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On well generated triangulated categories

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Contents

1. Introduction

One of the aims of the present thesis is to try and answer the question: what is the analogue, in the realm of triangulated categories, of the notion of Grothendieck category in the realm of abelian categories? The best way to proceed seemed to us that of lifting perhaps the most important theorem involving these notions from the abelian to the triangulated world. The theorem we are speaking of is due to Popescu-Gabriel:

Theorem 1.1 (Popescu-Gabriel [31]). *Let $\mathcal{T}$ be a Grothendieck category. Then the following statements are equivalent:*

(*i*) *$G \in \mathcal{T}$ is a generator of $\mathcal{T}$;*
(*ii*) *the functor* $\mathrm{Hom}(G,-) : \mathcal{T} \longrightarrow \mathrm{Mod}(A)$, *where* $A = \mathrm{Hom}(G,G)$, *is a localization.*

We refer to [30], [35], [23], [20] for complete proofs of the theorem.

In his book [27, Def. 1.15, p. 15], A. Neeman defined the class of *well generated triangulated categories*. It turns out that this class is a very good generalization to higher cardinals of the concept of *compactly generated triangulated category*. In fact, it preserves the most interesting properties, *e.g.* the validity of the Brown representability theorem [5] and of the Thomason localization theorem [36, Key Proposition 5.2.2, p. 338], and at the same time introduces new good features, such as the stability of the new class under localizations (assuming the quite weak hypothesis that the kernel of the localization functor is generated by a set of objects). H. Krause characterized the class of categories introduced by Neeman as follows [17]. Let $\mathcal{T}$ be a triangulated category with suspension functor Σ admitting arbitrary set-indexed coproducts. $\mathcal{T}$ is well generated in the sense of Krause [17] if and only if there exists a set $\mathcal{G}_0$ of objects with $\Sigma\mathcal{G}_0 = \mathcal{G}_0$ satisfying the conditions:

(G1) an object $X \in \mathcal{T}$ is zero provided that $\mathcal{T}(G,X) = 0$ for all G in $\mathcal{G}_0$;
(G2) for each family of morphisms $f_i : X_i \to Y_i$, $i \in I$, the induced map

$$\mathcal{T}(G, \coprod_{i\in I} X_i) \to \mathcal{T}(G, \coprod_{i\in I} Y_i)$$

is surjective for all $G \in \mathcal{G}_0$ provided that the maps

$$\mathcal{T}(G, X_i) \to \mathcal{T}(G, Y_i)$$

are surjective for all $i \in I$ and all $G \in \mathcal{G}_0$;
(G3) there is some regular cardinal α such that the objects $G \in \mathcal{G}_0$ are *α-small*, *i.e.* for each family of objects X_i, $i \in I$, of $\mathcal{T}$, each morphism

$$G \to \coprod_{i\in I} X_i$$

factors through a subsum $\coprod_{i\in J} X_i$ for some subset J of I of cardinality strictly smaller than α.

In the case $\alpha = \aleph_0$, the $\aleph_0$-*compact* objects are the *compact* objects of the classical literature [24], [25] and the definition of well generated category reduces to that of *compactly generated* one. Well generated triangulated categories arise very naturally when one localizes compactly generated ones, as it will be shown in detail in section 4.

For example, the unbounded derived category $\mathcal{D}(\mathrm{Sh}(X))$ of sheaves of abelian groups on a topological space X is well generated since it is a localization of the derived category of presheaves $\mathcal{D}(\mathrm{Presh}(X))$, which is compactly generated. However, Neeman shows in [28] that not all derived categories of sheaves are compactly generated. An example is the category $\mathcal{D}(\mathrm{Sh}(X))$ where X is a connected, non compact real manifold of dimension at least one; in this case, there do not exist non zero compact objects. In the same article, it is shown that the derived categories of Grothendieck categories are always well generated. Another large class of examples arises when one localizes the derived category $\mathcal{DA}$ of a small DG category $\mathcal{A}$ at the localizing subcategory generated by a set of objects. Indeed, since $\mathcal{DA}$ is a compactly generated triangulated category, such a localization is always well generated. Now we can state the main result of this thesis. It also gives a positive answer to Drinfeld's question [29] whether all well generated categories arise as localizations of module categories over DG categories, for the class of *algebraic* triangulated categories. Here algebraic means triangle equivalent to the stable category of a Frobenius category. One can show that each algebraic triangulated category is triangle equivalent to a full triangulated subcategory of the category up to homotopy of complexes over some additive category.

Theorem 1.2. *Let $\mathcal{T}$ be an algebraic triangulated category. Then the following statements are equivalent:*

- (i) *$\mathcal{T}$ is well generated;*
- (ii) *there is a small* DG *category $\mathcal{A}$ such that $\mathcal{T}$ is triangle equivalent to a localization of $\mathcal{DA}$ with respect to a localizing subcategory generated by a* set *objects.*

Moreover, if (i) holds and $\mathcal{G} \subseteq \mathcal{T}$ is a full triangulated subcategory stable under coproducts of strictly less than α factors and satisfying (G1), (G2) *and* (G3) *for some regular cardinal α, the functor*

$$\mathcal{T} \longrightarrow \mathrm{Mod}\,\mathcal{G}\,,\ X \longmapsto \mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{G}}$$

lifts to a localization $\mathcal{T} \longrightarrow \mathcal{D}(\widetilde{\mathcal{G}})$, where $\widetilde{\mathcal{G}}$ is a small DG *category such that $H^0(\widetilde{\mathcal{G}})$ is equivalent to $\mathcal{G}$.*

If $\mathcal{T}$ is compactly generated, the theorem yields a triangle equivalence $\mathcal{T} \longrightarrow \mathcal{DA}$, and we recover Theorem 4.3 of [14]. Note the structural similarity with the abelian case. One notable difference is that in the abelian case, one can work with a single generator whereas in the triangulated case, in general, it seems unavoidable to use a (small but usually infinite) triangulated subcategory.

An analogous result for topological triangulated categories has recently been proved by A. Heider [9].

The presence of a Quillen model category structure on the category of DG modules suggested to us that a completely different proof of the main theorem could be obtained without the use of the powerful theorem 4.10. In fact, this theorem contains more or less the same amount of information as the generalization given by Amnon Neeman [27, Thm. 4.4.9, p. 143] of the localization theorem of Thomason-Throbaugh [36, Key Proposition 5.2.2, p. 338]. We prove that this amount of information is also encoded in the exact and model category structures present on $\mathcal{CA}$. We report here the statement of theorem 5.5 for the convenience of the reader. Here the category $\mathcal{D}_\alpha\mathcal{A}$ is the localization of $\mathcal{DA}$ of item (*ii*) in theorem 1.2.

Theorem 1.3. *Let $\mathcal{A}$ be a homotopically α-cocomplete pretriangulated* DG *category. The α-continuous derived category of $\mathcal{A}$ is α-compactly generated by the images of the free* DG *modules $A^\wedge$, $A \in \mathcal{A}$. More precisely, the full subcategory $\mathcal{G}$ of $\mathcal{D}_\alpha\mathcal{A}$ formed by the images of the free* DG *modules $A^\wedge$, $A \in \mathcal{A}$, is a triangulated subcategory satisfying conditions* (G1)*,* (G2) *and* (G3) *of Krause's definition above (see also definition 3.1).*

In this new approach, we look at the derived category of the small DG category $\mathcal{A}$ as the homotopy category of the category of DG modules $\mathcal{CA}$ considered with the *projective model structure* (see subsection 6.1). This allows us to introduce, for all the infinite regular cardinals α, some subcategories of cofibrant DG modules of $\mathcal{CA}$ which can be considered as the DG analogues of the subcategories of α-compact objects of $\mathcal{DA}$. We call the objects of these subcategories *homotopically α-compact* DG *modules*. It turns out that these objects have a nice factorization property (theorem 6.31) which is the technical key-result to arrive at a new proof of theorem 1.3 (5.5 in the text).

It seems to us that this new approach is more suitable for further investigations.

1.1. **Organization of the thesis.** In section 2, we recall the main definitions and results about DG categories and categories of DG modules over them which will be used in the following chapters. Moreover, the construction of the derived category of a small DG category is introduced (subsection 2.2). Notions and results in this section are already known.

In section 3, we present some auxiliary results about well generated triangulated categories. After recalling the definition given by Krause (subsection 3.1), we establish a small set of conditions which allows us to show that two well generated triangulated categories are triangle equivalent (subsection 3.2).

In section 4, we recall some basic results about localizations of triangulated categories, localizations of well generated triangulated categories, and about their thick and (α-) localizing subcategories (where α is an infinite regular cardinal). Thick subcategories of triangulated categories are triangulated subcategories which contain the direct factors of their objects; (α-)localizing subcategories are thick subcategories stable under formation of all (α-)small coproducts of their objects. In subsection 4.3, we state a theorem concerning particular localizations of well generated triangulated categories, those which are triangle quotients by a subcategory *generated by a set*.

Section 5 contains the main theorem and can be considered the heart of the thesis. In subsection 5.1, DG categories enter the picture. Here we construct the *α-continuous derived category $\mathcal{D}_\alpha\mathcal{A}$* of a *homotopically α-cocomplete* small DG category $\mathcal{A}$. This construction enjoys a useful and beautiful property which is the key technical result for proving the main theorem of the thesis: Given a homotopically α-cocomplete pretriangulated DG category $\mathcal{A}$, we show that its α-continuous derived category $\mathcal{D}_\alpha\mathcal{A}$ is α-compactly generated by the images of the free DG modules. Our proof heavily uses theorem 4.10 of subsection 4.3 about localizations of well generated triangulated categories.

The categories $\mathcal{D}_\alpha \mathcal{A}$ turn out to be the prototypes of the α-compactly generated algebraic triangulated categories. This characterization is what we have called the Popescu-Gabriel theorem for triangulated categories. This is the main result of the thesis. We present it in subsection 5.3, after having introduced the notion of algebraic category in subsection 5.2. As an application, in subsection 5.4 we also give a result about compactifying subcategories of an algebraic well generated triangulated category. The notion of compactifying subcategory generalizes that of compactifying generator introduced by Lowen-Van den Bergh [21, Ch. 5] in the case of a Grothendieck abelian category.

In subsection 6.1, we recall some general notions and facts about the Quillen model structures before introducing the *projective model structure* present on the category of DG modules. In subsection 6.2 we look at the exact structure on the category $\mathcal{CA}$ in more detail. Moreover, the notion of homotopy for DG modules is introduced. In subsection 6.3 we study different closures of classes of cofibrant DG modules in $\mathcal{CA}$ and prove some useful properties of them. Then, we introduce the *homotopically α-small* and the *homotopically α-compact* DG *modules* (subsections 6.4 and 6.5). The last subsection 6.6 is dedicated to proving a key technical factorization result (theorem 6.31).

2. DG categories

In this section, the basic definitions and properties of DG categories will be briefly recalled. We refer to [14], [16] and [7] for in depth treatments.

2.1. **Notations.** Let k be a commutative ring. The tensor product over k will be denoted by $\otimes = \otimes_k$. A *k-category* $\mathcal{A}$ is a category enriched over the category of k-modules $\mathrm{Mod}(k)$ with k-linear associative composition maps

$$\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)\ ,\ (f,g) \mapsto fg.$$

The *category of graded k-modules* $\mathcal{G}(k)$ has the k-modules $V = \bigoplus_{p\in\mathbb{Z}} V^p$ as objects, with the usual shift $V[1]$ defined by $V[1]^p = V^{p+1}$, $p \in \mathbb{Z}$. The *tensor product* $V \otimes W$ of two graded k-modules V and W is the graded k-module with components

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q\ ,\ n \in \mathbb{Z}.$$

Given two graded k-modules V and V', the Hom-space $\mathcal{G}(k)(V,V')$ is a graded k-module whose n-th component

$$\mathcal{G}(k)(V,V')^n$$

consists of the *graded morphisms* $f : V \to V'$ of degree n, *i.e.* the k-linear morphisms such that $f(V^p) \subseteq V^{p+n}$, for all $p \in \mathbb{Z}$. The *tensor product* $f \otimes g$ of two maps $f : V \to V'$ and $g : W \to W'$ is defined as

$$(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes g(w)\ ,$$

where g is of degree p and v belongs to V^q. A *graded k-algebra* is a graded k-module A endowed with a multiplication morphism $A \otimes A \to A$ which is graded of degree 0, so that 'ordinary' k-algebras are graded k-algebras concentrated in degree 0.

A *differential graded (*=DG*) k-module* is a $\mathbb{Z}$-graded k-module V endowed with a *differential*, $d_V : V \to V$ of degree 1 such that $d_V^2 = 0$. Equivalently, V is a *complex* of k-modules. Remember that the *shifted* DG *module* $V[1]$ has differential $-d_V$. The *tensor product* of two DG k-modules is the graded module $V \otimes W$ with the differential $d_V \otimes \mathbf{1}_W + \mathbf{1}_V \otimes d_W$.

2.2. **Differential graded categories.** A *differential graded* or DG *category* is a k-category $\mathcal{A}$ enriched over the category of DG k-modules. This definition is equivalent to the following data:

- a class of objects $\mathrm{obj}(\mathcal{A})$;
- a small DG k-module $\mathcal{A}(X,Y)$ for any pair of objects X, Y in $\mathcal{A}$;
- associative unital composition maps given by morphisms of DG k-modules

$$\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z).$$

The typical example is the DG *category* $\mathcal{C}_{dg}(B)$, where B is a k-algebra. The objects of this category are the cochain complexes of right B-modules. For each pair of objects L and M, the Hom-space is defined as the DG k-module $\mathcal{C}_{dg}(B)(L,M)$ with components

$$\mathcal{C}_{dg}(B)(L,M)^n\ ,$$

whose elements are the morphisms of $f = (f^p)$, where, for each $p \in \mathbb{Z}$, the elements $f^p : L^p \to M^{p+n}$ are morphisms of B-modules, and whose differential is the commutator

$$d(f) = d_M \circ f - (-1)^n f \circ d_L .$$

The composition is the composition of graded maps. Now we can define the *category* $Z^0(\mathcal{A})$, which has the same objects as $\mathcal{A}$ and whose Hom-spaces are defined by

$$(Z^0\mathcal{A})(X,Y) = Z^0(\mathcal{A}(X,Y)) ,$$

where Z^0 is the kernel of $d : \mathcal{A}(X,Y)^0 \to \mathcal{A}(X,Y)^1$. In the same way, the *category* $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(H^0(\mathcal{A}))(X,Y) = H^0(\mathcal{A}(X,Y)) ,$$

where H^0 denotes the 0th homology of the complex $\mathcal{A}(X,Y)$. The *homology category* $H^*(\mathcal{A})$ is the graded category with the same objects as $\mathcal{A}$ and morphisms spaces $H^*\mathcal{A}(X,Y)$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be DG categories. A DG *functor*

$$F : \mathcal{A} \longrightarrow \mathcal{A}'$$

is given by the following data:

- a map
$$F : \mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{A}');$$
- morphisms of dg k-modules
$$F_{X,Y} : \mathcal{A}(X,Y) \longrightarrow \mathcal{A}(FX,FY) \ , \ X,Y \in \mathrm{obj}(\mathcal{A}) ,$$
compatible with the composition and the units.

Let $\mathcal{A}$ and $\mathcal{B}$ be two DG categories. For any pair of DG functors F, $G : \mathcal{A} \to \mathcal{B}$,

- a *graded morphism of DG functors* of degree n,
$$\phi : F \to G ,$$
is the data of a family of morphisms
$$\phi_X \in \mathcal{B}(FX,GX)^n ,$$
such that $(Gf)(\phi_X) = (\phi_Y)(Ff)$, for all $f \in \mathcal{A}(X,Y)$, $X,Y \in \mathcal{A}$;
- the differential of the *complex of graded morphisms* $\mathcal{H}om(F,G)$ is induced by the differential of $\mathcal{B}(FX,GX)$.

The set of *morphisms between the DG functors F and G* is defined as the set of cocycles $Z^0\mathcal{H}om(F,G)$.

A *quasi-equivalence* is a DG functor $F : \mathcal{A} \to \mathcal{A}'$ such that

- $F_{X,Y}$ is a quasi-isomorphism, for all objects X, Y of $\mathcal{A}$;
- the induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{A}')$ is an equivalence.

2.3. **DG modules.** A *right* DG $\mathcal{A}$-*module* over a small DG category $\mathcal{A}$ is a DG functor

$$M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k).$$

We can think of a right DG $\mathcal{A}$-module M as given by complexes $M(X)$ of k-modules, for each $X \in \mathrm{obj}(\mathcal{A})$, and by morphisms of complexes

$$M(Y) \otimes \mathcal{A}(X,Y) \to M(X)$$

compatible with compositions and units. The category $\mathcal{GA}$ of *graded* $\mathcal{A}$-*modules* is the category of functors

$$M : \mathcal{A}^{op} \to \mathcal{G}(k)$$

defined on the underlying graded category of $\mathcal{A}$ with values in the category $\mathcal{G}(k)$ of graded k-modules (*cf.* 2.1). An example is the *homology* $H^*(M)$ of a DG module M, *i.e.* the induced functor

$$H^*(\mathcal{A})^{op} \to \mathcal{G}(k)\ ,\ X \mapsto H^*(M(X)).$$

The *representable* $\mathcal{A}$-modules are those which are isomorphic to $X^\wedge = \mathcal{A}(-,X)$ for some object X of $\mathcal{A}$. The *category of* DG *modules* $\mathcal{CA}$ has as objects the DG $\mathcal{A}$-modules and as morphisms $L \to M$ the morphisms of DG functors.

Note that $\mathcal{CA}$ is an abelian category and that a morphism $L \to M$ is an epimorphism (respectively a monomorphism) iff it induces surjections (respectively injections) in each component of $L(X) \to M(X)$ for each object X of $\mathcal{A}$. A morphism $f : L \to M$ is a *quasi-isomorphism* if it induces an isomorphism in homology. We have $\mathcal{CA} = Z^0(\mathcal{C}_{dg}(\mathcal{A}))$, where the DG category $\mathcal{C}_{dg}(\mathcal{A})$ is defined by

$$\mathcal{C}_{dg}(\mathcal{A}) = \mathcal{H}om(\mathcal{A}^{op}, \mathcal{C}_{dg}(k)).$$

We write $\mathcal{H}om(L,M)$ for the complex of morphisms from L to M in $\mathcal{C}_{dg}(\mathcal{A})$. For each $X \in \mathcal{A}$, we have a natural isomorphism

$$\mathcal{H}om(X^\wedge, M) \xrightarrow{\sim} M(X). \tag{1}$$

The *category up to homotopy of* DG $\mathcal{A}$-*modules* is

$$\mathcal{H}(\mathcal{A}) = H^0(\mathcal{C}_{dg}(\mathcal{A})).$$

The isomorphism (1) yields the isomorphism

$$\begin{aligned} \mathcal{CA}(X^\wedge, M) &= Z^0(\mathcal{C}_{dg}(\mathcal{A}))(X^\wedge, M) \\ &= Z^0(\mathcal{C}_{dg}(\mathcal{A})(X^\wedge, M)) \\ &\xrightarrow{\sim} Z^0(M(X)) \\ &= (Z^0 M)(X)\ , \end{aligned} \tag{2}$$

for X and M as above. Similarly, we obtain

$$\mathcal{H}(\mathcal{A})(X^\wedge, M[n]) \xrightarrow{\sim} H^n(\mathcal{H}om(X^\wedge, M)) \xrightarrow{\sim} H^n M(X)\ , \tag{3}$$

where $n \in \mathbb{Z}$ and $M[n]$ is the *shifted* DG *module* $Y \mapsto M(Y)[n]$.

2.4. **The derived category.** Let us introduce the *derived category* $\mathcal{DA}$ by formally inverting the quasi-isomorphisms of $\mathcal{CA}$. The derived category can be made into a triangulated category by defining the distinguished triangles as the "images" of the short exact sequences of $\mathcal{CA}$

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$$

which split in the category of graded $\mathcal{A}$-modules. These sequences of $\mathcal{CA}$ are also called *conflations*. This means that the *deflation* p has a graded section s of degree 0 or, equivalently, the *conflation* i has a graded retraction r of degree 0. It turns out that, with this choice of conflations, the category of the DG modules $\mathcal{CA}$ has the structure of Frobenius category. Moreover, its stable category is equivalent to the (triangulated) category up to homotopy $\mathcal{H}(\mathcal{A})$ (*cf.* [10], [8], [14]). Bernhard Keller shows in [14] that the projection functor $\mathcal{HA} \to \mathcal{DA}$ admits a fully faithful left adjoint $\mathbf{p}$ and a fully faithful right adjoint $\mathbf{i}$. By using the embedding

$$\mathbf{p} : \mathcal{DA} \to \mathcal{H}(\mathcal{A})$$

and the isomorphisms (3), it is possible to show that, for an object X of $\mathcal{A}$ and a DG module M, the isomorphisms

$$\mathcal{DA}(X^\wedge, M[n]) \xrightarrow{\sim} \mathcal{H}(\mathcal{A})(X^\wedge, \mathbf{i}M[n]) \xrightarrow{\sim} H^n M(X) \tag{4}$$

hold, since $X^\wedge$ is cofibrant (6.1). The embedding $\mathbf{p} : \mathcal{DA} \to \mathcal{H}(\mathcal{A})$ is also the tool which makes it clear that the derived category is still an additive category. The derived category $\mathcal{DA}$ admits arbitrary coproducts, induced by the coproducts of the category $\mathcal{CA}$ of DG modules. Thanks to the isomorphisms

$$\mathcal{DA}(X^\wedge[n], M) \xrightarrow{\sim} H^{-n} M(X) \,, \tag{5}$$

obtained from (4), each DG module $X^\wedge[n]$, where X is an object of $\mathcal{A}$ and n an integer, is compact (*cf.* subsection 3.1 for the definition and subsection 4.2 for more details). The isomorphism (5) also shows that a DG module M vanishes in $\mathcal{DA}$ iff each morphism $X^\wedge[n] \to M$ vanishes. Thus the set $\mathcal{G}$ formed by the $X^\wedge[n]$, $X \in \mathcal{A}$, $n \in \mathbb{Z}$, is a set of compact generators (*cf.* subsection 4.2) for $\mathcal{DA}$.

3. Well generated triangulated categories

3.1. **Definitions of Krause and Neeman.** The notion of well generated triangulated category is due to A. Neeman [27, Def. 1.15, p. 15]. Instead of his original definition, we will use a characterization due to H. Krause [17] which is closer in spirit to the definition of Grothendieck abelian categories. We recall that a *regular* cardinal α is a cardinal which is *not* the sum of fewer than α cardinals, all smaller than α (see any standard reference about set theory for definitions and properties of ordinals and cardinals, a very readable one is [19]). In this thesis, we will usually assume that the cardinals we use are infinite and regular.

Definition 3.1. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and suspension functor Σ. Let α be an infinite regular cardinal. Then the category $\mathcal{T}$ is *α-compactly generated* if there exists a set of *α-good generators*, *i.e.* a set of objects $\mathcal{G}_0$ such that $\Sigma\mathcal{G}_0 = \mathcal{G}_0$, satisfying the conditions:

(G1) an object $X \in \mathcal{T}$ is zero if $\mathcal{T}(G, X) = 0$ for all G in $\mathcal{G}_0$;

(G2) for each family of morphisms $f_i : X_i \to Y_i$, $i \in I$, the induced map

$$\mathcal{T}(G, \coprod_{i\in I} X_i) \to \mathcal{T}(G, \coprod_{i\in I} Y_i)$$

is surjective for all $G \in \mathcal{G}_0$ if the maps

$$\mathcal{T}(G, X_i) \to \mathcal{T}(G, Y_i)$$

are surjective for all $i \in I$ and all $G \in \mathcal{G}_0$;

(G3) all the objects $G \in \mathcal{G}_0$ are *α-small*, *i.e.* for each family of objects X_i, $i \in I$, of $\mathcal{T}$, each morphism

$$G \to \coprod_{i\in I} X_i$$

factors through a subsum $\coprod_{i\in J} X_i$ for some subset J of I of cardinality strictly smaller than α.

A triangulated category is *well generated* [17] if there exists a regular cardinal δ such that it is δ-compactly generated.

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. We will say that condition (G4) holds for a class of objects $\mathcal{G}$ of $\mathcal{T}$ if the following holds:

(G4) for each family of objects X_i, $i \in I$, of $\mathcal{T}$, and each object $G \in \mathcal{G}$, each morphism

$$G \to \coprod_{i\in I} X_i$$

factors through a morphism $\coprod_{i\in I} \phi_i \colon \coprod_{i\in I} G_i \to \coprod_{i\in I} X_i$, with G_i in $\mathcal{G}$ for all $i \in I$.

Clearly, condition (G4) holds for the empty class and, if it holds for a family of classes, then it holds for their union. Thus, for a given regular cardinal α, there exists a unique maximal class satisfying (G4) and formed by α-small objects. Following Krause [17], we denote this class, and the triangulated subcategory on its objects, by $\mathcal{T}^\alpha$. Its objects are called the *α-compact* objects of $\mathcal{T}$.

Remark 3.2. This definition of $\mathcal{T}^\alpha$ is not identical to the one of Neeman [27, Def. 1.15, p. 15]. However, as shown in [17, Lemma 6], the two definitions are equivalent if the isomorphism classes of $\mathcal{T}^\alpha$ form a set. This always holds when $\mathcal{T}$ is well generated, *cf.* [17].

In the case $\alpha = \aleph_0$, the $\aleph_0$-*compact* objects are the objects usually called compact (also called small). We recall that an object K of $\mathcal{T}$ is called *compact* if the following isomorphism holds

$$\bigoplus_{i\in I} \mathcal{T}(K, X_i) \xrightarrow{\sim} \mathcal{T}(K, \coprod_{i\in I} X_i)\,,$$

where the objects X_i lie in $\mathcal{T}$ for all $i \in I$, and I is an arbitrary set. The triangulated category with coproducts $\mathcal{T}$ is usually called *compactly generated* if condition (G1) holds for a set $\mathcal{G}_0$ contained in the subcategory of compact objects $\mathcal{T}^c = \mathcal{T}^{\aleph_0}$. In the case $\alpha = \aleph_0$, the definition of well generated category specializes to that of compactly generated category.

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and $\mathcal{G}_0$ a small full subcategory of $\mathcal{T}$. Let $\mathcal{G} = \mathrm{Add}(\mathcal{G}_0)$ be the closure of $\mathcal{G}_0$ under arbitrary coproducts and direct factors. A functor $F : \mathcal{G}^{op} \to \mathcal{A}b$ is *coherent* [6], [1] if it admits a presentation

$$\mathcal{G}(-, G_1) \to \mathcal{G}(-, G_0) \to F \to 0$$

for some objects G_0 and G_1 of $\mathcal{G}$. Let $\mathrm{coh}(\mathcal{G})$ be the *category of coherent functors* on $\mathcal{G}$. It is a full subcategory of the *category* $\mathrm{Mod}\,\mathcal{G}$ of all additive functors $F : \mathcal{G}^{op} \to \mathcal{A}b$. Part c) of the following lemma appears in [18, Lemma 3], in a version with countable coproducts instead of arbitrary coproducts. We give a new, more direct proof.

Lemma 3.3.

a) *For each object X of $\mathcal{T}$, the functor $h(X)$ obtained by restricting $\mathcal{T}(-, X)$ to $\mathcal{G}$ is coherent.*

b) *The functor $\mathcal{G} \to \mathrm{coh}(\mathcal{G})$ taking G to $h(G)$ commutes with arbitrary coproducts.*

c) *Condition* (G2) *holds for $\mathcal{G}_0$ iff $h : \mathcal{T} \to \mathrm{coh}(\mathcal{G})$ commutes with arbitrary coproducts.*

Proof. a) We have to show that, for each $X \in \mathcal{T}$, the functor $\mathcal{T}(-, X)|_{\mathcal{G}}$, which, *a priori*, is in $\mathrm{Mod}\,\mathcal{G}$, is in fact coherent. We choose a morphism $\coprod_{i\in I} G_i \to X$, $G_i \in \mathcal{G}_0$, such that each $G \to X$, $G \in \mathcal{G}_0$, factors through a morphism $G_i \to X$. Then

$$\mathcal{T}(-, \coprod_{i\in I} G_i)|_{\mathcal{G}} \to \mathcal{T}(-, X)|_{\mathcal{G}}$$

is an epimorphism in $\mathrm{Mod}\,\mathcal{G}$. We form a distinguished triangle

$$X' \to \coprod_{i\in I} G_i \to X \to \Sigma X'$$

in $\mathcal{T}$. We can continue the construction and choose a morphism $\coprod_{i\in I'} G'_i \to X'$, $G'_i \in \mathcal{G}_0$, such that each $G \to X'$, $G \in \mathcal{G}_0$, factors through a morphism $G'_i \to X'$. Then the sequence

$$\mathcal{T}(-, \coprod_{i\in I'} G'_i)|_{\mathcal{G}} \to \mathcal{T}(-, \coprod_{i\in I} G_i)|_{\mathcal{G}} \to \mathcal{T}(-, X)|_{\mathcal{G}} \to 0$$

is a presentation of $\mathcal{T}(-,X)|_{\mathcal{G}}$.

b) Let $(G_i)_{i\in I}$ be a family of objects of $\mathcal{G}$. We have to show that the canonical morphism

$$\mathrm{coh}(\mathcal{G})(h(\coprod_{i\in I} G_i), F) \to \prod_{i\in I} \mathrm{coh}(\mathcal{G})(h(G_i), F)$$

is invertible for each coherent functor F. Since $h(G)$ is projective for each G in $\mathcal{G}$, it is enough to check this for representable functors F. For these, it follows from Yoneda's lemma and the definition of $\coprod_{i\in I} G_i$.

c) We suppose that (G2) holds for $\mathcal{G}_0$.

First step. For each family $(X_i)_{i\in I}$ of $\mathcal{T}$, the canonical morphism $\coprod_{i\in I} h(X_i) \xrightarrow{\varphi} h(\coprod_{i\in I} X_i)$ is an epimorphism. Indeed, for each $i \in I$, let $G_i \to X_i$ be a morphism such that

$$h(G_i) \to h(X_i)$$

is an epimorphism, where G_i belongs to $\mathcal{G}$. By b), the functor $h : \mathcal{G} \to \mathrm{coh}(\mathcal{G})$ commutes with coproducts. Thus, we obtain a commutative square

$$\begin{array}{ccc} \coprod_{i\in I} h(G_i) & \longrightarrow & \coprod_{i\in I} h(X_i) \\ \downarrow \wr & & \downarrow \varphi \\ h(\coprod_{i\in I} G_i) & \xrightarrow[\pi]{} & h(\coprod_{i\in I} X_i). \end{array}$$

By condition (G2), π is an epimorphism. Thus, φ is an epimorphism.

Second step. For each family $(X_i)_{i\in I}$ of $\mathcal{T}$, the canonical morphism $\coprod_{i\in I} h(X_i) \xrightarrow{\varphi} h(\coprod_{i\in I} X_i)$ is an isomorphism. Indeed, for each $i \in I$, we choose distinguished triangles

$$X_i' \to G_i \to X_i \to \Sigma X_i' ,$$

and morphisms $G_i' \to X_i'$, where $G_i \to X_i$ is as in the first step and G_i' belongs to $\mathcal{G}$, such that

$$h(G_i') \to h(X_i')$$

is an epimorphism. Then the sequence

$$0 \to h(\coprod_{i\in I} X_i') \xrightarrow[\iota]{} h(\coprod_{i\in I} G_i) \xrightarrow[\pi]{} h(\coprod_{i\in I} X_i) \to 0$$

is exact. Indeed, coproducts preserve distinguished triangles and h is cohomological since it is the composition of the Yoneda functor with the restriction functor $F \mapsto F|_{\mathcal{G}}$, which is clearly exact. In particular, ι is a monomorphism. Since the coproduct functor $\coprod_{i\in I}$ is right exact, the top morphism of the square

$$\begin{array}{ccc} \coprod_{i\in I} h(G_i') & \twoheadrightarrow & \coprod_{i\in I} h(X_i') \\ \downarrow \wr & & \downarrow \varphi \\ h(\coprod_{i\in I} G_i') & \longrightarrow & h(\coprod_{i\in I} X_i') \end{array}$$

is an epimorphism. By the first step, it follows that the morphism φ is an epimorphism. By b), the morphism $\coprod_{i\in I} h(G'_i) \to h(\coprod_{i\in I} G'_i)$ is an isomorphism. Therefore, the morphism

$$h(\coprod_{i\in I} G'_i) \to h(\coprod_{i\in I} X'_i)$$

is an epimorphism and the sequence

$$h(\coprod_{i\in I} G'_i) \to h(\coprod_{i\in I} G_i) \underset{\pi}{\to} h(\coprod_{i\in I} X_i) \to 0$$

is exact. The claim now follows from b) since, $\coprod_{i\in I}$ being a right exact functor, we have a diagram with exact rows

$$\begin{array}{ccccccc}
\coprod_{i\in I} h(G'_i) & \longrightarrow & \coprod_{i\in I} h(G_i) & \longrightarrow & \coprod_{i\in I} h(X_i) & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow \varphi & & \\
h(\coprod_{i\in I} G'_i) & \longrightarrow & h(\coprod_{i\in I} G_i) & \longrightarrow & h(\coprod_{i\in I} X_i) & \longrightarrow & 0.
\end{array}$$

We suppose now that h commutes with coproducts. We will show that condition (G2) holds for $\mathcal{G}$. Let $(f_i : X_i \to Y_i)_{i\in I}$ be a family of morphisms in $\mathcal{T}$ such that $\mathcal{T}(G, f_i) : \mathcal{T}(G, X_i) \to \mathcal{T}(G, Y_i)$ is surjective for all $i \in I$ and all $G \in \mathcal{G}_0$. Then $\mathcal{T}(\coprod_{l\in L} G_l, f_i) : \mathcal{T}(\coprod_{l\in L} G_l, X_i) \to \mathcal{T}(\coprod_{l\in L} G_l, Y_i)$ is surjective for all the families $(G_l)_{l\in L}$ of $\mathcal{G}_0$ and all $i \in I$, thanks to the isomorphisms $\mathcal{T}(\coprod_{l\in L} G_l, X_i) \to \prod_{l\in L} \mathcal{T}(G_l, X_i)$. Moreover, it is trivial to verify that $\mathcal{T}(A, f_i) : \mathcal{T}(A, X_i) \to \mathcal{T}(A, Y_i)$ is surjective, for all $i \in I$, for each direct factor A of any object $G \in \mathcal{G}_0$. Therefore, $\mathcal{T}(G, f_i)$ is surjective for all $i \in I$ and all $G \in \mathcal{G}$. Thus, $\mathcal{T}(-, X_i)|_{\mathcal{G}} \to \mathcal{T}(-, Y_i)|_{\mathcal{G}}$ is an epimorphism for all $i \in I$. The coproduct $\coprod_{i\in I} \mathcal{T}(-, X_i)|_{\mathcal{G}} \to \coprod_{i\in I} \mathcal{T}(-, Y_i)|_{\mathcal{G}}$ is still an epimorphism. Since h commutes with coproducts, it follows that $\mathcal{T}(G, \coprod_{i\in I} X_i) \to \mathcal{T}(G, \coprod_{i\in I} Y_i)$ is surjective for all $G \in \mathcal{G}$, in particular for all $G \in \mathcal{G}_0$. □

Consider a triangulated category $\mathcal{T}$ and a class of its objects $\mathcal{G}_0$, satisfying some or all the conditions of definition 3.1. It will be important for us to know if these conditions continue to hold for different closures of $\mathcal{G}_0$.

Proposition 3.4. *Let $\mathcal{T}$ be a* cocomplete *triangulated category,* i.e. *$\mathcal{T}$ admits all small coproducts. Let $\mathcal{G}_0$ be a class of objects in $\mathcal{T}$, stable under Σ and Σ^{-1}, satisfying conditions* (G2) *and* (G3) *of the definition 3.1. Let α be an infinite cardinal. Let $\mathcal{G}$ be the closure of $\mathcal{G}_0$ under Σ and Σ^{-1}, extensions and α-small coproducts. Then, conditions* (G3) *and* (G4) *hold for $\mathcal{G}$.*

Proof. (G3) We directly show that condition (G3) holds for shifts, α-coproducts and extensions of objects in $\mathcal{G}_0$.

Since the functor $\Sigma : \mathcal{T} \to \mathcal{T}$ is an equivalence, an object X of $\mathcal{T}$ is α-small iff ΣX is α-small. Thus, condition (G3) holds for all objects $\Sigma^n G, G \in \mathcal{G}_0, n \in \mathbb{Z}$.

Since α-small coproducts commute with α-filtered colimits, condition (G3) holds for α-small coproducts of objects of $\mathcal{G}_0$. Indeed, let $(G_j)_{j\in J}$, $|J| < \alpha$, be a family of α-small objects of $\mathcal{G}_0$ and let $(X_i)_{i\in I}$ be an arbitrary family of objects of $\mathcal{T}$. We have the following sequence of isomorphisms:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{T}}(\coprod_{j\in J} G_j, \coprod_{i\in I} X_i) & = & \prod_{j\in J} \mathrm{Hom}_{\mathcal{T}}(G_j, \coprod_{i\in I} X_i) \\ & = & \prod_{j\in J} \mathrm{colim}_{I'\subset I} \mathrm{Hom}_{\mathcal{T}}(G_j, \coprod_{i\in I'} X_i) \\ \mathrm{colim}_{I'\subset I} \mathrm{Hom}_{\mathcal{T}}(\coprod_{j\in J} G_j, \coprod_{i\in I'} X_i) & = & \mathrm{colim}_{I'\subset I} \prod_{j\in J} \mathrm{Hom}_{\mathcal{T}}(G_j, \coprod_{i\in I'} X_i) \,, \end{array}$$

where the cardinality of the subset I' is strictly smaller than α. The only non trivial isomorphism is the vertical third which holds since the cardinal α is supposed regular, hence the colimit is taken over an α-filtered set I.

Let us consider the (mapping) cone of an arbitrary morphism $G \to G'$ of $\mathcal{G}_0$

$$G \longrightarrow G' \longrightarrow C \longrightarrow \Sigma G.$$

We can form two long exact sequences by applying the cohomological functors

$$\mathrm{Hom}_{\mathcal{T}}(-, \coprod_{i\in I} X_i) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{T}}(-, \coprod_{i\in J} X_i)$$

to the last distinguished triangle. Now we consider the colimit over the subsets $J \subset I$ of cardinality strictly smaller than α of the long exact sequence induced by $\mathrm{Hom}_{\mathcal{T}}(-, \coprod_{i\in J} X_i)$. We obtain a long sequence which is still exact since we are using filtered colimits. There is a natural map of the two long exact sequences just formed. Let us represent a part of it in the following diagram, where we write col_J for $\mathrm{colim}_{J\subseteq I}$

$$\begin{array}{ccccccc} \mathrm{col}_J\,\mathcal{T}(G, \coprod_J X_i) & \longleftarrow & \mathrm{col}_J\,\mathcal{T}(G', \coprod_J X_i) & \longleftarrow & \mathrm{col}_J\,\mathcal{T}(C, \coprod_J X_i) & \longleftarrow & \mathrm{col}_J\,\mathcal{T}(\Sigma G, \coprod_J X_i) \\ \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr \\ \mathcal{T}(G, \coprod_I X_i) & \longleftarrow & \mathcal{T}(G', \coprod_I X_i) & \longleftarrow & \mathcal{T}(C, \coprod_I X_i) & \longleftarrow & \mathcal{T}(\Sigma G, \coprod_I X_i) \end{array}$$

The vertical arrows are isomorphisms since G, G' are in $\mathcal{G}_0$, and we have seen that ΣG is α-small. Thus, the third vertical arrow is an isomorphism by the Five-Lemma and C is α-small, too.

(G4) We call $\mathcal{U}$ the full subcategory of $\mathcal{T}$ formed by the objects $X \in \mathcal{G}$ which satisfy the following condition. Given a morphism

$$f : X \longrightarrow \coprod_{i\in I} Y_i \,,$$

where $(Y_i)_{i\in I}$ is a family of objects in $\mathcal{T}$, there exists a family $(X_i)_{i\in I}$ of objects of $\mathcal{G}$ and some morphisms $\varphi_i : X_i \to Y_i$ such that f factors as in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & \coprod_{i\in I} Y_i \\ & \searrow \qquad \nearrow_{\coprod_{i\in I} \varphi_i} & \\ & \coprod_{i\in I} X_i. & \end{array}$$

We shall show:

a) the subcategory $\mathcal{U}$ contains $\mathcal{G}_0$;
b) the subcategory $\mathcal{U}$ is stable under formation of α-coproducts;
c) the subcategory $\mathcal{U}$ is closed under Σ, Σ^{-1} and under extensions.

It follows by the properties a), b), c) that $\mathcal{U} = \mathcal{G}$, which shows that condition (G4) holds for $\mathcal{G}$.

a) Let G_0 be an object in $\mathcal{G}_0$ and $f : G_0 \to \coprod_{i\in I} Y_i$ a morphism in $\mathcal{T}$, where $(Y_i)_{i\in I}$ is a family of objects in $\mathcal{T}$. For every $i \in I$, let $(G_{ij} \to Y_i)_{j\in J_i}$, where $G_{ij} \in \mathcal{G}$, be a family of morphisms such that every morphism $G_0 \to Y_i$ factors through one of the morphisms $G_{ij} \to Y_i$. Then, the morphism

$$\varphi_i : \coprod_{j\in J_i} G_{ij} \longrightarrow Y_i$$

induces a surjection

$$\mathrm{Hom}_{\mathcal{T}}(G_0, \coprod_{j\in J_i} G_{ij}) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(G_0, Y_i) ,$$

for every $i \in I$. By (G2), the map

$$\mathrm{Hom}(G_0, \coprod_{i\in I}\coprod_{j\in J_i} G_{ij}) \longrightarrow \mathrm{Hom}(G_0, \coprod_{i\in I} Y_i)$$

is a surjection. Therefore, there exists a morphism

$$\widetilde{f} : G_0 \longrightarrow \coprod_{i\in I}\coprod_{j\in J_i} G_{ij}$$

such that the composition

$$G_0 \xrightarrow{\ \widetilde{f}\ } \coprod_{i\in I}\coprod_{j\in J_i} G_{ij} \xrightarrow{\coprod_{i\in I}\varphi_i} \coprod_{i\in I} Y_i$$

is equal to f. We have supposed that G is α-small (condition (G3) holds for $\mathcal{G}_0$). Therefore the morphism

$$G_0 \xrightarrow{\widetilde{f}} \coprod_{i\in I}\coprod_{j\in J_i} G_{ij} = \coprod_{(i,j)\in\mathcal{L}} G_{ij} ,$$

where $\mathcal{L}$ is the set of pairs (i, j) with $i \in I$ and $j \in J_i$, factors through the sub-sum

$$\coprod_{(i,j)\in\Lambda} G_{ij} = \coprod_{j\in\widetilde{I}}\coprod_{j\in\widetilde{J}_i} G_{ij} ,$$

where $\Lambda \subseteq \mathcal{L}$ is a subset of cardinality strictly smaller than α. Let $\widetilde{I}$ be the set of indices $i \in I$ such that Λ contains a pair of the form (i, j). Then $\widetilde{I}$ is of cardinality strictly smaller than α. Now for each $i \in \widetilde{I}$, let $\widetilde{J}_i$ be the set of indices $j \in J_i$ such that Λ contains the pair (i, j). Then each $\widetilde{J}_i$ is of cardinality strictly smaller than α. Now for $i \notin \widetilde{I}$, put $\widetilde{J}_i = \emptyset$. Then we have

$$\coprod_{(i,j)\in\Lambda} G_{ij} = \coprod_{i\in I}\coprod_{j\in\widetilde{J}_i} G_{ij}.$$

Let $Y_i = \coprod_{j \in \widetilde{J}_i} G_{ij}$. Then f factors as

$$G_0 \longrightarrow \coprod_{i\in I} Y_i \xrightarrow{\coprod_{i\in I}(\varphi_i|_{Y_i})} \coprod_{i\in I} X_i.$$

As $|\widetilde{J}| < \alpha$, Y_i lies in $\mathcal{G}$ for all $i \in I$.

b) Let $(U_j)_{j\in J}$ be a family of $\mathcal{U}$ where $|J| < \alpha$. Let

$$f : \coprod_{j\in J} U_j \longrightarrow \coprod_{i\in I} X_i\ ,$$

be a morphism in $\mathcal{T}$, where $(X_i)_{i\in I}$ is a family of $\mathcal{T}$. Let

$$f_j : U_j \longrightarrow \coprod_{i\in I} X_i$$

be the component of f associated to $j \in J$. For each $j \in J$, since U_j lies in $\mathcal{U}$, there exists a factorization

$$\begin{array}{ccc} U_j & \xrightarrow{\quad f_j \quad} & \coprod_{i\in I} X_i \\ & \searrow \qquad \nearrow_{\coprod_{i\in I}\varphi_{ji}} & \\ & \coprod_{i\in I} Y_{ji}\ , & \end{array}$$

where $(Y_{ji})_{i\in I}$ is a family of $\mathcal{G}$. Then, we have the factorization

$$\coprod_{j\in J} U_j \longrightarrow \coprod_{j\in J}\coprod_{i\in I} Y_{ji} \xrightarrow{\varphi} \coprod_{i\in I} X_i\ ,$$

which we can write as

$$\coprod_{j\in J} U_j \longrightarrow \coprod_{i\in I}\coprod_{j\in J} Y_{ji} \xrightarrow{\coprod_{i\in I}\varphi_i} \coprod_{i\in I} X_i\ ,$$

where $\coprod_{j\in J} Y_{ji}$ belongs to $\mathcal{G}$ since $|J| < \alpha$. Therefore, $\coprod_{j\in J} U_j$ lies in $\mathcal{U}$.

c) Clearly, $\mathcal{U}$ is stable under the action of Σ and Σ^{-1}. Let

$$X \longrightarrow X' \longrightarrow X'' \longrightarrow \Sigma X$$

be a distinguished triangle of $\mathcal{T}$ such that X, X' are in $\mathcal{U}$. Let

$$X'' \xrightarrow{f''} \coprod_{i\in I} Y_i$$

be a morphism of $\mathcal{T}$ where $(Y_i)_{i\in I}$ is a family of $\mathcal{T}$. We have the factorization

$$\begin{array}{ccc} X' & \longrightarrow & X'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\ \coprod_{i\in I} X'_i & \xrightarrow[\coprod_{i\in I}\varphi_i]{} & \coprod_{i\in I} Y_i\ , \end{array}$$

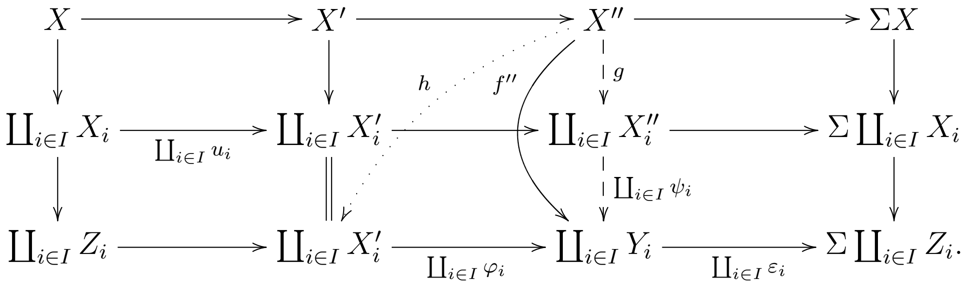
where $X' \in \mathcal{G}$ and $\varphi_i : X'_i \to Y_i$ are morphisms in $\mathcal{T}$ for all $i \in I$. We can extend each φ_i to a distinguished triangle, take coproducts over I and then complete the square above to a morphism of distinguished triangles (using axiom TR3 of triangulated categories):

$$\begin{array}{ccccccc}
X & \longrightarrow & X' & \longrightarrow & X'' & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle \Sigma f} \\
\coprod_{i\in I} Z_i & \longrightarrow & \coprod_{i\in I} X'_i & \xrightarrow[\coprod_{i\in I}\varphi_i]{} & \coprod_{i\in I} Y_i & \xrightarrow[\coprod_{i\in I}\varepsilon_i]{} & \Sigma\coprod_{i\in I} Z_i.
\end{array}$$

The objects X, X' and ΣX belong to $\mathcal{U}$. Thus, the morphisms f, f' and Σf above factor through a coproduct taken over I of objects in $\mathcal{G}$. We have the commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & X' & \longrightarrow & X'' & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & & & \downarrow \\
\coprod_{i\in I} X_i & \xrightarrow[\coprod_{i\in I} u_i]{} & \coprod_{i\in I} X'_i & & & & \Sigma\coprod_{i\in I} X_i \\
\downarrow & & \| & & & & \downarrow \\
\coprod_{i\in I} Z_i & \longrightarrow & \coprod_{i\in I} X'_i & \xrightarrow[\coprod_{i\in I}\varphi_i]{} & \coprod_{i\in I} Y_i & \xrightarrow[\coprod_{i\in I}\varepsilon_i]{} & \Sigma\coprod_{i\in I} Z_i\,,
\end{array}$$

where the morphisms $u_i : X_i \to X'_i$, $i \in I$, are in $\mathcal{G}$. Now, we extend the morphisms u_i to distinguished triangles and then form the distinguished triangle of coproducts over I. Successively, we form morphisms of distinguished triangles using axiom TR3 of triangulated categories, obtaining the maps g and $\coprod_{i\in I}\psi_i$ as shown in the diagram



Note that the subdiagram between X'' and $\coprod_{i\in I} Y_i$ does *not* commute, *i.e.* the composition $(\coprod_I \psi_i)\circ g$ is in general *not* equal to f''. Anyway, by composing with $\coprod_I \varepsilon_i$, we obtain

$$(\coprod_I \varepsilon_i)\circ(\coprod_I \psi_i)\circ g = (\coprod_I \varepsilon_i)\circ f''.$$

Therefore, by applying $\mathrm{Hom}_{\mathcal{T}}(X'', -)$ to the distinguished triangle in the third row of the last diagram, it is immediate that

$$(\coprod_I \psi_i)\circ g - f'' = (\coprod_I \varphi_i)\circ h\,,$$

for some morphism $h : X'' \to \coprod_{i\in I} X'_i$, as in the diagram. Then, the correct expression of f'' is

$$f'' = (\coprod_I \psi_i)\circ g + (\coprod_I \varphi_i)\circ(-h)$$

which shows that f'' factors as

$$X'' \xrightarrow[\begin{bmatrix} g \\ -h \end{bmatrix}]{} \coprod_{i\in I} X''_i \oplus \coprod_{i\in I} X'_i \xrightarrow[[\coprod_I \psi_i\,,\, \coprod_I \varphi_i]]{} \coprod_{i\in I} Y_i.$$

The previous factorization of f'' is trivially equivalent to the following

$$X'' \xrightarrow{\qquad} \coprod_{i\in I}(X''_i \oplus X'_i) \xrightarrow[\coprod_I[\psi_i\,,\,\varphi_i]]{} \coprod_{i\in I} Y_i.$$

Now, $X''_i \oplus X'_i$ is in $\mathcal{G}$ for all $i \in I$ by construction. □

There are two immediate and useful corollaries.

Corollary 3.5. *Let $\mathcal{T}$ be a cocomplete triangulated category. Let $\mathcal{G}_0$ be a class of objects in $\mathcal{T}$ satisfying all the conditions of the last proposition. Let α be an infinite cardinal. Let $\mathcal{G}$ be the closure of $\mathcal{G}_0$ under Σ and Σ^{-1}, extensions, α-small coproducts and direct factors,* i.e. *$\mathcal{G} = \langle \mathcal{G}_0 \rangle_\alpha$ in the notation of 4.2 below. Then, conditions* (G3) *and* (G4) *hold for $\mathcal{G}$.*

Proof. The proof of the preceding proposition works for (G4) if we verify that the subcategory $\mathcal{U}$ is also closed under direct factors, *i.e.* that it is thick (4.2).

Let U be an object in $\mathcal{U}$ and $U = U' \oplus U''$. Then, there is a section i of the projection $p : U \to U'$. Let $f : U' \to \coprod_{i\in I} W_i$ be a morphism in $\mathcal{T}$. The composition $f \circ p$ factors as

$$\begin{array}{ccccc} U & \overset{p}{\underset{i}{\rightleftarrows}} & U' & \xrightarrow{f} & \coprod_{i\in I} W_i \\ & {\scriptstyle g}\searrow & \Big\downarrow {\scriptstyle g\circ i} & \nearrow {\scriptstyle \coprod_{i\in I}\phi_i} & \\ & & \coprod_{i\in I} V_i\,, & & \end{array}$$

where the objects V_i are in $\mathcal{G}$ and the morphisms $\phi : V_i \to W_i$ in $\mathcal{T}$, for all $i \in I$. Then f also factors over $\coprod_{i\in I} V_i$, through the morphism $g \circ i$. Indeed, $f \circ p = (\coprod_I \phi_i) \circ g$, and $f \circ p \circ i = (\coprod_I \phi_i) \circ g \circ i$, but $p \circ i$ is the identity morphism of U'.

The proof of the preceding proposition works for (G3) if we verify that the direct factors of the objects in $\mathcal{G}_0$ are α-small, too. This requires the construction of a diagram structurally identical to the one above. Therefore, we omit it. □

Corollary 3.6. *Let α be an infinite regular cardinal. Let $\mathcal{T}$ be a triangulated category α-compactly generated by a set $\mathcal{G}_0$. Let $\mathcal{G}$ be the closure of $\mathcal{G}_0$ under Σ and Σ^{-1}, extensions and α-small coproducts. Let $\langle \mathcal{G}_0 \rangle_\alpha$ be the closure of $\mathcal{G}$ under direct factors. Then, $\mathcal{T}$ is α-compactly generated by both $\mathcal{G}$ and $\langle \mathcal{G}_0 \rangle_\alpha$.*

Proof. The condition (G1) clearly holds for both $\mathcal{G}$ and $\langle \mathcal{G}_0 \rangle_\alpha$, since they contain $\mathcal{G}_0$. The conditions (G3) and (G4) hold for $\mathcal{G}$ by proposition 3.4 and for $\langle \mathcal{G}_0 \rangle_\alpha$ by corollary 3.5. Moreover, condition (G4) easily implies (G2). □

3.2. **Equivalences of well generated triangulated categories.** This subsection is devoted to establishing a small set of conditions which allows us to show that two well generated triangulated categories are triangle equivalent.

Proposition 3.7. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulated categories admitting arbitrary set-indexed coproducts. Let α be a regular cardinal and $\mathcal{G} \subset \mathcal{T}$ and $\mathcal{G}' \subset \mathcal{T}'$ two α-*localizing *subcategories,* i.e. *thick and closed under formation of α-*small *coproducts (4.2). Suppose that $\mathcal{G}$ and $\mathcal{G}'$ satisfy conditions* (G1), (G2), (G3) *for the cardinal α. Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangle functor which commutes with all coproducts and induces an equivalence $\mathcal{G} \to \mathcal{G}'$. Then F is an equivalence of triangulated categories.*

Proof. 1^{st} *step*: The functor F induces an equivalence

$$\operatorname{Add}\mathcal{G} \longrightarrow \operatorname{Add}\mathcal{G}'.$$

As F commutes with coproducts and induces a functor $\mathcal{G} \to \mathcal{G}'$, F induces a functor $\operatorname{Add}\mathcal{G} \to \operatorname{Add}\mathcal{G}'$. Clearly, the induced functor is essentially surjective. Let us show that it is fully faithful. For any objects G and G' in $\operatorname{Add}\mathcal{G}$ we consider the map

$$F(G,G') : \mathcal{T}(G,G') \longrightarrow \mathcal{T}'(FG,FG').$$

By hypothesis, it is bijective if G and G' are in $\mathcal{G}$. Let G be in $\mathcal{G}$ and $G' = \coprod_{i\in I} G'_i$, where $(G'_i)_{i\in I}$ is a family in $\mathcal{G}$. Then, $F(G,G')$ is still bijective since we have the following sequence of isomorphisms

$$\begin{aligned}
\mathcal{T}(G,G') &= \mathcal{T}(G,\coprod_{i\in I} G'_i) \\
&= \operatorname{colim}_{J\subset I} \mathcal{T}(G,\coprod_{i\in J} G'_i) && (6)\\
&\xrightarrow{\sim} \operatorname{colim}_{J\subset I} \mathcal{T}'(F(G),F(\coprod_{i\in J} G'_i)) && (7)\\
&\xrightarrow{\sim} \operatorname{colim}_{J\subset I} \mathcal{T}'(F(G),\coprod_{i\in J} F(G'_i)) && (8)\\
&= \mathcal{T}'(F(G),\coprod_{i\in I} F(G'_i)) && (9)\\
\mathcal{T}'(F(G),F(G')) &\xrightarrow{\sim} \mathcal{T}'(F(G),F(\coprod_{i\in I} G'_i))\,, && (10)
\end{aligned}$$

where J runs through the subsets of cardinality strictly smaller than α of I. Here, we have used: (6) G is α-small; (7) $\mathcal{G}$ contains $\coprod_{i\in J} G_i$ since $\mathcal{G}$ is α-localizing; (8) F commutes with coproducts; (9) $F(G)$ is α-small; (10) F commutes with coproducts.

If G' is in $\operatorname{Add}\mathcal{G}$ and $G = \coprod_{i\in I} G_i$, where $(G_i)_{i\in I}$ is a family in $\mathcal{G}$, we have

$$\begin{aligned} \mathcal{T}(G, G') &= \mathcal{T}(\coprod_{i\in I} G_i, G') \\ &\xrightarrow{\sim} \prod_{i\in I} \mathcal{T}(G_i, G') \\ &\xrightarrow{\sim} \prod_{i\in I} \mathcal{T}'(F(G_i), F(G')) \\ &\xleftarrow{\sim} \mathcal{T}'(\coprod_{i\in I} F(G_i), F(G')) \\ \mathcal{T}'(F(G), F(G')) &= \mathcal{T}'(F(\coprod_{i\in I} G_i), F(G')). \end{aligned}$$

2^{nd} *step*: For each object G in $\mathcal{G}$ and each object X in $\mathcal{T}$, F induces a bijection

$$\mathcal{T}(G, X) \longrightarrow \mathcal{T}'(FG, FX).$$

Let $\mathcal{U}$ be the full subcategory of $\mathcal{T}$ formed by the objects X such that F induces a bijection

$$\mathcal{T}(G, X) \longrightarrow \mathcal{T}'(FG, FX)\,,$$

for each G in $\mathcal{G}$. Clearly, $\mathcal{U}$ is a triangulated subcategory. Let us show that $\mathcal{U}$ is stable under formation of coproducts. Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{U}$. We will show that the map

$$\mathcal{T}(G, \coprod_{i\in I} X_i) \longrightarrow \mathcal{T}'(FG, F(\coprod_{i\in I} X_i) = \mathcal{T}'(FG, \coprod_{i\in I} F(X_i))$$

is bijective. Let us show that it is surjective. Let

$$f : FG \longrightarrow \coprod_{i\in I} F(X_i)$$

a morphism in $\mathcal{T}'$. The condition (G4) holds for the subcategory $\mathcal{G}'$ by corollary 3.5. Therefore, as F is an equivalence $\mathcal{G} \to \mathcal{G}'$, there exists a family of objects $(G_i)_{i\in I}$ in $\mathcal{G}$ and a factorization of f

$$FG \xrightarrow[g]{\quad\quad} \coprod_{i\in I} F(G_i) \xrightarrow[\coprod_{i\in I} h_i]{\quad\quad} \coprod_{i\in I} F(X_i)\,,$$

for a family of morphisms $h_i : F(G_i) \to F(X_i)$. As each X_i is in $\mathcal{U}$, we have $h_i = F(k_i)$ for some morphisms $k_i : G_i \to X_i$. Since the object

$$\coprod_{i\in I} F(G_i) = F(\coprod_{i\in I} G_i)$$

is in $\operatorname{Add}\mathcal{G}'$ and F induces an equivalence

$$\operatorname{Add}\mathcal{G} \xrightarrow{\sim} \operatorname{Add}\mathcal{G}'\,,$$

there exists a morphism $l : G \to \coprod_{i\in I} G_i$ such that $F(l)$ gives g. Thus, f is the image of the composition

$$G \xrightarrow[l]{\quad\quad} \coprod_{i\in I} G_i \xrightarrow[\coprod_{i\in I} k_i]{\quad\quad} \coprod_{i\in I} X_i$$

under F. Let us show that it is injective. Let

$$f : G \longrightarrow \coprod_{i\in I} X_i$$

be a morphism such that $F(f) = 0$. As $\mathcal{G}$ has property (G4) by corollary 3.5, we have a factorization

$$G \xrightarrow[g]{} \coprod_{i\in I} G_i \xrightarrow[\coprod_{i\in I} h_i]{} \coprod_{i\in I} X_i\ ,$$

for a family of objects G in $\mathcal{G}$ and a family of morphisms $h_i : G_i \to X_i$. We have

$$F(\coprod_{i\in I} h_i) \circ F(g) = 0.$$

Let us extend the morphism $\coprod_{i\in I} h_i$ and form a distinguished triangle

$$\coprod_{i\in I} Y_i \xrightarrow[\coprod_{i\in I} k_i]{} \coprod_{i\in I} G_i \xrightarrow[\coprod_{i\in I} h_i]{} \coprod_{i\in I} X_i \longrightarrow \Sigma \coprod_{i\in I} Y_i.$$

There exists a morphism $m : FG \to \coprod_{i\in I} Y_i$ such that

$$F(\coprod_{i\in I} k_i) \circ m = F(g).$$

Note that each Y_i is in $\mathcal{U}$ since G_i and X_i are in $\mathcal{U}$. By the surjectivity already shown, we have

$$m = F(l)$$

for a morphism $l : G \to \coprod_{i\in I} Y_i$. Thus,

$$F(\coprod_{i\in I} k_i \circ l) = F(g).$$

As G and $\coprod_{i\in I} G_i$ are in $\operatorname{Add}\mathcal{G}$, it follows that

$$\coprod_{i\in I} k_i \circ l = g.$$

Thus,

$$f = (\coprod_{i\in I} h_i) \circ g = (\coprod_{i\in I} h_i) \circ (\coprod_{i\in I} k_i) \circ l = 0.$$

3^{rd} *step*: The functor F is fully faithful.

Let Y be an object in $\mathcal{T}$. Let $\mathcal{U}$ be the full subcategory of $\mathcal{T}$ formed by the objects X such that F induces a bijection

$$\mathcal{T}(X, Y) \longrightarrow \mathcal{T}'(FX, FY).$$

By the second step, $\mathcal{U}$ contains $\mathcal{G}$. Clearly, $\mathcal{U}$ is a triangulated subcategory. Let us show that $\mathcal{U}$ is stable under formation of coproducts. Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{U}$. Then we have

$$\begin{aligned}\mathcal{T}(\coprod_{i\in I} X_i, Y) &\xrightarrow{\sim} \prod_{i\in I} \mathcal{T}(X_i, Y)\\ &\xrightarrow{\sim} \prod_{i\in I} \mathcal{T}'(F(X_i), F(Y))\\ &= \mathcal{T}'(\coprod_{i\in I} F(X_i), F(Y)).\end{aligned}$$

Thus, $\coprod_{i\in I} X_i$ is indeed in $\mathcal{U}$. It is easy to see that $\mathcal{U}$ contains the direct factors of its objects. So, we have checked that $\mathcal{U}$ is an α-localizing subcategory of $\mathcal{T}$ and contains $\mathcal{G}$. By proposition 4.6 below, the smallest localizing subcategory containing $\mathcal{G}$ is the whole category $\mathcal{T}$. It follows that $\mathcal{U} = \mathcal{T}$.

4^{th} *step*: The functor F is essentially surjective.

The functor F induces an equivalence from $\mathcal{T}$ onto a localizing subcategory $\mathcal{V}$ of $\mathcal{T}'$ by the third step. Indeed, $\mathcal{V}$ is triangulated, stable under coproducts and thick since F is a triangle functor commuting with coproducts. It follows that $\mathcal{V} = \mathcal{T}'$, as $\mathcal{V}$ contains $\mathcal{G}'$, which generates $\mathcal{T}'$. □

It remains to find conditions such that the functor F of the preceding proposition commutes with coproducts. This is made in the following

Theorem 3.8. *Let α be a regular cardinal. Let $\mathcal{T}$ and $\mathcal{T}'$ be two cocomplete triangulated categories. Let $\mathcal{G} \subset \mathcal{T}$ and $\mathcal{G}' \subset \mathcal{T}'$ be two α-localizing subcategories, both of them satisfying conditions* (G1), (G2), (G3) *for α. Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangle functor. Suppose that F induces a functor*

$$\mathcal{G} \to \mathcal{G}'$$

which is essentially surjective and induces bijections

$$\mathcal{T}(G, X) \xrightarrow{\sim} \mathcal{T}'(FG, FX)$$

for all G in $\mathcal{G}$ and X in $\mathcal{T}$. Then F is an equivalence of triangulated categories.

Remark 3.9. We do not suppose that F commutes with coproducts.

Proof. 1^{st} *step*: For each family $(G_i)_{i\in I}$ in $\mathcal{G}$, the morphism

$$\coprod_{i\in I} F(G_i) \longrightarrow F(\coprod_{i\in I} G_i)$$

is invertible.

It is sufficient to show that, for all G' in $\mathcal{G}'$, the map

$$\mathcal{T}'(G', \coprod_{i\in I} F(G_i) \longrightarrow \mathcal{T}'(G', F(\coprod_{i\in I} G_i)$$

is bijective since $\mathcal{G}'$ verifies (G1). As $F : \mathcal{G} \to \mathcal{G}'$ is essentially surjective, it is sufficient to verify this for $G' = FG$ for all G in $\mathcal{G}$. Let $G \in \mathcal{G}$. We have

$$\mathcal{T}'(FG, \coprod_{i\in I} FG_i) = \mathrm{colim}_{J\subset I}\, \mathcal{T}'(FG, \coprod_{i\in J}^{\mathcal{T}'} FG_i) \tag{11}$$

$$= \mathrm{colim}_{J\subset I}\, \mathcal{T}'(FG, \coprod_{i\in J}^{\mathcal{G}'} FG_i) \tag{12}$$

$$= \mathrm{colim}_{J\subset I}\, \mathcal{T}(G, \coprod_{i\in J}^{\mathcal{G}} G_i) \tag{13}$$

$$= \mathcal{T}(G, \coprod_{i\in I} G_i)\,, \tag{14}$$

where J are subsets of I of cardinality strictly smaller than α. Here, we have used: (11) FG is α-small; (12) $\mathcal{G}'$ has α-small coproducts; (13) F induces an equivalence $\mathcal{G} \to \mathcal{G}'$; (14) G is α-small. On the other hand, we have

$$\mathcal{T}'(FG, F\coprod_{i\in I} G_i) \xleftarrow{\sim} \mathcal{T}(G, \coprod_{i\in I} G_i)\,,$$

by the hypothesis, with $X = \coprod_{i\in I} G_i$.

2^{nd} *step*: The functor F induces an equivalence $\mathrm{Add}\,\mathcal{G} \to \mathrm{Add}\,\mathcal{G}'$.

By the first step and the essential surjectivity of $F : \mathcal{G} \to \mathcal{G}'$, F induces an essentially surjective functor from $\mathrm{Add}\,\mathcal{G} \to \mathrm{Add}\,\mathcal{G}'$. By hypothesis, for G in $\mathcal{G}$ and X in $\mathrm{Add}\,\mathcal{G}$, F induces a bijection

$$\mathcal{T}(G, X) \longrightarrow \mathcal{T}'(FG, FX).$$

Let $(G_i)_{i\in I}$ be a family in $\mathcal{G}$ and X an object in $\mathrm{Add}\,\mathcal{G}$. Then,

$$\begin{array}{rcl} \mathcal{T}(\coprod_{i\in I} G_i, X) & \xrightarrow{\sim} & \prod_{i\in I} \mathcal{T}(G_i, X) \\ & \xrightarrow{\sim} & \prod_{i\in I} \mathcal{T}'(F(G_i), F(X)) \\ \mathcal{T}'(F(\coprod_{i\in I} G_i), F(X)) & = & \mathcal{T}'(\coprod_{i\in I} F(G_i), F(X)). \end{array}$$

Thus, F restricted to $\mathrm{Add}\,\mathcal{G}$ is fully faithful.

3^{rd} *step*: The functor F commutes with coproducts.

Let us consider the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F} & \mathcal{T}' \\ {\scriptstyle h_{\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle h'_{\mathcal{T}}} \\ \mathrm{coh}(\mathrm{Add}\,\mathcal{G}) & \xrightarrow[F^*]{\sim} & \mathrm{coh}(\mathrm{Add}\,\mathcal{G}'). \end{array}$$

We will show that it commutes up to isomorphism. Let X be an object in $\mathcal{T}$ and let

$$h(G_1) \longrightarrow h(G_0) \longrightarrow h(X) \longrightarrow 0$$

be a projective presentation, where G_1, G_0 are in $\operatorname{Add}\mathcal{G}$. Then, for all G in $\mathcal{G}$, we obtain an exact sequence

$$\mathcal{T}(G, G_1) \longrightarrow \mathcal{T}(G, G_0) \longrightarrow \mathcal{T}(G, X) \longrightarrow 0.$$

Therefore, the sequence

$$\mathcal{T}'(FG, FG_1) \longrightarrow \mathcal{T}'(FG, FG_0) \longrightarrow \mathcal{T}'(FG, FX) \longrightarrow 0$$

is exact (since isomorphic to the first). It follows that the sequence

$$h(FG_1) \longrightarrow h(FG_0) \longrightarrow h(FX) \longrightarrow 0$$

is exact (since the objects $h(FG)$, $G \in \mathcal{G}$, form a family of projective generators of $\operatorname{coh}(\operatorname{Add}\mathcal{G}'))$. Thus,

$$F^*(h(X)) = \operatorname{cok}(h(FG_1) \longrightarrow h(FG_0))$$

is indeed canonically isomorphic to $h(FX)$. To conclude, note that F^* (which is an equivalence!) and $h_{\mathcal{T}}$ commute with coproducts and that $h'_{\mathcal{T}}$ detects the isomorphisms.

4^{th} *step*: The claim follows thanks to the preceding proposition 3.7. □

4. Thick subcategories and localization of triangulated categories

We recall now some known results about the localizations of triangulated categories and about their thick subcategories, before stating the most important theorem of this section concerning the localization of *well generated* triangulated categories. For complete proofs of the cited results, we refer to Neeman's book [27, Ch. 2, p. 73] and the classical [38, Ch. 2.2, p. 111-133].

4.1. Localization of triangulated categories.

We begin with a collection of properties of the *triangle quotient* [38, Ch. 2.2, p. 111-133] of triangulated categories.

Proposition 4.1. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, let Φ be a* set *of morphisms in $\mathcal{T}$ and $\mathcal{N}$ the smallest triangulated subcategory of $\mathcal{T}$ containing the* $\mathrm{cone}(s)$*, with $s \in \Phi$, stable under arbitrary coproducts. Then the following assertions hold:*

a) *$\mathcal{T}/\mathcal{N}$ is a triangulated category and admits arbitrary coproducts;*
b) *the canonical functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ commutes with all coproducts;*
c) *the morphisms $Q(s)$ are invertible for all $s \in \Phi$;*
d) *if $F : \mathcal{T} \to \mathcal{S}$ is a triangle functor, where $\mathcal{S}$ is a triangulated category which admits all coproducts and the functor F commutes with all coproducts and makes every $s \in \Phi$ invertible in $\mathcal{S}$, then $F = \overline{F} \circ Q$ for a unique coproduct preserving triangle functor $\overline{F} : \mathcal{T}/\mathcal{N} \to \mathcal{S}$;*
e) *more precisely, if $\mathcal{S}$ is a triangulated category with arbitrary coproducts, there is an isomorphism of categories*

$$\mathcal{F}un_{cont}(\mathcal{T}/\mathcal{N}, \mathcal{S}) \xrightarrow{\sim} \mathcal{F}un_{cont,\Phi}(\mathcal{T}, \mathcal{S}) \,,$$

where $\mathcal{F}un_{cont}$ is the category of triangulated functors commuting with arbitrary coproducts, and $\mathcal{F}un_{cont,\Phi}$ is the category of the functors in $\mathcal{F}un_{cont}$ which have the additional property of making all $s \in \Phi$ invertible.

Proof. See Chapter 2 in [27] or [38, Ch. 2.2, p. 111-133]. We give only an argument for the commutativity of Q with coproducts because it is a general one, useful in other contexts. Let $\prod_I \mathcal{T}$ be the product category of copies of $\mathcal{T}$ indexed by I. Using the universality of coproducts it is easy to check that the functor $\coprod_{i \in I} : \prod_I \mathcal{T} \to \mathcal{T}$ which takes a family $(X_i)_{i \in I}$ to the coproduct $\coprod_{i \in I} X_i$ is left adjoint to the diagonal functor Δ. It is clear that $\Delta(\Phi) \subseteq \prod_I(\Phi)$ and that $\coprod_{i \in I}(\prod_I(\Phi)) \subseteq \Phi$. Therefore, the pair $\coprod_{i \in I} \dashv \Delta$ induces the following commutative diagram

$$\begin{array}{ccccc}
\prod_I \mathcal{T} & \xrightarrow{\;can\;} & (\prod_I \mathcal{T})[(\prod_I \Phi)^{-1}] & = & \prod_I(\mathcal{T}[\Phi^{-1}]) \\
{\scriptstyle \coprod_{i\in I}} \downarrow \uparrow {\scriptstyle \Delta} & & & & {\scriptstyle \coprod_{i\in I}} \downarrow \uparrow {\scriptstyle \Delta} \\
\mathcal{T} & & \xrightarrow{\;can\;} & & \mathcal{T}[\Phi^{-1}]
\end{array}$$

which entails the required commutativity of Q with all the coproducts over the set I. Of course, this construction is possible for every set I. □

The functor Q is usually called *(canonical) quotient functor*, even if $\mathcal{T}$ does not have coproducts. In general, the morphisms between two objects in a triangle quotient do not form a set. However, this is the case if the quotient functor Q admits a right adjoint Q_ρ, because Q_ρ is automatically fully faithful.

Definition 4.2. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. A triangle functor $F : \mathcal{T}' \to \mathcal{T}$ is a *localization functor* if it is fully faithful and admits a left adjoint functor.

If $F : \mathcal{T}' \to \mathcal{T}$ is a localization functor and F_λ is left adjoint, then F_λ induces an equivalence from the triangle quotient $\mathcal{T}/\ker(F_\lambda)$ to $\mathcal{T}'$. Via this equivalence, F_λ identifies with the quotient functor $\mathcal{T} \to \mathcal{T}/\ker(F_\lambda)$ and F with its right adjoint.

4.2. **Some thick subcategories of triangulated categories.** Let us recall that a full triangulated subcategory of a triangulated category is called *thick*, (*épaisse* in the French literature, *saturée* in the original definition in Verdier's thesis [38, 2.2.6, p. 114]) if it contains the direct factors of its objects. We remark that this property is automatically verified if the triangulated category has countable coproducts, since in this case idempotents splits (see [2], [27, Prop. 1.6.8, p. 65] for definitions and properties of idempotents in triangulated categories). Now we give definitions and notations about some important subcategories of a triangulated category $\mathcal{T}$ with arbitrary coproducts and suspension functor Σ. The best reference for this material is [27, Ch. 3-4]. We recall that a full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *localizing* if it is closed under arbitrary coproducts. It is called α*-localizing*, for a given regular cardinal α, if it is thick and closed under α*-coproducts* of its objects, *i.e.* coproducts of objects formed over a set of cardinality strictly smaller than α. We write $\langle \mathcal{S} \rangle_\alpha$ (resp. $\langle \mathcal{S} \rangle$) for the smallest α-localizing (resp. localizing) subcategory of $\mathcal{T}$ containing $\mathcal{S}$, where $\mathcal{S}$ is a set or a class of objects in $\mathcal{T}$ and α a *regular* cardinal. Note that in the above definitions the requirement that the subcategories are thick is necessary only for the case $\alpha = \aleph_0$, since for $\alpha > \aleph_0$ these subcategories are automatically thick as we underlined at the beginning of the section. In his book [27, Ch. 3-4] Neeman shows the very important properties of the previous subcategories and of the subcategories of the α-small objects $\mathcal{T}^{(\alpha)}$ and that of the α-compact objects $\mathcal{T}^\alpha$. They are triangulated, α-localizing and thick subcategories of $\mathcal{T}$ for $\alpha > \aleph_0$. There is the following filtration: if $\alpha \leq \beta$ then $\mathcal{T}^\alpha \subseteq \mathcal{T}^\beta$. Clearly $\langle \mathcal{S} \rangle = \bigcup_\alpha \langle \mathcal{S} \rangle_\alpha$, but if $\mathcal{S}$ contains a good set of generators for $\mathcal{T}$, then $\mathcal{T} = \bigcup_\alpha \langle \mathcal{S} \rangle_\alpha = \langle \mathcal{S} \rangle$, where α runs over all regular cardinals. If $\mathcal{T}$ is a well generated triangulated category, then $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$, where α runs over all regular cardinals.

We have the

Theorem 4.3 ([27, Lemma 3.2.10, p. 107])**.** *Let β be an infinite cardinal. Let $\mathcal{T}$ be a triangulated category closed under the formation of coproducts of fewer than β of its objects. Let $\mathcal{N}$ be a β-localizing subcategory of $\mathcal{T}$. Then $\mathcal{T}/\mathcal{N}$ is closed with respect to the formation of coproducts of fewer than β of its objects, and the universal functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ preserves coproducts.*

and its

Corollary 4.4 ([27, Cor. 3.2.11, p. 110])**.** *If $\mathcal{T}$ is a triangulated category with all coproducts and $\mathcal{N}$ is a localizing subcategory of $\mathcal{T}$, then $\mathcal{T}/\mathcal{N}$ is a triangulated* *category which admits all coproducts and the universal functor $\mathcal{T} \to \mathcal{T}/\mathcal{N}$ preserves coproducts.*

Now we state one of the major results in the theory of triangulated categories. This result has a long story (see for example [5], [26]), which comes from algebraic topology. We state it in the modern and general form Neeman gives it in his book, see [27, Thm. 1.17, p. 16] and [27, Thm. 8.3.3, p. 282] for a more general statement and the proof.

Theorem 4.5. *(Brown representability). Let $\mathcal{T}$ be a well-generated triangulated category. Let H be a contravariant functor $H : \mathcal{T}^{op} \to \mathcal{A}b$. The functor H is representable if and only if it is cohomological and takes coproducts in $\mathcal{T}$ to products of abelian groups.*

Let us now clarify the meaning of two similar but different notions. Let $\mathcal{T}$ be a triangulated category and $\mathcal{G}$ a set of objects in $\mathcal{T}$. We say that $\mathcal{T}$ is *generated by* $\mathcal{G}$ or, equivalently, that $\mathcal{G}$ *generates* $\mathcal{T}$ if $\mathcal{T} = \langle \mathcal{G} \rangle$. In contradistinction, we say that $\mathcal{G}$ is a *generating set* for $\mathcal{T}$ if condition (G1) of definition 3.1 holds for the subset $\mathcal{G}$ of $\mathcal{T}$. Let us recall condition (G1): an object $X \in \mathcal{T}$ is zero if and only if $\mathcal{T}(G, X) = 0$ for all G in $\mathcal{G}_0$ (we always assume that $\Sigma\mathcal{G}_0 = \mathcal{G}_0$). The former notion is stronger than the latter. If $\mathcal{G}$ generates $\mathcal{T}$, then $\mathcal{G}$ is a generating set for $\mathcal{T}$, whereas the converse is not true in general but holds if $\mathcal{G}$ is assumed further to be a $\aleph_1$*-perfect generating set for* $\mathcal{T}$ in the sense of Neeman (*cf.* [27, Ch. 8, Def. 8.1.2, p. 273]). Moreover, we give another link between the two notions, which is useful because it covers the case of well generated triangulated categories.

Proposition 4.6. *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{G}$ a generating set for $\mathcal{T}$,* i.e. *condition* (G1) *holds for $\mathcal{G}$. Then $\mathcal{G}$ generates $\mathcal{T}$,* i.e. $\langle \mathcal{G} \rangle = \mathcal{T}$.

Proof. Let us call $\mathcal{N}$ the subcategory $\langle \mathcal{G} \rangle$. Since $\mathcal{N}$ is a localizing subcategory generated by the *set* $\mathcal{G}$, it is well generated by corollary 4.12 below. Then, the Brown representability theorem (4.5) holds for $\mathcal{N}$. Therefore, for each object $X \in \mathcal{T}$, the functor $\mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{G}} : \langle \mathcal{G} \rangle^{op} \to \mathcal{A}b$, which is cohomological and sends coproducts into products, is representable. For each object X in $\mathcal{T}$ there exists an object $X_{\mathcal{N}}$ in $\mathcal{N}$ such that $\mathrm{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{G}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{N}}(-, X_{\mathcal{N}})$. Thus, we have obtained a functor i_ρ right adjoint to the fully faithful inclusion: $i : \mathcal{N} \to \mathcal{T}$. Consider now, for every $X \in \mathcal{T}$, the distinguished triangle in $\mathcal{T}$

$$ii_\rho X \longrightarrow X \longrightarrow Y \longrightarrow \Sigma ii_\rho X.$$

Applying to the triangle the covariant functor $\mathrm{Hom}_{\mathcal{T}}(iN, -)$, N an object of $\mathcal{N}$, we obtain a long exact sequence of abelian groups. Consider the part corresponding to the input triangle: The map from the first term $\mathrm{Hom}_{\mathcal{T}}(iN, ii_\rho X)$ to the second term $\mathrm{Hom}_{\mathcal{T}}(iN, X)$ is easily seen to be an isomorphism, since i is fully faithful and i_ρ is its right adjoint. Similarly, the map from the fourth to the fifth term is an isomorphism. Therefore, the third group $\mathrm{Hom}_{\mathcal{T}}(iN, Y)$ must be zero for all $N \in \mathcal{N}$. This forces the object Y to lie in $\mathcal{N}^\perp$. But $\mathcal{N}^\perp$ is zero. Indeed, condition (G1) holds for $\mathcal{G}$, *i.e.* $\mathcal{G}^\perp = 0$. Thus, the inclusion $\mathcal{G} \subseteq \mathcal{N}$ gives $\mathcal{N}^\perp \subseteq \mathcal{G}^\perp = 0$, *i.e.* $\mathcal{N}^\perp = 0$. Therefore, we have $Y = 0$. By the triangle above, this means $ii_\rho X \xrightarrow{\sim} X$, for all X in $\mathcal{T}$. It follows that i is an equivalence of categories, which gives $\mathcal{T} = \langle \mathcal{G} \rangle$. ☐

4.3. **Localization of well generated triangulated categories.** In this section, we will state a Theorem about particular localizations of well generated triangulated categories, those which are triangle quotients by a subcategory *generated by a set*. One could obtain this result using Thomason's powerful Theorem [36, Key Proposition 5.2.2, p. 338] in its generalized form given by Neeman in [27, Thm. 4.4.9, p. 143]. Neeman himself does this in [28] in proving that the derived category of a Grothendieck category is always a well generated triangulated category. We will give a more detailed and slightly different proof in order to make clear the machinery behind Thomason-Neeman's Theorem. Before doing this task we recall the key ingredient of the proof. We give a proof of this result to make clear that it works also with the definition 3.1 of Krause.

Theorem 4.7 ([27, Ch. 4, Thm. 4.3.3, p. 131])**.** *Let $\mathcal{T}$ be a triangulated category with small coproducts. Let β be a regular cardinal. Let $\mathcal{S}$ be some class of objects in $\mathcal{T}^\beta$. Let X be a β-compact object of $\mathcal{T}$,* i.e. *$X \in \mathcal{T}^\beta$, and let Z be an object of $\langle \mathcal{S} \rangle$. Suppose that $f : X \to Z$ is a morphism in $\mathcal{T}$. Then there exists an object $Y \in \langle \mathcal{S} \rangle_\beta$ so that f factors as $X \to Y \to Z$.*

Proof. Let $\mathcal{W}$ be the full subcategory of $\langle \mathcal{S} \rangle$ whose objects are all the $W \in \langle \mathcal{S} \rangle$ such that, for every object $X \in \mathcal{T}^\beta$ and every morphism $X \to W$, there does exist a factorization

$$\begin{array}{ccc} X & \longrightarrow & W \\ & \searrow \quad \nearrow & \\ & Y\,, & \end{array}$$

where the object Y lies in $\langle \mathcal{S} \rangle_\beta$. Clearly, it suffices to show that $\langle \mathcal{S} \rangle \subseteq \mathcal{W}$. We will show that $\mathcal{W}$ contains $\mathcal{S}$, that it is a triangulated subcategory and that it is closed under formation of arbitrary small coproducts. This is sufficient since $\langle \mathcal{S} \rangle$ is minimal with these properties. Note that $\mathcal{W}$ will also automatically be thick, since to be a localizing subcategory means to be α-localizing for every regular cardinal α.

It is clear that $\mathcal{W}$ contains $\mathcal{S}$. Indeed, we can factor any morphism $X \to Z$, with $Z \in \mathcal{S}$ through the identity morphism of Z, and Z lies in $\langle \mathcal{S} \rangle_\beta$.

Note that if W is in $\mathcal{W}$, then ΣW is in $\mathcal{W}$, too. Indeed, consider an arbitrary morphism

$$p : X \longrightarrow \Sigma W\,,$$

with X in $\mathcal{T}^\beta$. Since the functor Σ^{-1} is an endoequivalence of $\mathcal{T}$ the morphism p uniquely corresponds to a morphism $\Sigma^{-1}p : \Sigma^{-1}X \to W$. This morphism factors through an object $Y \in \langle \mathcal{S} \rangle_\beta$

$$\begin{array}{ccc} \Sigma^{-1}X & \xrightarrow{\Sigma^{-1}p} & W \\ & \searrow \quad \nearrow & \\ & Y\,, & \end{array}$$

as W lies in $\mathcal{W}$ and $\Sigma^{-1}X$ is in $\mathcal{T}^\beta$. Apply another time Σ^{-1} to get the factorization

$$\begin{array}{ccc} X & \xrightarrow{\;p\;} & \Sigma W \\ & \searrow \quad \nearrow & \\ & \Sigma Y. & \end{array}$$

As ΣY is in $\langle \mathcal{S} \rangle_\beta$, ΣW is in $\mathcal{W}$. Similarly, one sees that $\Sigma^{-1}W$ is in $\mathcal{W}$.

Consider an arbitrary morphism

$$h : W \longrightarrow W'$$

in $\mathcal{W}$. We want to show that there is a distinguished triangle in $\mathcal{W}$

$$(*) \qquad W \xrightarrow{\;h\;} W' \xrightarrow{\;l\;} W'' \xrightarrow{\;m\;} \Sigma W\ ,$$

which contains the morphism h.

Consider the distinguished triangle in $\langle \mathcal{S} \rangle$

$$W \xrightarrow{\;h\;} W' \xrightarrow{\;l'\;} S \xrightarrow{\;m'\;} \Sigma W\ ,$$

which contains the morphism h. We want to show that the object S lies in $\mathcal{W}$, too. Let us consider an arbitrary morphism

$$q : X \longrightarrow S\ ,$$

where X is an object in $\mathcal{T}^\beta$. By composition with m' we get the morphism

$$m' \circ q : X \longrightarrow \Sigma W.$$

Since $X \in \mathcal{T}^\beta$ and we have seen that $\Sigma W \in \mathcal{W}$ this morphism factors through an object Y' of $\langle \mathcal{S} \rangle_\beta$

$$\begin{array}{ccc} X & \xrightarrow{\;m'\circ q\;} & \Sigma W \\ & {\scriptstyle r}\searrow \quad \nearrow{\scriptstyle s} & \\ & Y'. & \end{array}$$

Therefore, the square

$$\begin{array}{ccc} X & \xrightarrow{\;r\;} & Y' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle s} \\ S & \xrightarrow[\;m'\;]{} & \Sigma W\ , \end{array}$$

is commutative. Using the axioms of triangulated categories we can extend this square to a morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\;r\;} & Y' & \xrightarrow{\;r'\;} & X' & \xrightarrow{\;r''\;} & \Sigma X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \Sigma q} \\ S & \xrightarrow[\;m'\;]{} & \Sigma W & \xrightarrow[\;m''\;]{} & \Sigma W' & \xrightarrow[\;m'''\;]{} & \Sigma S. \end{array}$$

Notice that the subcategory $\langle \mathcal{S} \rangle_\beta$ is contained in $\mathcal{T}^\beta$. Indeed, $\mathcal{T}^\beta$ is β-localizing, it contains $\mathcal{S}$, and $\langle \mathcal{S} \rangle_\beta$ is minimal with these properties. Hence the first distinguished triangle in the preceding diagram lies in $\mathcal{T}^\beta$. As $\Sigma W'$ is in $\mathcal{W}$ we have the factorization of t

$$\begin{array}{ccc} X' & \xrightarrow{\quad t \quad} & \Sigma W' \\ & {\scriptstyle p'}\searrow \quad \nearrow{\scriptstyle s'} & \\ & Y'' \, , & \end{array}$$

with Y'' in $\langle \mathcal{S} \rangle_\beta$. Form the composition

$$p' \circ r' : Y' \longrightarrow Y'' \, ,$$

and extend it to a distinguished triangle in $\langle \mathcal{S} \rangle_\beta$

$$\Sigma^{-1} Y''' \xrightarrow{\ q''\ } Y' \xrightarrow{\ p' \circ r'\ } Y'' \xrightarrow{\ q'\ } Y'''.$$

Putting together the preceding distinguished triangles we obtain the following morphisms of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\ r\ } & Y' & \xrightarrow{\ r'\ } & X' & \xrightarrow{\ r''\ } & \Sigma X \\ \Big\downarrow{\scriptstyle \Sigma^{-1} t'} & & \Big\| & & \Big\downarrow{\scriptstyle p'} & {\scriptstyle \varphi} & \Big\downarrow{\scriptstyle t'} \\ \Sigma^{-1} Y''' & \xrightarrow{\ q''\ } & Y' & \xrightarrow{\ p' \circ r'\ } & Y'' & \xrightarrow{\ q'\ } & Y''' \\ \Big\downarrow{\scriptstyle \Sigma^{-1} t''} & & \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle s'} & & \Big\downarrow{\scriptstyle t''} \\ S & \xrightarrow[\ m'\ ]{} & \Sigma W & \xrightarrow[\ m''\ ]{} & \Sigma W' & \xrightarrow[\ m'''\ ]{} & \Sigma S. \end{array} \qquad (\Sigma q : \Sigma X \to \Sigma S,\ \ \varphi : \Sigma X \dashrightarrow \Sigma W')$$

Indeed by axiom TR3, there are morphisms t' and t'' which make the diagram commute. Hence the morphisms $\Sigma^{-1} t'$ and $\Sigma^{-1} t''$ complete the diagram. Notice that $s' \circ p'$ equals t whereas $t'' \circ t'$ is *not* equal to Σq and so $\Sigma^{-1} t'' \circ \Sigma^{-1} t'$ is *not* equal to q. Anyway, the compositions $\Sigma m' \circ t'' \circ t'$ and $\Sigma m' \circ \Sigma q$ are equal. Therefore, by applying $\mathrm{Hom}_{\mathcal{T}}(\Sigma X, -)$ to the distinguished triangle in the third row of the last diagram, it is immediate that

$$t'' \circ t' - \Sigma q = m''' \circ \varphi \, ,$$

for some morphism $\varphi : \Sigma X \to \Sigma W'$, as in the diagram. Then, the correct expression of Σq is

$$\Sigma q = m''' \circ (-\varphi) + t'' \circ t'$$

which shows that Σq factors as

$$\Sigma X \xrightarrow[\begin{bmatrix} -\varphi \\ t' \end{bmatrix}]{} \Sigma W' \oplus Y''' \xrightarrow[{[m''' \, , \, t'']}]{} \Sigma S.$$

Since ΣX is in $\mathcal{T}^\beta$ and $\Sigma W'$ in $\mathcal{W}$, the morphism φ factors through an object $\overline{Y}$ lying in $\langle \mathcal{S} \rangle_\beta$

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\varphi} & \Sigma W' \\ & \searrow^{\overline{p}} \quad \nearrow_{\overline{q}} & \\ & \overline{Y}. & \end{array}$$

By composition, we have

$$\Sigma q = m''' \circ (\overline{q} \circ (-\overline{p})) + t'' \circ t' \ ,$$

which provides a further factorization of Σq through $\overline{Y} \oplus Y'''$

$$\Sigma X \xrightarrow{\begin{bmatrix} -\overline{p} \\ t' \end{bmatrix}} \overline{Y} \oplus Y''' \xrightarrow{[m''' \circ \overline{q} \ , \ t'']} \Sigma S.$$

Thus we have proved that the morphism $q : X \to S$ factors through the object $\Sigma^{-1}(\overline{Y} \oplus Y''')$ lying in $\langle \mathcal{S} \rangle_\beta$. Hence the object S lies in $\mathcal{W}$ and can be identified with the object $W'' \in \mathcal{W}$ in the distinguished triangle (*). Hence $\mathcal{W}$ is triangulated.

It remains to show that $\mathcal{W}$ is closed under formation of arbitrary small coproducts. Let I be an arbitrary set and W_i , $i \in I$, a family of objects of $\mathcal{W}$. Consider an arbitrary morphism

$$X \longrightarrow \coprod_{i \in I} W_i \ ,$$

with X in $\mathcal{T}^\beta$. The objects of $\mathcal{T}^\beta$ are β-small. Therefore, by condition (G3) of definition 3.1 we have the factorization

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{i \in I} W_i \\ & \searrow \quad \nearrow & \\ & \coprod_{i \in J} W_i \ , & \end{array}$$

where J is a subset of I of cardinality strictly smaller than α. Now we can apply condition (G4) to the morphism $X \to \coprod_{i \in J} W_i$. For all $i \in J$, there exist objects $X_i \in \mathcal{T}^\beta$ and morphisms $f_i : X_i \to W_i$ so that the following factorization holds

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{i \in J} W_i \\ & \searrow \quad \nearrow_{\coprod_{i \in J} f_i} & \\ & \coprod_{i \in J} X_i. & \end{array}$$

Each morphism f_i factors through an object Y_i lying in $\langle \mathcal{S} \rangle_\beta$, for all $i \in J$,

$$X_i \xrightarrow{\sigma_i} Y_i \xrightarrow{\rho_i} W_i.$$

Therefore we have the factorization

$$\begin{array}{ccc} \coprod_{i\in J} X_i & \xrightarrow{\coprod_{i\in J} f_i} & \coprod_{i\in J} W_i \\ & {\scriptstyle \coprod_{i\in J}\sigma_i} \searrow \quad \nearrow {\scriptstyle \coprod_{i\in J}\rho_i} & \\ & \coprod_{i\in J} Y_i. & \end{array}$$

Putting together the preceding factorizations we obtain the commutative triangle

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{i\in I} W_i \\ & \searrow \quad \nearrow & \\ & \coprod_{i\in J} Y_i. & \end{array}$$

Note that the object $\coprod_{i\in J} Y_i$ belongs to $\langle \mathcal{S}\rangle_\beta$ since it is a β-small coproduct of objects of $\langle \mathcal{S}\rangle_\beta$. This shows that $\coprod_{i\in I} W_i$ lies in $\mathcal{W}$. Thus, $\mathcal{W}$ equals $\langle \mathcal{S}\rangle$, and we are done. □

The power of this factorization property is seen at once, since it is the key to obtain the following results.

Corollary 4.8 ([27, Ch. 4, Lemma 4.4.5, p. 140 for item a) and Lemma 4.4.8, p. 142 for item b)])**.** *Let $\mathcal{T}$ be a triangulated category with small coproducts. Let $\mathcal{S}$ be some class of objects in $\mathcal{T}^\alpha$ for some infinite cardinal α. Let $\beta \geq \alpha$ be a regular cardinal. Then:*

a) *if $\langle \mathcal{S}\rangle = \mathcal{T}$, then the inclusion $\langle \mathcal{S}\rangle_\beta \subseteq \mathcal{T}^\beta$ is an equality;*
b) *let $\mathcal{N} = \langle \mathcal{S}\rangle$. Then there is an inclusion $\mathcal{N} \cap \mathcal{T}^\beta \subseteq \mathcal{N}^\beta$.*

Proof. a) Let X be an object of $\mathcal{T}^\beta$ and consider the identity map $\mathbf{1}_X : X \to X$. As X is at the same time in $\mathcal{T}^\beta$ and in $\langle \mathcal{S}\rangle$ we can apply the theorem 4.7 and factor $\mathbf{1}_X$ through some object $Y \in \langle \mathcal{S}\rangle_\beta$. Thus the object X is a direct factor of Y. Since $\langle \mathcal{S}\rangle_\beta$ is thick, we have $X \in \langle \mathcal{S}\rangle_\beta$.

b) Let K be an object of $\mathcal{N} \cap \mathcal{T}^\beta$. Then K is β-small as an object of $\mathcal{N}$ since the inclusion $\mathcal{N} \subseteq \mathcal{T}$ commutes with coproducts. Now, let $K \to \coprod_{i\in I} X_i$ be a morphism, where the objects X_i belong to $\mathcal{N}$. It factors through a morphism $\coprod_{i\in I} f_i : \coprod_{i\in I} K_i \to \coprod_{i\in I} X_i$, where the objects K_i belong to $\mathcal{T}^\beta$. By the theorem above, each morphism $K_i \to X_i$ factors through an object K_i' belonging to $\langle \mathcal{S}\rangle_\beta \subseteq \mathcal{N} \cap \mathcal{T}^\beta$. Therefore the class $\mathcal{N} \cap \mathcal{T}^\beta$ satisfies (G4) in $\mathcal{N}$ and we obtain the required inclusion. □

The next proposition states some useful properties of the images in the quotient category $\mathcal{T}/\mathcal{N}$ of the maps of the subcategories $\langle \mathcal{G}\rangle_\beta$ of $\mathcal{T}$ under the canonical quotient functor Q.

Proposition 4.9. *Let α be a regular cardinal. Let $\mathcal{T}$ be a triangulated category with small coproducts, generated by a class of objects $\mathcal{G} \subseteq \mathcal{T}^\alpha$. Let $\mathcal{S}$ be an arbitrary class of objects in $\mathcal{T}^\alpha$ and Q the canonical quotient functor*

$$Q : \mathcal{T} \to \mathcal{T}/\langle \mathcal{S}\rangle.$$

Let $\beta \geq \alpha$ be a regular cardinal. Then:

a) *each morphism $u : Q(G) \to Q(X)$, where G is an object of $\langle \mathcal{G} \rangle_\beta$ and $Q(X)$ an arbitrary object of $\mathcal{T}/\mathcal{N}$, is the equivalence class of a diagram in $\mathcal{T}$*

$$G \xleftarrow{\sim} G' \longrightarrow X\,,$$

where the object G' belongs to $\mathcal{T}^\beta = \langle \mathcal{G} \rangle_\beta$ and the arrow $\xrightarrow{\sim}$ means a morphism whose image under Q is invertible; in particular, the morphisms from $Q(G)$ to $Q(X)$ in $\mathcal{T}\langle \mathcal{S} \rangle$ form a set if $\mathcal{G}$ is a set;

b) *the image of $\langle \mathcal{G} \rangle_\beta$ under the (restriction of the) functor Q is a full triangulated subcategory of $\mathcal{T}/\mathcal{N}$;*

c) *if β is uncountable, then $\langle Q\mathcal{G} \rangle_\beta$ equals $Q(\langle \mathcal{G} \rangle_\beta)$. If β is countable, then $\langle Q\mathcal{G} \rangle_\beta$ equals the closure of $Q(\langle \mathcal{G} \rangle_\beta)$ under taking direct factors.*

Proof. a) Let $u : Q(G) \to Q(X)$ be a morphism in $\mathcal{T}/\mathcal{N}$. It is the equivalence class of a 'roof' diagram in $\mathcal{T}$

$$G \xleftarrow{\sim} T \longrightarrow X\,,$$

where the object T belongs to $\mathcal{T}$. We can form the distinguished triangle

$$N \longleftarrow G \xleftarrow{\sim} T \longleftarrow \Sigma^{-1} N\,,$$

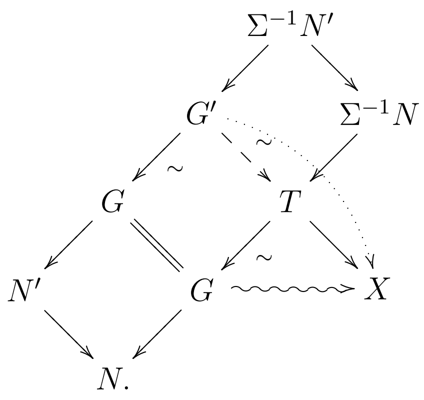
where N and $\Sigma^{-1}N$ lie in $\langle \mathcal{S} \rangle$. The object G is β-compact in $\mathcal{T}$. Therefore, we can apply theorem 4.7 to the morphism $N \leftarrow G$ and factor it as

$$N \longleftarrow N' \longleftarrow G\,,$$

where N' belongs to $\langle \mathcal{S} \rangle_\beta$. The class $\langle \mathcal{S} \rangle_\beta$ is contained in $\mathcal{T}^\beta$, since $\mathcal{S}$ is contained in $\mathcal{T}^\beta$ by the hypothesis. Therefore, we can complete the morphism $N' \leftarrow G$ to a distinguished triangle in $\mathcal{T}^\beta$

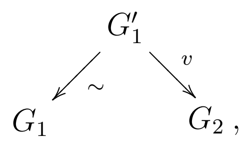
$$N' \longleftarrow G \xleftarrow{\sim} G' \longleftarrow \Sigma^{-1} N'$$

and deduce a map of distinguished triangles



adding the morphism $G' \to T$. The wavy arrow stands for the given morphism $Q(G) \to Q(X)$ in $\mathcal{T}/\mathcal{N}$, whereas the dotted arrow is the composition $G' \to T \xrightarrow{\sim} X$. The roof diagrams $G \xleftarrow{\sim} G' \to X$ and $G \xleftarrow{\sim} T \to X$ are clearly equivalent. We have supposed that $\mathcal{T}$ has small coproducts and that $\langle \mathcal{G} \rangle = \mathcal{T}$, with $\mathcal{G}$ contained in $\mathcal{T}^\alpha$, hence in $\mathcal{T}^\beta$. Therefore, $\langle \mathcal{G} \rangle_\beta = \mathcal{T}^\beta$ by point a) of corollary 4.8. This shows that G' also lies in $\langle \mathcal{G} \rangle_\beta$.

b) Clearly, the image of $\langle \mathcal{G} \rangle_\beta$ under Q is stable under Σ and Σ^{-1}. We have to show that it is stable under forming cones. Let G_1 and G_2 be two objects of $\langle \mathcal{G} \rangle_\beta$ and u a morphism from QG_1 to QG_2. By part a), the morphism u equals the equivalence class of a diagram



where G_1' belongs to $\langle \mathcal{G} \rangle_\beta$. Therefore, the cone C on v still belongs to $\langle \mathcal{G} \rangle_\beta$. Clearly, the cone on u is isomorphic to $Q(C)$, which still belongs to the image under Q of $\langle \mathcal{G} \rangle_\beta$.

c) Let $\mathcal{U}$ be the closure of $Q(\langle \mathcal{G} \rangle_\beta)$ under taking direct factors. We claim that $\mathcal{U}$ equals $\langle Q\mathcal{G} \rangle_\beta$ for all $\beta \geq \alpha$. Indeed, we have $Q(\langle \mathcal{G} \rangle_\beta) \subseteq \langle Q\mathcal{G} \rangle_\beta$ since Q is a triangle functor and commutes with arbitrary coproducts. It follows that $\mathcal{U} \subseteq \langle Q\mathcal{G} \rangle_\beta$ since $\langle Q\mathcal{G} \rangle_\beta$ is thick. For the reverse inclusion, we notice that $\mathcal{U}$ contains $Q\mathcal{G}$, that it is a triangulated subcategory since $Q(\langle \mathcal{G} \rangle_\beta)$ is a triangulated subcategory (by b), and that it is thick (by definition). We have thus proved the claim for countable β. Now suppose β is uncountable. Then $Q(\langle \mathcal{G} \rangle_\beta)$ is a triangulated subcategory stable under forming countable coproducts. Therefore, it is stable under taking direct factors (*cf.* 4.2) and thus equals $\mathcal{U} = \langle Q\mathcal{G} \rangle_\beta$. □

Now we can state the most important theorem of this section. This theorem has been inspired by Neeman's generalization to well generated categories [27, Thm. 4.4.9, p. 143] of Thomason-Trobaugh's theorem [36, Key Proposition 5.2.2, p. 338].

Theorem 4.10. *Let $\mathcal{T}$ be an α-compactly generated triangulated category and $\mathcal{G}$ a set of good generators for $\mathcal{T}$, contained in $\mathcal{T}^\alpha$. Let $\mathcal{S}$ be a* set *of objects contained in $\mathcal{T}^\gamma$, for some fixed regular cardinal γ. Let $\mathcal{N} = \langle \mathcal{S} \rangle$ and Q the canonical quotient functor*

$$Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}.$$

a) *The localizing triangulated subcategory $\mathcal{N}$ is the union*

$$\mathcal{N} = \bigcup_{\delta \geq \gamma} \mathcal{N}^\delta ,$$

where δ runs through the regular cardinals. Equivalently, $\mathcal{N}$ is given by the same union as above, formed over all regular cardinals;

b) *the subcategory $\mathcal{N}$ is δ-compactly generated for all regular cardinals $\delta \geq \gamma$ by the set $\langle \mathcal{S} \rangle_\gamma$;*

c) *the subcategory $Q(\langle \mathcal{G} \rangle_\beta)$ equals $\langle Q\mathcal{G} \rangle_\beta$ for $\beta > \aleph_0$ and its closure under taking direct factors equals $\langle Q\mathcal{G} \rangle_\beta$ for $\beta = \aleph_0$;*

d) *the quotient category $\mathcal{T}/\mathcal{N}$ is a δ-compactly generated triangulated category for all regular cardinals $\delta \geq \beta$, where $\beta = \sup(\alpha, \gamma)$, with set of good generators $Q(\langle \mathcal{G} \rangle_\beta)$.*

Proof. It is clearly sufficient to prove b) for $\delta = \gamma$ and c) for $\delta = \beta$.

a) The triangulated category $\mathcal{T}$ is well generated. Therefore, it is the union over all the regular cardinals σ of its subcategories $\mathcal{T}^\sigma$ [17, Corollary of Thm. A]. We know from the hypothesis that $\mathcal{S} \subseteq \mathcal{T}^\gamma$, hence $\mathcal{S} \subseteq \mathcal{N} \cap \mathcal{T}^\gamma$. Clearly, $\langle \mathcal{S} \rangle_\gamma \subseteq \mathcal{N} \cap \mathcal{T}^\gamma$, since $\langle \mathcal{S} \rangle_\gamma$ is the smallest γ-localizing subcategory of $\mathcal{T}$ containing the set $\mathcal{S}$. Moreover, $\mathcal{N} \cap \mathcal{T}^\gamma \subseteq \mathcal{N}^\gamma$ by point b) of corollary 4.8. Thus, we have the following sequence of inclusions:

$$\mathcal{S} \subseteq \langle \mathcal{S} \rangle_\gamma \subseteq \mathcal{N} \cap \mathcal{T}^\gamma \subseteq \mathcal{N}^\gamma.$$

Therefore, for each regular cardinal $\delta \geq \gamma$, we obtain $\langle \mathcal{S} \rangle_\delta = \mathcal{N} \cap \mathcal{T}^\delta = \mathcal{N}^\delta$, by point a) of corollary 4.8. The claim now follows by the equalities

$$\mathcal{N} = \mathcal{N} \cap \mathcal{T} = \mathcal{N} \cap (\bigcup_\lambda \mathcal{T}^\lambda) = \bigcup_\lambda (\mathcal{N} \cap \mathcal{T}^\lambda) = \bigcup_{\delta \geq \gamma} \mathcal{N}^\delta = \bigcup_\lambda \mathcal{N}^\lambda.$$

The two last equalities hold since the set of the subcategories $\mathcal{N}^\lambda$ is filtered over regular cardinals. This means that $\mathcal{N}^\alpha \subseteq \mathcal{N}^\beta$ if $\alpha \leq \beta$, for all regular cardinals α and β.

b) The isomorphism classes of the objects of the subcategory $\langle \mathcal{S} \rangle_\gamma$ form a set, since it is explicitly constructed from the objects in $\mathcal{S}$, which is also a set. Moreover, $\langle \mathcal{S} \rangle_\gamma$ is stable under shifts because it is triangulated. Let us show condition (G1). Let Y be an object of $\mathcal{N}$ such that $\mathrm{Hom}_\mathcal{N}(X, Y) = 0$ for all X in $\langle \mathcal{S} \rangle_\gamma$. Then, it is easy to check that this equality holds for X in $\langle \mathcal{S} \rangle$. In particular it holds for $X = Y$. Hence Y vanishes. Therefore, condition (G1) holds for $\langle \mathcal{S} \rangle_\gamma$. By the proof of point a), $\langle \mathcal{S} \rangle_\gamma = \mathcal{N}^\gamma$. Therefore, conditions (G2) and (G3) trivially hold by the definition of $\mathcal{N}^\gamma$.

c) All the conditions of proposition 4.9 hold. Thus, this point results from part c) of proposition 4.9.

d) The subcategory $\mathcal{N}$ is well generated by point b). Thus, the Brown representability theorem (4.5) holds for $\mathcal{N}$ and we conclude that the inclusion i of $\mathcal{N}$ into $\mathcal{T}$ admits a right adjoint i_ρ as in the proof of proposition 4.6. Now this implies that the quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ admits a right adjoint Q_ρ (which takes an object X to the cone of the adjunction morphism $ii_\rho X \to X$). The functor Q_ρ is a localization functor (4.2). Thus, it is fully faithful. Let us sum up the situation in the following diagram,

$$\mathcal{N} \underset{i_\rho}{\overset{i}{\rightleftarrows}} \mathcal{T} \underset{Q_\rho}{\overset{Q}{\rightleftarrows}} \mathcal{T}/\mathcal{N}.$$

We have to show that the conditions (G1), (G2) and (G3) of definition 3.1 hold for $Q(\langle \mathcal{G} \rangle_\beta)$. We begin by observing that the sets $\mathcal{G}$ and $\mathcal{S}$ are both contained in $\mathcal{T}^\beta$, since we have chosen $\beta = \sup(\alpha, \gamma)$. The condition (G1) holds even for the smaller set $Q\mathcal{G}$, hence for $Q(\langle \mathcal{G} \rangle_\beta)$. Indeed, suppose $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(Q\mathcal{G}, X) = 0$, for an arbitrary object X in $\mathcal{T}/\mathcal{N}$. By the adjunction, this is equivalent to $\mathrm{Hom}_{\mathcal{T}}(\mathcal{G}, Q_\rho(X)) = 0$. The condition (G1) holds for the set $\mathcal{G}$ in $\mathcal{T}$ and implies $Q_\rho(X) = 0$. Therefore, $X = QQ_\rho X = 0$, since QQ_ρ is naturally equivalent to the identity endofunctor of $\mathcal{T}/\mathcal{N}$. Thus, condition (G1) holds for the set $Q\mathcal{G}$. The subcategory $Q(\langle \mathcal{G} \rangle_\beta)$ contains its β-coproducts because Q commutes with all coproducts and its objects form a set. Therefore, conditions (G2) and (G4) are equivalent for $Q(\langle \mathcal{G} \rangle_\beta)$ (*cf.* [17, Lemma 4]). Let us now simultaneously show that conditions (G4) and (G3) hold for $Q(\langle \mathcal{G} \rangle_\beta)$. Consider a morphism $u : Q(G) \to \coprod_{i \in I} X_i$, where G is an arbitrary object in $\langle \mathcal{G} \rangle_\beta$. We know from point a) of proposition 4.9 that u is the equivalence class of a diagram in $\mathcal{T}$

$$\begin{array}{ccc} & G' & \\ \swarrow \sim & & \searrow f \\ G & & \coprod_{i \in I} X_i , \end{array}$$

where the object G' belongs to $\mathcal{T}^\beta = \langle \mathcal{G} \rangle_\beta$. The conditions (G3) and (G4) also hold for $\langle \mathcal{G} \rangle_\beta$, by corollary 3.6. Therefore, there exists a set $J \subset I$ of cardinality strictly smaller than β and a set of morphisms $(f_i : G_i \to X_i)_{i \in I}$, where G_i lies in $\langle \mathcal{G} \rangle_\beta$ for all $i \in J$, so that the morphism $f : G' \to \coprod_{i \in I} X_i$ factors through $\coprod_{i \in J} X_i$ (G3)

$$\begin{array}{ccc} G' & \xrightarrow{\quad f \quad} & \coprod_{i \in I} X_i \\ & \searrow \quad \nearrow & \\ & \coprod_{i \in J} X_i & \end{array}$$

and through the morphism $\coprod_{i\in I} f_i$ (G4)

$$\begin{array}{ccc} G' & \xrightarrow{\quad f \quad} & \coprod_{i\in I} X_i \\ & \searrow \qquad \nearrow_{\coprod_{i\in I} f_i} & \\ & \coprod_{i\in I} G_i. & \end{array}$$

The image under Q of the the last two diagrams shows that the morphism u factors in $\mathcal{T}/\mathcal{N}$ in the same way. Therefore, conditions (G3) and (G4) hold for $Q(\langle\mathcal{G}\rangle_\beta)$. $\square$

Remark 4.11. The construction of the cardinal β in the preceding proof is not optimized at all. In spite of the constructive proof, this result will be useful mainly for existence problems.

The next corollary is a result about the localization of well generated categories obtained by inverting a *set* of arrows, implicitly contained in Neeman's book [27].

Corollary 4.12. *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{N}$ a localizing triangulated subcategory of $\mathcal{T}$ generated by a* set *of objects $\mathcal{S}$. Then $\mathcal{N}$ and $\mathcal{T}/\mathcal{N}$ are well generated triangulated categories.*

Proof. Take the coproduct of all the objects in $\mathcal{S}$. Since $\mathcal{S}$ is a set, the coproduct will be in $\mathcal{T}^\gamma$ for some regular cardinal γ. Therefore, we have $\mathcal{S} \subseteq \mathcal{T}^\gamma$, because $\mathcal{T}^\gamma$ is thick in $\mathcal{T}$ and so contains the direct factors of its objects. Now apply theorem 4.10. $\square$

5. The Popescu-Gabriel theorem for triangulated categories

5.1. **The α-continuous derived category.** In this subsection, we construct the *α-continuous* derived category of a homotopically α-cocomplete DG category. This construction enjoys a useful and beautiful property. Given a homotopically α-cocomplete (*cf.* below) pretriangulated DG category $\mathcal{A}$, we will show that its α-continuous derived category $\mathcal{D}_\alpha\mathcal{A}$ is α-compactly generated by the free DG modules. The categories $\mathcal{D}_\alpha\mathcal{A}$ will be the prototypes of the α-compactly generated algebraic DG categories.

Definition 5.1. Let α be a regular cardinal and $\mathcal{A}$ a small DG k-category. We assume that $\mathcal{A}$ is *homotopically α-cocomplete*, *i.e.* that the category $H^0(\mathcal{A})$ admits all α-small coproducts. For each α-small family $(A_i)_{i\in I}$ of objects of $\mathcal{A}$, we write

$$\coprod_{i\in I}^{H^0} A_i$$

for their coproduct in $H^0(\mathcal{A})$. Each DG functor $M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k)$ induces a functor $H^0M : (H^0(\mathcal{A}))^{op} \to \mathcal{H}(k)$ and so we have a canonical morphism

$$(H^*M)(\coprod_{i\in I}^{H^0} A_i) \longrightarrow \prod_{i\in I}(H^*M)(A_i).$$

Let $\mathcal{D}\mathcal{A}$ be the derived category of $\mathcal{A}$. The *α-continuous derived category* $\mathcal{D}_\alpha\mathcal{A}$ is defined as the full subcategory of $\mathcal{D}\mathcal{A}$ whose objects are the DG functors M such that, for each α-small family of objects $(A_i)_{i\in I}$ of $\mathcal{A}$, the canonical morphism above is invertible.

Remark 5.2. All the small k-linear DG categories which are *α-cocomplete*, *i.e.* admit all α-small coproducts, are homotopically α-cocomplete. A partial converse is given in conjecture 5.4 below.

This definition describes $\mathcal{D}_\alpha\mathcal{A}$ as a subcategory of $\mathcal{D}\mathcal{A}$. One can give an equivalent definition in terms of a localization of $\mathcal{D}\mathcal{A}$, which yields a category $\mathcal{D}\mathcal{A}/\mathcal{N}$ triangle equivalent to $\mathcal{D}_\alpha\mathcal{A}$. For this, let us define some *sets* of morphisms in $\mathcal{C}\mathcal{A}$. We recall that the notation $A^\wedge$ means $\mathrm{Hom}_\mathcal{A}(-, A)$. Let Σ_0 be the set of all morphisms of $\mathcal{C}\mathcal{A}$

$$\sigma_\lambda : \coprod_{i\in I} A_i{}^\wedge \longrightarrow (\coprod_{i\in I}^{H^0} A_i)^\wedge\,,$$

where λ ranges over the *set* Λ of all families $(A_i)_{i\in I}$ in $\mathcal{A}$ of cardinality strictly smaller than α. We define Σ to be the set of cofibrations between cofibrant DG modules (see definitions in subsection 6.1)

$$\begin{bmatrix}\sigma_\lambda \\ -\iota\end{bmatrix} : \coprod_{i\in I} A_i{}^\wedge \rightarrowtail (\coprod_{i\in I}^{H^0} A_i)^\wedge \oplus I(\coprod_{i\in I} A_i{}^\wedge)\,,$$

where $\lambda \in \Lambda$ and, for each object X, the morphism $\iota : X \rightarrowtail IX$ is the inclusion of X into the cone over its identity morphism.

Remark that it is clear from the definition that the DG modules $\coprod_{i\in I} A_i{}^\wedge$ and $(\coprod_{i\in I}^{H^0} A_i)^\wedge \oplus I(\coprod_{i\in I} A_i{}^\wedge)$ are cofibrant. Moreover, the morphisms $[\sigma_\lambda, -\iota]^t$ are cofibrations by lemma 6.9.

We can also consider the set

$$\mathcal{M} = \{N_\lambda \rightarrowtail IN_\lambda \mid N_\lambda = \operatorname{cone}(\sigma_\lambda)\ ,\ \lambda \in \Lambda\}.$$

The cones over the morphisms in Σ_0, Σ or $\mathcal{M}$ generate the same localizing subcategory $\mathcal{N}$ of $\mathcal{DA}$ because the objects IX are contractible and thus become zero objects in $\mathcal{DA}$. The quotient functor

$$\mathcal{DA} \longrightarrow \mathcal{DA}/\mathcal{N}$$

induces an equivalence

$$\mathcal{D}_\alpha\mathcal{A} \xrightarrow{\sim} \mathcal{DA}/\mathcal{N}.$$

Following [7], we say that a DG category $\mathcal{A}$ is *pretriangulated* if the essential image of the Yoneda functor is a triangulated subcategory of the derived category $\mathcal{DA}$. In the case of pretriangulated DG categories, the definition of *quasi-equivalence of* DG *categories* of section 2.2 specializes to the following.

Definition 5.3. Let $\mathcal{A}$ and $\mathcal{A}'$ be pretriangulated DG categories. A DG *functor*

$$F : \mathcal{A} \longrightarrow \mathcal{A}'$$

is a *quasi-equivalence of pretriangulated* DG *categories* if the induced triangle functor

$$H^0(F) : H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{A}')$$

is an equivalence of triangulated categories.

The next conjecture states that a homotopically α-cocomplete pretriangulated DG category contains α-small coproducts up to a quasi-equivalence. This would establish the link with the article [34].

Conjecture 5.4 (Strictification Theorem for α-coproducts)**.** *Let $\mathcal{A}$ be a homotopically α-cocomplete pretriangulated* DG *category. Then, there exists a quasi-equivalence $\mathcal{A} \to \mathcal{A}'$, where $\mathcal{A}'$ is a pretriangulated* DG *category which is α-*cocomplete*.*

We now come to the result which motivated the definition of the α-continuous derived category. We have two proofs. In the first, presented here, the result follows as a corollary of the powerful theorem 4.10. In the second, we use the projective model category structure on $\mathcal{CA}$. It can be considered as a DG version of the theorem, since it does not make use of the triangulated structure present on the derived category at all (see section 6).

Theorem 5.5. *Let $\mathcal{A}$ be a homotopically α-cocomplete pretriangulated* DG *category. The α-continuous derived category of $\mathcal{A}$ is α-compactly generated by the images of the free* DG *modules $A^\wedge$, $A \in \mathcal{A}$. More precisely, the full subcategory $\mathcal{G}$ of $\mathcal{D}_\alpha\mathcal{A}$ formed by the images of the free* DG *modules $A^\wedge$, $A \in \mathcal{A}$, is a triangulated subcategory satisfying conditions* (G1)*,* (G2) *and* (G3) *of definition 3.1.*

Remark 5.6. We prove the theorem in the case where α is strictly greater than $\aleph_0$, the case $\alpha = \aleph_0$ being trivial. In fact, $\aleph_0$-coproducts are finite coproducts. Thus, the morphisms σ_λ above are isomorphisms already in $\mathcal{DA}$, and $\mathcal{D}_{\aleph_0}\mathcal{A}$ equals $\mathcal{DA}$.

Proof. This proof depends heavily on theorem 4.10. Therefore, let us explain how the notations correspond. The triangulated category $\mathcal{T}$ is $\mathcal{DA}$. The *set* $\mathcal{S}$ is formed by the cones on the following morphisms

$$\sigma_\lambda : \coprod_{i\in I}(A_i{}^\wedge) \longrightarrow (\overset{H^0}{\coprod_{i\in I}} A_i)^\wedge ,$$

where λ ranges over the *set* Λ of all families $(A_i)_{i\in I}$ in $\mathcal{A}$ of cardinality strictly smaller then α. The set $\mathcal{G}$ is formed by the free DG modules $A^\wedge$, $A \in \mathcal{A}$. It is contained in $\mathcal{T}^{\aleph_0}$, whereas $\mathcal{S}$ is contained in $\mathcal{T}^\alpha$. We have $\beta = \sup(\aleph_0, \alpha) = \alpha$. Let $\mathcal{N}$ be $\langle\mathcal{S}\rangle$ and Q the projection functor

$$Q : \mathcal{DA} \longrightarrow \mathcal{D}_\alpha\mathcal{A} \cong \mathcal{T}/\mathcal{N}.$$

Then, according to theorem 4.10, the α-continuous derived category $\mathcal{D}_\alpha\mathcal{A}$ is α-compactly generated by $\langle Q\mathcal{G}\rangle_\alpha$. Hence, the claim of the theorem is equivalent to the following claim: $\langle Q\mathcal{G}\rangle_\alpha = Q\mathcal{G}$ and the functor Q induces an equivalence $\mathcal{G} \xrightarrow{\sim} Q\mathcal{G}$. We begin with the equivalence $\mathcal{G} \xrightarrow{\sim} Q\mathcal{G}$. It amounts to the same as to show that the functor $Q|_\mathcal{G}$ is fully faithful. We know from the proof of the point d) of theorem 4.10 that Q admits a right adjoint Q_ρ (4.2). From the general theory of Bousfield localizations [27, Ch. 9], we have that $Q|_{\mathcal{N}^\perp} : \mathcal{N}^\perp \to \mathcal{D}_\alpha\mathcal{A}$ is an equivalence of triangulated categories. In particular $Q|_{\mathcal{N}^\perp}$ is fully faithful. Therefore, it is sufficient to show that $\mathcal{G}$ is contained in $\mathcal{N}^\perp$. By definition, $\mathcal{N}$ is the localizing subcategory generated by the cones $\mathrm{cone}(\sigma_\lambda)$, which we call $C_{(A_I)}$. We have to show that each $A^\wedge \in \mathcal{G}$ is right orthogonal to the objects $C_{(A_I)}$. By applying the cohomological functor $\mathrm{Hom}_{\mathcal{DA}}(-, \Sigma^n A^\wedge)$, $n \in \mathbb{Z}$, to the distinguished triangle

$$\coprod_{i\in I}(A_i{}^\wedge) \longrightarrow (\overset{H^0}{\coprod_{i\in I}} A_i)^\wedge \longrightarrow C_{(A_I)} \longrightarrow \Sigma\coprod_{i\in I}(A_i{}^\wedge) ,$$

it is clear that it is sufficient to show that the natural morphism

$$\mathrm{Hom}_{\mathcal{DA}}(\coprod_{i\in I}(A_i{}^\wedge), \Sigma^n A^\wedge) \longleftarrow \mathrm{Hom}_{\mathcal{DA}}((\overset{H^0}{\coprod_{i\in I}} A_i)^\wedge, \Sigma^n A^\wedge)$$

is an isomorphism for all $n \in \mathbb{Z}$. This follows from the following sequence of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{DA}}(\coprod_{i\in I}(A_i{}^\wedge), \Sigma^n A^\wedge) &\xrightarrow{\sim} \prod_{i\in I}\mathrm{Hom}_{\mathcal{DA}}(A_i{}^\wedge, \Sigma^n A^\wedge) \\
&= \prod_{i\in I} H^n\,\mathrm{Hom}_{\mathcal{A}}(A_i, A) \\
&\xleftarrow{\sim} H^n(\prod_{i\in I}\mathrm{Hom}_{\mathcal{A}}(A_i, A)) \\
\mathrm{Hom}_{\mathcal{DA}}((\overset{H^0}{\coprod_{i\in I}} A_i)^\wedge, \Sigma^n A^\wedge) &\xrightarrow{\sim} H^n(\mathrm{Hom}_{\mathcal{A}}(\overset{H^0}{\coprod_{i\in I}} A_i, A)).
\end{aligned}$$

The second and the last isomorphisms are justified by the following one

$$\mathrm{Hom}_{\mathcal{DA}}(A^\wedge, \Sigma^n B^\wedge) = H^n \mathrm{Hom}_{\mathcal{A}}(A, B) ,$$

valid for all A and B in $\mathcal{A}$. The third isomorphism is the fact that cohomology commutes with formation of products. For the fourth, we observe that the natural homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I}^{H^0} A_i, A) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(A_i, A)$$

is a homotopy equivalence, by the definition of $\coprod^{H^0}$. Therefore, it becomes invertible in cohomology.

It is trivial that $Q\mathcal{G}$ is stable under shifts. Moreover, it is automatically thick for $\alpha > \aleph_0$ (4.2). To prove that $Q\mathcal{G}$ equals $\langle Q\mathcal{G}\rangle_\alpha$ it is then sufficient to show that $Q\mathcal{G}$ is closed under α-coproducts and extensions. We have

$$\coprod_{i\in I}(QA_i{}^\wedge) \xrightarrow{\sim} Q(\coprod_{i\in I} A_i{}^\wedge) \xrightarrow{\sim} Q((\coprod_{i\in I}^{H^0} A_i)^\wedge) ,$$

where the cardinality of I is strictly smaller than α and the last isomorphism holds by the construction of $\mathcal{N}$. This shows that $Q\mathcal{G}$ is closed under formation of α-coproducts. Finally, $Q\mathcal{G}$ is stable under extensions in $\mathcal{D}_\alpha\mathcal{A}$. Indeed, $\mathcal{G}$ is stable under extensions in $\mathcal{DA}$ and hence in $\mathcal{N}^\perp$, since we have shown that $\mathcal{G}$ is contained in $\mathcal{N}^\perp$. We have also seen that the restriction $Q|_{\mathcal{N}^\perp}$ is an equivalence of the categories $\mathcal{N}^\perp$ and $\mathcal{D}_\alpha\mathcal{A}$. It follows that $Q\mathcal{G}$ is stable under extensions in $\mathcal{D}_\alpha\mathcal{A}$. □

5.2. **Algebraic triangulated categories.** Let us recall that an exact category $\mathcal{E}$ (6.2) is a Frobenius category if it has enough injectives, enough projectives, and the two classes of the injectives and projectives coincide. For all pairs of objects X, Y of $\mathcal{E}$, let $I_{\mathcal{E}}(X, Y)$ be the subgroup of the abelian group $\mathrm{Hom}_{\mathcal{E}}(X, Y)$ formed by the morphisms which factor over an injective-projective object of $\mathcal{E}$. The *stable category of* $\mathcal{E}$, written $\underline{\mathcal{E}}$, is the category which has the same objects as $\mathcal{E}$ and the morphisms

$$\mathrm{Hom}_{\underline{\mathcal{E}}}(X, Y) = \mathrm{Hom}_{\mathcal{E}}(X, Y)/I_{\mathcal{E}}(X, Y).$$

Definition 5.7. [16] An *algebraic* triangulated category is a k-linear triangulated category which is triangle equivalent to the stable category $\underline{\mathcal{E}}$ of some k-linear Frobenius category $\mathcal{E}$.

The class of algebraic triangulated categories is stable under taking triangulated subcategories and forming triangulated localizations (up to a set-theoretic problem). Examples abound since categories of complexes up to homotopy are algebraic. Therefore, the categories arising in homological contexts in algebra and geometry are algebraic. The area where one often encounters non algebraic triangulated categories is topology. In particular the stable homotopy category of spectra is not algebraic. More examples can be found in section 3.6 of [16].

5.3. **The main theorem.** We recall [14] that a *graded category* over a commutative ring k is a k-linear category $\mathcal{B}$ whose morphism spaces are $\mathbb{Z}$-graded k-modules

$$\mathrm{Hom}_{\mathcal{B}}(X,Y) = \coprod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(X,Y)^p$$

such that the composition maps

$$\mathrm{Hom}_{\mathcal{B}}(X,Y) \otimes_k \mathrm{Hom}_{\mathcal{B}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(Y,Z)$$

are homogeneous of degree 0, for all X, Y, Z in $\mathcal{B}$. Now we can state and prove the main theorem of this thesis.

Theorem 5.8. *Let $\mathcal{T}$ be a triangulated category. Then the following statements are equivalent:*

- (i) *$\mathcal{T}$ is algebraic and well generated;*
- (ii) *there is a small* DG *category $\mathcal{A}$ such that $\mathcal{T}$ is triangle equivalent to a localization of $\mathcal{D}\mathcal{A}$ with respect to a localizing subcategory generated by a* set *of objects.*

Moreover, if $\mathcal{T}$ is algebraic and α-compactly generated, and $\mathcal{U} \subset \mathcal{T}$ is a full triangulated subcategory stable under α-small coproducts and such that conditions (G1), (G2) *and* (G3) *of definition 3.1 hold for $\mathcal{U}$, then there is an associated localization functor (4.2) $\mathcal{T} \to \mathcal{D}\mathcal{A}$ for some small* DG *category $\mathcal{A}$ such that $H^*(\mathcal{A})$ is equivalent to the graded category $\mathcal{U}_{gr}$ whose objects are those of $\mathcal{U}$ and whose morphisms are given by*

$$\mathcal{U}_{gr}(U_1, U_2) = \bigoplus_{n\in\mathbb{Z}} \mathcal{T}(U_1, \Sigma^n U_2).$$

Proof. $(ii) \Longrightarrow (i)$: $\mathcal{T}$ is a localization of $\mathcal{D}\mathcal{A}$, *i.e.* there is a fully faithful functor

$$\mathcal{T} \overset{\widetilde{F}}{\hookrightarrow} \mathcal{D}\mathcal{A}\,,$$

admitting a left adjoint functor. The category $\mathcal{D}\mathcal{A}$ is algebraic. Triangulated subcategories of algebraic categories are algebraic, implying that $\mathcal{T}$ is algebraic, too. Moreover, $\mathcal{D}\mathcal{A}$ is compactly generated by the *set* ($\mathcal{A}$ is small)

$$\{\ X^\wedge[n] \mid n \in \mathbb{Z}\,,\ X \in \mathcal{A}\ \}$$

thanks to the isomorphism

$$\mathrm{Hom}_{\mathcal{D}\mathcal{A}}(X^\wedge[n], M) \xrightarrow{\sim} H^{-n}(M(X))\,,$$

where M is a DG module and X is an object of $\mathcal{A}$ (*cf.* 2.4). Therefore, $\mathcal{T}$ is well generated by Corollary 4.12, since it is assumed to be a localization generated by a set of the $\aleph_0$-compactly generated category $\mathcal{D}\mathcal{A}$.

$(i) \Longrightarrow (ii)$: for the sake of clarity, we will give the proof of this implication in several steps, after making the main construction.

Let $\mathcal{T}$ be an algebraic, well generated triangulated category, *i.e.* $\mathcal{T}$ is equivalent to $\underline{\mathcal{E}}$ for some Frobenius category $\mathcal{E}$. By the definition of well generated triangulated category (in the sense of Krause), there are a regular cardinal α and a set of α-good generators $\mathcal{G}_0 \subseteq \mathcal{T}$ such that $\Sigma\mathcal{G}_0 = \mathcal{G}_0$ and the conditions (G1), (G2) and (G3) of definition 3.1 hold. Let $\mathcal{G}$ be the closure of the set $\mathcal{G}_0$ under extensions and α-coproducts. The set $\mathcal{G}$ is stable under the suspension functor Σ of $\mathcal{T}$ and under its inverse. Therefore, it is a small triangulated subcategory of $\mathcal{T}$. Let us recall and summarize the properties which hold for $\mathcal{G}$.

(G0) The set $\mathcal{G}$ is a small full triangulated subcategory of $\mathcal{T}$, stable under the formation of all α-small coproducts;
(G1) the set $\mathcal{G}$ is a generating set for $\mathcal{T}$: An object $X \in \mathcal{T}$ is zero if $\mathrm{Hom}_{\mathcal{T}}(G, X) = 0$ for all G in $\mathcal{G}$;
(G3) all the objects $G \in \mathcal{G}$ are *α-small*: For each family of objects X_i, $i \in I$, of $\mathcal{T}$, we have $\mathrm{Hom}_{\mathcal{T}}(G, \coprod_I X_i) = \mathrm{colim}_{J \subset I} \mathrm{Hom}_{\mathcal{T}}(G, \coprod_J X_i)$, where the sets J have cardinality strictly smaller than α;
(G4) for each family of objects X_i, $i \in I$, of $\mathcal{T}$, and each object $G \in \mathcal{G}$, each morphism

$$G \longrightarrow \coprod_{i \in I} X_i$$

factors through a morphism $\coprod_{i \in I} \phi_i \colon \coprod_{i \in I} G_i \to \coprod_{i \in I} X_i$, with G_i in $\mathcal{G}$ for all $i \in I$.

Condition (G0) clearly holds for $\mathcal{G}$. Condition (G3) of definition 3.1 has just been rewritten using colimits. Condition (G4) holds for $\mathcal{G}$ by proposition 3.4. Note that conditions (G2) and (G4) are equivalent for $\mathcal{G}$. Indeed, we can apply [17, Lemma 4], since the set $\mathcal{G}$ has α-coproducts and its objects are α-small.

We may assume that the category $\mathcal{E}$ is of the form $Z^0(\widetilde{\mathcal{E}})$ for an exact DG category $\widetilde{\mathcal{E}}$ by the argument of the proof of theorem 4.4 of [14]. Let us recall that a DG category $\mathcal{A}$ is an *exact* DG *category* [15] if the full subcategory $Z^0(\mathcal{A})$ of $\mathcal{C}\mathcal{A}$ formed by the image of the Yoneda functor is closed under shifts and extensions (in the sense of the exact structure of subsection 2.4). Then, $H^0(\mathcal{A})$ becomes a triangulated subcategory of $\mathcal{H}(\mathcal{A})$ and the subcategory of the representable functors becomes a triangulated subcategory of $\mathcal{D}\mathcal{A}$. Thus, an exact DG category is also a pretriangulated DG category (*cf.* subsection 5.1). Let us now define a small full DG subcategory $\mathcal{A} \subset \widetilde{\mathcal{E}}$ as follows. For each isomorphism class of objects of $\mathcal{G}$, we choose a representative G and we denote by A_G the same object considered in the category $\widetilde{\mathcal{E}}$. By definition, these objects A_G are objects of $\mathcal{A}$. Then, clearly, the category $H^0(\mathcal{A})$ is a full subcategory of $H^0(\widetilde{\mathcal{E}}) = \underline{\mathcal{E}} = \mathcal{T}$ and it is equivalent to $\mathcal{G}$ by the functor sending A_G to G. In particular, $H^0(\mathcal{A})$ is a triangulated category and it admits all α-small coproducts. Thus, $\mathcal{A}$ is a homotopically α-cocomplete pretriangulated DG category. We define the functor

$$F : \mathcal{T} \longrightarrow \mathcal{D}_\alpha \mathcal{A}$$

by sending an object X of $\mathcal{T} = H^0(\widetilde{\mathcal{E}})$ to the DG module FX taking $A_G \in \mathcal{A}$ to $\mathrm{Hom}_{\widetilde{\mathcal{E}}}(G, X)$. A priori, FX lies in $\mathcal{D}\mathcal{A}$. Let us show that it belongs in fact to the full subcategory $\mathcal{D}_\alpha \mathcal{A}$ of $\mathcal{D}\mathcal{A}$. Let A_{G_i}, $i \in I$, be an α-small family in $\mathcal{A}$. Then the coproduct $\coprod_{i \in I}^{H^0} A_{G_i}$ of the A_{G_i} in $H^0(\mathcal{A})$ is isomorphic to $A_{\coprod_{i \in I} G_i}$. Thus, we have a quasi-isomorphism

$$(FX)(\coprod_{i\in I}^{H^0} A_{G_i}) = \mathrm{Hom}_{\widetilde{\mathcal{E}}}(\coprod_{i\in I}^{\mathcal{T}} G_i, X) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\widetilde{\mathcal{E}}}(G_i, X) = \prod_{i\in I}(FX)(A_{G_i}) \,,$$

induced by liftings to $\mathcal{E} = Z^0(\widetilde{\mathcal{E}})$ of the canonical morphisms $G_j \to \coprod_{i\in I}^{\mathcal{T}} G_i$ in $\mathcal{T}$, respectively by representatives in $Z^0(\mathcal{A})$ of the canonical morphisms $A_{G_j} \to \coprod_{i\in I}^{H^0} A_{G_i}$ in $H^0(\mathcal{A})$. For $A_G \in \mathcal{A}$, we have

$$FG = \mathrm{Hom}_{\widetilde{\mathcal{E}}}(-, G) = \mathrm{Hom}_{\mathcal{A}}(-, A_G) = A_G^{\wedge} \,,$$

which shows that F induces an essentially surjective functor from $\mathcal{G}$ to the full subcategory of the $A_G^{\wedge}$ in $\mathcal{D}_\alpha\mathcal{A}$. For A_G in $\mathcal{A}$ and X in $\mathcal{T}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_\alpha\mathcal{A}}(FG, FX) &= \mathrm{Hom}_{\mathcal{D}_\alpha\mathcal{A}}(A_G^{\wedge}, FX) \\ &= \mathrm{Hom}_{\mathcal{D}\mathcal{A}}(A_G^{\wedge}, FX) \\ &= H^0(FX(A_G)) \\ \mathrm{Hom}_{\mathcal{T}}(G, X) &= H^0(\widetilde{\mathcal{E}}(G, X)). \end{aligned}$$

We would like to apply theorem 3.8 to conclude that F is a triangle equivalence: In the notations of theorem 3.8, we take $\mathcal{T} = \mathcal{T}$, $\mathcal{G} = \mathcal{G}$, $\mathcal{T}' = \mathcal{D}_\alpha\mathcal{A}$ and $\mathcal{G}'$ to be the full subcategory on the objects $A_G^{\wedge}$ in $\mathcal{D}_\alpha\mathcal{A}$. By theorem 5.5, $\mathcal{T}'$ and $\mathcal{G}'$ do satisfy the hypothesis of theorem 3.8 and so F is indeed a triangle equivalence.

Now suppose that $\mathcal{T}$ is an algebraic well generated triangulated category. Let $\mathcal{U} \subset \mathcal{T}$ be a full small subcategory as in the last assertion of the theorem. Then the conditions (G0), (G1) and (G3) above hold for $\mathcal{G} = \mathcal{U}$. Moreover, condition (G4) holds for $\mathcal{G} = \mathcal{U}$ by [17, Lemma 4]. Therefore, we can construct a DG category $\mathcal{A}$ and an equivalence $F : \mathcal{T} \xrightarrow{\sim} \mathcal{D}_\alpha\mathcal{A}$ as above in the proof of the implication from i) to ii). Moreover, $H^*(\mathcal{A})$ equals $\mathcal{U}_{gr}$. Indeed, both have the same objects and we have

$$\begin{aligned} H^n(\mathcal{A})(A_{G_1}, A_{G_2}) &= \mathcal{H}\mathcal{A}(A_{G_1}^{\wedge}, \Sigma^n(A_{G_2}^{\wedge})) \\ &= \mathcal{H}\mathcal{A}(A_{G_1}^{\wedge}, (\Sigma^n A_{G_2})^{\wedge}) \\ &= H^0(\mathcal{A})(A_{G_1}, \Sigma^n A_{G_2}) \\ \mathcal{U}_{gr}(G_1, G_2)^n &= \mathcal{U}(G_1, \Sigma^n G_2). \end{aligned}$$

□

If $\mathcal{T}$ is compactly generated we recover a result obtained by B. Keller in [14, Thm. 4.3]:

Corollary 5.9. *Let $\mathcal{T}$ be an algebraic triangulated category. Then the following statements are equivalent:*

(*i*) *$\mathcal{T}$ is compactly generated;*
(*ii*) *$\mathcal{T}$ is equivalent to the derived category $\mathcal{D}\mathcal{A}$ for some small* DG *category $\mathcal{A}$.*

Proof. See remark 5.6 □

5.4. **Application.** We apply theorem 5.8 to a certain class of subcategories of algebraic triangulated categories we are going to define.

Definition 5.10. Let $\mathcal{T}$ be an algebraic triangulated category which is triangle equivalent to the stable category of the Frobenius category $\mathcal{E}$ and admits arbitrary coproducts. Let $\widetilde{\mathcal{E}}$ be a DG category (not necessarily small) such that $H^0(\widetilde{\mathcal{E}})$ is triangle equivalent to $\mathcal{T}$. Given a subcategory $\mathcal{G}$ of $\mathcal{T}$, let $\widetilde{\mathcal{G}}$ be the DG subcategory of $\mathcal{A}$ with the same objects as $\mathcal{G}$. Thus, the category $H^0(\widetilde{\mathcal{G}})$ is isomorphic to $\mathcal{G}$. We will say that $\mathcal{G}$ is a *compactifying* subcategory of $\mathcal{T}$ if it is small and the functor

$$\mathcal{T} \longrightarrow \mathcal{D}\widetilde{\mathcal{G}}\,,\ X \longmapsto \mathrm{Hom}_{\widetilde{\mathcal{E}}}(-,X)|_{\widetilde{\mathcal{G}}}$$

is fully faithful.

For example, W. T. Lowen and M. Van den Bergh proved in [21, Ch. 5] that, given a Grothendieck category $\mathcal{A}$ with a generator G, the one-object subcategory $\mathcal{G} = \{G\}$ of the derived category $\mathcal{D}\mathcal{A}$ is a compactifying subcategory. For this reason we call such a generator G *compactifying*.

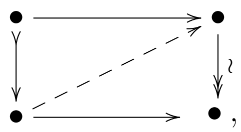
Theorem 5.11. *Let $\mathcal{T}$ be a well generated algebraic triangulated category. Then there is a regular cardinal α such that the subcategory* $\mathrm{sk}(\mathcal{T}^\beta)$ *formed by a system of representatives of the isomorphism classes of $\mathcal{T}^\beta$ is compactifying for each regular cardinal $\beta \geq \alpha$.*

Proof. Suppose that α is the first regular cardinal such that $\mathcal{T} = \langle \mathcal{T}^\alpha \rangle$. This cardinal exists because the category $\mathcal{T}$ is well generated. For each $\beta \geq \alpha$, the subcategory $\mathrm{sk}(\mathcal{T}^\beta)$ is small and satisfies conditions (G1), (G2) and (G3) of definition 3.1 by definition of the subcategory $\mathcal{T}^\beta$ and the filtration by increasing regular cardinals. Now the claim follows from the last part of theorem 5.8. □

6. DG enhancement of the α-continuous derived category

In this section, we give a characterization of the α-compact objects of $\mathcal{DA}$ (*cf.* definition 3.1) using the (projective) model category structure on $\mathcal{CA}$. In his book [11], Hirschhorn gives a definition of α-compact objects in a model category. In subsection 6.5, we compare the two notions. It turns out that our notion is not a particular case of Hirschhorn's.

6.1. **The (Quillen) model category structure on the category of DG modules.** Sometimes on a category one can put a *model structure*, *i.e.* the data of three special classes of morphisms, called fibrations, cofibrations and weak equivalences, which satisfy a given set of axioms [12]. This structure was invented by Quillen in the sixties [33] to give a formal general setting for abstract homotopy theory. In the last years people recognized more and more the foundational importance of this notion, and were lead to the definition of the modern *Model Categories*. Excellent sources for this material are the books [11], [12]. We recall that a model category is a category with all small limits and colimits which admits a model structure. Therefore, a model category always has an initial object $\emptyset$ (the colimit of the empty diagram) and one defines the *cofibrant objects* to be the objects P such that the unique morphism from $\emptyset$ to P is a cofibration. The *fibrant objects* are defined by the dual property. Let $\mathcal{A}$ be a small DG category. The (big) category of DG modules $\mathcal{CA}$ has (at least) two model category structures (*cf.* Thm. 3.2 in [16] which generalizes Thm. 2.3.11 in [12]). We are interested in the *projective structure*, where the *fibrations* are the epimorphisms (pointwise surjective maps), the *weak equivalences* are the quasi-isomorphisms and consequently the *cofibrations* are the morphisms having the *left lifting property (LLP)* with respect to the *trivial fibrations*. This means that, given the solid commutative square,



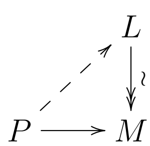
where the trivial fibration is the right vertical arrow and the cofibration is the left vertical one, there always exists a morphism (the dotted arrow) which makes the diagram commute.

An exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of DG modules is *graded split* if it splits in the category of graded $\mathcal{A}$-modules. A straightforward adaptation of proposition 2.3.9 in [12] allows us to characterize the cofibrations as the *graded split monomorphisms* with cofibrant cokernel.

In the projective model structure each object is fibrant and an object P is cofibrant if and only if it is cofibrant as a DG module, *i.e.* if and only if there is a commutative diagram



for every trivial fibration $L \overset{\sim}{\longrightarrow\!\!\!\!\to} M$ and every morphism $P \longrightarrow M$ in the category of DG modules. The subcategory of cofibrant DG modules can be described as the closure of the subcategory of the representables under left and right shifts, arbitrary small coproducts, extensions, and formation of direct factors [14, subsection 3.1].

Let us recall that in the category $\mathcal{CA}$, we have the usual notion of cochain homotopy. Indeed, we say that two morphisms $f, g : M \to N$ between DG modules are *cochain homotopic* and we write $f \approx g$ if there exists a morphism

$$h : M \longrightarrow N$$

of graded $\mathcal{A}$-modules, homogeneous of degree -1, such that

$$(*) \qquad f - g = d_N \circ h + h \circ d_M.$$

Notice that the morphism h is *not* a morphism of DG modules.

Clearly, we also have the notion of homotopy given by the model category structure which is present on $\mathcal{CA}$. We recall that a cylinder object for an object B in some fixed model structure is a factorization of the codiagonal morphism $\nabla_B = [\mathbf{1}_B, \mathbf{1}_B]$ through a cofibration and a weak equivalence as follows

$$\begin{array}{ccc} B \bigoplus B & \xrightarrow{\quad\nabla\quad} & B \\ & {\scriptstyle [i_0, i_1]} \searrow \quad \nearrow {\scriptstyle \sim} & \\ & \mathrm{cyl}(B). & \end{array}$$

Let f_0 and f_1 be two morphisms between B and an object Z. A *left homotopy* from f to g for some cylinder object $\mathrm{cyl}(B)$ is a morphism $H : \mathrm{cyl}(B) \to Z$ such that $H \circ i_0 = f_0$, $H \circ i_1 = f_1$, *i.e.* the diagram

$$\begin{array}{ccc} B \bigoplus B & \xrightarrow{\quad [f_0, f_1] \quad} & Z \\ & {\scriptstyle [i_0, i_1]} \searrow \quad \nearrow {\scriptstyle H} & \\ & \mathrm{cyl}(B) & \end{array}$$

commutes.

Under a mild condition, the notion of left homotopy for the projective model structure on $\mathcal{CA}$ reduces to that of cochain homotopy. We have the

Lemma 6.1. *Let f_0, $f_1 : M \to N$ be morphisms of* DG *modules, where we assume M to be cofibrant. Then, f_0 and f_1 are chain homotopic if and only if they are left homotopic in the sense of the projective model structure on $\mathcal{CA}$.*

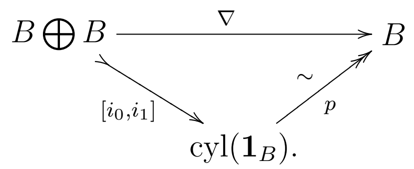
Proof. Let us fix notations. Given a morphism of DG modules $f : X \to Y$, its *(mapping) cone*, $\mathrm{cone}(f)$, is the DG module $Y \oplus \Sigma X$ whose differential is

$$d_{\mathrm{cone}(f)} = \begin{bmatrix} d_Y & \Sigma f \\ 0 & d_{\Sigma X} \end{bmatrix} = \begin{bmatrix} d_Y & \Sigma f \\ 0 & -\Sigma d_X \end{bmatrix}.$$

The *(mapping) cylinder* cyl(f) of f is the cone of the DG module morphism $[-\mathbf{1}_X, f]^t : X \to X \oplus Y$, *i.e.* the DG module $X \oplus Y \oplus \Sigma X$ whose differential is

$$d_{\mathrm{cyl}(f)} = \begin{bmatrix} d_X & 0 & -\Sigma \mathbf{1}_X \\ 0 & d_Y & \Sigma f \\ 0 & 0 & -\Sigma d_X \end{bmatrix}.$$

If B is a *cofibrant* DG module, then the cylinder cyl$(\mathbf{1}_B)$ is a cylinder object for B. Indeed, consider the factorization of the codiagonal morphism



Here, $i_0 = [\mathbf{1}_B, 0, 0]^t$, $i_1 = [0, \mathbf{1}_B, 0]^t$. It is easy to see that they are morphisms of DG modules. Thus, the DG module morphism $[i_0, i_1]$ sends each pair (x, y) in $B \oplus B$ to the triple $(x, y, 0)$. Here is the crucial point which we have to require that the DG module B is cofibrant for: in general the graded split monomorphism $[i_0, i_1]$ is *not* a cofibration if B is not cofibrant, since its cokernel ΣB may not be cofibrant.

The DG module morphism p is defined by $p(a, b, c) = a + b$, for each triple (a, b, c) in cyl$(\mathbf{1}_B)$. It is clear that $p \circ i_0 = p \circ i_1 = \mathbf{1}_M$ and that $p \circ [i_0, i_1] = [\mathbf{1}_M, \mathbf{1}_M] = \nabla$. Moreover, we remark that p is a fibration. We also have to show that p is a quasi-isomorphism. Let us define the morphisms

$$s_0, s_1 : \mathrm{cyl}(\mathbf{1}_M) \longrightarrow \mathrm{cyl}(\mathbf{1}_M)$$

of graded $\mathcal{A}$-modules, homogeneous of degree -1, by the formulas

$$s_0(x, y, z) = (0, z, y)\ ,\ s_1(x, y, z) = (z, 0, x).$$

An easy calculation gives

$$(i_0 \circ p)(x, y, z) = (x + y, 0, 0) = (\mathbf{1}_{\mathrm{cyl}(\mathbf{1}_B)} - (d_{\mathrm{cyl}(\mathbf{1}_B)} \circ s_0 + s_0 \circ d_{\mathrm{cyl}(\mathbf{1}_B)}))(x, y, z)$$

and

$$(i_1 \circ p)(x, y, z) = (0, x + y, 0) = (\mathbf{1}_{\mathrm{cyl}(\mathbf{1}_B)} + (d_{\mathrm{cyl}(\mathbf{1}_B)} \circ s_1 + s_1 \circ d_{\mathrm{cyl}(\mathbf{1}_B)}))(x, y, z).$$

Since $i_j \circ p$ and $p \circ i_j$, $j \in \{0, 1\}$, induce the identity morphisms on cohomology, i_0, i_1 and p are quasi-isomorphisms.

Note that, if a morphism f_0 is left homotopic to f_1, it is always possible to factor H through a cofibration $\mathrm{cyl}(B) \rightarrowtail \mathrm{cyl}(\mathbf{1}_{\mathrm{cyl}(B)})$ followed by a trivial fibration. Therefore, f_0 and f_1 are left homotopic if and only if there is a factorization through the DG module cyl$(\mathbf{1}_{\mathrm{cyl}(B)})$.

Now we can show the equivalence of the two notions of homotopy. We have already observed why the DG module M must be cofibrant.

Let the morphisms f_0 and f_1 from M to N be cochain homotopic. By definition, for each $X \in \mathcal{A}^{op}$ and each $i \in \mathbb{Z}$, there exists a morphism $h : M \to N$ of graded $\mathcal{A}$-modules, homogeneous of degree -1, such that the equality (*) holds. Define $H = [f_0, f_1, -h]$ from cyl$(\mathbf{1}_M)$ to N. Using (*) it is easy to see that H is a morphism of DG modules. Let the DG module morphisms i_0 and i_1 be as above. Clearly, we have $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$. Thus, H is a left homotopy between f_0 and f_1 since we have already seen that $\mathrm{cyl}(\mathbf{1}_M)$ is a cylinder object for M.

Conversely, suppose that two morphism f_0 and f_1 are left homotopic with respect to the projective model category structure. We have already observed that in this case we can replace the cylinder object $\mathrm{cyl}(M)$ which they factor through with the cylinder object $\mathrm{cyl}(\mathbf{1}_M)$. So, we can write $H = (\varphi, \psi, \rho)$, where $\rho : M \to N$ is a morphism of graded $\mathcal{A}$-modules, homogeneous of degree -1. By imposing the factorizations $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$, we get $\varphi = f_0$ and $\psi = f_1$. By imposing that H must be a morphism of DG modules, we get that φ and ψ also have to be morphisms of DG modules (but *not* ρ) and that

$$d_N \circ \rho = -(f_0 - f_1) - \rho \circ d_M.$$

This shows that we can take $-\rho : M \to N$ as cochain homotopy between f_0 and f_1. □

We have the dual notions of *path object* and *right homotopy*. For them, the dual statement of lemma 6.1 holds: Two morphisms between two DG modules whose codomain is fibrant are right homotopic if and only if they are cochain homotopic. When two morphisms are both left and right homotopic they are said *homotopic*.

Remark 6.2. Since in the projective model structure every DG module is fibrant, the notion of cochain homotopy and right homotopy coincide in $\mathcal{CA}$ (*cf.* also observations after theorem 2.3.11 in [12]). Therefore, two morphisms of $\mathcal{CA}$ between two DG modules whose domain is cofibrant are left homotopic if and only if they are right homotopic. This is always the case in the subcategory $\mathcal{CA}_{cof}$.

The next point to be recalled is the construction of the homotopy category of a model category, obtained by formally inverting the weak equivalences. In particular, we are interested in the derived category of the category of DG modules. Using Quillen's theorem [12, Thm. 1.2.10] it is possible to compare the model category construction of the derived category of DG modules with the one given in subsection 2.4. In the subcategory $\mathcal{CA}_{cof,fib}$ of fibrant and cofibrant objects of $\mathcal{CA}$ left and right homotopy coincide and homotopy is an equivalence relation [12, section 1.2]. Thus, forming the quotient is well defined. We have

$$\mathcal{DA} := \mathcal{CA}[qiso^{-1}] = \mathrm{Ho}(\mathcal{CA}) \xrightarrow{\sim} \mathcal{CA}_{cof,fib}/\sim,$$

where $\sim$ denotes the homotopy relation and the last two equivalences come from the definition of the homotopy category of a model category [12] and from Quillen's theorem respectively. So, using Quillen's important result, we get

$$\mathcal{DA} \xrightarrow{\sim} \mathcal{CA}_{cof,fib}/\sim.$$

In the projective model structure on the category of DG modules we have $\mathcal{CA}_{fib} = \mathcal{CA}$ and therefore $\mathcal{CA}_{cof,fib} = \mathcal{CA}_{cof}$. Thus, we have the remarkable equivalence

$$\mathcal{DA} \xrightarrow{\sim} \mathcal{CA}_{cof}/\sim,$$

which shows that we can construct the derived category $\mathcal{DA}$ identifying homotopic morphisms between cofibrant DG modules. Moreover, it becomes clear that each DG module is quasi-isomorphic to some cofibrant DG module.

We remark that the notion of cochain homotopy of DG modules gives us another interpretation of the category $\mathcal{HA}$. Indeed, cochain homotopy is an equivalence relation on $\mathcal{CA}$. Clearly, we have

$$\mathcal{HA} = \mathcal{CA}/\approx .$$

Now we can state two observations useful in the sequel. Let us give the

Definition 6.3. Let $\mathcal{A}$ be a small DG category. We write $\mathcal{D}_{dg}(\mathcal{A})$ for the full DG subcategory of $\mathcal{C}_{dg}(\mathcal{A})$ such that $Z^0(\mathcal{D}_{dg}(\mathcal{A}))$ is the subcategory of the cofibrant DG modules with respect to the projective model structure on $\mathcal{CA}$.

We have the following

Proposition 6.4. *Let $\mathcal{A}$ be a small* DG *category. Then $H^0(\mathcal{D}_{dg}(\mathcal{A}))$ is equivalent to the derived category $\mathcal{DA}$.*

Proof. We have the following equivalences

$$\begin{aligned}
H^0(\mathcal{D}_{dg}(\mathcal{A})) &= \mathcal{CA}_{cof}/\approx \\
&= \mathcal{CA}_{cof}/\sim \qquad (15)\\
&= \mathcal{CA}_{cof,fib}/\sim \\
&\xrightarrow{\sim} \mathrm{Ho}(\mathcal{CA}) \qquad (16)\\
&= \mathcal{CA}[qiso^{-1}] \\
&= \mathcal{DA}.
\end{aligned}$$

Here, everything is clear. We have used lemma 6.1 in (15) and Quillen's theorem about the homotopy category in (16). We remark that (16) is an equivalence and not an isomorphism. □

Proposition 6.5. *Let $\mathcal{A}$ be a small* DG *category and $\mathcal{CA}$ its category of* DG *modules. Let Σ be a class of cofibrations between cofibrant modules of $\mathcal{CA}$ and $\mathcal{N}$ the localizing subcategory of $\mathcal{DA}$ associated with the image of Σ,* i.e. *$\mathcal{N} = \langle \mathcal{S} \rangle$, where $\mathcal{S} = \{\mathrm{cone}(s) \mid s \in \Sigma\}$. Then every morphism ϕ in Σ-cell becomes invertible in the localization $\mathcal{DA}/\mathcal{N}$.*

Proof. We know from [12, Ch. 2] that such a morphism ϕ is a *transfinite composition* of pushouts of coproducts of elements of Σ such as

$$\begin{array}{ccc}
\coprod_{i\in I_k} A_i & \xrightarrow{\coprod_{i\in I_k} s_i} & \coprod_{i\in I_k} B_i \\
\downarrow & (PO) & \downarrow \\
X_k & \longrightarrow & X_{k+1} ,
\end{array}$$

i.e. a colimit of a λ-*sequence* of pushouts of cofibrations $\coprod_{i\in I_k} s_i$, $s_i \in \Sigma$, for some ordinal λ:

$$X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \cdots \rightarrowtail X_\omega \rightarrowtail X_{\omega+1} \rightarrowtail \cdots \rightarrowtail X_\beta \rightarrowtail \cdots ,$$

where $\beta < \lambda$.

We begin by considering the case of an ω-sequence, where ω is the colimit of the finite ordinals, *i.e.* we have $X_\omega = \mathrm{colim}_{k\in\mathbb{N}} X_k$. The cones of the morphisms $s_i \in \Sigma$

are in the subcategory $\mathcal{N}$ and are homotopically equivalent, hence quasi-isomorphic, to the cokernels $\operatorname{cok}(s_i)$, for all $i \in \mathbb{N}$. Indeed, $\operatorname{cone}(s_i) \cong \operatorname{cok}(s_i) \oplus IA_i$ (*cf.* lemma 6.9 below) and IA_i is a contractible cofibrant DG module, for all $i \in \mathbb{N}$ (see lemma 6.11):

$$A_i \overset{s_i}{\rightarrowtail} B_i \twoheadrightarrow B_i/A_i \sim \operatorname{cone}(s_i).$$

Thus, at the step k of the ω-sequence, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \coprod_{I_k} A_i & \overset{\coprod_{I_k} s_i}{\rightarrowtail} & \coprod_{I_k} B_i & \twoheadrightarrow & \operatorname{cok}(\coprod_{I_k} s_i) & \longrightarrow & 0 \\
 & & \downarrow & (PO) & \downarrow & & \downarrow\wr & & \\
0 & \longrightarrow & X_k & \rightarrowtail & X_{k+1} & \twoheadrightarrow & X_{k+1}/X_k & \longrightarrow & 0.
\end{array}$$

Here, $\operatorname{cok}(\coprod_{I_k} s_i) \in \mathcal{N}$ implies $X_{k+1}/X_k \in \mathcal{N}$. Therefore, the morphisms $X_k \rightarrowtail X_{k+1}$, $k \in \mathbb{N}$, are invertible in the localized category $\mathcal{DA}/\mathcal{N}$. Therefore, by induction, the image of every composition $X_0 \rightarrowtail X_i$, $i \in \mathbb{N}$, is invertible in $\mathcal{DA}/\mathcal{N}$. We can take the colimit over $\mathbb{N}$ of the horizontal exact sequences of the diagram

$$\begin{array}{ccccc}
X_0 & \rightarrowtail & X_k & \twoheadrightarrow & N_k \\
\| & & \downarrow & & \downarrow \\
X_0 & \rightarrowtail & X_{k+1} & \rightarrowtail & N_{k+1} \\
 & & \downarrow & & \downarrow \\
 & & \operatorname{cok}(\coprod_{I_k} s_i) & = & \operatorname{cok}(\coprod_{I_k} s_i)\,,
\end{array}$$

and get

$$\begin{array}{ccccc}
\operatorname{colim}_{i\in\mathbb{N}} X_0 & \rightarrowtail & \operatorname{colim}_{i\in\mathbb{N}} X_i & \twoheadrightarrow & \operatorname{colim}_{i\in\mathbb{N}} N_i \\
\| & & \| & & \| \\
X_0 & \rightarrowtail & X_\omega & \twoheadrightarrow & N_\omega\,,
\end{array}$$

where each $N_i \in \mathcal{N}_\Sigma$ by recurrence. Then we find $N_\omega := \operatorname{colim}_{i\in\mathbb{N}} N_i = \operatorname{cok}\Phi$ lying in $\mathcal{N}$, where Φ is the morphism defined in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \coprod N_i & \overset{\Phi}{\rightarrowtail} & \coprod N_i & \twoheadrightarrow & \operatorname{colim} N_i = \operatorname{cok}\Phi & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & N_i & \xrightarrow{\begin{bmatrix} \mathbf{1} \\ -\phi_i \end{bmatrix}} & N_i \oplus N_{i+1} & & & &
\end{array}$$

where $\phi_i : N_i \to N_{i+1}$, proving that the colimit $X_0 \to X_\omega$ becomes an isomorphism in $\mathcal{DA}/\mathcal{N}$.

In order to generalize the result obtained for ω to higher ordinals it is useful to remember the pair of adjoint functors:

$$\mathcal{D}(kI \otimes \mathcal{A}) \underset{\Delta}{\overset{\operatorname{colim}_I}{\rightleftarrows}} \mathcal{DA}\,,$$

where kI is the k-linear category associated to a filtered category I. The functor Δ is the diagonal functor. The category $\mathcal{D}(kI \otimes \mathcal{A})$ is isomorphic to the localization of the category $\mathcal{C}(\mathcal{A})^I$ with respect to the class of morphisms $L \to M$ such that $L_i \to M_i$ is a quasi-isomorphism of $\mathcal{CA}$ for all $i \in I$. The functor colim_I is exact since I is filtered, thus it induces the derived left adjoint functor of Δ, $\mathbf{L}\operatorname{colim}_I$, which is naturally equivalent to colim_I. In other words, it suffices to show that: if $N \in \mathcal{D}(kI \otimes \mathcal{A})$ such that $N_i \in \mathcal{N}$, for all $i \in I$, then $\operatorname{colim}_I N \in \mathcal{N}$. We define the triangulated subcategory of $\mathcal{D}(kI \otimes \mathcal{A})$ stable under coproducts

$$\mathcal{N}_I = \{N \in \mathcal{D}(kI \otimes \mathcal{A}) \mid N_i \in \mathcal{N} \ , \ \forall i \in I\} \subseteq \mathcal{D}(kI \otimes \mathcal{A}).$$

Recall that colim_I commutes with coproducts because it has a right adjoint Δ. Now it is sufficient to find a class $\Xi \subseteq \mathcal{N}_I$ such that $\mathcal{N}_I$ is the smallest triangulated subcategory stable under coproducts of $\mathcal{D}(kI \otimes \mathcal{A})$ containing Ξ and to show that $\operatorname{colim}_I N \in \mathcal{N}$, for all $N \in \Xi$. This is sufficient since, if $\mathcal{N}$ contains the colimits of the objects in Ξ, then it also contains the colimits of the objects in $\langle \Xi \rangle = \mathcal{N}_I$. We can consider, for all $i \in I$, the evaluation functor ev_i and its left adjoint $- \otimes i$

$$\begin{array}{ccc} L \otimes i & \mathcal{D}(kI \otimes \mathcal{A}) & X \\ \uparrow & {\scriptstyle -\otimes i}\uparrow \ \downarrow{\scriptstyle \mathrm{ev}_i} & \downarrow \\ L & \mathcal{DA} & X_i \end{array}$$

where the symbol of tensor product $\otimes$ is only utilized as a convention for

$$(L \otimes i)(j) := \coprod_{I(i,j)} L \ , \ \forall i, j \in I \ ,$$

where $I(i,j)$ denotes the set of morphisms in I from i to j. Then $\operatorname{colim}_{j \in I}(L \otimes i)(j) \xrightarrow{\sim} L$ since I is filtered. Let us give an example to be clear. If $I = \mathbb{N}$, we could have

$$\begin{array}{ccccccccccc} X & = & (X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & ....) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \vdots \\ Y \otimes 2 & = & (\ 0 & \longrightarrow & 0 & \longrightarrow & Y & = & Y & = & ....) \end{array}$$

and then $(Y \otimes 2)(4) := \coprod_{\mathbb{N}[2,4]} Y = Y$. According to the previous definition for the 'tensored' objects, we will let $X(j) = \coprod_{k \in I(0,j)} X_k$ for a generic object X of $\mathcal{D}(kI \otimes \mathcal{A})$. The searched class is $\Xi := \{N \otimes i \mid N \in \mathcal{N} \ , \ i \in I\}$. Indeed, take an object $Y \in \mathcal{N}_I$, the unit of the adjunction $Y_i \otimes i \to Y$ yields the short exact sequence in $\mathcal{C}(\mathcal{A})^I$

$$0 \longrightarrow \ker \Psi \longrightarrow \textstyle\coprod_{i \in I}(Y_i \otimes i) \xrightarrow{\Psi} Y \longrightarrow 0 \ ,$$

which splits pointwise since for each $j \in I$ the epimorphism

$$(\textstyle\coprod_{i \in I}(Y_i \otimes i))_j \xrightarrow{\Psi_j} Y_j$$

splits. This implies that $\ker\Psi$ belongs to $\mathcal{N}_I$ too, since the sequence gives rise to a distinguished triangle of $\mathcal{D}(kI\otimes\mathcal{A})$ and Y and the coproduct of the $Y_i\otimes i$ belong to $\mathcal{N}_I$. Then we can iterate the construction and obtain an acyclic resolution of Y

$$\cdots\longrightarrow C^{-2}\longrightarrow C^{-1}\longrightarrow C^{0}\twoheadrightarrow Y$$

by means of a cochain complex $C=C^*$ of DG modules. Notice that for each $j\in I$, the induced complex

$$\cdots\longrightarrow C_j^{-2}\longrightarrow C_j^{-1}\longrightarrow C_j^{0}\longrightarrow Y_j\longrightarrow 0$$

is contractible. For a complex

$$K:\qquad \cdots\longrightarrow K^p\longrightarrow K^{p+1}\longrightarrow\cdots$$

of DG $\mathcal{A}$-modules we define the associated total DG $\mathcal{A}$-module $\mathrm{Tot}^{\oplus}(K)$ as the DG functor

$$A\longmapsto \mathrm{Tot}^{\oplus}(K(A))\,,\ A\in\mathcal{A}\,,$$

where $\mathrm{Tot}^{\oplus}(K(A))$ denotes the sum-total complex. For the complex C above, we define the object $\mathrm{Tot}^{\oplus}(C)$ of $\mathcal{C}(\mathcal{A})^I$ to be the functor

$$j\longmapsto \mathrm{Tot}^{\oplus}(C_j)\,,\ j\in I.$$

Then, for each $j\in I$, the natural morphism $\mathrm{Tot}^{\oplus}(C)\to Y$ induces a homotopy equivalence

$$\mathrm{Tot}^{\oplus}(C_j)\longrightarrow Y_j.$$

Therefore, the morphism $\mathrm{Tot}^{\oplus}(C)\to Y$ is a quasi-isomorphism in $\mathcal{C}(\mathcal{A})^I$. Moreover, $\mathrm{Tot}^{\oplus}(C)$ lies in $\langle\Xi\rangle$ because of Milnor's telescope argument, showing that $\mathcal{N}_I=\langle\Xi\rangle$. Indeed, one can think of the double complex C as $C=\mathrm{colim}_{p\in\mathbb{Z}}\,C^{[-p}$, where $C^{[-p}:=\coprod_{i\geq -p}C^{i,*}$, which yields the filtration

$$C^{[0}\subset C^{[-1}\subset C^{[-2}\subset\ \ldots\,,$$

where all the inclusions split in $(\mathcal{GA})^I$. Then, the first two objects of the sequence

$$0\longrightarrow\coprod_{p\in\mathbb{N}}C^{[-p}\xrightarrow{\ \Phi\ }\coprod_{p\in\mathbb{N}}C^{[-p}\longrightarrow C\longrightarrow 0$$

lie in $\langle\Xi\rangle$. Moreover, the sequence is split exact in $(\mathcal{GA})^I$ and hence exact. This forces the third object to lie in $\langle\Xi\rangle$ too, since $\langle\Xi\rangle$ is a thick subcategory. □

6.2. **The Frobenius structure on the category of DG modules.** In this subsection we recall the Frobenius structure present on the category $\mathcal{CA}$ and study the relation with the projective model structure that we have seen in the preceding subsection.

We recall the definition of exact category. It differs from Quillen's original definition [32], but it is equivalent to that one [13, App. A].

Definition 6.6. Let $\mathcal{A}$ be an additive category. A pair $(i;d)$ of composable morphisms

$$X\overset{i}{\rightarrowtail}Y\overset{d}{\twoheadrightarrow}Z$$

is *exact* if i is a kernel of d and d a cokernel of i. The category $\mathcal{A}$ endowed with a class $\mathcal{E}$ of exact pairs closed under isomorphism and satisfying the following axioms (Ex0), (Ex1), (Ex2) and (Ex2$^{\mathrm{op}}$), is an *exact category*. The pairs (i, d) in the class $\mathcal{E}$ are by definition the *conflations*. The first and second components i and d of the conflations (i, d) are, respectively, the *inflations* and the *deflations*.

(Ex0) The identity morphism of 0 is a deflation;

(Ex1) the composition of two deflations is a deflation;

(Ex2) for each $f \in \mathrm{Hom}_{\mathcal{A}}(Z', Z)$ and each deflation $d \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$, there is a cartesian square (pull-back)

$$\begin{array}{ccc} Y' & \overset{d'}{\dashrightarrow\!\!\!\!\rightarrow} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \overset{d}{\twoheadrightarrow} & Z \end{array},$$

where d' is a deflation;

(Ex2$^{\mathrm{op}}$) for each $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$ and each inflation $i \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, there is a cocartesian square (push-out)

$$\begin{array}{ccc} X & \overset{i}{\rightarrowtail} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \overset{i'}{\dashrightarrow} & Y' \end{array},$$

where i' is an inflation.

We recall some very useful properties from the first proposition in [13, App. A], which permit to work easily with inflations, deflations and conflations.

a) For all X and Z in $\mathcal{E}$, the pair

$$X \xrightarrow[{[\mathbf{1}_X, 0]^t}]{} X \oplus Z \xrightarrow[{[0, \mathbf{1}_X]}]{} Z$$

is a conflation.

b) In the setting of Ex2 the pair

$$Y' \xrightarrow[{[-d', f']^t}]{} Z' \oplus Y \xrightarrow[{[f, d]}]{} Z$$

is a conflation.

c) In the setting of Ex2$^{\mathrm{op}}$ the pair

$$X \xrightarrow[{[-i, f]^t}]{} Y \oplus X' \xrightarrow[{[f', i']}]{} Y'$$

is a conflation.

We have the following

Lemma 6.7. *Let $\mathcal{E}$ be an exact category, where idempotents split. Let*

$$(*) \qquad X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$$

be a conflation of $\mathcal{E}$. Suppose that there is an isomorphism

$$Z \overset{\sim}{\longleftarrow} Z' \oplus P\,,$$

where P is a projective object. Then, the sequence (*) is isomorphic to the direct sum of the conflation

$$0 \rightarrowtail P \xrightarrow{\mathbf{1}_P} \twoheadrightarrow P$$

with a conflation

$$X \overset{i'}{\rightarrowtail} Y' \overset{p'}{\twoheadrightarrow} Z'.$$

Proof. Let

$$[u, v] : Z' \oplus P \xrightarrow{\sim} Z$$

be the given isomorphism. Since P is projective, there exists $\widetilde{v} : P \to Y$ such that $p \circ \widetilde{v} = v$. Therefore we obtain a morphism of conflations

$$\begin{array}{ccccc} 0 & \overset{0}{\rightarrowtail} & P & \overset{\mathbf{1}_P}{\twoheadrightarrow} & P \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \widetilde{v}} & & \downarrow{\scriptstyle v} \\ X & \underset{i}{\rightarrowtail} & Y & \underset{p}{\twoheadrightarrow} & Z. \end{array}$$

Let

$$\begin{bmatrix} u' \\ v' \end{bmatrix} : Z \xrightarrow{\sim} Z' \oplus P$$

be the inverse of the given isomorphism $[u, v]$. We obtain a morphism of conflations

$$\begin{array}{ccccc} X & \overset{i}{\rightarrowtail} & Y & \overset{p}{\twoheadrightarrow} & Z' \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle v' \circ p} & & \downarrow{\scriptstyle v} \\ 0 & \underset{0}{\rightarrowtail} & P & \underset{\mathbf{1}_P}{\twoheadrightarrow} & P. \end{array}$$

The composition of the two morphisms of conflations that we have obtained above is the identity morphism of the conflation

$$0 \overset{0}{\rightarrowtail} P \overset{\mathbf{1}_P}{\twoheadrightarrow} P.$$

Indeed, $v' \circ v = \mathbf{1}_P$ and $(v' \circ p) \circ \widetilde{v} = v' \circ v = \mathbf{1}_P$, since

$$\begin{bmatrix} u' \\ v' \end{bmatrix} \circ [u, v] = \begin{bmatrix} u' \circ u & u' \circ v \\ v' \circ u & v' \circ v \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{Z'} & 0 \\ 0 & \mathbf{1}_P \end{bmatrix}.$$

It follows that the conflation (*) is the direct sum of the conflation

$$0 \overset{0}{\rightarrowtail} P \overset{\mathbf{1}_P}{\twoheadrightarrow} P$$

and the cokernel of the morphism of conflations $(0, \widetilde{v}, v)$. This cokernel exists (since the idempotents split) and it is isomorphic to the conflation

$$X \rightarrowtail Y' \twoheadrightarrow Z'$$

for a direct factor Y' of Y. □

Remark 6.8. Clearly, the dual of the preceding lemma is also true.

An exact category $\mathcal{E}$ is a Frobenius category if it has enough injectives, enough projectives, and the two classes of the injectives and projectives coincide. For all pairs of objects X, Y of $\mathcal{E}$, let $I_{\mathcal{E}}(X,Y)$ be the subgroup of the abelian group $\mathrm{Hom}_{\mathcal{E}}(X,Y)$ formed by the morphisms which factor over an injective-projective object of $\mathcal{E}$. The *stable category of* $\mathcal{E}$, written $\underline{\mathcal{E}}$, is the category which has the same objects as $\mathcal{E}$ and the morphisms

$$\mathrm{Hom}_{\underline{\mathcal{E}}}(X,Y) = \mathrm{Hom}_{\mathcal{E}}(X,Y)/I_{\mathcal{E}}(X,Y).$$

The stable category $\underline{\mathcal{E}}$ is a triangulated category [8]. The suspension functor Σ is defined by choosing a *standard conflation*

$$X \rightarrowtail IX \twoheadrightarrow \Sigma X$$

for each object X, where IX is an injective-projective object of $\mathcal{E}$.

By definition, the distinguished triangles of $\underline{\mathcal{E}}$ are those isomorphic to a triangle

$$X \xrightarrow{\bar{i}} Y \xrightarrow{\bar{p}} Z \xrightarrow{\bar{e}} \Sigma X \,,$$

where the morphisms $\bar{i}$, $\bar{p}$, $\bar{e}$ are the equivalence classes of the morphisms i, p, e obtained from the conflations (i,p) which fit into the commutative diagram in $\mathcal{E}$

$$\begin{array}{ccccc} X & \overset{i}{\rightarrowtail} & Y & \overset{p}{\twoheadrightarrow} & Z \\ {\scriptstyle \mathbf{1}_X}\Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle e} \\ X & \rightarrowtail & IX & \twoheadrightarrow & \Sigma X. \end{array}$$

Let us consider the important example of the Frobenius category $\mathcal{E} = \mathcal{CA}$ whose conflations are defined to be the graded split short exact sequences [10], [14]. For each DG module X in $\mathcal{CA}$ we write IX for the (mapping) cone on the identity morphism of X and PX for the (mapping) cocone on the same morphism. The objects IX and PX are injective-projective DG modules for each $X \in \mathcal{CA}$. We can choose

$$X \underset{[\mathbf{1}_X, d_X]^t}{\rightarrowtail} IX \underset{[-d_X, \mathbf{1}_X]}{\twoheadrightarrow} \Sigma X$$

as standard conflation. By dualizing the definition of the suspension functor Σ we obtain the loop (or cosuspension) functor Ω. These two endofunctors of $\mathcal{CA}$, Σ and Ω, are respectively the shift functor in the right and left directions. With this choice, Σ and Ω are inverse to each other. The (triangulated) stable category $\underline{\mathcal{CA}}$ is the category $\mathcal{HA}$ of DG modules up to homotopy.

We list some useful properties of the functors I, P, Σ, Ω. Since Σ and Ω are exact functors, I and P are exact, too. By the isomorphisms

$$(*) \qquad \mathrm{Hom}_{\mathcal{CA}}(IX,Y) = \mathrm{Hom}_{\mathcal{GA}}(X,Y) = \mathrm{Hom}_{\mathcal{CA}}(X,PY)\,,$$

for all X, Y in $\mathcal{CA}$, we have that I is left adjoint to P. Moreover, I and P commute with arbitrary small coproducts. In particular, for each regular cardinal α, they commute with α-small coproducts. It is clear that $I\Omega = P$ and $P\Sigma = I$. Moreover, applying Σ to the standard conflation for X, we have the conflation

$$\Sigma X \rightarrowtail \Sigma IX \twoheadrightarrow \Sigma^2 X\,,$$

which equals the standard conflation for ΣX

$$\Sigma X \rightarrowtail I\Sigma X \twoheadrightarrow \Sigma^2 X.$$

This shows that I commutes with Σ. Similarly, Ω and P commute.

As an application of property c) above we show that the morphisms $[\sigma_\lambda, -\iota]^t$ in subsection 5.1 are cofibrations. This follows immediately by the following

Lemma 6.9. *Let A and B be cofibrant* DG *modules. Let ι be the inclusion $A \rightarrowtail IA$. For any morphism $f : A \to B$ the morphism*

$$A \xrightarrow[{[f,-\iota]^t}]{} B \oplus IA$$

is a cofibration.

Proof. Indeed, the morphism $[f, -\iota]^t$ can be constructed from the cocartesian square

$$\begin{array}{ccc} A & \overset{\iota}{\rightarrowtail} & IA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \underset{\iota'}{\rightarrowtail} & C \end{array},$$

where ι' is an inflation. By item c) after definition 6.6 the pair

$$A \xrightarrow[{[f,-\iota]^t}]{} B \oplus IA \xrightarrow[{[\iota',f']}]{} C$$

is a conflation. Therefore, the morphism $[f, -\iota]^t$ is an inflation. Hence, it is a monomorphism. Moreover, the inclusion ι has a retraction r in $\mathcal{GA}$, which in its turn induces a retraction in $\mathcal{GA}$ of $[f, -\iota]^t$ by composition with the projection

$$B \oplus IA \twoheadrightarrow IA.$$

Thus, the conflation above splits and the cokernel C of $[f, -\iota]^t$ is $B \oplus \Sigma A$, which is a cofibrant DG module since we assume A and B cofibrant. It follows that the morphism $[f, -\iota]^t$ is a cofibration. □

In order to study the relation between injective-projective *with respect to the exact structure* and (cochain) contractible DG modules, we introduce the following

Definition 6.10. (Cochain) homotopically equivalent and (cochain) contractible DG modules.

a) Two DG modules M and N are *cochain homotopically equivalent* or, equivalently, have the same *cochain homotopy type* if there exist two DG morphisms $f : M \to N$ and $g : N \to M$ such that the compositions $g \circ f$ and $f \circ g$ are respectively cochain homotopic to the identity morphisms of M and N. Briefly, we write $M \approx N$. If M and N are cofibrant, M and N are cochain homotopically equivalent if and only if they are also *homotopically equivalent* or, equivalently, have the same *homotopy type* with respect to the projective model structure on $\mathcal{CA}$ (*cf.* remark 6.2). Briefly, we write $M \sim N$.

b) A DG module M is *cochain contractible* if it is cochain homotopically equivalent to the zero DG module (*i.e.* if its identity morphism $\mathbf{1}_M$ is cochain null-homotopic, *i.e.* cochain homotopical to the zero morphism). Briefly, we write $M \approx 0$. If M is cofibrant and cochain contractible, then it is also *contractible* with respect to the projective model structure on $\mathcal{CA}$ (*cf.* remark 6.2). Briefly, we write $M \sim 0$.

There is a very useful characterization of contractible DG modules.

Lemma 6.11. *Let F be a* DG *module. Let $\mathcal{CA}$ be endowed with the projective model structure. Then the following assertions are equivalent:*

(*i*) *F is cochain contractible;*
(*ii*) *F is injective-projective with respect to the exact structure;*
(*iii*) *F is a direct factor of a* DG *module IX,* i.e. *the mapping cone on the identity morphism $\mathbf{1}_X$, for some* DG *module X.*

Proof. (*i*) $\Longrightarrow$ (*iii*) : Let h be a contracting cochain homotopy for $\mathbf{1}_F$. We have

$$d_F \circ h + h \circ d_F = \mathbf{1}_F.$$

Define

$$r : IF \twoheadrightarrow F$$

to be the morphism $[\mathbf{1}_F, h]$. Then it is easy to check that r is a morphism of DG modules and that r is a retraction of the canonical morphism

$$i = [\mathbf{1}_F, 0]^t : F \rightarrowtail IF.$$

(*iii*) $\Longrightarrow$ (*ii*) : It suffices to show that if X is a DG module, then IX is projective with respect to the exact structure. For this, it suffices to show that the functor $\mathrm{Hom}_{\mathcal{CA}}(IX, -)$ takes graded split short exact sequences to exact sequences. But this is clear from the isomorphism of functors (*)

$$\mathrm{Hom}_{\mathcal{CA}}(IX, -) \cong \mathrm{Hom}_{\mathcal{GA}}(X, -).$$

(*ii*) $\Longrightarrow$ (*i*) : Since F is injective with respect to the exact structure, every inflation

$$F \rightarrowtail IF$$

splits in the category $\mathcal{CA}$. Therefore F is a direct factor of IF. As IF is the cone of the identity morphism of F, it is cochain contractible. Thus, F is cochain contractible as well as any direct factor of a cochain contractible DG module. □

We will frequently use the following particular case of the preceding lemma: A cofibrant DG module is contractible if and only if it is a direct factor of the cone of the identity morphism of some cofibrant DG module.

Notice that the null-homotopic morphisms are exactly those that factor through an injective-projective object.

6.3. **Different closures of representable DG modules in $\mathcal{CA}$.** In this subsection, we introduce different closures of the class of representable DG modules in $\mathcal{CA}$ with respect to the exact structure and study the relations among them and with the analogous triangulated subcategories of $\mathcal{DA}$.

Definition 6.12. Let $\mathcal{A}$ be a small DG category. Let $\mathcal{U}$ be a class of DG modules of $\mathcal{CA}$. We define the *subcategory* $[\,\mathcal{U}\,]_\alpha$ to be the closure of $\mathcal{U}$ in $\mathcal{CA}$ under

a) the actions of Σ and Ω;
b) formation of α-small coproducts;
c) formation of graded split extensions;
d) formation of direct factors.

The *subcategory* $[\,\mathcal{U}\,]$ is the closure of $\mathcal{U}$ in $\mathcal{CA}$ under the operations of points a), c), d) and formation of arbitrary set-indexed coproducts.

Let $\mathcal{G}$ be the subcategory of $\mathcal{CA}$ formed by the representable objects. In this section we are interested in some closures of $\mathcal{G}$. Therefore, we begin with the following simple

Lemma 6.13. *The following equality holds*

$$[\,\mathcal{G}\,] = \bigcup_\alpha [\,\mathcal{G}\,]_\alpha\,,$$

where α runs through the infinite regular cardinals.

Proof. Clearly $[\,\mathcal{G}\,]$ contains $\bigcup_\alpha[\,\mathcal{G}\,]_\alpha$, as it contains each $[\,\mathcal{G}\,]_\alpha$. We have to show the reverse inclusion. The category $\bigcup_\alpha[\,\mathcal{G}\,]_\alpha$ contains $\mathcal{G}$, is stable under left and right shifts, extensions and formation of direct factors of its objects. Therefore, it suffices to check that it is also closed under arbitrary set-indexed coproducts, by the minimality of $[\,\mathcal{G}\,]$ for these properties. Let $(X_i)_{i\in I}$ be an arbitrary family of DG modules contained in $\bigcup_\alpha[\,\mathcal{G}\,]_\alpha$, indexed over the *set* I. Let the cardinality of I be β and suppose that for each $i \in I$ the DG module X_i lies in $[\mathcal{G}]_{\beta_i}$, for some cardinal β_i. Let γ be a regular cardinal greater than the sum of β and the β_i's. The DG module X_i is in $[\mathcal{G}]_\gamma$ for all $i \in I$. Moreover, since $\gamma > \beta$, the coproduct of the X_i's, that is of β objects of $[\mathcal{G}]_\gamma$, must lie in $[\mathcal{G}]_\gamma$. Hence, it lies in $\bigcup_\alpha[\,\mathcal{G}\,]_\alpha$. ☐

Notation 6.14. Let $\mathcal{C}$ be a class of cofibrant DG modules of $\mathcal{CA}_{cof}$. We denote $\widetilde{\mathcal{C}}$ the class of cofibrant DG modules which are homotopically equivalent to some DG module of $\mathcal{C}$.

For example, let $\mathcal{G}_0$ be the full subcategory of $\mathcal{CA}_{cof}$ whose unique object is the zero DG module. Then $\widetilde{\mathcal{G}_0}$ is the full subcategory of contractible cofibrant DG modules.

Lemma 6.15. *The class formed by the objects of $\widetilde{\mathcal{G}_0}$ is stable under*

a) *the action of Σ and Ω;*
b) *formation of arbitrary set-indexed coproducts;*
c) *graded split extensions;*
d) *formation of direct factors.*

That is, $\widetilde{\mathcal{G}_0}$ equals its closure $[\,\widetilde{\mathcal{G}_0}\,]$.

Proof. a) Let C be a contractible object of $\mathcal{CA}_{cof}$. By lemma 6.11, C is a direct factor of IX, for some DG module X. Therefore, the cofibrant DG module ΣC is a direct factor of $I(\Sigma X)$, since I commutes with Σ.

b) Suppose that $(C_i)_{i\in I}$ is a family of contractible DG modules of $\mathcal{CA}_{cof}$, with I a set of arbitrary cardinality. The coproduct $\coprod_{i\in I} C_i$ lies in $\mathcal{CA}_{cof}$ and is a direct factor of $I(\coprod_{i\in I} X_i)$, since I commutes with coproducts.

c) Let C and C' be contractible objects of $\mathcal{CA}_{cof}$. Their extension is the cofibrant DG module E which fits in the conflation

$$C \rightarrowtail E \twoheadrightarrow C'.$$

This conflation splits since C is injective and C' is projective. This is also a particular case of lemma 6.7. Therefore E equals $C \oplus C'$ which is a contractible DG module of $\mathcal{CA}_{cof}$.

d) Let C be a contractible DG module of $\mathcal{CA}_{cof}$. Then C is a direct factor of IX, for some DG module X. Suppose that D is a direct factor of C. Clearly D is a direct factor of IX, too. Hence it is a contractible DG module of $\mathcal{CA}_{cof}$. □

In the following we will always work with small DG categories in order to ensure that the (large) category of DG modules has small Hom-sets. Nevertheless, some results would be true even if we had not made this smallness assumption.

We need to prove that the closures $\widetilde{(-)}$ and $[\,-\,]_\alpha$ of a class of cofibrant DG modules commute for every regular cardinal α. It turns out that this property is crucial for the constructions that we will do in the sequel.

Proposition 6.16. *Let $\mathcal{S}$ be a class of cofibrant* DG *modules. Let α be an arbitrary regular cardinal. Then we have*

$$\widetilde{[\,\mathcal{S}\,]_\alpha} \subseteq [\,\widetilde{\mathcal{S}}\,]_\alpha \quad \textit{and} \quad \widetilde{[\,\mathcal{S}\,]} = [\,\widetilde{\mathcal{S}}\,].$$

The subcategory $[\,\widetilde{\mathcal{S}}\,]_\alpha$ is stable under homotopy equivalences and its image in $\mathcal{HA}$ is stable under formation of direct factors.

Moreover, if $\alpha > \aleph_0$, we have the equality

$$\widetilde{[\,\mathcal{S}\,]_\alpha} = [\,\widetilde{\mathcal{S}}\,]_\alpha.$$

Proof. We begin by showing that $\widetilde{[\,\mathcal{S}\,]_\alpha} \subseteq [\,\widetilde{\mathcal{S}}\,]_\alpha$, for all the regular cardinals α.

Since it is clear that $[\,\widetilde{\mathcal{S}}\,]_\alpha$ contains $[\,\mathcal{S}\,]_\alpha$, it is sufficient to verify that $[\,\widetilde{\mathcal{S}}\,]_\alpha$ is stable under homotopy equivalences. Indeed, the subcategory $\widetilde{[\,\mathcal{S}\,]_\alpha}$ is minimal for these properties.

Actually, it turns out that it is enough to verify that the following property holds:

- the subcategory $[\,\widetilde{\mathcal{S}}\,]_\alpha$ is stable under adding objects of the form IZ, where Z is an arbitrary cofibrant DG module.

Indeed, let X be in $[\,\widetilde{\mathcal{S}}\,]_\alpha$ and let X' be a cofibrant DG module homotopically equivalent to X. We have to show that X' is in $[\,\widetilde{\mathcal{S}}\,]_\alpha$, too.

By definition there are morphisms of DG modules $f : X' \to X$ and $g : X \to X'$ so that $f \circ g \sim \mathbf{1}_X$ and $g \circ f \sim \mathbf{1}'_X$. The cone of the morphism f is given by the push-out

$$\begin{array}{ccc} X' & \stackrel{i}{\rightarrowtail} & IX' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \stackrel{i'}{\rightarrowtail} & \mathrm{cone}(f)\,, \end{array}$$

where i is the canonical inclusion. Here, i' is a graded split monomorphism (since it is obtained by a push-out from a graded split monomorphism) and f' is still a homotopy equivalence (since it is obtained by a push-out along a graded split monomorphism). By point c) after definition 6.6 we have the graded split exact sequence (conflation)

$$X' \xrightarrow[{[f,-i]^t}]{} X \oplus IX' \xrightarrow[{[i',f']}]{} \mathrm{cone}(f).$$

As $\mathrm{cone}(f)$ is homotopically equivalent to IX', it is contractible and cofibrant. By lemma 6.11 it is projective, too. Therefore the last conflation splits in $\mathcal{CA}$ and X' is a direct factor of $X \oplus IX'$. Thus, since the subcategory $[\,\widetilde{\mathcal{S}}\,]_\alpha$ is stable under formation of direct factors, in order to show that X' lies in $[\,\widetilde{\mathcal{S}}\,]_\alpha$ it suffices to verify that the subcategory $[\,\widetilde{\mathcal{S}}\,]_\alpha$ is stable under sums with DG modules of the form IZ, for any cofibrant DG module Z.

It remains to verify that the subcategory $[\,\widetilde{\mathcal{S}}\,]_\alpha$ has property $\bullet$.

We begin and remark that $\widetilde{\mathcal{S}}$ is stable under adding objects IZ, where Z is an arbitrary cofibrant DG module. Then, we verify that, given a subcategory $\mathcal{C}$ which is stable under adding objects IZ (Z an arbitrary cofibrant DG module), the same property holds for

a) the closure of $\mathcal{C}$ under Σ and Ω, say $\mathcal{C}_{\Sigma\Omega}$;
b) the closure of $\mathcal{C}$ under α-small coproducts, say $\mathcal{C}_{\coprod_\alpha}$;
c) the closure of $\mathcal{C}$ under graded split extensions, say $\mathcal{C}_{\mathrm{Ext}}$;
d) the closure of $\mathcal{C}$ under direct factors, say $\mathcal{C}_{\ominus}$.

This guarantees that the closure $[\,\widetilde{\mathcal{S}}\,]_\alpha$ has the property $\bullet$, too.

Point a) : Let A be an object in $\mathcal{C}$, Z a cofibrant DG module, and consider the sum $\Sigma A \oplus IZ$. We can write (*cf.* considerations after lemma 6.7) $\Sigma A \oplus IZ$ as

$$\begin{aligned} \Sigma A \oplus IZ &= \Sigma A \oplus I\Sigma\Omega Z \\ &= \Sigma A \oplus \Sigma I \Omega Z \\ &= \Sigma(A \oplus I(\Omega Z)). \end{aligned}$$

As $A \in \mathcal{C}$ and ΩZ is cofibrant, the object $A \oplus I(\Omega Z)$ is in $\mathcal{C}$, by the assumed property of $\mathcal{C}$. It follows that $\Sigma(A \oplus I(\Omega Z))$ lies in $\mathcal{C}_{\Sigma\Omega}$.

Point b) : Let I be a set of cardinality strictly smaller than α. Let A_i, $i \in I$, be a family of cofibrant DG modules in $\mathcal{C}$. Consider the sum of the α-small coproduct $\coprod_{i\in I} A_i$ with the object IZ, where Z is an arbitrary cofibrant DG module. We can write

$$(\coprod_{i\in I} A_i) \oplus IZ = (\coprod_{i\in I\setminus\{i_0\}} A_i) \oplus (A_{i_0} \oplus IZ)\,,$$

for an arbitrary element $i_0 \in I$. By the assumed property, $A_{i_0} \oplus IZ$ is in $\mathcal{C}$. Therefore, the term on the right in the last equation is an α-small coproduct of objects of $\mathcal{C}$. Hence $(\coprod_{i\in I} A_i) \oplus IZ$ lies in $\mathcal{C}_{\coprod_\alpha}$.

Point c) : Let A and B be two cofibrant DG modules lying in the subcategory $\mathcal{C}$. Consider the sum of an extension E of A and B with an object IZ, where Z is a cofibrant DG module. We think of IZ as of the split extension of the DG modules IZ and 0. Then, by adding these two extensions, we get the conflation

$$A \oplus IZ \rightarrowtail E \oplus IZ \twoheadrightarrow B.$$

Since $A \oplus IZ$ is in $\mathcal{C}$ by the assumed property, the object $E \oplus IZ$ lies in $\mathcal{C}_{\mathrm{Ext}}$.

Point d) : Let F be a direct factor of an object A lying in $\mathcal{C}$. Consider the object $F \oplus IZ$, where Z is a cofibrant DG module. Clearly, $F \oplus IZ$ is a direct factor of $A \oplus IZ$. Since this object is in $\mathcal{C}$ by the assumed property, the object $F \oplus IZ$ lies in $\mathcal{C}_{\ominus}$.

Now we show that, if α is strictly greater than $\aleph_0$, then $[\,\widetilde{\mathcal{S}}\,]_\alpha \subseteq \widetilde{[\,\mathcal{S}\,]_\alpha}$.

It is clear that $\widetilde{[\,\mathcal{S}\,]_\alpha}$ contains $\widetilde{\mathcal{S}}$. Therefore it suffices to show that $\widetilde{[\,\mathcal{S}\,]_\alpha}$ is stable under Σ, Ω, α-small coproducts, graded split extensions and direct factors. Indeed, $[\,\widetilde{\mathcal{S}}\,]_\alpha$ is minimal for these properties.

The functors Σ and Ω preserve homotopy equivalences since they are DG functors. The same is true for arbitrary small coproducts. Let us consider a graded split extension of two objects X, X' in $\widetilde{[\,\mathcal{S}\,]_\alpha}$

$$X \overset{}{\underset{i}{\rightarrowtail}} E \underset{p}{\twoheadrightarrow} Y.$$

By definition of $\widetilde{[\,\mathcal{S}\,]_\alpha}$ there are two objects X', Y' in $[\,\mathcal{S}\,]_\alpha$ which are respectively homotopically equivalent to X and Y. Let $u : X \to X'$ and $v' : Y \to Y'$ be two homotopy equivalences. Let us form the push-out of the morphisms i and u

$$\begin{array}{ccccc} X & \overset{i}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} & & \| \\ X' & \underset{}{\overset{i'}{\rightarrowtail}} & F & \overset{p'}{\twoheadrightarrow} & Y. \end{array}$$

Here i' is a graded split monomorphism (since it is obtained by a push-out from a graded split monomorphism) and u' is still a homotopy equivalence (since it is obtained by a push-out along a graded split monomorphism). Now we form the pull-back along v' to obtain the diagram

$$\begin{array}{ccccc} X' & \overset{i'}{\rightarrowtail} & F & \overset{p'}{\twoheadrightarrow} & Y \\ \| & & \uparrow{\scriptstyle u''} & & \uparrow{\scriptstyle v'} \\ x' & \overset{i''}{\rightarrowtail} & G & \overset{p''}{\twoheadrightarrow} & Y'. \end{array}$$

Then u'' is a homotopy equivalence, as in the case of push-out. By the transitivity of homotopy equivalence among cofibrant DG modules we get that E is homotopically equivalent to G. Since G is in $[\,\mathcal{S}\,]_\alpha$, as an extension of objects in $[\,\mathcal{S}\,]_\alpha$, it follows that E lies in $\widetilde{[\,\mathcal{S}\,]_\alpha}$, *i.e.* $\widetilde{[\,\mathcal{S}\,]_\alpha}$ is stable under graded split extensions.

Let F be a direct factor of an object X lying in $\widetilde{[\mathcal{S}]_\alpha}$. Thus, there are morphisms $i : F \to X$ and $r : X \to F$ such that $r \circ i = \mathbf{1}_F$. Let e be the composed morphism $i \circ r$. The morphism $e : X \to X$ is idempotent. Moreover, there exist a DG module $X' \in [\mathcal{S}]_\alpha$ and two morphisms $f : X \to X'$ and $g : X' \to X$ so that $g \circ f \sim \mathbf{1}_X$ and $f \circ g \sim \mathbf{1}_{X'}$. This situation induces the endomorphism $e' := f \circ i \circ r \circ g$ of X'. Note that e' is not idempotent, nevertheless it is "homotopically idempotent", *i.e.* e'^2 is homotopically equivalent to the identity morphism of X'. This implies that in the homotopy category $[\mathcal{S}]_\alpha/\sim \subseteq \mathcal{HA}$ the equivalence class $\overline{e'}$ is a strict idempotent. Since in $\mathcal{HA}$ idempotents split for α strictly greater than $\aleph_0$, we get a direct factor $\overline{F'}$ of the image of X' in $[\mathcal{S}]_\alpha/\sim$ which is isomorphic to the image of F. Clearly there is a representative DG module F' in $\widetilde{[\mathcal{S}]_\alpha}$ which is homotopically equivalent to F. It follows by transitivity that F is homotopically equivalent to a DG module lying in $[\mathcal{S}]_\alpha$. Hence F lies in $\widetilde{[\mathcal{S}]_\alpha}$ and we are done.

Clearly the equality $\widetilde{[\mathcal{S}]} = [\widetilde{\mathcal{S}}]$ also holds. □

The following lemma gives a characterization of the subcategories $[\widetilde{\mathcal{S}}]_\alpha$ and $[\widetilde{\mathcal{S}}]$ as the closures of the union of $\mathcal{S}$ with the contractible DG modules. It is in these terms that we will often consider the subcategories $[\widetilde{\mathcal{S}}]_\alpha$ and $[\widetilde{\mathcal{S}}]$ in the sequel.

Lemma 6.17. *Let α be an (infinite) regular cardinal. Let $\mathcal{A}$ be a* DG *category and $\mathcal{S}$ a class of cofibrant* DG *modules of $\mathcal{A}$. Then, the following equalities hold*

$$[\widetilde{\mathcal{S}}]_\alpha = [\mathcal{S} \cup \widetilde{\mathcal{G}}_0]_\alpha \quad , \quad [\widetilde{\mathcal{S}}] = [\mathcal{S} \cup \widetilde{\mathcal{G}}_0].$$

Remark that it is equivalent to take the closures of $\mathcal{S} \cup \widetilde{\mathcal{G}}_0$ in $\mathcal{CA}$ or in $\mathcal{CA}_{cof}$, since $\mathcal{CA}_{cof}$ is closed under the operations $[-]_\alpha$ and $[-]$.

Proof. We give only the proof for the closure $[-]_\alpha$, as it also works for $[-]$.

Let us begin by showing that $[\widetilde{\mathcal{S}}]_\alpha$ contains $[\mathcal{S} \cup \widetilde{\mathcal{G}}_0]_\alpha$. Remark that the subcategory $[\mathcal{S}]_\alpha$ contains the zero object, which is the sum of the empty family. Therefore, the subcategory $\widetilde{[\mathcal{S}]_\alpha}$ contains the subcategory of contractible cofibrant DG modules $\widetilde{\mathcal{G}}_0$. By the first part of proposition 6.16 the subcategory $[\widetilde{\mathcal{S}}]_\alpha$ contains $\widetilde{[\mathcal{S}]_\alpha}$, for all regular cardinals α. It follows that $[\widetilde{\mathcal{S}}]_\alpha$ contains $\widetilde{\mathcal{G}}_0$. Hence the subcategory $[\widetilde{\mathcal{S}}]_\alpha$ also contains the union $\mathcal{S} \cup \widetilde{\mathcal{G}}_0$. Since the subcategory $[\mathcal{S} \cup \widetilde{\mathcal{G}}_0]_\alpha$ is the smallest full DG subcategory of $\mathcal{CA}$ containing $\mathcal{S} \cup \widetilde{\mathcal{G}}_0$ and stable under Σ, Ω, α-small coproducts, graded split extensions and direct factors, the claim follows.

For the reverse inclusion, it suffices to show that $[\mathcal{S} \cup \widetilde{\mathcal{G}}_0]_\alpha$ contains $\widetilde{\mathcal{S}}$ and that it is stable under right and left shifts, formation of α-small coproducts, extensions, direct factors. By definition, $[-]_\alpha$ is the closure under these operations. Thus, it remains only to prove that $[\mathcal{S} \cup \widetilde{\mathcal{G}}_0]_\alpha$ contains $\widetilde{\mathcal{S}}$.

Consider an arbitrary DG module $X \in \widetilde{\mathcal{S}}$, *i.e.* X is cofibrant and $X \sim X'$, for some $X' \in \mathcal{S}$. Then, there are two morphisms $f : X \to X'$ and $g : X' \to X$ whose compositions $f \circ g$ and $g \circ f$ are respectively homotopical to the identity morphisms of X' and X. By point c) after definition 6.6, from the cocartesian square (push-out)

$$\begin{array}{ccc} X' & \xrightarrow{\iota} & IX' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{\iota'} & \mathrm{cone}(g)\ , \end{array}$$

where ι is the inclusion inflation and ι' is an inflation, too, we have the conflation

$$X' \xrightarrow[{[g,-\iota]^t}]{} X \oplus IX' \xrightarrow[{[\iota',g']}]{} \mathrm{cone}(g).$$

We observe that the morphism $[g, -\iota]^t$ is a quasi-isomorphism. Therefore, the cohomology of the DG module cone(g) is zero. Since cone(g) is also cofibrant, by Quillen's theorem about the homotopy category of a model category, it is contractible, too. This shows that cone(g) lies in $\widetilde{\mathcal{G}}_0$. As $X' \in \mathcal{S}$, the extension $X \oplus IX'$ lies in $[\,\mathcal{S} \cup \widetilde{\mathcal{G}}_0\,]_\alpha$. Thus, the direct factor X is in $[\,\mathcal{S} \cup \widetilde{\mathcal{G}}_0\,]_\alpha$ and the assertion follows. □

Now we can consider the subcategories $[\,\widetilde{\mathcal{G}}\,]_\alpha$ and $[\,\widetilde{\mathcal{G}}\,]$ of $\mathcal{C}\mathcal{A}_{cof}$, where $\mathcal{G}$ is the class of representable DG modules. These subcategories turn out to be the DG analogues of the triangulated subcategories $\langle\mathcal{G}\rangle_\alpha$ and $\langle\mathcal{G}\rangle$ of the derived category $\mathcal{D}\mathcal{A}$, which in their turn coincide respectively with the subcategories of α-compact objects $(\mathcal{D}\mathcal{A})^\alpha$ and all of $\mathcal{D}\mathcal{A}$ (*cf.* proposition 4.6 and corollary 4.8). We will show that a version 'up to homotopy' of the conditions (G3) and (G4), which were used to define $(\mathcal{D}\mathcal{A})^\alpha$, holds for the category $[\,\widetilde{\mathcal{G}}\,]_\alpha$.

Remark that, for $\mathcal{S} = \mathcal{G}$, lemma 6.17 gives the useful relation

$$[\,\widetilde{\mathcal{G}}\,]_\alpha = [\,\mathcal{G} \cup \widetilde{\mathcal{G}}_0\,]_\alpha\ , \qquad [\,\widetilde{\mathcal{G}}\,] = [\,\mathcal{G} \cup \widetilde{\mathcal{G}}_0\,].$$

The link with the derived category is given by the inversion of quasi-isomorphisms.

Proposition 6.18. *Let $\mathcal{A}$ be a* DG *category and $\mathcal{G}$ the subcategory of the representable* DG *modules.*

a) *Let α be a regular cardinal strictly greater than $\aleph_0$. Then the subcategory of the α-compact objects of the derived category $\mathcal{D}\mathcal{A}$ is equivalent to the localization of $[\,\widetilde{\mathcal{G}}\,]_\alpha$ at the class of quasi-isomorphisms or, equivalently, to the quotient category of $[\,\widetilde{\mathcal{G}}\,]_\alpha$ by the homotopy relation.*

b) *Let α equal the regular cardinal $\aleph_0$. Then the subcategory of the ($\aleph_o$-)compact objects of the derived category $\mathcal{D}\mathcal{A}$ is equivalent to the idempotent completion of the localization of $[\,\widetilde{\mathcal{G}}\,]_{\aleph_0}$ at the class of quasi-isomorphisms or, equivalently, of the quotient category of $[\,\widetilde{\mathcal{G}}\,]_{\aleph_0}$ by the homotopy relation.*

Proof. By item a) of corollary 4.8 with $\alpha = \aleph_0$, we have that the equality $(\mathcal{D}\mathcal{A})^\alpha = \langle\mathcal{G}\rangle_\alpha$ holds for all regular cardinals. We recall that $\langle\mathcal{G}\rangle_\alpha$ is the closure in $\mathcal{D}\mathcal{A}$ of $\mathcal{G}$ under right and left shifts, α-small coproducts, extensions (in the triangulated sense) and direct factors. Therefore the triangulated subcategory $\langle\mathcal{G}\rangle_\alpha$ is triangle equivalent to the closure $\mathcal{U}$ in $\mathcal{C}\mathcal{A}_{cof}/\sim$ of $\mathcal{G}$ under the same operations.

a) To conclude we have to show that, for all the regular cardinals strictly greater than $\aleph_0$, the subcategory $\mathcal{U}$ equals $[\,\widetilde{\mathcal{G}}\,]_\alpha/\sim$, *i.e.* the image of $[\,\widetilde{\mathcal{G}}\,]_\alpha$ in $\mathcal{C}\mathcal{A}_{cof}/\sim$. Let us consider the projection functor

$$\mathcal{CA}_{cof} \xrightarrow[\pi]{} \mathcal{CA}_{cof}/\sim .$$

The image of $\mathcal{G}$ under π is clearly contained in $\mathcal{U}$. Therefore $\mathcal{U}$ also contains the image of $\widetilde{\mathcal{G}}$, as the two images are the same in $\mathcal{DA}$. Hence we have

$$\pi([\,\widetilde{\mathcal{G}}\,]_\alpha) \subseteq \mathcal{U}\,,$$

since $\mathcal{U}$ is stable under Σ, Ω, α-small coproducts, triangle extensions and direct factors. We remark that this inclusion holds for all regular cardinals α.

For the reverse inclusion, it suffices to show that $\pi([\,\widetilde{\mathcal{G}}\,]_\alpha)$ contains $\mathcal{G}$ and that it is stable under Σ, Ω, α-small coproducts, triangle extensions and direct factors.

Here, the non trivial properties to verify are the stability of $\pi([\,\widetilde{\mathcal{G}}\,]_\alpha)$ under triangle extensions and direct factors.

Let us remark that $\pi([\,\widetilde{\mathcal{G}}\,]_\alpha)$ is stable under isomorphisms in $\mathcal{CA}_{cof}/\sim$ since $[\,\widetilde{\mathcal{G}}\,]_\alpha$ is stable under homotopy equivalences as it equals $\widetilde{[\,\mathcal{G}\,]_\alpha}$, by the part of proposition 6.16 concerning the regular cardinals $\alpha > \aleph_0$. Moreover, $\pi([\,\widetilde{\mathcal{G}}\,]_\alpha)$ is stable under cones. Therefore it is stable under triangle extensions (recall that the cone construction is unique only up to isomorphisms).

Since, for $\alpha > \aleph_0$ the subcategory $\pi([\,\widetilde{\mathcal{G}}\,]_\alpha)$ is idempotent complete in $\mathcal{CA}/\sim$, it is closed under direct factors (*cf.* [2]). This shows the claimed reverse inclusion, hence the equivalence of point a) in the statement, for all the reguler cardinals strictly greater than $\aleph_0$.

b) Let $\mathcal{C}^{\ominus}$ denote the idempotent completion of a (sub-)category $\mathcal{C}$. We have to show that

$$\pi([\,\widetilde{\mathcal{G}}\,]_{\aleph_0})^{\ominus} = \mathcal{U}.$$

As we already know that the inclusion "$\subseteq$" also holds in the case $\alpha = \aleph_0$, we need to show the reverse inclusion only.

The subcategory $\pi([\,\widetilde{\mathcal{G}}\,]_{\aleph_0})^{\ominus}$ contains $\mathcal{G}$, is clearly stable under Σ, Ω, finite coproducts and direct factors. It remains to verify stability under extensions. For this, as the subcategory $\pi([\,\widetilde{\mathcal{G}}\,]_{\aleph_0})$ is closed under extensions, it suffices to show that the direct factors of objects of $\pi([\,\widetilde{\mathcal{G}}\,]_{\aleph_0})$ are closed under extensions.

Let the diagram $f' : X' \to Y'$ be a direct factor of the diagram $f : X \to Y$ of $\pi([\,\widetilde{\mathcal{G}}\,]_{\aleph_0})$. Then, the identity of the diagram $f' : X' \to Y'$ factors as

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

Let us extend the rows of this diagram to distinguished triangles. We obtain the commutative diagram

$$
\begin{array}{ccccccc}
X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \\
{\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle \Sigma i}\downarrow \\
X & \xrightarrow{f} & Y & \longrightarrow & Z & \xrightarrow{f} & \Sigma X \\
{\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle \Sigma p}\downarrow \\
X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'.
\end{array}
$$

Here, the dashed arrows come from axiom (T3) of triangulated categories. The composition $r \circ k$ is an isomorphism, since $p \circ i = \mathbf{1}_{X'}$, $q \circ j = \mathbf{1}_{Y'}$, $\Sigma p \circ \Sigma i = \mathbf{1}_{\Sigma X'}$, but it is *not* necessarily the identity of Z'. Anyway, there exists an inverse h of $r \circ k$. Thus, the composition of k with the morphism $h \circ r : Z \to Z'$ is the identity of Z'. This shows that Z' is a direct factor of Z. As $\pi([\widetilde{\mathcal{G}}]_{\aleph_0})$ is stable under extensions, the object Z is in $\pi([\widetilde{\mathcal{G}}]_{\aleph_0})$. It follows that Z' is in $\pi([\widetilde{\mathcal{G}}]_{\aleph_0})^{\ominus}$ and we are done. $\square$

Remark 6.19. Note that the reason of the different behaviour in the case $\alpha = \aleph_0$ is due to the fact that idempotents in $[\widetilde{\mathcal{G}}]_\alpha$ may not split in this case.

Moreover, there are inclusions $[\mathcal{G}]_\alpha \subseteq [\mathcal{G}]_\beta$ and $[\widetilde{\mathcal{G}}]_\alpha \subseteq [\widetilde{\mathcal{G}}]_\alpha$, for all regular cardinals $\alpha \leq \beta$, analogous to those of the triangulated subcategories $\langle \mathcal{G} \rangle_\alpha$ of $\mathcal{DA}$.

6.4. **The homotopically α-small DG modules.** In this subsection, we define and study the DG analogue $(\widetilde{\text{G3}})$ of condition (G3), which was used in the definition of the subcategory $(\mathcal{DA})^\alpha$ of α-compact objects of the (triangulated) derived category $\mathcal{DA}$.

Definition 6.20. Let $\mathcal{A}$ be a DG category. A cofibrant DG module M is *homotopically α-small* if the following condition holds

$(\widetilde{\text{G3}})$ for each family of DG modules X_i, $i \in I$, of $\mathcal{CA}_{cof}$ and each morphism

$$f : M \longrightarrow \coprod_{i \in I} X_i \, ,$$

there exist a subset $J \subseteq I$ of cardinality strictly smaller than α and a factorization

$$
\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & \coprod_{i\in I} X_i \\
& \searrow \qquad \nearrow {\scriptstyle \coprod_{i\in I} f_i} & \\
& \coprod_{i\in I} \widetilde{X}_i \, , &
\end{array}
$$

where the $\widetilde{X}_i$ are cofibrant DG modules such that: If $i \in J$, then the morphism $f_i : \widetilde{X}_i \to X_i$ is a homotopy equivalence, whereas, if $i \in I \setminus J$, then $\widetilde{X}_i$ is a contractible cofibrant DG module, *i.e.* , by lemma 6.11, a direct factor of an object of the form IZ, for some cofibrant DG module Z.

Remark 6.21. The DG modules $\widetilde{X}_i$ and the morphisms f_i which are required to exist in the definition of the homotopically α-small DG modules are of course not unique.

Note that condition ($\widetilde{\text{G3}}$) 'reduces' to condition (G3) of definition 3.1 if we neglect the homotopy equivalences. This fact justifies the term "homotopically" that we have just introduced in definition 6.20.

The relation between conditions (G3) and ($\widetilde{\text{G3}}$) is clarified in the next

Proposition 6.22. *Let X be a cofibrant* DG *module. Then property* (G3) *holds for the image of X in $\mathcal{DA}$ if and only if property* ($\widetilde{\text{G3}}$) *holds for X in $\mathcal{CA}_{cof}$.*

Proof. Suppose that property (G3) holds for X in $\mathcal{DA}$. Let X_i, $i \in I$, be a family of objects in $\mathcal{CA}_{cof}$ and

$$f : X \longrightarrow \coprod_{i\in I} X_i$$

a morphism of $\mathcal{CA}_{cof}$. Then the image of f in $\mathcal{DA}$ factors as

$$X \xrightarrow{\quad\gamma\quad} \coprod_{i\in J} X_i \xrightarrow{\quad can\quad} \coprod_{i\in I} X_i$$

for a subset J of I, of cardinality strictly smaller than α and a morphism γ of $\mathcal{DA}$. Since X is cofibrant (and all objects of $\mathcal{CA}$ are fibrant), the morphism γ lifts to a morphism g of $\mathcal{CA}$ and the difference $f - can \circ g$ is null-homotopic. Then the difference factors as

$$X \xrightarrow{\quad h\quad} \coprod_{i\in I} PX_i \xrightarrow{\coprod_{i\in I} p_{X_i}} \coprod_{i\in I} X_i$$

for some morphism h of $\mathcal{CA}$ and the canonical projections $p_{X_i} : PX_i \to X_i$. We get the factorization

$$X \xrightarrow{\quad [g,h]^t\quad} (\coprod_{i\in J} X_i) \oplus (\coprod_{i\in I} PX_i) \xrightarrow{[can, \coprod_{i\in I} p_{X_i}]} \coprod_{i\in I} X_i$$

of f in $\mathcal{CA}$. Clearly, if we put $\widetilde{X_i} = X_i \oplus PX_i$ for $i \in J$ and $\widetilde{X_i} = PX_i$ for $i \notin J$ we get a factorization as required by ($\widetilde{\text{G3}}$).

Conversely, suppose that ($\widetilde{\text{G3}}$) holds for X. Let X_i, $i \in I$, be a family in $\mathcal{DA}$ and

$$\varphi : X \longrightarrow \coprod_{i\in I} X_i$$

a morphism od $\mathcal{DA}$. Since X is cofibrant, φ lifts to a morphism

$$f : X \longrightarrow \coprod_{i\in I} X_i$$

of $\mathcal{CA}$. Let

$$X \longrightarrow \coprod_{i\in I} \widetilde{X_i} \longrightarrow \coprod_{i\in I} X_i$$

be a factorization as in ($\widetilde{\text{G3}}$). Then for $i \in J$, the morphism $\widetilde{X_i} \to X_i$ becomes an isomorphism in $\mathcal{DA}$ and for $i \notin J$, the object $\widetilde{X_i}$ vanishes in $\mathcal{DA}$. Clearly this implies that φ factors through $\coprod_{i\in J} X_i$ in $\mathcal{DA}$. $\square$

From propositions 6.22 and 6.18 we get, as a corollary, the following result.

Theorem 6.23. *Let $\mathcal{A}$ be a* DG *category. Let α be an infinite regular cardinal. Let $[\,\widetilde{\mathcal{G}}\,]_\alpha$ be as in definition 6.12 and notation 6.14. Then, condition* $(\widetilde{\mathrm{G3}})$ *holds for the subcategory $[\,\widetilde{\mathcal{G}}\,]_\alpha$.*

Proof. Suppose that α is an infinite regular cardinal and let G be in $[\,\widetilde{\mathcal{G}}\,]_\alpha$. By proposition 6.18 we have that the image of G in $\mathcal{DA}$ lies in $(\mathcal{DA})^\alpha$. So, G has property (G3) in $\mathcal{DA}$. By proposition 6.22, this implies that G has property $\widetilde{\mathrm{G3}}$ in $\mathcal{CA}_{cof}$. □

6.5. **The homotopically α-compact DG modules.** In this subsection, we define and study the DG analogue $(\widetilde{\mathrm{G4}})$ of condition (G4), which was used in the definition of the subcategory $(\mathcal{DA})^\alpha$ of α-compact objects of the (triangulated) derived category $\mathcal{DA}$.

Definition 6.24. Let $\mathcal{A}$ be a DG category and $\mathcal{C}$ a class of cofibrant DG modules in $\mathcal{CA}$. We say that condition $(\widetilde{\mathrm{G4}})$ holds for $\mathcal{C}$ if

$(\widetilde{\mathrm{G4}})$ for each object $C \in \mathcal{C}$, for each family of DG modules X_i, $i \in I$, of $\mathcal{CA}_{cof}$ and for any morphism of DG modules

$$f : C \longrightarrow \coprod_{i \in I} X_i \, ,$$

there exist morphisms $\phi_i : C_i \to X_i$, $i \in I$, such that the following factorization holds

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & \coprod_{i\in I} X_i \\ & \searrow \qquad \nearrow{\scriptstyle \coprod_{i\in I}\phi_i} & \\ & \coprod_{i\in I} C_i \, , & \end{array}$$

where the DG modules C_i lie in $\widetilde{\mathcal{C}}$, for all $i \in I$.

Theorem 6.28 below claims that, for all regular cardinals α strictly greater than $\aleph_0$, condition $(\widetilde{\mathrm{G4}})$ holds for the subcategory $[\,\widetilde{\mathcal{G}}\,]_\alpha$ of $\mathcal{CA}_{cof}$. We remark that this is the smallest subcategory of $\mathcal{CA}$ closed under operations a), b), c), d) of definition 6.12, which contains the representables $\mathcal{G}$, the contractibles $\widetilde{\mathcal{G}}_0$ and for which both conditions $(\widetilde{\mathrm{G3}})$ and $(\widetilde{\mathrm{G4}})$ hold. Therefore, in analogy with the triangulated case, we call the objects of $[\,\widetilde{\mathcal{G}}\,]_\alpha$ *homotopically α-compact* DG *modules.*

Remark 6.25. Note that condition $(\widetilde{\mathrm{G4}})$ 'reduces' to condition (G4) of definition 3.1 if we neglect the homotopy equivalences. This fact justifies the term "homotopically" that we have just introduced.

The definition above is different from the definition of α-compact objects in a model category that Hirschhorn gives in his book (*cf.* definitions 10.8.1, 11.4.1 and 12.1.1 in [11]). In order to see what the α-compact objects of Hirschhorn's definition are in our case we recall that the model category $\mathcal{CA}$ is *cofibrantly generated* [12].

For each integer $n \in \mathbb{Z}$ and each object X of $\mathcal{A}$, we denote by $S^n \otimes X^\wedge$ the DG module $\Sigma^n X^\wedge$ and by $D^n \otimes X^\wedge$ the cone on the identity of $S^{n-1} \otimes X^\wedge$. The set $\mathcal{I}$ of *generating cofibrations* consists of the cofibrations

$$S^{n-1} \otimes X^\wedge \rightarrowtail D^n \otimes X^\wedge \ , \ n \in \mathbb{Z} \ , \ X \in \mathcal{A}.$$

The set $\mathcal{I}$ of *generating trivial cofibrations* consists of the cofibrations

$$0 \rightarrowtail D^n \otimes X^\wedge \ , \ n \in \mathbb{Z} \ , \ X \in \mathcal{A}.$$

Let λ be an α-filtered ordinal, where α is a regular cardinal. Let

$$s : X \rightarrowtail \mathrm{colim}_{\sigma < \lambda} X_\sigma$$

be a morphism in $\mathcal{I}$-cell (*cf.* [12] and the proof of proposition 6.5). A DG module K is α-compact according to Hirschhorn's definition if, given an arbitrary morphism

$$l : K \longrightarrow \mathrm{colim}_{\sigma < \lambda} X_\sigma \ ,$$

the following conditions are satisfied:

- There is a morphism in $\mathcal{I}$-cell

  $$t : X \rightarrowtail \mathrm{colim}_{\sigma < \lambda} X'_\sigma \ ,$$

  i.e. t is the transfinite composition of push-outs of type

  $$\begin{array}{ccc} \coprod_{j \in \Lambda'_\sigma} A_j & \rightarrowtail & \coprod_{j \in \Lambda'_\sigma} B_j \\ {\scriptstyle h'_\sigma} \downarrow & & \downarrow {\scriptstyle n'_\sigma} \\ X'_\sigma & \rightarrowtail & X'_{\sigma+1} \ , \end{array}$$

  such that the morphisms $A_j \rightarrowtail B_j$ are in $\mathcal{I}$, the sets Λ'_σ are subsets of their analogues Λ_σ for the morphism s and the morphisms h'_σ, n'_σ are the restrictions of their analogues h_σ, n_σ for the morphism s.
- The cardinality of the union of the set Λ_σ, for σ running through the ordinals strictly smaller than λ, is strictly smaller than α.
- There is a morphism of λ-sequences $(m_\sigma : X'_\sigma \to X_\sigma)_{\sigma < \lambda}$ such that the following factorization holds

  $$\begin{array}{ccc} K & \xrightarrow{\quad l \quad} & \mathrm{colim}_{\sigma<\lambda} X_\sigma \\ & \searrow \quad \nearrow {\scriptstyle \mathrm{colim}_{\sigma<\lambda} m_\sigma} & \\ & \mathrm{colim}_{\sigma<\lambda} X'_\sigma . & \end{array}$$

The main differences between the two definitions are:

(i) Hirschhorn's definition states a property about single objects, whereas our definition states a property about the class of α-compact objects;

(ii) Hirschhorn's definition depends on the set of generating cofibrations $\mathcal{I}$, whereas our definition is independent from this set;

(iii) there are contractible DG modules which are not α-compact in the sense of Hirschhorn but they are all homotopically α-compact in the sense of our definition.

Question 6.1. *We are working to answer the following questions:*

1) *Are the α-compact objects in the sense of Hirschhorn homotopically α-compact?*
2) *Is each homotopically α-compact object homotopically equivalent to an α-compact object in the sense of Hirschhorn?*

The relation between conditions (G3) and $(\widetilde{\mathrm{G3}})$ is clarified in the next

Proposition 6.26. *Let $\mathcal{C}$ be a class of cofibrant* DG *modules. Then condition* $(\widetilde{\mathrm{G4}})$ *holds for $\mathcal{C}$ in $\mathcal{CA}_{cof}$ if and only if condition* (G4) *holds for the image of $\mathcal{C}$ in $\mathcal{DA}$.*

Proof. Suppose that condition $(\widetilde{\mathrm{G4}})$ holds for $\mathcal{C}$. Let

$$\varphi : C \longrightarrow \coprod_{i\in I} X_i$$

be a morphism of $\mathcal{DA}$. We may assume without loss of generality that the X_i are cofibrant objects. Since C is cofibrant, φ lifts to a morphism f of $\mathcal{CA}_{cof}$. Now we factorize f as in $(\widetilde{\mathrm{G4}})$ and take the image of this factorization in $\mathcal{DA}$.

Conversely, suppose that condition (G4) holds for the image of $\mathcal{C}$ in $\mathcal{DA}$. Let

$$f : C \longrightarrow \coprod_{i\in I} X_i$$

be a morphism of $\mathcal{CA}$ where C belongs to $\mathcal{C}$ and the X_i are cofibrant DG modules. Let

$$\overline{f} = (\coprod_{i\in I} \overline{\varphi_i}) \circ \overline{g}$$

be a factorization as in (G4) of the image of f in $\mathcal{DA}$ through some objects C_i contained in the image of $\mathcal{C}$ in $\mathcal{DA}$. Since C and the C_i are cofibrant, we can lift the morphisms $\overline{g}$ and $\overline{\varphi_i}$ to morphisms g and φ_i of $\mathcal{CA}$. Then the difference

$$f - (\coprod_{i\in I} \varphi_i) \circ g$$

is null-homotopic and factors as

$$C \xrightarrow{\ h\ } \textstyle\coprod_{i\in I} PX_i \xrightarrow{\ \coprod_{i\in I} p_{X_i}\ } \textstyle\coprod_{i\in I} X_i$$

for some morphism h. Clearly, if we put $C_i' = C_i \oplus PX_i$, we obtain a factorization

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & \coprod_{i\in I} X_i \\ & \searrow \quad \nearrow & \\ & \coprod_{i\in I} C_i' , & \end{array}$$

where the C_i' are in $\widetilde{\mathcal{C}}$, as required in condition $(\widetilde{\mathrm{G4}})$. □

Remark 6.27. A class $\mathcal{C}$ of cofibrant DG modules and its closure $\widetilde{\mathcal{C}}$ under homotopy equivalences have the same image in $\mathcal{DA}$. Thus, by proposition 6.26, condition $(\widetilde{\mathrm{G4}})$ holds for $\mathcal{C}$ if and only if it holds for $\widetilde{\mathcal{C}}$.

From propositions 6.26 and 6.18 we get, as a corollary, the following result.

Theorem 6.28. *Let $\mathcal{A}$ be a* DG *category. Let α be a regular cardinal. Let $[\widetilde{\mathcal{G}}]_\alpha$ be as in definition 6.12 and notation 6.14. Then, condition $(\widetilde{\text{G4}})$ holds for the subcategory $[\widetilde{\mathcal{G}}]_\alpha$.*

Proof. By proposition 6.18 we have that the idempotent completion of the image of $[\widetilde{\mathcal{G}}]_\alpha$ in $\mathcal{DA}$ is $(\mathcal{DA})^\alpha$. As condition (G4) holds for $(\mathcal{DA})^\alpha$ in $\mathcal{DA}$, it also holds for the image of $[\widetilde{\mathcal{G}}]_\alpha$. Thus, by proposition 6.26 condition $\widetilde{\text{G4}}$ holds for $[\widetilde{\mathcal{G}}]_\alpha$ in $\mathcal{CA}_{cof}$. □

After the previous results we can establish the characterization announced at the beginning of this section

Theorem 6.29. *Let α be any regular cardinal of cardinality strictly greater than $\aleph_0$. Then the subcategory $[\widetilde{\mathcal{G}}]_\alpha$ is maximal in the class $\mathcal{P}$ of subcategories of $\mathcal{CA}_{cof}$ whose objects have property $(\widetilde{\text{G3}})$ and for which property $(\widetilde{\text{G4}})$ holds.*

Proof. Suppose that $\mathcal{B}$ is a subcategory of $\mathcal{CA}_{cof}$ properly containing $[\widetilde{\mathcal{G}}]_\alpha$ and that $\mathcal{B}$ lies in the class $\mathcal{P}$. By proposition 6.22, the objects contained in the image $\pi(\mathcal{B})$ of $\mathcal{B}$ in $\mathcal{DA}$ have property (G3). By proposition 6.26 property (G4) holds for $\pi(\mathcal{B})$ in $\mathcal{DA}$. Thus, $\pi(\mathcal{B})$ is contained in the subcategory $(\mathcal{DA})^\alpha$ of the α-compact objects of $\mathcal{DA}$, since $(\mathcal{DA})^\alpha$ is maximal among the strictly full subcategories of $\mathcal{DA}$ with respect to these properties (*cf.* [27]). The strictly full subcategories of $\mathcal{DA}$ are in bijection, through the functors π and π^{-1}, with the full subcategories of $\mathcal{CA}_{cof}$ stable under homotopy equivalences. Since $[\widetilde{\mathcal{G}}]_\alpha$ is stable under homotopy equivalences (proposition 6.16) and $\pi([\widetilde{\mathcal{G}}]_\alpha)$ is equivalent to $(\mathcal{DA})^\alpha$ (item a) in proposition 6.18), the subcategory $\mathcal{B}$ should be contained in $[\widetilde{\mathcal{G}}]_\alpha$ against the initial assumption. The claim follows by contradiction. □

Remark 6.30. If $\alpha = \aleph_0$, then some compact objects of $\mathcal{DA}$ may not be images of homotopically compact DG modules (*cf.* item b) in proposition 6.18).

6.6. **The key-lemma : Morphisms factor through $[\widetilde{\mathcal{S}}]_\alpha$.** In this subsection we can use all the material developed in the preceding subsections in order to state a theorem about factorization of certain morphisms. This theorem is the key result which allow us to prove the important result 5.5, which says that the α-continuous derived category is well generated and that its α-good generators are the images of the free modules of $\mathcal{DA}$ under the quotient functor of a certain localization. It turns out that this is one of the crucial ingredients for proving the main theorem 5.8. For this reason we call the factorization result of this subsection "the key-lemma". This is the homotopical DG version of theorem 4.7.

Theorem 6.31. *Let $\mathcal{A}$ be a* DG *category. Let β be an (infinite) regular cardinal. Let $\mathcal{S}$ be some class of cofibrant* DG *modules of $[\widetilde{\mathcal{G}}]_\beta$. Let X be a β-homotopically compact cofibrant* DG *module,* i.e. *$X \in [\widetilde{\mathcal{G}}]_\beta$, and let Z be an object of $[\widetilde{\mathcal{S}}]$. Suppose that $f : X \to Z$ is a morphism in $\mathcal{CA}_{cof}$. Then there exists an object $Y \in [\widetilde{\mathcal{S}}]_\beta$ so that f factors as $X \to Y \to Z$.*

Proof. Let $\mathcal{U}$ be the full subcategory of $\mathcal{CA}_{cof}$ whose objects are the cofibrant DG modules U which satisfy the follwing condition : For all the objects $P \in [\widetilde{\mathcal{G}}]_\beta$ and all the morphisms $f : P \to U$, there is some DG module Y contained in $[\widetilde{\mathcal{S}}]_\beta$ so that f factors as $P \to Y \to U$.

In order to prove the theorem, it is sufficient to show that $[\widetilde{\mathcal{S}}]$ is contained in $\mathcal{U}$. The subcategory $[\widetilde{\mathcal{S}}]$ is the smallest full subcategory of $\mathcal{CA}_{cof}$ containing $\widetilde{\mathcal{S}}$ and closed under left and right shifts, arbitrary set-indexed coproducts, extensions, direct factors. Therefore, we have to show that $\mathcal{U}$ contains $\widetilde{\mathcal{S}}$, and that the following assertions hold:

a) $\mathcal{U}$ is stable under the action of Σ and Ω;
b) $\mathcal{U}$ is closed under formation of arbitrary set-indexed coproducts;
c) $\mathcal{U}$ is closed under extensions;
d) $\mathcal{U}$ is stable under formation of direct factors.

We begin with the inclusion of $\widetilde{\mathcal{S}}$ in $\mathcal{U}$. Let S be some DG module of $\widetilde{\mathcal{S}}$ and P an arbitrary one of $[\widetilde{\mathcal{G}}]_\beta$. There is the trivial factorization

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & S \\ & {\scriptstyle f}\searrow \quad \nearrow\!\!\!\!=\, {\scriptstyle \mathbf{1}_S} & \\ & S\,, & \end{array}$$

for any morphism f. The DG module S lies in $[\widetilde{\mathcal{G}}]_\beta$, since $\widetilde{\mathcal{S}}$ is contained in $[\widetilde{\mathcal{G}}]_\beta$ by hypothesis. Thus, S belongs to $\mathcal{U}$, too.

Now we will check assertions a), b), c), d).

a) Let U be a DG module of $\mathcal{U}$. Consider ΣU and an arbitrary morphism

$$P \xrightarrow{\quad f \quad} \Sigma U\,,$$

where P is an arbitrary DG module of $[\widetilde{\mathcal{G}}]_\beta$. This morphism corresponds to the morphism

$$\Omega P \xrightarrow{\quad \Omega f \quad} U\,,$$

where ΩP is in $[\widetilde{\mathcal{G}}]_\beta$, too. Therefore, we have the following factorization of Ωf

$$\begin{array}{ccc} \Omega P & \xrightarrow{\quad \Omega f \quad} & U \\ & \searrow \qquad \nearrow & \\ & Y\,, & \end{array}$$

where the DG module Y is in $[\widetilde{\mathcal{S}}]_\beta$. This factorization diagram corresponds to the following one

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & \Sigma U \\ & \searrow \qquad \nearrow & \\ & \Sigma Y\,, & \end{array}$$

where ΣY lies in $[\widetilde{\mathcal{S}}]_\beta$, by the definition of $[\widetilde{\mathcal{S}}]_\beta$. Thus, ΣU is in $\mathcal{U}$. Similarly, one can prove that ΩU is in $\mathcal{U}$, too. This shows point a).

b) Let U_i , $i \in I$, be a family of arbitrary DG modules lying in $\mathcal{CA}_{cof}$. Let P be some DG module of $[\,\widetilde{\mathcal{G}}\,]_\beta$. We consider an arbitrary morphism

$$P \xrightarrow{\quad f \quad} \coprod_{i\in I} U_i.$$

We can apply condition $(\widetilde{\mathrm{G3}})$. Thus, there exists a subset $J \subseteq I$ of cardinality strictly smaller than β such that, in the factorization

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & \coprod_{i\in I} U_i \\ & {\scriptstyle \widetilde{f}} \searrow \quad \nearrow {\scriptstyle \coprod_{i\in I} \varphi_i} & \\ & \coprod_{i\in I} \widetilde{U}_i\ , & \end{array}$$

if $i \in J$, then $\varphi_i : \widetilde{U}_i \to U_i$ is a homotopy equivalence, whereas, if $i \in I \setminus J$, then the cofibrant DG module $\widetilde{U}_i$ is contractible. Clearly, the morphism $\widetilde{f}$ factors as

$$\begin{array}{ccc} P & \xrightarrow{\quad \widetilde{f} \quad} & \coprod_{i\in I} \widetilde{U}_i \\ & {\scriptstyle [\widetilde{f}_J\,,\,\widetilde{f}_{I\setminus J}]^t} \searrow \quad \nearrow {\scriptstyle \coprod_{i\in J} \mathbf{1}_{\widetilde{U}_i} \oplus \coprod_{i\in I\setminus J} \mathbf{1}_{\widetilde{U}_i}} & \\ & \coprod_{i\in J} \widetilde{U}_i \oplus \coprod_{i\in I\setminus J} \widetilde{U}_i\ , & \end{array}$$

where $\widetilde{f}_J$ is the morphism whose components are those of $\widetilde{f}$ from P to $\widetilde{U}_i$, for $i \in J$. The analogous notation is adopted for $\widetilde{f}_{I\setminus J}$. Thus, we can apply condition $(\widetilde{\mathrm{G4}})$ to the morphism $\widetilde{f}_J$. This gives the following factorization of $\widetilde{f}_J$

$$\begin{array}{ccc} P & \xrightarrow{\quad \widetilde{f}_J \quad} & \coprod_{i\in J} \widetilde{U}_i \\ & {\scriptstyle g_J} \searrow \quad \nearrow {\scriptstyle \coprod_{i\in J} \psi_i} & \\ & \coprod_{i\in J} Z_i\ , & \end{array}$$

where Z_i belongs to $[\,\widetilde{\mathcal{G}}\,]_\beta$, for all $i \in J$. For any $i \in J$, the DG modules U_i are in $\mathcal{U}$, hence we can factor the compositions $\varphi_i \circ \psi_i : Z_i \to U_i$ through DG modules W_i of $[\,\widetilde{\mathcal{S}}\,]_\beta$ as in the following diagram

$$\begin{array}{ccc} Z_i & \xrightarrow{\quad \varphi_i \circ \psi_i \quad} & U_i \\ & {\scriptstyle \alpha_i} \searrow \quad \nearrow {\scriptstyle \beta_i} & \\ & W_i\ , & \end{array}$$

for all $i \in J$. Thus, we have shown that f factors as

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & \coprod_{i\in I} U_i \\ & {\scriptstyle [(\coprod_{i\in J}\alpha_i)\circ g_J\,,\ \widetilde{f}_{I\setminus J}]^t} \searrow \quad \nearrow {\scriptstyle \coprod_{i\in J}\beta_i \oplus \coprod_{i\in I\setminus J}\varphi_i} & \\ & \coprod_{i\in J} W_i \oplus \coprod_{i\in I\setminus J} \widetilde{U}_i\ , & \end{array}$$

where: If $i \in J$, then $W_i \in [\,\widetilde{\mathcal{S}}\,]_\beta$; whereas, if $i \in I \setminus J$, then $\widetilde{U}_i$ is a contractible cofibrant DG module and thus lies in $\widetilde{\mathcal{G}}_0 \subseteq [\,\widetilde{\mathcal{S}}\,]_\beta$. Indeed, when we check the commutativity of the triangle above, we get the following sequence of equalities:

$$\begin{aligned} & (\coprod_{i\in J}\beta_i \oplus \coprod_{i\in I\setminus J}\varphi_i) \circ [(\coprod_{i\in J}\alpha_i)\circ g_J\ ,\ \widetilde{f}_{I\setminus J}]^t \\ = \ & [\coprod_{i\in J}\beta_i \circ ((\coprod_{i\in J}\alpha_i)\circ g_J)\ ,\ (\coprod_{i\in I\setminus J}\varphi_i)\circ \widetilde{f}_{I\setminus J}]^t \\ = \ & [(\coprod_{i\in J}\beta_i \circ \coprod_{i\in J}\alpha_i)\circ g_J\ ,\ f_{I\setminus J}]^t \\ = \ & [(\coprod_{i\in J}(\beta_i\circ\alpha_i))\circ g_J\ ,\ f_{I\setminus J}]^t \\ = \ & [(\coprod_{i\in J}(\varphi_i\circ\psi_i))\circ g_J\ ,\ f_{I\setminus J}]^t \\ = \ & [\coprod_{i\in J}\varphi_i \circ ((\coprod_{i\in J}\psi_i)\circ g_J)\ ,\ f_{I\setminus J}]^t \\ = \ & [(\coprod_{i\in J}\varphi_i)\circ \widetilde{f}_J\ ,\ f_{I\setminus J}]^t \\ = \ & [f_J\ ,\ f_{I\setminus J}]^t \\ = \ & f. \end{aligned}$$

Here, the morphisms f_J and $f_{I\setminus J}$ are induced by the following trivial factorization of f

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & \coprod_{i\in I} U_i \\ & {\scriptstyle [f_J\,,\ f_{I\setminus J}]^t} \searrow \quad \nearrow {\scriptstyle \coprod_{i\in J}\mathbf{1}_{U_i} \oplus \coprod_{i\in I\setminus J}\mathbf{1}_{U_i}} & \\ & \coprod_{i\in J} U_i \oplus \coprod_{i\in I\setminus J} U_i. & \end{array}$$

Since $\coprod_{i\in J} W_i$ and $\coprod_{i\in I\setminus J}\widetilde{U}_i$ lie in $[\,\widetilde{\mathcal{S}}\,]_\beta$, their sum will lie, too. This proves that $\coprod_{i\in I} U_i \in \mathcal{U}$. Thus, point b) holds.

c) Let U and U' be two DG modules lying in $\mathcal{U}$. We want to show that an arbitrary extension E of U and U'

$$U \rightarrowtail^{i} E \twoheadrightarrow^{d} U' ,$$

lies in $\mathcal{U}$. Let us consider an arbitrary morphism

$$Q \xrightarrow{f} E ,$$

where the DG module Q lies in $[\widetilde{\mathcal{G}}]_\beta$. Then, we can factor the composition $d \circ f$ as in the following commutative square

$$\begin{array}{ccc} Q & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ E & \underset{d}{\twoheadrightarrow} & U' , \end{array}$$

with the DG module Y lying in $[\widetilde{\mathcal{S}}]_\beta$. Let us form the pull-back of the deflation $h : PY \twoheadrightarrow Y$ and of the morphism $e : Q \longrightarrow Y$

$$\begin{array}{ccc} X' & \overset{j}{\dashrightarrow} & PY \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\ Q & \underset{e}{\longrightarrow} & Y , \end{array}$$

where $X' \in [\widetilde{\mathcal{G}}]_\beta$. Indeed, if we complete the square above to a morphism of conflations, we get the commutative diagram

$$\begin{array}{ccccc} \Omega Y & \rightarrowtail & X' & \overset{k}{\twoheadrightarrow} & Q \\ {\scriptstyle \mathbf{1}_{\Omega Y}}\| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle e} \\ \Omega Y & \rightarrowtail & PY & \overset{h}{\twoheadrightarrow} & Y , \end{array}$$

where Q and ΩY are in $[\widetilde{\mathcal{G}}]_\beta$. Therefore, X' will, too, since $[\widetilde{\mathcal{G}}]_\beta$ is stable under extensions. By the fact that the square on the right of the diagram above is cartesian, we can form the conflation

$$X' \overset{[-k\,,\,j]^t}{\rightarrowtail} Q \oplus PY \overset{[e\,,\,h]}{\twoheadrightarrow} Y.$$

Thus, since i is the kernel of the morphism d, there exists a morphism l which makes the following diagram commute

$$\begin{array}{ccccc} X' & \overset{[-k\,,\,j]^t}{\rightarrowtail} & Q \oplus PY & \overset{[e\,,\,h]}{\twoheadrightarrow} & Y \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle [f\,,\,f']} & & \downarrow{\scriptstyle g} \\ U & \underset{i}{\rightarrowtail} & E & \underset{d}{\twoheadrightarrow} & U' , \end{array}$$

where the morphism $f' : PY \to E$ exists thanks to the projectivity of PY. Since $X' \in [\widetilde{\mathcal{G}}]_\beta$ and $U \in \mathcal{U}$, the morphism l factors through a DG module Y' which lies in $[\,\widetilde{\mathcal{S}}\,]_\beta$, as in the commutative triangle

$$\begin{array}{ccc} X' & \xrightarrow{\quad l \quad} & U \\ & {\scriptstyle m}\searrow \qquad \nearrow{\scriptstyle n} & \\ & Y'. & \end{array}$$

Now, form the push-out

$$\begin{array}{ccc} X' & \xrightarrow{[-k\,,\,j]^t} & Q \oplus PY \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle [q\,,\,p]} \\ Y' & \xrightarrow[r]{} & N\,. \end{array}$$

By extending it to a morphism of conflations

$$\begin{array}{ccccc} X' & \xrightarrow{[-k\,,\,j]^t} & Q \oplus PY & \xrightarrow{[e\,,\,h]} & Y \\ {\scriptstyle m}\downarrow & PO & \downarrow{\scriptstyle [q\,,\,p]} & & \|{\scriptstyle \mathbf{1}_Y} \\ Y' & \xrightarrow[r]{} & N & \xrightarrow[s]{} & Y\,, \end{array}$$

we can see that the DG module N lies in $[\,\widetilde{\mathcal{S}}\,]_\beta$, since it is an extension of two DG modules, Y' and Y, both lying in $[\,\widetilde{\mathcal{S}}\,]_\beta$. The claim is that the conflation $(r\,,\ s)$ factors the morphism of conflation $(l\,,\ [f\,,\ f']\,,\ g)$ above. Indeed, since the diagram on the left of the preceding diagram is cocartesian, we get the morphism t which makes the following diagram commute

$$\begin{array}{ccccc} X' & \xrightarrow{[-k\,,\,j]^t} & Q \oplus PY & & \\ {\scriptstyle m}\downarrow & & {\scriptstyle [q\,,\,p]}\downarrow & \searrow{\scriptstyle [f\,,\,f']} & \\ Y' & \xrightarrow[r]{} & N & \overset{t}{\cdots\!\!\searrow} & \\ & \searrow_{i\circ n} & & & E. \end{array}$$

The last diagram immediately implies the claimed factorization

$$\begin{array}{ccccc} X' & \xrightarrow{[-k\,,\,j]^t} & Q \oplus PY & \xrightarrow{[e\,,\,h]} & Y \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle [q\,,\,p]} & & \|{\scriptstyle \mathbf{1}_Y} \\ Y' & \xrightarrow[r]{} & N & \xrightarrow[s]{} & Y \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle g} \\ U & \xrightarrow[i]{} & E & \xrightarrow[d]{} & U'\,, \end{array}$$

where the composition $t \circ [q\,,\ p] = [t \circ q\,,\ t \circ p]$ equals $[f\,,\ f']$. We can write the morphism f as the composition

$$Q \xrightarrow{[\mathbf{1}_Q\,,\,0]^t} Q \oplus PY \xrightarrow{[f\,,\,f']} E.$$

As the morphism $[f\,,\ f']$ factors through N, the morphism f factors through N, too, as in the commutative diagram

$$\begin{array}{ccccc} Q & \overset{[\mathbf{1}_Q\,,\,0]^t}{\rightarrowtail} & Q \oplus PY & \overset{[f\,,\,f']}{\twoheadrightarrow} & E \\ & \overset{q}{\searrow} & \downarrow{\scriptstyle [q\,,\,p]} & \overset{t}{\nearrow} & \\ & & N. & & \end{array}$$

This shows that the searched factorization of f is the following,

$$Q \xrightarrow{\ q\ } N \xrightarrow{\ t\ } E.$$

As $N \in [\,\widetilde{\mathcal{S}}\,]_\beta$, it follows that $E \in \mathcal{U}$ and point c) holds.

d) Let U be some DG module in $\mathcal{U}$. Let Z be a direct factor of U. This means that there exist an inclusion $i : Z \rightarrowtail U$ and a retraction $r : U \longrightarrow Z$ so that $r \circ i = \mathbf{1}_Z$. Let P be an arbitrary DG module lying in $[\,\widetilde{\mathcal{G}}\,]_\beta$ and

$$f : P \longrightarrow Z$$

an arbitrary morphism. Then, we can factor the composition $i \circ f$ through a DG module Y of $[\,\widetilde{\mathcal{S}}\,]_\beta$ as in the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ i\circ f\ } & U \\ & {\scriptstyle f'}\searrow \quad \nearrow{\scriptstyle f''} & \\ & Y. & \end{array}$$

This easily implies that f factors through Y, too,

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & Z \\ & {\scriptstyle f'}\searrow \quad \nearrow{\scriptstyle r\circ f''} & \\ & Y. & \end{array}$$

Indeed, the last triangle is commutative, since we have

$$(r \circ f'') \circ f' = r \circ (f'' \circ f') = r \circ (i \circ f) = (r \circ i) \circ f = \mathbf{1}_Z \circ f = f.$$

Hence, point d) and the theorem follow. $\square$

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