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Sur la K-théorie des catégories hermitiennes

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Contents

1 Introduction (en français)

Après l'introduction de K_0 par Grothendieck [BS], la K-théorie algébrique s'est développée d'abord pour la catégorie des modules projectifs de type fini sur un anneau A. (Tous nos anneaux sont associatifs et unitaires, et tous nos modules sont des modules à droite.) Ceci est une catégorie additive. Une catégorie additive peut être considérée ou comme catégorie monoïdale symétrique (parfois on appelle une telle catégorie une ACU-catégorie tensorielle) ou comme catégorie exacte (où la structure exacte est donnée par les suites exactes courtes scindées). Quillen [Q1, Q2] a défini la K-théorie pour ces deux classes de catégories via les constructions $S^{-1}S$ et Q. Il a démontré que les deux définitions coïncident pour les catégories additives et qu'elles généralisent la construction Plus [Lo1]. La construction Q permet de démontrer des théorèmes importants, par exemple les théorèmes de localization et de résolution.

Supposons maintenant que nos modules projectifs sur l'anneau A sont munis d'une forme bilinéaire symétrique (ou de manière plus générale d'une forme ϵ-hermitienne). Notre objectif est de décrire la K-théorie de cette catégorie que nous baptiserons la “K-théorie hermitienne”. La K-théorie hermitienne devrait également exister dans des situations plus générales, par exemple pour la catégorie des fibrés vectoriels sur un schéma munis d'une forme bilinéaire symétrique.
En comparant la K-théorie classique avec la K-théorie hermitienne, on constate des différences assez importantes: Ainsi, les théorèmes cités ci-dessus n'ont pas d'analogue en K-théorie hermitienne. Pour une catégorie exacte, la K-théorie hermitienne n'est même pas definie. Il y a une définition seulement pour les catégories additives ainsi que quelques résultats dans le cas particulier des modules projectifs munis d'une forme hermitienne. Dans cette thèse, nous allons définir la K-théorie hermitienne pour une catégorie exacte $\mathcal{C}$ comme étant les groupes d'homotopie de l'espace classifiant d'une certaine catégorie $\mathcal{L}(\mathcal{C}_h)$ (Définition 5.10). Nous allons également définir la $\mathcal{U}$-théorie de $\mathcal{C}_h$ comme étant les groupes d'homotopie de l'espace des lacets de l'espace classifiant d'une catégorie $\mathcal{W}(\mathcal{C}_h)$. Ensuite nous pouvons établir une fibration homotopique induite par le foncteur hyperbolique H

$$\Omega BQ(\mathcal{C}) \xrightarrow{H_*} B\mathcal{L}(\mathcal{C}_h) \to B\mathcal{W}(\mathcal{C}_h)$$

qui généralise le cas additif (Théorème 5.7). Comme application principale, nous démontrons un Théorème de Localisation pour la K-théorie hermitienne d'un anneau de Dedekind A, par rapport à un systeme multiplicatif S (Théorème 8.7). Plus précisement, nous démontrons l'existence d'une fibration homotopique

$$B\mathcal{L}(P(A)_h) \to B\mathcal{L}(P(S^{-1}A)_h) \to B\mathcal{W}((\mathcal{T}_S)_h),$$

où $\mathcal{T}_S$ est la catégorie des A-modules de S-torsion. Par conséquent, nous avons une suite exacte longue

$$... \to U_n((\mathcal{T}_S)_h) \to K_n(P(A)_h) \to K_n(P(S^{-1}A)_h) \to ...$$

En plus, nous démontrons un Théorème de Dévissage (Théorème 9.5) qui nous permet de remplacer $U_n((\mathcal{T}_S)_h)$ par $\oplus U_n(P(A/\wp)_h)$ où on prend la somme sur tous les ideaux maximaux.
Pour A l'anneau des entiers dans un corps de nombres F, nous pouvons calculer la $\mathcal{U}$-théorie de ses corps residuels (Corollaire 9.9) en utilisant les calculs de Quillen et Friedlander, et nous en déduisons (Théorème 9.11) par exemple

$${}_1K_n^h(A) \overset{\cong}{\to} {}_1K_n^h(F) \quad \forall\, n \equiv 3, 4\, mod\, 8;$$

$${}_{-1}K_n^h(A) \overset{\cong}{\to} {}_{-1}K_n^h(F) \quad \forall\, n \equiv 0, 7\, mod\, 8.$$

Ce texte est structuré de la manière suivante:
Dans la section 3, nous rappelons quelques faits sur la K-théorie hermitienne des modules, tout en prenant un point de vue qui nous permet aisément de généraliser aux catégories avec dualité. En suivant [CL2],[Sch3], nous introduisons une catégorie $\mathcal{W}({}_\epsilon P(A)_h)$ qui donne un délaçage de la fibre homotopique du foncteur hyperbolique de la K-théorie classique vers la K-théorie hermitienne (Théorème 3.15) du moins si 2 est une unité dans notre anneau.
La section 4 traite de la K-théorie hermitienne des catégories additives avec dualité. La philosophie est que la plupart des choses qui sont vraies pour la K-théorie hermitienne des anneaux est encore vraie pour la K-théorie hermitienne des catégories additives.
La K-théorie hermitienne d'une catégorie exacte avec dualité et ses propriétés fondamentales mentionnées ci-dessus sont décrites dans la section 5.
Dans la section 6, nous établissons un Théorème de Localisation pour la K-théorie hermitienne des catégories additives (Corollaire 6.7) qui nous permet de définir leur K-théorie hermitienne négative.
Dans la section 7, nous construisons une catégorie additive simpliciale avec dualité qui donne un modèle à la Waldhausen $|iR_*^h\mathcal{C}|$ pour la $\mathcal{U}$-théorie d'une catégorie exacte $\mathcal{C}$ avec dualité.
La section 8 contient le Théorème de Localisation pour la K-théorie hermitienne des anneaux de Dedekind (et de manière plus générale pour les anneaux héréditaires, en particulier pour un anneau de groupe AG avec G un groupe fini et A un anneau de Dedekind dans lequel l'ordre du groupe est une unité). Plusieurs stratégies et difficultés pour le démontrer sont esquissées. La démonstration sera donnée dans la section 11.

Dans la section 9, nous démontrons un Théorème de Dévissage qui nous permet de simplifier la fibration homotopique ci-dessus en remplaçant $\mathcal{W}((\mathcal{T}_S)_h)$ par $\oplus\mathcal{W}(P(A/\wp)_h)$ (Théorème 9.5). Comme nous pouvons calculer les groupes d'homotopie de cette catégorie si $A/\wp$ est fini (prenons les entiers dans un corps de nombres, par exemple), notre résultat de localisation plutôt abstrait devient très concret et nous permet de voir dans quels degrés l'inclusion $A \to Quot(A)$ induit un isomorphisme en K-théorie hermitienne (Théorème 9.11).
Dans la section 10 nous évoquons d'autres problèmes, en particulier ceux qui apparaissent si nous remplaçons notre catégorie exacte $\mathcal{C}$ par la catégorie des complexes de chaines $Ch^b(\mathcal{C})$.
La section 11 contient la démonstration du Théorème de Localisation 8.7.

Les résultats principaux de la section 5 ont été annoncés dans [Ho]. Les sections 6,7 et 11 sont un travail en commun avec Marco Schlichting, bien que je sois entièrement responsable de la présentation et de tous ses défauts.

2 Introduction

After K_0 was introduced by Grothendieck [BS], algebraic K-theory first developed for the category of finitely generated projective modules over a ring A. (All rings we are dealing with are supposed to be associative and with a unit, and all modules are right modules.) This is an additive category. An additive category can be regarded as a symmetric monoidal category (also called a ACU-tensor category) or as an exact category (where the exact structure is given by the split exact sequences). Quillen [Q1, Q2] defined K-theory for both kinds of categories via the constructions $S^{-1}S$ and Q. He proved that the two definitions coincide for additive categories and that they generalize the Plus construction [Lo1]. The Q-construction allows us to prove important theorems such as those linked to localization and resolution etc.

Assume now that our projective modules over a given ring A are equipped with a symmetric bilinear (or more generally with an ϵ-hermitian) form. We want to describe the K-theory of this category which we will call "hermitian K-theory". Hermitian K-theory should also cover more general situations, for example the category of vector bundles over a scheme equipped with a bilinear symmetric form.
Comparing classical K-theory and hermitian K-theory, the situation is quite different. We have no hermitian analogues of the theorems in classical K-theory cited above. For an exact category in general, the hermitian K-theory was not even defined. We only had a definition for additive categories and very few results in the special case of projective modules equipped with a hermitian form. In this thesis, we will give a general definition of the hermitian K-theory of an exact category $\mathcal{C}$ as the homotopy groups of the classifying space of a certain category $\mathcal{L}(\mathcal{C}_h)$ (see Definition 5.10). We can also define the $\mathcal{U}$-theory of $\mathcal{C}_h$ as the homotopy groups of the loop space of the classifying space of the category $\mathcal{W}(\mathcal{C}_h)$. Then we can establish a homotopy fibration induced by the hyperbolic functor H

$$\Omega BQ(\mathcal{C}) \xrightarrow{H_*} B\mathcal{L}(\mathcal{C}_h) \to B\mathcal{W}(\mathcal{C}_h)$$

which generalizes the additive case (see Theorem 5.7). As an important application, we prove a Localization Theorem for the hermitian K-theory of a Dedekind ring A and a multiplicative subset S (see Theorem 8.7). We prove the existence of a homotopy fibration

$$B\mathcal{L}(P(A)_h) \to B\mathcal{L}(P(S^{-1}A)_h) \to B\mathcal{W}((\mathcal{T}_S)_h)$$

where $\mathcal{T}_S$ is the category of S-torsion modules. Consequently, we get a long exact sequence

$$\ldots \to U_n((\mathcal{T}_S)_h) \to K_n(P(A)_h) \to K_n(P(S^{-1}A)_h) \to \ldots .$$

Moreover, we can prove a Dévissage theorem (Theorem 9.5) which allows us to replace $U_n((\mathcal{T}_S)_h)$ by $\oplus U_n(P(A/\wp)_h)$ where the sum is taken over all prime ideals $\wp$ different from (0).
If A is the ring of integers in a number field F, we can compute the $\mathcal{U}$-theory of these finite residue fields (Corollary 9.9) using calculations of Quillen and Friedlander, and we obtain among other things

$$ {}_1K_n^h(A) \stackrel{\cong}{\to} {}_1K_n^h(F) \ \ \forall\, n \equiv 3, 4\, mod\, 8 $$

$$ {}_{-1}K_n^h(A) \stackrel{\cong}{\to} {}_{-1}K_n^h(F) \ \ \forall\, n \equiv 0, 7\, mod\, 8 $$

(see Theorem 9.11).

This text is organized as follows:
In section 3, we recall some facts about the K-theory of hermitian modules, taking a point of view which allows us to immediately generalize to a categorical viewpoint. In particular, following [CL2],[Sch3], we introduce a category $\mathcal{W}({}_\epsilon P(A)_h)$ which gives a delooping of the homotopy fiber of the hyperbolic functor from classical K-theory to hermitian K-theory (Theorem 3.15) at least if 2 is a unit in our ring.
Section 4 deals with the hermitian K-theory of additive categories with duality. The philosophy is that most what is true for the hermitian K-theory of rings is still true for the hermitian K-theory of additive categories with duality.
The hermitian K-theory of an exact category with duality and its basic properties as mentioned above are described in section 5.
In section 6, we state the localization theorem for the K-theory of hermitian additive categories (Corollary 6.7) which allows us to define their negative K-theory.
In section 7, we construct a simplicial additive category with duality which yields a Waldhausen-like model $|iR_*^h\mathcal{C}|$ for the $\mathcal{U}$-theory of an exact category $\mathcal{C}$ with duality.
Section 8 contains the Localization Theorem for the hermitian K-theory of Dedekind rings (and more generally for hereditary rings, in particular for group rings AG where G is a finite group and A is a Dedekind ring in which the order of the group is a unit). Some strategies and difficulties of possible proofs are sketched. The proof will be the subject of section 11.
In section 9, we prove a Dévissage theorem which allows to simplify the above homotopy fibration replacing $\mathcal{W}((\mathcal{T}_S)_h)$ by $\oplus\mathcal{W}(P(A/\wp)_h)$ (Theorem 9.5). As we can calculate the homotopy groups of this category if $A/\wp$ is finite (consider the integers in a number field, for example), our somewhat abstract localization result becomes very concrete and allows us to see in which degrees the inclusion $A \to Quot(A)$ induces an isomorphism in hermitian K-theory (Theorem 9.11).

In section 10 we sketch some further questions, in particular those arising when we replace an exact category $\mathcal{C}$ by the category of chain complexes $Ch^b(\mathcal{C})$.
Section 11 contains the proof of the Localization Theorem 8.7.

The main results of section 5 have been announced in [Ho]. Sections 6,7 and 11 are joint work with Marco Schlichting, but I am entirely responsible for the presentation and all the defects it has.

3 The hermitian K-theory of rings

Before we start with the K-theory of hermitian modules, we recall some basic facts (without proofs) about simplicial sets and classifying spaces. A topologist might skip this part.

A simplicial set is by definition a functor $F : \Delta^{op} \to \underline{Set}$. Recall that Δ is the category whose objects are the non-negative integers n, considered as an ordered set $n = \{0 < 1 < ... < n\}$, and a morphism $f : m \to n$ is a monotonic map, i.e. $i \leq j$ implies $f(i) \leq f(j)$. In particular, we have the face maps $\delta^i : n \to n+1$ (the i tells us which element is not in the image) and the degeneracy maps $\sigma^j : n \to n-1$ (where the j indicates the element onto which two elements are mapped). In fact, up to permutation any map is a unique composition of these maps. We write $\Delta^{op}\underline{Set}$ for the category of simplicial sets. Then we have a pair of adjoint functors

$$| \; | : \Delta^{op}\underline{Set} \leftrightarrows \underline{Top} : Sing.$$

Here $| \; |$ is the so-called geometric realization functor and $Sing = Hom_{\underline{Top}}(\Delta^*_{top}, \;)$ where Δ^n_{top} is the topological standard n-simplex. Moreover, both categories can be equipped with the structure of a closed model category, and the above pair of adjoint functors respects enough of this structure that by a theorem of Quillen it becomes an equivalence of the homotopy categories. Next, we have the nerve functor

$$N : \underline{small\,cat} \to \Delta^{op}\underline{Set} \; .$$

For a small category $\mathcal{C}$, the set $N\mathcal{C}_m$ contains all the diagrams of compositions of m composable morphism of $\mathcal{C}$. The face maps correspond to the composition of maps and the degeneracy maps are given by inserting identity maps. The classifying space $B\mathcal{C}$ of a small category $\mathcal{C}$ is defined by $B\mathcal{C} := |N\mathcal{C}|$. Observe that this generalizes the classical definition of the classifying space of a discrete group when we consider it as a category with a single object. From now on, when we talk about realizations, we assume that the categories we are dealing with are small. To simplify our notations, we will use the same symbol for a category, its nerve and its classifying space.

Let A be a ring equipped with an anti-involution, i.e., a morphism of rings $\bar{\ } : A \to A^{op}$ such that $\bar{\bar{a}} = a \;\forall\, a \in A$. Furthermore, choose $\epsilon \in center(A)$ with $\epsilon\bar{\epsilon} = 1$. Consider a finitely generated projective A-module M.

Definition 3.1 *If M is equipped with a sesquilinear non-degenerate form Φ (anti-linear in the first component) such that $\Phi(m,n) = \overline{\Phi(n,m)}\epsilon \;\forall\; m,n \in M$, we say that (M,Φ) is an ϵ-hermitian module. The dual of M is defined by ${}^tM = Hom_{A,anti}(M,A)$, i.e., $f \in {}^tM \Leftrightarrow f(ma) = \bar{a}f(m)$. This is an A-module via $fa(m) = f(m)a$.*

With Φ is associated $\phi : M \to {}^tM$ via $\phi(m) = \Phi(\ ,m)$. One observes that ϕ is an isomorphism and that ${}^t\phi = \bar{\epsilon}\phi$; more precisely, the triangle

$$\begin{array}{ccc} M & \xrightarrow{\overline{ev}} & {}^{t^2}M \\ & {\scriptstyle \bar{\epsilon}\phi} \searrow \quad \swarrow {\scriptstyle {}^t\phi} & \\ & {}^tM & \end{array} \qquad (*)$$

commutes (with the canonical isomorphism $\overline{ev}(m)(f) = \overline{f(m)}\ \forall\, m \in M,\ f \in {}^tM$). It follows that the category of ϵ-hermitian modules over A and the unitary morphisms (i.e. those respecting the form - it would be better to call them metric) is canonically isomorphic to the following category ${}_{\epsilon}P(A)_h$:

Definition 3.2 *Let ${}_{\epsilon}P(A)_h$ be the category of hermitian modules relative to $(A,\ {}^t\ ,\ ^-\ ,\epsilon)$. An object is an isomorphism $\phi : M \overset{\cong}{\to} {}^tM$ such that (*) commutes. A morphism $\alpha : (M,\phi) \to (N,\psi)$ is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi}\downarrow{\scriptstyle \cong} \\ {}^tM & \xleftarrow[{}^t\alpha]{} & {}^tN \end{array}$$

Remark: We can also drop the condition that the morphisms ϕ and ψ are isomorphisms. Then we obtain the category $P(A)_{hd}$ of all hermitian objects, including the degenerate ones. Of course, $P(A)_{hd}$ contains $P(A)_h$ as a full subcategory.

Examples: For A commutative, $^- = Id$ and ϵ =1 (resp.-1) we obtain the theory of (anti-)symmetric forms. When $A = \mathbf{C}$ is the complex numbers, the complex conjugation is an anti-involution. If A is a commutative ring and G is a group, then the group ring AG is equipped with an anti-involution sending ag to ag^{-1}.
$({}_{\epsilon}P(A)_h, \oplus)$ is a symmetric monoidal category via $(M,\phi) \oplus (N,\psi) = (M \oplus N, \phi \oplus \psi)$, so its K-theory is defined by Quillen's $S^{-1}S$-construction (see [Q2] or p.16):

Definition 3.3

$$K_n({}_{\epsilon}P(A)_h) := \pi_n(\mathrm{Iso}({}_{\epsilon}P(A)_h)^{-1}\,\mathrm{Iso}({}_{\epsilon}P(A)_h))\ \forall n \geq 0$$

One often writes ${}_{\epsilon}K_n^h(P(A))$ or even ${}_{\epsilon}K_n^h(A)$ instead of $K_n({}_{\epsilon}P(A)_h)$. If the ϵ is understood, it is sometimes dropped. Many people write $(M,\phi)\perp(N,\psi)$ instead of $(M,\phi) \oplus (N,\psi)$.
Let us recall some facts from the classical K-theory of finitely generated projective modules $P(A)$. Any projective module is a direct summand of a free module; in other words, the subcategory $F(A)$ of the objects $A^n,\ n \geq 0$

is cofinal in $P(A)$. This cofinality implies, among other things, that:
$\bullet K_1(P(A)) \cong H_1(\text{colim } Aut(A^n), \mathbf{Z})$ (in the classical notation, one writes $GL(A)$ instead of colim $Aut(A^n)$)
$\bullet K_2(P(A)) \cong H_2([GL(A), GL(A)], \mathbf{Z})$
$\bullet K_n(P(A)) \cong \pi_n(BGL(A)^+ \times K_0(P(A)))$ (in the classical notation, one writes $K_n(A)$ instead of $K_n(P(A))$)
$\bullet BGL(A)^+ \times K_0(A) \simeq \Omega(BGL(\Sigma A)^+ \times K_0(\Sigma A))$ (see [Wag], ΣA is the algebraic suspension of the ring A: Its elements are the infinite matrices over A having only a finite number of non-zero elements in any row and any column, divided by those matrices having only a finite number of non-zero elements.).
It also implies the existence of a "Volodin model" [Su]. Furthermore, the description using the Plus construction can be used to calculate the K-theory of a finite field [Q3] and the free part of the K-theory of an algebraic number field [Bo].

In the category ${}_\epsilon P(A)_h$, we would like to have a cofinal subcategory which behaves like the free modules in $P(A)$ and enables us to prove similar results. For this, it is necessary to introduce hyperbolic and metabolic modules:

Definition 3.4 *A module (P, λ) is hyperbolic if there is a module M and an isomorphism $(P, \lambda) \cong (M \oplus {}^tM, \left(\begin{smallmatrix} 0 & 1 \\ \epsilon & 0 \end{smallmatrix}\right)) =: (H(M), \mu_M)$. It is metabolic if there is a (possibly degenerate) hermitian module (M, ϕ) and an isomorphism $(P, \lambda) \cong (M \oplus {}^tM, \left(\begin{smallmatrix} \phi & 1 \\ \epsilon & 0 \end{smallmatrix}\right))$*

Lemma 3.5 *a) Any hermitian module (M, ϕ) is a direct summand of a metabolic module: $(M, \phi) \oplus (M, -\phi) \cong (M \oplus {}^tM, \left(\begin{smallmatrix} \phi & 1 \\ \epsilon & 0 \end{smallmatrix}\right))$*
b) If there exists $\lambda \in center(A)$ such that $\lambda + \bar{\lambda} = 1$, then any hermitian module is a direct summand of a hyperbolic module: $(M, \phi) \oplus (M, -\phi) \cong (H(M), \mu_M)$

Proof: a)

$$\begin{array}{ccc} M \oplus M & \xrightarrow{\left(\begin{smallmatrix} 1 & -1 \\ 0 & \phi \end{smallmatrix}\right)} & M \oplus {}^tM \\ {\scriptstyle \left(\begin{smallmatrix} \phi & 0 \\ 0 & -\phi \end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} \phi & 1 \\ \epsilon & 0 \end{smallmatrix}\right)} \\ {}^tM \oplus {}^tM & \xleftarrow{\left(\begin{smallmatrix} 1 & 0 \\ -1 & {}^t\phi \end{smallmatrix}\right)} & {}^tM \oplus M \end{array}$$

commutes.
b)

$$\begin{array}{ccc} M \oplus {}^tM & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ \lambda\phi & 1 \end{smallmatrix}\right)} & M \oplus {}^tM \\ {\scriptstyle \left(\begin{smallmatrix} \phi & 1 \\ \epsilon & 0 \end{smallmatrix}\right)}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 1 \\ \epsilon & 0 \end{smallmatrix}\right)} \\ {}^tM \oplus M & \xleftarrow{\left(\begin{smallmatrix} 1 & \overline{\epsilon\lambda}\phi \\ 0 & 1 \end{smallmatrix}\right)} & {}^tM \oplus M \end{array}$$

commutes. □

We will always say "if 2 is invertible" even if weaker conditions (i.e. the existence of a λ such that $\lambda + \bar{\lambda} = 1$ or the fact that the hyperbolic modules form a cofinal subcategory of ${}_{\epsilon}P(A)_h$) are sufficient in many proofs. In this case, we can conclude, as for classical K-theory (replacing A^n by $(H(A^n), \mu_{A^n})$), that:

•$K_1({}_{\epsilon}P(A)_h) \cong H_1(\text{colim } Aut(H(A^n), \mu_{A^n}), \mathbf{Z})$ (one often writes ${}_{\epsilon}O(A)$ instead of colim $Aut(H(A^n), \mu_{A^n})$)

•$K_2({}_{\epsilon}P(A)_h) \cong H_2([{}_{\epsilon}O(A), {}_{\epsilon}O(A)], \mathbf{Z})$

•$K_n({}_{\epsilon}P(A)_h) \cong \pi_n(B{}_{\epsilon}O(A)^+ \times K_0({}_{\epsilon}P(A)_h))$ In the notation of [Ka2], one sets ${}_{\epsilon}L_n(A) := \pi_n(B{}_{\epsilon}O(A)^+ \times K_0({}_{\epsilon}P(A)_h)$ independent of the fact whether 2 is invertible or not. This should not be confused with the L-theory of Wall, Ranicki and Mischenko, see the first remark below.

•$B{}_{\epsilon}O(A)^+ \times K_0({}_{\epsilon}P(A)_h) \simeq \Omega(B{}_{\epsilon}O(\Sigma A)^+ \times K_0({}_{\epsilon}P(\Sigma A)_h))$

and that a "Volodin model" can be constructed [So]. Furthermore, it is possible to calculate the K-theory of a finite field [Fr] and the free part of the K-theory of an algebraic number field [Bo].

Remark: The idea of defining the L-theory (a good reference is [Ra2]) of a ring A is as follows: Consider the abelian monoid of n-dimensional "Poincaré complexes" of projective modules of finite type over A together with an n-cycle playing the role of the ϵ-hermitian form. Then we divide out by a certain "cobordism" relation to get the group $L_n(A, \epsilon)$. The geometric interest in these L-groups is that they contain obstruction classes in the surgery theory of a Poincaré space X when $A = \mathbf{Z}\pi_1(X)$. We have periodicity in L-theory [Ra1], i.e $L_n(A, \epsilon) \cong L_{n+2}(A, -\epsilon)$. This should be compared with the higher Witt groups (see Definition 3.13) where the same is true up to 2-torsion [Ka6], i.e. ${}_{\epsilon}W_n(P(A)) \otimes \mathbf{Z}[1/2] \cong {}_{-\epsilon}W_{n+2}(P(A)) \otimes \mathbf{Z}[1/2]$. Moreover, these two theories coincide up to 2-torsion [Lo2]: $L_n(A, \epsilon) \otimes \mathbf{Z}[1/2] \cong {}_{\epsilon}W_n(P(A)) \otimes \mathbf{Z}[1/2]$.

Remark: Instead of looking at hermitian modules, we can deal with the (symmetric monoidal) category of quadratic forms over a given ring: consider the sesquilinear non-degenerate forms $Sesq(M)$ on a finitely generated projective module M. We have a morphism $T_\epsilon : Sesq(M) \to Sesq(M)$ induced by $T_\epsilon(\Phi)(m, n) = \overline{\Phi(n, m)}\epsilon$. Then the quadratic forms on M (more precisely, the classes of their associated bilinear forms) are equal to $coker(T_\epsilon - 1)$ while the ϵ-hermitian forms are equal to $ker(1 - T_\epsilon)$. The bilinearization $T_\epsilon + 1$ induces a morphism $coker(1 - T_\epsilon) \to ker(1 - T_\epsilon)$ which is an isomorphism if 2 is invertible [Wall]. In the category of quadratic forms, the hyperbolic objects always form a cofinal subcategory. Therefore the Plus construction always gives a model for its K-theory. To study the differences between the categories of hermitian modules and that of quadratic modules and their K-theories, one introduces form parameters [Bak].

The category ${}_{\epsilon}P(A)_h$ and the underlying category $P(A)$ are related by the forgetful functor and the hyperbolic functor

$$F : {}_{\epsilon}P(A)_h \to P(A) \ \ \textit{and}$$

$$H : \mathrm{Iso}(P(A)) \to \mathrm{Iso}({}_{\epsilon}P(A)_h)$$

given by $H(M) = (H(M), \mu_M)$ on the objects and $H(f) = (\left(\begin{smallmatrix} f & 0 \\ o & {}^t f^{-1} \end{smallmatrix} \right))$ on the morphisms. These functors induce functors F_* and H_* between the classifying spaces for the K-theories.

Definition 3.6

$$\mathcal{U}({}_{\epsilon}P(A)_h) := \mathrm{hofib}(H_*, 0)$$
$$\mathcal{V}({}_{\epsilon}P(A)_h) := \mathrm{hofib}(F_*, 0)$$
$${}_{\epsilon}U_n(A) := \pi_n(\mathcal{U}({}_{\epsilon}P(A)_h))$$

Here, $\mathrm{hofib}(f, y)$ is the homotopy fiber of $f : X \to Y$ over a point $y \in Y$, i.e., the homotopy limit of the diagram $\{y\} \to Y \stackrel{f}{\leftarrow} X$. Consequently, these four spaces form a homotopy cartesian square, i.e., $\mathrm{hofib}(f, y) \simeq \{y\} \times_Y Y^I \times_Y X$. It follows that we get a long exact homotopy sequence. In general, $\mathrm{hofib}(f, y)$ depends on y, but any path connecting y and y' in Y gives a homotopy equivalence of the homotopy fibers over these two points. That's why we often suppress y in $\mathrm{hofib}(f, y)$ if Y is connected or if f is a morphism of H-groups.
Recall that a homotopy cartesian square remains homotopy cartesian if we replace one of the four spaces by a space having the same homotopy type. We say that $F \stackrel{g}{\to} X \stackrel{f}{\to} Y$ is a homotopy fibration if it fits in a homotopy cartesian square where the fourth space is contractible. This is equivalent to saying that $f \circ g$ is homotopic to a constant map $\tilde{y}$ for some $y \in Y$ and the induced map (depending on the choice of the contraction) $F \to \ \mathrm{hofib}(f, y)$ is a homotopy equivalence.

Theorem 3.7 *("Fundamental Theorem")*
There is a homotopy equivalence

$$\mathcal{V}({}_{\epsilon}P(A)_h) \simeq \Omega(\mathcal{U}({}_{-\epsilon}P(A)_h))$$

if 2 is invertible.

Proof: See [Ka7]. The main idea is that the homotopy groups of both spaces are equipped with a $K_0(\ {}_1(P(\mathbf{Z}[1/2])_h))$-module structure and the isomorphism between the homotopy groups is given by the multiplication with an element of this ring. □

Karoubi and Giffen [CL2] proposed a category $\mathcal{W}({}_{\epsilon}P(A)_h)$ whose loop space is a model for $\mathcal{U}({}_{\epsilon}P(A)_h)$ if 2 is invertible:

Definition 3.8 *Let $\mathcal{W}({}_{\epsilon}P(A)_h)$ be the category whose objects are the same as those of ${}_{\epsilon}P(A)_h$. For the morphisms, we set $\mathcal{W}({}_{\epsilon}P(A)_h)((P,\lambda),(M,\psi)) = \{$equivalence classes of $(q,(L,\phi),i)\}$, where a representative is given by a commutative diagram (in $P(A)$)*

$$\begin{array}{ccccc} P & \overset{q}{\twoheadleftarrow} & L & \overset{i}{\rightarrowtail} & M \\ {\scriptstyle\lambda}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\phi} & & {\scriptstyle\cong}\downarrow{\scriptstyle\psi} \\ {}^tP & \overset{{}^tq}{\rightarrowtail} & {}^tL & \overset{{}^ti}{\twoheadleftarrow} & {}^tM \end{array}$$

with q a split epimorphism and i a split monomorphism. We also demand that the inclusion of $L_M^{\perp} := ker({}^ti \circ \psi)$ in M factors over L; in other words we have a commutative triangle

$$\begin{array}{ccc} & L_M^{\perp} & \\ \swarrow\!\!{}^{\dashrightarrow} & & \searrow \\ L & \overset{i}{\rightarrowtail} & M \end{array}$$

Consequently, $L^0 := ker\,\phi \cong ker\,q \cong L_M^{\perp}$. We say that L is a "sur-lagrangian" (and we often write $L^{\perp}$ instead of $L_M^{\perp}$). $(q,(L,\phi),i)$ and $(q',(L',\phi'),i')$ are equivalent if there is an isomorphism in $P(A)$ $\alpha : L \to L'$ such that $q = q' \circ \alpha$ and $i = i' \circ \alpha$. The composition of morphism is explained in the following lemma.

In fact, the condition that $L^{\perp}$ is included in L is equivalent to saying that the outer rectangle is a bicartesian square, and this will be used in the proof of the following lemma:

Lemma 3.9 *$\mathcal{W}({}_{\epsilon}P(A)_h)$ is a category with the composition law specified below.*

Proof: Given two morphisms $[v,(L_1,\phi_1),i] \in \mathcal{W}({}_{\epsilon}P(A)_h)((M,\psi),(N,\chi))$ and $[u,(L_2,\phi_2),j] \in \mathcal{W}({}_{\epsilon}P(A)_h)((N,\chi),(P,\lambda))$, we set $L := L_1 \times_N L_2$. Then their composition is $[(v \circ \tilde{u}),(L, {}^t\tilde{i} \circ \phi_2 \circ \tilde{i}),(j \circ \tilde{i})]$ defined by the diagram

$$\begin{array}{ccccc} L & \overset{\tilde{i}}{\rightarrowtail} & L_2 & \overset{j}{\rightarrowtail} & P \\ {\scriptstyle\tilde{u}}\downarrow & cart & \downarrow{\scriptstyle u} & & \\ L_1 & \overset{i}{\rightarrowtail} & N & & \\ {\scriptstyle v}\downarrow & & & & \\ M & & & & \end{array} \qquad (+)$$

and one checks that this is well-defined. It remains to show that $L_P^{\perp}$ factors over L. Consider the following commutative diagram (in which we have suppressed the isomorphims ψ, χ and λ):

$$\begin{array}{ccccc}
L & \overset{\tilde{i}}{\rightarrowtail} & L_2 & \overset{j}{\rightarrowtail} & P \\
{\scriptstyle \tilde{u}}\downarrow & (1) & \downarrow{\scriptstyle u} & (2) & \downarrow{\scriptstyle {}^t j} \\
L_1 & \overset{i}{\rightarrowtail} & N & \overset{{}^t u}{\rightarrowtail} & {}^t L_2 \\
{\scriptstyle v}\downarrow & (3) & {\scriptstyle {}^t i}\downarrow & (4) & \downarrow{\scriptstyle {}^t \tilde{i}} \\
M & \overset{{}^t v}{\rightarrowtail} & {}^t L_1 & \overset{{}^t \tilde{u}}{\rightarrowtail} & {}^t L
\end{array}$$

The square (1) is not only cartesian, but also cocartesian (see the appendix of [Ke1]), so the square (4) is bicartesian. Similary, one can show that the squares (2) and (3) are also bicartesian. It follows that the big square composed of four bicartesian squares is bicartesian. This implies that $ker({}^t\tilde{i} \circ {}^t j)$ factors over L. □

Instead of looking at the morphisms induced by "sur-lagrangians", we can consider "sub-lagrangians" (also called "totally isotropic" modules):

Definition 3.10 *Let $\mathcal{W}'({}_{\epsilon}P(A)_h)$ be the category whose objects are the same as those of ${}_{\epsilon}P(A)_h$. For the morphisms, we set $\mathcal{W}'({}_{\epsilon}P(A)_h)((P,\lambda),(M,\psi)) = \{$equivalence classes of $(p,(T^{\perp},\chi),j)\}$ where $T^{\perp}$ is the orthogonal of some object T and a representative is given by a commutative diagram (in $P(A)$)*

$$\begin{array}{ccccc}
P & \overset{p}{\twoheadleftarrow} & T^{\perp} & \overset{j}{\rightarrowtail} & M \\
{\scriptstyle \lambda}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \chi} & & {\scriptstyle \cong}\downarrow{\scriptstyle \psi} \\
{}^t P & \overset{{}^t p}{\rightarrowtail} & {}^t T^{\perp} & \overset{{}^t j}{\twoheadleftarrow} & {}^t M
\end{array}$$

with p a split epimorphism and j a split monomorphism. We also demand that T is an isotropic sub-module of M; in other words we have a commutative triangle:

$$\begin{array}{ccc}
 & T & \\
\swarrow\!\!{\scriptstyle -\,-} & & \searrow \\
T^{\perp}_M & \overset{j}{\rightarrowtail} & M
\end{array}$$

Lemma 3.11 *We have an isomorphism of categories $F: \mathcal{W}({}_{\epsilon}P(A)_h) \overset{\cong}{\rightarrow} \mathcal{W}'({}_{\epsilon}P(A)_h)$ given by $F([q,(L,\phi),i]) = [q\circ\gamma,((L^{\perp})^{\perp},\gamma\circ\phi\circ\gamma),i\circ\gamma]$ with $\gamma:(L^{\perp})^{\perp} \overset{\cong}{\rightarrow} L$.*

Proof: Because $L^{\perp} \overset{\iota}{\rightarrowtail} M \overset{{}^t i\circ\psi}{\twoheadrightarrow} {}^t L$ is exact, $L \overset{i}{\rightarrowtail} M \overset{{}^t \iota\circ\psi}{\twoheadrightarrow} {}^t L^{\perp}$ is also exact. It follows that there is an isomorphism $\gamma:(L^{\perp})^{\perp} \overset{\cong}{\rightarrow} L$ over M and that $F([q,(L,\phi),i])$ is a morphism in $\mathcal{W}'({}_{\epsilon}P(A)_h)$. We can define $G: \mathcal{W}'({}_{\epsilon}P(A)_h) \rightarrow \mathcal{W}({}_{\epsilon}P(A)_h)$ in a dual way setting $G([p,(T^{\perp},\chi),j]) := [p,(T^{\perp},\chi),j]$. One easily checks that F are G are well-defined and inverses to each other. □

Remark: The proofs of Lemma 3.9 and Lemma 3.11 show that $\mathcal{W}({}_\epsilon P(A)_h)$ is isomorphic to the category ${}_\epsilon WP(A)$ in [Ur].

Proposition 3.12 *a) Suppose we have a K such that $(P,\lambda)\oplus(H(K),\mu_K)\cong(M,\psi)$. Then there is a morphism from (P,λ) to (M,ψ) in $\mathcal{W}({}_\epsilon P(A)_h)$.*
b) If 2 is invertible and there is a morphism in $\mathcal{W}({}_\epsilon P(A)_h)$ from (P,λ) to (M,ψ). Then there exists a K such that $(P,\lambda)\oplus(H(K),\mu_K)\cong(M,\psi)$.

Proof: a) The class of $(\pi,(P\oplus K,\lambda\oplus 0),\iota)$ is such a morphism.
b) Let $[(q,(L,\phi),i)]$ be a morphism. Choose a section s of q. Then P is a direct summand of M as a hermitian module (see [Kn, II.2.5.2] where it is shown that if we have a non-degenerate hermitian object with a subobject on which the form is non-degenerate, then this is a direct summand as a hermitian subobject. Here we use the fact that 2 is invertible, which implies that any hermitian module can be considered as the equivalence class of a module equipped with a bilinear form). So the problem reduces to showing that the hermitian modules which are connected with 0 by a morphism are precisely the metabolic modules (which are hyperbolic if 2 is invertible). In this case, $(M,\psi)\cong(L^\perp\oplus{}^tL^\perp,\ \left(\begin{smallmatrix} a & b\\ \epsilon\,{}^tb & d\end{smallmatrix}\right)\ )$. As $a=0$ and b is an isomorphism (because $L\oplus{}^tL \xrightarrow{[0\ b]} {}^tL$ is a split epimorphism with kernel L), conjugation with $\left(\begin{smallmatrix} b^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)$ gives the desired module. $\square$

We therefore see that $\mathcal{W}({}_\epsilon P(A)_h)$ is isomorphic to the category ${}_\epsilon\hat{W}(A)$ in [CL2]. Following [Ka2], we define the higher Witt groups:

Definition 3.13

$$W_n({}_\epsilon P(A)_h) := \mathit{coker}\,(K_n(P(A)) \xrightarrow{H_*} K_n({}_\epsilon P(A)_h)).$$

We will often write $W({}_\epsilon P(A)_h)$ instead of $W_0({}_\epsilon P(A)_h)$ for the classical Witt group.
Let ${}_\epsilon P(A)_H$ be the full subcategory of objects in ${}_\epsilon P(A)_h$ isomorphic to a hyperbolic object. Because H_* factors via the epimorphism $K_0(P(A))\to K_0({}_\epsilon P(A)_H)$, we have $W({}_\epsilon P(A)_h)\cong \mathit{coker}\,(K_0({}_\epsilon P(A)_H)\to K_0({}_\epsilon P(A)_h))$

Lemma 3.14

$$\pi_0(\mathcal{W}({}_\epsilon P(A)_h))\cong W({}_\epsilon P(A)_h)$$

Proof: Any element of $W({}_\epsilon P(A)_h)$ can be represented by a difference $(P,\lambda)-(M,\mu)$ where (M,μ) is metabolic (Lemma 3.5 a)). One can show [QSS, Lemma 5.4] that the metabolic modules are zero in $W({}_\epsilon P(A)_h)$ even if 2 is not invertible. This implies the following:
(P,ϕ) and (Q,ψ) are in the same connected component of $\mathcal{W}({}_\epsilon P(A)_h)$

$\Longleftrightarrow$ There exist metabolic modules (M,μ) and (N,ν) such that $(P,\phi)\oplus(M,\mu)\cong(Q,\psi)\oplus(N,\nu)$ ("$\Rightarrow$" is a consequence of [QSS, 5.3] which says the following: If (L,λ) is a totally isotropic submodule of (M,ψ), then $(M,\psi)\oplus(\overline{L^\perp/L,-\lambda|_{L^\perp}})$ is metabolic.)
$\Longleftrightarrow$ The classes of (P,ϕ) and (Q,ψ) in $W({}_\epsilon P(A)_h)$ are equal. □

Let $\mathcal{W}({}_\epsilon P(A)_H)$ be the full subcategory of $\mathcal{W}({}_\epsilon P(A)_h)$ of objects isomorphic to a hyperbolic object. We let $\mathcal{S}=\mathrm{Iso}(P(A))$, $\mathcal{S}_H=\mathrm{Iso}({}_\epsilon P(A)_H)\cong \mathrm{Iso}(\mathcal{W}({}_\epsilon P(A)_H))$ and $\mathcal{S}_h=\mathrm{Iso}({}_\epsilon P(A)_h)\cong\mathrm{Iso}(\mathcal{W}({}_\epsilon P(A)_h))$.
We have $\theta=\psi\circ\mathcal{S}^{-1}\tau:\mathcal{S}_h\longrightarrow\mathcal{W}({}_\epsilon P(A)_h)$ where the action of $\mathcal{S}$ on $\mathcal{S}_h$ is given by the hyperbolic functor. Here τ is the inclusion of $\mathcal{S}_h$ in $\mathcal{W}({}_\epsilon P(A)_h)$ and $\psi:\mathcal{S}^{-1}\mathcal{W}({}_\epsilon P(A)_h)\overset{\simeq}{\to}\mathcal{W}({}_\epsilon P(A)_h)$ exist because the action of $\mathcal{S}$ on $\mathcal{W}({}_\epsilon P(A)_h)$ is trivial (hence ψ is a homotopy equivalence given by the projection on the level of objects). Then hermitian K-theory, the underlying classical K-theory and the $\mathcal{W}$-construction are related by the following result of M. Schlichting.

Theorem 3.15 *If 2 is invertible, then there is a homotopy fibration*

$$\mathcal{S}^{-1}\mathcal{S}\overset{\mathcal{S}^{-1}H}{\longrightarrow}\mathcal{S}_h^{-1}\mathcal{S}_h\overset{\theta}{\longrightarrow}\mathcal{W}({}_\epsilon P(A)_h)$$

where the action of $\mathcal{S}$ on $\mathcal{S}_h$ is given by the hyperbolic functor.

Proof: The proof can be found in [Sch3]; it uses the Waldhausen-like model of section 6 and Karoubi's Fundamental Theorem (Theorem 3.7) and establishes an additivity theorem for hermitian K-theory. □

Remark: In [CL2, 3.1-3.4] R.Charney and R.Lee believed that they had a proof for Theorem 3.15. They wanted to show that all base changes of the functor $\mathcal{S}^{-1}\tau$ are $H_*(\ ,\mathbf{Z})$-isomorphisms and then show by a rather explicit calculation that we have a homotopy equivalence $\mathcal{S}^{-1}\mathcal{S}\to\mathcal{S}^{-1}(0\downarrow\tau)$. To prove that all base changes are isomorphisms in homology (and hence homotopy equivalences as they are morphisms of H-groups), they finally needed to show that the action induced by the switch $\sigma^*:(P\oplus P\downarrow\tau)\to(P\oplus P\downarrow\tau)$ is the identity in homology. They claimed that σ^* acts as an inner automorphism, which is not true.

Remark: The interest of Charney and Lee in the category $\mathcal{W}({}_\epsilon P(A)_H)$ comes from geometry: Let $\Sigma_n=\{\tau\,\epsilon\,M(n,\mathbf{C})|\ {}^t\tau=\tau, Im(\tau)>0\}$ the Siegel space of dimension n. The symplectic group $Sp_{2n}(\mathbf{Z})={}_{-1}O_{n,n}(\mathbf{R})=Aut(H(\mathbf{R}^n),\mu_{\mathbf{R}^n})$ acts on Σ_n transitively. The action of $\Gamma_n={}_{-1}O_{n,n}(\mathbf{Z})$ even extends to the Satake compactification of this space. We can construct full subcategories $\mathcal{W}_n$ of $\mathcal{W}({}_{-1}P(\mathbf{Z}))$ with $H_i(\mathcal{W}_n,\mathbf{Q})\cong H_i(\mathcal{W}_{n+1},\mathbf{Q})$ for $n>i$ and we can establish an isomorphism $H_*(\mathcal{W}_n,\mathbf{Q})\overset{\cong}{\to}H_*(\Sigma_n^*/\Gamma_n,\mathbf{Q})$(see [CL1, CL2] for more details).

4 The hermitian K-theory of additive categories

First, let us recall the definition of a symmetric monoidal category and its K-theory as defined by Quillen [Q2].

Definition 4.1 *A symmetric monoidal category* $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ *is a category* $\mathcal{C}$ *together with a functor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *and an object* $1_{\mathcal{C}}$ *such that we have natural isomorphisms* $1_{\mathcal{C}} \otimes A \cong A \cong A \otimes 1_{\mathcal{C}}$*,* $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ *and* $A \otimes B \cong B \otimes A$ *for any objects* A, B, C *of* $\mathcal{C}$ *such that certain diagrams [Q2, p.218] commute.*

Any additive category is a symmetric monoidal category, of course (take $\otimes = \oplus$), but there are other important examples as we will see below.
Given two symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we define an action of $\mathcal{C}$ on $\mathcal{D}$ to be a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ fulfilling some obvious conditions. We then define the category $<\mathcal{C}, \mathcal{D}>$. Its objects are the objects of $\mathcal{D}$. A morphism from D to D' is an equivalence class of a pair (C, f) with $f : F(C, D) \to D'$. For example, we have the diagonal action of $\mathcal{C}$ on $\mathcal{C} \times \mathcal{C}$, and we set $\mathcal{C}^{-1}\mathcal{C} := <\mathcal{C}, \mathcal{C} \times \mathcal{C}>$.

Definition 4.2 *Let* $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ *be a symmetric monoidal category and* $\mathcal{S} = Iso(\mathcal{C})$ *be the subcategory of* $\mathcal{C}$ *with the same objects, and whose morphisms are the isomorphisms of* $\mathcal{C}$*. Then we define the K-theory of* $\mathcal{C}$ *by*

$$K_n(\mathcal{C}) = \pi_n(\mathcal{S}^{-1}\mathcal{S}) \quad \forall\, n \geq 0.$$

Let C be a category equipped with a duality functor $^t : \mathcal{C} \to \mathcal{C}^{op}$ and a natural isomorphism $\eta : Id_{\mathcal{C}} \to {}^t\circ^t$ satisfying ${}^t\eta_M \circ \eta_{\,{}^tM} = Id_{\,{}^tM}$ for all objects M of $\mathcal{C}$. We "choose" $^-$ and ϵ as in the case of modules; in general, the choice will often be $^- = Id$ and $\epsilon = 1$ (or $\epsilon =$ -1 if $\mathcal{C}$ is additive). More precisely, one should consider the category of additive categories with duality and ask that functors in this category commute with the identification isomorphisms up to natural isomorphism; see [Sch2] for the details. As before, we will suppress the isomorphism η. This is justified by the fact that any category with duality $(\mathcal{C}, {}^t, \eta)$ is equivalent to a category with duality where the isomorphism between the identity functor and the bidual is given by the identity, see [Sch2] for the details. We can then define:

Definition 4.3 *Let* $\mathcal{C}_h$ *be the category of hermitian objects relative to* $(\mathcal{C}, {}^t, {}^-, \epsilon)$*. An object is an isomorphism* $\phi : M \overset{\cong}{\to} {}^tM$ *such that* ${}^t\phi \circ \eta_M = \bar{\epsilon}\phi$*. A morphism* $\alpha : (M, \phi) \to (N, \psi)$ *is a commutative square*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi}\downarrow{\scriptstyle \cong} \\ {}^tM & \xleftarrow{{}^t\alpha} & {}^tN \end{array}$$

Remark: As before (see Definition 3.4) we have the notion of a hyperbolic object and we write $\mathcal{C}_H$ for the full subcategory of hyperbolic objects in $\mathcal{C}_h$. We can also drop the condition that the morphisms ϕ and ψ are isomorphisms. Then we obtain the category $\mathcal{C}_{hd}$ of all hermitian objects, including the degenerate ones. Of course, $\mathcal{C}_{hd}$ contains $\mathcal{C}_h$ as a full subcategory.

If $\mathcal{C}$ is additive (resp. symmetric monoidal) and ^{t} respects this structure, we call $\mathcal{C}_h$ the hermitian category associated to $\mathcal{C}$. This category $\mathcal{C}_h$ is still symmetric monoidal, so its K-theory is defined by the above $S^{-1}S$-construction. A hermitian exact category is no longer exact.

Examples: 1) $\mathcal{C} = {}_\epsilon P(A)_h$ is an additive category with duality $Hom_A(\ ,A)$.
2) The category of finite abelian groups together with $^t = Hom_{\mathbf{Z}}(\ ,\mathbf{Q}/\mathbf{Z})$ gives a hermitian exact category. This example will be extended to torsion modules over Dedeking rings in section 9.
3) Let $(X,\mathcal{O}_X)$ be a scheme. Then the locally free $\mathcal{O}_X$-sheaves of finite rank $Vect(X)$ form an exact category with $Hom_{\mathcal{O}_X}(\ ,\mathcal{O}_X)$ as duality functor.
4) $\mathcal{C}\times\mathcal{C}^{op}$ admits a duality functor sending $(A,B) \overset{(f,g^{op})}{\longrightarrow} (C,D)$ to $(B,A) \overset{(g^{op},f)}{\longrightarrow} (D,C)$ in $(\mathcal{C}\times\mathcal{C}^{op})^{op} \cong \mathcal{C}^{op}\times\mathcal{C}$. If $\mathcal{C}$ is exact, $\mathcal{C}\times\mathcal{C}^{op}$ is also exact: (f,g^{op}) is an admissible monomorphism if f is an admissible monomorphism and g is an admissible epimorphism in $\mathcal{C}$ etc.

For the rest of this chapter, let us fix a hermitian additive category $\mathcal{C}_h$ and its full subcategory of hyperbolic objects $\mathcal{C}_H$.
Obviously, **all of parts 3.4-3.14 (except 3.7) remains true for hermitian additive categories**, replacing "module" by "object", $P(A)$ by $\mathcal{C}$ and ${}_\epsilon P(A)_h$ by $\mathcal{C}_h$. (In 3.5 b), "If there exists $\lambda \in center(A)$ such that $\lambda + \bar{\lambda} = 1$" has to be replaced by "If for any object E, there exists $\lambda : {}^tM \to {}^tM$ such that ${}^t(\lambda\circ\phi) + \lambda\circ\phi = 1\ \forall\phi = {}^t\phi : M \overset{\cong}{\to} {}^tM$"). We still write $\mathcal{S}_H = \mathrm{Iso}(\mathcal{C}_H) \cong \mathrm{Iso}(\mathcal{W}(\mathcal{C}_H))$ and $\mathcal{S}_h = \mathrm{Iso}(\mathcal{C}_h) \cong \mathrm{Iso}(\mathcal{W}(\mathcal{C}_h))$.
Schlichting's proves Theorem 3.15 for additive categories as well using that we have the Fundamental Theorem 3.7 also for additive categories. This is the case, as there is a way to generalize certain isomorphisms from the hermitian K-theory of rings to additive pseudo-abelian categories (also called "karoubian", "idempotent complete" or "saturated"). Recall that after [Ka5], a category is called pseudo-abelian if for any object E and any projection $p^2 = p \in End(E)$, $ker\, p$ exists. Moreover, for any additive category there is a pseudo-abelian category $\widetilde{\mathcal{C}}$ whose objects are couples (E,p) where E is an object of $\mathcal{C}$ and $p^2 = p \in End(E)$. Then we have a full inclusion $\mathcal{C} \rightarrowtail \widetilde{\mathcal{C}}$ induced by $E \mapsto (E,1_E)$. M. Schlichting drew my attention to the idea of the following proposition which also applies to more general situations such as homotopy cartesian squares. We write <u>$add\,cat$</u> for the category of (small) additive categories and $F(R)$ for the category of finitely generated

free R-modules.

Proposition 4.4 *Let D and E be two functors $\underline{add\,cat} \to \Delta^{op}\underline{Set}$ commuting with filtered colimits and sending equivalences of categories to homotopy equivalences of simplicial sets. Suppose that there is a natural transformation $\phi : D \longrightarrow E$ such that $D\,P(R) \overset{\phi_{P(R)}}{\longrightarrow} E\,P(R)$ is a homotopy equivalence for any ring R. Then $D\,\mathcal{A} \overset{\phi_{\mathcal{A}}}{\to} E\,\mathcal{A}$ is a homotopy equivalence for any pseudo-abelian category $\mathcal{A}$.*

Proof: We say that an additive category Θ is finitely generated if there is an object A_Θ such that any object of Θ is a direct summand of $(A_\Theta)^n$ for some n. Define $R := End(A_\Theta)$. We have inclusions $F(R) \rightarrowtail \Theta \rightarrowtail P(R)$. (given by the isomorphism of the subcategory of $P(R)$ consisting of the single object R and the subcategory of $P(A_\Theta)$ consisting of the single object A_Θ.) The pseudo-abelianization $\tilde{}$ gives $P(R) \simeq \widetilde{F(R)} \rightarrowtail \tilde{\Theta} \rightarrowtail \widetilde{P(R)} \simeq P(R)$ and we therefore get an equivalence of categories $P(R) \simeq \tilde{\Theta}$. It follows that ϕ is a homotopy equivalence for any finitely generated pseudo-abelian category. Let $\underline{fg\,\mathcal{A}}$ be the category of finitely generated subcategories of $\mathcal{A}$. Then this implies $hocolim_{\underline{fg\,\mathcal{A}}} D(\tilde{\Theta}) \simeq hocolim_{\underline{fg\,\mathcal{A}}} E(\tilde{\Theta})$ and consequently $colim_{\underline{fg\,\mathcal{A}}} D(\tilde{\Theta}) \simeq colim_{\underline{fg\,\mathcal{A}}} E(\tilde{\Theta})$ because our colimits are filtered [BK, XII,3.5]. Finally, *colim* and $\tilde{}$ commute and any additive category is the filtered (see the following remark) colimit of its finitely generated subcategories. □

Replacing $End(A_\Theta)$ by $End(A_\Theta \oplus {}^tA_\Theta)$ in the above argument, we can also prove that any duality functor on any finitely generated additive pseudo-abelian category is isomorphic to $Hom_R(\;\;, R)$ on $P(R)$.

Remark: We say that a category (and in particular a partially ordered set) is "cofiltered" (or "right filtered") if *i*) it is non-empty, *ii*) for any two objects, there is a "bigger" object and *iii*) for any pair of morphisms between two objects there is a morphism which coequalizes them (Quillen [Q1] calls such a category "filtering"). We also have the dual notion of "filtered" (or "left filtered"). Nevertheless, we say "filtered colimits" as in [BK] instead of "cofiltered colimits".
Of course, we want to have our results not only for pseudo-abelian categories, but for additive categories in general. For an arbitrary additive category $\mathcal{C}$ we observe that the inclusion $\mathcal{C} \rightarrowtail \tilde{\mathcal{C}}$ is cofinal (because $(A,p) \oplus (A, 1-p) \cong (A,1)$). This implies that the inclusion $(\mathcal{S}^{-1}\mathcal{S})_0 \to (\tilde{\mathcal{S}}^{-1}\tilde{\mathcal{S}})_0$ induces a homotopy equivalence where $\mathcal{D}_0$ always stands for the connected component of 0 of a category $\mathcal{D}$[Q2, p.221 and 224].

In fact, two years ago I was not aware of the mistake in the proof of [CL2]. I generalized their result (which was only valid for rings) to additive categories [Ho] using essentially Proposition 4.4, cofinality and the following:

Lemma 4.5 *The inclusion $\mathcal{W}(\mathcal{C}_H) \to \mathcal{W}(\mathcal{C}_h)_0$ is a homotopy equivalence.*

Proof: Let $\mathcal{W}_N$ be the full subcategory of $\mathcal{W}(\mathcal{C}_h)_0$ of objects (P, λ) such that $(P, \lambda) \oplus (H(N), \mu_N)$ is hyperbolic. Then we have that $\forall\, M, N$ objects of $\mathcal{C}$, the inclusion $\iota : \mathcal{W}_M \to \mathcal{W}_{M\oplus N}$ is a homotopy equivalence. To see this, let $\theta : \mathcal{W}_{M\oplus N} \to \mathcal{W}_M$ be given by $\theta(P, \lambda) = (P \oplus H(N), \lambda \oplus \mu_N)$ and $\theta(f) = f \oplus id_{H(N)}$. We then have a natural transformation $\phi : Id_{\mathcal{W}_M} \Rightarrow \theta \circ \iota$ defined by $\phi_{(P,\lambda)} = [(p, (P\oplus N, \lambda\oplus 0), i)]$ and a similair natural transformation $Id_{\mathcal{W}_{M\oplus N}} \Rightarrow \iota \circ \theta$. Let $\underline{sym\, mon\, cat}$ be the category of (small) symmetric monoidal categories. For any abelian monoid A, we write $\hat{A}$ for the category whose objects are the elements of A and $\hat{A}(a, b) = \{C | a + c = b\}$. Then this implies that the natural transformation $\Delta_0 \Rightarrow \mathcal{W}$ is a homotopy equivalence for any object of $\widehat{\pi_0(\mathcal{S})}$ where $\Delta_0 : \widehat{\pi_0(\mathcal{S})} \to \underline{sym\, mon\, cat}$ is the constant functor $\Delta_0(M) = \mathcal{W}(\mathcal{C}_H)$. The second functor is given by $\mathcal{W}(N) = \mathcal{W}_N$ on objects and on morphisms it is the inclusion of categories. With [BK, XII,3.5] and Lemma 3.14 we can conclude that $\mathcal{W}(\mathcal{C}_H) \simeq hocolim_{\widehat{\pi_0(\mathcal{S})}} \Delta_0 \simeq hocolim_{\widehat{\pi_0(\mathcal{S})}} \mathcal{W} \simeq colim_{\widehat{\pi_0(\mathcal{S})}} \mathcal{W} \simeq \mathcal{W}(\mathcal{C}_h)_0$. □

Remark: The category $<\mathcal{S}, \mathcal{S}_h>$ (see [Q2]) and the double mapping cone of Thomason [Th2] give two other models for the homotopy type of $\mathcal{W}(\mathcal{C}_h)$; and the latter one fits the above homotopy fibration even if 2 is not invertible. The advantage of our category is that it generalizes well to exact categories, as we will see in the next section.

Another model for $\mathcal{W}(\mathcal{C}_h)$ in the additive case can be constructed in terms of “pseudo-simplicial symmetric monoidal categories” using [Ja2, Corollary 4.8]. In the same article (p.192), Jardine suggests a definition of étale hermitian K-theory with finite coefficients for schemes, using a globally fibrant replacement (with respect to the closed model structure of [Ja1]) of the presheaf of hermitian K-theory as defined in this section. We could replace the étale topology by the Zariski or the Nisnevish topology and work with integral coefficients as well, compare with the last paragraph of section 10.

5 The hermitian K-theory of exact categories

Recall the definition of an exact category due to [Q1] [Ke1]:

Definition 5.1 *An exact category $\mathcal{C}$ is an additive category, together with a class of sequences $\{A \rightarrowtail B \twoheadrightarrow C\}$, called "exact sequences". We say that $A \rightarrowtail B$ is an "admissible monomorphism" and that $B \twoheadrightarrow C$ is an "admissible epimorphism". They fulfill the following axioms:*
i) The class of admissible monomorphisms is closed under composition and under cobase-change.
ii) The class of admissible epimorphisms is closed under composition and under base-change.
iii) Any sequence isomorphic to an exact sequence is exact. Any sequence of the form

$$A \overset{\iota_1}{\rightarrowtail} A \oplus B \overset{\pi_2}{\twoheadrightarrow} B$$

is exact.
iv) In any exact sequence $A \rightarrowtail B \twoheadrightarrow C$, A is a kernel for $B \twoheadrightarrow C$ and C is a cokernel for $A \rightarrowtail B$.

In fact, the definition we give here is due to Keller who simplified the original definition of Quillen, showing that the following axiom (and its dual) is a consequence of the other axioms: If an admissible monomorphism $i : C \rightarrowtail D$ can be factored as $i = v \circ u$ such that u has a cokernel in $\mathcal{C}$, then u is an admissible monomorphism.

Any additive category can be considered as an exact category, taking the split monomorphisms and epimorphisms to be the admissible ones. Of course, in general there are other choices for families of admissible short exact sequences. We will frequently use "additive" as a synonym for an exact category in which every exact sequence splits. Another example for exact categories is to take a convenient subcategory of an abelian category and taking all monomorphisms and epimorphisms to be admissible. Following Quillen, we then associate to any exact category $\mathcal{C}$ a category $Q\mathcal{C}$ which has the same objects as $\mathcal{C}$. A morphism from C to D in $Q\mathcal{C}$ is given by the equivalence class of a diagramm $C \twoheadleftarrow E \rightarrowtail D$.

Definition 5.2 *Let $\mathcal{C}$ be an exact category. Then its K-theory is defined by*

$$K_n(\mathcal{C}) = \pi_n(\Omega Q\mathcal{C}) \ .$$

It is this Q-construction which allows Quillen to prove his theorems on resolution, localization and dévissage. His proof that the $\mathcal{S}^{-1}\mathcal{S}$-construction and the Q-construction coincide for additive categories and that they generalize the Plus construction for projective modules of finite type over a given ring can be found in [Q2].

Let us fix an exact category $\mathcal{C}$ with duality. Obviously, **all of parts 3.5-3.12 a) (except 3.7) remains true for hermitian exact categories**, replacing "module" by "object", "split monomorphism" by "admissible monomorphism", "split epimorphism" by "admissible epimorphism", $P(A)$ by $\mathcal{C}$ and ${}_{\epsilon}P(A)_h$ par $\mathcal{C}_h$.
The associated hermitian category to an exact category with duality is no longer exact or Waldhausen; it does not even have a final object. Therefore, its K-theory is not defined in general. If all short exact sequences split, we can define the K-theory as (the homotopy groups of) $\mathcal{S}_h^{-1}\mathcal{S}_h$. But in all other cases, this definition would be as bad as taking $\mathcal{S}^{-1}\mathcal{S}$ to define classical K-theory of an exact category in which not all short exact sequences split.
The aim of this chapter is to define for any exact category $\mathcal{C}$ with duality a category $\mathcal{L}(\mathcal{C}_h)$ such that there is a homotopy fibration containing $\Omega Q(\mathcal{C})$ and $\mathcal{L}(\mathcal{C}_h)$ and which coincides with the one induced by the hyperbolic functor as given in corollary 4.8 if the exact structure is given by the additive structure (i.e. any short exact sequence splits). In particular, we would have a homotopy equivalence $\mathcal{L}(\mathcal{C}_h) \simeq \mathcal{S}_h^{-1}\mathcal{S}_h$ in this case. Furthermore, we would like to have (as it is known for modules, see [Ka7]) $\mathcal{L}((\mathcal{C}\times\mathcal{C}^{op})_h) \simeq \Omega Q(\mathcal{C})$. If such a category $\mathcal{L}(\mathcal{C}_h)$ exists, it seems to be justified to call it "the hermitian K-theory of the hermitian exact category $\mathcal{C}_h$". By the way, we should mention that up to now, everything we have called "hermitian K-theory" was a special case of classical K-theory for symmetric monoidal categories.

For any exact category $\mathcal{C}$ with duality, we define its Witt group by $W(\mathcal{C}_h) := \pi_0(\mathcal{W}(\mathcal{C}_h))$. As we now consider short exact sequences which do not split in general, the fact that (M,ϕ) contains a "lagrangian" $L = L^{\perp} \rightarrowtail M$ no longer implies an isomorphism $(M,\phi) \cong (L^{\perp} \oplus {}^tL^{\perp}, \begin{pmatrix} 0 & 1 \\ \epsilon & d \end{pmatrix}$). We therefore have to distinguish these two classes of objects and call them metabolic and split metabolic, respectively. Lemma 5.3 of [QSS] implies that given a morphism from (P,λ) to (M,ϕ) in $\mathcal{W}(\mathcal{C}_h)$, the object $(M,\phi)\oplus(P,-\lambda)$ is metabolic. It follows that $W(\mathcal{C}_h)$ still is the monoid of isomorphism classes of hermitian objects, divided by the equivalence relation generated by identifying the metabolic objects with 0.

Lemma 5.3 *The functor $\mathcal{H}$ defined below is an equivalence of categories*

$$\mathcal{H} : Q(\mathcal{C}) \to \mathcal{W}((\mathcal{C}\times\mathcal{C}^{op})_h)$$

Proof: We set $\mathcal{H}(A) = ((A,A),(1,1^{op}))$ and $\mathcal{H}[A \overset{p}{\twoheadleftarrow} B \overset{i}{\rightarrowtail} C] =$ the class of

$$\begin{array}{ccccc}
(A,A) & \overset{(p,\tilde{i}^{op})}{\twoheadleftarrow} & (B, C \sqcup_B A) & \overset{(i,\tilde{p}^{op})}{\rightarrowtail} & (C,C) \\
{\scriptstyle (1,1^{op})}\downarrow & & \downarrow & & \downarrow{\scriptstyle (1,1^{op})} \\
(A,A) & \overset{(\tilde{i},p^{op})}{\rightarrowtail} & (C \sqcup_B A, B) & \overset{(\tilde{p},i^{op})}{\twoheadleftarrow} & (C,C)
\end{array}$$

where $\tilde{i}$ and $\tilde{p}$ are defined by the bicartesian square

$$\begin{array}{ccc}
B & \overset{i}{\rightarrowtail} & C \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\
A & \overset{\tilde{i}}{\rightarrowtail} & C \sqcup_B A
\end{array}$$

As the objects of $(\mathcal{C} \times \mathcal{C}^{op})_h$ are necessarily of the form $(E,F) \overset{(f,f^{op})}{\longrightarrow} (F,E)$, the isomorphism $[((1,1^{op}),((E,F),(f,f^{op})),(f,1^{op})] \in \mathcal{W}((\mathcal{C}\times\mathcal{C}^{op})_h)(((E,F),(f,f^{op})),\mathcal{H}(F))$ shows that $\mathcal{H}$ is essentially surjective. It is clearly faithful. Finally, $\mathcal{H}$ is full because any morphism in $\mathcal{W}((\mathcal{C}\times\mathcal{C}^{op})_h)$ is given by a cartesian square. □

Definition 5.4 *Let $\mathcal{F}(\mathcal{C}_h)$ be the homotopy fiber of the forgetful functor $\mathcal{W}(\mathcal{C}_h) \overset{F}{\longrightarrow} Q(\mathcal{C})$.*

This is only a topological space, not a category. It enables us to give a model for the K-theory of an exact category:

Proposition 5.5 *There is a homotopy equivalence*

$$\Omega Q(\mathcal{C}) \simeq \mathcal{F}((\mathcal{C} \times \mathcal{C}^{op})_h)$$

Proof: We have the following diagram

$$\begin{array}{ccccc}
\mathcal{F}((\mathcal{C}\times\mathcal{C}^{op})_h) & \overset{f}{\longrightarrow} & \mathcal{W}((\mathcal{C}\times\mathcal{C}^{op})_h) & \overset{F}{\longrightarrow} & Q(\mathcal{C}\times\mathcal{C}^{op}) \\
 & & & & {\scriptstyle \phi}\downarrow{\scriptstyle \cong} \\
 & & {\scriptstyle \mathcal{H}}\uparrow{\scriptstyle \simeq} & & Q(\mathcal{C})\times Q(\mathcal{C}^{op}) \\
 & & & & {\scriptstyle \tau}\downarrow{\scriptstyle \cong} \\
 & & Q(\mathcal{C}) & \overset{\Delta}{\dashrightarrow} & Q(\mathcal{C})\times Q(\mathcal{C})
\end{array}$$

which is commutative: $\tau \circ \phi \circ F \circ \mathcal{H}(P \overset{q}{\twoheadleftarrow} L \overset{j}{\rightarrowtail} M) = \tau \circ \phi((P,P) \overset{(q,\tilde{j}^{op})}{\twoheadleftarrow} (L, M \sqcup_L P) \overset{(j,\tilde{q}^{op})}{\rightarrowtail} (M,M)) = (P,P) \overset{(q,q)}{\twoheadleftarrow} (L,L) \overset{(j,j)}{\rightarrowtail} (M,M) = \Delta(P \overset{q}{\twoheadleftarrow} L \overset{j}{\rightarrowtail} M)$ where Δ is the diagonal map, ϕ is the canonical isomorphism and τ is the equivalence of categories defined in [Q1]. The existence of the homotopy fibration $\mathcal{F}((\mathcal{C}\times\mathcal{C}^{op})_h) \to Q(\mathcal{C}) \overset{\Delta}{\to} Q(\mathcal{C})\times Q(\mathcal{C})$ implies $\mathcal{F}((\mathcal{C}\times\mathcal{C}^{op})_h) \simeq \mathrm{hofib}(\Delta) \simeq \Omega Q(\mathcal{C})$ □

To find a category whose classifying space is homotopy equivalent to $\mathcal{F}(\mathcal{C}_h)$ (which will be finally given in Definition 5.10) we remark first of all that a cartesian square in the category of categories is still cartesian in the category of simplicial sets because *nerve* : $\underline{cat} \to \Delta^{op}\underline{Set}$ has a left adjoint [Th1] and therefore respects limits.
Recall [Q1, p.93] that a functor $g : B \to C$ is called "prefibered" in the sense of Quillen if for any object c of C the inclusion $g^{-1}(c) \to (c \downarrow g)$ has a right adjoint, $g^{-1}(c)$ being the subcategory of morphisms of B mapping to id_C.

Proposition 5.6 *Given a diagram in the category of categories*

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ A \times_C B & \longrightarrow & B \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g} \\ A & \longrightarrow & C \end{array}$$

where the square is cartesian, (realizations of) A and C are connected or H-groups and $F \to B \xrightarrow{g} C$ is a homotopy fibration such that g is fibered in the sense of Quillen [Q1] and g fullfills the conditions of Quillen's theorem B (i.e., all base changes are homotopy equivalences). Then g^* is also fibered and $F \to A \times_C B \xrightarrow{g^*} A$ is a homotopy fibration.
Moreover, if B is contractible, then $A \times_C B \xrightarrow{g*} A \to C$ is also a homotopy fibration.

Proof: Following [SGA1, VI,6.9], if g is fibered, then g^* is also fibered. A trivial calculation shows that if g fullfills the conditions of theorem B, then so does g^*. It follows that $F \simeq \text{hofib}(g) \simeq g^{-1}(c) \cong g^{*-1}(a) \simeq \text{hofib}(g^*)$ $\forall a \in A$ such that $f(a) = c$.
For the second assertion, replace $B \xrightarrow{g} C$ by a (Serre) fibration p:

$$\begin{array}{ccc} B & \xrightarrow{\simeq} & B \times_C C^I \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle p} & \\ & C & \end{array}$$

Then we have to check that $A \times_C B \to A \times_C C^I \times_C B$ is a homotopy equivalence. This is a consequence of the five lemma applied to the long exact homotopy sequences of the homotopy fibrations $F \to A \times_C B \to A$ and $F \to A \times_C C^I \times_C B \to A$ □

Of course, we want to apply this proposition when $A \to C$ is the forgetful functor $F : \mathcal{W}(\mathcal{C}_h) \to Q(\mathcal{C})$. Let $\mathcal{E} = \mathcal{E}(\mathcal{C})$ be the category in [Q2]. Its objects are the short exact sequences in $\mathcal{C}$, and a morphism from $A \rightarrowtail B \twoheadrightarrow C$ to $A' \rightarrowtail B' \twoheadrightarrow C'$ is given by the isomorphism class of a commutative diagram

$$\begin{array}{ccccc}
A & \rightarrowtail & B & \twoheadrightarrow & C \\
\uparrow & & \| & & \uparrow \\
A' & \rightarrowtail & B & \twoheadrightarrow & C_0 \\
\| & & \downarrow & \square & \downarrow \\
A' & \rightarrowtail & B' & \twoheadrightarrow & C'
\end{array}$$

where $\square$ stands for a bicartesian square. Observe that saying that this square is bicartesian is not an extra condition but a consequence of the diagram. $\mathcal{S} = \mathrm{Iso}\,\mathcal{C}$ acts on $\mathcal{E}$ by $D + (A \rightarrowtail B \twoheadrightarrow C) = D \oplus A \rightarrowtail D \oplus B \twoheadrightarrow C$. Following [Q2], we know that the projection on the third component $g : \mathcal{S}^{-1}\mathcal{E} \to Q(\mathcal{C})$ fullfills all the conditions required in Proposition 5.5 if all short exact sequences in $\mathcal{C}$ split. It follows that we have a homotopy fibration

$$\mathcal{S}^{-1}\mathcal{E}_0 \xrightarrow{\iota} \mathcal{S}^{-1}\mathcal{E} \times_{Q(\mathcal{C})} \mathcal{W}(\mathcal{C}_h) \xrightarrow{g*} \mathcal{W}(\mathcal{C}_h)$$

where $\mathcal{S}^{-1}\mathcal{E}_0 := g^{-1}(0)$. The somewhat surprising fact now is that this homotopy fibration induced by the forgetful functor $F : \mathcal{W}(\mathcal{C}_h) \to Q(\mathcal{C})$ coincides with the homotopy fibration of the previous chapter induced by the hyperbolic functor. More precisely, we have the following theorem:

Theorem 5.7 *Let $\mathcal{C}_h$ be a hermitian exact category in which all short exact sequences split and 2 is invertible. Then we have a diagram of categories and functors*

$$\begin{array}{ccccc}
\mathcal{S}^{-1}\mathcal{S} & \xrightarrow{\mathcal{S}^{-1}H} & \mathcal{S}^{-1}\mathcal{S}_h & \xrightarrow{\theta} & \mathcal{W}(\mathcal{C}_h) \\
{\scriptstyle j}\downarrow & & {\scriptstyle J}\downarrow & & \| \\
\mathcal{S}^{-1}\mathcal{E}_0 & \xrightarrow{\iota} & \mathcal{S}^{-1}\mathcal{E} \times_{Q(\mathcal{C})} \mathcal{W}(\mathcal{C}_h) & \xrightarrow{g*} & \mathcal{W}(\mathcal{C}_h)
\end{array}$$

where the vertical morphisms are homotopy equivalences, the rows are homotopy fibrations, the right-hand square commutes and the left-hand square commutes up to homotopy.

Proof: Observe first of all that we have a homotopy equivalence $\mathcal{S}^{-1}\mathcal{S}_h \xrightarrow{\simeq} \mathcal{S}_h^{-1}\mathcal{S}_h$ if 2 is invertible by cofinality. We let j be the trivial equivalence of categories given by $(A, B) \to (A, B \overset{1}{\rightarrowtail} B \twoheadrightarrow 0)$. The functor J is defined on the objects by $J(A, (M, \phi)) := (A, 0 \rightarrowtail M \overset{1}{\twoheadrightarrow} M, (M, \phi))$. A morphism from $(A, (M, \phi))$ to $(A', (M', \phi')$ given by $(C, C \oplus A \xrightarrow{\cong} A', (H(C), \mu_C) \oplus (M, \phi) \xrightarrow{\cong} (M', \phi'))$ is mapped by J to $(C, C \oplus A \xrightarrow{\cong} A, \gamma, \delta)$

where γ is given by

$$\begin{array}{ccccc}
C & \rightarrowtail & C \oplus M & \twoheadrightarrow & M \\
\uparrow & & \| & & \uparrow \\
0 & \rightarrowtail & C \oplus M & \twoheadrightarrow & C \oplus M \\
\| & & \downarrow & \square & \downarrow \\
0 & \rightarrowtail & M' & \twoheadrightarrow & M'
\end{array}$$

and δ is given by

$$\begin{array}{ccccc}
M & \twoheadleftarrow & C \oplus M & \rightarrowtail & M' \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle 0\oplus\phi} & & \downarrow{\scriptstyle \phi'} \\
{}^tM & \rightarrowtail & {}^tC \oplus {}^tM & \twoheadleftarrow & {}^tM'
\end{array}$$

It is obvious that the right-hand square commutes. For the left-hand side, it is straightforward to check that there is a natural transformation $\tau : \iota \circ j \Rightarrow J \circ \mathcal{S}^{-1}H$ defined by $\tau_{(A,B)} = (0, A \xrightarrow{1} A, \alpha, \beta)$
where α is given by

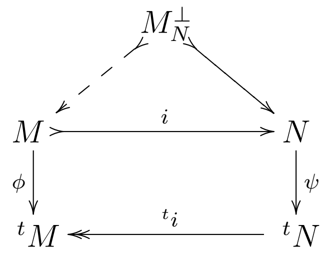
$$\begin{array}{ccccc}
B & \rightarrowtail & B & \twoheadrightarrow & 0 \\
\uparrow & & \| & & \uparrow \\
0 & \rightarrowtail & B & \twoheadrightarrow & B \\
\| & & \downarrow & \square & \downarrow \\
0 & \rightarrowtail & H(B) & \twoheadrightarrow & H(B)
\end{array}$$

and β is given by

$$\begin{array}{ccccc}
0 & \twoheadleftarrow & B & \rightarrowtail & H(B) \\
\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mu_B} \\
0 & \rightarrowtail & {}^tB & \twoheadleftarrow & H(B)
\end{array}$$

The upper row is a homotopy fibration by Theorem 3.15 and the lower one by Proposition 5.6. Finally, the five lemma applied to the long exact homotopy sequences implies that J is also a homotopy equivalence. $\square$

Remark: In the appendix of [CL1], Charney and Lee give an alternative description of the category $\mathcal{E} \times_{Q(\mathcal{C})} \mathcal{W}(\mathcal{C}_h)$ which they called E_{sp}. Moreover, they prove that $\pi_*(\mathcal{S}^{-1}E_{sp}) \otimes \mathbf{Q} \cong \pi_*(\mathcal{S}^{-1}\mathcal{S}_h) \otimes \mathbf{Q}$ for $\mathcal{C}_h = {}_{-1}P(\mathbf{Z})_h$.
Karoubi [unpublished] introduced the following category: An object is a morphism $M \xrightarrow{\phi} {}^tM$ with ${}^t\phi \circ \eta_M = \bar{\epsilon}\phi$ such that $M^0 \rightarrowtail M$ is an admissible monomorphism. A morphism from from (M,ϕ) to $(N,\psi))$ is a commutative diagram



The interested reader can check that $\mathcal{E} \times_{Q(\mathcal{C})} \mathcal{W}(\mathcal{C}_h)$ is equivalent to this category where the equivalence is induced by sending $(A \rightarrowtail B \overset{p}{\twoheadrightarrow} C, (C, \lambda))$ to $(B, {}^tp \circ \lambda \circ p)$. Hence the "localization" by $\mathcal{S}$ of these categories yields alternative models for the hermitian K-theory of an additive category (but not for an exact category in general).

We now give the definition of the hermitian K-theory of an exact category which generalizes the hermitian K-theory of an additive category:

Definition 5.8 *For any hermitian exact category $\mathcal{C}_h$ in which 2 is invertible, we define its hermitian K-theory by*

$$K_n^h(\mathcal{C}_h) := \pi_n(\mathcal{F}(\mathcal{C}_h)) \ \forall n \geq 0$$

and its $\mathcal{U}$-theory by

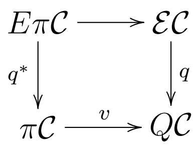
$$\mathcal{U}(\mathcal{C}_h) := \Omega\mathcal{W}(\mathcal{C}_h)$$

$$U_n(\mathcal{C}_h) := \pi_{n+1}(\mathcal{W}(\mathcal{C}_h)) \ \forall n \geq 0$$

Looking at the long exact sequence of homotopy groups for a fibration, we immediatly get a long exact sequence

$$\ldots \to U_n(\mathcal{C}_h) \to K_n(\mathcal{C}) \overset{H_*}{\to} K_n^h(\mathcal{C}_h) \to U_{n-1}(\mathcal{C}_h) \to K_{n-1}(\mathcal{C}) \overset{H_*}{\to} K_{n-1}^h(\mathcal{C}_h) \to \ldots$$

where the morphism H is defined by topology, but we stick to this notation as it coincides with the hyperbolic functor in the additive case. Proposition 5.5 allows us to construct categories whose classifying spaces are homotopy equivalent to $\mathcal{F}(\mathcal{C}_h)$. For this, we need contractible categories which are fibered over $Q(\mathcal{C})$ such that all base changes are homotopy equivalences. Giffen [Gif] constructs two such categories. As I am not quite convinced by the proof of his "big K-construction", I will restrict my attention to his "small K-construction". Giffen first defines the category $\pi\mathcal{C} := (Q\mathcal{C} \times Q\mathcal{C}) \times_{Q\mathcal{C}} \mathcal{E}\mathcal{C}$ where the morphisms to $Q\mathcal{C}$ are given by the direct sum and the projection q to the third component, respectively. He then defines $E\pi\mathcal{C}$ by the cartesian square of fibered functors



where v is defined by the projection to the second factor $Q\mathcal{C} \times Q\mathcal{C} \to Q\mathcal{C}$. More explicitly, an object of $E\pi\mathcal{C}$ is given by $U \twoheadleftarrow M \twoheadrightarrow V \twoheadleftarrow N$ such that the induced morphism $M \twoheadrightarrow U \oplus V$ is also an admissible epimorphism. Using Quillen's Theorem A, Giffen concludes that $E\pi\mathcal{C}$ is contractible. Let $u : \pi\mathcal{C} \to Q\mathcal{C}$ be induced by the projection to the first factor $Q\mathcal{C} \times Q\mathcal{C} \to Q\mathcal{C}$. Let $u_\pi := u \circ q^*$ and $k\mathcal{C} := u_\pi^{-1}(0)$.

Proposition 5.9 *[Gif, 3.4] For any exact category $\mathcal{C}$, there is a homotopy fibration*

$$k\mathcal{C} \to E\pi\mathcal{C} \xrightarrow{u_\pi} Q\mathcal{C}$$

with contractible total space $E\pi\mathcal{C}$.

Proof: We want to apply Quillen's Theorem B. As u_π is fibered, we have to show that all the base changes are homotopy equivalences which is done in [Gif, 3.3]. □

Definition 5.10 *Let $\mathcal{L}(\mathcal{C}_h)$ be the category defined by*

$$\mathcal{L}(\mathcal{C}_h) := E\pi\mathcal{C} \times_{Q\mathcal{C}} \mathcal{W}(\mathcal{C}_h)$$

Corollary 5.11 *There is a homotopy equivalence*

$$\mathcal{L}(\mathcal{C}_h) \simeq \mathcal{F}(\mathcal{C}_h)$$

Proof: As u_π is fibered, we can apply proposition 5.6. □

Of course, the interested reader can try to give a more beautiful description of this category; one might also look out for categories other than $E\pi\mathcal{C}$ fullfilling the conditions of Proposition 5.6.

It should also be pointed out that there is a Waldhausen model in $\mathcal{U}$-theory ${}_\epsilon s^e\mathcal{C}$ (see [SY]) together with a forgetful functor to the classical Waldhausen model $s^e\mathcal{C}$ (see [Wald]) which coincides with our forgetful functor $\mathcal{W}(\mathcal{C}_h) \to Q\mathcal{C}$. This implies a homotopy equivalence
$\mathcal{L}(\mathcal{C}_h) \simeq \pi_n(\mathrm{hofib}({}_\epsilon s^e\mathcal{C} \xrightarrow{forget} s^e\mathcal{C}))$.

6 Localization I: Negative hermitian K-theory of additive categories

The following section is joint work with Marco Schlichting. Recall the following localization theorem of Pedersen and Weibel for additive categories ([PW], see also [Ka1]):

Theorem 6.1 *Let $\mathcal{B}$ be an additive category and $\mathcal{A}$ a pseudo-abelian full subcategory such that $\mathcal{B}$ is $\mathcal{A}$-filtered in the sense of Karoubi. Then there is a homotopy fibration*

$$\mathrm{Iso}(\mathcal{A})^{-1}\,\mathrm{Iso}(\mathcal{A}) \to \mathrm{Iso}(\mathcal{B})^{-1}\,\mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{B}/\mathcal{A})^{-1}\,\mathrm{Iso}(\mathcal{B}/\mathcal{A}).$$

Here $\mathcal{B}/\mathcal{A}$ is the category with the same objects as $\mathcal{B}$. The morphisms from U to V in $\mathcal{B}/\mathcal{A}$ are those in $\mathcal{B}$ modulo the ones factoring through an object of $\mathcal{A}$. As we want to prove a hermitian analogue of this theorem, we give a sketch of the proof:

Proof: One can show that $\mathcal{B}/\mathcal{A}$ is the localization of $\mathcal{B}$ with respect to those morphisms which are a composition of split monomorphisms with cokernel in $\mathcal{A}$ and split epimorphisms with kernel in $\mathcal{A}$. We even have left and right calculus of fractions (see [GZ] for the definitions). The proof of Pedersen and Weibel uses Thomason's double mapping cone (see [Th2, 5.1] and Definition 6.2 below) and the following two properties which are a consequence of the assumptions (see [Sch1, Lemma 3.16,i)-iv)]):
($P1$) Any isomorphism $U \xrightarrow{\sim} V$ in $\mathcal{B}/\mathcal{A}$ is represented by a fraction $U \overset{\sim}{\longleftarrow\!\!\prec} W \overset{\sim}{\rightarrowtail} V$ where $\overset{\sim}{\rightarrowtail}$ denotes a split monomorphism whose cokernel lies in $\mathcal{A}$.
($P2$) Given two morphisms $f : U \overset{\sim}{\rightarrowtail} V$ and $g : U \overset{\sim}{\rightarrowtail} V$ in $\mathcal{B}$ such that $f = g$ in $\mathcal{B}/\mathcal{A}$, then there is a $h : W \overset{\sim}{\rightarrowtail} U$ such that $f \circ h = g \circ h$ in $\mathcal{B}$.
Observe that the $\overset{\sim}{\rightarrowtail}$ in (P2) are not morphisms in $<\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{B}>$ as we did not choose a splitting. We denote by $(i, p) : U \overset{\sim}{\longrightarrow\!\!\!\!\!+\!\!\!\!\!\!\rightarrow} V$ a "direct morphism" from U to V. This means by definition that $i : U \overset{\sim}{\rightarrowtail} V$ as above and p is a retraction of i. Hence (i, p) is a morphism in $<\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{B}>$. As the dual statement of (P2) for split epimorphisms is also true, (P2) remains true when we replace the $\overset{\sim}{\rightarrowtail}$ by $\overset{\sim}{\longrightarrow\!\!\!\!\!+\!\!\!\!\!\!\rightarrow}$. □

We will always note the objects of $\mathcal{B}$ by $T, U, V...$ and those of $\mathcal{A}$ by $A, B, C....$ We will use the double mapping cone of Thomason [Th2, 5.1] only in the special case of a symmetric monoidal functor $\mathcal{A} \to \mathcal{B}$ and call it the "mapping cone" in this case:

Definition 6.2 *Let*

$$F : \mathrm{Iso}\,\mathcal{A} \to \mathrm{Iso}\,\mathcal{B}$$

be a symmetric monoidal functor. Then the mapping cone $\mathcal{T} = \mathcal{T}(F)$ is the symmetric monoidal category with the same objects as $\mathcal{A} \times \mathcal{B}$. A morphism $(A, U) \to (B, V)$ is given by the equivalence class of $(C, D, \alpha : A \to C \oplus B, \beta : F(D) \oplus U \to V)$ (see [Th2] for the equivalence relation).

In fact, this mapping cone is a special case of Thomason's double mapping cone. Schlichting [Sch1, Remarque 14.15] gave an easier proof of Theorem 6.1 which also needs the mapping cone and the above two properties. Instead of giving a hermitian version of the original proof of Pedersen and Weibel - which is possible as well - we will work with this simplified version. It is the following:
We first observe that the mapping cone $\mathcal{T}$ of $\mathrm{Iso}\,\mathcal{A} \to \mathrm{Iso}\,\mathcal{B}$ is homotopy equivalent to $<\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{B}>$. In fact, one can check that the fibers $(U \downarrow p)$ of the evident projection $p : \mathcal{T} \to <\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{B}>$ are all contractible because the functor $<\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{A}>^{op} \to (U \downarrow p)$ given by $A \mapsto ((A, U), 1_U)$ has a right adjoint and $<\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{A}>^{op}$ is contractible because it has a final object. Thus p is a homotopy equivalence by Quillen's theorem A [Q1]. Applying [Th2, Lemma 2.3], it remains to show that the morphism $\sigma : <\mathrm{Iso}\,\mathcal{A}, \mathrm{Iso}\,\mathcal{B}> \to \mathrm{Iso}(\mathcal{B}/\mathcal{A})$ induced by the identity on the objects is a homotopy equivalence. But applying (P1), (P2) and the statement dual to (P2) yields that the fibers $(U \downarrow \sigma)$ are cofiltered (in the sense of [Q1]) and hence contractible.

Pedersen and Weibel then define for every additive category $\mathcal{A}$ an additive category $\mathcal{CA}$ whose classifying space for its K-theory is contractible and such that $\mathcal{CA}$ is $\mathcal{A}$-filtered. In fact, they work with a slightly modified version of Karoubi's $\mathcal{CA}$ (which they call $\mathcal{C}_+\mathcal{A}$). An object of $\mathcal{CA}$ is given by a sequence $A = (A_0, A_1, ...)$ of objects of $\mathcal{A}$, and a morphism $f : A \to B$ is given by a matrix of morphisms $f_{ji} : A_i \to B_j$ in $\mathcal{A}$ such that there exists an n with $f_{ji} = 0$ whenever $|j - i| > n$, and the compositon is just defined by matrix multiplication. Define $\mathcal{SA}$=$\mathcal{CA}/\mathcal{A}$ to be the "suspension" of $\mathcal{A}$, the quotient $\mathcal{CA}/\mathcal{A}$ defined as in Theorem 6.1.. Then $\mathrm{Iso}(\mathcal{SA})^{-1}\,\mathrm{Iso}(\mathcal{SA})$ is a delooping of $\mathrm{Iso}(\mathcal{A})^{-1}\,\mathrm{Iso}(\mathcal{A})$ by their localization theorem if $\mathcal{A}$ is pseudo-abelian. Iterating this process allows to define the negative K-theory of an additive category $\mathcal{A}$, following Karoubi [Ka1].

Definition 6.3 *The negative K-theory of an additive category $\mathcal{A}$ is defined by*

$$K_{-n}(\mathcal{A}) := K_1(\mathcal{S}^{n+1}\mathcal{A}) \;\forall\; n \geq 0.$$

The hermitian analogue of Theorem 6.1 will allow us to define the negative K-theory of a hermitian additive category.

Compared with ordinary K-theory, the situation in hermitian K-theory is slightly more complicated. Roughly speaking, we have to decide whether to replace $\mathcal{B}$ by $\mathcal{B}_H$ or by $\mathcal{B}_h$ (the category of hyperbolic resp. all hermitian objects in $\mathcal{B}$, see Definition 4.3). Roughly spoken, the category $\mathcal{B}_H$ is too small, as the induced form on the cokernel of a monomorphism between hyperbolic objects is not necessarily hyperbolic. The category $\mathcal{B}_h$ is too big, as we do not know how to find a W as in (P1) and (P2) which is equipped with a non-degenerate form. We therefore introduce an intermediate category:

Definition 6.4 *Let $\widetilde{\mathcal{B}_H}^{\mathcal{A}_h}$ be the full subcategory of $\mathcal{B}_h$ of those objects (U,λ) such that there is an (A,α) in $\mathcal{A}_h$ with $(U,\lambda)\oplus(A,\alpha)$ an object of $\mathcal{B}_H$.*

We will need the following, which is trivial if $\mathcal{B}=\mathcal{CA}$:

Lemma 6.5 *For any direct morphism $Y \overset{\sim}{\rightarrowtail} H(X)$, there is a direct morphism $H(Z) \overset{\sim}{\rightarrowtail} Y$ such that the composition $H(Z) \overset{\sim}{\rightarrowtail} H(X)$ is induced by the hyperbolic functor.*

Proof: The section $Y\oplus A \overset{\cong}{\to} X\oplus {}^tX$ and the filtration on X (as $\mathcal{B}$ is $\mathcal{A}$-filtered) yields a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle \left(\begin{smallmatrix}0\\ \hat\alpha\end{smallmatrix}\right)}\downarrow & \overset{\cong}{\swarrow} & \downarrow{\scriptstyle {}^t\beta} \\ Z\oplus B & \xrightarrow[\left(\begin{smallmatrix}0 & {}^t\hat\beta\end{smallmatrix}\right)]{} & {}^tA \end{array}$$

where $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right) : A \to Y\oplus A \overset{\cong}{\to} X\oplus {}^tX$. Then one checks that the monomorphism $Z\oplus {}^tZ \to Z\oplus {}^tZ\oplus B\oplus {}^tB \overset{\cong}{\to} X\oplus {}^tX$ factors over Y such that everything commutes. □

Theorem 6.6 *Let $\mathcal{B}$ be an additive category with duality functor in which 2 is invertible. Let $\mathcal{A}$ be a pseudo-abelian subcategory closed under the duality functor such that $\mathcal{B}$ is $\mathcal{A}$-filtered. Then we have a homotopy fibration*

$$\mathrm{Iso}(\mathcal{A}_h)^{-1}\,\mathrm{Iso}(\mathcal{A}_h) \to \mathrm{Iso}(\widetilde{\mathcal{B}_H}^{\mathcal{A}_h})^{-1}\,\mathrm{Iso}(\widetilde{\mathcal{B}_H}^{\mathcal{A}_h}) \to \mathrm{Iso}((\mathcal{B}/\mathcal{A})_H)^{-1}\,\mathrm{Iso}((\mathcal{B}/\mathcal{A})_H)$$

Proof: As above in the additive case, we will show that the fibers of $\sigma : <\mathrm{Iso}(\mathcal{A}_h), \mathrm{Iso}(\widetilde{\mathcal{B}_H}^{\mathcal{A}_h})> \to \mathrm{Iso}((\mathcal{B}/\mathcal{A})_H)$ are cofiltered. Let $H(X)$ be an object of $(\mathcal{B}/\mathcal{A})_H$.

i) As $H(X)$ is also an object of $\widetilde{\mathcal{B}_H}^{\mathcal{A}_h}$, $(\sigma \downarrow H(X))$ is non-empty.

ii) Given two objects (U,ϕ) and (V,ψ) of $\widetilde{\mathcal{B}_H}^{\mathcal{A}_h}$ equipped with an isomorphism in $(\mathcal{B}/\mathcal{A})_H$ to $H(X)$, we can apply (P1) and (P2) to get a common subobject (without form) W of U and V. Applying (P2) again, we can assume that $\phi|_W = \psi|_W$. By Lemma 6.5, we can choose $W = H(Z)$. Applying (P2) and the lemma again, we find a commun hermitian subobject $(H(T),\mu_T)$ of

(U,ϕ) and (V,ψ). Finally, observe that the induced form on the complement in $\mathcal{A}$ is also non-degenerate as 2 is invertible [Kn, II,2.5.2].
iii) Given again two objects (U,ϕ) and (V,ψ) of $\widetilde{\mathcal{B}_H}^{\mathcal{A}_h}$ equipped with an isomorphism in $(\mathcal{B}/\mathcal{A})_H$ to $H(X)$, together with two split monomorphisms $g_1, g_2 : (U,\phi) \overset{\sim}{\rightarrowtail} (V,\psi)$ respecting the forms. Then we first equalize g_1 and g_2 without forms and then proceed as in ii), using both Lemma 6.5 and the dual statement of Lemma 6.5. □

Corollary 6.7 *Under the hypotheses of Theorem 6.6, we have a homotopy fibration*

$$\mathrm{Iso}(\mathcal{A}_h)^{-1}\,\mathrm{Iso}(\mathcal{A}_h) \to \mathrm{Iso}(\mathcal{B}_h)^{-1}\,\mathrm{Iso}(\mathcal{B}_h) \to \mathrm{Iso}((\mathcal{B}/\mathcal{A})_h)^{-1}\,\mathrm{Iso}((\mathcal{B}/\mathcal{A})_h)$$

Proof: We will write $\mathcal{K}(\mathcal{C})$ instead of $\mathrm{Iso}(\mathcal{C})^{-1}\,\mathrm{Iso}(\mathcal{C})$. Assume that we have full inclusions of symmetric monoidal categories $\mathcal{C} \subset \mathcal{D} \subset \mathcal{E}$ where $\mathcal{C} \subset \mathcal{E}$ is cofinal (which implies that $\pi_i(\mathcal{K}(\mathcal{C})) \to \pi_i(\mathcal{K}(\mathcal{E}))$ is an isomorphism for $i \geq 1$ and a monomorphism for $i = 0$). Then $\mathcal{D} \subset \mathcal{E}$ is also cofinal and hence $\pi_i(\mathcal{K}(\mathcal{C})) \to \pi_i(\mathcal{K}(\mathcal{D}))$ is an isomorphism for $i \geq 1$ and a monomorphism for $i = 0$. This implies in particular that $\Omega\mathcal{K}(\mathcal{S}\mathcal{A})_H \simeq \mathcal{K}(\mathcal{A}_h)$ when we apply Theorem 6.6 to $\mathcal{A} \subset \mathcal{C}\mathcal{A}$, consider the inclusions $\mathcal{C}\mathcal{A}_H \subset \widetilde{\mathcal{C}\mathcal{A}_H}^{\mathcal{A}_h} \subset \mathcal{C}\mathcal{A}_h$ and recall that $\mathcal{K}(\mathcal{C}\mathcal{A}_H) \simeq \mathcal{K}(\mathcal{C}\mathcal{A}_h) \simeq \{pt\}$. We write $\widehat{\mathcal{B}}^{\mathcal{A}}$ for the full subcategory of those objects X of the pseudo-abelianization $\widehat{\mathcal{B}}$ for which there exists an object A of $\widehat{\mathcal{A}}$ with $A \oplus X$ an object of $\mathcal{B}$. As $\mathcal{B}$ is $\mathcal{A}$-filtered, $\mathcal{S}\mathcal{B}$ is $\mathcal{S}\mathcal{A}$- filtered [Ka1, p.152] and $\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}}$ is $\widehat{\mathcal{S}\mathcal{A}}$-filtered [Sch1, Lemma 3.8]. Our Theorem gives a homotopy fibration $\mathcal{K}(\widehat{\mathcal{S}\mathcal{A}_h}) \to \mathcal{K}(\widetilde{(\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}})_H}^{\widehat{\mathcal{S}\mathcal{A}_h}}) \to \mathcal{K}(\mathcal{S}(\mathcal{B}/\mathcal{A})_H)$. Here we use the equivalences $\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}}/\widehat{\mathcal{S}\mathcal{A}} \simeq \mathcal{S}\mathcal{B}/\mathcal{S}\mathcal{A} \simeq \mathcal{S}(\mathcal{B}/\mathcal{A})$, the first equivalence is trivial and the second is shown in [Ka1, p.154]. Applying the loop space functor Ω to this homotopy fibration, we are reduced to establish two homotopy equivalences $\Omega\mathcal{K}(\widehat{\mathcal{S}\mathcal{A}_h}) \simeq \Omega\mathcal{K}(\mathcal{S}\mathcal{A}_H)$ and $\Omega\mathcal{K}(\mathcal{S}\mathcal{B}_H) \simeq \Omega\mathcal{K}(\widetilde{(\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}})_H}^{\widehat{\mathcal{S}\mathcal{A}_h}})$. The first homotopy equivalence follows from the cofinal inclusions $\mathcal{S}\mathcal{A}_H \subset \widehat{\mathcal{S}\mathcal{A}}_H \subset \widehat{\mathcal{S}\mathcal{A}}_h$. For the second, consider the inclusions $\mathcal{S}\mathcal{B}_H \subset (\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}})_H \subset \widehat{\mathcal{S}\mathcal{B}}_H$ and $(\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}})_H \subset \widetilde{(\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}})_H}^{\widehat{\mathcal{S}\mathcal{A}_h}} \subset (\widehat{\mathcal{S}\mathcal{B}}^{\mathcal{S}\mathcal{A}})_h$ and use the cofinality arguments given at the beginning of the proof. □

Using the suspension functor $\mathcal{S}$, we see that this is in fact a homotopy fibration of non-connective spectra. The following definition is implicitly in [KV], although they only consider the case $P(A)$.

Definition 6.8 *Let $\mathcal{A}$ be a pseudo-abelian category with duality functor. Then we define*

$$K_{-n}(\mathcal{A}_h) := K_1((\mathcal{S}^{n+1}\mathcal{A})_h)\ \forall n \geq 0$$

The resulting long exact sequence extends the 5-term exact sequence of Karoubi and Villamayor [KV, Théorème 3.4].
Of course, we can now also define negative $\mathcal{U}$-theory.
Recently, Schlichting [Sch1] gave a non-connective delooping of the K-theory of an exact category in general. Imitating his construction, the corresponding theorems in the hermitian setting remain to be proved.

7 A Waldhausen-like model for $\mathcal{U}$-theory

The following section is joint work with M. Schlichting.
As before, we assume that the identification between the identity and the bidual is given by the identity, and we often suppress the duality functors in our notations when we write down a functor between categories with duality. Recall [Wald] that for any exact category (and more generally for any "Waldhausen category") $\mathcal{E}$, Waldhausen constructs a simplicial category $iS_*\mathcal{E}$ whose classifying space is homotopy equivalent to $Q\mathcal{E}$. We will now do something similar for exact categories with dualitiy. (A main advantage of Waldhausen's construction is that iterating it yields deloopings and hence a spectrum, but as this feature will not carry over to the following hermitian setting, we won't discuss it.)
Let $(\mathcal{E},\ ^t)$ be an exact category with duality. We will construct an associated simplicial category with duality $R_*\mathcal{E}$. Consequently, we obtain an associated hermitian simplicial category $R^h_*\mathcal{E}$. Consider the category $\mathbf{n} = \{n' < (n-1)' < ... < 0' < 0 < ... < (n-1) < n\}$ and the category $\mathcal{I}(n)$ of arrows of $\mathbf{n}$, i.e. its objects are couples $(p,q) \in \mathbf{n} \times \mathbf{n}$ with $p \leq q$. There is one morphisms from m to n in $\mathcal{I}(n)$ if $m \leq n$ and no morphism otherwise.

Definition 7.1 *Let $(\mathcal{E},\ ^t)$ be an exact category with duality. $(R_*\mathcal{E},^d)$ is the following simplicial additive category with duality: Its objects are functors $A : \mathcal{I}(n) \to \mathcal{E}$ where all the sequences $A_{pq} \rightarrowtail A_{pr} \twoheadrightarrow A_{qr}$ are amissible short exact sequences $\forall\ p,q,r \in \mathbf{n}$ and morphisms are natural transformations. The dual of an object is given dy $A^d_{p,q} :=\ ^tA_{q',p'}$ where by definition $p'' = p$, and the dual of a morphism is also given by taking the pointwise dual and reindexing. Finally, the face maps δ_p are given by eliminating all objects A_{qr} where $q = p$ or $q = p'$ or $r = p$ or $r = p'$, and the degeneracies are given by adding identities at the convenient places. We further set*

$$R^h_*\mathcal{E} := (R_*\mathcal{E})_h$$

and

$$iR^h_*\mathcal{E} := \mathrm{Iso}(R^h_*\mathcal{E}).$$

It is straightforward to check that $(R_*\mathcal{E},^d)$ is a simplicial category with duality. Observe also that in particular $A_{pp} = 0$.

An object A_{**} of $R_n\mathcal{E}$ is given by a diagramm

$$
\begin{array}{ccccccccccccc}
A_{n',n'} & \rightarrowtail & A_{n',(n-1)'} & \rightarrowtail & \cdots & & A_{n',0'} & \rightarrowtail & A_{n',0} & \rightarrowtail & \cdots & & A_{n',n} \\
& & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
& & A_{(n-1)',(n-1)'} & \rightarrowtail & \cdots & & A_{(n-1)',0'} & \rightarrowtail & A_{(n-1)',0} & \rightarrowtail & \cdots & & A_{(n-1)',n} \\
& & & & & & \downarrow & & \downarrow & & & & \downarrow \\
& & & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
& & & & & & A_{0',0'} & \rightarrowtail & A_{0',0} & \rightarrowtail & \cdots & & A_{0',n} \\
& & & & & & & & \downarrow & & & & \downarrow \\
& & & & & & & & A_{0,0} & \rightarrowtail & \cdots & & A_{0,n} \\
& & & & & & & & & & & & \downarrow \\
& & & & & & & & & & & & A_{1,n} \\
& & & & & & & & & & \cdots & & \cdots \\
& & & & & & & & & & \cdots & & A_{n-1,n} \\
& & & & & & & & & & & & \downarrow \\
& & & & & & & & & & & & A_{n,n}
\end{array}
$$

From this point of view, a morphism f from A_{**} to B_{**} is nothing else but a collection of pointwise morphisms making everything commutative.

We discover an alternative description for the $\mathcal{U}$-theory of an exact category with duality:

Proposition 7.2 *For any hermitian exact category $\mathcal{E}_h$, we have a homotopy equivalence of simplicial sets*

$$\mathcal{W}(\mathcal{E}_h) \simeq iR^h_*\mathcal{E}$$

where on the right hand side we take the diagonal of the bisimplicial set.

Proof: Forgetting the choice of the cokernels, we easily see that our simplicial category $iR^h_*\mathcal{E}$ is equivalent to the simplicial category $i\,{}_\epsilon S^e_*\mathcal{E}$ of Shapiro and Yao [SY]. They show that this category is degreewise equivalent to the simplicial category associated to $\mathcal{W}(\mathcal{E}_h)$. In fact, this is just the classical argument of Waldhausen [Wald]. $\square$

We note that $R_n\mathcal{E}$ is an additive category with duality. This means that although the associated hermitian simplicial category $iR^h_*\mathcal{E}$ we obtain in the end is equivalent to the simplicial category given by Shapiro and Yao, we exhibit an underlying stronger and more conceptional structure. This will become very important in the sequel.

Remark: Recently, Weiss and Williams [WW] defined L-theory (compare with our remark preceding Definition 3.6) for Waldhausen categories with a "Spanier-Whitehead-product" which plays more or less the role of a duality functor. To see if this equals up to 2-torsion our higher Witt groups (see Definition 9.10) as is known for $P(R)$ remains an open problem.

8 Localization II: Hermitian K-theory of hereditary rings

Let A be a hereditary ring in which 2 is invertible, equipped with the trivial involution (i.e. the identity) and fix a central multiplicative subset $S \subset A - \{0\}$ containing no zero divisors. Recall that a ring A is called (right) hereditary if any sub-A-module of a projective (right) module is projective (equivalently, if any module of finite type has a projective resolution of length at most one). An integral domain is hereditary if and only if it is a Dedekind ring. See Proposition 8.8. for other interestin hereditary rings.

Let $\mathcal{T}_S$ denote the category of S-torsion A-modules of finite type (those M such that $S^{-1}M = 0$ or equivalently such that $M \otimes_A S^{-1}A = 0$). Karoubi [Ka3] observed that this category is equipped with the duality functor $Hom_A(\ \ , S^{-1}A/A) = Ext^1_A(\ \ , A)$ and made the following definition:

Definition 8.1 *Let $\mathcal{C}$ be an additive category with duality or the exact category $\mathcal{T}_S$. We define $U(\mathcal{C})$ to be the free group generated by isomorphism classes of triples (M, L_1, L_2) where M is an object of $\mathcal{C}_h$, and L_1 and L_2 are Lagrangians (i.e., equal to their orthogonal in M) of M, divided by the following relations:*
i) $(M, L_1, L_2) + (M', L_1', L_2') \sim (M \oplus M', L_1 \oplus L_1', L_2 \oplus L_2')$
ii) $(M, L_1, L_2) + (M, L_2, L_3) \sim (M, L_1, L_3)$
iii) $(M, L_1, L_2) \sim (L^\perp/L, L_1/L, L_2/L)$ where L is an isotropic submodule contained in L_1 and L_2.

Lemma 8.2 *Let $\mathcal{C}$ be an exact category with duality in which all admissible short exact sequences split or the exact category $\mathcal{T}_S$ with duality $Ext^1_A(\ \ , A)$ with 2 invertible in both cases. Then we have an isomorphism*

$$f : U(\mathcal{C}) \xrightarrow{\cong} \pi_1(\mathcal{W}(\mathcal{C}_h)) = U_0(\mathcal{C}_h).$$

Proof: Let $(M, L_1, L_2) \in U(\mathcal{C})$. Then L_1 and L_2 define morphisms α_1 and α_2 in $\mathcal{W}(\mathcal{C}_h)$ from 0 to M. We define f by sending (M, L_1, L_2) to the loop $\alpha_2^{-1} \circ \alpha_1$. One checks that f is well-defined. To construct an inverse of f, we have to show that the group $\pi_1(\mathcal{W}(\mathcal{C}_h))$ is generated by loops of the form $\alpha_2^{-1} \circ \alpha_1$. For $\mathcal{C} = \mathcal{T}_S$, this is a consequence of the fact that any stably metabolic object is metabolic [BLLV, Corollary A.14], hence any object has a morphism from the zero object. For an additive category $\mathcal{C}$, consider a loop of the form $\alpha_2^{-1} \circ \beta_2 \circ \beta_1^{-1} \circ \alpha_1$ (the argument for bigger loops is similar). Here α_i is a morphism from 0 to T_i, $i = 1, 2$. As we are in the additive case, β_1 and β_2 are induced by adding hyperbolic objects H_1 and H_2. Then we define γ_1 by adding H_2 to T_1 and γ_2 by adding H_1 to T_2. We have an isomorphism $T_1 \oplus H_2 \cong T_2 \oplus H_1$ and the loop $(\gamma_2 \circ \alpha_2)^{-1} \circ (\gamma_1 \circ \alpha_1)$ is equivalent to $\alpha_2^{-1} \circ \beta_2 \circ \beta_1^{-1} \circ \alpha_1$. □

Then we can reformulate Karoubi's exact localization sequence:

Theorem 8.3 *Let A be a Dedekind ring in which 2 is invertible, equipped with the trivial involution (i.e. the idendity) and fix a multiplicative subset $S \subset A - \{0\}$. Let $\mathcal{T}_S$ denote the category of S-torsion A-modules of finite type equipped with the duality functor $Hom_A(\ \ , S^{-1}A/A)$. Then there exists an exact sequence*

$$K_1({}_\epsilon P(A)_h) \to K_1({}_\epsilon P(S^{-1}A)_h) \to {}_\epsilon U_0(\mathcal{T}_S)$$

$$\to K_0({}_\epsilon P(A)_h) \to K_0({}_\epsilon P(S^{-1}A)_h) \to {}_\epsilon W((\mathcal{T}_S)_h)$$

Proof: Karoubi ([Ka3] and [Ka4]) established the two exact sequences $K_1({}_\epsilon P(A)_h) \to K_1({}_\epsilon P(S^{-1}A)_h) \to {}_\epsilon U(\mathcal{T}_S) \to K_0({}_\epsilon P(A)_h)$ $\to K_0({}_\epsilon P(S^{-1}A)_h)$ and $W({}_\epsilon P(A)_h) \to W({}_\epsilon P(S^{-1}A)_h) \to W({}_\epsilon(\mathcal{T}_S)_h)$. Lemma 8.2 and some easy diagram-chasing give the desired result. □

We will prove that this exact sequence is part of a long exact sequence induced by a homotopy fibration

$$\mathrm{Iso}({}_\epsilon P(A)_h)^{-1}\,\mathrm{Iso}({}_\epsilon P(A)_h) \to \mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)^{-1}\,\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$$

$$\to \mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)\ .$$

Our theorem answers the localization conjecture of Karoubi [Ka3, 3.2]. Of course, we would like to have a more general localization for an abelian category and a Serre subcategory. Looking at Quillen's localization theorem [Q1, Theorem 5], we know this generalized conjecture to be true if our exact category is $\mathcal{C} \times \mathcal{C}^{op}$.

To define a functor $\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h) \to \mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)$, we have to thicken up $\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$. Before we do this, let us recall something about "lattices" (see also [MH]). Let (E, ϕ) be an object of $P(S^{-1}A)_h$. A lattice L for E is a finitely generated A-submodule of E such that the inclusion $L \to E$ induces an isomorphism of $S^{-1}A$-modules $S^{-1}L \to E$. Furthermore, if L is equipped with a bilinear form Φ, we define the dual lattice of L by $L^\sharp := \{x \in E \mid \Phi(x, l) \in A\, \forall\, l \in L\}$.

Definition 8.4 *Let $\mathrm{Iso}({}_\epsilon\widetilde{P(S^{-1}A)}_h)$ be the category where an object is a couple $((E, \phi), L)$. Here (E, ϕ) is an object of $\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$ and L is a lattice for E contained in its dual lattice $L^\sharp$. A morphism from $((E, \phi), L)$ to $((F, \psi), M)$ is given by a morphism $g : (E, \phi) \to (F, \psi)$ in $\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$ such that $g^{-1}(M)$ is contained in L.*

To simplify our notations, we will write from now on ${}_\epsilon\mathcal{P} := \mathrm{Iso}({}_\epsilon P(A)_h)$ and ${}_\epsilon\tilde{\mathcal{P}}_S := \mathrm{Iso}({}_\epsilon\widetilde{P(S^{-1}}A)_h)$.

Lemma 8.5 *The forgetful functor*

$$f : {}_\epsilon\tilde{\mathcal{P}}_S \to \mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$$

is a homotopy equivalence. The induced functor

$${}_\epsilon\mathcal{P}^{-1} f : {}_\epsilon\mathcal{P}^{-1}\,{}_\epsilon\tilde{\mathcal{P}}_S \to {}_\epsilon\mathcal{P}^{-1}\,\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$$

is a homotopy equivalence.

Proof: One first checks that f is prefibered in the sense of Quillen. By Theorem A it is therefore sufficient to show that $f^{-1}(E,\phi)$ is contractible for any object (E,ϕ) of $\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)$. But this category is nonempty (there is always a lattice contained in its dual lattice: take any lattice and intersect it with its dual), and it is cofiltered because the intersection of two lattices is a lattice (observe that over a Dedekind ring any module without torsion is projective). The second homotopy equivalence follows from $H_*({}_\epsilon\mathcal{P}^{-1}\,{}_\epsilon\tilde{\mathcal{P}}_S) \cong \pi_0({}_\epsilon\mathcal{P})^{-1}H_*({}_\epsilon\tilde{\mathcal{P}}_S) \cong \pi_0({}_\epsilon\mathcal{P})^{-1}H_*(\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h)) \cong H_*({}_\epsilon\mathcal{P}^{-1}\,\mathrm{Iso}({}_\epsilon P(S^{-1}A)_h))$ □

For the rest of this section, $\rightarrowtail$ and $\twoheadrightarrow$ stand for monomorphisms and epimorphisms in the abelian category of A-modules of finite type. Monomorphisms are therefore not split in general even if source and target are objects of $P(A)$.
Using some calculations in the appendix of [BLLV], we can make the following:

Definition 8.6 *Let* $\theta : <{}_\epsilon\mathcal{P}, {}_\epsilon\tilde{\mathcal{P}}_S> \to \mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)$ *be the functor sending* $((E,\phi),L)$ *to* $(L^\sharp/L, \overline{\phi|_{L^\sharp}})$*. The morphism* $((E,\phi),L) \to ((F,\psi),M)$ *in* $<{}_\epsilon\mathcal{P}, {}_\epsilon\tilde{\mathcal{P}}_S>$ *represented by* $[(P,\lambda), g : (S^{-1}P \oplus E, S^{-1}\lambda \oplus \phi) \xrightarrow{\cong} (F,\psi)]$ *such that* $g^{-1}(M) \subset P \oplus L$ *is sent to*

$$\begin{array}{ccccc}
L^\sharp/L & \twoheadleftarrow & (P\oplus L)^\sharp/M & \overset{\bar g}{\rightarrowtail} & M^\sharp/M \\
\bar\phi \downarrow \cong & & \downarrow & & \cong \downarrow \bar\psi \\
{}^t(L^\sharp/L) & \rightarrowtail & {}^t((P\oplus L)^\sharp)/M) & \overset{{}^t\bar g}{\twoheadleftarrow} & {}^t(M^\sharp/M)
\end{array}$$

using the canonical isomorphism $L^\sharp/L \cong (P\oplus L)^\sharp/(P\oplus L)$.

Assume that the functor above induces a homotopy equivalence on the zero components $\theta : <{}_\epsilon\mathcal{P}, {}_\epsilon\tilde{\mathcal{P}}_S>_0 \xrightarrow{\cong} \mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)_0$. All morphisms in ${}_\epsilon\tilde{\mathcal{P}}_S$ are monomorphisms and the functor $\oplus((E,\phi),L) : {}_\epsilon\mathcal{P} \to {}_\epsilon\tilde{\mathcal{P}}_S$ is faithful. Following Grayson [Q2], this gives us a homotopy fibration

$${}_\epsilon\mathcal{P}^{-1}\,{}_\epsilon\mathcal{P} \to {}_\epsilon\mathcal{P}^{-1}\,{}_\epsilon\tilde{\mathcal{P}}_S \to <{}_\epsilon\mathcal{P}, {}_\epsilon\tilde{\mathcal{P}}_S>

Then we could apply Lemma 8.5 and would have the desired homotopy fibration. The first idea to prove that θ is a homotopy equivalence is to apply Quillen's Theorem A. Hence we are reduced to show that $(P \downarrow \theta)$ is contractible $\forall\, P \in {}_\epsilon\tilde{\mathcal{P}}_S$. This is obvious in the case $P = 0$ because 0 is an initial object in this category. This is a consequence of the following useful construction in the special case $T = 0$: We can form the pull-back of lattices along morphisms of $\mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)$. More precisely, consider the following diagram

$$\begin{array}{ccccc}
 & & L & = & L \\
 & & \downarrow & & \downarrow \\
P & = & P & \rightarrowtail & L^\sharp \\
\downarrow & & \downarrow & \square & \downarrow \\
T & \twoheadleftarrow & X & \rightarrowtail & L^\sharp/L \\
\tau \downarrow \cong & & \square & & \cong \downarrow \overline{\phi} \\
{}^tT & \rightarrowtail & {}^tX & \twoheadleftarrow & {}^t(L^\sharp/L)
\end{array}$$

We set $U := ker(P \twoheadrightarrow T)$. Fitting together this short exact sequence and the two bicartesian squares, we get a short exact sequence $U \rightarrowtail L^\sharp \twoheadrightarrow {}^tX$. But $L \rightarrowtail P \twoheadrightarrow X$ is also exact, and therefore $P^\sharp \rightarrowtail L^\sharp \twoheadrightarrow {}^tX$ is exact, too (see [Ka3, p.362]). It follows that $P = U^\sharp$.

Now we consider a morphism β of $\mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)$ given by $(0,0) \twoheadleftarrow (X,\chi) \rightarrowtail (T,\tau)$. Following Karoubi (see [Ka4, appendix 3], here we need that 2 is invertible), we can choose a lattice (K,κ) with $K \subset K^\sharp$ such that $(K^\sharp/K, \overline{\kappa}) = (T,\tau)$. Let $U = X \times_T K^\sharp$ be the induced self-dual lattice obtained by the above construction. Then $[U, id_{S^{-1}K}]$ maps under θ to β. Now fix a morphism β of $\mathcal{W}({}_\epsilon(\mathcal{T}_S)_h)$ given by $(0,0) \twoheadleftarrow (X,\chi) \rightarrowtail (T,\tau)$. The above construction for $T = 0$ shows that P is a self-dual lattice for (E,ϕ). In other words, $(P, \phi|_P)$ is an object of $P(A)_h$ such that $(S^{-1}P, S^{-1}\phi|_P) = (E,\phi)$. Hence there is a morphism from 0 to $((E,\phi), L)$ given by $[(P,\phi), (S^{-1}P,\phi) \oplus (0,0) \overset{id}{\to} (E,\phi)]$ such that $id^{-1}(L) \subset P$. This is the only morphism because the pull-back P is unique up to unique isomorphism.

But there is no reason why the other categories $(P \downarrow \theta)$ should be contractible. We might try to prove that all the base changes induce homotopy equivalences. But then we have the same problem as in section 3, see the remark following Theorem 3.15: we have to show that the endofunctor induced by the switch $\sigma^* : (P \oplus P \downarrow \theta) \to (P \oplus P \downarrow \theta)$ is homotopic to the identity. Forgetting about the forms, this is essentially how Gersten [Ge] tried to prove a localization theorem for ordinary K-theory. But his proof contains a mistake, as he claims the existence of a certain "natural transformation" which does not exist. Then Grayson [Gr] filled the gap, introducing an H-

space structure on $< P \downarrow \theta >$ and exploiting this to show contractibility. Unfortunately, it is not possible to introduce a similar product structure in our case with forms.
Quillen's proofs [Q1],[Q2] do not carry over either.
Another idea would be to prove Karoubi's Fundamental Theorem (Theorem 3.7), and then proceed by induction as we will do in the next section to prove devissage (Theorem 9.5). In fact, using essentially Theorem 8.3 we can show that θ is a monomorphism for π_0 and an isomorphism for π_1 which is enough to start the induction. But we do not have the Fundamental Theorem for exact categories in general. To generalize Karoubi's proof of it, we would need a generalization of Corollary 6.7 which we do not have. Another idea of proving the Fundamental Theorem would be to use our Waldhausen model of section 7 which is degreewise additive. Roughly speaking, we hence have the Fundamental Theorem degreewise and then have to show that after realization (i.e. applying the diagonal functor) we still have it. We can work either in the category of topological spaces or in the category of spectra. In **Top**, the Bousfield-Friedlander Theorem [BF, Theorem B.4] gives us two conditions which implies that a degreewise homotopy fibration of simplicial spaces remains a homotopy fibration after realization. The first condition is always true in the case of H-groups. But the second says that we need to have degreewise π_0-surjectivity of the second morphism which is not true in our case.
In the category of spectra, the realization of a degreewise homotopy fibration always remains a homotopy fibration. But we have to work with nonconnective spectra because this is where we can apply the Fundamental Theorem. Hence we have degreewise negative homotopy groups which might change the positive homotopy groups after realization.

The proof of the localization theorem we finally give uses much heavier tools.

Theorem 8.7 *For any hereditary ring A in which 2 is invertible and for any central multiplicative subset $S \subset A$ containing no zero divisors, we have a homotopy fibration*

$$\mathrm{Iso}({}_{\epsilon}P(A)_h)^{-1}\,\mathrm{Iso}({}_{\epsilon}P(A)_h) \to \mathrm{Iso}({}_{\epsilon}P(S^{-1}A)_h)^{-1}\,\mathrm{Iso}({}_{\epsilon}P(S^{-1}A)_h)$$
$$\to \mathcal{W}({}_{\epsilon}(\mathcal{T}_S)_h)$$

Consequently, we have a long exact sequence

$$\ldots K_n({}_{\epsilon}P(A)_h) \to K_n({}_{\epsilon}P(S^{-1}A)_h) \to U_{n-1}({}_{\epsilon}(\mathcal{T}_S)_h)$$
$$\to K_{n-1}({}_{\epsilon}P(A)_h) \to K_{n-1}({}_{\epsilon}P(S^{-1}A)_h)\ldots$$

$$...K_0({}_\epsilon P(A)_h) \to K_0({}_\epsilon P(S^{-1}A)_h) \to W({}_\epsilon(\mathcal{T}_S)_h)$$

Note that the morphism $K_0({}_\epsilon P(S^{-1}A)_h) \to W({}_\epsilon(\mathcal{T}_S)_h)$ *will not be surjective in general.*

For the proof of Theorem 8.7, see section 11.

Remark: In the next section, we will prove a Dévissage Theorem (Theorem 9.5) which gives us a more concrete description of the $\mathcal{U}$-theory of the category of torsion modules.
The most interesting example of non-commutative hereditary rings is probably the following:

Proposition 8.8 *Let* G *be a finite group and* A *be a Dedekind ring in which* $|G|$ *is a unit. Then the group ring* AG *is hereditary.*
Hence we have a homotopy fibration

$$\operatorname{Iso}({}_\epsilon P(AG)_h)^{-1}\operatorname{Iso}({}_\epsilon P(AG)_h) \to \operatorname{Iso}({}_\epsilon P(S^{-1}AG)_h)^{-1}\operatorname{Iso}({}_\epsilon P(S^{-1}AG)_h)$$
$$\to \mathcal{W}({}_\epsilon(\mathcal{T}_S^G)_h)$$

where the anti-involution on AG *is given by* $\overline{ag} = ag^{-1}L$ *and* $S \subset center(A)$ *containing no zero divisors.*

Proof: We need some facts about orders and group rings that can be found in [CR]. Let $F = Quot(A)$. Recall that an A-order Λ in a finite-dimensional F-algebra B is by definition a subring of B such that Λ is a projective A-module of finite type and $F \cdot \Lambda = B$. Maschke's theorem [CR, 3.14] tells us that FG is semi-simple. As $|G|$ is a unit in A, AG is a maximal A-order in FG [CR, 27.1]. As FG is semi-simple and finite-dimensional over F, this implies that AG is (left and right) hereditary [CR, 26.12]. □

If we do not assume that $|G|$ is a unit in A, then AG is not hereditary. In fact, the theorem of Stallings and Swan [St] [Sw] tells us that the trivial $\mathbf{Z}G$-module $\mathbf{Z}$ admits a projective resolution of length one if and only if G is a free group.

Recall that the category $\mathcal{L}(\mathcal{C}_h)$ (see Definition 5.10) yields a model for the hermitian K-theory of $\mathcal{C}$.

Corollary 8.9 *For any hereditary ring* A *in which 2 is invertible and for any central multiplicative subset* $S \subset A$ *containing no zero-divisors, there is a homotopy fibration*

$$\mathcal{L}({}_\epsilon(\mathcal{T}_S)_h) \to \mathcal{U}({}_{-\epsilon}P(A)_h) \to \mathcal{U}({}_{-\epsilon}P(S^{-1}A)_h)$$

Consequently, we have a long exact sequence

$$\dots\ {}_{-\epsilon}U_n(A) \to {}_{-\epsilon}U_n(S^{-1}A) \to K^h_{n-1}({}_\epsilon(\mathcal{T}_S)_h) \to {}_{-\epsilon}U_{n-1}(A) \to {}_{-\epsilon}U_{n-1}(S^{-1}A)$$

$$\ldots \to K_0^h({}_\epsilon(\mathcal{T}_S)_h) \to {}_{-\epsilon}U_0(A) \to {}_{-\epsilon}U_0(S^{-1}A)$$

Note that the morphism ${}_{-\epsilon}U_0(A) \to {}_{-\epsilon}U_0(S^{-1}A)$ will not be surjective in general.

Proof: We define $\tilde{\mathcal{P}_S}$ to be the category where an object is given by $(E, K \subset L)$ where E is an object of $P(S^{-1}A)$ and $K \subset L$ is an inclusion of lattices for E. A morphism from $(E, K \subset L)$ to $(E', K' \subset L')$ is an isomorphism $g : E \xrightarrow{\cong} E'$ such that we have a chain of inclusions $g^{-1}(K') \subset K \subset L \subset g^{-1}(L')$. We set $\mathcal{P} := \mathrm{Iso}(P(A))$. As in Lemma 8.5, we can show that the forgetful functor $f : \tilde{\mathcal{P}_S} \to \mathrm{Iso}(P(S^{-1}A))$ is a homotopy equivalence. We have the following commutative diagram including five homotopy fibrations

$$\begin{array}{ccccc}
\mathcal{V}({}_\epsilon P(A)_h) & & \mathcal{V}({}_\epsilon P(S^{-1}A)_h) & & \mathcal{L}({}_\epsilon(\mathcal{T}_S)_h) \\
\downarrow & & \downarrow & & \downarrow \\
{}_\epsilon\mathcal{P}^{-1}\,{}_\epsilon\mathcal{P} & \xrightarrow{(f_A)_h} & {}_\epsilon\mathcal{P}^{-1}\,{}_\epsilon\tilde{\mathcal{P}_S} & \xrightarrow{\theta} & \mathcal{W}({}_\epsilon(\mathcal{T}_S)_h) \\
\downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F} \\
\mathcal{P}^{-1}\mathcal{P} & \longrightarrow & \mathcal{P}^{-1}\tilde{\mathcal{P}_S} & \xrightarrow{\eta} & Q(\mathcal{T}_S)
\end{array}$$

Here η is given by θ and forgetting the hermitian forms which coincides with θ applied to the hermitian category $(P(S^{-1}A) \times P(S^{-1}A)^{op})_h$ via the equivalence of Lemma 5.3. Applying the previous Theorem 8.7 and Theorem 3.7 yields a homotopy fibration $\Omega(\mathcal{U}({}_{-\epsilon}P(A)_h)) \to \Omega(\mathcal{U}({}_{-\epsilon}P(S^{-1}A)_h)) \to \mathcal{L}({}_\epsilon(\mathcal{T}_S)_h)$. The left-hand square is a square of infinite loop spaces, therefore a delooping gives the desired result. More precisely, we can replace in the above diagram the ring A by its algebraic suspension ΣA. Setting $X := \mathrm{hofib}((f_{\Sigma A})_h)$, we have ${}_\epsilon\mathcal{W}(\mathcal{T}_S)_0 \simeq B\Omega\,{}_\epsilon\mathcal{W}(\mathcal{T}_S)_0 \simeq B\Omega X_0 \simeq X_0$. Let $K_0^h(f_{\Sigma A})$ be the K_0 of the symmetric monoidal functor $(f_{\Sigma A})_h$ as defined by Bass [Bass, chapter 7]. As we have $\pi_0(X) \cong K_0^h(f_{\Sigma A})$, it remains to show that there exists an isomorphism $\pi_0({}_\epsilon\mathcal{W}(\mathcal{T}_S)) \xrightarrow{\cong} K_0^h(f_{\Sigma A})$. This is a consequence of Bass localization sequence, Karoubi's localization sequence for Witt groups [Ka4, Théorème 2.8] and the fact that $K_0(\Sigma A) = 0$ for any regular ring A [Bass, p.685].
(Observe that the category $\tilde{\mathcal{P}_S}$ and the functor η are quite similar to the category $\mathcal{U}$ and the functor δ used by Gersten [Ge] when he tries to to prove his localization theorem for the K-theory of rings.) □

Observe that localization hence implies an analogue for exact categories of Karoubi's Fondamental Theorem $\Omega\,{}_\epsilon\mathcal{U}(\mathcal{T}_S) \simeq {}_{-\epsilon}\mathcal{V}(\mathcal{T}_S)$ for the exact category $\mathcal{T}_S$ of torsion modules over a hereditary ring; see the proof of Theorem 9.5 for a more detailed statement.

9 Dévissage and calculations

The idea of dévissage is that under some conditions, instead of dealing with a huge abelian category it might be sufficient to consider the subcategory of semi-simple objects. Recall Quillen's Dévissage Theorem in K-theory [Q1]:

Theorem 9.1 *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a subcategory closed under subobjects, quotients and finite products. Suppose that any object M of $\mathcal{A}$ has a finite filtration $0 = M_0 \subset M_1 ... \subset M_n = M$ such that M_j/M_{j-1} is in $\mathcal{B}$ for each j. Then the inclusion functor induces a homotopy equivalence $Q(\mathcal{B}) \xrightarrow{\simeq} Q(\mathcal{A})$.*

Let A be a Dedekind ring, $S = A - \{0\}$ and $F = S^{-1}A = Quot(A)$. Then applying this theorem to the category $\mathcal{T}_S$ of A-modules of finite type of S-torsion yields a homotopy fibration $\oplus_{(0)\neq\wp\in Spec(A)} Q(P(A/\wp)) \to Q(P(A)) \to Q(P(F))$ where the direct sum means that all morphisms between the different subcategories are 0. If A is such that F is a number field, then Quillen's calculation of the K-theory of finite fields gives rise to exact sequences ($n \geq 2$)

$$0 \to K_{2n}(A) \to K_{2n}(F) \to \oplus K_{2n-1}(A/\wp) \xrightarrow{\iota} K_{2n-1}(A) \to K_{2n-1}(F) \to 0\ .$$

Moreover, a theorem of Soulé [Sou] (whose proof uses calculations in étale cohomology) tells us that $\iota = 0$, hence the above sequence splits into two shorter pieces.
We want to proceed in a similar manner for hermitian K-theory. That is, we will prove a dévissage theorem for hermitian K-theory and $\mathcal{U}$-theory of the category of torsion modules over a hereditary ring and then try to make some more precise statements for the hermititian K-theory of the integers in a number field, using the calculations of the hermitian K-theory of the finite residue fields [Q3],[Fr].

For the rest of this section, we consider the following situation: G is a finite group, A is a Dedekind ring in which 2 and $|G|$ are invertible (hence the group ring AG is hereditary, cf. Theorem 8.8) and $F = Quot(A)$. Let $\mathcal{T}_S$ and $\mathcal{T}_S^G$ be the abelian categories of A- resp. AG-modules of finite type and of S-torsion, where $S = A - \{0\} \subset A{<}1{>} \subset\ center(AG)$. We have the obvious forgetful functors $P(AG) \to P(A)$ and $\mathcal{T}_S^G \to \mathcal{T}_S$. Finally, $\wp$ denotes a prime ideal in A different from (0).

Definition 9.2 *An object M of $\mathcal{T}_S$ is called a $\wp$-torsion module if there exists a positive integer n such that $m\wp^n = 0\,\forall\, m \in M$. We denote the subcategory of $\wp$-torsion modules by $\mathcal{T}_\wp$.*

Lemma 9.3 *We have a decomposition of hermitian categories*

$$(\mathcal{T}_S^G)_h \cong \oplus_{(0)\neq\wp\in Spec(A)} (\mathcal{T}_\wp^G)_h.$$

By this, again we mean that any object decomposes as a direct sum of objects lying in these subcategories, and all the morphisms between objects from different subcategories are 0*.*

Proof: This is well known for the underlying abelian categories if G is the trivial group [Bass, p.509]. The decomposition is closed under the duality functor and clearly respects the G-module structure. □

Remark: When we are interested in taking a smaller S, then we have to take the sum over all prime ideals containing at least one element of S.

We remark that $P((A/\wp)G)$ is precisely the full subcategory of the semisimple objects of $\mathcal{T}_\wp^G$ (projectivity is automatic since our ring is semisimple). In order to avoid too many parentheses, we write $A/\wp G$ instead of $(A/\wp)G$ from now on. On this subcategory $P(A/\wp G)$ we have two duality functors $Hom_{AG}(\ \ , S^{-1}AG/AG)$ and $Hom_{A/\wp G}(\ \ , A/\wp G)$ which fortunately coincide:

Lemma 9.4 *For any object V of $P(A/\wp G)$, there is a canonical isomorphism*

$$Hom_{AG}(V, S^{-1}AG/AG) \xrightarrow{\cong} Hom_{A/\wp G}(V, A/\wp G).$$

Proof: We only do the case $G = \{1\}$, the general case is analogous. As both duality functors are additive, it is enough to consider the case $V = A/\wp$. Following Karoubi [Ka3, Théorème 1.1 et Remarque 1], we know that $Hom_A(A/\wp, S^{-1}A/A) = Hom_{A/\wp}(A/\wp, A/pA)$ for any element $p \in \wp$ with $p(A/\wp) = 0$. Choose $p \in \wp - \wp^2$. As we have unique decomposition of prime ideals and the chinese remainder theorem, the decomposition of the previous lemma implies $Hom_{A/\wp}(A/\wp, A/pA) = Hom_{A/\wp}(A/\wp, A/\wp)$. □

The main result of this section will be the following:

Theorem 9.5 *("Dévissage") The inclusion of hermitian categories*

$$P(A/\wp G)_h \xrightarrow{f} (\mathcal{T}_\wp^G)_h$$

induces isomorphisms $\forall\, n \geq 0$

$$_\epsilon U_n(P(A/\wp G)_h) \xrightarrow{_\epsilon U_n(f)} {}_\epsilon U_n((\mathcal{T}_\wp^G)_h)$$

$$_\epsilon K_n^h(P(A/\wp G)_h) \xrightarrow{_\epsilon K_n^h(f)} {}_\epsilon K_n^h((\mathcal{T}_\wp^G)_h)$$

The idea of the proof is the following: Prove it in low degrees, recall that it is true for ordinary K-theory and then make an induction. In fact, this strategy is inspired by [Ka8].

Proposition 9.6 *The inclusion of hermitian abelian categories*

$$P(A/\wp G)_h \xrightarrow{f} (\mathcal{T}_\wp^G)_h$$

induces isomorphisms

$$_\epsilon W(P(A/\wp G)_h) \xrightarrow{_\epsilon W(f)} {_\epsilon W}((\mathcal{T}_\wp^G)_h)$$

$$_\epsilon U_0(P(A/\wp G)_h) \xrightarrow{_\epsilon U_0(f)} {_\epsilon U_0}((\mathcal{T}_\wp^G)_h)$$

$$_\epsilon K_0^h(P(A/\wp G)_h) \xrightarrow{_\epsilon K_0^h(f)} {_\epsilon K_0^h}((\mathcal{T}_\wp^G)_h)$$

Proof: As AG is noetherian, any object of $\mathcal{T}_\wp^G$ is of finite length [Ei, p.76]. Then one can show [QSS, Theorem 6.12] that for any object (T,τ) of $(\mathcal{T}_\wp^G)_h$ there is a totally isotropic subobject U such that $U^\perp/U$ is semisimple, hence $(U^\perp/U, \overline{\tau})$ is in $P(A/\wp G)_h$. We then can apply [QSS, Theorem 6.9] to show that $_\epsilon W(f)$ is an isomorphism.
Next, consider the commutative square

$$\begin{CD}
K_0(A/\wp G) @>{g_{A/\wp G}}>> {_\epsilon U_0}(A/\wp G)\\
@V{K_0(f)}V{\cong}V @VV{_\epsilon U_0(f)}V\\
K_0(\mathcal{T}_\wp^G) @>{g_{\mathcal{T}_\wp^G}}>> {_\epsilon U_0}(\mathcal{T}_\wp^G)
\end{CD}$$

as established in [Ka3, p.392]. Then it is shown that $g_{\mathcal{T}_\wp^G}$ is an epimorphism (Karoubi's argument carries over to group rings as he only needs heredity). It follows that $_\epsilon U_0(f)$ is an epimorphism. Moreover, essentially the same argument (observe that $A/\wp G$ is semisimple) shows that $g_{A/\wp G}$ is an epimorphism. Now consider the following diagrams of short exact sequences, induced by the hyperbolic functor and the forgetful functor, respectively:

$$\begin{CD}
{_\epsilon U_0}(A/\wp G) @>>> K_0(A/\wp G) @>>> {_\epsilon K_0^h}(A/\wp G) @>>> {_\epsilon W}(A/\wp G) @>>> 0\\
@VV{_\epsilon U_0(f)}V @V{\cong}V{K_0(f)}V @VV{_\epsilon K_0^h(f)}V @V{\cong}V{_\epsilon W(f)}V\\
{_\epsilon U_0}(\mathcal{T}_\wp^G) @>>> K_0(\mathcal{T}_\wp^G) @>>> {_\epsilon K_0^h}(\mathcal{T}_\wp^G) @>>> {_\epsilon W}(\mathcal{T}_\wp^G) @>>> 0
\end{CD}$$

and

$$\begin{CD}
{_{-\epsilon} K_0^h}(A/\wp G) @>>> K_0(A/\wp G) @>>> {_\epsilon U_0}(A/\wp G) @>>> 0\\
@VV{_{-\epsilon} K_0^h(f)}V @V{\cong}V{K_0(f)}V @VV{_\epsilon U_0(f)}V\\
{_{-\epsilon} K_0^h}(\mathcal{T}_\wp^G) @>>> K_0(\mathcal{T}_\wp^G) @>>> {_\epsilon U_0}(\mathcal{T}_\wp^G) @>>> 0
\end{CD}$$

The five lemma shows that $_\epsilon K_0^h(f)$ and hence $_\epsilon U_0(f)$ are isomorphims. □

Remark: We observe that the epimorphism $g_{A/\wp G}$ together with the fact that $K_{-1}(R) = K_0(\Sigma R) = 0$ for any regular ring R implies that $_\epsilon K_{-1}^h(A/\wp G) =$

${}_\epsilon W_{-1}(A/\wp G) = 0$. Instead of using the five lemma, one can make some concrete calculations (inspired by [Ka3, p.392]) if G is the trivial group. Consider the following two exact sequences

$$K_1(A/\wp) \xrightarrow{H_1} {}_\epsilon K_1^h(A/\wp) \longrightarrow {}_\epsilon U_0(A/\wp) \longrightarrow K_0(A/\wp) \xrightarrow{H_0} {}_\epsilon K_0^h(A/\wp)$$

and

$${}_\epsilon K_1^h(A_\wp) \longrightarrow {}_\epsilon K_1^h(F) \longrightarrow {}_\epsilon U_0(\mathcal{T}_\wp) \longrightarrow {}_\epsilon K_0^h(A_\wp) \longrightarrow {}_\epsilon K_0^h(F)$$

As $A/\wp$ is a field, H_0 is a monomorphism and the first exact sequence decomposes (i.e. one morphism is 0). If $\epsilon = -1$, it is classical that ${}_\epsilon K_1^h(A/\wp) = 0$, hence ${}_\epsilon U_0(A/\wp) = {}_\epsilon U_0(\mathcal{T}_\wp) = 0$. If $\epsilon = 1$, we know that ${}_\epsilon W(A_\wp) \to {}_\epsilon W(F)$ is a monomorphism [MH], and we have monomorphisms $K_0(A_\wp) \to {}_\epsilon K_0^h(A_\wp)$ and $K_0(F) \to {}_\epsilon K_0^h(F)$ so ${}_\epsilon K_0^h(A_\wp) \to {}_\epsilon K_0^h(F)$ is a monomorphism by the five lemma and the second exact sequence also decomposes. The following generalities can be found in detail in [Kn, chapter IV]. For any commutative ring R, we have a morphism ${}_\epsilon K_1^h(R) \to Disc(R) \times \mathbf{Z}/2$. Here $Disc(R)$ is the abelian group of discriminant modules over R. Then this morphism is given by the spinor norm SN on the first component and by the determinant on the second. If R is principal, $Disc(R) \cong R^*/(R^*)^2$. Recall that there is also the determinant morphism $det : K_1(R) \to R^*$. The following square commutes for any projective R-module P of finite type [Kn, p.236]

$$\begin{array}{ccc}
Aut(P) & \xrightarrow{\quad H \quad} & Aut(H(P), \mu_P) \\
\downarrow{\scriptstyle det} & & \downarrow{\scriptstyle SN} \\
R^* & \xrightarrow{\quad \pi \quad} & R^*/(R^*)^2
\end{array}$$

It follows that for $\epsilon = 1$ we have ${}_\epsilon U_0(A/\wp) \cong {}_\epsilon U_0(\mathcal{T}_\wp) \cong \mathbf{Z}/2$.

Now we are ready to prove the Dévissage Theorem:

Proof of Theorem 9.5 Consider the following two diagrams of long exact sequences

$$\begin{array}{ccccc}
{}_\epsilon U_{n+1}(A/\wp G) & \longrightarrow & K_{n+1}(A/\wp G) & \xrightarrow{H_*} & {}_\epsilon K_{n+1}^h(A/\wp G) \\
\downarrow{\scriptstyle {}_\epsilon U_{n+1}(f)} & & \cong\downarrow{\scriptstyle K_{n+1}(f)} & & \downarrow{\scriptstyle {}_\epsilon K_{n+1}^h(f)} \\
{}_\epsilon U_{n+1}(\mathcal{T}_\wp^G) & \longrightarrow & K_{n+1}(\mathcal{T}_\wp^G) & \xrightarrow{H_*} & {}_\epsilon K_{n+1}^h(\mathcal{T}_\wp^G)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & {}_\epsilon U_n(A/\wp G) & \longrightarrow & K_n(A/\wp G) \\
 & \cong\downarrow{\scriptstyle {}_\epsilon U_n(f)} & & \cong\downarrow{\scriptstyle K_n(f)} \\
\longrightarrow & {}_\epsilon U_n(\mathcal{T}_\wp^G) & \longrightarrow & K_n(\mathcal{T}_\wp^G)
\end{array}$$

and

$$\begin{array}{ccccc}
{}_{-\epsilon}K^h_{n+1}(A/\wp G) & \xrightarrow{F_*} & K_{n+1}(A/\wp G) & \longrightarrow & {}_{\epsilon}U_{n+1}(A/\wp G) \\
\downarrow {}_{-\epsilon}K^h_{n+1}(f) & & \cong\downarrow K_{n+1}(f) & & \downarrow {}_{\epsilon}U_{n+1}(f) \\
{}_{-\epsilon}K^h_{n+1}(\mathcal{T}^G_\wp) & \xrightarrow{F_*} & K_{n+1}(\mathcal{T}^G_\wp) & \longrightarrow & {}_{\epsilon}U_{n+1}(\mathcal{T}^G_\wp)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & {}_{-\epsilon}K^h_{n}(A/\wp G) & \xrightarrow{F_*} & K_{n}(A/\wp G) \\
 & \cong\downarrow {}_{-\epsilon}K^h_{n}(f) & & \cong\downarrow K_{n}(f) \\
\longrightarrow & {}_{-\epsilon}K^h_{n}(\mathcal{T}^G_\wp) & \xrightarrow{F_*} & K_{n}(\mathcal{T}^G_\wp)
\end{array}$$

In the first diagram, both long exact sequences are induced by the homotopy fibration associated to the hyperbolic functor, therefore everything commutes. The upper line in the second diagram is given by the forgetful functor and Theorem 3.7. For the lower line, consider the homotopy fibration induced by the hyperbolic funtor ${}_{\epsilon}\mathcal{U}(A) \xrightarrow{f} \mathcal{K}(A) \xrightarrow{H} {}_{\epsilon}\mathcal{L}(A)$. Then we use the localization Theorems 8.7, 8.9 and the localization theorem in ordinary K-theory (which is a consequence of either 8.7 or 8.9 when applied to $\mathcal{C} \times \mathcal{C}^{op}$). Then link these three homotopy fibrations together by the hyperbolic functor as described above.Theorem 5.7 tells us that f is nothing else than ΩF where $F : {}_{\epsilon}\mathcal{W}(P(A)_h) \to Q(P(A))$ is the forgetful functor. Looking at Karoubi's proof of Theorem 3.7 [Ka7, Paragraphe III] we see that the morphism in the upper row $K_n(A/\wp G) \to {}_{\epsilon}U_n(A/\wp G)$ is given by the "hyperbolic functor", i.e., $\mathcal{K}(P(A/\wp G)) \simeq \mathcal{U}({}_{\epsilon}P(A/\wp G \times A/\wp G^{op})_h) \to \mathcal{U}({}_{\epsilon}P(M_2(A/\wp G))_h) \simeq \mathcal{U}({}_{\epsilon}P(A/\wp G)_h)$. Hence the second diagram commutes as well.
Now we proceed by induction. Recall that we have dévissage for $n = 0$ by Proposition 9.6. Assume now that we have dévissage for n as marked in the diagrams above. It remains to apply the five lemma a couple of times simultaneously for ϵ and $-\epsilon$: By the first diagram, $K^h_{n+1}(f)$ is an epimorphism. Hence the second diagram shows that U_{n+1} is an isomorphism, and looking again at the first diagram, it follows that K^h_{n+1} is also an isomorphism. □

This allows us to state the following simplification of the localization Theorems of the previous chapter (we still write $\mathcal{K}(\mathcal{C})$ for $\mathrm{Iso}(\mathcal{C})^{-1}\,\mathrm{Iso}(\mathcal{C})$):

Corollary 9.7 *Let G be a finite group and A a Dedekind ring in which* 2 *and $|G|$ are units. Then we have two homotopy fibrations*

$$\mathcal{K}({}_{\epsilon}P(AG)_h) \to \mathcal{K}({}_{\epsilon}P(S^{-1}AG)_h) \to \oplus\mathcal{W}({}_{\epsilon}P(A/\wp G)_h)$$

and

$$\oplus\mathcal{L}({}_{\epsilon}P(A/\wp G)_h) \to \mathcal{U}({}_{-\epsilon}P(AG)_h) \to \mathcal{U}({}_{-\epsilon}P(S^{-1}AG)_h)$$

Consequently, we get two long exact sequences

$$\dots K_n({}_{\epsilon}P(AG)_h) \to K_n({}_{\epsilon}P(S^{-1}AG)_h) \to \oplus U_{n-1}({}_{\epsilon}(P(A/\wp G)_h))$$

$$\rightarrow K_{n-1}({}_{\epsilon}P(AG)_h) \rightarrow K_{n-1}({}_{\epsilon}P(S^{-1}AG)_h)...$$
$$...K_0({}_{\epsilon}P(AG)_h) \rightarrow K_0({}_{\epsilon}P(S^{-1}AG)_h) \rightarrow \oplus W({}_{\epsilon}(P(A/\wp G)_h))$$

and

$$... {}_{-\epsilon}U_n(AG) \rightarrow {}_{-\epsilon}U_n(S^{-1}AG) \rightarrow \oplus K^h_{n-1}({}_{\epsilon}(P(A/\wp G)_h)) \rightarrow {}_{-\epsilon}U_{n-1}(AG)$$
$$\rightarrow {}_{-\epsilon}U_{n-1}(S^{-1}AG)... \rightarrow K^h_0({}_{\epsilon}(P(A/\wp G)_h)) \rightarrow {}_{-\epsilon}U_0(AG) \rightarrow {}_{-\epsilon}U_0(S^{-1}AG)$$

Note that the morphisms ${}_{-\epsilon}U_0(AG) \rightarrow {}_{-\epsilon}U_0(S^{-1}AG)$ *and* $K_0({}_{\epsilon}P(S^{-1}AG)_h) \rightarrow \oplus W({}_{\epsilon}(P(A/\wp G)_h))$ *will not be surjective in general.*

Proof: Use Theorem 8.7, Corollary 8.9 and Theorem 9.5. □

Remark: Using dévissage for Balmer's Witt groups (which will be introduced in the next section) and the results of section 11, in particular Lemma 11.2, one easily sees that this long exact localization sequence extends to negative U- and K^h-groups.

We also obtain this way higher degree Gysin morphisms generalizing the Gysin morphism ${}_{\epsilon}U_0(A/\wp) \rightarrow {}_{\epsilon}K^h_0(A)$ as given in [Ka3, Appendice 4]. Of course, we would like to have such morphisms in more general situations.

For the rest of this section, assume that G is the trivial group and all the residue fields $A/\wp$ have only a finite number of elements. This is the case when F is an algebraic number field. The K-theory and the hermitian K-theory of finite fields have been computed by Quillen and Friedlander.

Theorem 9.8 *Let $n > 0$ be an integer and F_q the finite field with q elements, q odd. Then the K-theory of F_q is given by*

$$K_{2n}(F_q) = 0 \qquad K_{2n-1}(F_q) \cong \mathbf{Z}/(q^n - 1).$$

The hermitian K-theory K^h_n behaves in a periodic way of period 8. It is given by

$${}_1K^h_{8n}(F_q) \cong \mathbf{Z}/2 \qquad {}_1K^h_{8n+1}(F_q) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2 \qquad {}_1K^h_{8n+2}(F_q) \cong \mathbf{Z}/2$$
$${}_1K^h_{8n+3}3(F_q) \cong \mathbf{Z}/(q^{((8n+3)+1)/2} - 1) \qquad {}_1K^h_{8n+4}(F_q) \cong 0 \qquad {}_1K^h_{8n+5}(F_q) \cong 0$$
$${}_1K^h_{8n+6}(F_q) \cong 0 \qquad {}_1K^h_{8n+7}(F_q) \cong \mathbf{Z}/(q^{((8n+7)+1)/2} - 1)$$
$${}_{-1}K^h_{8n}(F_q) \cong 0 \qquad {}_{-1}K^h_{8n+1}(F_q) \cong 0 \qquad {}_{-1}K^h_{8n+2}(F_q) \cong 0$$
$${}_{-1}K^h_{8n+3}(F_q) \cong \mathbf{Z}/(q^{((8n+3)+1)/2} - 1) \qquad {}_{-1}K^h_{8n+4}(F_q) \cong \mathbf{Z}/2 \qquad {}_{-1}K^h_{8n+5}(F_q) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$$
$${}_{-1}K^h_{8n+6}(F_q) \cong \mathbf{Z}/2 \qquad {}_{-1}K^h_{8n+7}(F_q) \cong \mathbf{Z}/(q^{((8n+7)+1)/2} - 1)$$

Proof: [Q3] and [Fr]. □

This allows us to calculate the $\mathcal{U}$-theory of a finite field:

Corollary 9.9 *The $\mathcal{U}$-theory U_n of a finite field F_q (q odd) can be described in a periodic way of period 8, it is given by*

$$ {}_1U_{8n}(F_q) \cong \mathbf{Z}/2 \qquad {}_1U_{8n+1}(F_q) \cong \mathbf{Z}/(q^{((8n+1)+1)/2}-1) \qquad {}_1U_{8n+2}(F_q) \cong 0 $$

$$ {}_1U_{8n+3}(F_q) \cong 0 \qquad {}_1U_{8n+4}(F_q) \cong 0 \qquad {}_1U_{8n+5}(F_q) \cong \mathbf{Z}/(q^{((8n+5)+1)/2}-1) $$

$$ {}_1U_{8n+6}(F_q) \cong \mathbf{Z}/2 \qquad {}_1U_{8n+7}(F_q) \cong Gr_4 $$

$$ {}_{-1}U_{8n}(F_q) \cong 0 \qquad {}_{-1}U_{8n+1}(F_q) \cong \mathbf{Z}/(q^{((8n+1)+1)/2}-1) \qquad {}_{-1}U_{8n+2}(F_q) \cong \mathbf{Z}/2 $$

$$ {}_{-1}U_{8n+3}(F_q) \cong Gr_4 \qquad {}_{-1}U_{8n+4}(F_q) \cong \mathbf{Z}/2 \qquad {}_{-1}U_{8n+5}(F_q) \cong \mathbf{Z}/(q^{((8n+5)+1)/2}-1) $$

$$ {}_{-1}U_{8n+6}(F_q) \cong 0 \qquad {}_{-1}U_{8n+7}(F_q) \cong 0 $$

where Gr_4 equals $\mathbf{Z}/4$ or $\mathbf{Z}/2 \oplus \mathbf{Z}/2$.

Proof: Let $\epsilon = 1$. Consider the long exact sequences (HS) and (FS) associated to the hyperbolic functor and the forgetful functor where we use the calculations of the above theorem

$$ 0 \to \mathbf{Z}/2 \to {}_1U_{8n+7}(F_q) \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1) \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1) $$
$$ \to {}_1U_{8n+6}(F_q) \to 0 \to 0 \to {}_1U_{8n+5}(F_q) \to \mathbf{Z}/(q^{((8n+5)+1)/2}-1) \to 0 $$
$$ \to {}_1U_{8n+4}(F_q) \to 0 $$

and

$$ 0 \to \mathbf{Z}/(q^{((8(n+1)+1)+1)/2}-1) \to {}_1U_{8(n+1)+1}(F_q) \to 0 \to 0 \to {}_1U_{8(n+1)}(F_q) $$
$$ \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1) \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1) \to {}_1U_{8n+7}(F_q) \to \mathbf{Z}/2 \to 0 $$
$$ \to {}_1U_{8n+6}(F_q) \to \mathbf{Z}/2 \oplus \mathbf{Z}/2 \to \mathbf{Z}/(q^{((8n+5)+1)/2}-1) $$
$$ \to {}_1U_{8n+5}(F_q) \to \mathbf{Z}/2 \to 0 \to {}_1U_{8n+4}(F_q) \to \mathbf{Z}/(q^{((8n+3)+1)/2}-1) $$
$$ \to \mathbf{Z}/(q^{((8n+3)+1)/2}-1) \to {}_1U_{8n+3}(F_q) \to 0 \to 0 \to {}_1U_{8n+2}(F_q) \to 0 $$

We have ${}_1U_{8n+5}(F_q) \cong \mathbf{Z}/(q^{((8n+5)+1)/2}-1)$ and ${}_1U_{8n+4}(F_q) \cong 0$ by (HS), ${}_1U_{8n+6}(F_q) \cong \mathbf{Z}/2$, ${}_1U_{8n+3}(F_q) \cong 0$, ${}_1U_{8n+2}(F_q) \cong 0$ and ${}_1U_{8n+1}(F_q) \cong \mathbf{Z}/(q^{((8n+1)+1)/2}-1)$ by (FS) (still true for $n = 0$ because ${}_1U_0(F_q) \cong \mathbf{Z}/2$ and $K_0(F_q) \to K_0^h(F_q)$ is a monomorphism), ${}_1U_{8n+7}(F_q) \cong Gr_4$ by (HS) and ${}_1U_{8n}(F_q) \cong \mathbf{Z}/2$ by (FS) (still true for $n = 0$).
Let $\epsilon = -1$. Consider the long exact sequences (HS) and (FS) associated to

the hyperbolic functor and the forgetful functor where we use the calculations of the above theorem

$$\begin{aligned}0 \to\ {}_{-1}U_{8(n+1)}(F_q) \to 0 \to 0 \to\ {}_{-1}U_{8n+7}(F_q) \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1)\\ \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1) \to\ {}_{-1}U_{8n+6}(F_q) \to 0 \to \mathbf{Z}/2 \to\ {}_{-1}U_{8n+5}(F_q)\\ \to \mathbf{Z}/(q^{((8n+5)+1)/2}-1) \to \mathbf{Z}/2 \oplus \mathbf{Z}/2 \to\ {}_{-1}U_{8n+4}(F_q) \to 0 \to \mathbf{Z}/2\\ \to\ {}_{-1}U_{8n+3}(F_q) \to \mathbf{Z}/(q^{((8n+3)+1)/2}-1) \to \mathbf{Z}/(q^{((8n+3)+1)/2}-1)\\ \to\ {}_{-1}U_{8n+2}(F_q) \to 0 \to 0 \to\ {}_{-1}U_{8n+1}(F_q) \to \mathbf{Z}/(q^{((8n+1)+1)/2}-1) \to 0\end{aligned}$$

and

$$\begin{aligned}0 \to\ {}_{-1}U_{8(n+1)}(F_q) \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1) \to \mathbf{Z}/(q^{((8n+7)+1)/2}-1)\\ \to\ {}_{-1}U_{8n+7}(F_q) \to 0 \to 0 \to\ {}_{-1}U_{8n+6}(F_q) \to 0\\ \to \mathbf{Z}/(q^{((8n+5)+1)/2}-1) \to\ {}_{-1}U_{8n+5}(F_q) \to 0\\ \to 0 \to\ {}_{-1}U_{8n+4}(F_q) \to \mathbf{Z}/(q^{((8n+3)+1)/2}-1)\\ \to \mathbf{Z}/(q^{((8n+3)+1)/2}-1) \to\ {}_{-1}U_{8n+3}(F_q) \to \mathbf{Z}/2 \to 0\end{aligned}$$

We have ${}_{-1}U_{8n+1} \cong \mathbf{Z}/(q^{((8n+1)+1)/2}-1)$ and ${}_{-1}U_{8n} \cong 0$ by (HS) (still true for $n=0$), ${}_{-1}U_{8n+7} \cong 0$, ${}_{-1}U_{8n+6} \cong 0$ and ${}_{-1}U_{8n+5} \cong \mathbf{Z}/(q^{((8n+5)+1)/2}-1)$ by (FS), ${}_{-1}U_{8n+4} \cong \mathbf{Z}/2$ by (HS), ${}_{-1}U_{8n+3} \cong Gr_4$ by (FS) and ${}_{-1}U_{8n+2} \cong \mathbf{Z}/2$ by (HS). □

These calculations allow us to compare the hermitian K-theory, the $\mathcal{U}$-theory and the higher Witt groups of A and F. Our definition of higher Witt groups for an hermitian exact category is the obvious generalization of Definition 3.13:

Definition 9.10 *The higher Witt groups of a hermitian exact category $\mathcal{C}_h$ are defined by*

$${}_\epsilon W_n(\mathcal{C}_h) := coker(K_n(\mathcal{C}) \xrightarrow{H_n} {}_\epsilon K_n^h(\mathcal{C}_h))\ \forall\ n \in \mathbf{N}$$

and even $\forall\ n \in \mathbf{Z}$ if $\mathcal{C}$ is split exact.

It follows immediately that our Dévissage Theorem 9.5 also applies to higher Witt groups. Observe that ${}_\epsilon W_0(\mathcal{C}_h)$ equals ${}_\epsilon W(\mathcal{C}_h)$ as defined at the beginning of section 5.

In the following theorem, we only state the isomorphisms which are immediate from the above results. The reader can easily verify that we have many other monomorphisms and epimorphisms and we have information about the kernel and the cokernel. Moreover, other known results as known calculations in low degrees, Soulé's theorem mentioned above, etc., might allow us to give stronger results.

Theorem 9.11 *Let A be a Dedekind ring in which 2 is a unit such that $F = Quot(A)$ is an algebraic number field. Then we have isomorphisms*

$$ {}_1K_n^h(A) \stackrel{\cong}{\to} {}_1K_n^h(F) \quad \forall\, n \equiv 3, 4 \, mod \, 8 $$

$$ {}_1U_n(A) \stackrel{\cong}{\to} {}_1U_n(F) \quad \forall\, n \equiv 1, 2 \, mod \, 8 $$

$$ {}_1W_n(A) \stackrel{\cong}{\to} {}_1W_n(F) \quad \forall\, n \equiv 3 \, mod \, 8 $$

$$ {}_{-1}K_n^h(A) \stackrel{\cong}{\to} {}_{-1}K_n^h(F) \quad \forall\, n \equiv 0, 7 \, mod \, 8 $$

$$ {}_{-1}U_n(A) \stackrel{\cong}{\to} {}_{-1}U_n(F) \quad \forall\, n \equiv 5, 6 \, mod \, 8 $$

$$ {}_{-1}W_n(A) \stackrel{\cong}{\to} {}_{-1}W_n(F) \quad \forall\, n \equiv 7 \, mod \, 8 $$

Proof: Use Corollary 9.7, Theorem 9.8 and Corollary 9.9. For the ${}_{-1}K_0^h$-isomorphism, recall that ${}_{-1}U_0(A/\wp) = 0$ (see the remark following Proposition 11.6) and that ${}_{-1}K_0^h(F) = \mathbf{Z}$ [Ka2, p.306], thus everything is stably hyperbolic and ${}_{-1}K_0^h(A) = \mathbf{Z}$ as well. □

Recall that the K-theory of the integers A in an algebraic number field is a finitely generated abelian group [Q4], and we know the free part of $K_n(A)$ [Bo]. Calculating the odd torsion means proving the Quillen-Lichtenbaum Conjectures [Li]. Concerning the 2-torsion, we know [HS] that if A is not exceptional, then there are infinitely many $\wp$ such that $K_i(A) \otimes \mathbf{Z}_{(2)} \to K_i(A/\wp) \otimes \mathbf{Z}_{(2)}$ is a split epimorphism for $i > 0$. In hermitian K-theory, we still know the free part of ${}_\epsilon K_n^h(A)$ [Bo]. Recently it has been proved that if A is exceptional, then there are infinitely many $\wp$ such that ${}_1K_i^h(A) \otimes \mathbf{Z}_{(2)} \to {}_1K_i^h(A/\wp) \otimes \mathbf{Z}_{(2)}$ is a split epimorphism for $i > 0$ [Ham].

10 Some further remarks and open problems

In the last three sections, we established deloopings, localization and dévissage for hermitian K-theory, which were all known in ordinary K-theory. Any time we look at a construction or a theorem in ordinary K-theory, we may ask the question: Does it still hold in hermitian K-theory? In general, the lack of a reasonable duality functor might prevent us from even defining the hermitian K-theory of an exact category. But in the case of $P(A)$ we have a duality functor, and many techniques and results of ordinary K-theory carry over to hermitian K-theory. We already mentioned some of them on page 8, moreover we have still a multiplicative structure [Lo1] which we did not investigate at all in this thesis.

The next step is to consider those results that are not covered by the plus construction, the category of vector bundles over a scheme being the most interesting example. Here we really need the techniques developped in our thesis. The proof of the localization sequence (Theorem 8.7) also needed the general definition of the hermitian K-theory of an exact category with duality as given in section 5.
When we want to prove a theorem in hermitian K-theory that we know to be true in ordinary K-theory, we can of course try to translate the proof to the hermitian setting. But we can also use Karoubi's strategy (prove it for ordinary K-theory and for hermitian K-theory in low degrees, then proceed by induction), as we did to prove the Dévissage Theorem. But this strategy heavily relies on his Fundamental theorem ([Ka7] and Theorem 3.7) which we only have for rings. To prove a homotopy equivalence $\Omega\,{}_{\epsilon}\mathcal{U}(\mathcal{C}_h) \simeq {}_{-\epsilon}\mathcal{V}(\mathcal{C}_h)$ for an exact category $\mathcal{C}$ in general would hence allow us to apply this strategy in many other cases. But the proof would require explicit deloopings of the hermitian K-theory (as established in Corollary 6.5 for additive categories) for exact categories in general.
Schlichting [Sch1] recently defined the negative K-theory of an exact category and proved that some classical theorems (additivity, resolution, localization) remain true for negative K-theory. It is natural to ask wether all this can be done for negative hermitian K-theory as well.

Let us consider the category of all modules of finite type $M(A)$ over a given ring A or more generally the category of all coherent sheaves $Coh(X)$ on a scheme X. It is not clear how to define a duality functor and hence the associated hermitian category. On the full subcategory of S-torsion modules $\mathcal{T}_S \subset M(A)$, we have $Hom_A(\ \ , S^{-1}A/A) = Ext^1_A(\ \ , A)$ as a duality functor if A is hereditary, and on the subcategory $P(A)$ we have $Hom_A(\ \ , A)$. To define a duality on $M(A)$ (and more general for quasi-coherent sheaves instead of vector bundles) might be easier when we replace our category by its bounded derived category and using the duality suggested in [Har, chapter 5]. More precisely, he considers the derived category (bounded below) of $\mathcal{O}_X$-modules $D^+(\mathcal{O}_X - mod)$ over a given noetherian scheme X and the full subcategory of complexes with coherent cohomology and finite injective dimension $D^+_{c,fid}(\mathcal{O}_X - mod)$. He then introduces the notion of a "dualizing complex":

Definition 10.1 *Fixing a complex $R^* \in D^+(\mathcal{O}_X - mod)$, we say that $F^* \in D(\mathcal{O}_X - mod)$ is reflexive with respect to R^* if the natural map $\eta : F^* \to \mathbf{R}\underline{Hom}^*(\mathbf{R}\underline{Hom}^*(F^*, R^*), R^*)$ is an isomorphism in $D(\mathcal{O}_X - mod)$.*
If there exist an $R^* \in D^+(\mathcal{O}_X - mod)$ such that any $F^* \in D^+_{c,fid}(\mathcal{O}_X - mod)$ is reflexive with respect to R^*, then we call R^* a "dualizing complex".

Then it is shown that $\mathcal{O}_X$, considered as a complex concentrated in degree 0, is a dualizing complex when our noetherian scheme X is regular. The idea is that it suffices to show that $\mathcal{O}_X$ is $\mathcal{O}_X$-reflexive, as any coherent sheaf is locally a quotient of a free sheaf.
We can therefore "generalize" our duality from vector bundles to coherent sheaves. As an illustration, consider the category of finitely generated abelian groupes $M(\mathbf{Z})$. The complex $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ is an injective resolution of $\mathbf{Z}$, and given a finite abelian group, all morphisms to $\mathbf{Q}$ are 0, of course. Hartshorne also proved that O_X is still a dualizing complex when our noetherian scheme X is Gorenstein which means that all its local rings have a finite injective resolution. In the case of a non-commutative ring, the Gorenstein condition and the existence of a dualizing complex is less clear.

As Thomason did for ordinary K-theory [TT], one might hope to define the hermitian K-theory of the category of bounded chain complexes and to prove that this coincides with our definition. Recall that Gillet [Gil] and Thomason defined for any exact category $\mathcal{C}$ a Waldhausen structure on the category of bounded complexes $Ch^b(\mathcal{C})$, and Gillet (generalizing a proof of Waldhausen) proved that the inclusion in degree 0 induces a homotopy equivalence $\mathcal{K}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{K}(Ch^b(\mathcal{C}))$. Here $\mathcal{K}$ stands for Waldhausen K-theory resp. Thomason's variant of the K-theory of a complicial biWaldhausen category [TT, 1.2.11,1.11.7]. He also proved that the following important result:

Theorem 10.2 *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor of exact categories such that the induced functor on the bounded derived categories $D^b(F) : D^b(\mathcal{C}) \to D^b(\mathcal{D})$ is an equivalence of categories. Then $Q(F) : Q(\mathcal{C}) \to Q(\mathcal{D})$ is a homotopy equivalence.*

Proof: [TT, theorem 1.9.8] □

Although the η defined above is not an isomorphism but only a quasi-isomorphism in $Ch^b_{c,fid}(\mathcal{O}_X - mod)$, we can still define an associated "hermitian Waldhausen category". If we want to imitate Thomason's proofs, we will need an additivity theorem and a localization theorem. It might therefore be useful to consider a Waldhausen-like model of the hermitian K-theory (see [SY] for such a model in $\mathcal{U}$-theory). Restricting to the case of a noetherian regular scheme, we then should have homotopy equivalences $\mathcal{L}(Vect(X)_h) \overset{\simeq}{\to} \mathcal{L}(Coh(\mathcal{O}_X)_h) \overset{\simeq}{\to} \mathcal{L}(Ch^b_{c,fid}(\mathcal{O}_X - mod)$. It will probably be easier to attack these questions before in $\mathcal{U}$-theory and then deduce the statement on $\mathcal{L}(\mathcal{C}_h) \simeq \mathrm{hofib}(\mathcal{W}(\mathcal{C}_h) \overset{F}{\to} Q(\mathcal{C}))$.

Trying to give a direct definition of the hermitian K-theory of a triangulated category without using the fact that it is the derived category of an exact category is as complicated as in ordinary K-theory. There is a definition suggested by Neeman for the K-theory of a triangulated category. But recently Schlichting [Sch5] proved that it is impossible to give a definition of the K-theory of a triangulated category that simultaneously generalizes Quillen's definition for exact categories and has the property that a short exact sequence of triangulated categories yields a long exact sequence of K-groups.
When we are only interested in Witt groups, such a definition is possible due to the work of Balmer. In fact in the case of triangulated categories with duality functor, both Balmer [Bal2] and Youssin [Yo] suggest a definition for the Witt group of a triangulated category $\mathcal{T}$ with a duality functor ^{t}. Balmer's definition of the Witt group $W_B(\mathcal{T}_h)$ is to take the monoid of hermitian objets and then to divide by the "metabolic objects" in the sense of triangulated categories. By definition L is a lagrangian of (P, ϕ) if we have a commutative diagram $(*)$ below with $Q = 0$ such that the rows and columns are exact triangles. This implies more generally that whenever we have a commutative diagram where the rows and columns are exact triangles

$$\begin{array}{ccccccc}
T^{-1}({}^tM) & \xrightarrow{\nu_0} & L & \xrightarrow{\nu_1} & P & \xrightarrow{\nu_2} & {}^tM \\
\downarrow {\scriptstyle T^{-1}({}^t\eta_0)} & & \downarrow {\scriptstyle \eta_0} & & {\scriptstyle \cong}\downarrow {\scriptstyle \phi} & & \downarrow {\scriptstyle {}^t\eta_0} \\
T^{-1}({}^tL) & \xrightarrow[T^{-1}({}^t\nu_0)]{} & M & \xrightarrow[{}^t\nu_2]{} & {}^tP & \xrightarrow[{}^t\nu_1]{} & {}^tL \\
\downarrow {\scriptstyle {}^t\eta_2} & & \downarrow {\scriptstyle \eta_1} & & & & \\
{}^tQ & \xleftarrow[\cong]{\psi} & Q & & & & \\
\downarrow {\scriptstyle {}^t\eta_1} & & \downarrow {\scriptstyle \eta_2} & & & & \\
{}^tM & \xrightarrow[T(\nu_0)]{} & T(L) & & & &
\end{array} \qquad (*)$$

such that $\nu_2 \circ \phi^{-1} \circ {}^t\nu_2 = -{}^t\eta_1 \circ \psi \circ \eta_1$, then $(P, \phi) = (Q, -\psi)$ in $W_B(\mathcal{T}_h)$. In this situation, we say that L is a sub-lagrangian of (P, ϕ); compare this with the Definitions 3.8 and 3.10. Balmer also introduced "shifted" Witt groups, observing that if ^{t} is a duality on a triangulated category $\mathcal{T}$ with translation functor T, then $T^n \circ {}^t$ is also a duality functor.

Definition 10.3 *Let $\mathcal{T}$ be a triangulated category with translation functor T and duality ^{t}. Then $W_B^n(\mathcal{T}_h)$ is by definition the monoid of isomorphism classes of objects of $\mathcal{T}_h$ relative to the duality functor $T^n \circ {}^t$, divided by the metabolic objects.*

We write $W_B(\mathcal{T}_h)$ for $W_B^0(\mathcal{T}_h)$. Balmer proves that this Witt group generalizes the Witt group of a hermitian exact category we defined at the beginning of section 5 (and hence in Definition 9.10) if $\mathcal{C}$ is semi-saturated. (Recall that "semi-saturated" means that any morphism having a right inverse is an admissible epimorphism. In particular, any pseudo-abelian category is semi-saturated.)

Theorem 10.4 *Let $\mathcal{C}$ be an exact category in which 2 is invertible together with a duality functor and let $D^b(\mathcal{C})$ be the derived category with the induced duality functor, i.e. $({}^t(C_*))_n = {}^tC_{-n}$ and similar for the face maps and morphisms. Then the inclusion in degree 0 induces a morphism*

$$W(\mathcal{C}_h) \to W_B(D^b(\mathcal{C})_h)$$

which is an isomorphism if $\mathcal{C}$ is semi-saturated.

Proof: See [Bal2, 4.3]. □

Observe that the "duality functor" of Hartshorne $(\mathbf{R}\underline{Hom}^*(\ \ , \mathcal{O}_X))$ for noetherian regular separated schemes becomes the duality functor of Balmer when we restrict to the category of vector bundles and invert the quasi-isomorphisms, hence pass to the derived category.
Recall that we can also define K_0 of a triangulated category (take the Grothendieck group of the abelian monoid of objects and divide out by the triangles) such that this "generalizes" the definition for exact categories when we consider the bounded derived category. This suggest a definition of K_0 for a hermitian triangulated category:

Definition 10.5 *For any triangulated category $\mathcal{T}$ with a duality functor ^{t}, we define $K_0^h(\mathcal{T}_h)$ to be the Grothendieck group of the abelian monoid of isomorphism classes of objects of $\mathcal{T}_h$, divided by the relation $(P, \phi) \sim (Q, -\psi) \oplus (H(L), \mu_L)$ whenever we have a diagram (*).*

Proposition 10.6 *Let $\mathcal{C}$ be an exact category in which 2 is invertible together with a duality functor and let $D^b(\mathcal{C})$ be the derived category with the* *induced duality functor, i.e.* $({}^t(C_*))_n = {}^tC_{-n}$ *and similar for the face maps and morphisms. Then the inclusion in degree 0 induces a morphism*

$$\iota : K_0^h(\mathcal{C}_h) \to K_0^h(D^b(\mathcal{C})_h)$$

which is an epimorphism if $\mathcal{C}$ *is semi-saturated.*

Proof: We claim that there is a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
K_0(\mathcal{C}) & \longrightarrow & K_0^h(\mathcal{C}_h) & \longrightarrow & W(\mathcal{C}_h) & \longrightarrow & 0 \\
\cong\downarrow & & \downarrow\iota & & \cong\downarrow & & \\
K_0(D^d(\mathcal{C})) & \xrightarrow{H} & K_0^h(D^b(\mathcal{C})_h) & \longrightarrow & W^B(D^b(\mathcal{C})_h) & \longrightarrow & 0
\end{array}$$

By Theorem 5.7 $K_0(\mathcal{C}_h)$ is nothing else but the Grothendieck group of the underlying hermitian additive category divided by the relation $(P,\phi) \sim (L^\perp/L, \overline{\phi}) \oplus (H(L), \mu_L)$ whenever L is a sublagrangian of (P,ϕ) (in [QSS], this group is denoted by $GH_0^\epsilon(\mathcal{C})$). This shows that ι is well-defined. Next, we show that H is well-defined. Assume that $A \xrightarrow{i} B \xrightarrow{p} C \xrightarrow{s} T(A)$ is an exact triangle. Then we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
T^{-1}(C) \oplus {}^tT(A) & \xrightarrow{T^{-1}(s)\oplus {}^ts} & A \oplus {}^tC & \xrightarrow{i\oplus {}^tp} & H(B) & \xrightarrow{p\oplus {}^ti} & C \oplus {}^tA \\
id\downarrow & & \downarrow id & & \downarrow \mu_B & & \downarrow id \\
T^{-1}(C) \oplus {}^tT(A) & \xrightarrow{T^{-1}(s)\oplus {}^ts} & A \oplus {}^tC & \xrightarrow{i\oplus {}^tp} & H(B) & \xrightarrow{p\oplus {}^ti} & C \oplus {}^tA \\
0\downarrow & & \downarrow 0 & & & & \\
0 & \xleftarrow[0]{} & 0 & & & & \\
0\downarrow & & \downarrow 0 & & & & \\
C \oplus {}^tA & \xrightarrow[s\oplus T({}^tc)]{} & T(A) \oplus T({}^tC) & & & &
\end{array}$$

such that both morphisms from $A\oplus {}^tC$ to ${}^tA \oplus C$ are 0, hence $(H(B), \mu_B) = (H(A \oplus {}^tC), \mu_{A\oplus {}^tC}) \cong (H(A), \mu_A) \oplus (H(C), \mu_C)$ in $K_0^h(D^b(\mathcal{C})_h)$ by definition.

The upper row of the diagram is exact by definition, and the lower row is exact because any object in $K_0^h(D^b(\mathcal{C})_h)$ which is stably metabolic is already metabolic [Bal1, Theorem 3] and therefore equivalent to an object in the image of H. Finally, the five lemma shows that ι is an epimorphism. □

I conjecture that ι is in fact an isomorphism.

Balmer uses his definition to prove a 12-term localization exact sequence for Witt groups [Bal1] which we will use in the next section. In this localization sequence there do also appear the “shifted Witt groups” defined above which are of period four. He explained to me the idea of the proof that for additive categories these equal the higher L-groups of Ranicki et al. (see our remark in section 3) and hence - up to 2-torsion - Karoubi's higher Witt groups. But there is still no written proof. There seems to be a generalization of L-theory to exact categories due to C. Walter which should still coincide with Balmer's Witt groups. This should also be compared with the work of [WW], see our remark at the end of section 7.

One might conjecture that for exact categories our higher Witt groups (see Definition 9.10) coincide with Balmer's shifted Witt groups when we tensorize with $\mathbf{Z}[1/2]$. Nevertheless, there are interesting elements in the 2-torsion of our higher Witt groups which do not exist in the periodic shifted Witt groups of Balmer. For example, the element of order 2 in $W_1(F)$ for F a field corresponding to the determinant is 0 in $W_B^{-1}(F)$. See also Lemma 11.2 for a comparison of his shifted Witt groups with our negative ones.

Another open problem is to investigate when the "isomorphism conjecture" for the assembly maps in hermitian K-theory of group rings holds. In fact, one observes that our non-connective spectrum for the hermitian K-theory of section 6 fits well in the framework of [DL].

Finally, let us say that it is a very tempting problem to construct a spectrum representing hermitian K-theory of a (regular) scheme in the $\mathbf{A}^1$-homotopy category of Morel and Voevodsky, as can be done for ordinary algebraic K-theory [MV]. The key geometric properties of hermitian K-theory needed to establish its representability can be proved for regular affine varieties using the tools of the next section. To prove them for regular varieties in general, we will probably need again an analogon of the techniques of [TT]. In particular, we do not know yet wether replacing the presheaf $\mathcal{K}^h$ by a fibrant sheaf with respect to the closed model structure of [MV] yields the same definition for the hermitian K-theory of regular varieties as our definition of section 5.

11 The proof of the Localization Theorem

This section is joint work with M. Schlichting. We will prove the Localization Theorem 8.7 which we repeat for the reader's convenience:

Theorem 11.1 *Let A be a hereditary ring and assume that 2 is invertible. Then for any central multiplicative subset $S \subset A$ containing no zero divisors, we have a homotopy fibration*

$$i({}_{\epsilon}P(A)_h)^+ \to i({}_{\epsilon}P(S^{-1}A)_h)^+ \to \mathcal{W}({}_{\epsilon}(\mathcal{T}_S)_h)$$

Consequently, we have a long exact sequence

$$\ldots K_n({}_{\epsilon}P(A)_h) \to K_n({}_{\epsilon}P(S^{-1}A)_h) \to U_{n-1}({}_{\epsilon}(\mathcal{T}_S)_h)$$

$$\to K_{n-1}({}_{\epsilon}P(A)_h) \to K_{n-1}({}_{\epsilon}P(S^{-1}A)_h)\ldots$$

$$\ldots K_0({}_{\epsilon}P(A)_h) \to K_0({}_{\epsilon}P(S^{-1}A)_h) \to W({}_{\epsilon}(\mathcal{T}_S)_h)$$

Note that the morphism $K_0({}_{\epsilon}P(S^{-1}A)_h) \to W({}_{\epsilon}(\mathcal{T}_S)_h)$ will not be surjective in general.

To simplify notations, we write $i\mathcal{C}^+$ instead of $\operatorname{Iso}\mathcal{C}^{-1}\operatorname{Iso}\mathcal{C}$ for any symmetric monoidal category $\mathcal{C}$.

The idea of the proof is the following: Consider the following diagram induced by a diagram of simplicial additive categories with duality (the first one being constant)

$$(iP(A))_h^+ \xrightarrow{\iota_*} |(iG_*)_h^+| \xrightarrow{\pi_*} |(iR_*\mathcal{T}_S)_h^+|$$

where G_* is a simplicial version of the category of lattices (see Definition 11.3). Recall (Theorem 7.2) that $\Omega|(iR_*\mathcal{T}_S)_h| \simeq \mathcal{U}(\mathcal{T}_S)$. Now G_k and $R_k\mathcal{T}_S$ are additive categories whose K_{-1} will vanish, so by Lemma 11.2 the degreewise localization sequence for hermitian K-theory reduces at $n = -1$ to the localization sequence for Balmer's Witt groups whose hypotheses we can actually prove to be fullfilled (Proposition 11.4). Further calculations and Karoubi induction will show that we have in fact for k fixed a localization homotopy fibration in hermitian K-theory. To see that we still have a homotopy fibration after realization, we need a π_0-surjectivity condition (cf [BF] and section 8). This difficulty will be solved by Lemma 11.7. Finally, we have to identify the realization of the degreewise hermitian K-theory of the "simplicial lattices" G_* with the hermitian K-theory of the localized ring $S^{-1}A$ (see Proposition 11.8).

Definition 3.13 for additive categories with duality allows us also to define Karoubi's negative higher Witt groups (Definition 9.10) thanks to the results of section 6. The following result identifies Balmer's shifted triangulated Witt groups (Definition 10.3) and Karoubi's negative Witt groups of regular rings (not only up to 2-torsion):

Lemma 11.2 *Let $\mathcal{A}$ be an additive pseudo-abelian category with duality with $K_n(\mathcal{A}) = 0 \, \forall N \leq n < 0$. Then $W_n(\mathcal{A}) = W_B^{-n}(\mathcal{A}) \, \forall N \leq n \leq 0$.*

Proof: For $n = 0$ this is [Bal2]. Let $\mathcal{CA}$ and $\mathcal{SA}$ be the cone and suspension of $\mathcal{A}$ as in section 6. By [CP],[Sch1], we always have a sequence of triangulated categories which is exact up to direct factors (i.e. after pseudo-abelianization, see p. 19)

$$(*) \qquad D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{CA}) \longrightarrow D^b(\mathcal{SA})$$

Since $\mathcal{A}$ is pseudo-abelian, $D^b(\mathcal{A})$ is pseudo-abelian (see [BSch], we have $D^b(\widetilde{\mathcal{E}}) \simeq \widetilde{D^b(\mathcal{E})}$ for any exact category $\mathcal{E}$). In particular, $D^b(\mathcal{A})$ is the kernel of $D^b(\mathcal{CA}) \longrightarrow D^b(\mathcal{SA})$. Since $K_{-1}(\mathcal{A}) = 0$ by hypothesis, $D^b(\mathcal{SA}) \to D^b(\widetilde{\mathcal{SA}})$ is an equivalence of triangulated categories. This is a consequence of the following fact: if $\mathcal{V} \subset \mathcal{W}$ is a triangulated full cofinal subcategory of a triangulated category $\mathcal{W}$ and if the inclusion induces an isomorphism $K_0(\mathcal{V}) \to K_0(\mathcal{W})$ then $\mathcal{V} \subset \mathcal{W}$ is an equivalence of triangulated categories, see for instance [Th3, theorem 2.1]. In our situation $0 = K_{-1}(\mathcal{A}) = K_0(\widetilde{\mathcal{SA}}) = K_0(D^b(\widetilde{\mathcal{SA}}))$. Hence the claimed equivalence of triangulated categories. Therefore, $D^b(\mathcal{SA})$ is pseudo-abelian, and $(*)$ is an exact sequence of triangulated categories. Applying Balmer's 12-term exact localization sequence [Bal1, Theorem 6.2] and using the fact that $W_B^i(\mathcal{CA}) = 0$ for all $i \in \mathbf{N}$ (because $\mathcal{CA}$ is flasque), we see that $W_B^i(\mathcal{SA}) = W_B^{i+1}(\mathcal{A})$. For Karoubi's Wittgroups we have $W_{-i}(\mathcal{SA}) = W_{-i-1}(\mathcal{A})$. For $i = 0$ we find $W_{-1}(\mathcal{A}) = W_0(\mathcal{SA}) = W_B^0(\mathcal{SA}) = W_B^1(\mathcal{A})$. The result then follows by induction. □

Definition 11.3 *Let G_* be the following simplicial additive category with duality:*
Its objects are triples (P_*, T_{**}, ϕ_*) where P_* is a functor $P_* : \mathbf{n} \to (P(A))$ (recall that $\mathbf{n}$ is the category $\mathbf{n} = n' < \ldots < 0' < 0 < \ldots < n$), T_{**} is an object of $R_*\mathcal{T}_S$ and ϕ_* is a natural transformation $\phi_* : P_* \to T_{n'*}$ such that $P_k \rightarrowtail P_l \twoheadrightarrow T_{kl}$ is a short exact sequence $\forall k \leq l$. Morphisms are natural transformations. The simplicial structure and the duality are induced by those of $R_*\mathcal{T}_S$ (recall that if $P_k \rightarrowtail P_l \twoheadrightarrow T_{kl}$ is a short exact sequence, then $Hom_A(P_l, A) \rightarrowtail Hom_A(P_k, A) \twoheadrightarrow Ext^1_A(T_{kl}, A)$ is also a short exact sequence).

We have simplicial functors of simplicial additive categories with duality

$$P(A) \stackrel{\iota_*}{\to} G_* \stackrel{\pi_*}{\to} R_* \mathcal{T}_S$$

where $P(A)$ is the constant simplicial category. The functor ι_k is given by the constant inclusion mapping P to $(P_*, 0_{**}, 0_*)$ and π_k is the projection mapping (P_*, T_{**}, ϕ_*) to T_{**}.

Proposition 11.4 *The simplicial functors constructed above induce* $\forall k \geq 0$ *a short exact sequence of triangulated categories*

$$D^b(P(A)) \stackrel{D^b(\iota_k)}{\to} D^b(G_k) \stackrel{D^b(\pi_k)}{\to} D^b(R_k \mathcal{T}_S).$$

Proof: We have to show that a) $D^b(P(A))$ is equivalent to the full subcategory of objects of $D^b(G_k)$ whose image in $D^b(R_k \mathcal{T}_S)$ is isomorphic to 0 and b) that we have an equivalence of categories $D^b(G_k)/D^b(P(A)) \simeq D^b(R_k \mathcal{T}_S)$. First observe that for any additive category $\mathcal{A}$, the bounded derived category $D^b(\mathcal{A})$ is just the bounded homotopy category (i.e., $Ch^b(\mathcal{A})$ modulo chain homotopies). To prove a), we first consider the case $k = 0$. An object of $D^b(G_0)$ is just a bounded complex of short exact sequences $M_* \rightarrowtail N_* \twoheadrightarrow T_*$. with M_k and N_k projective and T_k a $S-$torsion module $\forall k$. Assume that T_* is contractible. We pretend that this object is then homotopy equivalent to $M_* \rightarrowtail M_* \twoheadrightarrow 0_*$, i.e., is in the image of $D^b(P(A))$. Consider the cone of $(M_* \rightarrowtail M_* \twoheadrightarrow 0_*) \to (M_* \rightarrowtail N_* \twoheadrightarrow T_*)$. We have to construct a null-homotopy of this chain complex in G_0 which is nothing else but a compatible choice (all squares commute) of null homotopies of the three chain complexes. This can be done applying Lemma 11.5 below which was shown to me by P. Balmer.

For $k > 0$, we first use the lemma to construct a null homotopy for P_k using the one given on $T_{k'k}$. The others are then again a consequence of functoriality, and they are compatible because the null homotopies of the T_{ij} are.

To prove b), check that the functor $F : Ch^b(G_k) \to Ch^b(R_k \mathcal{T}_S)$ is full and essentially surjective, hence $\overline{F} : D^b(G_k)/D^b(P(A)) \to D^b(R_k \mathcal{T}_S)$ given by the universal property is also full and essentially surjective.

To prove that $\overline{F}$ is faithful, fix a morphism $a : A_* \to A'_*$ in $Ch^b(G_k)$ mapping to 0 in $D^b(R_k \mathcal{T}_S)$. Hence $F(a)$ factors through a contractible object in $Ch^b(R_k \mathcal{T}_S)$. Choose a preimage B_* in $Ch^b(G_k)$ of this contractible object. Following Lemma 11.5., B_* is homotopy equivalent to an element in $Ch^b(P(A))$. Choosing arbitrary liftings of the morphisms $A_* \stackrel{b}{\to} B_* \stackrel{c}{\to} A'_*$, we do not have $a = c \circ b$ in general. But the composition of $A_* \stackrel{(1,b)}{\longrightarrow} A_* \oplus B_* \stackrel{(a,-c)}{\longrightarrow} A'_*$ maps to 0 in $Ch^b(R_k \mathcal{T}_S)$. It therefore suffices to consider maps a with $F(a) = 0$ already in $Ch^b(R_k \mathcal{T}_S)$, and those obviously factor through objects of $Ch^b(P(A))$. □

Lemma 11.5 *Any bounded complex of short exact sequences* $(P_* \rightarrowtail Q_* \twoheadrightarrow T_*)$ *with* P_* *and* T_* *contractible and* P_k *and* Q_k *projective* $\forall k$ *is itself contractible.*

Proof: Consider the following diagram

$$
\begin{array}{ccccc}
P_{n+1} & \overset{f_{n+1}}{\rightarrowtail} & Q_{n+1} & \overset{g_{n+1}}{\twoheadrightarrow} & T_{n+1} \\
\downarrow{\scriptstyle a_{n+1}} & & \downarrow{\scriptstyle b_{n+1}} & & {\scriptstyle t_{n+1}}\uparrow\downarrow{\scriptstyle c_{n+1}} \\
P_n & \overset{f_n}{\rightarrowtail} & Q_n & \overset{g_n}{\twoheadrightarrow} & T_n \\
{\scriptstyle r_n}\uparrow\downarrow{\scriptstyle a_n} & & {\scriptstyle s_n}\uparrow\downarrow{\scriptstyle b_n} & & {\scriptstyle t_n}\uparrow\downarrow{\scriptstyle c_n} \\
P_{n-1} & \overset{f_{n-1}}{\rightarrowtail} & Q_{n-1} & \overset{g_{n-1}}{\twoheadrightarrow} & T_{n-1}
\end{array}
$$

where r_j, s_j and t_j are the null-homotopies ($id_{P_j} = a_{j+1} \circ r_{j+1} + r_j \circ a_j$ etc) which are compatible for $j \leq n$ by induction. Given t_{n+1}, we then have to construct a morphism s_{n+1} such that $g_{n+1} \circ s_{n+1} = t_{n+1} \circ g_n$ and that $b_{n+1} \circ s_{n+1} + s_n \circ b_n = 1_{Q_n}$. As Q_n is projective, there is a morphism $v : Q_n \to Q_{n+1}$ such that $t_{n+1} \circ g_n = g_{n+1} \circ v$. Define $e = 1_{Q_n} - s_n \circ b_n - b_{n+1} \circ v$. Then check that $b_n \circ e = 0$ and $g_n \circ e = 0$. As $f_n = ker(g_n)$, there exists a $m : Q_n \to P_n$ such that $e = f_n \circ m$. Let $l_j : P_{j-1} \to P_j$ be a null homotopy defined on the whole complex P_* (which is not assumed to be compatible with anything else). We have $id_{P_n} - r_n \circ a_n = (a_{n+1} \circ l_{n+1} + l_n \circ a_n) \circ (id_{P_n} - r_n \circ a_n) = a_{n+1} \circ l_{n+1} \circ (id_{P_n} - r_n \circ a_n)$ using $a_n \circ r_n = id_{P_{n-1}} - r_{n-1} \circ a_{n-1}$. Setting $k := l_{n+1} \circ (id_{P_n} - r_n \circ a_n)$ and $s_{n+1} := v + f_{n+1} \circ k \circ m$ we are done (check that $id_{P_n} = r_n \circ a_n + a_{n+1} \circ k$ and thus $e = b_{n+1} \circ f_{n+1} \circ k \circ m$). Then we obtain a contraction of P_* compatible with the others by functoriality of the kernel. □

We will now construct an additive category $C(\iota_k) := G_k \amalg_{P(A)} \mathcal{C}P(A)$ associated to the additive functor $\iota_k : P(A) \to G_k$ where $\mathcal{C}$ is the cone of section 6. This will be a kind of K-theoretic cofiber of ι_k. Recall that a preadditive category is a category whose Hom-sets are abelian groups and composition is bilinear. In contrast to the more restrictive definition of an additive category, we do not demand the existence of finite sums. We let $\underline{pr\,add\,cat}$ be the category of small preadditive categories. The construction above will be carried out in two steps. First, we work on the level of preadditive categories. Second we define a functor $L : \underline{pr\,add\,cat} \to \underline{add\,cat}$.

Assume that we have a functor $F : \mathcal{D} \to \underline{pr\,add\,cat}$ where $\mathcal{D}$ might be any small category. We define the colimit of this functor by $Obj(\mathrm{colim}_{\mathcal{D}} F) = \mathrm{colim}_{d \in \mathcal{D}} Obj(F(d))$. For two objects $[A]$ and $[B]$ of $\mathrm{colim}_{\mathcal{D}} F$ represented by $A \in F_a$ and B in F_b, $a, b \in \mathcal{D}$, we have

$$\mathrm{colim}_{\mathcal{D}} F(A,B) = \bigoplus_J F_{d_1}(X_1,Y_1) \otimes \ldots \otimes F_{d_n}(X_n,Y_n)/\sim$$

with $J = \{n \geq 1, d_j \in Obj(\mathcal{D}), X_j, Y_j \ objects\ of\ F_{d_j}, 1 \leq j \leq n, [X_1] = [A], [Y_i] = [X_{i+1}], 1 \leq i \leq n, [Y_n] = [B]\ in\ Obj(\operatorname{colim}_{\mathcal{D}} F)\}$ and the equivalence relation is generated by $f \otimes g \sim f \circ g$ when the source of f is the target of g and $f \sim F(\gamma)(f)$ for γ a morphism in $\mathcal{D}$.

In our case $\mathcal{D} = \{* \leftarrow * \rightarrow *\}$ and the functor $P(A) \rightarrow \mathcal{C}P(A)$ is a full inclusion. Hence $Obj(C(\iota_k)) = (Obj(\mathcal{C}P(A)) - Obj(P(A))) \amalg Obj(G_k)$. We obtain a commutative diagram of additive categories with duality

$$\begin{array}{ccc} P(A) & \xrightarrow{\iota_k} & G_k \\ \downarrow j & cocart & \downarrow \bar{j} \\ \mathcal{C}P(A) & \xrightarrow{\bar{\iota_k}} & C(\iota_k) \\ \downarrow \pi & & \downarrow \bar{\pi} \\ \mathcal{S}P(A) & = & \mathcal{S}P(A) \end{array}$$

where $\bar{\pi}$ is induced by the zero map $G_k \rightarrow \mathcal{S}P(A)$ and the universal property of $C(\iota_k)$. As j is a full inclusion, we do not need to consider morphisms in $F_{P(A)}$ (see our definition of the morphisms in $\operatorname{colim}_{\mathcal{D}} F$). Thus any morphism in $C(\iota_k)$ from $\bar{j}(B)$ to $\bar{\iota_k}(U)$ (with B an object in G_k and U in G_k) can be represented as $\Sigma_i u_i \otimes b_i$ with $B \stackrel{b_i}{\rightarrow} \iota_k(P_i)$ a morphism in G_k, $j(P_i) \stackrel{u_i}{\rightarrow} U$ a morphism in $\mathcal{C}P(A)$ and P_i an object in $P(A)$. Moreover, $\mathcal{C}P(A)$ is a full subcategory of $C(\iota_k)$.

Now comes the second step. We define the functor $L : \underline{pr\,add\,cat} \rightarrow \underline{add\,cat}$ as follows: an object of $L\mathcal{A}$ is given by an $l \in \mathbf{N}$ and an object A_i in $\mathcal{A}$ for all $0 \leq i \leq l$. We write $(A_1, ..., A_l)$ for this object. A morphism from $(A_1, ..., A_l)$ to $(B_1, ..., B_j)$ is given by a collection of morphisms $\phi_{j,i} : A_i \rightarrow B_j$ for all pair $(0,0) \leq (i,j) \leq (l,m)$. The composition is defined as usual for matrices. We also allow the 0-tuple $0 = ()$. (For example, the category of free R-modules of finite type is equivalent to $L(R)$.) We have an inclusion $\mathcal{A} \rightarrow L\mathcal{A}$ given by $A \mapsto 0(A)$. For any small category $\mathcal{D}$ and any functor $F : \mathcal{D} \rightarrow \underline{pr\,add\,cat}$ we have canonical functors

$$\operatorname{colim}_{\mathcal{D}} F \stackrel{i}{\rightarrow} \operatorname{colim}_{\mathcal{D}} LF \stackrel{j}{\rightarrow} L \operatorname{colim}_{\mathcal{D}} F$$

Thew composition $j \circ i$ is fully faithful and injective on the objects. Hence i is faithful and injective on objects as well. Calculations similar to the above ones show that i is also full. Any object of LF_d is a direct sum of objects of F_d. As additive functors preserve direct sums, any obect of $\operatorname{colim}_D LF$ is a direct sum of objects in the image of i. Putting all this together, we obtain an equivalence of categories $\operatorname{colim}_{\mathcal{D}} LF \stackrel{\simeq}{\rightarrow} L \operatorname{colim}_{\mathcal{D}} F$. Hence, we assume from now on that $C(\iota_k)$ is additive.

Both columns of our diagram fulfill the conditions of Corollary 6.7. To see this for the second column, observe that any object in the image of $\bar{j}$ is filtered by itself. Now fix $\bar{\iota_k}(U)$ and a $P(A)$-filtration of U given by $\{(p_i, s_i) : P_i \twoheadrightarrow U, i \in I\}$. We claim that its image under $\bar{\iota_k}$ is a G_k-filtration of $\bar{\iota_k}(U)$. The image of a filtered category is a filtered category. Consider a morphism from B to $\iota_k(U)$ represented by $\phi = \Sigma_{j=1}^n f_j \otimes g_j \in C(\iota_k)(\bar{\iota_k}(U), B)$. The f_j have sources in $P(A)$ and target U. As U is $P(A)$-filtered, we have $f_j = s \circ f'_j$ where $(p, s) : P \twoheadrightarrow U$ and hence

$$\phi = \Sigma_{j=1}^n s \circ f'_j \otimes g_j = s \otimes (\Sigma_{j=1}^n \iota_k(f'_j) \circ g_j) = \bar{\iota_k}(s) \circ g$$

where $g = \Sigma_{j=1}^n \iota_k(f'_j) \circ g_j$ a morphism in G_k. The case of the other factorization into an admissible epimorphism and a morphism in G_k is similar. Hence by Corollary 6.7 we obtain a homotopy fibration

$$iP(A)_h^+ \to i(G_k)_h^+ \to i\widetilde{C(\iota_k)}_h^+$$

where as before $\widetilde{C(\iota_k)}$ stands for the pseudo-abelianization of $C(\iota_k)$.

Lemma 11.6 *The functor $\widetilde{C(\iota_k)} \to R_k\mathcal{T}_S$ given by the universal property induces a homotopy equivalence $i\widetilde{C(\iota_k)}_h^+ \simeq iR_k^h\mathcal{T}_S^+$.*

Proof: Consider the functor $C(\iota_k) \to R_k\mathcal{T}_S$ defined by the universal property of the pushout. As $R_k\mathcal{T}_S$ is pseudo-abelian, we have an induced functor $\phi : \widetilde{C(\iota_k)} \to R_k\mathcal{T}_S$. Applying the Fundamental Theorem for additive pseudo-abelian categories, we have two commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
{}_\epsilon U_0(\widetilde{C(\iota_k)}) & \longrightarrow & K_0(\widetilde{C(\iota_k)}) & \xrightarrow{H_*} & {}_\epsilon K_0^h(\widetilde{C(\iota_k)}) & \longrightarrow & {}_\epsilon W_0(\widetilde{C(\iota_k)}) & \longrightarrow & 0 \\
\downarrow {\scriptstyle {}_\epsilon U_0(\phi)} & & \downarrow {\scriptstyle K_0(\phi)} & & \downarrow {\scriptstyle {}_\epsilon K_0^h(\phi)} & & \downarrow {\scriptstyle {}_\epsilon W_0(\phi)} & & \\
{}_\epsilon U_0(R_k\mathcal{T}_S) & \longrightarrow & K_0(R_k\mathcal{T}_S) & \xrightarrow{H_*} & {}_\epsilon K_0^h(R_k\mathcal{T}_S) & \longrightarrow & {}_\epsilon W_0(R_k\mathcal{T}_S) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
{}_\epsilon K_0^h(\widetilde{C(\iota_k)}) & \xrightarrow{F_*} & K_0(\widetilde{C(\iota_k)}) & \longrightarrow & {}_{-\epsilon}U_0(\widetilde{C(\iota_k)}) & \longrightarrow & {}_\epsilon K_{-1}^h(\widetilde{C(\iota_k)}) & \xrightarrow{F_*} & K_{-1}(\widetilde{C(\iota_k)}) \\
\downarrow {\scriptstyle {}_\epsilon K_0^h(\phi)} & & \downarrow {\scriptstyle K_0(\phi)} & & \downarrow {\scriptstyle {}_{-\epsilon}U_0(\phi)} & & \downarrow {\scriptstyle {}_\epsilon K_{-1}^h(\phi)} & & \downarrow {\scriptstyle {}_\epsilon K_{-1}^h(\phi)} \\
{}_\epsilon K_0^h(R_k\mathcal{T}_S) & \xrightarrow{F_*} & K_0(R_k\mathcal{T}_S) & \longrightarrow & {}_{-\epsilon}U_0(R_k\mathcal{T}_S) & \longrightarrow & {}_\epsilon K_{-1}^h(R_k\mathcal{T}_S) & \xrightarrow{F_*} & K_{-1}(R_k\mathcal{T}_S)
\end{array}$$

We want to show that $K_n(\phi)\ \forall\, n \geq -1, W_0(\phi)$ and $K_{-1}^h(\phi)$ are isomorphisms. This would imply that $K_0^h(\phi)$ is also an isomorphism, and the Karoubi induction we already used in the proof of Theorem 9.5 would yield that $K_n^h(\phi)$ is an isomorphism for any $n \geq -1$. That $K_n(\phi)$ is an isomorphism for $n \geq 0$ follows from comparing the two long exact sequences of K-groups we get from [PW],[TT, 1.8.2, 1.11.7] and observing that both $K_0(G_k) \to K_0(\widetilde{C(\iota_k)})$ and $K_0(G_k) \to K_0(R_k\mathcal{T}_S)$ are surjective because $K_{-1}(P(A)) = 0$ for A a regular ring [Bass, p.685].

As $R_k\mathcal{T}_S$ is the underlying additive category of an exact category with admissible kernels, we have $K_{-1}(R_k\mathcal{T}_S) = 0$ (see [Sch4]). But K_{-1} of $\widetilde{C(\iota_k)}$ also vanishes because it is part of a long exact sequence of K-groups in positive and negative degrees [PW],[Sch1, Theorem 7.12] with $K_{-1}(G_k) = 0$ (again this is a consequence of [Sch4]) and $K_{-1}(\widetilde{\mathcal{S}P(A)}) \cong K_{-2}(P(A)) = 0$. Applying Lemma 11.2, we are reduced to show that $W^0_B(\phi)$ and $W^1_B(\phi)$ are isomorphisms. Using [Sch1, Theorem 10.1] and Proposition 11.4, we get two exact sequences of triangulated categories

$$D^b(G_k) \to D^b(\widetilde{C(\iota_k)}) \to D^b(\widetilde{SP(A)})$$

and

$$D^b(P(A)) \to D^b(G_k) \to D^b(R_k\mathcal{T}_S)$$

and hence by the localization exact sequence [Bal1, Theorem 6.2] we have long exact sequences for W_B. Again we want to conclude by the five lemma. As W_B only depends on the bounded derived category and D^b commutes with the pseudo-abelianization [BSch] it is sufficient to prove that $D^b(C(\iota_k))$ is pseudo-abelian. For this, consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
K_1(\mathcal{S}P(A)) & \longrightarrow & K_0(D^b(G_k)) & \longrightarrow & K_0(D^b(C\iota_k)) & \longrightarrow & K_0(D^b(\mathcal{S}P(A))) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
K_1(\widetilde{\mathcal{S}P(A)}) & \longrightarrow & K_0(\widetilde{D^b(G_k)}) & \longrightarrow & K_0(\widetilde{D^b(C\iota_k)}) & \longrightarrow & K_0(D^b(\widetilde{\mathcal{S}P(A)})).
\end{array}$$

By cofinality, α is an isomorphism, and β is an isomorphism again by [BSch] because G_k is pseudo-abelian. As $\mathcal{C}P(A) \to \mathcal{S}P(A)$ is surjective on objects, $K_0(D^b(\mathcal{S}P(A))) = 0$, and $K_0(D^b(\widetilde{\mathcal{S}P(A)})) = 0$ because $K_0(D^b(\widetilde{\mathcal{S}P(A)})) \cong K_0(\widetilde{\mathcal{S}P(A)}) \cong K_{-1}(P(A)) = 0$ as A is regular. Hence γ is an isomorphism, and by [Th3, theorem 2.1] we know that $D^b(C(\iota_k))$ is already pseudo-abelian.
□

Summing up, we have a homotopy fibration

$$(iP(A))^+_h \xrightarrow{\iota_k} (iG_k)^+_h \xrightarrow{\pi_k} (iR^h_k\mathcal{T}_S)^+ \;\forall k.$$

Now we would like to apply the Bousfield-Friedlander Theorem [BF, Theorem B.4] to conclude that the realization

$$(iP(A))^+_h \xrightarrow{\iota_*} |(iG_*)^+_h| \xrightarrow{\pi_*} |(iR_*\mathcal{T}_S)^+_h|$$

is still a homotopy fibration. For this we have to check the two conditions of [BF]. The "π_*-Kan-condition" holds because we are dealing with simplicial H-groups. But the morphism $\pi_0((iG_k)^+_h) \xrightarrow{\pi_k} \pi_0((iR_k\mathcal{T}_S)^+_h)$ might not be surjective in general. But if we write $\hat{\mathcal{T}}_k$ for the full subcategory of $i(R_k(\mathcal{T}_S)_h)^+$ consisting of those components lying in the image of $i(G_k)^+_h \to i(R_k(\mathcal{T}_S)_h)^+$, then we can prove the following:

Lemma 11.7 *There is a homotopy fibration*

$$|\hat{\mathcal{T}}_*| \to |i(R_*(\mathcal{T}_S)_h)^+| \to |L_*|$$

where L_* is a constant simplicial group.

Proof: We define $L_k := coker(\pi_0(\hat{\mathcal{T}}_k) \to \pi_0(i(R_k(\mathcal{T}_S)_h)^+)$. By the Bousfield-Friedlander Theorem, we have a homotopy fibration $|\hat{\mathcal{T}}_*| \to |i(R_*(\mathcal{T}_S)_h)^+| \to |L_*|$. Denote by L_k^W the cokernel of $W_0(G_k) \to W_0(R_k(\mathcal{T}_S))$. The induced map on the cokernels $L_k \to L_k^W$ is an isomorphism since $K_0(G_k) \to K_0(R_k(\mathcal{T}_S))$ is an epimorphism. Consider the following (up to sign) commutative diagram

$$\begin{array}{ccccc} W_0(G_k) & \longrightarrow & W_0(R_k(\mathcal{T}_S)) & \xrightarrow{\delta_k} & W_B^1(P(A)) \\ & & \downarrow{\scriptstyle \alpha} & & \| \\ W_0(P(S^{-1}A)) & \longrightarrow & W_0(\mathcal{T}_S) & \xrightarrow{\delta} & W_B^1(P(A)) \end{array}$$

where the $\mathcal{T}_S$ in the bottom row is equipped with the non-split exact structure. Both rows are exact by Balmer's localization theorem [Bal1] because we have exact sequences of bounded derived categories (Proposition 11.4 and [Ke2, Example p.17], beware of the degree shift depending on the different dualities on $\mathcal{T}_S$). The map α is nothing else but the projection on the $T_{0'0}$-entry and its hermitian form, hence is surjective. It follows that $L_k \cong L_k^W = im(\delta_k) = im(\delta)$ does not depend on k. Considering L_0 as a constant simplicial group, this implies morover that $L_0 \to L_*$ is an isomorphism of simplicial groups. □

Hence $\pi_n(|\hat{\mathcal{T}}_*|) \to \pi_n(|i(R_*(\mathcal{T}_S))_h^+|)$ is an isomorphism $\forall n \geq 1$ and a monomorphism whose cokernel is the image of δ if $n = 0$. Observe that the isomorphism (by the five lemma) $W_0(C(\iota_k)) \cong W_0(R_k(\mathcal{T}_S))$ shows that the same argument yields a homotopy fibration $|\hat{C}_*| \to |i(C(\iota_*))_h^+| \to |L_*|$ where $\hat{C}_k$ is the full subcategory of the components lying in the image of $i(G_k)_h^+ \to i(C(\iota_k))_h^+$.

We also observe [Sch3] that in our case realization commutes with group completion. More precisely, we have $|(iG_*)_h^+| \simeq |(iG_*)_h|^+$ and $|(iR_*\mathcal{T}_S)_h^+| \simeq |(iR_*\mathcal{T}_S)_h|^+ \simeq |(iR_*\mathcal{T}_S)_h|$ as the latter one is already an H-group.

So the last step of the proof is the following

Proposition 11.8 *There is a homotopy equivalence*

$$iP(S^{-1}A)_h \simeq |i(G_*)_h|.$$

Proof: For any category with duality $\mathcal{C}$, we denote by $n_*\mathcal{C}$ its nerve (a simplicial set), and by $N_*\mathcal{C}$ the simplicial category whose objects in degree k are chains of morphisms of length k and morphisms are commutative diagrams. Further, $N_*^{iso}\mathcal{C}$ is the full simplicial subcategory of $N_*\mathcal{C}$ whose objects are those diagrams containing only isomorphisms. We write $\mathcal{D}_*^e$ for the edgewise subdivision of a simplicial category $\mathcal{D}_*$. Finally, recall that for a category $\mathcal{C}$ with duality, $\mathcal{C}_{hd}$ denotes the category of all hermitian objects including the degenerate ones. Writing $\{\overset{\sim}{\rightarrowtail}\}P(A)$ for the subcategory of $P(A)$ whose morphisms are monomorphisms with S-torsion as cokernel, we have $i(G_*)_h = (i(N_*^e\{\overset{\sim}{\rightarrowtail}\}P(A)))_{hd}$ and $n_*((\{\overset{\sim}{\rightarrowtail}\}P(A))_{hd}) \simeq diag \circ n_* N_\sharp^{iso}((\{\overset{\sim}{\rightarrowtail}\}P(A))_{hd}) = diag \circ n_\sharp i N_*[(\{\overset{\sim}{\rightarrowtail}\}P(A))_{hd}]$.
Now we have the following functor of simplicial categories $F : (i(N_k^e\{\overset{\sim}{\rightarrowtail}\}P(A)))_{hd} = iN_k[(\{\overset{\sim}{\rightarrowtail}\}P(A))_{hd}]$ which is defined on objects by mapping

$$
\begin{array}{ccccccccccc}
P_{k'} & \overset{\sim}{\underset{\beta_{k'}}{\rightarrowtail}} & P_{(k-1)'} & \cdots & P_{0'} & \overset{\sim}{\underset{\beta_{0'}}{\rightarrowtail}} & P_0 & \cdots & P_{k-1} & \overset{\sim}{\underset{\beta_{k-1}}{\rightarrowtail}} & P_k \\
{\scriptstyle \psi_{k'}=\psi_k^*}\downarrow\cong & & \cong\downarrow{\scriptstyle \psi_{(k-1)'}=\psi_{k-1}^*} & & {\scriptstyle \psi_{0'}=\psi_0^*}\downarrow\cong & & \cong\downarrow{\scriptstyle \psi_0=\psi_{0'}^*} & & {\scriptstyle \psi_{k-1}=\psi_{(k-1)'}^*}\downarrow\cong & & {\scriptstyle \psi_k=\psi_{k'}^*}\downarrow\cong \\
P_k^* & \overset{\sim}{\underset{\beta_{(k-1)^*}}{\rightarrowtail}} & P_{k-1}^* & \cdots & P_0^* & \overset{\sim}{\underset{\beta_{0'}^*}{\rightarrowtail}} & P_{0'}^* & \cdots & P_{(k-1)'}^* & \overset{\sim}{\underset{\beta_{k'}^*}{\rightarrowtail}} & P_{k'}^*
\end{array}
$$

to

$$
\begin{array}{ccccccc}
P_{k'} & \overset{\sim}{\underset{\beta_{k'}}{\rightarrowtail}} & P_{(k-1)'} & \cdots & P_{1'} & \overset{\sim}{\underset{\beta_{1'}}{\rightarrowtail}} & P_{0'} \\
{\scriptstyle \psi_k\beta_{k-1}\dots\beta_{k'}}\downarrow\sim & & \sim\downarrow{\scriptstyle \psi_{k-1}\beta_{k-2}\dots\beta_{(k-1)'}} & & {\scriptstyle \psi_1\beta_0\dots\beta_{1'}}\downarrow\sim & & \sim\downarrow{\scriptstyle \psi_0\beta_{0'}} \\
P_{k'}^* & \overset{\sim}{\underset{\beta_{k'}^*}{\leftarrowtail}} & P_{(k-1)'}^* & \cdots & P_{1'}^* & \overset{\sim}{\underset{\beta_{1'}^*}{\leftarrowtail}} & P_{0'}^*
\end{array}
$$

and this forces the definition of F on morphisms (i.e., $(f_j : P_j \to Q_j)_{k' \le j \le k}$ is sent to $(f_j : P_j \to Q_j)_{k' \le j \le 0})$. One checks that this is fully faithful and essentially surjective.
Finally, we see that the localization functor $(\{\overset{\sim}{\rightarrowtail}\}P(A))_{hd} \to (iP(S^{-1}A))_{hd} = iP(S^{-1}A)_h$ is a homotopy equivalence. This follows from Quillen's Theorem A and the argument of Lemma 8.5 $\square$

12 Conclusion (en français)

Dans les sections 6,7 et 8 nous avons établi le délaçage pour une catégorie additive hermitienne ainsi qu'un théorème de localisation et un théorème de dévissage pour la K-théorie hermitienne des anneaux héréditaires. Les analogues de ces résultats en K-théorie ordinaire sont connus depuis longtemps. Chaque fois que nous avons une construction ou un théorème en K-théorie ordinaire, la question de l'existence d'un résultat similaire en K-théorie hermitienne se pose. Bien sûr ceci nécessite d'abord l'existence d'un foncteur de dualité qui existe pour la catégorie des modules projectifs de type fini ou encore pour les fibrés vectoriels sur un schéma, mais qui n'existe pas pour la catégorie des modules de type fini ou des faisceaux cohérents en général.

Deux résultats essentiels de ma thèse sont la "bonne" définition de la K-théorie hermitienne d'une catégorie exacte avec dualité - en utilisant le Théorème 5.7 - et la description de la $\mathcal{U}$-théorie avec un modèle à la Waldhausen (Proposition 7.2). Le Théorème de Localisation après Dévissage (Theorem 9.7) nous donne une suite exacte longue dont tous les objects sont définis depuis longtemps (Karoubi avait conjecturé l'exactitude de cette suite déjà dans [Ka3]). Mais pour la démonstration (qui passe par la catégorie des modules de torsion) nous avons eu besoin de la théorie pour des catégories exactes avec dualité en toute généralité comme nous l'avons développée dans les paragraphe 5 et 7. Il s'agit d'un très joli exemple où la généralisation et les considérations plutôt abstraites impliquent finalement des résultats très concrets comme l'isomorphisme (cf Théorème 9.11)

$$ {}_1K_n^h(A) \overset{\cong}{\to} {}_1K_n^h(F) \quad \forall n \equiv 3,4 \, mod \, 8 $$

où A est l'anneau des entiers dans un corps de nombres F.

Pour conclure, nous allons indiquer quelques problèmes ouverts qui nous semblent intéressants. Au lieu de donner une liste des théorèmes en K-théorie ordinaire qu'on souhaiterait établir en K-théorie hermitienne, je voudrais dire quelques mots sur les stratégies différentes pour les démontrer. D'abord, il serait bien de démontrer le Théorème 3.7 pour des catégories exactes avec dualité en général ce qui nous permettrait d'appliquer la stratégie de Karoubi, comme nous l'avons fait pour démontrer le Théorème 9.5. Le problème principal pour le démontrer semble l'absence des délaçages explicites de la K-théorie hermitienne pour les catégories exactes avec dualité. Ensuite, il sera souvent plus commode de travailler avec notre description à la Waldhausen qui nous permet de reduire beaucoup de choses des catégories exactes aux catégories additives dont la K-théorie hermitienne est beaucoup mieux comprise (Théorème Fondamental, délaçage etc).

Enfin, on pourrait passer d'une catégorie exacte avec dualité $(\mathcal{C},\ ^t)$ à sa catégorie dérivée bornée $D^b(\mathcal{C})$ (qui est muni d'une structure triangulée et d'une dualité induite). Parfois, on a un foncteur de dualité sur $D^b(\mathcal{C})$ même si un tel n'existait pas pour la catégorie exacte de départ (cf Définition 10.1 et [Har]). On aimerait alors exploiter la machinerie de Thomason [TT] pour les "catégories (bicompliciales) de Waldhausen avec dualité".

Balmer [Bal1][Bal2] a défini des groupes de Witt pour une catégorie triangulée avec dualité tels que son groupe de Witt $W_B(D^b(\mathcal{C})_h)$ coïncide avec notre groupe de Witt $W(\mathcal{C}_h)$ (cf [Bal2] et Théorème 10.3). Il est naturel de se demander si ses groupes de Witt translatés coïncident avec nos groupes de Witt supérieurs (Définition 9.10) à la 2-torsion près. Notre démonstration a d'ailleurs utilisé certains résultats sur ces groupes de Witt triangulaires.

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