \underset{s}{\overset{\mu}{\rightleftarrows}} & X \\ & & \big\downarrow{\scriptstyle q} & & \\ & & X & & \end{array}$$

where p and q are the projections, μ is the action map and s is the section of μ defined by $x \mapsto (e, x)$. This induces the following diagram of G-equivariant coherent modules on $G \times X$:

$$\begin{array}{ccc} \mu^*\Omega^1_X & \xrightarrow{d\mu} & \Omega^1_{G\times X} = p^*\Omega^1_G \oplus q^*\Omega^1_X \\ & \searrow & \downarrow \\ & & p^*\Omega^1_G \end{array}$$

Taking the pull-back by s of the diagonal morphism above, we obtain a morphism

$$d\mu^1_{X,G} \quad : \quad \Omega^1_X \longrightarrow s^*p^*\Omega^1_G = \mathfrak{g}^\vee \otimes \mathcal{O}_X$$

which, as a complex in degree $[0, 1]$, will be called $M^1_{X,G}$. For all $k \in \mathbb{N}$ we also define (cf. appendix C) $M^k_{X,G} = \wedge^k(M^1_{X,G})$.

Definition 4.1.1 *The G-equivariant module $\Omega^k_{X,G} = \mathrm{H}^0\big(M^k_{X,G}\big)$ is called the module of horizontal k-forms. We denote by $\Omega_{X,G}$ the graded algebra $\oplus_{k\geq 0}\Omega^k_{X,G}$.*

The module $\Omega^k_{X,G}$ is therefore the kernel of

$$\begin{aligned} d\mu^k_{X,G} \quad &: \quad \Omega^k_X \longrightarrow \Omega^{k-1}_X \otimes \mathfrak{g}^\vee \\ d\mu^k_{X,G}(df_1 \wedge \ldots \wedge df_k) \quad &= \quad \sum_{i=1}^{k} (-1)^{k-i} df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k \otimes d\mu^1_{X,G}(df_i), \end{aligned}$$

and we see that horizontal forms are those forms whose interior product with any vector field induced by the group action vanishes.

The preceeding construction is natural in X. Thus, for any equivariant map $f : X \longrightarrow Y$ the cotangent morphism induces $f^*M^k_{Y,G} \longrightarrow M^k_{X,G}$ and therefore morphisms $f^*\Omega^k_{Y,G} \longrightarrow \Omega^k_{X,G}$. It is also clear from the construction that if the action of G is trivial then $d\mu^1_{X,G} = 0$ and consequently we have $\Omega^k_{X,G} = \Omega^k_X$. From these remarks, we deduce :

Proposition 4.1.2 *Let $\pi : X \longrightarrow Y$ be a G-invariant morphism, then the cotangent morphism $d\pi : \pi^*\Omega_Y \longrightarrow \Omega_X$ factors through $\Omega_{X,G} \subset \Omega_X$.*

4.1.3 Remark This last proposition applies in particular when π is a categorical quotient of X. Assume that X is affine and that G is a reductive linear group. Let $\pi : X \longrightarrow Y$ be the quotient of X. By (4.1.2) there is a morphism $\pi^*\Omega_Y \longrightarrow \Omega_{X,G}$ and therefore a morphism

$$(d\pi)^G \quad : \quad \Omega_Y \longrightarrow (\Omega_{X,G})^G$$

of coherent modules on Y. Under the additional assumption that X is smooth, then $(\Omega_{X,G})^G$ is a torsion-free module and by ([Bri98, Theorem 1]) the morphism $(d\pi)^G$ is generically an isomorphism. Consequently, the kernel of $(d\pi)^G$ is exactly the torsion of Ω_Y and we have an inclusion : $\bar{\Omega}_Y \subseteq (\Omega_{X,G})^G$.

Some lemmas that we shall need later :

Lemma 4.1.4 *Let $f : X \longrightarrow Y$ be an equivariant map of G-schemes. Assume that the adjoint morphism $\Omega_Y \longrightarrow f_*\Omega_X$ is injective. Then the diagram :*

$$\begin{array}{ccc} \Omega_{Y,G} & \longrightarrow & \Omega_Y \\ \downarrow & & \downarrow \\ f_*\Omega_{X,G} & \longrightarrow & f_*\Omega_X \end{array}$$

is a fiber product diagram where all the morphisms are injective.

In other words, under the assumption, a differential form is horizontal if and only if its pull-back is.

Proof of 4.1.4 The statement is an easy consequence of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{Y,G} & \longrightarrow & \Omega_Y & \xrightarrow{d\mu_{Y,G}} & \Omega_Y \otimes \mathfrak{g}^\vee \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f_*\Omega_{X,G} & \longrightarrow & f_*\Omega_X & \xrightarrow{f_*d\mu_{X,G}} & f_*\Omega_X \otimes \mathfrak{g}^\vee \end{array}$$

where the two vertical morphisms on the left are injective by assumption. ■

Lemma 4.1.5 *Let G be an algebraic group and $f : X \longrightarrow Y$ be a principal G-fibration. Then the natural morphism $df : f^*\Omega_Y \longrightarrow \Omega_{X,G}$ is an isomorphism.*

One is reduced to proving the statement in the case of a trivial G-fibration where this is obvious.

Lemma 4.1.6 *Let G and H be algebraic groups acting on a scheme X. The natural commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{X,G} \otimes \mathfrak{h}^\vee & \longrightarrow & \Omega_X \otimes \mathfrak{h}^\vee & \xrightarrow{d\mu_{X,G}\otimes\mathfrak{h}^\vee} & \Omega_X \otimes \mathfrak{g}^\vee \otimes \mathfrak{h}^\vee \\ & & \uparrow & & \uparrow{\scriptstyle d\mu_{X,H}} & & \uparrow{\scriptstyle d\mu_{X,H}\otimes\mathfrak{g}^\vee} \\ 0 & \longrightarrow & \Omega_{X,G} & \longrightarrow & \Omega_X & \xrightarrow{d\mu_{X,G}} & \Omega_X \otimes \mathfrak{g}^\vee \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Omega_{X,G} \cap \Omega_{X,H} & \longrightarrow & \Omega_{X,H} & \longrightarrow & \Omega_{X,H} \otimes \mathfrak{g}^\vee \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

has exact rows and columns. Moreover, it induces an exact sequence :

$$0 \longrightarrow \Omega_{X,G} \cap \Omega_{X,H} \longrightarrow \Omega_X \longrightarrow \Omega_X \otimes (\mathfrak{g}^\vee \oplus \mathfrak{h}^\vee).$$

Observe, that we did not assume that the actions of G and H on X commute, therefore this diagram is only separately G and H-equivariant, but, in general, not $G \times H$-equivariant.

4.2 The Euler derivation

We go on using the notations of section 4.1. Let $T = \mathbb{G}_m = \mathrm{Spec}(\mathbf{k}[\lambda, \lambda^{-1}])$ be a one-dimensional torus with Lie algebra $\mathfrak{t}$ and X an affine T-scheme. We recall that since T is abelian, the adjoint representation is trivial, i.e. $\mathfrak{t}$ is a trivial T-module. We fix once for all an isomorphism $\mathbf{k} \simeq \mathfrak{t}$ via the left-invariant derivation $\lambda\frac{\partial}{\partial\lambda}$. Composing the dual of this last isomorphism with $d\mu^1_{X,T}$ we obtain a derivation on X :

$$\mathrm{e}_{X,T} \quad : \quad \Omega^1_X \longrightarrow \mathcal{O}_X$$

called the Euler derivation. We write e for $\mathrm{e}_{X,T}$ when no confusion can occur. The morphism $\mathrm{e}_{X,T}$ can also be understood as a complex in degree $[-1, 0]$ that we denote by E^1_X. Since X is affine, we have $X = \mathrm{Spec}(A)$ with A a graded ring. The grading of A corresponds to the weight for the T-action : A section f of $\mathcal{O}_X$ is said to be homogeneous of weight w if $\mu^* f = \lambda^w q^* f$. If f is homogeneous of weight w, we set $|f| = w$.

Proposition 4.2.1 *Let f be an homogeneous section of $\mathcal{O}_X$. Then :*

$$\mathrm{e}(df) = |f|f.$$

Proof of 4.2.1 Let $w = |f|$. We have :

$$\begin{aligned} \mathrm{e}(df) &= \lambda\frac{\partial}{\partial\lambda} d\mu^1_{X,T}(df) \\ &= \lambda\frac{\partial}{\partial\lambda} s^*(w\lambda^{w-1} f.d\lambda) \\ &= \lambda\frac{\partial}{\partial\lambda} s^*(w\lambda^w f.\frac{d\lambda}{\lambda}) \\ &= \lambda\frac{\partial}{\partial\lambda}(wf.\frac{d\lambda}{\lambda}) \\ &= wf \end{aligned}$$

as expected. ■

For $n \geq 1$ we consider the complex $\mathrm{S}^n(E^1_X)$ (cf. appendix C). This is a complex concentrated in degree $[-n, 0]$, whose term of degree $-k$ is Ω^k_X and with differential in degree $-k$ given by :

$$\mathrm{d}(df_1 \wedge \ldots \wedge df_k) = \sum_{i=1}^{k} (-1)^{k-i} \mathrm{e}(df_i) df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k.$$

Hence for $n \gg 0$ (for instance, $n \geq \beta(\Omega^1_X)$) we see that the complex $\mathrm{S}^n(E^1_X)$ is independent of n. We thus have constructed a complex

$$\cdots \longrightarrow \Omega^{k+1}_X \longrightarrow \Omega^k_X \longrightarrow \cdots \longrightarrow \Omega^1_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

that we will denote by (Ω_X, e). With this notation we mean that we have extended e to Ω_X. The exterior differential algebra (Ω_X, d) is also graded by weight : A section α of (Ω_X, d) is homogeneous of weight w if $\mu^*\alpha = \lambda^w q^*\alpha$. The following properties are then easy to check :

Proposition 4.2.2 *Let α and β be homogeneous sections of Ω_X.*

(i) The forms $d\alpha$ and $\mathrm{e}(\alpha)$ are homogeneous and $|d\alpha| = |\mathrm{e}(\alpha)| = |\alpha|$.

(ii) The form $\alpha \wedge \beta$ is homogeneous and $|\alpha \wedge \beta| = |\alpha| + |\beta|$.

(iii) The algebra Ω_X is generated by the differentials of homogeneous sections of $\mathcal{O}_X$.

(iv) $\mathrm{Ker}\,(\mathrm{e}) = \Omega_{X,T}$.

Proposition 4.2.3 *For any homogeneous k-forms α, we have :*

$$[\mathrm{e}, d]\alpha = (-1)^k |\alpha|\alpha.$$

Let $c \geq 0$. The operators e and d preserve the filtration by codimension of the support and therefore they induce operators on $\Gamma_c\Omega_X$ and $\Gamma_{(c)}\Omega_X$ that we again denote by e and d. Moreover, since $\Gamma_c\Omega_X$ and $\Gamma_{(c)}\Omega_X$ are also T-equivariant, the statement above remains true for these modules.

Proof of 4.2.3 We can assume that $\alpha = f_0 df_1 \wedge \ldots \wedge df_k$ and compute :

$$\begin{aligned}
[\mathrm{e}, d](f_0 df_1 \wedge \ldots \wedge df_k) &= \ldots \\
&= \mathrm{e}(df_0 \wedge df_1 \wedge \ldots \wedge df_k) - d(\sum_{i=1}^{k} (-1)^{k-i} f_0 \mathrm{e}(df_i) df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k) \\
&= (-1)^k \mathrm{e}(df_0) df_1 \wedge \ldots \wedge df_k + \sum_{i=1}^{k} (-1)^{k-i} \mathrm{e}(df_i) df_0 \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k \\
&\quad - \sum_{i=1}^{k} (-1)^{k-i} \mathrm{e}(df_i) df_0 \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k \\
&\quad - \sum_{i=1}^{k} (-1)^{k-i} f_0 |f_i| df_i \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k \\
&= (-1)^k \mathrm{e}(df_0) df_1 \wedge \ldots \wedge df_k - \sum_{i=1}^{k} (-1)^{k-i} |f_i| f_0 df_i \wedge df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k \\
&= (-1)^k |f_0| f_0 df_1 \wedge \ldots \wedge df_k + \sum_{i=1}^{k} (-1)^k |f_i| f_0 df_1 \wedge \ldots \wedge df_i \wedge \ldots \wedge df_k \\
&= (-1)^k (|f_0| + |f_1| + \ldots + |f_k|) f_0 df_1 \wedge \ldots \wedge df_k \\
&= (-1)^k |\alpha|\alpha.
\end{aligned}$$

■

Proposition 4.2.4 *The submodule $(\Omega_{X,T})^T \subseteq (\Omega_X)^T$ is stable by the exterior derivative of Ω_X.*

Proof of 4.2.4 Keeping in mind that T-invariants are precisely homogeneous sections of null weight, the result is a direct consequence of (4.2.3) and (4.2.2 (iv)). ■

4.3 Horizontal differentials : Poincaré lemmas

Proposition 4.3.1 *Let G be a reductive algebraic group and X an affine G-scheme. Then the submodule $(\Omega_{X,G})^G \subset (\Omega_X)^G$ is stable by the exterior derivative of Ω_X.*

4.3.2 Remark This statement holds more generally for G a linear algebraic group. But its proof would require an algebraic construction of the Lie derivative that we did not explain here. The proof would run as follows : For $v \in \mathfrak{g}$, denotes by L_v the Lie derivative and by $< v, \cdot >$ the interior product. Then, for any section α of Ω_X we have the relation :

$$\mathrm{L}_v\alpha = d < v, \alpha > + < v, d\alpha > .$$

The statement therefore follows from the observation that L_v vanishes on $(\Omega_X)^G$.

Proof of 4.3.1 We recall $(\Omega_X)^G$ is obviously stable by exterior differentiation. Since G is reductive, on can find one-dimensional subtori $T_1, \ldots, T_d$ of G such that $\mathfrak{g} = \mathfrak{t}_1 \oplus \ldots \oplus \mathfrak{t}_d$. Then, by (4.1.6), we have :

$$\Omega_{X,G} \quad = \quad \Omega_{X,T_1} \cap \ldots \cap \Omega_{X,T_d}.$$

And therefore

$$\begin{aligned}(\Omega_{X,G})^G &= (\Omega_X)^G \cap \Omega_{X,T_1} \cap \ldots \cap \Omega_{X,T_d} \\ &= (\Omega_X)^G \cap (\Omega_{X,T_1})^{T_1} \cap \ldots \cap (\Omega_{X,T_d})^{T_d}.\end{aligned}$$

By (4.2.4), all the terms in the intersection above are stable by d, so we can conclude that $(\Omega_{X,G})^G$ is stable by d too. ■

Proposition 4.3.3 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme. Then $\Omega_{X,G}$ is stable by $\mathrm{e} = \mathrm{e}_{X,T}$. We write $(\Omega_{X,G}, \mathrm{e})$ for this subcomplex of (Ω_X, e).*

Therefore, if $c \geq 0$ is an integer, $\Gamma_c \Omega_{X,G}$ is also stable by e and therefore there is an induced endomorphism on $\Gamma_{(c)}\Omega_{X,G}$.

Corollary 4.3.4 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme. Let α be a homogeneous section (with respect to the T-action) of $(\Omega^k_{X,G})^G$. Then*

$$[\mathrm{e}, d]\alpha = (-1)^k |\alpha| \alpha.$$

Clearly, we again have a similar statement for $\Gamma_c(\Omega_{X,G})^G$, $\Gamma_{(c)}(\Omega_{X,G})^G$, $(\Gamma_c\Omega_{X,G})^G$ or $(\Gamma_{(c)}\Omega_{X,G})^G$.

Proof of 4.3.3 Let $\mathrm{e} = \mathrm{e}_{X,T}$ and consider the diagram

$$\begin{array}{ccc} \mathfrak{g}^\vee \otimes \mathcal{O}_X & & \\ \uparrow {\scriptstyle d\mu^1_{X,G}} & & \\ \Omega^1_X & \xrightarrow{\mathrm{e}} & \mathcal{O}_X \end{array}$$

as the following complex of objects of $\mathcal{C}(G \times T{-}\mathcal{O}_X{-}\mathrm{mod.})$ in degree $[-1, 0]$:

$$\cdots \longrightarrow 0 \longrightarrow M^1_{X,G} \longrightarrow \mathcal{O}_X \longrightarrow 0 \longrightarrow \cdots$$

As in the construction of (Ω_X, e) we consider the n-fold symmetric product of this last complex (of complexes). For $n \gg 0$, it is independent of n. Hence we obtain a complex of complexes :

$$\cdots \longrightarrow M^{k+1}_{X,G} \longrightarrow M^k_{X,G} \longrightarrow \cdots \longrightarrow M^1_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

which, as a bicomplex, writes down :

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & & & \\ & & \uparrow & & \uparrow & & & & \\ \cdots & \longrightarrow & \Omega^{k-1}_X \otimes \mathrm{S}^2(\mathfrak{g}^\vee) & \longrightarrow & \Omega^{k-1}_X \otimes \mathrm{S}^2(\mathfrak{g}^\vee) & \longrightarrow \cdots \longrightarrow & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \Omega^k_X \otimes \mathfrak{g}^\vee & \xrightarrow{\mathrm{e}\otimes\mathfrak{g}^\vee} & \Omega^{k-1}_X \otimes \mathfrak{g}^\vee & \longrightarrow \cdots \longrightarrow & \mathfrak{g}^\vee \otimes \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow {\scriptstyle d\mu^{k+1}_{X,G}} & & \uparrow {\scriptstyle d\mu^k_{X,G}} & & \uparrow {\scriptstyle d\mu^1_{X,G}} & & \uparrow \\ \cdots & \longrightarrow & \Omega^{k+1}_X & \xrightarrow{\mathrm{e}} & \Omega^k_X & \longrightarrow \cdots \longrightarrow & \Omega^1_X & \xrightarrow{\mathrm{e}} & \mathcal{O}_X \longrightarrow 0 \end{array}$$

Recall that $\mathrm{Ker}\,(d\mu^k_{X,G}) = \Omega^k_{X,G}$. Thus, taking homology along the columns in vertical degree 0 gives the result. ∎

Proposition 4.3.5 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme. Then*

$$\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big) = \mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)^T .$$

Let $c \geq 0$. Then the same relation holds for $\Gamma_c(\Omega_{X,G})^G$, $\Gamma_{(c)}(\Omega_{X,G})^G$, $(\Gamma_c\Omega_{X,G})^G$ and $(\Gamma_{(c)}\Omega_{X,G})^G$.

Proof of 4.3.5 Let α be a homogeneous section of $(\Omega^k_{X,G})^G \cap \mathrm{Ker}\,(\mathrm{e})$. Then by (4.3.4) we have $\mathrm{e}d\alpha = (-1)^k|\alpha|\alpha$. Therefore if $|\alpha| \neq 0$ the class of α in $\mathrm{H}_k\Big((\Omega_{X,G})^G, \mathrm{e}\Big)$ vanishes. Since $\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)^T$ is a direct factor of $\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)$, the equality is proved. ■

Lemma 4.3.6 *Let X be a quasi-conical affine T-scheme. Then the pull-back morphism for the quotient map $X \longrightarrow X/\!\!/T$ induces isomorphisms :*

$$\Omega_{X/\!\!/T} \xrightarrow{\sim} (\Omega_{X,T})^T \xrightarrow{\sim} (\Omega_X)^T \subset \Omega_X .$$

Proof of 4.3.6 Easy, by arguments on weights. ■

Proposition 4.3.7 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme, quasi-conical with respect to the T-action. Then the natural morphism*

$$\Omega_{X/\!\!/T,G} \longrightarrow (\Omega_{X,G})^T$$

induced by the G-equivariant map $X \longrightarrow X/\!\!/T$, is an isomorphism.

Proof of 4.3.7 By (4.3.6) the hypotheses of (4.1.4) are satisfied for the map $X \longrightarrow X/\!\!/T$. Taking T-invariants in the diagram of (4.1.4) together with the isomorphism $\Omega_{X/\!\!/T} \xrightarrow{\sim} (\Omega_X)^T$ gives the result. ■

Proposition 4.3.8 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme, quasi-conical with respect to the T-action. Let $d \geq c \geq 0$. There are isomorphisms of exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Gamma_d\Omega_{X,G})^T & \longrightarrow & (\Gamma_c\Omega_{X,G})^T & \longrightarrow & (\Gamma_c/\Gamma_d\,\Omega_{X,G})^T & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{H}\Big((\Gamma_d\Omega_{X,G})^T, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c\Omega_{X,G})^T, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c/\Gamma_d\,\Omega_{X,G})^T, \mathrm{e}\Big) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Gamma_d\Omega_{X,G})^{G\times T} & \longrightarrow & (\Gamma_c\Omega_{X,G})^{G\times T} & \longrightarrow & (\Gamma_c/\Gamma_d\,\Omega_{X,G})^{G\times T} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{H}\Big((\Gamma_d\Omega_{X,G})^G, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c\Omega_{X,G})^G, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c/\Gamma_d\,\Omega_{X,G})^G, \mathrm{e}\Big) & \longrightarrow & 0
\end{array}$$

Proof of 4.3.8 By (4.3.7) we have $(\Gamma_c\Omega_{X,G})^T \subset \Omega_{X/\!\!/T}$. Therefore e vanishes for all the complexes involved in the first isomorphism and this proves the first statement. For the second one, take G-invariants in the first diagram and use (4.3.5). ∎

One might understand the next two statements as a natural generalisation, with e and d exchanged, of the Poincaré Lemma to singular varieties with reductive group action :

Corollary 4.3.9 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme, quasi-conical with respect to the T-action. Then the G-equivariant map $X \longrightarrow X/\!\!/T$ induces an isomorphism*

$$(\Omega_{X/\!\!/T,G})^G \xrightarrow{\sim} \mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big).$$

Corollary 4.3.10 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme, quasi-conical with respect to the T-action and such that $X/\!\!/T = \mathrm{Spec}(\mathbf{k})$. Then*

$$\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big) = \mathrm{H}\Big((\bar{\Omega}_{X,G})^G, \mathrm{e}\Big) = \mathbf{k}.$$

In particular, in the case of a trivial action of G, under the preceding hypotheses we have exact complexes

$$\cdots \longrightarrow \Omega^n_X \longrightarrow \cdots \longrightarrow \Omega^1_X \longrightarrow \mathcal{O}_X \longrightarrow \mathbf{k} \longrightarrow 0$$

$$0 \longrightarrow \bar{\Omega}^n_X \longrightarrow \cdots \longrightarrow \bar{\Omega}^1_X \longrightarrow \mathcal{O}_X \longrightarrow \mathbf{k} \longrightarrow 0$$

4.4 Absolutely regular horizontal differentials

In this section, we merge the construction of horizontal differentials and the content of appendix B.2.

Let X be a G-scheme and $f : \tilde{X} \longrightarrow X$ a G-equivariant desingularisation. We denote by $\tilde{\Omega}_{X,G}$ the sheaf $f_*\Omega_{\tilde{X},G}$. This definition is independent of the choice of f, as in the non-equivariant case, since two equivariant resolutions of singularities can be covered by a third one.

By construction, we have natural equivariant morphisms

$$\Omega_{X,G} \longrightarrow \tilde{\Omega}_{X,G} \longrightarrow i_*\Omega_{X_{\mathrm{smth}},G}$$

where i is the inclusion $X_{\mathrm{smth}} \subset X$. Therefore, when X is reduced, we have :

$$\Omega_{X,G} \longrightarrow \bar{\Omega}_{X,G} \subset \tilde{\Omega}_{X,G} \subset i_*\Omega_{X_{\mathrm{smth}},G}.$$

Proposition 4.4.1 *Let $f : X \longrightarrow Y$ be an equivariant dominant morphism. Then we have a commutative diagram*

$$\begin{array}{ccc} \Omega_{X,G} & \longrightarrow & \tilde{\Omega}_{X,G} \\ \uparrow & & \uparrow \\ f^*\Omega_{Y,G} & \longrightarrow & f^*\tilde{\Omega}_{Y,G} \end{array}$$

Proposition 4.4.2 *Let $f : X \longrightarrow Y$ be an invariant dominant morphism. Then we have a commutative diagram*

$$\begin{array}{ccc} \Omega_{X,G} & \longrightarrow & \tilde{\Omega}_{X,G} \\ \uparrow & & \uparrow \\ f^*\Omega_{Y} & \longrightarrow & f^*\tilde{\Omega}_{Y} \end{array}$$

Proposition 4.4.3 *Let $f : X \longrightarrow Y$ be a proper equivariant birational morphism. Then the morphism $\tilde{\Omega}_{Y,G} \longrightarrow f_*\tilde{\Omega}_{X,G}$ is an isomorphism.*

With this at hand, we can give a partial answer to the question raised by M. Brion ([Bri98, after Theorem 2]) :

Proposition 4.4.4 *Let G be a reductive algebraic group, X be a smooth affine G-scheme and $\pi : X \longrightarrow Y$ the quotient. There is a sequence of inclusions :*

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq \omega_Y$$

which are isomorphisms on the smooth locus of Y. Let $n = \dim(Y)$, then there are isomorphisms :

$$\tilde{\Omega}^n_Y \xrightarrow{\sim} (\Omega^n_{X,G})^G \xrightarrow{\sim} \omega^n_Y.$$

Proof of 4.4.4 Since $\Omega_{X,G} = \tilde{\Omega}_{X,G}$, by (4.4.2) we have inclusions $\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G$ of torsion-free modules. Moreover, by the theorem of Brion ([Bri98, Theorem 1]), these are isomorphisms outside the closed subset Y_{sing}, therefore outside a closed subset of codimension ≥ 2. Thus the modules involved have isomorphic biduals and we obtain :

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq {\Omega_Y}^{\vee\vee} = \omega_Y.$$

The second statement is then a direct consequence of the fact that Y has rational singularities ([Bou87]). Indeed, this implies that $\tilde{\Omega}^n_Y \xrightarrow{\sim} \omega^n_Y$. ■

4.4.5 Remark If one assume that all the points of X are strongly stable for the action of G, i.e., that for all closed points $x \in X$, the orbit Gx is closed and the stabilizer G_x is finite, then there are isomorphisms

$$\tilde{\Omega}_Y \xrightarrow{\sim} (\Omega_{X,G})^G \xrightarrow{\sim} \omega_Y.$$

To prove this, one can assume that the group G is already finite (use the Etale Slice Theorem as in the last reduction step in (4.5) below). With this assumption made it is easily seen that $\Omega_{X,G} = \Omega_X$ (here $\mathfrak{g} = (0)$) and that consequently $(\Omega_X)^G = \omega_Y$. It therefore remains to see that $\tilde{\Omega}_Y = (\Omega_X)^G$. This can be done as follows.

We have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\pi}} & \tilde{Y} \\ \downarrow g & & \downarrow f \\ X & \xrightarrow{\pi} & Y \end{array}$$

where f is a resolution of singularities for Y and $\tilde{X}$ is the normalization of the component birational to X in $X \times_Y \tilde{Y}$. The group G acts naturally on $\tilde{X}$ and the map $\tilde{\pi}$ is the quotient morphism. We thus have a morphism

$$\Omega_{\tilde{Y}} \longrightarrow (\tilde{\pi}_* \tilde{\Omega}_{\tilde{X}})^G$$

induced by $\tilde{\pi}$. Since $\tilde{X}$ is normal it is an isomorphism in codimension 1 and since $\Omega_{\tilde{Y}}$ is locally free it is in fact an isomorphism (recalling that $\tilde{\Omega}_{\tilde{X}}$ is torsion-free). Consequently, we have

$$\tilde{\Omega}_Y = f_* \Omega_{\tilde{Y}} = f_* (\tilde{\pi}_* \tilde{\Omega}_{\tilde{X}})^G = (\pi_* g_* \tilde{\Omega}_{\tilde{X}})^G = (\pi_* \Omega_X)^G.$$

This proves our claim.

4.5 Invariant horizontal differentials and smoothness

In this section we give proofs for the results stated in the introduction :

Theorem 4.5.1 *Let G be a reductive algebraic group acting on a smooth affine variety X and let $\pi : X \longrightarrow Y$ the quotient. If the morphism $(d\pi)^G$ is surjective in a fixed degree $k \leq \dim(Y)$, then Y is smooth in codimension $k+1$.*

Theorem 4.5.2 *Let Y be the quotient of a smooth affine variety by a reductive algebraic group . If the fundamental class morphism $c_Y : \Omega_Y \longrightarrow \omega_Y$ is surjective in a fixed degree $k \leq \dim(Y)$, then Y is smooth in codimension $k+1$.*

Proof of 4.5.1 We suppose that the morphism $(d\pi)^G : \Omega_Y \longrightarrow (\Omega_{X,G})^G$ is surjective in degree k, i.e. that we have a surjection $\Omega_Y^k \longrightarrow (\Omega_{X,G}^k)^G$ and we want to prove that under this hypothesis the singular locus of Y has codimension $> k+1$.

Etale slices

Quite generally, let $H \longrightarrow G$ be a map of reductive algebraic groups and W an affine H-scheme together with an H-equivariant map $j : W \longrightarrow X$. We let $G \times H$ act on $G \times W$ in the following way : $(g, h)(g', w) = (gg'h^{-1}, hw)$ and denote by $f : G \times W \longrightarrow G \times_H W$ the quotient by $1 \times H$. Observe that since $1 \times H$ acts freely on $G \times W$, the map f is a principal fibration and therefore is smooth. We obtain commutative diagram of $G \times H$-schemes :

$$\begin{array}{ccccc} G \times W & \xrightarrow{f} & G \times_H W & \xrightarrow{\bar{\mu}(G \times_H j)} & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ W & \longrightarrow & W/\!\!/H & \longrightarrow & X/\!\!/G \end{array} \tag{3.1}$$

where the vertical maps are quotients by G, the horizontal maps in the left-square are quotients by $1 \times H$ and $\bar{\mu}$ is the factorization of the $1 \times H$-invariant map μ ($1 \times H$ acts trivially on X).

For $y \in Y$ a closed point, we denote by $T_y \subset X_y$ the unique closed orbit over y. Let $x \in T_y$ be a closed point with (necessarily) reductive stabilizer $H = G_x$. The Etale Slice theorem of Luna ([Lun73, pp 96–99]), asserts the following : There exists a smooth locally closed, H-stable subvariety W of X such that $x \in W$, $G.W$ is an open set and such that in the natural commutative diagram (3.1) the right-square is cartesian with etale horizontal maps (i.e. an etale base change diagram). Moreover, letting $N = N_{T_y/X}(x)$ be the normal space at x of the orbit T_y, understood geometrically as a rational representation of H, there is a natural map of H-schemes $\rho : W \longrightarrow N$, etale at 0, which induces a commutative diagram :

$$\begin{array}{ccccc} G \times_H N & \xleftarrow{G \times_H \rho} & G \times_H W & \xrightarrow{\bar{\mu}(G \times_H j)} & X \\ \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ N/\!\!/H & \longleftarrow & W/\!\!/H & \longrightarrow & X/\!\!/G \end{array} \tag{3.2}$$

where the two squares are cartesian and the horizontal maps are etale neighbourhoods.

Stratification by slice type

We again refer to ([Lun73, pp 100–102]). Let $H \subseteq G$ be a reductive subgroup and N an H-module. We have a commutative diagram :

$$\begin{array}{ccc} G \times N & \xrightarrow{f} & G \times_H N \\ & \searrow & \downarrow \\ & & G/H \end{array} \tag{3.3}$$

which realizes $G \times_H N$ as the total space of a G-equivariant vector bundle over the affine homogeneous space G/H with fiber at 1 equals to N. Conversely let N be a G-equivariant vector bundle over an affine G-homogeneous base T. Let $t \in T$ be a closed point then $N(t)$ is a G_t-module and G_t is reductive. Thus we have an equivalence between the set $\{(H, N)\}$ up to conjugacy and the isomorphism classes of G-equivariant vector bundles over affine homogeneous bases. We denote by $\mathcal{M}(G)$ any of those sets and classes by brackets $[\,]$.

By the preceding, we thus have a map $\mu : Y(\bar{\mathbf{k}}) \longrightarrow \mathcal{M}(G)$ which sends y to the isomorphism class $[N_{T_y/X} \longrightarrow T_y]$ or equivalently to the "conjugacy class" $[H, N]$ with the notations of the preceding section. Let $\nu \in \mathcal{M}(G)$, then the set $\mu^{-1}(\nu)$ is a locally closed subset of Y, smooth with its reduced scheme structure. We will denote by Y_ν this smooth locally closed subscheme of Y. Moreover the collection $\{Y_\nu\}_{\nu \in \mathcal{M}(G)}$ is a finite stratification of Y (in particular μ has finite image). Therefore, the map μ can be extended to all the points of Y : Let $Z \subset Y$ be an irreducible closed subset, then there exists a unique $\nu \in \mathcal{M}(G)$ such that $Z \cap Y_\nu$ is dense in Z and one can set $\mu(Z) = \nu$. Observe that $\mu(Z)$ is the slice type of a general point of Z.

Another important fact about μ is that it is compatible with strongly etale (also called excellent) morphisms : Given such a map φ between smooth affine G-schemes, we have $\mu(\varphi /\!\!/ G) = \mu$.

We now look closer to G-schemes of the kind $G \times_H N$ and their quotients by G. Write N_H for the canonical complementary submodule to N^H in N : $N = N^H \times N_H$. Then in the construction of $G \times_H N$, N^H is a trivial H-module and therefore the diagram obtained when W is replaced by N in the left square of (3.1) reads :

$$\begin{array}{ccc} N^H \times (G \times N_H) & \xrightarrow{f} & N^H \times (G \times_H N_H) \\ \downarrow p & & \downarrow \phi \\ N^H \times N_H & \xrightarrow{\psi} & N^H \times (N_H /\!\!/ H) \end{array} \tag{3.4}$$

Let $\nu \in \mathcal{M}(G)$ be the class of (H, N), then $((G \times_H N)/\!\!/ G)_\nu = N^H \times 0 \subseteq N/\!\!/H$. One can convince oneself of this fact through the description of $G \times_H N$ as an equivariant vector bundle over G/H.

Reduction to an isolated singularity

First, it is harmless to assume that the singular locus of Y, Y_{sing} is irreducible. Let $\mu(Y_{\text{sing}}) = \nu = [H, N]$ and let $y \in Y_{\text{sing}} \cap Y_\nu$ be a general closed point. By standard etale base change arguments in the diagram (3.2), our hypothesis and our conclusion hold for π at y if and only if they respectively hold for ϕ at 0. We can therefore assume that $X = G \times_H N$, $\pi = \phi$ and $Y = N/\!\!/H$.

Now, with the notations of (3.4), it is clear that $Y_{\text{sing}} = N^H \times (N_H/\!\!/H)_{\text{sing}}$. On the other hand $Y_\nu = N^H \times 0$ and, since $\mu(Y_{\text{sing}}) = \nu$, the closed subset Y_ν should cut a dense open set on Y_{sing}. Consequently, we must have $Y_\nu = Y_{\text{sing}}$ and thus $(N_H/\!\!/H)_{\text{sing}} = 0$.

Let $\pi_H : X_H = G \times_H N_H \longrightarrow Y_H = N_H/\!\!/H$ be the quotient map by G, then clearly $\pi = N^H \times \pi_H$. Let k be an integer, then the map $(d\pi)^G$ is diagonal with respect to the decompositions :

$$\begin{aligned} \left(\Omega^k_{X,G}\right)^G &= \bigoplus_{i=0}^{k} \Omega^i_{N^H} \boxtimes \left(\Omega^{k-i}_{X_H,G}\right)^G \\ \Omega^k_Y &= \bigoplus_{i=0}^{k} \Omega^i_{N^H} \boxtimes \Omega^{k-i}_{Y_H} \end{aligned}$$

Therefore $(d\pi)^G$ is surjective in degree k if and only if $(d\pi_H)^G$ is surjective in all degrees $k - \dim\left(N^H\right), \ldots, k$.

To conclude, we can therefore make the extra assumption that $Y = X/\!\!/G = N/\!\!/H$ has only an isolated singularity at 0. And one should notice that the theorem remains in fact only to be proved when $k = \dim(Y) - 1$ or $\dim(Y)$, since, otherwise ($k < \dim(Y) - 1$) the statement is obviously true.

Reduction to the case of a representation

We keep in mind all the identifications and assumptions made previously. Recalling diagram (3.4) and applying lemmas (4.1.5) and (4.1.6) to the fibration f, we have an exact sequence

$$0 \longrightarrow f^*\Omega_{G\times_H N,G} \longrightarrow \Omega_{G\times N,G} \longrightarrow \Omega_{G\times N,G} \otimes \mathfrak{h}^\vee.$$

Taking G-invariants together with lemma (4.1.5) for p leads to the exact sequence :

$$0 \longrightarrow \left(f^*\Omega_{G\times_H N,G}\right)^G \longrightarrow \Omega_N \longrightarrow \Omega_N \otimes \mathfrak{h}^\vee$$

Therefore, we have proved that $\left(f^*\Omega_{G\times_H N,G}\right)^G = \Omega_{N,H}$. Taking H-invariants, we obtain

$$\left(\Omega_{N,H}\right)^H = \left(f^*\Omega_{G\times_H N,G}\right)^{G\times H} = \left(\Omega_{G\times_H N,G}\right)^G.$$

One can then conclude, that the hypothesis and the conclusion of the theorem hold for ϕ if and only if they respectively hold for ψ. Thus we are reduced to prove the theorem in the case where X is a rational representation of G with $X/\!\!/G$ having only an isolated singularity at the origin.

Conclusion

Carrying on, X is now a rational G-module with quotient $\pi : X \longrightarrow Y$, such that Y has only an isolated singularity at the origin. We recall the hypothesis in the theorem : The morphism $(d\pi)^G$ is surjective in degree $k \leq \dim(Y)$. We must prove that Y is smooth in codimension $k+1$. Thus we have to prove that if $k = \dim(Y)$ or $\dim(Y) - 1$ then Y is smooth.

The one dimensional torus $T = \mathbb{G}_m$ acts on X by homothety and this action commutes with the action of G. Thus X is a $G \times T$ scheme and Y is a T-scheme. Both X and Y are conical and $X/\!\!/T = Y/\!\!/T = \mathrm{Spec}(\mathbf{k})$.

Let $n = \dim(Y)$. Applying (4.3.10) to X and Y we obtain an injective morphism of exact complexes (the kernel of $(d\pi)^G$ is exactly the torsion of Ω_Y, cf. remark 4.1.3) :

$$\begin{array}{ccccccccccccccccc}
(\Omega_{X,G})^G & & 0 & \to & (\Omega^n_{X,G})^G & \to & (\Omega^{n-1}_{X,G})^G & \to & \cdots & \to & (\Omega^1_{X,G})^G & \to & \mathcal{O}_Y & \to & \mathbf{k} & \to & 0 \\
{\scriptstyle (d\pi)^G}\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \| & & \| & & \\
\bar{\Omega}_Y & & 0 & \longrightarrow & \bar{\Omega}^n_Y & \longrightarrow & \bar{\Omega}^{n-1}_Y & \longrightarrow & \cdots & \longrightarrow & \bar{\Omega}^1_Y & \longrightarrow & \mathcal{O}_Y & \to & \mathbf{k} & \to & 0
\end{array}$$

From this diagram, we deduce that if $(d\pi)^G$ is surjective in degree $n-1$, then it is also surjective in degree n. Therefore we have an isomorphism $\bar{\Omega}^n_Y \xrightarrow{\sim} (\Omega^n_{X,G})^G$. Moreover, by proposition (4.4.4) we know that $(\Omega^n_{X,G})^G = \omega^n_Y = \Omega^{n\,\vee\vee}_Y$. Thus $\bar{\Omega}^n_Y$ is a reflexive module.

Recall that by the theorem of Boutot ([Bou87]), Y has rational singularities and in particular is normal and Cohen-Macaulay and that ω^n_Y is then the dualizing module of Y. The fundamental class map c ([KW88, 5.2 p 91, 5.15 p 99], [EZ78] and appendix B), in degree n, factors through :

$$\begin{array}{ccc}
\Omega^n_Y & \xrightarrow{c} & \omega^n_Y \\
\downarrow & \nearrow & \\
\bar{\Omega}^n_Y & &
\end{array}$$

But $\bar{\Omega}^n_Y$ is reflexive and, since Y is normal, c is an isomorphism in codimension 1. Therefore c is necessarily surjective. We now invoke a theorem of Kunz and Waldi ([KW88, 5.22 p 107]) to conclude that Y is smooth.

The proof of theorem 4.5.1 is complete. ∎

Proof of 4.5.2 Using (4.5.1), we can a give a straightforward proof of the result : By (4.4.4) the hypotheses of (4.5.1) are satisfied for the same integer k. ∎

4.6 Additional results and remarks for finite groups

Let G be a finite group acting on a quasi-projective scheme X and let $\pi : X \longrightarrow Y$ be the quotient. We fix some terminology and recall some more or less classical results.

For an element $g \in G$, we denote the closed subscheme of g-fixed points by X^g and for a point $x \in X$, we denote its stabilizer (also called isotropy subgroup) by G_x. We then define a increasing filtration of G by normal subgroups in the following way : For $k \geq 0$ an integer we set $G^k = < g \in G, \operatorname{codim}(X^g) \leq k >$. In particular G^1 is the subgroup generated by the *pseudo-reflections* in G. For a point $x \in X^g$, if $\operatorname{codim}(X^g, x) \leq 1$ then g is said to be a *pseudo-reflection at* x. When X is smooth, this condition is satisfied if and only if locally at x, the diagonal form of g is of the kind $(\zeta, 1, \ldots, 1)$ for some root of unity ζ. Clearly g is a pseudo-reflection if and only if it is a pseudo-reflection at all the points of X^g.

When $G^1 = (1)$ one says that G is a *small* group of automorphisms of X. When $G = G^1$ one says that G is generated by pseudo-reflections. We now recall the classical

Theorem (Shephard-Todd, Chevalley, Serre) *With the preceding notations, the following conditions are equivalent :*

(i) The quotient Y is smooth.

(ii) For all $x \in X$, the group G_x is generated by the pseudo-reflections at x.

(iii) The $\mathcal{O}_Y$-module $\pi_* \mathcal{O}_X$ is locally free.

Thus, the study of quotients of smooth varieties by finite groups reduces to the study of quotients of smooth varieties by small finite groups of automorphisms : Indeed, the theorem above implies that G/G^1 is a small group of automorphisms of the smooth variety X/G^1. It is also clear that, for local questions, by the Etale Slice Theorem (see (4.5)) one is reduced to study the case where X is a rational representation of G.

In connection with the question raised in the introduction, J. Fogarty proved the following :

Theorem ([Fog88, Lemma 5]) *With the preceding notations and assuming moreover that the group G is abelian, the following conditions are equivalent :*

(i) The quotient Y is smooth.

(ii) The morphism $\Omega^1_Y \longrightarrow (\Omega^1_X)^G$ is surjective.

Note that since G is a finite group $\Omega_{X,G} = \Omega_X$.

At an early stage of our work we also obtained the following :

Proposition 4.6.1 *With the preceding notations and assuming moreover that the group G is abelian, the following conditions are equivalent :*

(i) The quotient Y is smooth.

(ii) The morphism $\Omega^2_Y \longrightarrow (\Omega^2_X)^G$ is surjective.

4.6.2 Remark With the theorem of Fogarty we can easily exhibit a counter example for the converse of theorems 4.5.1 and 4.5.2. Let ζ be a non-trivial root of unity and let G be the cyclic group $< \zeta \mathrm{Id} > \subset \mathrm{GL}(\mathbb{C}^3)$. Then the quotient $Y = \mathbb{C}^3/G$ has an isolated singularity and is therefore smooth in codimension 2. However, by this theorem, we know that the morphism $\Omega^1_Y \longrightarrow (\Omega^1_{\mathbb{C}^3})^G$ can't be surjective.

Proof of 4.6.1 We can first assume that X is a rational representation of G and choose a system of coordinates $X_1, \ldots, X_n$ which diagonalize the action of G. Then $X = \mathrm{Spec}(S)$, $S = \mathbf{k}[X_1, \ldots, X_n]$ and $Y = \mathrm{Spec}(S^G)$.

We also have that $\Omega^1_X = (X_1) \oplus \ldots \oplus (X_n)$ as G-equivariant $\mathcal{O}_X$-modules. Therefore

$$\Omega^2_X = \bigoplus_{i<j} (X_i X_j).$$

Observe also that $(\Omega^1_X)^G = (X_1)^G \oplus \ldots \oplus (X_n)^G$. Hence, we can write

$$\bigwedge^2 (\Omega^1_X)^G = \left(\bigoplus_i \bigwedge^2 (X_i)^G \right) \oplus \left(\bigoplus_{i<j} (X_i)^G \otimes (X_j)^G \right)$$

In this expression, the first factor in the right term is a torsion submodule. Indeed the $(X_i)^G$'s are generically free of rank one.

The surjectivity of $\Omega^2_Y \longrightarrow (\Omega^2_X)^G$ implies that the natural morphism

$$\bigwedge^2 (\Omega^1_X)^G \longrightarrow (\Omega^2_X)^G$$

is surjective also. Consequently, for all i, j such that $i \neq j$ the multiplication morphism

$$(X_i)^G \otimes (X_j)^G \longrightarrow (X_i X_j)^G = (X_i)^G \cap (X_j)^G$$

is surjective. In other words, for all i, j such that $i \neq j$, we have

$$(X_i)^G . (X_j)^G = (X_i X_j)^G = (X_i)^G \cap (X_j)^G.$$

Let $i = 1, \ldots, n$. The ideal $(X_i)^G$ is generated by monomials. Let $f \in (X_i)^G$ be a monomial of minimal degree. Then $f = X_i g$ for some monomial $g \in S$. Let $j \neq i$ and suppose that $X_j \mid g$. Then $f \in (X_i X_j)^G$ and therefore $f \in (X_i)^G . (X_j)^G$. But this is impossible since f was supposed to be of minimal degree. Thus we see that $f = \lambda X_i^{d_i}$ for some integer $d_i > 0$ and some scalar λ. Moreover, by this argument, we have that $X_i^{d_i}$ is the unique, up to scalar multiplication, monomial of minimal degree in $(X_i)^G$. We are now going to prove by induction on degree that $(X_i^{d_i}) = (X_i)^G$. As before, let $f = X_i g \in (X_i)^G$ be a monomial. If for some $j \neq i$, $X_j \mid g$ then again we can write $f = ab$ with monomials $a \in (X_i)^G$ and $b \in (X_j)^G$ and such that $\deg a < \deg f$. By induction we can conclude that $(X_i)^G = (X_i^{d_i})$ for all $i = 1, \ldots, n$.

Now, we have that

$$(X_1, \ldots, X_n) = (X_1) + \ldots + (X_n)$$

and therefore that

$$(X_1, \ldots, X_n)^G = (X_1)^G + \ldots + (X_n)^G.$$

Consequently $S^G = \mathbf{k}[X_1^{d_1}, \ldots, X_n^{d_n}]$ and the proposition is proved. ■

5 Complete intersections

Let A be an algebra essentially of finite type over a perfect field $\mathbf{k}$, and for $i \in \mathbb{N}$, let $\Omega^i = \bigwedge^i \Omega^1_{A/\mathbf{k}}$, where $\Omega^1_{A/\mathbf{k}}$ is the module of Kähler differentials. Represent A as the quotient of a regular algebra R, essentially of finite type over $\mathbf{k}$:

$$0 \longrightarrow I \longrightarrow R \longrightarrow A \longrightarrow 0$$

Then, we have a presentation :

$$I/I^2 \longrightarrow \Omega^1_{R/\mathbf{k}} \otimes A \longrightarrow \Omega^1_{A/\mathbf{k}} \longrightarrow 0.$$

The following is well known ([Fer67]) : *If A is reduced then A is locally complete intersection if and only if* $\mathrm{proj.dim}\,\Omega^1 \leq 1$ and for such a reduced complete intersection the presentation above is a projective resolution of Ω^1.

Assume now that A is a complete intersection. Then, obviously, A is reduced if and only if A is smooth in codimension 0, if and only if Ω^0 $(= A)$ is torsion-free in codimension 0 and these equivalent conditions imply $\mathrm{proj.dim}\,\Omega^1 \leq 1$. No less obviously, A is smooth in codimension 1 if and only if Ω^0 is torsion-free in codimension 0 and Ω^1 is torsion-free in codimension 1. In this section we prove the following (see below for conventions and notations) :

Theorem 5.0.3 *Let A be an algebra essentially of finite type over a perfect field $\mathbf{k}$, which is locally a complete intersection. Let $0 \leq k < \mathrm{char}^*\mathbf{k}$ be an integer. Then the following conditions are equivalent :*

(i) A is smooth in codimension k.

(ii) For all $i = 0 \ldots k$, Ω^i is torsion-free in codimension i.

Under these conditions we have that $\mathrm{proj.dim}\ \Omega^i \leq i$, *for all* $1 \leq i \leq \inf\{k+1, \mathrm{char}^*\mathbf{k} - 1\}$. *Moreover, if* $\mathrm{proj.dim}\ \Omega^i < i$ *for some i in this range then A is smooth.*

The following corollary relates the codimension of the singular locus and the possibility to extend differentials on the smooth locus :

Proposition 5.0.4 *Let A be an algebra essentially of finite type over a perfect field $\mathbf{k}$, which is reduced and locally a complete intersection. Let $U \subseteq \mathrm{Spec}(A)$ be its smooth locus and let $0 \leq k < \mathrm{char}^*\mathbf{k}$ be an integer. Then the following conditions are equivalent :*

(i) A is smooth in codimension k.

(ii) For all $0 \leq i < k$, the restriction map $\Omega^i \to \mathrm{H}^0(U, \Omega^i)$ is surjective.

Notations and conventions Let A a nœtherian commutative ring and M a finitely generated A-module. We recall that an element $x \in M$ is said to be a torsion element if there exists $f \in A$, such that $f \neq 0$ and $fx = 0$. If the only torsion element in M is 0, then one says that M is torsion-free. By saying M *is torsion-free in codim* k we mean that M_P is a torsion free A_P-module for all prime ideals $P \subset A$ of height $\leq k$. If $x \in \mathrm{Spec}(A)$, we denote by $\mathbf{k}(x)$ the residual field at x. We denote by $\beta(M)$ the upper-semicontinuous function $\mathrm{Spec}(A) \to \mathbb{N}$, $x \mapsto \mathrm{rank}_{\mathbf{k}(x)} M \otimes \mathbf{k}(x)$. When A is a local ring and $F \in \mathcal{C}(A)$ a complex, we say that F is *minimal* if $\mathrm{Im}\,\mathrm{d} \subseteq \mathfrak{m}_A F$. For $\mathbf{k}$ a field, we also define for convenience : $\mathrm{char}^*\mathbf{k} = \mathrm{char}\,\mathbf{k}$, if $\mathrm{char}\,\mathbf{k} > 0$ and $\mathrm{char}^*\mathbf{k} = +\infty$, if $\mathrm{char}\,\mathbf{k} = 0$.

Proof of 5.0.3 Let $k \geq 0$. Assume first that A is smooth in codimension k. Then for all $i \geq 0$, Ω^i is locally free in codimension k and therefore torsion-free in codimension k. This proves $(i) \Rightarrow (ii)$.

We now prove that (i) implies the two statements concerning projective dimension and smoothness. So we assume $0 \leq k < \mathrm{char}^*\mathbf{k}$ and A smooth in codimension k. We may also assume that A is local. Since $k \geq 0$ we know that $\mathrm{proj.dim}\ \Omega^1 \leq 1$. Let F be a minimal free resolution of Ω^1 of length ≤ 1 and let $i = 1 \ldots \inf\{k+1, \mathrm{char}^*\mathbf{k} - 1\}$. Then $\mathrm{T}_i F$ (cf. apppendix C) is a complex of free modules of length $\leq k+1$, with $\mathrm{H}_j(\mathrm{T}_i F)$ supported in codimension $\geq k+1$ for all $j > 0$. A direct application of Peskine and Spziro's acyclicity lemma ([PS73]), shows that $\forall j > 0$, $\mathrm{H}_j(\mathrm{T}_i F) = 0$. By the construction of (C.2) we have :

$$\mathrm{H}_j\left(\bigwedge^i F\right) = \mathrm{H}_j\left(e_{(\bar{i})}\mathrm{T}_i F\right) = e_{(\bar{i})}\mathrm{H}_j(\mathrm{T}_i F)\,.$$

Thus $\forall j > 0$, $\mathrm{H}_j\left(\bigwedge^i F\right) = 0$ and

$$\mathrm{H}_0\left(\bigwedge^i F\right) = e_{(\bar{i})}\mathrm{H}_0(\mathrm{T}_i F) = e_{(\bar{i})}\mathrm{T}_i\mathrm{H}_0(F) = e_{(\bar{i})}\mathrm{T}_i\Omega^1 = \Omega^i.$$

Therefore $\bigwedge^i F$ is a free resolution of Ω^i of length $\leq i$ and, moreover, a minimal one, since it is a direct subfactor of $\mathrm{T}_i F$. We now make explicit the expected last term of this resolution :

$$\begin{aligned}\left(\bigwedge^i F\right)_i &= \ldots \\ &= e_{(\bar{i})}\left(\mathrm{T}_i F\right)_i \\ &= e_{(\bar{i})}\left(\mathrm{T}_i F[-i]\right)_0 \\ &= e_{(\bar{i})}\left(\mathrm{T}_i(F[-1]) \otimes_{\mathbb{Z}} \epsilon\right)_0 \\ &= e_{(\bar{i})}\left((\mathrm{T}_i(F[-1]))_0 \otimes_{\mathbb{Z}} \epsilon\right)\end{aligned}$$

$$\begin{aligned} &= e_{(\bar{i})}\,(\mathrm{T}_i(F_1)\otimes_{\mathbb{Z}}\epsilon) \\ &= e_{(i)}(\mathrm{T}_i(F_1)) \\ &= \mathrm{S}^i F_1. \end{aligned}$$

Since $\bigwedge^i F$ is minimal, its length equals proj.dim Ω^i. Therefore, if proj.dim $\Omega^i < i$, we must have $\mathrm{S}^i F_1 = 0$ and this in turn implies that $F_1 = 0$. In this case Ω^1 is free and A is smooth.

To prove that $(ii) \Rightarrow (i)$ we proceed by induction on k. The case $k = 0$ is clear, so we may assume that the theorem is proved for $k-1 \geq 0$ and that $k < \mathrm{char}^*\mathbf{k}$. We may localise at a prime ideal of height $k > 0$ and then assume that A is a complete intersection local ring of dimension k. By the induction hypothesis, A is smooth in codimension $k-1$ and proj.dim $\Omega^k \leq k$. If Ω^k is torsion-free then depth $\Omega^k > 0$, and the Auslander-Buchsbaum formula implies that proj.dim $\Omega^k < k$. By what has already been proved, we can conclude that A is smooth. The theorem is proved. ∎

Proof of 5.0.4 Let $X = \mathrm{Spec}(A)$ and Z be the singular locus of X.

We first prove that $(i) \Rightarrow (ii)$. So we assume that A is smooth in codim $k \geq 1$. By (5.0.3) we know that proj.dim $\Omega^i \leq i$ for all $i < k$. Since codim $Z \geq k+1$, by the Auslander-Buchsbaum formula, we have that $\mathrm{depth}_Z\ \Omega^i \geq k+1-i \geq 2$ for all $i < k$. Therefore $\mathrm{H}^1_Z(\Omega^i) = 0$ for $i < k$ and (ii) holds.

Since X is reduced, we know that codim $Z \geq 1$. If $k = 1$, then (ii) implies that $\mathrm{H}^1_Z(\mathcal{O}_X) = 0$ and Z can't have a component of codimension 1. Now, if $k \geq 2$, by induction we can assume that codim $Z \geq k$. Let $P \in Z$ then by (5.0.3) we have that $\mathrm{H}^0_Z(\Omega^{k-1}) = 0$ and that $\mathrm{proj.dim}_P\ \Omega^{k-1} = k-1$ and by (ii), $\mathrm{H}^1_Z(\Omega^{k-1}) = 0$. Thus $\mathrm{depth}_P\ \Omega^{k-1} \geq 2$ and by Auslander-Buchsbaum again :

$$\text{height } P = \mathrm{proj.dim}_P\ \Omega^{k-1} + \mathrm{depth}_P\ \Omega^{k-1} \geq k-1+2 = k+1.$$

This concludes the proof. ∎

Appendices

A Actions of algebraic groups

For the definition of categorical quotients and other related notions we refer to ([MF82]). Let us recall the following from ([LP97]) :

Definition A.0.5 *Let G be an algebraic group acting on a scheme X. A good quotient of X by G is a G-invariant morphism $\pi : X \longrightarrow Y$ such that*

(i) The morphism π is surjective and affine.

(ii) The canonical morphism $\mathcal{O}_Y \longrightarrow (\pi_* \mathcal{O}_X)^G$ is an isomorphism.

(iii) The image by π of any G-invariant closed subset is a closed subset.

(iv) The morphism π separates disjoint G-invariant closed subsets.

If X admits a good quotient we may denote it by $\pi : X \longrightarrow X/\!\!/G$.

Proposition A.0.6 *Let $\pi : X \longrightarrow Y$ be a good quotient. Then*

(i) The quotient $\pi : X \longrightarrow Y$ is a categorical quotient.

(ii) The topology of Y is the quotient topology.

(iii) There is one and only one closed orbit in each fiber of π.

Then, one has the "Hilbert-Mumford theorem" :

Theorem A.0.7 ([MF82, Theorem 1.1, p 27]) *Let X be an affine scheme and G be a reductive algebraic group acting on X. Then the map $\pi : X \longrightarrow Y = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)^G)$ is a good quotient of X by G. Moreover, if X is of finite type over the ground field then Y is also.*

An important corollary (when working in the category of schemes of finite type over a field) of this result is that the invariants of an equivariant coherent $\mathcal{O}_X$-module are a coherent $\mathcal{O}_Y$-module.

B Regular and absolutely regular differentials

B.1 Regular differentials

Regular differentials together with duality theory have been studied by many authors but from different viewpoints. The main results that we need are found in the book of Kunz and Waldi ([KW88]), but we feel that the very general and explicit construction of regular differentials in this book (where the construction is local and relative from the beginning) asks a lot of the (lazy) reader, and therefore does not "specialize" easily to a convenient tool in the common case of schemes of finite type over a field.

Thus we choose the following path : We take the theory of the residual complex and fundamental class as exposed in the work of El Zein ([EZ78]) as a "black box" and rephrase, with a view toward Kunz and Waldi's theory of regular differentials, the results and constructions of El Zein. We do not intend to say anything new here and all the subsequent claims are implicitely proved in El Zein's article ([EZ78]). In fact, this approach was inspired to us by the work of Kersken ([Ker83b, Ker83a, Ker84]).

Construction

Let $\mathbf{k}$ be a field of characteristic 0. For any scheme X of finite type over $\mathbf{k}$, there exists a *residual complex* K_X ([Har66]). This is a complex of injective $\mathcal{O}_X$-modules concentrated in degree $[-\dim(X), 0]$, the image of which in the derived category is the *dualizing complex*.

Let $n = \dim(X)$. We denote by ω_X^n the module $\mathrm{H}^0(\mathrm{K}_X[-n])$. If X is smooth, K_X is the Cousin resolution of $\Omega_X^n[n]$. If $i : X \longrightarrow Y$ is an embedding of X into a smooth Y then $\mathrm{K}_X = i^!\mathrm{K}_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathrm{K}_Y)$. If $\pi : X \longrightarrow Y$ is a finite surjective morphism then the complexes K_X and $\pi^!\mathrm{K}_Y$ are quasi-isomorphic and therefore $\omega_X^n \simeq \pi^!\omega_Y^n$. Moreover, the formation of the residual complex commutes with restriction to an open set. Thus, for a general X, ω_X^n has the S_2 property and coincides with Ω_X^n at the smooth points of X. Consequently, if X is normal then there is a natural isomorphism $\Omega_X^{n\,\vee\vee} \xrightarrow{\sim} \omega_X^n$.

The complex K_X is exact in degrees $\neq \dim(X)$ if and only if X is equidimensionnal and Cohen-Macaulay. In this case, the module ω_X^n is the *dualizing module* (usually denoted ω_X).

Now, following El Zein, let $\mathrm{K}_X^{*,\cdot} = \underline{\mathrm{Hom}}(\Omega_X, \mathrm{K}_X)$. It is a bigraded object, where the $*$ (resp. the $\cdot$) corresponds to degrees in Ω_X (resp. in K_X), concentrated in degrees $[-\infty, 0] \times [-\dim(X), 0]$. We now explain how one can put on $\mathrm{K}_X^{*,\cdot}$ a structure of complex of right differential graded Ω_X-modules concentrated in degree $[-\dim(X), 0]$.

The left Ω_X-module structure of Ω_X given by exterior product induces an obvious right Ω_X-module structure on $\mathrm{K}_X^{*,p} = \underline{\mathrm{Hom}}(\Omega_X, \mathrm{K}_X^p)$ and the differential δ of K_X induces an Ω_X-linear differential : $\delta' = \underline{\mathrm{Hom}}(\Omega_X, \delta)$.

The non-trivial point is the existence for all p of a differential endo-operator d' of order ≤ 1 and $*$-degree 1 on $\mathrm{K}_X^{*,p}$ satisfying the conditions

(i) $\delta'.d' = d'.\delta'$.

(ii) $d'(\phi.\alpha) = \phi.(d\alpha) + (-1)^q(\mathrm{d}'\phi).\alpha$, for $\alpha \in \Omega_X^q$ and $\phi \in \mathrm{K}_X^{*,p}$.

The construction of d' is explained in ([EZ78, 2.1.2]), the proof of (ii) follows from the lemma ([EZ78, 2.1.2, Lemme], be aware that there is a misprint in this paper : The logical section 2.1.2 is labelled 3.1.2) and the remarks following the proof of this lemma. Finally, (i) is a direct consequence of ([EZ78, 2.1, Proposition]) and ([EZ78, 2.1.2, Proposition]). We want to insist on the fact that, even in the smooth case, the operator d' is not the naive (and above all, meaningless) "$\underline{\mathrm{Hom}}(d, \mathrm{K}_X)$". We can now define the module of *regular differential forms* : $\omega_X = \mathrm{H}^{*,0}(\mathrm{K}_X^{*,\cdot}[-n,-n])$. Thus, ω_X is a right differential graded Ω_X-module and one has $\omega_X^k = \underline{\mathrm{Hom}}(\Omega_X^{n-k}, \omega_X^n)$.

When X is normal and equidimensional, the isomorphism $\Omega_X^{n\,\vee\vee} \xrightarrow{\sim} \omega_X^n$ therefore induces an isomorphism $\Omega_X{}^{\vee\vee} \xrightarrow{\sim} \omega_X$. Thus, in this case, it is easily seen that this construction coincides with that of Kunz and Waldi ([KW88, 3.17, Theorem]). Note also that, when X is normal, ω_X is a reflexive module.

The fundamental class

The *fundamental class* is constructed and studied by El Zein in ([EZ78, 3.1, Théorème]). The fundamental class is defined as a global section C_X of $\mathrm{K}_X^{*,\cdot}$ (as a bigraded object) satisfying $d'\mathrm{C}_X = \delta'\mathrm{C}_X = 0$. When X is equidimensional of dimension n, the fundamental class is homogeneous of degree $(-n,-n)$. In general, the contribution to C_X of an m-dimensional irreducible component of X is homogeneous of degree $(-m,-m)$ (cf. the next section). Let X be an n-dimensionnal scheme. By this observation, since $\delta'\mathrm{C}_X = 0$, we have an induced cohomology class $\mathrm{c}_X \in \omega_X^0$. Then, right multiplication defines a morphism

$$\begin{array}{ccc} \Omega_X & \longrightarrow & \omega_X \\ \alpha & \longmapsto & \mathrm{c}_X.\alpha \end{array}$$

of differential graded Ω_X-modules, thanks to the relation $d'\mathrm{c}_X = 0$. We again denote by c_X this morphism and also call it the fundamental class morphism.

To be a little more explicit, $\mathrm{c}_X \in \mathrm{H}^0(X, \mathrm{K}_X^{*,\cdot}[-n,-n]) = \mathrm{Hom}(\Omega_X^n, \omega_X^n)$ and the fundamental class morphism in degree k is the composition

$$\Omega_X^k \longrightarrow \mathrm{Hom}(\Omega^{n-k}, \Omega_X^n) \longrightarrow \mathrm{Hom}(\Omega^{n-k}, \omega_X^n) \simeq \omega_X^k.$$

When X is normal and equidimensional, the morphism c_X can be identified with the natural morphism $\Omega_X \longrightarrow \Omega_X{}^{\vee\vee} \simeq \omega_X$.

We can now state the following fundamental theorem of Kunz and Waldi :

Theorem ([KW88, 5.22, p107]) *Let X be an equidimensional Cohen-Macaulay reduced scheme of finite type over* $\mathbf{k}$ *and let $n = \dim(X)$. Then the support of* $\mathrm{Coker}\,(\mathrm{c}_X)^n$ *is precisely the singular locus of X.*

The trace map for regular differentials

Let $f : X \longrightarrow Y$ be a proper morphism, then the trace morphism $\mathrm{Tr}f : f_*\mathrm{K}_X^{*,\cdot} \longrightarrow \mathrm{K}_Y^{*,\cdot}$ is obtained by the composition of the natural morphism $\Omega_Y \longrightarrow f_*\Omega_X$ with the trace morphism for residual complexes $f_*\mathrm{K}_X \longrightarrow \mathrm{K}_Y$. We thus have a well defined trace morphism $\mathrm{Tr}f : f_*\omega_X \longrightarrow \omega_Y$ vanishing if $\dim(X) \neq \dim(Y)$.

Assume that f is birational, i.e., that there exists a dense open subset $V \subset Y$ such that the induced morphism $f^{-1}(V) \longrightarrow V$ be an isomorphism. Then, by ([EZ78, 3.1, Théorème]) the trace morphism $\mathrm{Tr}f : f_*\mathrm{K}_X^{*,\cdot} \longrightarrow \mathrm{K}_Y^{*,\cdot}$ sends C_X to C_Y. Consequently, under these hypotheses we have a commutative diagram :

$$\begin{array}{ccc} f_*\Omega_X & \xrightarrow{\mathrm{c}_X} & f_*\omega_X \\ \uparrow & & \downarrow{\scriptstyle \mathrm{Tr}f} \\ \Omega_Y & \xrightarrow{\mathrm{c}_Y} & \omega_Y \end{array}$$

Let X be a scheme and $X_1, \ldots, X_k$ its irreducible components with their reduced structure and inclusions $j_i : X_i \subset X$. Then by construction ([EZ78, p37]) $\mathrm{C}_X = \sum_i e_{X_i}(X)\mathrm{Tr}j_i(\mathrm{C}_{X_i})$, where $e_{X_i}(X) = \mathrm{length}(\mathcal{O}_{X,X_i})$, the multiplicity of X along X_i. Thus, $\mathrm{c}_X = \sum_i e_{X_i}(X)\mathrm{Tr}j_i(\mathrm{c}_{X_i})$.

Assume now that $f : X \longrightarrow Y$ is a finite dominant morphism between integral schemes then by ([EZ78, 3.1, Proposition 2]) we have that $\mathrm{Tr}f(\mathrm{C}_X) = \deg(f)\mathrm{C}_Y$. We therefore have a commutative diagram :

$$\begin{array}{ccc} f_*\Omega_X & \xrightarrow{\mathrm{c}_X} & f_*\omega_X \\ \uparrow & & \downarrow{\scriptstyle \mathrm{Tr}f} \\ \Omega_Y & \xrightarrow{\deg(f)\mathrm{c}_Y} & \omega_Y \end{array}$$

B.2 Absolutely regular differentials

Let X be a scheme and $f : \tilde{X} \longrightarrow X$ a desingularisation (if X is not reduced, by this, we mean a desingularisation of X_{red}). We recall that the $\mathcal{O}_X$-module $f_*\Omega_{\tilde{X}}$ is independent of the choice of f, we denote it by $\tilde{\Omega}_X$. It is usually called the module of *absolutely regular differentials*, or sometimes, when X is a normal variety, the module of *Zariski differentials*. By construction, we have natural morphisms

$$\Omega_X \longrightarrow \tilde{\Omega}_X \longrightarrow i_*\Omega_{X_{\mathrm{smth}}}$$

where i is the inclusion $X_{\mathrm{smth}} \subset X$. Therefore, when X is reduced, we have :

$$\Omega_X \longrightarrow \bar{\Omega}_X \subset \tilde{\Omega}_X \subset i_*\Omega_{X_{\mathrm{smth}}}.$$

In general, we also have a commutative diagram :

$$\begin{array}{ccc} f_*\Omega_{\tilde{X}} & = & f_*\omega_{\tilde{X}} \\ \uparrow & & \downarrow{\scriptstyle \mathrm{Tr}f} \\ \Omega_X & \xrightarrow{\mathrm{c}_X} & \omega_X \end{array}$$

and consequently, a sequence of morphisms

$$\Omega_X \longrightarrow \tilde{\Omega}_X \longrightarrow \omega_X.$$

We recall some elementary facts about absolutely regular differentials :

Proposition B.2.1 *Let $f : X \longrightarrow Y$ be a dominant morphism. Then we have a commutative diagram*

$$\begin{array}{ccc} \Omega_X & \longrightarrow & \tilde{\Omega}_X \\ \uparrow & & \uparrow \\ f^*\Omega_Y & \longrightarrow & f^*\tilde{\Omega}_Y \end{array}$$

Proposition B.2.2 *Let $f : X \longrightarrow Y$ be a proper birational morphism. Then we have a commutative diagram*

$$\begin{array}{ccccc} f_*\Omega_X & \longrightarrow & f_*\tilde{\Omega}_X & \longrightarrow & f_*\omega_X \\ \uparrow & & \uparrow & & \downarrow {\scriptstyle \mathrm{Tr} f} \\ \Omega_Y & \longrightarrow & \tilde{\Omega}_Y & \longrightarrow & \omega_Y \end{array}$$

where the rows are factorisations of the respective fundamental class morphisms.

This last statement is a tautology :

Proposition B.2.3 *Let $f : X \longrightarrow Y$ be a proper birational morphism. Then the morphism $\tilde{\Omega}_Y \longrightarrow f_*\tilde{\Omega}_X$ is an isomorphism.*

C Schur functors for complexes

If S is a ring and G a group, we denote by $S[G]$ the group algebra. If $\mathcal{A}$ is an abelian category, we also denote by $\mathcal{A}[G]$ the abelian category of objects of $\mathcal{A}$ with a G-action and equivariant morphisms between them. We call $\mathrm{Res}_{\mathcal{A}}$ the forgetful functor from $\mathcal{A}[G]$ to $\mathcal{A}$. We denote by $\mathcal{C}(\mathcal{A})$ the category of complexes. Remark that $\mathcal{C}(\mathcal{A})[G] = \mathcal{C}(\mathcal{A}[G])$. We recall that $\mathcal{A}$ is said to be an S-linear category if it is abelian and if we are given a morphism of $\mathbb{Z}$-algebras $S \to \mathrm{End}(M)$ for each object M of $\mathcal{A}$ such that morphisms in $\mathcal{A}$ are S-linear for this datum. Observe that if $\mathcal{A}$ is S-linear then $\mathcal{A}[G]$ is S-linear but not $S[G]$-linear in general. However we do have natural morphisms $S[G] \to \mathrm{End}(\mathrm{Res}_{\mathcal{A}}N)$ for all N in $\mathcal{A}[G]$. Consequently, for any $f \in S[G]$, left multiplication by f defines an endomorphism of the functor $\mathrm{Res}_{\mathcal{A}}$. The image ($\mathcal{A}$ is abelian) of this last endomorphism is thus a well defined subfunctor of $\mathrm{Res}_{\mathcal{A}}$ that we will write f.

C.1 Monoidal symmetric abelian categories

Let $\mathcal{A}$ be an abelian category with a tensor product (also called a monoidal category, we refer to [Mac71, Chapter XI]) noted $\otimes$ (or $\otimes_{\mathcal{A}}$ when there may be ambiguities) and a braiding : $\gamma_{M,N} : M \otimes N \simeq N \otimes M$. If $M_1, \dots, M_n$ and M are objects of $\mathcal{A}$ we define inductively : $M_1 \otimes \dots \otimes M_n = (M_1 \otimes \dots \otimes M_{n-1}) \otimes M_n$ and $M^{\otimes n} = M^{\otimes(n-1)} \otimes M$. In order to give a clear statement of the theorem below, we denote by W_n the functor from $\mathcal{A}^n$ to $\mathcal{A}$ wich sends $(M_1, \dots, M_n)$ to $M_1 \otimes \dots \otimes M_n$ and by Δ_n the "diagonal embedding" of $\mathcal{A}$ into $\mathcal{A}^n$. Observe also that Σ_n, the symmetric group of $\{1 \dots n\}$, left-acts functorially in an obvious way on $\mathcal{A}^n$: If $\sigma \in \Sigma_n$, we write σ for the functor which sends $(M_1, \dots, M_n)$ to $(M_{\sigma^{-1}(1)}, \dots, M_{\sigma^{-1}(n)})$. With these notations, the braiding can be seen as the choice of an isomorphism $\gamma : \mathrm{W}_2 \longrightarrow \mathrm{W}_2(1,2)$.

Definition C.1.1 *When the diagrams*

$$
\begin{array}{ccc}
M_1 \otimes M_2 & \xrightarrow{\gamma_{M_1,M_2}} & M_2 \otimes M_1 \\
& \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle \gamma_{M_2,M_1}} \\
& & M_1 \otimes M_2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M_1 \otimes M_2 \otimes M_3 & \xrightarrow{\gamma_{M_1 \otimes M_2, M_3}} & M_3 \otimes M_1 \otimes M_2 \\
{\scriptstyle \mathrm{Id} \otimes \gamma_{M_2,M_3}} \searrow & & \nearrow {\scriptstyle \gamma_{M_1,M_3} \otimes \mathrm{Id}} \\
& M_1 \otimes M_3 \otimes M_2 &
\end{array}
$$

are commutative for all triples of object, $\mathcal{A}$ is said to be monoidal symmetric.

In this case, one has a "coherence theorem" :

Theorem C.1.2 ([Mac71, Theorem 1, p.253]) *Let $\mathcal{A}$ be a monoidal symmetric category. There exists a unique collection of isomorphisms of functors*

$$\left\{ \sigma_* : \mathrm{W}_n \longrightarrow \mathrm{W}_n \sigma \right\}_{\sigma \in \Sigma_n}$$

such that any "permutation" of the factors given by successive application of the braiding is in the collection.

The uniqueness implies that for $\sigma, \tau \in \Sigma_n$ the following diagram is commutative

$$\begin{array}{ccc} \mathrm{W}_n & \xrightarrow{\sigma_*} & \mathrm{W}_n\sigma \\ & \searrow^{(\tau\sigma)_*} & \downarrow{\scriptstyle \tau_*\sigma} \\ & & \mathrm{W}_n\tau\sigma \end{array}$$

and that $1_* = \mathrm{Id}$! Since $\forall\sigma \in \Sigma_n$, $\sigma\Delta_n = \Delta_n$, we can make the following

Definition C.1.3 *The functor* $\mathrm{W}_n\Delta_n$ *is naturally endowed with a* Σ_n *transformation group. We define* T_n *to be the associated functor from* $\mathcal{A}$ *to* $\mathcal{A}[\Sigma_n]$.

Clearly, one has : $\mathrm{Res}_{\mathcal{A}}\mathrm{T}_n = \mathrm{W}_n\Delta_n = (*)^{\otimes n}$. We also define :

Definition C.1.4 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian monoidal symmetric categories and* F *an additive functor from* $\mathcal{A}$ *to* $\mathcal{B}$. *If there exists a natural isomorphism* $\Psi : \mathrm{W}_2\mathrm{F}^2 \longrightarrow \mathrm{FW}_2$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{W}_2\mathrm{F}^2 & \xrightarrow{\Psi} & \mathrm{FW}_2 \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{F}(\gamma)} \\ \mathrm{W}_2(1,2)\mathrm{F}^2 & \xrightarrow{\Psi(1,2)} & \mathrm{FW}_2(1,2) \end{array}$$

commutes, then one says that F *is a strong monoidal symmetric, or tensor-preserving, functor.*

Proposition C.1.5 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian monoidal symmetric categories and* F *a tensor-preserving functor between them. Let* $n = n_1 + n_2$ *and* $\Sigma_{n_1} \times \Sigma_{n_2} \subset \Sigma_n$ *the natural inclusion. Then one has :*

(i) $\mathrm{T}_n\mathrm{F} = \mathrm{F}[\Sigma_n]\mathrm{T}_n$.

(ii) $\mathrm{T}_{n_1} \boxtimes_{\mathcal{A}} \mathrm{T}_{n_2} = \mathrm{Res}_{\mathcal{A}[\Sigma_{n_1}\times\Sigma_{n_2}]}\mathrm{T}_n$.

In the proposition above $\boxtimes_{\mathcal{A}}$ is defined in the same way as the external product of linear group representations : Let G and H be groups. If M is in $\mathcal{A}[G]$ and N in $\mathcal{A}[H]$ then $G \times H$ acts on $(\mathrm{Res}_{\mathcal{A}}M) \otimes_{\mathcal{A}} (\mathrm{Res}_{\mathcal{A}}N)$ and this defines the bifunctor $\boxtimes_{\mathcal{A}}$ from $\mathcal{A}[G] \times \mathcal{A}[H]$ to $\mathcal{A}[G \times H]$. The proof of (C.1.5) is then quite obvious.

C.2 Schur functors.

Let $\mathbf{k}$ be a field and $\mathcal{A}$ an $\mathbf{k}$-linear monoidal symmetric category. In this section we review quickly and rephrase in our context the construction of the Schur functors as explained in [FH91, I.6].

We first recall that partitions $\lambda\ : \lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq 1$ of n are in bijection with Young diagrams of weight n and that with any such diagram we can associate its

Young symmetrizer $c_\lambda \in \mathbb{Z}[\Sigma_n]$. The Young symmetrizer satisfies $c_\lambda^2 = n_\lambda c_\lambda$ with n_λ a positive integral divisor of $n!$. In full generality, the Schur functor $\mathbb{S}_\lambda$ from $\mathcal{A}$ to $\mathcal{A}$ is defined as $\mathbb{S}_\lambda = c_\lambda \mathrm{T}_n$.

Consequently, under the assumption that n_λ is invertible in $\mathbf{k}$, $e_\lambda = c_\lambda / n_\lambda$ is an idempotent in $\mathbf{k}[\Sigma_n]$. And therefore, if N is an object in $\mathcal{A}[\Sigma_n]$, $c_\lambda N$ is naturally a direct subfactor of $\mathrm{Res}_{\mathcal{A}} N$. In particular, in this case, we see that $\mathbb{S}_\lambda$ is a direct subfunctor of $(*)^{\otimes n}$. When $n < \mathrm{char}^* \mathbf{k}$, we will be mainly concerned with the partitions (n) and $(1, \ldots, 1) = (\bar{n})$ ($\bar{\lambda}$ denoting the conjugate partition of λ) of n. For these, we have :

$$c_{(n)} = \sum_{\sigma \in \Sigma_n} \sigma \quad n_{(n)} = n! \quad \mathbb{S}_{(n)}(M) = e_{(n)} \mathrm{T}_n(M) = \mathrm{S}^n(M)$$

$$c_{(\bar{n})} = \sum_{\sigma \in \Sigma_n} \epsilon(\sigma)\sigma \quad n_{(\bar{n})} = n! \quad \mathbb{S}_{(\bar{n})}(M) = e_{(\bar{n})} \mathrm{T}_n(M) = \bigwedge^n (M)$$

the symmetric and exterior n-fold power of M.

C.3 Application to categories of complexes

In this section $\mathcal{A}$ denotes an abelian monoidal symmetric category. We recall that the complex $M \otimes_{\mathcal{C}(\mathcal{A})} N$ is constructed by setting :

$$(M \otimes_{\mathcal{C}(\mathcal{A})} N)_d = \bigoplus_{p+q=d} M_p \otimes_{\mathcal{A}} N_q$$

and

$$\mathrm{d}_d = \bigoplus_{p+q=d} \mathrm{d}_p \otimes_{\mathcal{A}} \mathrm{Id} + (-1)^p \mathrm{Id} \otimes_{\mathcal{A}} \mathrm{d}_q.$$

Proposition C.3.1 *The category $\mathcal{C}(\mathcal{A})$ is monoidal symmetric with the usual tensor product of complexes and the braiding $\gamma_{M,N} : M \otimes_{\mathcal{C}(\mathcal{A})} N \to N \otimes_{\mathcal{C}(\mathcal{A})} M$ defined by :*

$$(\gamma_{M,N})_d = \bigoplus_{p+q=d} (-1)^{pq} \gamma_{M_p, N_q}$$

for M, N in $\mathcal{C}(\mathcal{A})$. Furthermore, the functor i from $\mathcal{A}$ to $\mathcal{C}(\mathcal{A})$, which makes of an object of $\mathcal{A}$ a complex concentrated in degree 0, is tensor-preserving.

Proof of C.3.1 It is an easy calculation : Check that the γ's are merely morphims of complexes and that they satisfy the conditions in (C.1.1). ■

Corollary C.3.2 *The category $\mathcal{C}^k(\mathcal{A})$ is monoidal symmetric.*

where the category $\mathcal{C}^k(\mathcal{A})$ of k-complexes is defined as usual by $\mathcal{C}^k = \mathcal{C}\mathcal{C}^{k-1}$. We will use the convenient and self-explanatory multi-indices notation : For M in $\mathcal{C}^k(\mathcal{A})$, $(M_p)_{p_1,\ldots,p_{k-1}} = M_{p_1,\ldots,p_{k-1},p}$ and therefore for N in $\mathcal{C}^{k-1}(\mathcal{A})$, $(iN)_0 = N$. We will also write $\mathrm{d}^{(i)}$ for the differential along the i-th axis. Continuing with our notational remarks, for integers $1 \leq l \leq k$, $p \in \mathbb{Z}^l$ and M in $\mathcal{C}^k(\mathcal{A})$, we may write M_p for $(\ldots((M_{p_l})_{p_{l-1}})\ldots)_{p_1}$, this standing in $\mathcal{C}^{k-l}(\mathcal{A})$. We define $M_{|p} = (i^l M_p)[p]$.

We can make more explicit the structure of the tensor product in $\mathcal{C}^k(\mathcal{A})$: Let $M, N \in \mathcal{C}^k(\mathcal{A})$ and $d \in \mathbb{Z}^k$, then

$$(M \otimes_{\mathcal{C}^k(\mathcal{A})} N)_d = \bigoplus_{p+q=d} M_p \otimes_{\mathcal{A}} N_q$$

with differentials

$$\mathrm{d}_d^{(i)} = \bigoplus_{p+q=d} \mathrm{d}_p^{(i)} \otimes_{\mathcal{A}} \mathrm{Id} + (-1)^{p_i} \mathrm{Id} \otimes_{\mathcal{A}} \mathrm{d}_q^{(i)}$$

and braiding

$$(\gamma_{M,N})_d = \bigoplus_{p+q=d} (-1)^{p.q} \gamma_{M_p,N_q}$$

where $p.q = p_1q_1 + \ldots + p_kq_k$.

We now study the behaviour of total complex functors with respect to the tensor product. We first define tot from $\mathcal{C}^2(\mathcal{A})$ to $\mathcal{C}(\mathcal{A})$:

$$\begin{aligned} \mathrm{tot}(M)_p &= \bigoplus_{p_1+p_2=p} M_{p_1,p_2} \\ \mathrm{d}_p &= \bigoplus_{p_1+p_2=p} \mathrm{d}^{(1)}_{p_1,p_2} + (-1)^{p_1} \mathrm{d}^{(2)}_{p_1,p_2} \end{aligned}$$

Let $k \geq 1$. We can then define tot from $\mathcal{C}^{k+1}(\mathcal{A})$ to $\mathcal{C}^k(\mathcal{A})$ with the preceding construction, using that $\mathcal{C}^{k+1}(\mathcal{A}) = \mathcal{C}^2\mathcal{C}^{k-1}(\mathcal{A})$ and $\mathcal{C}^k(\mathcal{A}) = \mathcal{C}\mathcal{C}^{k-1}(\mathcal{A})$.

It is easily checked that $\mathrm{tot}\, i = \mathrm{Id}$. Our functor tot should not be confused with the functor $\mathcal{C}^{k-1}(\mathrm{tot})$: tot from $\mathcal{C}^{k+1}(\mathcal{A})$ to $\mathcal{C}^k(\mathcal{A})$ totalizes on the last two indices whereas $\mathcal{C}^{k-1}(\mathrm{tot})$ does on the first two. With this (maybe unusual) definition, the truly total complex functor from $\mathcal{C}^{k+1}(\mathcal{A})$ to $\mathcal{C}(\mathcal{A})$ is $\mathrm{Tot} = \mathrm{tot}^k$.

Proposition C.3.3 *The functor* tot *is tensor preserving.*

Proof of C.3.3 We only have to consider the case of tot from $\mathcal{C}^2(\mathcal{A})$ to $\mathcal{C}(\mathcal{A})$. Let M and N be in $\mathcal{C}^2(\mathcal{A})$. We have :

$$\begin{aligned} (\mathrm{tot}(M) \otimes \mathrm{tot}(N))_d &= \ldots \\ \bigoplus_{(p_1+p_2)+(q_1+q_2)=d} M_{p_1,p_2} \otimes N_{q_1,q_2} &= \bigoplus_{(p_1+q_1)+(p_2+q_2)=d} M_{p_1,p_2} \otimes N_{q_1,q_2} \\ \ldots &= (\mathrm{tot}(M \otimes N))_d. \end{aligned}$$

The differential for $\operatorname{tot}(M)\otimes\operatorname{tot}(N)$ is :

$$\mathrm{d}_d=\bigoplus_{(p_1+p_2)+(q_1+q_2)=d}(-1)^{p_1+p_2+q_1}\mathrm{Id}\otimes\mathrm{d}^{(2)}+(-1)^{p_1+p_2}\mathrm{Id}\otimes\mathrm{d}^{(1)}+(-1)^{p_1}\mathrm{d}^{(2)}\otimes\mathrm{Id}+\mathrm{d}^{(1)}\otimes\mathrm{Id}$$

and the braiding $\gamma:\operatorname{tot}(M)\otimes\operatorname{tot}(N)\longrightarrow\operatorname{tot}(N)\otimes\operatorname{tot}(M)$ writes down

$$\gamma_d=\bigoplus_{(p_1+p_2)+(q_1+q_2)=d}(-1)^{(p_1+q_1)(p_2+q_2)}\gamma_{M_{p_1,p_2},N_{q_1,q_2}}.$$

The differential for $\operatorname{tot}(M\otimes N)$ is :

$$\mathrm{d}_d=\bigoplus_{(p_1+q_1)+(p_2+q_2)=d}(-1)^{p_1+q_1+p_2}\mathrm{Id}\otimes\mathrm{d}^{(2)}+(-1)^{p_1}\mathrm{Id}\otimes\mathrm{d}^{(1)}+(-1)^{p_1+q_1}\mathrm{d}^{(2)}\otimes\mathrm{Id}+\mathrm{d}^{(1)}\otimes\mathrm{Id}$$

and $\operatorname{tot}(\gamma)$:

$$\operatorname{tot}(\gamma)_d=\bigoplus_{(p_1+q_1)+(p_2+q_2)=d}(-1)^{p_1q_1+p_2q_2}\gamma_{M_{p_1,p_2},N_{q_1,q_2}}.$$

We define $\Psi:\operatorname{tot}(M)\otimes\operatorname{tot}(N)\longrightarrow\operatorname{tot}(M\otimes N)$ by

$$\Psi_d=\bigoplus_{(p_1+p_2)+(q_1+q_2)=d}(-1)^{p_2q_1}\mathrm{Id}.$$

Direct calculations show then that Ψ is actually a morphism of complexes and that the condition of (C.1.4) is satisfied. ∎

Corollary C.3.4 *There is a natural isomorphism :*

$$\operatorname{tot}\mathrm{T}_n=\mathrm{T}_n\operatorname{tot}.$$

Proposition C.3.5 *There is a natural isomorphism in $\mathcal{C}(\mathcal{A}[\Sigma_n])$:*

$$\mathrm{T}_n(*[1])=(\mathrm{T}_n(*))[n]\otimes_{\mathbb{Z}}\epsilon.$$

Where ϵ is the alternating representation.

Proof of C.3.5 One has :

$$\left(M[1]^{\otimes n}\right)_d=\bigoplus_{\sum(p_i+1)=d}M_{p_1}\otimes\ldots\otimes M_{p_n}=\left(M^{\otimes n}\right)[n]_d.$$

The differential for the complex on the left-hand side is :

$$\begin{aligned}\mathrm{d}_d &= \bigoplus_{\sum(p_i+1)=d}\sum_{i=1}^{n}-(-1)^{(p_1+1)+\ldots+(p_{i-1}+1)}\mathrm{Id}\otimes\ldots\otimes\mathrm{d}_{p_i}\otimes\ldots\otimes\mathrm{Id}\\ &= \bigoplus_{\sum p_i=d-n}\sum_{i=1}^{n}(-1)^{i}(-1)^{p_1+\ldots+p_{i-1}}\mathrm{Id}\otimes\ldots\otimes\mathrm{d}_{p_i}\otimes\ldots\otimes\mathrm{Id}.\end{aligned}$$

And for the complex on the right-hand side we have :

$$\mathrm{d}_d \quad = \quad \bigoplus_{\sum p_i = d-n} \sum_{i=1}^{n} (-1)^n (-1)^{p_1+\ldots+p_{i-1}} \mathrm{Id} \otimes \ldots \otimes \mathrm{d}_{p_i} \otimes \ldots \otimes \mathrm{Id}.$$

So that setting

$$(\Psi_M)_d = \bigoplus_{\sum p_i = d-n} (-1)^{nd} (-1)^{\sum i.p_i} \mathrm{Id}$$

defines an isomorphism $\Psi_M : \; M[1]^{\otimes n} \longrightarrow (M^{\otimes n})\,[n]$.

It remains to show that $\sigma \Psi_M \sigma^{-1}[n] = \epsilon(\sigma)\Psi_M$, and it is sufficient to prove it for $\sigma = (j, j+1), 0 \leq j < n$:

$$\begin{aligned}
&\big((j, j+1)\Psi_M (j, j+1)^{-1}[n]\big)_d = \ldots \\
&= \bigoplus_{\sum p_i = d-n} (-1)^{(p_{j+1}+1)(p_j+1)}.(-1)^{nd}(-1)^{p_1+\ldots+j.p_{j+1}+(j+1).p_j+\ldots+n.p_n}.(-1)^{p_j p_{j+1}} \mathrm{Id} \\
&= \bigoplus_{\sum p_i = d-n} (-1).(-1)^{nd}(-1)^{p_1+\ldots+j.p_j+(j+1).p_{j+1}+\ldots+n.p_n} \mathrm{Id} \\
&= (-1)\,(\Psi_M)_d\,.
\end{aligned}$$

This concludes the proof. ■

Proposition C.3.6 *Let M be in $\mathcal{C}(\mathcal{A})$. There is a natural decomposition :*

$$\mathrm{S}^n(M)_d = \bigoplus_{k>0} \bigoplus_{\substack{n_1 d_1+\ldots+n_k d_k = d \\ n_1+\ldots+n_k = n \\ d_1 > \ldots > d_k \\ n_1,\ldots,n_k > 0}} \bigotimes_{i=1}^{k} \mathrm{S}^{n_i}(M_{|d_i})_{n_i d_i}.$$

The same holds true when replacing symmetric by exterior power.

C.3.7 Example Consider a map beetwen objects of $\mathcal{A}$, $M_1 \xrightarrow{f} M_0$ as a complex, M in degree $[-1, 0]$, then the n-fold symmetric power $\mathrm{S}^n(M)$ of this complex will be :

$$0 \longrightarrow \bigwedge^n(M_1) \longrightarrow \cdots \longrightarrow \bigwedge^k(M_1) \otimes \mathrm{S}^{n-k}(M_0) \longrightarrow \cdots \longrightarrow \mathrm{S}^n(M_0) \longrightarrow 0.$$

In the case where $M_1 = A^n$ and $M_0 = A$ (A a ring) one therefore recovers the Koszul complex.

Proposition C.3.8 *Assume that $\otimes_{\mathcal{A}}$ is right-exact and let F be in $\mathcal{C}_{\geq 0}(\mathcal{A})$. Then*

(i) $\mathrm{H}_0(\mathrm{T}_n F) = \mathrm{T}_n(\mathrm{H}_0(F))$.

(ii) If F is an acyclic complex of flat objects then $\mathrm{T}_n F$ is acyclic.

Proof of C.3.8 By a repeated use of the convergent spectral sequence for the tensor product of two complexes in $\mathcal{C}^-(\mathcal{A})$. ■

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