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THESE

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Sujet:

Délaçage de la K-théorie des catégories exactes et K-groupes négatifs

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Und wo sich Wort und Ton gesellt,
Wo Lied erklingt, Kunst sich entfaltet,
Wird jedesmal der Sinn der Welt,
Des ganzen Daseins neu gestaltet...

Hermann Hesse

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Contents

1 Introduction (en français)

Motivation. Soit X un schéma et U un sous-schéma ouvert. En général, le morphisme de restriction $K_0(X) \to K_0(U)$ n'est pas surjectif. Pour mesurer son conoyau, Bass [Bas68] a définit les K-groupes négatifs pour les anneaux. Indépendamment, Karoubi ([Kar68]) a donné une définition d'une K-théorie négative pour les catégories additives (appelée par la suite exactes scindées). Pour les schémas, ces groupes ont été introduits par Thomason et Trobaugh ([TT90]). Dans le cas des variétés lisses ou des anneaux réguliers, ces invariants sont nuls. Mais très peu de résultats sont connus dans le cas non régulier. Que dit la K-théorie négative sur les singularités? Par exemple, une conjecture de Weibel ([Wei80]) datant de 1980 qui dit que $K_{-i}(R) = 0$, $i > d$, pour les anneaux noethériens de dimension de Krull d n'est toujours pas démontrée. On renvoie le lecteur à [Wei] pour les progrès les plus récents qui vont dans cette direction.

Comme groupes d'obstructions, les $K_{-i}(\mathbb{Z}G)$, pour G un groupe (de présentation finie), sont d'un interêt particulier en topologie géométrique. Ici une conjecture de Hsiang de 1983 ([Hsi84]) dit que $K_{-i}(\mathbb{Z}G) = 0$ for $i > 1$. C'est toujours un problème ouvert.

Beaucoup de progrès en K-théorie supérieure pour les anneaux et les schémas sont dus à l'introduction en K-théorie des concepts de catégories exactes par Quillen et de catégories dérivées par Thomason et Trobaugh. Il nous semble naturel de chercher une définition des K-groupes négatifs pour les catégories exactes ou même dérivées qui généraliseraient les définitions pour les anneaux et les schémas.

Dans cette thèse nous proposons une définition des K-groupes négatifs pour des catégories exactes quelconques.

Résumé des résultats. Première partie (section 3-12). Dans la première partie de la thèse nous construisons pour une catégorie exacte quelconque $\mathcal{E}$ une catégorie exacte $S\mathcal{E}$, appelée suspension de $\mathcal{E}$, telle que l'espace de la K-théorie $K(\tilde{\mathcal{E}})$ de la complétion idempotente $\tilde{\mathcal{E}}$ de $\mathcal{E}$ (voir l'appendice de [TT90]) a le même type d'homotopie que $\Omega K(S\mathcal{E})$. Ainsi on obtient un Ω-spectre $\mathbb{K}(\mathcal{E})$ en posant $\mathbb{K}(\mathcal{E})_n = K(\widetilde{S^n\mathcal{E}})$. Ses groupes d'homotopie positifs sont les groupes de la K-théorie de Quillen. Mais en général il est non-connexe. C'est pourquoi notre construction donne une définition d'une K-théorie négative comme groupes d'homotopie négatifs du spectre $\mathbb{K}$. Nos groupes de K-théorie inférieure coïncident avec ceux de Bass [Bas68]. Plus généralement ils sont les mêmes que ceux définis dans [Kar70] et [PW89] pour les catégories exactes scindées. Ceci donne une première réponse à une question posée par Quillen en 1974 pendant le congrès international de Vancouver ([Qui75]). On résout aussi en partie le problème 5 de la Lake Louise Problem Session ([JS89]).

Dans la section 3 on introduit le concept de “left” ou “right s-filtering” qui est une condition sur les sous-catégories exactes $\mathcal{A}$ d'une catégorie exact $\mathcal{U}$ pour pouvoir construire le quotient exact $\mathcal{U}/\mathcal{A}$ au sens catégoriel. Il se révèle (5.1) que dans cette situation on a une fibration homotopique d'espaces de K-théorie

$$K(\tilde{\mathcal{A}}) \to K(\tilde{\mathcal{U}}) \to K(\widetilde{\mathcal{U}/\mathcal{A}}).$$

Dans la section 4 on montre que toute catégorie exacte $\mathcal{E}$ peut tre plongée dans une catégorie exacte “flasque” $\mathcal{FE}$ dont l'espace de K-théorie est contractile. Ce plongement est fonctoriel et remplit les conditions “left” et/ou “right s-filtering”. C'est pourquoi la K-théorie de $S\mathcal{E} = \mathcal{FE}/\mathcal{E}$ est un délaçage de $K(\tilde{\mathcal{E}})$.

La fibration homotopique ci-dessus est généralisée dans (7.12) en une fibration homotopique de spectres non-connexes de $I\!\!K$-théorie. On étend les théorèmes de résolution et d'additivité de Quillen à la K-théorie négative (8.3, 8.1) et on démontre que la K-théorie négative des catégories abéliennes noethériennes est triviale (9.4). Nous conjecturons que nos groupes de K-théorie négatifs coïncident avec ceux définis pour les schémas par Thomason et Trobaugh dans [TT90], au moins si le schéma a une famille ample de fibrés vectoriels. Dans 11.3 nous esquissons une idée d'une démonstration. Pour tout foncteur exact f on construit dans 7.17 une catégorie exacte $C(f)$ dont la $I\!\!K$-théorie est la fibre homotopique de $I\!\!K(f)$. La suite exacte longue en K-théorie positive et négative associée à la localisation d'un anneau par rapport à un ensemble multiplicatif de dénominateurs à gauche qui sont des non-zéro diviseurs à droite (voir [Car80]) rentre naturellement dans notre cadre (12.3).

Dans le travail de Carlsson et Pedersen sur la conjecture de Novikov, certaines catégories additives représentant la K-homologie ont joué un rôle important. On donne une généralisation aux catégories exactes de ces catégories additives. Plus précisément, comme cela a été fait dans [PW89] pour les catégories exactes scindées, on construit des catégories exactes qui représentent une théorie homologique pour les complexes simpliciaux finis à coéfficients dans le spectre non-connexe $I\!\!K(\mathcal{E})$ (12.7).

On dit qu'une catégorie additive est karoubienne si tout projecteur possède un noyau. Comme corollaire (10.2) d'une deuxième démonstration de 5.1 on trouve que la catégorie dérivée bornée d'une catégorie exacte karoubienne est karoubienne.

Les méthodes essentielles que nous utilisons sont le calcule des fractions dans la section 3 pour construire des catégories quotients, la construction $S.$ de Waldhausen ([Wal85]) dans la démonstration de la proposition clé 5.1 et la catégorie des spectres symétriques de Hovey, Shipley et Smith qui nous permet de construire un foncteur des petites catégories exactes vers les spectres symétriques représentant la $I\!K$-théorie et se comportant bien par rapport aux produit. En particulier, nous avons un foncteur monoïdale symétrique de la catégorie des anneaux vers la catégorie des spectre symétrique représentant la $I\!K$-théorie (section 7). Ceci est essentiel dans les démonstrations des sections 9 et 12.

Un dernier mot de motivation, la démonstration du théorème de périodicité en K-théorie hermitienne de Karoubi ([Kar80b], [Kar80a]) dépend beaucoup de la construction d'une suspension pour les anneaux délaçant la K-théorie. Si on veut généraliser ce théorème aux catégories exactes avec dualité et si on veut suivre la méthode de la démonstration de Karoubi on a besoin d'une construction de suspension pour les catégories exactes généralisant la suspension des anneaux. On espère que notre construction (4.7) va se révéler utile dans ce context. Pour des resultats récents sur la K-théorie hermitienne des catégories exactes avec dualité on renvoie le lecteur à [Hor].

Deuxième partie (section 13 et 14). Dans la deuxième partie de la thèse nous montrons que dans beaucoup de cas la complétion en groupe du H-espace $B\mathcal{S}$, $\mathcal{S}$ étant une catégorie monoïdale symétrique $\mathcal{S} = Iso(\mathcal{S})$, peut être obtenue comme la construction plus appliquée au classifiant d'un certain groupe $G(\mathcal{S})$ associé à $\mathcal{S}$. Ceci est une généralisation de la construction $BGL(R)^+$ bien connue en K-théorie algébrique. Ce résultat permet de donner une démonstration "élémentaire" d'un théorème de Pedersen-Weibel (theorem 5.3 [PW89]) qui n'utilise que des théorèmes classiques de la topologie algébrique et de la théorie homotopique.

Troisième partie (section 15). Dans la dernière partie on propose une définition d'une K-théorie algébrique bivariante pour les anneaux et les schémas qui satisfait la propriété d'excision en les deux variables. On montre que notre définition appliquée aux C^*-algèbres donne dans beaucoup de cas la KK-théorie de Kasparov, ce qui justifie notre définition d'une K-théorie algébrique bivariante.

2 Introduction (in English)

Motivation. Let X be a scheme and U an open subscheme. In general, the restriction map $K_0(X) \to K_0(U)$ is not surjective. In order to measure the lack of surjectivity, Bass [Bas68] defined negative K-groups of rings. Independently, Karoubi ([Kar68]) gave a definition of negative K-theory for additive (henceforth called split exact) categories. For schemes these groups have been introduced by Thomason and Trobaugh ([TT90]). In the case of smooth schemes or regular rings these invariants vanish. However, only a few computations are known in the non regular case. What does negative K-theory tell us about singularities? For instance, a 1980 conjecture of Weibel ([Wei80]) which states that $K_{-i}(R) = 0$, $i > d$, for a noetherian ring R of Krull dimension d is still open. We refer the reader to [Wei] for some recent progress in this direction.

As obstructions, the groups $K_{-i}(\mathbb{Z}G)$, G being a (finitely presented) group, are of particular interest in geometric topology. For example, Hsiang conjectured in 1983 ([Hsi84]) that $K_{-i}(\mathbb{Z}G) = 0$ for $i > 1$. This is still an open problem.

We owe much of the progress in higher K-theory of rings and schemes to the use of the concepts of exact categories (e.g. Quillen) and of derived categories (e.g. Waldhausen, Gillet, Thomason-Trobaugh). It therefore is natural to ask for a definition of negative K-groups of exact or even derived categories in general extending the definitions for rings and schemes.

In our thesis we propose a definition of negative K-groups for arbitrary (small) exact categories.

Synopsis of the results. Part I (section 3-12). In the first part of this thesis we construct for any exact category $\mathcal{E}$ an exact category $S\mathcal{E}$, called suspension of $\mathcal{E}$, such that the K-theory space $K(\tilde{\mathcal{E}})$ of the idempotent completion $\tilde{\mathcal{E}}$ of $\mathcal{E}$ (see appendix of [TT90]) has the same homotopy type as $\Omega K(S\mathcal{E})$. We therefore obtain a (in general non-connective) Ω-spectrum $\mathbb{K}(\mathcal{E})$ by setting $\mathbb{K}(\mathcal{E})_n = K(\widetilde{S^n\mathcal{E}})$ whose positive homotopy groups are the Quillen K-groups of $\mathcal{E}$. Our construction gives a definition of negative algebraic K-groups of an arbitrary exact category which coincides with Bass' negative K-theory of a ring A if $\mathcal{E}$ is $\mathcal{P}(A)$, the (split exact) category of finitely generated projective A modules (11.1). More generally, for any split exact category $\mathcal{E}$, our negative K-groups agree with those defined in [Kar70] and [PW89]. This gives a first answer to a question Quillen raised in 1974 at the international Congress of Vancouver ([Qui75]). It also answers part of problem 5 of the Lake Louise Problem Session ([JS89]).

In section 3 we introduce the concept of "left" or "right s-filtering", a condition on an exact subcategory $\mathcal{A}$ of an exact category $\mathcal{U}$ needed to construct the categorical exact quotient $\mathcal{U}/\mathcal{A}$. It turns out (5.1) that in this situation we have a homotopy fibration of K-theory spaces

$$K(\tilde{\mathcal{A}}) \to K(\tilde{\mathcal{U}}) \to K(\widetilde{\mathcal{U}/\mathcal{A}}).$$

In section 4 we show that any exact category $\mathcal{E}$ can be embedded in a functorial, left and/or right s-filtering way into a "flasque" exact category $\mathcal{FE}$ whose K-theory space is contractible. Hence the K-theory of $S\mathcal{E} = \mathcal{FE}/\mathcal{E}$ is a one-fold delooping of $K(\tilde{\mathcal{E}})$.

We prove a localization theorem (7.12) associating a homotopy fibration of $\mathbb{K}$-theory spectra to any left or right s-filtering inclusion. We extend Quillen's resolution and additivity theorems to negative K-theory (8.3, 8.1) and show that the negative K-theory of noetherian abelian categories is trivial (9.4). We conjecture that our negative K-theory of the exact category of vector bundles over a scheme coincides with the negative K-theory defined by Thomason and Trobaugh in [TT90] if the scheme admits an "ample family of line bundles". In 11.3 we outline an idea of a proof. For an exact functor f, we construct in 7.17 an exact category $C(f)$ whose $\mathbb{K}$-theory is the homotopy cofiber of $\mathbb{K}(f)$. The long exact sequence in positive and negative K-theory associated to the localization of a ring with respect to a left denominator set of right non zero divisors (see [Car80]) is a direct consequence of our setting (12.3).

In the work of Carlsson and Pedersen on the Novikov conjecture, certain additive categories representing K-homology were playing an important role. We give a generalization of these additive categories to exact categories, i.e. as in the spirit of [PW89], we construct in 12.7 a homology theory on finite simplicial complexes with coefficients in the non-connective $\mathbb{K}$-theory spectrum of an exact category $\mathcal{E}$ (12.7).

As a byproduct of a second proof of 5.1 we obtain that the bounded derived category of an idempotent complete exact category is idempotent complete (10.2).

The methods we are using are the calculus of fractions in section 3 to construct quotient categories, Waldhausen's $S.$-construction ([Wal85]) in proving the key proposition 5.1 and the category of symmetric spectra of Hovey, Shipley and Smith ([HSS]) with its closed model structure which enables us to construct a functor from exact categories to spectra representing our $\mathbb{K}$-theory and behaving well with respect to products. In particular, we have a strictly symmetric monoidal functor from rings to symmetric spectra representing our $\mathbb{K}$-theory (section 7). This is essential for the proofs in section 9 and 12.

We remark that Karoubi's proof of his periodicity theorem in hermitian K-theory ([Kar80b], [Kar80a]) depends on the existence of the suspension of rings delooping the K-theory. If one wants to extend this theorem to exact categories with dualities and proceed along the lines of Karoubi's proof, a suspension construction of exact categories is needed. We hope that our construction (4.7) will do this job as it is a generalization of the suspension of rings. For some recent results on hermitian K-theory of exact categories see [Hor].

Part II (section 13 and 14). In the second part of our thesis we show that in many cases the group completion of the H-space $B\mathcal{S}$, $\mathcal{S}$ being a symmetric monoidal category $\mathcal{S} = Iso(\mathcal{S})$, can be obtained as the plus construction applied to the classifying space of a certain group $G(\mathcal{S})$ associated with $\mathcal{S}$. This generalizes the well known $BGL(R)^+$ construction. Using this result, we give an "elementary", homotopy theoretical proof of a theorem of Pedersen-Weibel (theorem 5.3 [PW89]).

Part III (section 15). In the last part we propose a definition of a bivariant algebraic K-theory for rings and schemes satisfying excision in both variables. We show that if we apply our definition to C^*-algebras, we obtain in many cases Kasparov's KK-theory, hereby justifying the above definition.

PART I: Negative K-theory of exact categories

3 Filtering categories and quotients

In this section we define right resp. left filtering exact subcategories, special monomorphisms and epimorphisms. Those concepts are essentiel in constructing quotient exact categories. We develop the algebraic properties needed in the following sections.

3.1 An *exact category* $\mathcal{U}$ is a pointed additive category, i.e. a category equipped with a chosen zero object called basepoint, together with a set of sequences $\{\ U \rightarrowtail V \twoheadrightarrow W\ \}$ of maps of $\mathcal{U}$, called exact sequences, satisfying a), b), c) and d) below. In an exact sequence $U \overset{i}{\rightarrowtail} V \overset{p}{\twoheadrightarrow} W$ the map i is called an admissible monomorphism and the map p an admissible epimorphism.

a) The set of admissible monomorphisms is closed under composition. The cobase change of an admissible monomorphism along an arbitrary map exists and is again an admissible monomorphism.

b) The set of admissible epimorphisms is closed under composition. The base change of an admissible epimorphism along an arbitrary map exists and is again an admissible epimorphism.

c) Any sequence isomorphic to an exact sequence is exact. Any sequence of the form

$$U \overset{\begin{pmatrix}1\\0\end{pmatrix}}{\rightarrowtail} U \oplus W \overset{(0\ 1)}{\twoheadrightarrow} W$$

is exact.

d) In an exact sequence $U \overset{i}{\rightarrowtail} V \overset{p}{\twoheadrightarrow} W$ the map i is a kernel for p and p is a cokernel for i.

An additive functor between exact categories is exact if it sends exact sequences to exact sequences and the basepoint to the basepoint.

Usually one demands two further axioms, but as shown in appendix A of [Kel90] they follow from a)-d). That's why we state it as a lemma.

3.2 Lemma. *Let $\mathcal{U}$ be an additive category satisfying a)-d) above. If the admissible monomorphism $i : U \rightarrowtail V$ can be factored as $i = \beta \circ \alpha$ and α has a cokernel then α is an admissible monomorphism.*

Dually, if the admissible epimorphism $p : V \twoheadrightarrow W$ *can be factored as* $p = \beta \circ \alpha$ *and* β *has a kernel then* β *is an admissible epimorphism.*

Convention: If $\mathcal{A}$ is a full subcategory of a category $\mathcal{U}$, then $A \in \mathcal{A}$ means A is isomorphic to an object of $\mathcal{A}$.

An exact subcategory $\mathcal{A}$ of $\mathcal{U}$ is a full subcategory of $\mathcal{U}$ which is exact such that a sequence in $\mathcal{A}$ is exact if and only if it is exact in $\mathcal{U}$ and which is extension closed in $\mathcal{U}$. We write objects of $\mathcal{A}$ as A, B, C,... and objects of $\mathcal{U}$ as U, V, W,...

3.3 Recall from [TT90] appendix A the *idempotent completion* $\widetilde{\mathcal{A}}$ of an exact category $\mathcal{A}$. It is an idempotent complete category, i.e. every idempotent in $\widetilde{\mathcal{A}}$ has a kernel. Its objects are pairs (A, p) with A an object of A and $p : A \to A$ an idempotent, i.e. $p^2 = p$. A morphism from (A, p) to (B, q) is a morphism $f : A \to B$ in $\mathcal{A}$ such that $f = f \circ p = q \circ f$. Composition is composition in $\mathcal{A}$. $\widetilde{\mathcal{A}}$ is obviously an additive category. It is an exact category in which a sequence $A \xrightarrow{i} B \xrightarrow{p} C$ is exact iff there exist objects D, E of $\widetilde{\mathcal{A}}$ such that the sequence

$$D \oplus A \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & i \\ 0 & 0 \end{pmatrix}} D \oplus B \oplus E \xrightarrow{\begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}} C \oplus E$$

is exact in $\mathcal{A}$. It is straight forward to verify the axioms 3.1 a)-d). $\mathcal{A} \mapsto \widetilde{\mathcal{A}}$ clearly is a functor from exact categories to idempotent complete exact categories. In an idempotent complete exact category any morphism α for which there is a β with $\beta \circ \alpha$ an admissible monomorphism always has a cokernel and is therefore by lemma 3.2 an admissible monomorphism itself. Dually, a map β for which there is a map α with $\beta \circ \alpha$ an admissible epimorphism is an admissible epimorphism itself. We remark that $\mathcal{A}$ is cofinal in $\widetilde{\mathcal{A}}$, i.e. it is extension closed in $\widetilde{\mathcal{A}}$ and to every object A of $\widetilde{\mathcal{A}}$ there is an object B of $\widetilde{\mathcal{A}}$ such that $A \oplus B$ is isomorphic to an object of $\mathcal{A}$.

3.4 Definition. Let $\mathcal{A}$ be an exact subcategory of $\mathcal{U}$. The exact category $\widetilde{\mathcal{U}}^{\mathcal{A}}$ is the exact subcategory of $\widetilde{\mathcal{U}}$ generated by $\mathcal{U}$ and $\widetilde{\mathcal{A}}$. Its objects are those U of $\widetilde{\mathcal{U}}$ such that there is an A of $\widetilde{\mathcal{A}}$ with $U \oplus A$ in $\mathcal{U}$. It is immediate to see that $\widetilde{\mathcal{U}}^{\mathcal{A}}$ is extension closed in $\widetilde{\mathcal{U}}$ and is therefore an exact category. A sequence $U \xrightarrow{i} V \xrightarrow{p} W$ is exact in $\widetilde{\mathcal{U}}^{\mathcal{A}}$ iff there exist objects A, B of $\widetilde{\mathcal{A}}$ such that the sequence

$$A \oplus U \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & i \\ 0 & 0 \end{pmatrix}} A \oplus V \oplus B \xrightarrow{\begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}} W \oplus B$$

is exact in $\mathcal{U}$. Clearly, $\mathcal{U}$ is cofinal in $\widetilde{\mathcal{U}}^{\mathcal{A}}$ which is cofinal in $\widetilde{\mathcal{U}}$.

3.5 Definition. An exact subcategory $\mathcal{A}$ of an exact category $\mathcal{U}$ is said to be *strictly right filtering* if $\mathcal{A}$ is closed under taking admissible subobjects and admissible quotients in $\mathcal{U}$ and if every map $U \to A$ from an object U of $\mathcal{U}$ to an object A of $\mathcal{A}$ factors through an object B of $\mathcal{A}$ such that the arrow $U \to B$ is an admissible epimorphism

$$\begin{tikzcd} U \arrow[r] \arrow[dr, two heads] & A \\ & B. \arrow[u] \end{tikzcd}$$

$\mathcal{A}$ is called *strictly left filtering* in $\mathcal{U}$ if $\mathcal{A}^{op}$ is strictly right filtering in $\mathcal{U}^{op}$. $\mathcal{A}$ is called *strictly filtering* in $\mathcal{U}$ if $\mathcal{A}$ is strictly left filtering and strictly right filtering in $\mathcal{U}$. $\mathcal{A}$ is called *(right, left) filtering* in $\mathcal{U}$ if $\widetilde{\mathcal{A}}$ is strictly (right, left) filtering in $\widetilde{\mathcal{U}}^{\mathcal{A}}$.

3.6 Definition. Let $\mathcal{A}$ be an exact subcategory of the exact category $\mathcal{U}$. If A is an object of $\mathcal{A}$ and U one of $\mathcal{U}$, an admissible monomorphism $i : A \rightarrowtail U$ is called *special* if there exists an admissible epimorphism $p : U \twoheadrightarrow B$ to an object B of $\mathcal{A}$ such that $p \circ i$ is an admissible monomorphism of $\mathcal{A}$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[dr, tail] & U \arrow[d, two heads] \\ & B. \end{tikzcd}$$

An admissible epimorphism $U \twoheadrightarrow A$ is called *special* if there exists an admissible monomorphism $i : B \rightarrowtail U$ from an object B of $\mathcal{A}$ such that $p \circ i$ is an admissible epimorphism in $\mathcal{A}$

$$\begin{tikzcd} U \arrow[r, two heads] & A \\ B. \arrow[u, tail] \arrow[ur, two heads] \end{tikzcd}$$

3.7 Definition. Let $\mathcal{A}$ be an exact subcategory of the exact category $\mathcal{U}$. Then $\mathcal{A}$ is called *strictly right s-filtering* in $\mathcal{U}$ if $\mathcal{A}$ is strictly right filtering in $\mathcal{U}$ and every admissible monomorphism $A \rightarrowtail U$ from an object of $\mathcal{A}$ to an object of $\mathcal{U}$ is special. $\mathcal{A}$ is called *strictly left s-filtering* in $\mathcal{U}$ if $\mathcal{A}^{op}$ is strictly right s-filtering in $\mathcal{U}^{op}$. $\mathcal{A}$ is called *strictly s-filtering* in $\mathcal{U}$ if $\mathcal{A}$ is strictly left s-filtering and strictly right s-filtering in $\mathcal{U}$. $\mathcal{A}$ is called *(right, left) s-filtering* if $\widetilde{\mathcal{A}}$ is strictly (right, left) s-filtering in $\widetilde{\mathcal{U}}^{\mathcal{A}}$.

3.8 Lemma. *Strictly (right, left) s-filtering subcategories are (right, left) s-filtering.*

Proof. Let $\mathcal{A}$ be strictly right s-filtering in $\mathcal{U}$. We have to show that $\widetilde{\mathcal{A}}$ is strictly right s-filtering in $\widetilde{\mathcal{U}}^{\mathcal{A}}$.

a) We claim that every U of $\widetilde{\mathcal{U}}^{\mathcal{A}}$ fits into an exact sequence $V \overset{i}{\rightarrowtail} U \overset{q}{\twoheadrightarrow} D$ with V in $\mathcal{U}$ and D in $\widetilde{\mathcal{A}}$. With U in $\widetilde{\mathcal{U}}^{\mathcal{A}}$ we find an A of $\widetilde{\mathcal{A}}$ such that $U \oplus A$ is in $\mathcal{U}$ (3.4). Let A' be an object of $\widetilde{\mathcal{A}}$ such that $A' \oplus A$ is an $\mathcal{A}$. The $\mathcal{U}$-map $0 \oplus 1_A : U \oplus A \to A' \oplus A$ factors through an object B of $\mathcal{A}$ (3.5), $0 \oplus 1_A = (\bar{\alpha}, \alpha) \circ p$ with $p : U \oplus A \to B$ an admissible epimorphism of $\mathcal{U}$, $\bar{\alpha} : B \to A'$ and $\alpha : B \to A$. The map $(0, 1_A) : A \to U \oplus A$ induces a section of $\alpha : B \to A$ which therefore is an admissible epimorphism of $\widetilde{\mathcal{A}}$. By the snake lemma we have a $\widetilde{\mathcal{U}}^{\mathcal{A}}$-exact sequence $ker(p) \rightarrowtail U \twoheadrightarrow ker(\alpha)$. By construction, $D = ker(\alpha)$ is an object of $\widetilde{\mathcal{A}}$ and $V = ker(p)$ is an object of $\mathcal{U}$.

b) $\widetilde{\mathcal{A}}$ is closed under admissible subobjects and admissible quotients in $\widetilde{\mathcal{U}}^{\mathcal{A}}$. Let $0 \to U \to A \to V \to 0$ be a short exact sequence in $\widetilde{\mathcal{U}}^{\mathcal{A}}$. By definition there are objects B, C of $\widetilde{\mathcal{A}}$ such that $U \oplus B$ and $V \oplus C$ are objects of $\mathcal{U}$. Since $\mathcal{U}$ is extension closed in $\widetilde{\mathcal{U}}^{\mathcal{A}}$, the object $W := A \oplus B \oplus C$ has to lie in $\mathcal{U}$ and is a direct factor of an object D of $\mathcal{A}$. The monic arrow $W \to D$ factors by the right filtering property through an object E of $\mathcal{A}$ such that $W \to E$ is an admissible epimorphism, hence an isomorphism. Since $\mathcal{A}$ is closed under admissible subobjects and admissible quotients, $U \oplus B$ and $V \oplus C$ are objects of $\mathcal{A}$. It follows $U, V \in \widetilde{\mathcal{A}}$.

c) $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{U}}^{\mathcal{A}}$ is strictly right filtering. To see this, take a $\widetilde{\mathcal{U}}^{\mathcal{A}}$-map $f : U \to A$ with U in $\mathcal{U}$, A in $\widetilde{\mathcal{A}}$ and let A' an object of $\widetilde{\mathcal{A}}$ with $A' \oplus A$ in $\mathcal{A}$. Let $V \overset{i}{\rightarrowtail} U \overset{q}{\twoheadrightarrow} D$ as in a). Then the map $(0, f \circ i) : V \to A' \oplus A$ factors through an object C of $\mathcal{A}$ as $(0, g) \circ p = (0, f \circ i)$ with p an admissible epimorphism of $\mathcal{U}$. Let E be a pushout of p along i and $\bar{i} : C \to E$ be induced by i, $\bar{p} : U \to E$ induced by p. $\bar{p}$ is an admissible epimorphism of $\widetilde{\mathcal{U}}^{\mathcal{A}}$ since p is and since i is an admissible monomorphism. E is an object of $\widetilde{\mathcal{A}}$ as it is an extension of C and D. The pushout property of E induces a map $h : E \to A$ with $f = h \circ \bar{p}$.

d) Every admissible monomorphism $i : A \rightarrowtail U$ from an object A of $\widetilde{\mathcal{A}}$ to U of $\widetilde{\mathcal{U}}^{\mathcal{A}}$ is special. In order to show this, let A' in $\widetilde{\mathcal{A}}$ with $A' \oplus A$ in $\mathcal{A}$. Then there is a B of $\widetilde{\mathcal{A}}$ such that

$$A' \oplus A \xrightarrow{j := \begin{pmatrix} 1 & 0 \\ 0 & i \\ 0 & 0 \end{pmatrix}} A' \oplus U \oplus B$$

is an admissible monomorphism of $\mathcal{U}$ from an object of $\mathcal{A}$ to an object of $\mathcal{U}$. By hypothesis, it is special. So there is an admissible epimorphism $p : A' \oplus U \oplus B \to C$ to an object C of $\mathcal{A}$ such that $p \circ j$ is an admissible monomorphism. By part c) $p \circ (0, 1_U, 0) : U \to C$ factors through an object D of $\widetilde{\mathcal{A}}$ as $p \circ (0, 1_U, 0) = r \circ q$ with $q : U \to D$ an admissible epimorphism of $\widetilde{\mathcal{U}}^{\mathcal{A}}$. $q \circ i$ is an admissible monomorphism since $r \circ q \circ i$ is and $\widetilde{\mathcal{A}}$ idempotent complete (3.3). $\square$

3.9 *Example.* Let $\mathcal{A}$ be a Serre subcategory of an abelian category $\mathcal{U}$. Then $\mathcal{A}$ is filtering in $\mathcal{U}$. The factorizations required in definition 3.5 are given by the factorization of a map through its image.

3.10 *Example.* Any filtering subcategory in the sense of Karoubi ([Kar70]) or Pedersen-Weibel ([PW89]) is s-filtering. This is because split admissible monomorphisms and split admissible epimorphisms are obviously special, take $A = B$ in definition 3.6.

3.11 *Example.* (see p.233 of [Gra76]). Let R be a ring with unit, S a set of central non zero divisors in R, $\mathcal{H}_S$ be the exact category of S-torsion finitely presented left R-modules of projective dimension at most 1 as an exact subcategory of the category of all left R-modules, and let $\mathcal{P}$ be the exact category of finitely presented left R-modules M of projective dimension at most 1 such that $S^{-1}M$ is isomorphic to $S^{-1}P$ for some finitely generated projective left R-module P. Then $\mathcal{H}_S$ is right s-filtering in $\mathcal{P}$. A more detailed analysis of this example will be given in 12.

3.12 *Example.* Let R be a noetherian ring. Then the category of left modules of finite type over R, $\mathcal{M}_{ft}(R)$, is a left s-filtering subcategory of the category $\mathcal{M}(R)$ of all left R-modules. The filtering property is clear since it is a Serre subcategory. For a surjective map $U \to M$ to a finitely generated module M, we lift a finite set of generators for M to U. This generates a finitely generated submodule of U which still surjects onto M.

3.13 *Example.* In section 4 we construct several left s-, right s- and s-filtering embeddings $\mathcal{A} \subset \mathcal{F}\mathcal{A}$ of any exact category $\mathcal{A}$ into an exact category $\mathcal{F}\mathcal{A}$ whose K-theory space is contractible.

3.14 In order to state our next lemma we recall the definition of a *calculus of right fractions* for additive categories (see [Wei94] and [GZ67]). Let $\mathcal{C}$ be a small additive category, w a set of morphisms in $\mathcal{C}$ called weak equivalences. w is said to satisfy a calculus of right fractions if

i) w is closed under composition. Any identity morphism is in w.

ii) Any diagram $X \xrightarrow{u} Y \xleftarrow{s} Z$ with s a weak equivalence can be completed to a commutative diagram

$$\begin{tikzcd} W \arrow[r, "v"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

with t a weak equivalence.

iii) Given a morphisms $f : X \to Y$. If there is a weak equivalence s with $sf = 0$, there exists a weak equivalence t with $ft = 0$.

If the set of weak equivalences in an additive category $\mathcal{C}$ satisfies a calculus of right fractions we can give a very explicit construction of the additive category $\mathcal{C}[w^{-1}]$ obtained from $\mathcal{C}$ by formally inverting the weak equivalences. It has the same objects as $\mathcal{C}$ and a morphism in $\mathcal{C}[w^{-1}]$ can be written as fs^{-1} with $s \in w$ and composition is calculated using 3.14 i)-iii). We have a functor $\mathcal{C} \to \mathcal{C}[w^{-1}]$ which is universal in the following sense: functors from $\mathcal{C}[w^{-1}]$ to a category correspond bijectively to functors from $\mathcal{C}$ to that category sending weak equivalences to isomorphisms. Furthermore, the functor $\mathcal{C} \to \mathcal{C}[w^{-1}]$ preserves cartesian squares.

3.15 Let $\mathcal{U}$ be an exact category. For an integer n Waldhausen defined in [Wal85] an exact category $S_n\mathcal{U}$ which is equivalent to the category whose objects are sequences of admissible monomorphisms $U = (\, U_{01} \rightarrowtail U_{02} \rightarrowtail \cdots \rightarrowtail U_{0n}\,)$. Morphisms are commutative diagrams in $\mathcal{U}$. A sequence $U \to V \to W$ in $S_n\mathcal{U}$ is exact if it is pointwise exact, i.e. if $U_{0i} \to V_{0i} \to W_{0i}$ is exact in $\mathcal{U}$ for $1 \le i \le n$. Sometimes we may write $I_n\mathcal{U}$ instead of $S_n\mathcal{U}$ and $P_n\mathcal{U}$ for $I_n(\mathcal{U}^{op})^{op}$. We also introduce the category $I_\infty\mathcal{U}$. Its objects are infinite sequences of admissible monomorphisms $X_0 \rightarrowtail X_1 \rightarrowtail X_2 \rightarrowtail \cdots$ and morphisms are natural transformations. $I_\infty\mathcal{U}$ is an exact category in which a sequence is exact if it is pointwise exact in $\mathcal{U}$. We write $P_\infty\mathcal{U}$ for $I_\infty(\mathcal{U}^{op})^{op}$. Its objects are sequences of admissible epimorphisms and maps are commutative diagrams. We remark that $S_n\mathcal{U}$, $I_n\mathcal{U}$ and $P_n\mathcal{U}$ are equivalent exact categories, whereas $I_\infty\mathcal{U}$ is in general not equivalent to $P_\infty\mathcal{U}$.

3.16 Lemma. *Let $\mathcal{A}$ be an idempotent complete exact category which is strictly right s-filtering in $\mathcal{U}$ (3.7) and let w be the set of $\mathcal{U}$-morphisms which are (finite) compositions of admissible monomorphisms whose cokernels are in $\mathcal{A}$ and admissible epimorphisms with kernel in $\mathcal{A}$. Call a map in w a weak equivalence. Then*

i) The set of weak equivalences w satisfies a calculus of right fractions.

ii) (compare 3.2) Let $\alpha : A \to B$ be an $\mathcal{A}$-morphism such that there is a $\mathcal{U}$-morphism $\varphi : X \to A$ with $\alpha \circ \varphi$ an admissible epimorphism. Then α is an admissible epimorphism.

iii) Any diagram $X \overset{u}{\underset{\sim}{\rightarrowtail}} Y \overset{s}{\underset{\sim}{\twoheadleftarrow}} Z$ can be completed to a commutative diagram

$$\begin{tikzcd}
W \arrow[r, tail, "v", "\sim"'] \arrow[d, two heads, "t"', "\wr"] & Z \arrow[d, two heads, "\wr"', "s"] \\
X \arrow[r, tail, "\sim", "u"'] & Y.
\end{tikzcd}$$

Here $\overset{\sim}{\rightarrowtail}$ *means "admissible monomorphism with cokernel in* $\mathcal{A}$*".*

iv) For any weak equivalence s *there is an admissible monomorphism* i *with cokernel in* $\mathcal{A}$ *such that* $s \circ i$ *is an admissible monomorphism with cokernel in* $\mathcal{A}$*.*

v) $S_n\mathcal{A}$ *is strictly right s-filtering in* $S_n\mathcal{U}$*,* $I_\infty\mathcal{A}$ *is strictly right s-filtering in* $I_\infty\mathcal{U}$*.*

vi) Any diagram $X \overset{u}{\rightarrowtail} Y \underset{\sim}{\overset{s}{\leftarrowtail}} Z$ *can be completed to a commutative diagram*

$$\begin{tikzcd} W \arrow[r, tail, "v"] \arrow[d, tail, "t"', "\wr"] & Z \arrow[d, tail, "s", "\wr"'] \\ X \arrow[r, tail, "u"'] & Y. \end{tikzcd}$$

vii) The set of weak equivalences w *is saturated, i.e. any* $\mathcal{U}$*-morphism whose image in* $\mathcal{U}[w^{-1}]$ *is an isomorphism lies in* w*.*

viii) The category $\mathcal{U}[w^{-1}]$ *has a canonical structure of an exact category in which a sequence is exact iff it is isomorphic to the image of an exact sequence of* $\mathcal{U}$*.* $\mathcal{U}[w^{-1}]$ *has the universal property that exact functors from* $\mathcal{U}[w^{-1}]$ *to an exact category correspond bijectively to exact functors from* $\mathcal{U}$ *to this category vanishing on* $\mathcal{A}$*. We write* $\mathcal{U}/\mathcal{A}$ *for the category* $\mathcal{U}[w^{-1}]$ *where we identify the image of* $\mathcal{A}$ *with the basepoint of* $\mathcal{U}/\mathcal{A}$*.*

ix) $S_n\mathcal{A}$ *is strictly right s-filtering in* $S_n\mathcal{U}$*. The canonical functor* $S_n\mathcal{U}/S_n\mathcal{A} \to S_n(\mathcal{U}/\mathcal{A})$ *is an equivalence of exact categories.*

Proof. We begin with i). We have to verify 3.14 i)-iii). 3.14 i) is clear by definition of w. To see 3.14 ii) resp. iii) write $s = s_1 s_2 ... s_n$ where s_i is an admissible monomorphism with cokernel in $\mathcal{A}$ or an admissible epimorphism with kernel in $\mathcal{A}$. The proof proceeds by induction on n. $n = 0$ is trivial. The induction step consists of proving 3.14 ii) resp. iii) when s is an admissible monomorphism with cokernel in $\mathcal{A}$ or an admissible epimorphism with kernel in $\mathcal{A}$. For 3.14 ii) if s is an admissible epimorphism with kernel in $\mathcal{A}$ the pullback of s along u does the job. If s is an admissible monomorphism with cokernel in $\mathcal{A}$, let $p : Y \twoheadrightarrow A$ denote its cokernel. So A is an object of $\mathcal{A}$. By the right filtering property 3.5 $pu : X \to A$ can be written as aq where $q : X \twoheadrightarrow B$ is an admissible epimorphism, B an object of $\mathcal{A}$ and $a : B \to A$. Let $t : W \rightarrowtail X$ be the kernel of $q : X \twoheadrightarrow B$ and let $v : W \longrightarrow Z$ the induced map. By definition, t is a weak equivalence. For 3.14 iii), if s is an admissible monomorphism and $sf = 0$ then $f = 0$. If s is an admissible epimorphism with kernel in $\mathcal{A}$ let $i : A \rightarrowtail Y$ be the kernel of s, so A is an object of $\mathcal{A}$. Since $sf = 0$ there exists a (unique) $g : X \longrightarrow A$ such that $ig = f$. By the right filtering property 3.5 g is the composition of an admissible epimorphism $p : X \twoheadrightarrow B$ with target in $\mathcal{A}$ and a map $a : B \longrightarrow A$ in $\mathcal{A}$. Let $t : U \rightarrowtail X$ be the kernel of p. By definition, the cokernel of t lies in $\mathcal{A}$ and $ft = iapt = 0$.

Next we show ii). Let W be the pull back of $\alpha \circ \varphi$ along α

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d, "q"'] & X \arrow[d, "\alpha\circ\varphi"] \\ A \arrow[r, "\alpha"'] & B. \end{tikzcd}$$

The map $\varphi : X \to A$ and the universal property of a pull back induce a unique map $s : X \to W$ such that $r \circ s = 1_X$ and $q \circ s = \varphi$. $u := 1_W - sr$ is an idempotent of W. By the right filtering property 3.5 we can write $q \circ u$ as $\gamma \circ p$ with $p : W \to C$ an admissible epimorphism having its target in $\mathcal{A}$. Let $i : ker(p) \to W$ be a kernel for p then $u \circ i = 0$ since W is a pullback and $r \circ u \circ i = r(1_W - sr)i = 0$ and $q \circ u \circ i = \gamma \circ p \circ i = 0$. Therefore, there is a $t : C \to W$ such that $t \circ p = u$. Since $\mathcal{A}$ **is idempotent complete** the idempotent $p \circ u \circ t$ on C has an image lying in $\mathcal{A}$. Now, $ut : Im(p \circ u \circ t) \to Im(u)$ has inverse pu (in the idempotent completion of $\mathcal{U}$). That's why the idempotent u has an image lying in $\mathcal{A}$. $Im(u)$ is a kernel for r and also a kernel for α. By 3.2 α is an admissible epimorphism.

Now we prove iii). Let $p : Y \to A$ be a cokernel for u and $r : Y \to C$ be a cokernel for s. So A, C are objects of $\mathcal{A}$. The induced map $(p, r) : Y \to A \oplus C$ factors by the right filtering property 3.5 as $(p, r) = (\alpha, \gamma) \circ q$ with $q : Y \to B$ an admissible epimorphism whose target lies in $\mathcal{A}$, $\alpha : B \to A$ and $\gamma : B \to C$. By ii) α and γ are admissible epimorphisms. Let W be a kernel for q. Then the admissible epimorphisms α resp. γ induce the admissible monomorphisms t resp. v whose cokernels are the kernels of α resp. γ and lie therefore in $\mathcal{A}$.

Here is a proof of iv). We decompose s as in i) and show the assertion by induction. Using iii), we see that it suffices to prove it for $s : X \to Y$ an admissible epimorphism with kernel in $\mathcal{A}$. Let $l : A \to X$ be a kernel of s, A is an object of $\mathcal{A}$. Since $\mathcal{A}$ is strictly right s-filtered in $\mathcal{U}$, l is a special admissible monomorphism (3.6), hence there are an admissible monomorphism $j : A \to B$ and an admissible epimorphism $p : X \to B$ such that $p \circ l = j$. Let $q : B \to C$ denote a cokernel of j, $r : Y \to C$ denote the map induced by p and $i : U \to X$ a kernel for p, then by the snake lemma r is an admissible epimorphism and $s \circ i : U \to Y$ is a kernel for r and therefore an admissible monomorphism.

v) is shown as follows. a) We claim that any diagram $B_0 \overset{i}{\rightarrowtail} U \xrightarrow{\alpha} A_1$ with B_0 and A_1 in $\mathcal{A}$ can be completed to the commutative diagram

$$\begin{tikzcd} B_0 \arrow[r, tail, "i"] \arrow[d, tail] & U \arrow[d, "\alpha"] \arrow[dl, two heads, "p"'] \\ B_1 \arrow[r, "\beta"'] & A_1 \end{tikzcd}$$

with B_1 in $\mathcal{A}$. Since i is special, there is an admissible epimorphism $q : U \to C$ with target in $\mathcal{A}$ such that $q \circ i$ is an admissible monomorphism. As in the proof of iii) we find an admissible epimorphism $p : U \to B_1$ and factorizations $q = c \circ p$ and $\alpha = \beta \circ p$. By 3.2 $p \circ i$ is an admissible monomorphism since $c \circ p \circ i = q \circ i$ is and $\mathcal{A}$ **is idempotent complete** (3.3).

b) Next we verify that $S_2\mathcal{A}$ is strictly right filtering in $S_2\mathcal{U}$. Let $f = (f_0, f_1, f_2)$ be a map of exact sequences from the exact sequence $X = (X_0 \overset{j_X}{\rightarrowtail} X_1 \overset{q_X}{\twoheadrightarrow} X_2)$ representing an object of $S_2\mathcal{U}$ to the $S_2\mathcal{A}$-object $A = (A_0 \overset{j_A}{\rightarrowtail} A_1 \overset{q_A}{\twoheadrightarrow} A_2)$. By the right filtering property 3.5 we can write f_0 as $\beta_0 \circ r_0$ such that $r_0 : X_0 \to B_0$ is an admissible epimorphism with target in $\mathcal{A}$. Let U be a push-out of j_X along r_0 and let $i : B_0 \to U$ resp. $\bar{r} : X_1 \to U$ be the induced map by j_X resp. r_0. i is an admissible monomorphism and $\bar{r}$ is an admissible epimorphism. The universal property of pushouts gives us a map $\alpha : U \to A_1$ making all possible diagrams commute. Applying a) gives B_1, p and β as in a). $j_B := p \circ i$ is an admissible monomorphism. Let $q_B : B_1 \to B_2$ be a cokernel for j_B, then we have induced maps $\beta_2 : B_2 \to A_2$ and $r_2 : X_2 \to B_2$ and r_2 is by snake lemma applied to the map $(1_{B_0}, p, r_2)$ of exact sequences an admissible epimorphism. That's why $r = (r_0, p \circ \bar{r}, r_2)$ is an admissible epimorphism in $S_2\mathcal{U}$ from X to B and $f = (\beta_0, \beta, \beta_2) \circ r$. We can repeat this procedure showing that $I_n\mathcal{A}$ is right filtered in $I_n\mathcal{U}$ for $n = 1, 2, ..., \infty$.

c) It remains to show that every admissible monomorphism from an object of $I_n\mathcal{A}$ to an object of $I_n\mathcal{U}$ is special, $n = 1, 2, ..., \infty$. It suffices to prove that given an admissible monomorphism $l = (l_0, l_1) : (A_0 \overset{a}{\rightarrowtail} A_1) \to (U_0 \overset{u}{\rightarrowtail} U_1)$ in $S_2\mathcal{U}$ and a $\mathcal{U}$-admissible epimorphism $p_0 : U_0 \twoheadrightarrow B_0$ such that $p_0 \circ l_0$ is an admissible monomorphism, there is a $S_2\mathcal{U}$-admissible epimorphism $p = (p_0, p_1) : (U_0 \rightarrowtail U_1) \to (B_0 \rightarrowtail B_1)$ such that $p \circ l$ is an admissible monomorphism in $S_2\mathcal{A}$. That's what we are going to prove now. Let $\bar{U}$ be the pushout of a along l_0 and call the induced maps $\bar{l_0} : A_1 \to \bar{U}$ and $\bar{a} : U_0 \to \bar{U}$ which are $\mathcal{U}$-admissible monomorphisms. The universal property of the pushout gives factorizations $u = \bar{u} \circ \bar{a}$ and $l_1 = \bar{u} \circ \bar{l_0}$ where $\bar{u} : \bar{U} \to U_1$ is an admissible monomorphism because l is an admissible monomorphism in $S_2\mathcal{U}$. Let C be the pushout of $\bar{a}$ along p_0, the admissible monomorphism $c : B_0 \to C$ and the admissible epimorphism $q : \bar{U} \to C$ the induced maps. Since the composition of two pushouts is a pushout, C is the pushout of a along $p_0 \circ l_0$. Hence C is an object of $\mathcal{A}$ and $q \circ \bar{l_0}$ is an admissible monomorphism of $\mathcal{A}$. Let V be the pushout of $\bar{u}$ along q and call the induced admissible monomorphism $v : C \to V$ and the induced admissible epimorphism $\bar{q} : U_1 \to V$. Now, $v : C \to V$ is a special monomorphism, therefore there is an admissible epimorphism $r : V \to B_1$ such that $r \circ v : C \to B_1$ is an admissible monomorphism in $\mathcal{A}$. Let $p_1 = r \circ \bar{q}$, then, by construction, $p = (p_0, p_1) : (\, U_0 \rightarrowtail U_1 \,) \to (\, B_0 \rightarrowtail B_1 \,)$ is a $S_2\mathcal{U}$-admissible epimorphisms such that $p \circ l$ is an admissible monomorphism in $S_2\mathcal{A}$.

Now we show vi). Let $p : Y \to A$ be a cokernel for s. Then $(p \circ u, u)$ is a map in $S_2\mathcal{U}$ from $X \overset{u}{\rightarrowtail} Y$ to the $S_2\mathcal{A}$-object $A \xrightarrow{1} A$. So by v) it can be written as $(\alpha_0, \alpha_1) \circ (q_0, q_1)$ with $(q_0, q_1) : (\, X \overset{u}{\rightarrowtail} Y \,) \to (\, B_0 \overset{j_B}{\rightarrowtail} B_1 \,)$ an admissible epimorphism in $S_2\mathcal{U}$. Next we let $(t, l) : (\, W \overset{v_0}{\rightarrowtail} U \,) \to (\, X \overset{u}{\rightarrowtail} Y \,)$ be a kernel of (q_0, q_1) in $S_2\mathcal{U}$. In particular, v_0 is an admissible monomorphism. Furthermore, let $v_1 : U \to Z$ be the map induced by id_Y on the kernel of q_1 and p. Since by ii) α_1 is an admissible epimorphism, v_1 is an admissible monomorphism and we can take v as $v_1 \circ v_0$.

We prove vii) as follows. a) We claim that any idempotent $p : U \to U$ in $\mathcal{U}$ which becomes zero in $\mathcal{U}[w^{-1}]$ possesses an image lying in $\mathcal{A}$. Since p is zero in $\mathcal{U}[w^{-1}]$ there is an admissible monomorphism i with cokernel in $\mathcal{A}$ such that $p \circ i = 0$ (i), iv)). Let $q : U \to A$ be a cokernel for i. Then there is a map $s : A \to U$ such that $p = s \circ q$. $q \circ p \circ s$ is an idempotent on A which has an image in $\mathcal{A}$ since $\mathcal{A}$ **is idempotent complete**. Now, $p \circ s : Im(q \circ p \circ s) \to Im(p)$ has inverse $q \circ p$. So p has an image isomorphic to $Im(q \circ p \circ s)$ which lies in $\mathcal{A}$.

b) Let $f : X \to Y$ be a $\mathcal{U}$-map which becomes an isomorphism in $\mathcal{U}[w^{-1}]$ and let gs^{-1} be an inverse to f, s a weak equivalence. By the calculus of right fractions and iv) we can assume s an admissible monomorphism with cokernel in $\mathcal{A}$ and $f \circ g = s$ in $\mathcal{U}$. Let U be a push out of the diagram $X \xleftarrow{g} Z \overset{s}{\rightarrowtail} Y$ as in the diagram

$$\begin{tikzcd} Z \arrow[r, tail, "s"] \arrow[d, "g"'] & Y \arrow[d, "\bar{g}"] \\ X \arrow[r, tail, "t"'] \arrow[ur, "f"] & U. \end{tikzcd}$$

Using the universal property of pushouts, f induces a map $h : U \to Y$ such that $h \circ \bar{g} = 1_Y$ and $f = h \circ t$. Since g and hence $\bar{g}$ are $\mathcal{U}[w^{-1}]$-isomorphisms, the idempotent $1 - \bar{g} \circ h$ is zero in $\mathcal{U}[w^{-1}]$ and has therefore by a) an image in $\mathcal{A}$ which is also a kernel for h. So h is an admissible epimorphism with kernel in $\mathcal{A}$, t is an admissible monomorphism with cokernel in $\mathcal{A}$ and $f = h \circ t$.

We show viii). a) Given a map $f = (f_0, f_1, f_2)$ of diagrams in $\mathcal{U}$ from $Y_0 \overset{y_{01}}{\leftarrowtail} Y_1 \xrightarrow{y_{21}} Y_2$ to $X_0 \overset{x_{01}}{\leftarrowtail} X_1 \xrightarrow{x_{21}} X_2$, such that $f_i : Y_i \to X_i$ is an admissible monomorphism with cokernel in $\mathcal{A}$, $i = 0, 1, 2$. Let Y_3 resp. X_3 be a pushout of the diagrams Y_* resp. X_*. We claim that the induced map $f_3 = push(f_*) : Y_3 \to X_3$ is a weak equivalence. Since pushouts along admissible monomorphisms and pullbacks along admissible epimorphisms of weak equivalences are weak equivalences and since a map of exact sequences is always a juxtaposition of a pushout and a pullback square as in the following picture, it suffices to show that the quotient map $Y_3/Y_2 \to X_3/X_2$ is a weak equivalence

$$\begin{tikzcd}
Y_2 \arrow[r, tail, "y_{32}"] \arrow[dd, tail, "f_2"', "\wr"] \arrow[dr, phantom, "push"] & Y_3 \arrow[r, two heads] \arrow[d, tail, "\wr"] & Y_3/Y_2 \arrow[dd] \\
 & \bullet \arrow[ur, two heads] \arrow[d] \arrow[dr, phantom, "pull"] & \\
X_2 \arrow[r, tail, "x_{32}"'] \arrow[ur, tail] & X_3 \arrow[r, two heads] & X_3/X_2.
\end{tikzcd}$$

now, the map $Y_3/Y_2 \to X_3/X_2$ can be identified with the map $Y_0/Y_1 \to X_0/X_1$ which fits into the diagram

$$\begin{tikzcd}
Y_1 \arrow[r, tail, "y_{01}"] \arrow[d, equal] & Y_0 \arrow[r, two heads] \arrow[d, tail, "f_0"', "\wr"] \arrow[dr, phantom, "push"] & Y_0/Y_1 \arrow[d, tail, "\wr"] \\
Y_1 \arrow[r, tail, "f_0 y_{01}"] \arrow[d, tail, "f_1"', "\wr"] & X_0 \arrow[r, two heads] \arrow[d, equal] & U \arrow[d, two heads, "\wr"] \\
X_1 \arrow[r, tail, "x_{01}"'] & X_0 \arrow[r, two heads] & X_0/X_1.
\end{tikzcd}$$

The kernel of $U \to X_0/X_1$ is the cokernel of f_1 and lies therefore in $\mathcal{A}$. Hence $Y_0/Y_1 \to X_0/X_1$ is a weak equivalence.

b) We claim that the functor $\mathcal{U} \to \mathcal{U}[w^{-1}]$ preserves pushouts along admissible monomorphisms. Given a pushout square in $\mathcal{U}$

$$\begin{tikzcd}
X_1 \arrow[r, "x_{21}"] \arrow[d, tail, "x_{01}"'] & X_2 \arrow[d, tail, "x_{32}"] \\
X_0 \arrow[r, "x_{30}"] & X_3
\end{tikzcd}$$

and $\mathcal{U}[w^{-1}]$-maps $g_i \circ s_i^{-1} : X_i \to T$, $i = 0, 2$, such that $g_0 s_0^{-1} \circ x_{01} = g_2 s_2^{-1} \circ x_{21}$ in $\mathcal{U}[w^{-1}]$. By iv) we can suppose s_i an admissible monomorphism with cokernel in $\mathcal{A}$. Taking the pushout of s_i along g_i we can write $g_i s_i^{-1} = \bar{s}_i^{-1} \circ \bar{g}_i$ with $\bar{s}_i$ an admissible monomorphism having the same cokernel as s_i.

Let U be the pushout of $\bar{s}_0$ and $\bar{s}_2$

$$\begin{tikzcd} T \arrow[r, tail, "\bar{s}_0", "\sim"'] \arrow[d, tail, "\bar{s}_2"', "\wr"] & \bullet \arrow[d, tail, "t_0"', "\wr"] \\ \bullet \arrow[r, tail, "\sim", "t_2"'] & U. \end{tikzcd}$$

Then $t_0 \circ \bar{g}_0 \circ x_{01} = t_2 \circ \bar{g}_2 \circ x_{21}$ in $\mathcal{U}[w^{-1}]$. By the calculus of right fractions and iv) there is an admissible monomorphism $f_1 : Y_1 \to X_1$ with cokernel in $\mathcal{A}$ such that $t_0 \circ \bar{g}_0 \circ x_{01} \circ f_1 = t_2 \circ \bar{g}_2 \circ x_{21} \circ f_1$ in $\mathcal{U}$. Let Y_3 be the pushout of $x_{01} \circ f_1$ along $x_{21} \circ f_1$ and let $f_3 : Y_3 \to X_3$ be the induced map which by a) is a weak equivalence. The latter pushout gives by its universal property a map $g : Y_3 \to U$. The $\mathcal{U}[w^{-1}]$-morphism $u = (t_0\bar{s}_0)^{-1} g f_3^{-1} : X_3 \to T$ satisfies $u \circ x_{30} = g_0 s_0^{-1}$ and $u \circ x_{32} = g_2 s_2^{-1}$ in $\mathcal{U}[w^{-1}]$. We show uniqueness by supposing given a $\mathcal{U}[w^{-1}]$-map $gs^{-1} : X_3 \to T$ such that $gs^{-1} \circ x_{30} = 0$ and $gs^{-1} \circ x_{32} = 0$. As above we can suppose s an admissible monomorphism with cokernel in $\mathcal{A}$ and by taking the pushout of s along g we can suppose $s = 1$. By the calculus of right fractions and iv) the equations $g \circ x_{30} = 0$ and $g \circ x_{32} = 0$ in $\mathcal{U}[w^{-1}]$ give admissible monomorphisms with cokernel in $\mathcal{A}$ $f_i : Y_i \to X_i$, $i = 0, 2$ such that $g \circ x_{3i} \circ f_i = 0$. By i), iii) and vi) there are an admissible monomorphism $f_1 : Y_1 \to X_1$, a map $y_{21} : Y_1 \to Y_2$ such that $x_{21} \circ f_1 = f_2 \circ y_{21}$ and an admissible monomorphism $y_{01} : Y_1 \to Y_0$ such that $x_{01} \circ f_1 = f_0 \circ y_{01}$. Let Y_3 be the pushout of y_{01} along y_{21}. By a) the induced map $f_3 : Y_3 \to X_3$ is a weak equivalence. By the uniqueness of the map $g \circ f_3 : Y_3 \to T$ in $\mathcal{U}$ we have $g \circ f_3 = 0$, so $g = 0$ in $\mathcal{U}[w^{-1}]$.

c) We claim the set of admissible monomorphisms in $\mathcal{U}[w^{-1}]$ is closed under composition. By vii) we have to show that $\beta \circ ts^{-1} \circ \alpha$ is up to fractions an admissible monomorphism if α, β are admissible monomorphisms in $\mathcal{U}$ and s,t are weak equivalences. By iv) we can assume s, t being admissible monomorphisms with cokernel in $\mathcal{A}$. Composing t and β we get an admissible monomorphism and taking the pushout of βt along s we get rid of s.

d) The cobase change of an admissible monomorphism along an arbitrary map exists and is again an admissible monomorphism. This is because any $\mathcal{U}[w^{-1}]$-diagram $X_0 \leftarrowtail X_1 \longrightarrow X_2$ is isomorphic to the image of such a diagram coming from $\mathcal{U}$ which can be seen using iv) and appropriate $\mathcal{U}$-pushouts. By b) $\mathcal{U} \to \mathcal{U}[w^{-1}]$ preserves pushouts along admissible monomorphisms.

e) The set of admissible epimorphisms is closed under composition. By vii) we have to show that $\beta \circ ts^{-1} \circ \alpha$ is up to fractions an admissible epimorphism if α, β are admissible epimorphisms in $\mathcal{U}$ and s,t are weak equivalences. By iv) we can assume s, t being admissible monomorphisms with cokernel in $\mathcal{A}$. Taking the pullback of α along s we can assume $s = 1$. Let $t : X \to Y$ and let $i : ker(\beta) \to Y$ be a kernel for $\beta : Y \to Z$. By vi) there are an admissible monomorphism $r : U \to ker(\beta)$ with cokernel in $\mathcal{A}$ and an admissible monomorphism $j : U \to X$ such that $ir = tj$. Write $p : X \to X/U$

for a cokernel of j and $u : X/U \to Z$ for the map induced by t. Using a) we see that u is a weak equivalence. Evidently $p \circ \alpha$ is an admissible epimorphism.

f) The base change of an admissible epimorphism along an arbitrary map exists and is again an admissible epimorphism because a calculus of right fractions preserves pullbacks.

g) In an exact sequence $U \overset{i}{\rightarrowtail} V \overset{p}{\twoheadrightarrow} W$ the map i is a kernel for p and p is a cokernel for i. This is because $\mathcal{U} \to \mathcal{U}[w^{-1}]$ preserves pullbacks of admissible epimorphisms and pushouts of admissible monomorphisms. The other axioms are trivially satisfied. The universal property is obvious by construction.

It remains to show ix). By v) it remains to show that the functor $g : S_n\mathcal{U}/S_n\mathcal{A} \to S_n(\mathcal{U}/\mathcal{A})$ given by the universal properties is an equivalence of exact categories. We only prove it for $n = 2$. The general case is analog. It is easy to see that an object of $S_2(\mathcal{U}/\mathcal{A})$ is isomorphic to an object of $S_2\mathcal{U}/S_2\mathcal{A}$. We therefore have to show that g is full and faithful.

a) We claim that a $\mathcal{U}$-commutative diagram

$$\begin{tikzcd} X \arrow[r, tail, "y"] \arrow[d, "\alpha", "\wr"'] & Y \arrow[d, "\beta", "\wr"'] \\ U \arrow[r, tail, "v"] & V \end{tikzcd}$$

with y, v admissible monomorphisms and α, β weak equivalences is a weak equivalence for $S_2\mathcal{A} \subset S_2\mathcal{U}$ as a morphism from y to v. Let Z be a pushout of α along y, the induced maps being $\bar{\alpha} : Y \to Z$, $z : U \to Z$, $\gamma : Z \to V$. Evidently, $(\alpha, \bar{\alpha})$ is a weak equivalence for $S_2\mathcal{A} \subset S_2\mathcal{U}$. By vii) γ is a weak equivalence for $\mathcal{A} \subset \mathcal{U}$. Therefore $\gamma = \gamma_0 \circ \gamma_1 \circ ... \gamma_n$ with γ_i an admissible monomorphism with cokernel in $\mathcal{A}$, in this case (id_U, γ_i) has cokernel in $S_2\mathcal{A}$, or an admissible epimorphism with kernel in $\mathcal{A}$, but then (id_U, γ_i) has kernel in $S_2\mathcal{A}$.

b) The faithfulness of the functor g is a direct consequence of a). We show that g is full. Given $(f_0 s_0^{-1}, f_1 s_1^{-1}) : (\, U_0 \overset{u}{\rightarrowtail} U_1 \,) \to (\, X_0 \overset{x}{\rightarrowtail} X_1 \,)$ in $S_2\mathcal{U}/\mathcal{A}$. By iv) we can suppose $s_1 : V_1 \to U_1$ and $s_0 : \bar{V} \to U_0$ admissible monomorphism with cokernel on $\mathcal{A}$. Using vi) and iii) we find an admissible monomorphism $t : V_0 \to \bar{V}$ with cokernel in $\mathcal{A}$ and an admissible monomorphism $v : V_0 \to V_1$ such that $s_1 \circ v = u \circ s_0 \circ t$ and $x \circ f_0 \circ t = f_1 \circ v$. Since by a) $(s_0 t, s_1)$ is a weak equivalence for $S_2\mathcal{A} \subset S_2\mathcal{U}$, the $S_2\mathcal{U}/\mathcal{A}$-map $(f_0 s_0^{-1}, f_1 s_1^{-1}) = (f_0 t, f_1) \circ (s_0 t, s_1)^{-1}$ is in the image of g. Hence g is an equivalence of categories. The proof that the two exact structures coincide makes repeated use of iii), iv) and vi) and the ideas of a) and b). We omit it here. $\square$

The author doesn't have an argument that if $\mathcal{A}$ is strictly right s-filtering in $\mathcal{U}$ then the induced functor $I_\infty(\mathcal{U})/I_\infty(\mathcal{A}) \to I_\infty(\mathcal{U}/\mathcal{A})$ is an equivalence of categories.

3.17 Lemma. *Let $\mathcal{A}$ be right s-filtering in $\mathcal{U}$. Then*

i) there is a quotient category $\mathcal{U}/\mathcal{A}$ and an exact functor $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ having the universal property that exact functors from $\mathcal{U}$ to an exact category $\mathcal{E}$ sending $\mathcal{A}$ to the basepoint correspond bijectively to exact functors from $\mathcal{U}/\mathcal{A}$ to $\mathcal{E}$,

ii) $I_\infty(\mathcal{A})$ is right s-filtering in $I_\infty(\mathcal{U})$.

Proof. i) follows from the corresponding statements in lemma 3.16 applied to the strictly right s-filtering inclusion $\widetilde{\mathcal{A}}$ in $\widetilde{\mathcal{U}}^{\mathcal{A}}$. In ii) we use that $\widetilde{I_\infty(\mathcal{A})} = I_\infty(\widetilde{\mathcal{A}})$. We remark that $\mathcal{U}/\mathcal{A}$ is equivalent to $\widetilde{\mathcal{U}}^{\mathcal{A}}/\widetilde{\mathcal{A}}$. □

3.18 Lemma. *Let $\mathcal{A}$ be right s-filtering in $\mathcal{U}$. Then $\widetilde{\mathcal{A}}$ is closed under taking admissible subobjects and admissible quotients in $\widetilde{\mathcal{U}}$.*

Proof. The exact functor $\widetilde{\mathcal{U}}^{\mathcal{A}} \to \mathcal{U}/\mathcal{A}$ extends to an exact functor $\widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}/\mathcal{A}}$ (3.3). Therefore, if for an idempotent $p : U \to U$ of $\mathcal{U}$, $Im(p)$ is an admissible subobject or an admissible quotient of an object of $\widetilde{\mathcal{A}}$, $Im(p)$ is a zero object in $\widetilde{\mathcal{U}/\mathcal{A}}$. So $p = 0$ in $\mathcal{U}/\mathcal{A}$. By part a) of the proof of 3.16 vii), $Im(p)$ is an object of $\widetilde{\mathcal{A}}$. □

3.19 Lemma. *Given a commutative square of exact categories*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{V} \end{tikzcd}$$

in which $\mathcal{A}$ is idempotent complete, the horizontal exact functors are strictly right s-filtering and the vertical functors are strictly left s-filtering. Suppose further that an object of $\mathcal{V}$ which is isomorphic to an object of $\mathcal{U}$ and isomorphic to an object of $\mathcal{B}$ is itself isomorphic to an object of $\mathcal{A}$. Then the induced functor $\mathcal{B}/\mathcal{A} \to \mathcal{V}/\mathcal{U}$ is strictly right s-filtering, $\mathcal{U}/\mathcal{A} \to \mathcal{V}/\mathcal{B}$ is strictly left s-filtering and there is a natural equivalence of exact categories $(\mathcal{V}/\mathcal{U})/(\mathcal{B}/\mathcal{A}) \simeq (\mathcal{V}/\mathcal{B})/(\mathcal{U}/\mathcal{A})$.

Proof. We will only prove that $\mathcal{B}/\mathcal{A} \to \mathcal{V}/\mathcal{U}$ is strictly right s-filtering.

a) We claim that for every $\widetilde{\mathcal{V}}^{\mathcal{U}}$-map $f : B \to V$ from an object B of $\mathcal{B}$ to an object V of $\widetilde{\mathcal{V}}^{\mathcal{U}}$ which is a $\mathcal{V}/\mathcal{U}$- isomorphism there is a $\widetilde{\mathcal{V}}^{\mathcal{U}}$-map and $\mathcal{V}/\mathcal{U}$-isomorphism $v : V \to B'$ to an object B' of $\mathcal{B}$ such that $v \circ f$ is an admissible epimorphism of $\mathcal{B}$ with kernel in $\mathcal{A}$. Hence $v \circ f$ is an $\mathcal{B}/\mathcal{A}$-isomorphism and also a $\mathcal{V}/\mathcal{U}$-isomorphism. This proves that $\mathcal{B}/\mathcal{A} \to \mathcal{V}/\mathcal{U}$ is fully faithful. We show the claim as follows.

By 3.16 iv)$^{\mathrm{op}}$ we can suppose f an admissible epimorphism of $\widetilde{\mathcal{V}}^{\mathcal{U}}$ with kernel in $\widetilde{\mathcal{U}}$. The functor $\mathcal{U} \to \mathcal{V}$ is strictly left s-filtering, so by part a)$^{\mathrm{op}}$ of the proof of lemma 3.8 we can assume V in $\mathcal{V}$ and f still an admissible epimorphism with kernel in $\widetilde{\mathcal{U}}$. Let $i : U \to B$ be a kernel of f and U' an object of $\widetilde{\mathcal{U}}$ with $U \oplus U'$ in $\mathcal{U}$. Using that $\mathcal{U} \to \mathcal{V}$ is strictly left s-filtering, we see that the map $(i, 0) : U \oplus U' \to B$ factors through an object A of $\mathcal{U}$ as $(i, 0) = j \circ (u, 0)$ with j an admissible monomorphism of $\mathcal{V}$. Since $\mathcal{B}$ is closed under admissible subobjects and admissible quotients in $\mathcal{V}$, the cokernel B' of j is in $\mathcal{B}$, the $\mathcal{U}$-object A is in $\mathcal{B}$ and by the hypothesis of the lemma it is indeed isomorphic to an object of $\mathcal{A}$. The map $u : U \to A$ induces a map on the quotients $v : V \to B'$ which is a $\mathcal{V}/\mathcal{U}$-isomorphism. The kernel of $v \circ f$ is A.

b) The rest is straight forward using a), the calculus of left fractions with respect to admissible epimorphisms whose kernels are in $\widetilde{\mathcal{U}}$ and 3.18. □

4 Flasque exact categories

In this section we define flasque exact categories and construct several right s-, left s- and s-filtering embeddings of an arbitrary exact category into flasque exact categories. At the end we derive some useful right s-filtering embeddings we will need in the next sections. For a more conceptual approach we refer the reader to section 6.

4.1 Definition. (flasque exact category) An exact category $\mathcal{E}$ is said to be *flasque* if there exist an exact functor $S : \mathcal{E} \to \mathcal{E}$ and a natural equivalence of functors $S \oplus id_{\mathcal{E}} \simeq S$.

4.2 Remark. Any additive functor from exact categories to some additive category will necessarily send flasque categories to zero since it's identity map will be sent to the zero map.

The categories $\mathcal{F}\mathcal{A}$ and $S\mathcal{A}$

4.3 For any eaxct category $\mathcal{A}$ and any small category $\mathcal{D}$, the category of functors $\mathcal{A}^{\mathcal{D}}$ from $\mathcal{D}$ to $\mathcal{A}$ is an exact category if we declare a sequence $F \to G \to H$ of functors to be exact if it is pointwise exact, i.e. for all objects $D \in \mathcal{D}$ the sequence $F(D) \to G(D) \to H(D)$ is exact in $\mathcal{A}$. Let $\mathcal{N}$ be the partially ordered set consisting of a copy of $\mathbb{N}$ endowed with its usual order and a copy of $\mathbb{N}^{op}$ where we declare $n < m^{op}$ for all $n \in \mathbb{N}$ and $m^{op} \in \mathbb{N}^{op}$. So the category $\mathcal{A}^{\mathcal{N}}$ is an exact category. Let $\mathcal{F}_0\mathcal{A}$ be the full subcategory of $\mathcal{A}^{\mathcal{N}}$ whose objects are the functors $G : \mathcal{N} \to \mathcal{A}$ such that $G(n \to m)$ is an admissible monomorphism, $G(n^{op} \to m^{op})$ is an admissible epimorphism, there exists a $d \in \mathbb{N}$ with $G(n \to m^{op})$ an admissible monomorphism if $m \geq n + d$ and $G(n \to m^{op})$ an admissible epimorphism if $n \geq m + d$. $\mathcal{F}_0\mathcal{A}$ is extension closed in $\mathcal{A}^{\mathcal{N}}$ since in the category of morphisms extensions of admissible monomorphisms are admissible monomorphisms and extensions of admissible epimorphisms are admissible epimorphisms. Therefore, $\mathcal{F}_0\mathcal{A}$ is an exact category.

Example. Given a sequence $A = (A_0, A_1, ...)$ of objects in $\mathcal{A}$. We can associate an object $PW(A) \in \mathcal{F}_0\mathcal{A}$ by setting $PW(A)(n) = PW(A)(n^{op}) = \bigoplus_{0 \leq i \leq n} A_i$, $n^{(op)} \to m^{(op)}$ is sent to the partial injections and projections. Here $d = 0$.

Given two functors $s, t : \mathcal{N} \to \mathcal{N}$ there is at most one natural transformation between them, say $s => t$ and we write $\eta_{t,s}$ for it. We have a functor $m : \mathcal{N} \to \mathcal{N}$ sending $n \mapsto n + 1$ and $n^{op} \mapsto n^{op}$. Similarly, we have a functor $e : \mathcal{N} \to \mathcal{N}$ sending $n \mapsto n$ and $n^{op} \mapsto (n + 1)^{op}$. Notice that there are natural transformations $1_{\mathcal{N}} => m$ and $e => 1_{\mathcal{N}}$. e and m commute. We write e_i for the

i-times composition e^i of e with itself, the same applies to m_i. For $(i,j) \leq (k,l)$ we have inclusions $\mathcal{F}_0\mathcal{A}(e_i^*G, m_j^*H) \hookrightarrow \mathcal{F}_0\mathcal{A}(e_k^*G, m_l^*H)$ induced by the natural transformations $e_k => e_i$ and $m_j => m_l$. Then we define the category $\mathcal{F}\mathcal{A}$ in the following way. The objects are the same as the objects of $\mathcal{F}_0\mathcal{A}$ and morphisms are defined by

$$\mathcal{F}\mathcal{A}(G,H) = colim_{i,j}\mathcal{F}_0\mathcal{A}(e_i^*G, m_j^*H).$$

Given two $\mathcal{F}\mathcal{A}$-morphisms $[\varphi] : G \to H$ and $[\psi] : H \to K$ represented by the $\mathcal{F}_0\mathcal{A}$-morphism $\varphi : e_i^*G \to m_j^*H$ and $\psi : e_k^*H \to m_l^*K$ their composition is defined as to be $[\psi] \circ [\varphi] : G \to K$ represented by $m_j^*\psi \circ e_k^*\varphi : e_{i+k}^*G \to e_k^*m_j^*H = m_j^*e_k^*H \to m_{j+l}^*K$. One checks that this is well defined. Clearly, $\mathcal{F}_0\mathcal{A}$ is a subcategory of $\mathcal{F}\mathcal{A}$.

4.4 Lemma. *$\mathcal{F}\mathcal{A}$ is an exact category in which a sequence is exact iff it is isomorphic to an exact sequence lying in $\mathcal{F}_0\mathcal{A}$.*

Proof. i) We first remark that the $\mathcal{F}_0\mathcal{A}$-maps $G \to m_i^*G$ and $e_j^*G \to G$ induced by the natural transformations $1_\mathcal{N} => m_i$ and $e_j => 1_\mathcal{N}$ are $\mathcal{F}\mathcal{A}$-isomorphism.

ii) We claim that the functor $\mathcal{F}_0\mathcal{A} \to \mathcal{F}\mathcal{A}$ preserves pushouts of arbitrary maps along admissible monomorphisms and pullbacks of arbitrary maps along admissible epimorphisms. We will only prove the first part. Given an $\mathcal{F}_0\mathcal{A}$-pushout

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\delta"] \\ C \arrow[r, tail, "\gamma"] & D \end{tikzcd}$$

and two $\mathcal{F}\mathcal{A}$-maps $[\varphi] : B \to E$ and $[\psi] : C \to E$ represented by the $\mathcal{F}_0\mathcal{A}$-maps $\varphi : e_i^*B \to m_j^*E$ and $\psi : e_{i'}^*C \to m_{j'}^*E$ such that $[\psi] \circ [\beta] = [\varphi] \circ [\alpha]$. By replacing i, i' and j, j' by their maximum, respectively, we can assume $i = i'$ and $j = j'$. Since $e_i^* : \mathcal{F}_0\mathcal{A} \to \mathcal{F}_0\mathcal{A}$ is an exact functor we have a $\mathcal{F}_0\mathcal{A}$ commutative diagram

$$\begin{tikzcd} e_i^*A \arrow[r, tail, "e_i^*\alpha"] \arrow[d, "e_i^*\beta"'] & e_i^*B \arrow[d, "e_i^*\delta"'] \arrow[ddr, bend left, "\varphi"] & \\ e_i^*C \arrow[r, tail, "e_i^*\gamma"] \arrow[drr, bend right, "\psi"'] & e_i^*D & \\ & & m_j^*E \end{tikzcd}$$

in which the upper left square is a $\mathcal{F}_0\mathcal{A}$-pushout. By the universal property of a pushout there is a $\mathcal{F}_0\mathcal{A}$-map $\phi : e_i^*C \to m_j^*E$ such that $\phi \circ e_i^*\delta = \varphi$ and $\phi \circ e_i^*\gamma = \psi$. ϕ defines a $\mathcal{F}\mathcal{A}$-map $[\phi] : D \to E$

such that $[\phi] \circ [\delta] = [\varphi]$ and $[\phi] \circ [\gamma] = [\psi]$. We show uniqueness of $[\phi]$ by assuming given the $\mathcal{FA}$-map $[\phi] : D \to E$ represented by the $\mathcal{F}_0\mathcal{A}$-map $\phi : e_i^*C \to m_j^*E$ such that $[\phi] \circ [\delta] = 0$ and $[\phi] \circ [\gamma] = 0$. Then we have a $\mathcal{F}_0\mathcal{A}$ commutative diagram

$$\begin{tikzcd}
e_i^*A \arrow[r, tail, "e_i^*\alpha"] \arrow[d, "e_i^*\beta"'] & e_i^*B \arrow[d, "e_i^*\delta"'] \arrow[ddr, bend left, "0"] & \\
e_i^*C \arrow[r, tail, "e_i^*\gamma"] \arrow[drr, bend right, "0"'] & e_i^*D \arrow[dr, "\phi"] & \\
& & m_j^*E
\end{tikzcd}$$

in which the upper left square is a $\mathcal{F}_0\mathcal{A}$-pushout. But then $\phi = 0$ by the universal property of this pushout. This shows the claim.

iii) Next we show that the composition of admissible monomorphisms in $\mathcal{FA}$ is an admissible monomorphism. More precisely, given $\mathcal{F}_0\mathcal{A}$-admissible monomorphism $\alpha : A \rightarrowtail B$ and $\beta : B' \rightarrowtail C$ and a $\mathcal{FA}$-isomorphism $[\varphi] : B \to B'$ with inverse $[\phi] : B' \to B$ represented by $\phi : e_i^*B' \to m_j^*B$ then $[\beta] \circ [\varphi] \circ [\alpha]$ is up to $\mathcal{FA}$- isomorphism a $\mathcal{F}_0\mathcal{A}$-admissible monomorphism. Let D be the $\mathcal{F}_0\mathcal{A}$-pushout of ϕ along $e_i^*\beta$, then by the preceding paragraph $e_i^*C \to D$ is a $\mathcal{FA}$-isomorphism since ϕ is. We have a $\mathcal{FA}$ commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "\alpha"] \arrow[d, "\wr"] & B \arrow[r, dashed, "{[\varphi]}"] \arrow[d, "\wr"] & B' \arrow[r, tail, "\beta"] & C \\
m_j^*A \arrow[r, tail, "m_j^*\alpha"] & m_j^*B \arrow[dr] & e_i^*B' \arrow[l, "\phi"', "\sim"] \arrow[r, tail, "e_i^*\beta"] \arrow[u, "\wr"] & e_i^*C \arrow[u, "\wr"] \arrow[dl, "\sim"] \\
& & D &
\end{tikzcd}$$

in which the solid arrows are $\mathcal{F}_0\mathcal{A}$-maps, solid squares commute in $\mathcal{F}_0\mathcal{A}$, $\xrightarrow{\sim}$ indicate $\mathcal{FA}$-isomorphisms, the dashed arrow is a $\mathcal{FA}$-map and the non labeled horizontal maps are induced by the corresponding natural transformation of endofunctors of $\mathcal{N}$. This shows that $[\beta] \circ [\varphi] \circ [\alpha]$ is up to $\mathcal{FA}$-isomorphism a $\mathcal{F}_0\mathcal{A}$ admissible monomorphism. The proof that composition of admissible epimorphism in $\mathcal{FA}$ is an admissible epimorphism is similar.

iv) Next we show that pushouts of admissible monomorphisms along arbitrary maps exist in $\mathcal{FA}$ and are admissible monomorphisms. Given $[\psi] : A \to B$ and a $\mathcal{FA}$-admissible monomorphism $[\varphi] : A \to C$. Changing $[\varphi]$ up to isomorphism we can assume $[\varphi]$ represented by the $\mathcal{F}_0\mathcal{A}$ admissible monomorphism $\varphi : A \rightarrowtail C$ and $[\psi]$ represented by the $\mathcal{F}_0\mathcal{A}$ morphism $\psi : e_i^*A \to m_j^*B$. Consider the $\mathcal{FA}$-

commutative diagram

$$\begin{tikzcd} B \arrow[d, "\wr"] & A \arrow[l, dashed, "{[\psi]}"'] \arrow[r, tail, "\varphi"] & C \\ m_j^*B & e_i^*A \arrow[l, "\psi"] \arrow[u, "\wr"'] \arrow[r, tail, "e_i^*\varphi"'] & e_i^*C. \arrow[u, "\wr"'] \end{tikzcd}$$

By ii) the pushout of $e_i^*\varphi$ along ψ exists in $\mathcal{FA}$ and is a $\mathcal{FA}$ admissible monomorphism. This proves the claim. The proof that pullbacks along admissible epimorphisms exist in $F\mathcal{A}$ is similar and we therefore omit it.

All other axioms for exact categories are trivially satisfied. This finishes the proof of lemma 4.4. $\square$

There is a functor $c : \mathcal{A} \to \mathcal{FA}$ which factors through $\mathcal{F}_0\mathcal{A}$ and sends an object A of $\mathcal{A}$ to the constant diagram: $cA(n) = cA(n^{op}) = A$ and every $\mathcal{N}$ morphism is sent to the identity.

4.5 Lemma. *The constant functor $c : \mathcal{A} \to \mathcal{FA}$ makes $\mathcal{A}$ into a strictly s-filtering exact subcategory of $\mathcal{FA}$.*

Proof. Clearly, c makes $\mathcal{A}$ into a full additive subcategory of $\mathcal{FA}$.

i) We claim, if G is $\mathcal{FA}$-isomorphic to cA for some object A of $\mathcal{A}$ then there are $i, j \in \mathbb{N}$ such that $e_i^*m_j^*G$ is $\mathcal{F}_0\mathcal{A}$-isomorphic to cA. Given $\varphi : e_l^*cA \to m_j^*G$ and $\psi : e_i^*G \to m_k^*cA$ representing $[\varphi] : cA \to G$ and its inverse $[\psi] : G \to cA$. This means that the identity decomposes as $1 = m_j^*\psi \circ e_i^*\varphi : cA = e_{i+l}^*cA \to e_i^*m_j^*G \to m_{j+k}^*cA = A$. So cA is a direct factor of $e_i^*m_j^*G$ in (the idempotent completion of) $\mathcal{F}_0\mathcal{A}$ whose inclusion in $e_i^*m_j^*G$ becomes an isomorphism in $\mathcal{FA}$. Since $\mathcal{F}_0\mathcal{A} \to \mathcal{FA}$ is faithful, the complement of cA in $e_i^*m_j^*G$ must be zero in $\mathcal{F}_0\mathcal{A}$ and, therefore, $e_i^*m_j^*G$ is $\mathcal{F}_0\mathcal{A}$-isomorphic to cA.

ii) Using i) we see that $\mathcal{A}$ is extension closed in $\mathcal{FA}$ since it is extension closed in $\mathcal{F}_0\mathcal{A}$.

iii) $\mathcal{A}$ is closed under taking admissible subobjects and admissible quotient objects in $\mathcal{FA}$. Let G be a $\mathcal{FA}$ admissible quotient object of A. By i) we can assume G a $\mathcal{F}_0\mathcal{A}$-admissible quotient object of cA: $\phi : cA \twoheadrightarrow G$. We have $\phi(i) \circ cA(0 \to i) = G(0 \to i) \circ \phi(i)$ for $i \in \mathbb{N}$. But $G(0 \to i)$ is an admissible monomorphism, $cA(0 \to i) = 1_A$ and $\phi(i)$ and $\phi(0)$ are admissible epimorphisms, so $G(0 \to i)$ is an ($\mathcal{A}$-) isomorphism. We also have the equality $\phi(j^{op}) \circ cA(0 \to j^{op}) = G(0 \to j^{op}) \circ \phi(0)$. There is a $d \in \mathbb{N}$ such that $G(0 \to j^{op})$ is an admissible monomorphism for $j \geq d$. By the same argument as above we find that $G(0 \to j^{op})$ is an ($\mathcal{A}$-) isomorphism. So e_d^*G is $\mathcal{F}_0\mathcal{A}$-isomorphic to $cG(0)$ and therefore G is $\mathcal{FA}$-isomorphic to $cG(0)$. $\mathcal{A}$ is closed under taking admissible quotient objects in $\mathcal{FA}$. The proof for admissible subobjects is analogue.

iv) Since 0 is an initial object of $\mathcal{N}$ we have a $\mathcal{F}_0\mathcal{A}$-map $cG(0) \to G$. There is a $d \in I\!N$ such that $G(0) \to e_d^*G(i)$ is an admissible monomorphism in $\mathcal{A}$, i.e. $cG(0) \to e_d^*G$ is an $\mathcal{A}^{\mathcal{N}}$ admissible monomorphism. But its quotient $e_d^*G/G(0)$ lies even in $\mathcal{F}_0\mathcal{A}$, so the sequence $cG(0) \to e_d^*G(i) \to e_d^*G/G(0)$ is exact in $\mathcal{F}_{(0)}\mathcal{A}$. Therefore, $cG(0) \to G$ is an admissible monomorphism in $\mathcal{FA}$. By using the $\mathcal{FA}$-isomorphisms m_i^* we see that this argument shows that $cG(i) \to G$ is an admissible monomorphism in $\mathcal{FA}$. Similarly, the maps $G \to cG(j^{op})$ are admissible epimorphisms in $\mathcal{FA}$. Since for $i \in I\!N$ there is a $j > i$ such that $G(i \to j^{op})$ is an admissible monomorphism in $\mathcal{A}$ and hence $cG(i) \to cG(j^{op})$ an $\mathcal{FA}$ admissible monomorphism, the maps $cG(i) \to G$ are even special admissible monomorphisms. By the dual argument $G \to cG(j^{op})$ is a special admissible epimorphism. Every $\mathcal{FA}$-map $cA \to G$ resp. $G \to cA$ factors through some special monomorphism $cG(i) \to G$ resp. special epimorphism $G \to cG(j^{op})$. It follows that every admissible monomorphism from $\mathcal{A}$ to $\mathcal{FA}$ resp. admissible epimorphism from $\mathcal{FA}$ to $\mathcal{A}$ is special. This finishes the proof of lemma 4.5.

□

4.6 Lemma. *$\mathcal{FA}$ is flasque.*

Proof. We have a shift functor $s : \mathcal{FA} \to \mathcal{FA}$ defined by $sG(n^{(op)}) = G((n-1)^{(op)})$ for $n \neq 0$ and $G(0^{(op)}) = 0$. Since $e^*m^* \circ s = 1_{\mathcal{FA}}$ we have a natural equivalence of functors $1_{\mathcal{FA}} => s$. Now the functor $S := \bigoplus_{k\geq 0} s^k : \mathcal{FA} \to \mathcal{FA}$ make sense and is even exact. We then have $1 \oplus S \simeq 1 \oplus s \circ S = S$ and so $\mathcal{FA}$ is flasque. This proves the lemma and also finishes the proof of proposition 4.7. □

Remark. The assignment PW extends to a functor $PW : \mathcal{C}_{I\!N}\mathcal{A} \to \mathcal{FA}$ in the obvious way (for the category $\mathcal{C}_{I\!N}\mathcal{A}$ see [PW89].

4.7 Proposition/Definition. *The constant functor makes $\mathcal{A}$ into a strictly s-filtering subcategory of the flasque exact category $\mathcal{FA}$. The exact category $S\mathcal{A} := \mathcal{FA}/\mathcal{A}$ is called suspension of $\mathcal{A}$.*

4.8 Remark. The virtue of the category $\mathcal{FA}$ is its symmetry $\mathcal{F}(\mathcal{A}^{op}) = (\mathcal{FA})^{op}$. For if $\mathcal{A}$ is an exact category with duality, $\mathcal{FA}$ and $S\mathcal{A}$ are naturally exact categories with duality as well. This is not true any more for the flasque exact categories we will construct now.

The categories $\mathcal{F}_I\mathcal{A}$, $\mathcal{F}_P\mathcal{A}$, $S_I\mathcal{A}$ and $S_P\mathcal{A}$

4.9 We proceed now to give some simplified versions of flasque embeddings. They are not as symmetric as the $\mathcal{FA}$ and $S\mathcal{A}$ constructions but have the technical advantage of simplicity. They are essential in the proof of 5.5. We use the categories $I_\infty\mathcal{A}$ and $P_\infty\mathcal{A}$ (see 3.15) as substitutes for $\mathcal{F}_0\mathcal{A}$. By definition, $I_\infty\mathcal{A}$ is an exact subcategory of $\mathcal{A}^{\mathbb{N}}$ and $P_\infty\mathcal{A}$ is an exact subcategory of $\mathcal{A}^{\mathbb{N}^{\mathrm{op}}}$. As in the proof of 4.7 we have a functor $m : \mathbb{N} \to \mathbb{N}$, $n \mapsto n+1$ and a natural transformation $\mu : 1_{\mathbb{N}} => m$. Similarly, we have a functor $e : \mathbb{N}^{\mathrm{op}} \to \mathbb{N}^{\mathrm{op}}$, $n^{op} \mapsto (n+1)^{op}$ and a natural transformation $\varepsilon : e => 1_{\mathbb{N}^{\mathrm{op}}}$. We write m_i for the i-times composition m^i of m with itself, the same applies to e_i. Then we define

$$\mathcal{F}_I\mathcal{A}(G,H) = colim_i I_\infty\mathcal{A}(G, m_j^* H),$$

$$\mathcal{F}_P\mathcal{A}(G,H) = colim_i P_\infty\mathcal{A}(e_i^* G, H).$$

The set of all natural transformations $S = \{\mu^{i-j}G : m_j^*G => m_i^*G \;/\; j \leq i,\; G \in I_\infty(\mathcal{A})\}$ satisfies a calculus of left fractions and $\mathcal{F}_I\mathcal{A} = [S^{-1}]I_\infty(\mathcal{A})$. A similar argument applies to $\mathcal{F}_P\mathcal{A}$. We endow $\mathcal{F}_I\mathcal{A}$ with a structure of an exact category in which a sequence is exact iff it is isomorphic to the image of an exact sequence of $I_\infty(\mathcal{A})$ under the inclusion $I_\infty(\mathcal{A}) \to \mathcal{F}_I\mathcal{A}$. Similarly, we endow $\mathcal{F}_P\mathcal{A}$ with the set of exact sequences coming from $P_\infty\mathcal{A}$. The functor sending an object of $\mathcal{A}$ to the constant diagram of $I_\infty(\mathcal{A})$ resp. of $P_\infty(\mathcal{A})$ induces exact functors $c : \mathcal{A} \to \mathcal{F}_I\mathcal{A}$ and $c : \mathcal{A} \to \mathcal{F}_P\mathcal{A}$.

4.10 Lemma. *The categories $\mathcal{F}_I\mathcal{A}$ and $\mathcal{F}_P\mathcal{A}$ are flasque exact categories. The constant functor $c : \mathcal{A} \to \mathcal{F}_I\mathcal{A}$ makes $\mathcal{A}$ into a strictly left s-filtering subcategory. The constant functor $c : \mathcal{A} \to \mathcal{F}_P\mathcal{A}$ makes $\mathcal{A}$ into a strictly right s-filtering subcategory. There are exact functors $\mathcal{F}\mathcal{A} \to \mathcal{F}_I\mathcal{A}$ and $\mathcal{F}\mathcal{A} \to \mathcal{F}_P\mathcal{A}$ natural in $\mathcal{A}$ and compatible with the inclusion of $\mathcal{A}$ into the corresponding flasque categories.*

Proof. The proof is a simpler version of the proof of lemma 4.7. For a proof in a more general context see section 6. The functors $\mathcal{F}\mathcal{A} \to \mathcal{F}_I\mathcal{A}$ and $\mathcal{F}\mathcal{A} \to \mathcal{F}_P\mathcal{A}$ are induced by the inclusions of $\mathbb{N} \to \mathcal{N}$ and $\mathbb{N}^{op} \to \mathcal{N}$ respectively. □

4.11 Lemma. *If $\mathcal{A}$ is idempotent complete and strictly right s-filtering in $\mathcal{U}$, then $\mathcal{F}_I\mathcal{A}$ is strictly right s-filtering in $\mathcal{F}_I\mathcal{U}$.*

Proof. By lemma 3.16 v) $I_\infty(\mathcal{A})$ is strictly right s-filtering in $I_\infty(\mathcal{U})$. The lemma follows by the calculus of fraction (4.9) and the fact that an object $U = (U_0 \to U_1 \to ...)$ of $I_\infty(\mathcal{U})$ which is $\mathcal{F}_I\mathcal{U}$-isomorphic to an object $A = (A_0 \to A_1 \to ...)$ of $I_\infty(\mathcal{A})$ is already $I_\infty(\mathcal{U})$-isomorphic to A. To see the last claim, since the functor m_i^* preserves objects of $I_\infty(\mathcal{A})$ we only have to investigate the case of a $I_\infty(\mathcal{U})$-morphism $f : U \to A$ which is a $\mathcal{F}_I\mathcal{U}$-isomorphism. Let $\mu^i(U)^{-1}g$, $g : A \to m_i^*U$, be a $\mathcal{F}_I\mathcal{U}$-inverse for f. Then $g \circ f = \mu^i(U)$. It follows that $U(k \to k+i)$ factors through A_i. Therefore

$U(k \to k+i) = 0$ in $\mathcal{U}/\mathcal{A}$. Fix k. By lemma 3.16 i), iv) there is an admissible monomorphism $\alpha : V \to U_k$ with cokernel in $\mathcal{A}$ such that $U(k \to k+i) \circ \alpha = 0$ in $\mathcal{U}$. Since $U(k \to k+i)$ is itself an admissible monomorphism this is only possible if $V = 0$ in $\mathcal{U}$ and hence U_k is an object of $\mathcal{A}$. Therefore, U is an object of $I_\infty(\mathcal{A})$. □

4.12 Definition. For $\mathcal{A}$ an exact category, we let $S_I(\mathcal{A}) = \mathcal{F}_I(\mathcal{A})/\mathcal{A}$ and $S_P(\mathcal{A}) = \mathcal{F}_P(\mathcal{A})/\mathcal{A}$.

4.13 Lemma. *Let $\mathcal{A}$ right s-filtering in $\mathcal{U}$. Then*

i) $S_I(\mathcal{A})$ is right s-filtering in $S_I(\mathcal{U})$

ii) $\mathcal{U}/\mathcal{A}$ is left s-filtering in $\mathcal{F}_I(\mathcal{U})/\mathcal{F}_I(\mathcal{A})$ and the quotient of the latter category by the first is equivalent to $S_I(\mathcal{U})/S_I(\mathcal{A})$, naturally in $\mathcal{A}$ and $\mathcal{U}$.

Proof. We can suppose $\mathcal{A}$ idempotent complete and strictly right s-filtering in $\mathcal{U}$. We have a commutative diagram of exact categories

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathcal{F}_I\mathcal{A} \arrow[r] & \mathcal{F}_I\mathcal{U} \end{tikzcd}$$

in which the first horizontal line is strictly right s-filtering by the lemma's hypothesis, the second horizontal line is strictly right s-filtering by 4.11, the vertical lines are strictly left s-filtering by 4.7. i) and ii) then follow from 3.19. □

The categories $\mathcal{C}_1\mathcal{A}$, $\mathcal{F}_2(\mathcal{A})$ and $S_2(\mathcal{A})$

4.14 The only use of the categories we are going to construct now is their occurrence in the proof of 5.5. The functor I_∞ applied twice to an exact category $\mathcal{A}$ yields an exact category $I^2_\infty(\mathcal{A})$ whose objects are a collection of objects $\{A(n)\}_{n\in\mathbb{N}\times\mathbb{N}}$ of $\mathcal{A}$ and an admissible monomorphism $A(n \to m) : A(n) \rightarrowtail A(m)$ whenever $n = (n_1, n_2) \le m = (m_1, m_2)$, i.e. $n_1 \le m_1$ and $n_2 \le m_2$, such that for $i, j, k, l \in \mathbb{N}$ the map $A(i+l, j) \oplus_{A(i,j)} A(i, j+k) \to A(i+l, j+k)$ is an admissible monomorphism. In fact, $I^2_\infty(\mathcal{A})$ is the exact subcategory of $\mathcal{A}^{\mathbb{N}\times\mathbb{N}}$ consisting of those objects. As in the previous constructions, for $k \in \mathbb{N}\times\mathbb{N}$ we have functors $m_k : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N} :\ n \mapsto n+k$ and a unique natural transformation $m_k => m_l$ whenever $k \le l$. We define the category $\mathcal{F}_2(\mathcal{A})$ as the category having the same objects as $I^2_\infty(\mathcal{A})$ and where the morphisms are

$$\mathcal{F}_2(\mathcal{A})(G, H) = colim_{k\in\mathbb{N}\times\mathbb{N}} I^2_\infty\mathcal{A}(G, m_k^* H).$$

The usual arguments developed in Lemma 4.4 and Lemma 4.6 show that $\mathcal{F}_2(\mathcal{A})$ is a flasque exact category whose exact structure is induced from $I^2_\infty \mathcal{A} \to \mathcal{F}_2(\mathcal{A})$. Let $\mathcal{C}_1\mathcal{A}$ be the full subcategory of $\mathcal{F}_2(\mathcal{A})$ whose objects are those A such that $A(i+l,j) \oplus_{A(i,j)} A(i,j+k) \to A(i+l,j+k)$ is an isomorphism. We have a functor $c_1 : \mathcal{F}_I(\mathcal{A}) \to \mathcal{C}_1\mathcal{A}$ sending the object $A = (A(0) \to A(1) \to ...)$ of $\mathcal{F}_I(\mathcal{A})$ to $c_1A(i,j) = A(i)$.

4.15 Proposition/Definition. *i) $\mathcal{C}_1\mathcal{A}$ is a strictly left s-filtering subcategory of $\mathcal{F}_2(\mathcal{A})$. We define $S_2(\mathcal{A}) := \mathcal{F}_2(\mathcal{A})/\mathcal{C}_1\mathcal{A}$.*

ii) $c_1 : \mathcal{F}_I(\mathcal{A}) \to \mathcal{C}_1\mathcal{A}$ is left s-filtering with quotient $S_I(\mathcal{A})$.

Proof. Section 6. □

4.16 Lemma. *Let $\mathcal{A}$ be right s-filtering in $\mathcal{U}$. Then*

i) $\mathcal{C}_1\mathcal{A}$ is right s-filtering in $\mathcal{C}_1\mathcal{U}$,

ii) $\mathcal{C}_1\mathcal{U}/\mathcal{C}_1\mathcal{A}$ is left s-filtering in $\mathcal{F}_2(\mathcal{U})/\mathcal{F}_2(\mathcal{A})$ with quotient $S_2(\mathcal{U})/S_2(\mathcal{A})$ and

iii) $c_1 : \mathcal{F}_I(\mathcal{U})/\mathcal{F}_I(\mathcal{A}) \to \mathcal{C}_1\mathcal{U}/\mathcal{C}_1\mathcal{A}$ is left s-filtering with quotient $S_I(\mathcal{U})/S_I(\mathcal{A})$.

The proof is similar to 4.13 and we omit the details.

5 Localization and negative K-theory

In this section we will give a first definition of negative K-groups of an exact category $\mathcal{E}$ (5.3) by explicitly constructing a non-connective Ω-spectrum whose positive homotopy groups coincide with Quillen's K-groups and whose zero's homotopy group is $K_0(\tilde{\mathcal{E}})$. The key proposition for the delooping result is 5.1. Lemma 5.5 is at the heart of the proof of theorem 7.12 which extends the homotopy fibration of proposition 5.1 to a homotopy fibration of non-connective $I\!K$-theory spectra.

5.1 Proposition. *Let $\mathcal{A}$ be an idempotent complete strictly right s-filtering exact subcategory of the exact category $\mathcal{U}$ (3.7). Then the sequence of exact functors $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of K-theory spaces*

$$K(\mathcal{A}) \to K(\mathcal{U}) \to K(\mathcal{U}/\mathcal{A}).$$

Proof. For any exact functor $f : \mathcal{A} \to \mathcal{B}$ between categories with cofibrations and weak equivalences Waldhausen constructs a simplicial category with cofibrations and weak equivalences $S.(f : \mathcal{A} \to \mathcal{B})$ and a homotopy fibration

$$wS.\mathcal{B} \to S.S.(f : \mathcal{A} \to \mathcal{B}) \to wS.S.\mathcal{A}$$

(see proposition 1.5.5 of [Wal85]). In our application f is the inclusion $\mathcal{A} \subset \mathcal{U}$ of an idempotent complete strictly right s-filtering subcategory $\mathcal{A}$ of an exact category $\mathcal{U}$ and w is the set of isomorphisms. We write i instead of w. Then $S.(\mathcal{A} \subset \mathcal{U})$ is a simplicial exact category for which $S_q(\mathcal{A} \subset \mathcal{U})$ is equivalent to the exact category whose objects are sequences of admissible monomorphisms with cokernel in $\mathcal{A}$, $U = (U_0 \overset{\sim}{\rightarrowtail} U_1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} U_q)$. Morphisms are commutative diagrams in $\mathcal{U}$. A sequence $U \to V \to W$ in $S_q(\mathcal{A} \subset \mathcal{U})$ is exact iff it is pointwise exact, i.e. if $U_i \to V_i \to W_i$ is exact in $\mathcal{U}$ for $0 \le i \le q$. Using Waldhausen's fibration proposition, 5.1 follows once we have proven the next lemma.

5.2 Lemma. *The map*

$$iS.S.(\mathcal{A} \subset \mathcal{U}) \to iS.S.(0 \subset \mathcal{U}/\mathcal{A})$$

induced by the quotient map $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ is a homotopy equivalence.

Assuming the lemma for a moment and going back to the proof of 5.1, we have a commutative diagram

$$\begin{tikzcd} iS.\mathcal{A} \arrow[r] & iS.\mathcal{U} \arrow[r] \arrow[d] & iS.S.(\mathcal{A} \subset \mathcal{U}) \arrow[d, "\wr"] \\ & iS.\mathcal{U}/\mathcal{A} \arrow[r, "\sim"] & iS.S.(0 \subset \mathcal{U}/\mathcal{A}) \end{tikzcd}$$

in which the first horizontal line is a homotopy fibration by the Waldhausen fibration proposition. The map $iS.\mathcal{U}/\mathcal{A} \to iS.S.(0 \subset \mathcal{U}/\mathcal{A})$ is a homotopy equivalence (for instance by appealing again to Waldhausen fibration proposition) and $iS.S.(\mathcal{A} \subset \mathcal{U}) \to iS.S.(0 \subset \mathcal{U}/\mathcal{A})$ is a homotopy equivalence by lemma 5.2.

Proof (of Lemma 5.2). We are going to use that the order doesn't matter when realizing multisimplicial sets and that a map of simplicial categories is a homotopy equivalence if it is degree wise a homotopy equivalence. Let $\mathcal{C}$ be a small category, s a set of morphisms in $\mathcal{C}$ closed under composition and containing all isomorphisms. We write $s\mathcal{C}$ for the category which has the same objects as $\mathcal{C}$ and where a morphism is a $\mathcal{C}$-morphism lying in s. Moreover, we write $\mathcal{N}_p^s\mathcal{C}$ for the category whose objects are sequences $C_0 \to C_1 \to \cdots \to C_p$ of maps lying in s and whose morphisms are commutative diagrams in $\mathcal{C}$. If we consider $s\mathcal{C}$ as a constant simplicial category, the canonical map of simplicial categories $s\mathcal{C} \to s\mathcal{N}_p^{iso}\mathcal{C}$, given by $C \mapsto (C \overset{id}{\to} C \overset{id}{\to} \dots \overset{id}{\to} C)$ on objects, is a homotopy equivalence since it is degree wise an equivalence of categories.

If $\mathcal{A}$ is an idempotent complete strictly right s-filtering exact subcategory of $\mathcal{U}$ we write w_q for the set of admissible monomorphisms in $S_q\mathcal{U}$ with cokernel in $S_q\mathcal{A}$. Now, the category $iS_pS_q(\mathcal{A} \subset \mathcal{U})$ is equivalent to $i\mathcal{N}_q^{w_p}S_p\mathcal{U}$. By realizing in different orders, we see that for fixed p $i\mathcal{N}_*^{w_p}S_p\mathcal{U}$ and $w_p\mathcal{N}_*^{iso}S_p\mathcal{U}$ have isomorphic realizations. As pointed out before, $w_pS_p\mathcal{U} \to w_p\mathcal{N}_*^{iso}S_p\mathcal{U}$ is a homotopy equivalence of simplicial categories. The same applies to the trivial (strictly right s-filtering) subcategory 0 of $\mathcal{U}/\mathcal{A}$. So the lemma is proven once we see that for every p the functor $f_p : w_pS_p\mathcal{U} \to iS_p(\mathcal{U}/\mathcal{A})$ is a homotopy equivalence. In light of 3.16 ix) it is enough to prove it for $f = f_1$. But for every object X of $iS_1(\mathcal{U}/\mathcal{A}) = i(\mathcal{U}/\mathcal{A})$, the obviously non-empty category $(f \downarrow X)$ is cofiltered (and therefore contractible) by 3.16 iv) and the calculus of fraction. So Quillen's theorem A applies and we are done. $\square$ $\square$

5.3 Definition. For $i \geq 0$ we (re)define

$$K_{-i}(\mathcal{E}) := K_0(\widetilde{S^i\mathcal{E}}).$$

Proposition 5.1 together with 4.7 enables us to construct an Ω-spetrum whose n-th space is $K(\widetilde{S^n\mathcal{E}})$ and whose negative homotopy groups are the groups defined in 5.3. The structure maps are the homotopy equivalences $K(\widetilde{S^n\mathcal{E}}) \simeq \Omega K(\widetilde{S^{n+1}\mathcal{E}})$ given by proposition 5.1 and the flasqueness of $\mathcal{F}$. Unfortunately, the construction of the structure maps is not very functorial. For a functorial version we refer the reader to section 7.

5.4 Definition. Let $\mathcal{E}$ be an exact category. A *flasque resolution* of $\mathcal{E}$ is a sequence of left or right s-filtering embeddings $\mathcal{E}_i \subset \mathcal{F}_i$, $i \in I\!N$, such that $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{E}_{i+1}$ is equivalent to $\mathcal{F}_i/\mathcal{E}_i$ and such that the categories $\mathcal{F}_i$ are flasques.

There is a canonical flasque resolution for $\mathcal{E}$ an exact category. We take $\mathcal{E}_i = S^i(\mathcal{E})$ and $\mathcal{F}_i = \mathcal{F}(S^i(\mathcal{E}))$. Then we have $K_{-i}(\mathcal{E}) := K_0(\widetilde{\mathcal{E}_i})$ using the last flasque resolution. It will follow from the next lemma that we can calculate negative K-groups using arbitrary flasque resolutions.

5.5 Lemma. *Let $\mathcal{A}$ be right or left s-filtering in $\mathcal{U}$. Then for any $n \in I\!N$, the sequence of exact categories*

$$\widetilde{S^n_*\mathcal{A}} \to \widetilde{S^n_*\mathcal{U}}^{S^n_*\mathcal{A}} \to S^n_*(\mathcal{U}/\mathcal{A})$$

induces a homotopy fibration of their K-theory spaces, $* \in \{I, P, \emptyset\}$.

Proof. In this proof we call a sequence of exact categories $\mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C}$ a homotopy fibration if $b \circ c = 0$ and if it induces a homotopy fibration of K-theory spaces $K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$. An exact functor is called a homotopy equivalence if it is after applying the functor K.

We first remark that an exact functor $\tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$ is a homotopy equivalence iff $S_*\mathcal{A} \to S_*\mathcal{B}$ is, $* \in \{\emptyset, I, P\}$. This is because we have a map of homotopy fibrations

$$\begin{array}{ccccc}
\tilde{\mathcal{A}} & \longrightarrow & \widetilde{\mathcal{F}_*\mathcal{A}}^{\mathcal{A}} & \longrightarrow & S_*\mathcal{A} \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{\mathcal{U}} & \longrightarrow & \widetilde{\mathcal{F}_*\mathcal{U}}^{\mathcal{U}} & \longrightarrow & S_*\mathcal{U}
\end{array}$$

with connected base spaces, and contractible total spaces.

We will show by induction on n the following statements.

a_n) Let $\mathcal{A}$ be right s-filtering in $\mathcal{U}$. Then $\widetilde{S^n_I\mathcal{A}} \to \widetilde{S^n_I\mathcal{U}}^{S^n_I\mathcal{A}} \to S^n_I(\mathcal{U}/\mathcal{A})$ is a homotopy fibration.

b_n) For any exact category $\mathcal{A}$, the exact functors $\widetilde{S^n_I\mathcal{A}} \leftarrow \widetilde{S^n\mathcal{A}} \to \widetilde{S^n_P\mathcal{A}}$ are homotopy equivalences.

c_n) Let $\mathcal{A}$ be right s-filtering in $\mathcal{U}$. Then $\widetilde{S^n_P\mathcal{A}} \to \widetilde{S^n_P\mathcal{U}}^{S^n_P\mathcal{A}} \to S^n_P\mathcal{U}/\mathcal{A}$ is a homotopy fibration.

d_n) Let $\mathcal{A}$ be right s-filtering in $\mathcal{U}$. Then the exact functor $S^n_I(S_I\mathcal{U}/S_I\mathcal{A}) \to S^{n+1}_I(\mathcal{U}/\mathcal{A})$ is a homotopy equivalence.

The functor in d_n) is induced by $S_I\mathcal{U}/S_I\mathcal{A} \to S_I(\mathcal{U}/\mathcal{A})$ defined by lemma 4.13.

The claims a_0) and c_0) are proposition 5.1 applied to $\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{U}}^{\mathcal{A}}$. The functors in b_0) are the identity functors. We only have to prove d_0). This follows from the map of homotopy fibrations (see 5.1, 4.13 ii) and 4.10)

$$\begin{tikzcd}
\widetilde{\mathcal{U}/\mathcal{A}} \arrow[r] \arrow[d, equal] & \widetilde{(\mathcal{F}_I\mathcal{U})/(\mathcal{F}_I\mathcal{A})}^{\mathcal{U}/\mathcal{A}} \arrow[r] \arrow[d] & (S_I\mathcal{U})/(S_I\mathcal{A}) \arrow[d] \\
\widetilde{\mathcal{U}/\mathcal{A}} \arrow[r] & \widetilde{\mathcal{F}_I(\mathcal{U}/\mathcal{A})}^{\mathcal{U}/\mathcal{A}} \arrow[r] & S_I(\mathcal{U}/\mathcal{A})
\end{tikzcd}$$

with flasque, hence contractible total spaces and connected base spaces. Since it is a homotopy equivalence on the fibers, it is also one on the base spaces.

Let $n \geq 1$. Now we proceed to show that a_n)-d_n) follow from the previous cases. By lemma 4.13 i) and proposition 5.1 the following sequence is a homotopy fibration

$$\widetilde{S^n_I\mathcal{A}} \to \widetilde{S^n_I\mathcal{U}}^{S^n_I\mathcal{A}} \to (S^n_I\mathcal{U})/(S^n_I\mathcal{A}).$$

In order to show that $(S^n_I\mathcal{U})/(S^n_I\mathcal{A}) \to S^n_I(\mathcal{U}/\mathcal{A})$ is a homotopy equivalence it suffices to verify that the functors $S^k_I((S^{n-k}_I\mathcal{U})/(S^{n-k}_I\mathcal{A})) \to S^{k+1}_I((S^{n-k-1}_I\mathcal{U})/(S^{n-k-1}_I\mathcal{A}))$ are homotopy equivalences for $k = 0, ...n-1$. But this is d_k) applied to the inclusion $S^{n-k-1}_I\mathcal{A} \to S^{n-k-1}_I\mathcal{U}$ which by iterated use of lemma 4.13 i) is right s-filtering. This proves a_n).

Next, we apply a_n) to the right s-filtered inclusions $\mathcal{A} \to \mathcal{F}_P\mathcal{A}$ and $\mathcal{A} \to \mathcal{F}\mathcal{A}$. The functor $\mathcal{F}\mathcal{A} \to \mathcal{F}_P\mathcal{A}$ induces a map of homotopy fibrations

$$\begin{tikzcd}
\widetilde{S^n_I\mathcal{A}} \arrow[r] \arrow[d, equal] & \widetilde{S^n_I\mathcal{F}\mathcal{A}}^{S^n_I\mathcal{A}} \arrow[r] \arrow[d] & S^n_I(S\mathcal{A}) \arrow[d] \\
\widetilde{S^n_I\mathcal{A}} \arrow[r] & \widetilde{S^n_I\mathcal{F}_P\mathcal{A}}^{S^n_I\mathcal{A}} \arrow[r] & S^n_I(S_P\mathcal{A})
\end{tikzcd}$$

with connected base and contractible total spaces, since $\mathcal{F}_P\mathcal{A}$ and $\mathcal{F}\mathcal{A}$ are flasque. Hence the functor $S^n_I(S\mathcal{A}) \to S^n_I(S_P\mathcal{A})$ is a homotopy equivalence. By the remark at the beginning of the proof, $\widetilde{S^{n-1}_I S\mathcal{A}} \to \widetilde{S^{n-1}_I S_P\mathcal{A}}$ is a homotopy fibration. Applying b_{n-1}) to $S\mathcal{A}$ yields a homotopy equivalence $\widetilde{S^n\mathcal{A}} \to \widetilde{S^{n-1}_I S\mathcal{A}}$. Applying b_{n-1}) to $S_P\mathcal{A}$, we find the following homotopy equivalences $\widetilde{S^{n-1}_I S_P\mathcal{A}} \leftarrow \widetilde{S^{n-1} S_P\mathcal{A}} \to \widetilde{S^n_P\mathcal{A}}$. This proves the right homotopy equivalence of b_n). The other part follows from the dual situation.

c_n) is a direct consequence of a_n) and b_n) using that $K_0(\widetilde{S^n_P\mathcal{U}}^{S^n_P\mathcal{A}}) = K_0(\widetilde{S^n\mathcal{U}}^{S^n\mathcal{A}}) = K_0(\widetilde{S^n_I\mathcal{U}}^{S^n_I\mathcal{A}})$. To see the last claim we only show the second equality, i.e. the exact functor $\widetilde{S^n\mathcal{U}}^{S^n\mathcal{A}} \to \widetilde{S^n_I\mathcal{U}}^{S^n_I\mathcal{A}}$ is a K_0-isomorphism. There is a commutative diagram

$$\begin{tikzcd}
K_0(\widetilde{S^n\mathcal{A}}) \arrow[r, two heads] \arrow[d] & K_0(\widetilde{S^n\mathcal{U}}^{S^n\mathcal{A}}) \arrow[r, tail] \arrow[d] & K_0(\widetilde{S^n\mathcal{U}}) \arrow[d] \\
K_0(\widetilde{S^n_I\mathcal{A}}) \arrow[r, two heads] & K_0(\widetilde{S^n_I\mathcal{U}}^{S^n_I\mathcal{A}}) \arrow[r, tail] & K_0(\widetilde{S^n_I\mathcal{U}})
\end{tikzcd}$$

in which the right and left vertical maps are isomorphisms by b_n). The left horizontal maps are surjective since $K_0(S_{(I)}(\ )) = 0$ and the definition 3.4. The right horizontal maps are injective by cofinality. Hence the middle vertical map is an isomorphism.

We have a commutative diagram of exact categories

$$\begin{tikzcd}
S_I^{n-1}(\widetilde{(S_I\mathcal{U})/(S_I\mathcal{A})}) \arrow[r] \arrow[dd, bend right, "\delta"'] & S_I^{n-1}\mathcal{F}_I(\widetilde{(S_I\mathcal{U})/(S_I\mathcal{A})})^w \arrow[r] & S_I^n((S_I\mathcal{U})/(S_I\mathcal{A})) \\
S_I^{n-1}(\widetilde{(\mathcal{C}_1\mathcal{U})/(\mathcal{C}_1\mathcal{A})}) \arrow[r] \arrow[u, "\alpha"] \arrow[d, "\gamma"'] & S_I^{n-1}(\widetilde{(\mathcal{F}_2\mathcal{U})/(\mathcal{F}_2\mathcal{A})})^w \arrow[r] \arrow[u] \arrow[d] & S_I^{n-1}((S_2\mathcal{U})/(S_2\mathcal{A})) \arrow[u, "\beta"'] \arrow[d, "\beta"'] \\
S_I^n(\widetilde{\mathcal{U}/\mathcal{A}}) \arrow[r] & S_I^n(\widetilde{(\mathcal{F}_I\mathcal{U})/(\mathcal{F}_I\mathcal{A})})^w \arrow[r] & S_I^n((S_I\mathcal{U})/(S_I\mathcal{A}))
\end{tikzcd}$$

in which all maps are induced by taking quotients. We wrote $\widetilde{\mathcal{V}}^w$ instead of $\widetilde{\mathcal{V}}^{\mathcal{B}}$. The first two horizontal lines are homotopy fibrations by the dual of c_{n-1}), 4.10 and 4.16 ii). The third horizontal line is a homotopy fibration by the dual of c_n) and 4.13 ii). By flasqueness, the total spaces are all contractible. The base spaces are connected. We prove below that α is a homotopy equivalence. Suppose it is, then β is a homotopy equivalence as well, and therefore γ and δ are homotopy equivalences. By the remark at the beginning of the proof, d_n) follows.

We are left with proving that α is a homotopy equivalence. Applying the dual of c_n to the left s-filtering inclusion $(\mathcal{F}_I\mathcal{U})/(\mathcal{F}_I\mathcal{A}) \to (\mathcal{C}_1\mathcal{U})/(\mathcal{C}_1\mathcal{A})$ of lemma 4.16 iii) yields a homotopy fibration

$$S_I^n(\widetilde{(\mathcal{F}_I\mathcal{U})/(\mathcal{F}_I\mathcal{A})}) \to S_I^n(\widetilde{(\mathcal{C}_1\mathcal{U})/(\mathcal{C}_1\mathcal{A})})^w \to S_I^n((S_I\mathcal{U})/(S_I\mathcal{A}))$$

with contractible fiber and connected base space, hence connected total space. Then by cofinality, the inclusion between the two connected spaces $S_I^n((\mathcal{C}_1\mathcal{U})/(\mathcal{C}_1\mathcal{A})) \to S_I^n(\widetilde{(\mathcal{C}_1\mathcal{U})/(\mathcal{C}_1\mathcal{A})})^w$ is a homotopy equivalence and so $S_I^n((\mathcal{C}_1\mathcal{U})/(\mathcal{C}_1\mathcal{A})) \to S_I^n((S_I\mathcal{U})/(S_I\mathcal{A}))$ is a homotopy equivalence, too. By the remark at the beginning of the proof α is a homotopy equivalence as well. $\square$

Lemma 5.5 basically says that a left or right s-filtering inclusion induces a homotopy fibration of non-connective spectra (defined after 5.3) extending proposition 5.1 to negative K-groups. We will make this statement precise in section 7.

6 The categories $\mathcal{C}_{\mathcal{P}}(\mathcal{E})$

Pedersen and Weibel ([PW89]) have constructed additive categories $\mathcal{C}_M(\mathcal{A})$ for M a metric space and $\mathcal{A}$ an additive category. Those categories have found applications in the work of Carlsson and Pedersen on the Novikov conjecture. They are also the first step in constructing their additive categories $\mathcal{C}_p(\mathcal{A})$ whose K-theories are p-fold non-connective deloopings of the K-theory of $\mathcal{A}$. Furthermore, taking $M = \mathcal{O}(K)$, the open cone over the simplicial complex K, they obtain models for the K-homology of K with coefficients in the non-connective K-theory spectrum of $\mathcal{A}$. In this section we give generalizations of their categories $\mathcal{C}_M(\mathcal{A})$ to exact categories by replacing M by subposets of $\mathbb{N}^k$ which will allow us to construct analogues of $\mathcal{C}_p(\mathcal{A})$ and $\mathcal{C}_{\mathcal{O}(K)}(\mathcal{A})$. Besides their construction, the main result of this section is a certain Mayer-Vietoris principle in proposition 6.18. Those categories are used in the next section to construct a functor from exact categories to symmetric spectra which behaves well with respect to products. In 12.5 and 12.7 we extend the results of Pedersen and Weibel to exact categories.

6.1 Localizing exact categories by 2-actions. We conceptualize the approach of chapter 4. Let M be commutative monoid with zero object and let $\mathcal{E}$ be an exact category. A 2-action μ of M on $\mathcal{E}$ is given by an exact functor $m. : \mathcal{E} \to \mathcal{E},\ E \mapsto m.E, \alpha \mapsto m.\alpha$ for each $m \in M$ such that $0. = id_{\mathcal{E}}$ and such that $(m+n). = m. \circ n.$ and by natural transformations $k_{(\ )} : id_{\mathcal{E}} => k.$ such that $0_E = id_E$ for E an object of $\mathcal{E}$ and such that $m.(k_E) = k_{m.E}$, $k_{m.E} \circ m_E = (m+k)_E$. Let $\hat{M}$ be the translation category associated with M. Its objects are the elements of M and a morphism from m to n is an element $k \in M$ with $n = m + k$. Composition in $\hat{M}$ corresponds to addition in M. Since M is commutative, the translation category $\hat{M}$ is a filtered category. We define a new category $\mu^{-1}\mathcal{E}$. Its objects are the objects of $\mathcal{E}$. Morphisms between two objects A, B of $\mu^{-1}\mathcal{E}$ are

$$\mu^{-1}\mathcal{E}(A,B) = colim_{m \in \hat{M}} \mathcal{E}(A, m.B).$$

where for a $\hat{M}$-morphism $k : m \to m+k$, the map $\mathcal{E}(A, m.B) \to \mathcal{E}(A, (m+k).B)$ is induced by $k_{m.B} : m.B \to (m+k).B$. Given two $\mu^{-1}\mathcal{E}$-morphism $[\varphi] : A \to B$ and $[\psi] : B \to C$ represented by the $\mathcal{E}$-morphism $\varphi : A \to m.B$ and $\psi : B \to n.C$ their composition is defined as to be $[\psi] \circ [\varphi] : A \to C$ represented by the composition $m.\psi \circ \varphi : A \to (m+n).C$ of φ and $m.\psi : m.B \to (m+n).C$. One checks that this is well defined. We have a functor $\mathcal{E} \to \mu^{-1}\mathcal{E}$ which is the identity on objects and on morphisms it sends $\alpha : E \to F$ to $\alpha : E \to 0.F = F$.

6.2 Lemma/Definition. *Call a sequence in $\mu^{-1}\mathcal{E}$ exact if it is isomorphic to the image of an exact sequence of $\mathcal{E}$ under the functor $\mathcal{E} \to \mu^{-1}\mathcal{E}$. Then, equipped with these exact sequences, $\mu^{-1}\mathcal{E}$ is an exact category.*

Proof. We first remark that the morphisms $k_{m.E} : m.E \to (m+k).E$ are isomorphisms in $\mu^{-1}\mathcal{E}$. We claim that the functor $\mathcal{E} \to \mu^{-1}\mathcal{E}$ preserves pushouts of arbitrary maps along admissible monomorphisms and pullbacks of arbitrary maps along admissible epimorphisms. The main argument is that for any $m \in M$, the functor $m : \mathcal{E} \to \mathcal{E} : E \mapsto m.E$ preserves such pushout and pullback diagrams as it is an exact functor. We will only show the statement about pullbacks. Given a $\mathcal{E}$-pullback

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\delta"] \\ C \arrow[r, "\gamma"] & D \end{tikzcd}$$

and two $\mu^{-1}\mathcal{E}$-maps $[\varphi] : E \to B$ and $[\psi] : E \to C$ represented by the $\mathcal{E}$-maps $\varphi : E \to m.B$ and $\psi : E \to n.C$ such that $[\gamma] \circ [\psi] = [\delta] \circ [\varphi]$. By composing φ with $n_{m.B}$ and ψ with $m_{n.C}$, we can replace m and n by $m+n$ and we therefore assume $m = n$. We have $[\gamma] \circ [\psi] = [m.\gamma \circ \psi]$ and $[\delta] \circ [\varphi] = [m.\delta \circ \varphi]$. By the equality $[m.\gamma \circ \psi] = [m.\delta \circ \varphi]$ there is a $k \in M$ such that $k_{m.D} \circ m.\gamma \circ \psi = k_{m.D} \circ m.\delta \circ \varphi$. Replacing ψ and φ by $k_{m.C} \circ \psi$ and $k_{m.B} \circ \varphi$ we can suppose $\varphi : E \to m.B$, $\psi : E \to m.C$ and $m.\gamma \circ \psi = m.\delta \circ \varphi$. Since the functor $m : \mathcal{E} \to \mathcal{E} : E \mapsto m.E$ is exact, the lower right square of the diagram

$$\begin{tikzcd} E \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\psi"'] & & \\ & m.A \arrow[r, "m.\alpha"] \arrow[d, "m.\beta"] & m.B \arrow[d, "m.\delta"] \\ & m.C \arrow[r, "m.\gamma"] & m.D \end{tikzcd}$$

is a pullback square. Its universal property together with $m.\gamma \circ \psi = m.\delta \circ \varphi$ provides us a map $\phi : E \to m.A$ such that $\varphi = m.\alpha \circ \phi$ and $\psi = m.\beta \circ \phi$, hence $[\varphi] = [\alpha] \circ [\phi]$ and $[\psi] = [\beta] \circ [\phi]$. Unicity uses a similar argument.

We indicate only how to check the axiom 3.1 a). The verification of the other axioms are either trivial or similar to a). Composition of admissible monomorphisms in $\mu^{-1}\mathcal{E}$ is an admissible monomorphism. More precisely, given $\mathcal{E}$-admissible monomorphism $\alpha : A \rightarrowtail B$ and $\beta : B' \rightarrowtail C$ and a $\mu^{-1}\mathcal{E}$-isomorphism $[\varphi] : B \to B'$ with inverse $[\phi] : B' \to B$ represented by $\phi : B' \to m.B$, then $[\beta] \circ [\varphi] \circ [\alpha]$ is up to $\mu^{-1}\mathcal{E}$-isomorphism a $\mathcal{E}$-admissible monomorphism. Let D be the $\mathcal{E}$-pushout of ϕ along β, then by the preceding claim $C \to D$ is a $\mu^{-1}\mathcal{E}$-isomorphism since a pushout of an isomorphism (e.g. ϕ) is an isomorphism. We have a $\mu^{-1}\mathcal{E}$-commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[d, "m_A"', "\wr"] & B \arrow[r, dashed, "{[\varphi]}"] \arrow[d, "\wr"', "m_B"] & B' \arrow[r, tail, "\beta"] \arrow[d, "\wr"', "\phi"] & C \arrow[d, "\wr"] \\ m.A \arrow[r, tail, "m.\alpha"] & m.B \arrow[r, equal] & m.B \arrow[r, tail] & D \end{tikzcd}$$

in which the solid arrows are $\mathcal{E}$-maps, solid squares commute in $\mathcal{E}$, $\xrightarrow{\sim}$ indicates $\mu^{-1}\mathcal{E}$-isomorphisms, the dashed arrow is a $\mu^{-1}\mathcal{E}$-map. This shows that $[\beta]\circ[\varphi]\circ[\alpha]$ is up to $\mu^{-1}\mathcal{E}$-isomorphism a $\mathcal{E}$-admissible monomorphism. Next we show that a pushout of admissible monomorphisms along an arbitrary map exists in $\mu^{-1}\mathcal{E}$ and is an admissible monomorphisms. Given $[\psi]: A \to B$ and a $\mu^{-1}\mathcal{E}$-admissible monomorphism $[\varphi]: A \to C$. Changing $[\varphi]$ up to isomorphism we can assume $[\varphi]$ represented by the $\mathcal{E}$-admissible monomorphism $\varphi : A \rightarrowtail C$ and $[\psi]$ represented by the $\mathcal{E}$-morphism $\psi : A \to m.B$. Let D be a push-out of φ along ψ. By the claim at the beginning of the proof, D is also a $\mu^{-1}\mathcal{E}$-pushout of φ along ψ, hence a $\mu^{-1}\mathcal{E}$-pushout of $[\varphi]$ along $[\psi]$. This shows a). □

6.3 Remark. The category $\mu^{-1}\mathcal{E}$ is a localization of $\mathcal{E}$ with respect to the set of morphisms of the form $k_{m.E} : m.E \to (m+k).E$.

6.4 The category $\mathcal{P}(\mathcal{E})$. Let $\mathcal{P}$ be a locally finite poset, i.e. a partially ordered set in which every element has a finite number of predecessors. As usual, we regard $\mathcal{P}$ as a small category whose objects are the elements of $\mathcal{P}$. There is exactly one morphism from x to y if $x \le y$, otherwise there is no morphism. If $\mathcal{S}$ is a subset of $\mathcal{P}$, we say that $\mathcal{S}$ is saturated if with every element of $\mathcal{S}$ all of its predecessors are also in $\mathcal{S}$. The union and intersection of saturated subsets are always saturated. Let $\mathcal{E}$ be an exact category, we write $\mathcal{P}(\mathcal{E})$ for the full subcategory of the category of functors $G : \mathcal{P} \to \mathcal{E}$ such that for all finite saturated $\mathcal{S} \subset \mathcal{S}' \subset \mathcal{P}$

$$colim_{x\in\mathcal{S}} G(x) \rightarrowtail colim_{x\in\mathcal{S}'} G(x) \tag{6.5}$$

is a admissible monomorphism in $\mathcal{E}$. There is something to say about the existence of the colimits in (6.5). Suppose they exist, then for $\mathcal{S}, \mathcal{S}' \subset \mathcal{P}$ finite, saturated we have a cocartesian square

$$\begin{tikzcd} colim_{\mathcal{S}\cap\mathcal{S}'} G \arrow[r] \arrow[d] & colim_{\mathcal{S}'} G \arrow[d] \\ colim_{\mathcal{S}} G \arrow[r] & colim_{\mathcal{S}\cup\mathcal{S}'} G. \end{tikzcd} \tag{6.6}$$

Using induction on the size of $\mathcal{T}$ (e.g. by picking a maximal element m of $\mathcal{T}$ and writing (6.6) with $\mathcal{S} = \{x \le m\}$ and $\mathcal{S}' = \mathcal{T} - \{m\}$, remark that $colim_\mathcal{S} G = G(m)$), condition (6.5) and (6.6), we see that $colim_\mathcal{T} G$ can be constructed by iterated pushouts along admissible monomorphisms, and therefore exists.

6.7 Remark. In order to show that a functor $G : \mathcal{P} \to \mathcal{E}$ lies in $\mathcal{P}(\mathcal{E})$ it suffices to verify (6.5) for the sets $\mathcal{S}_x = \{y < x\} \subset \mathcal{S'}_x = \{y \le x\}$, $x \in \mathcal{P}$. This already implies 6.5.

6.8 Definition. A commutative square in an exact category

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is called inclusion square if all maps are admissible monomorphisms and the map $B \sqcup_A C \to D$ is an admissible monomorphism.

The category of functors $G : \mathcal{P} \to \mathcal{E}$ is an exact category in which a sequence is exact iff it is pointwise exact, i.e. evaluated in each $x \in \mathcal{P}$.

6.9 Lemma. *$\mathcal{P}(\mathcal{E})$ is extension closed in the category of functors $G : \mathcal{P} \to \mathcal{E}$. This makes $\mathcal{P}(\mathcal{E})$ into an exact category. Furthermore, if $0 \to G \to H \to L \to 0$ is exact in $\mathcal{P}(\mathcal{E})$, then for all finite saturated $\mathcal{S} \subset \mathcal{P}$ we have an exact sequence*

$$0 \to colim_{t \in \mathcal{S}} G(t) \to colim_{t \in \mathcal{S}} H(t) \to colim_{t \in \mathcal{S}} L(t) \to 0.$$

Proof. This follows by induction using (6.6), that extensions of admissible monomorphisms are admissible monomorphisms and that extensions of pushouts along admissible monomorphisms are pushouts along admissible monomorphism. □

6.10 Lemma. *Let $\mathcal{P}$, $\mathcal{Q}$ be locally finite posets and $\mathcal{E}$ an exact category then $(\mathcal{P} \times \mathcal{Q})(\mathcal{E}) = \mathcal{P}(\mathcal{Q}(\mathcal{E}))$. In particular, the morphism $G \to H$ is an admissible monomorphism in $\mathcal{P}(\mathcal{E})$ iff $(G \to H)$ is an object of $(\mathcal{P} \times \{0 < 1\})(\mathcal{E})$.*

Proof. Since both categories are extension closed fully exact subcategories of $\mathcal{E}^{\mathcal{P} \times \mathcal{Q}}$ we only have to verify that they share the same objects. The functor $G : \mathcal{P} \times \mathcal{Q} \to \mathcal{E}$ is an object of $\mathcal{P}(\mathcal{Q}(\mathcal{E}))$ iff for every $p \in \mathcal{P}$ the functor $G(p,\ ) : \mathcal{Q} \to \mathcal{E}$ is an object of $\mathcal{Q}(\mathcal{E})$ and the map $colim_{p'<p} G(p',\ ) \to G(p,\ )$ is an admissible monomorphism in $\mathcal{Q}(\mathcal{E})$. This is the case iff for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ the following diagram is an inclusion square in $\mathcal{E}$

$$\begin{tikzcd} colim_{\{p'<p,\ q'<q\}} G(p', q') \arrow[r] \arrow[d] & colim_{p'<p} G(p', q) \arrow[d] \\ colim_{q'<q} G(p, q') \arrow[r] & G(p, q). \end{tikzcd}$$

But this is the same as $colim_{(p',q')<(p,q)} G(p', q') \to G(p, q)$ being an admissible monomorphism for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, i.e. G is an object of $(\mathcal{P} \times \mathcal{Q})(\mathcal{E})$. □

We give a simple criterion for a map of colimits to be an isomorphism.

6.11 Lemma. *Let $\mathcal{E}$ be an exact category, let $f : \mathcal{Q} \to \mathcal{P}$ be a map of locally finite posets and let $G : \mathcal{P} \to \mathcal{E}$ a functor. Suppose further*

i) the colimit of G over $\mathcal{P}$ is formed by iterated pushouts along admissible monomorphism and

ii) for every $x \in \mathcal{P}$, the poset $f^{-1}(x)$ has a smallest element $f^{-1}(x)_{\min}$ satisfying the condition that $f^{-1}(x)_{\min} \leq f^{-1}(y)_{\min}$ whenever $x \leq y$.

Then $\bar{f} : colim_{\mathcal{Q}} G \circ f \to colim_{\mathcal{P}} G$ is an isomorphism, in particular, the colimit over $\mathcal{Q}$ exists.

Proof. Let $i : \mathcal{E} \to \mathcal{A}$ be a full embedding of $\mathcal{E}$ into an abelian category $\mathcal{A}$ as an exact subcategory. This is possible, see appendix of [TT90] or [Kel90]. Then by i), $colim_{\mathcal{P}} G = colim_{\mathcal{P}} iG$, the latter colimit being constructed in $\mathcal{A}$. If a colimit over objects of $\mathcal{E}$ constructed in $\mathcal{A}$ lies in $\mathcal{E}$, it is also the colimit as constructed in $\mathcal{E}$. It therefore suffices to show that $\bar{f}$ is an isomorphism when the colimit over $\mathcal{Q}$ is constructed in $\mathcal{A}$. The assignment $x \in \mathcal{P} \mapsto f^{-1}(x)_{\min}$ induces by ii) a functor $g : \mathcal{P} \to Q$ with $f \circ g = 1$. Hence, $\bar{g} : colim_{\mathcal{P}} G \to colim_{\mathcal{Q}} G \circ f$ satisfies $\bar{f} \circ \bar{g} = 1$. We show the surjectivity of $\bar{g}$ which insures that $\bar{f}$ is an isomorphism. $colim_{\mathcal{Q}} G \circ f$ is a quotient of $\bigoplus_{q \in \mathcal{Q}} G(f(q))$. Now, the element $x \in G(f(q))$ in the summand corresponding to q is equivalent to $x \in G(f(q))$ in the summand corresponding to $f^{-1}(f(q))_{\min}$ which lies in the image of $\bar{g}$. □

6.12 Here is an application of 6.11. Let $\mathcal{S} \subset \mathcal{P}$ be s saturated subposet, $\mathcal{E}$ exact. There is a forgetful functor $r : \mathcal{P}(\mathcal{E}) \to \mathcal{S}(\mathcal{E})$. It has a left adjoint $l : \mathcal{S}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$ which sends G to lG with

$$lG(x) = colim_{\{y \leq x,\ y \in \mathcal{S}\}} G(y).$$

The only thing to verify is that lG is really an object of $\mathcal{P}(\mathcal{E})$, i.e. that for $x \in \mathcal{P}$

$$colim_{y<x} lG(y) \to lG(x)$$

is an admissible monomorphism. The right hand term is a colimit over $\{z \leq x,\ z \in \mathcal{S}\}$, the left hand term is a colimit induced by the map of posets

$$\{(y, z) \mid y < x,\ y \in \mathcal{P},\ z \leq y,\ z \in \mathcal{S}\} \to \{z < x,\ z \in \mathcal{S}\} :\ (y, z) \mapsto z$$

which satisfies the hypothesis of 6.11, the minimal element of the fiber over z being (z, z). Hence the left hand term is in fact a colimit over $\{z < x,\ z \in \mathcal{S}\}$. Since G is an object of $\mathcal{S}(\mathcal{E})$, the map claimed to be an admissible monomorphism really is one. So lG is an object of $\mathcal{P}(\mathcal{E})$ and l is well defined.

We obviously have $r \circ l = id$. By the adjointness of r and l, it follows that l makes $\mathcal{S}(\mathcal{E})$ into a full subcategory of $\mathcal{P}(\mathcal{E})$. l is exact by 6.9.

6.13 Lemma. *Let $\mathcal{P}$ be a locally finite poset, $\mathcal{S}$ a saturated subposet, and let $\mathcal{E}$ be an exact category. Then the left adjoint $l : \mathcal{S}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$ to the forgetful functor r makes $\mathcal{S}(\mathcal{E})$ into a full subcategory of $\mathcal{P}(\mathcal{E})$*

i) closed under extensions, admissible subobjects and admissible quotients, and such that

ii) the adjunction map $l \circ r(G) \to G$ is an admissible monomorphism in $\mathcal{P}(\mathcal{E})$.

Proof. We already know that l makes $\mathcal{S}(\mathcal{E})$ into a full subcategory of $\mathcal{P}(\mathcal{E})$. Let $0 \to F \to G \to H \to 0$ exact in $\mathcal{P}(\mathcal{E})$ and $x \in \mathcal{P}$. $L \in \mathcal{P}(\mathcal{E})$ is isomorph to an object of $\mathcal{S}(\mathcal{E})$ iff $lrL \to L$ is an isomorphism. By lemma 6.9 we have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & colim_{\{t\in\mathcal{S},\ t\leq x\}}F(t) \arrow[r] \arrow[d, "f"] & colim_{\{t\in\mathcal{S},\ t\leq x\}}G(t) \arrow[r] \arrow[d, "g"] & colim_{\{t\in\mathcal{S},\ t\leq x\}}H(t) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & F(x) \arrow[r] & G(x) \arrow[r] & H(x) \arrow[r] & 0
\end{tikzcd}$$

in which the first horizontal line is nothing else than $0 \to lrF(x) \to lrG(x) \to lrH(x) \to 0$. The vertical maps are admissible monomorphisms, so by the snake lemma g is an isomorphism iff f and h are. This shows i).

The morphism $lrG \to G$ is pointwise an admissible monomorphism, so we can define $H : \mathcal{P} \to \mathcal{E}$ by $H(x) = G(x)/lrG(x)$. We have to verify that $H \in \mathcal{P}(\mathcal{E})$, in other words that for $x \in \mathcal{P}$ the map $colim_{y<x}H(y) \to H(x)$ is an admissible monomorphism, i.e.

$$\begin{tikzcd}
colim_{y<x}lrG(y) \arrow[r] \arrow[d] & colim_{y<x}G(y) \arrow[d] \\
lrG(x) \arrow[r] & G(x)
\end{tikzcd}$$

is an inclusion square. The upper left corner is $colim_{y<x}colim_{\{t\in\mathcal{S},\ t\leq y\}}G(t)$, it is a colimit over the poset $\{(y,t) \mid t \leq y < x,\ t \in \mathcal{S}\}$ which by $(y,t) \mapsto t$ surjects onto $\{t \in \mathcal{S} |\ t < x\}$. This surjection verifies the hypothesis of lemma 6.11, hence the upper left corner is a colimit over $\{t \in \mathcal{S} |\ t < x\}$ which is the intersection of the indexing posets of the colimits in the upper right corner and the lower left corner. The union of the latter two indexing sets is contained in $\{t \leq x\}$, the indexing set of the lower right corner. By 6.5 and 6.6, the square is an inclusion square. $\square$

6.14 Corollary. *$l : \mathcal{S}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$ is strictly left s-filtering.* $\square$

We endow $I\!N$ with the usual structure of a (locally finite) poset given by the natural order, and we give $I\!N^k$, $k = 1, 2, ..., \infty$, the structure of a product poset. The poset $I\!N^\infty$ is the countable sum of $I\!N$.

6.15 Lemma. *Let $\mathcal{P} \subset I\!N^k$ be a saturated subposet, $\mathcal{E}$ an exact category, and let $\mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$ be the isomorphism closure of $\mathcal{P}(\mathcal{E})$ in $I\!N^k(\mathcal{E})$, i.e. the full subcategory of $I\!N^k(\mathcal{E})$ of objects G such that $lr(G) \to G$ is an isomorphism. Then for $z \in I\!N^k$, $G \in \mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$, the object $z.G$ defined by $z.G(x) = G(x+z)$ is an object of $\mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$. For $z \le z'$ we have a natural transformation $z.G => z'.G$ which in $x \in I\!N^k$ is $G(x+z) \to G(x+z')$ induced by $x+z \le x+z'$. This induces a 2-action μ of the abelian monoid $I\!N^k$ on $\mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$.*

Proof. The abelian monoid obviously 2-acts on $I\!N^k(\mathcal{E})$ in the way described in the lemma. So we only have to check that it restricts to $\mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$, i.e. that for $x \in I\!N^k$ we have an isomorphism

$$g : colim_{\{y \le x,\ y \in \mathcal{P}\}} z.G(y) \to z.G(x).$$

The left hand colimit is a colimit induced by the map of posets

$$f : \{(t,y) |\ y, t \in \mathcal{P},\ y \le x,\ t \le y+z\} \to \{t \in \mathcal{P} |\ t \le x+z\} :\ (t,y) \mapsto t$$

whose target is the right hand indexing set. The minimal element of $f^{-1}(t)$ is $(t, (t-z)_+)$, the i-th coordinate of $(t-z)_+$ being $max(t_i - z_i, 0)$. By lemma 6.11 g is an isomorphism. $\square$

6.16 Definition. We set $\mathcal{C}_{\mathcal{P}}(\mathcal{E}) = \mu^{-1}\mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$. Its structure as an exact category has been defined in 6.2.

Since the 2-action μ fixes the category $\mathcal{C}^0_{\mathcal{P}}(\mathcal{E})$ for every saturated $\mathcal{P} \subset I\!N^k$, we have full inclusions $l_{\mathcal{P}\subset I\!N^k}\mathcal{C}_{\mathcal{P}}(\mathcal{E}) \to \mathcal{C}_{I\!N^k}(\mathcal{E})$. Henceforth we consider $\mathcal{C}_{\mathcal{P}}(\mathcal{E})$ as included in $\mathcal{C}_{I\!N^k}(\mathcal{E})$. The functors $l_{\mathcal{P}\subset I\!N^k}$ are then identities. The same applies to the inclusions $\mathcal{C}^0_{\mathcal{P}}(\mathcal{E}) \subset \mathcal{C}^0_{I\!N^k}(\mathcal{E})$, and we write $r_{\mathcal{P}}$ for the right adjoint of the last inclusion.

6.17 Lemma. *For $\mathcal{S} \subset \mathcal{P} \subset I\!N^k$ the full inclusion $\mathcal{C}_{\mathcal{S}}(\mathcal{E}) \to \mathcal{C}_{\mathcal{P}}(\mathcal{E})$ is strictly left s-filtering.*

Proof. We claim that a $\mathcal{C}^0_{I\!N^k}(\mathcal{E})$-map $\varphi : G \to H$ with $G \in \mathcal{C}^0_{\mathcal{S}}(\mathcal{E})$ which is a $\mathcal{C}_{I\!N^k}(\mathcal{E})$-isomorphism has necessarily its target $H \in \mathcal{C}^0_{\mathcal{S}}(\mathcal{E})$. The claim together with corollary 6.14 proves the lemma. So we will prove the claim. The map φ is a $\mathcal{C}_{I\!N^k}(\mathcal{E})$-isomorphism if there is an inverse $\mathcal{C}^0_{I\!N^k}(\mathcal{E})$-map $\psi : H \to z.G$

fitting into a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "z_G"'] & H \arrow[d, "z_H"] \arrow[dl, "\psi"] \\
z.G \arrow[r, "z.\varphi"'] & z.H.
\end{tikzcd}$$

We have to show that the adjunction map $r_{\mathcal{S}}H \to H$ is a $\mathcal{C}^0_{\mathbb{N}^k}(\mathcal{E})$-isomorphism. By lemma 6.13 ii) this is an admissible monomorphism. So we will show that its quotient $L = H/r_{\mathcal{S}}H$ is zero. Since G is an element of $\mathcal{C}^0_{\mathcal{S}}(\mathcal{E})$, the map $\varphi : G \to H$ factors through $r_{\mathcal{S}}H$ as $r_{\mathcal{S}}(\varphi)$ composed with the adjunction map $adj : r_{\mathcal{S}}H \to H$. Applying $z.$ to this factorization, composing with ψ and the adjunction map yields a commutative diagram

$$\begin{tikzcd}
r_{\mathcal{S}}H \arrow[rr, "z_{r_{\mathcal{S}}H}"] \arrow[d, "adj"'] & & z.r_{\mathcal{S}}H \arrow[d, "z.(adj)"] \\
H \arrow[r, "\psi"'] & z.G \arrow[r, "z.\varphi"'] \arrow[ur, "z.r_{\mathcal{S}}(\varphi)"] & z.H.
\end{tikzcd}$$

The lower horizontal map is $z.\varphi \circ \psi = z_H$, so the upper horizontal map is $z_{r_{\mathcal{S}}H}$ as indicated, since all maps of the outer square are monics. The diagram shows that the map z_H lifts to $z.r_{\mathcal{S}}H$. It follows that the quotient map $z_L : L \to z.L$ is zero. But z_L is always monic because evaluated in $x \in \mathbb{N}^k$ it is an admissible monomorphism of $\mathcal{E}$, hence $L = 0$ and $H \in \mathcal{C}^0_{\mathcal{S}}(\mathcal{E})$. $\square$

6.18 Proposition. *Given saturated subposets $\mathcal{S}, \mathcal{T} \subset \mathbb{N}^k$, all inclusion functors in the commutative square*

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{C}_{\mathcal{S}}(\mathcal{E}) \arrow[d] \\
\mathcal{C}_{\mathcal{T}}(\mathcal{E}) \arrow[r] & \mathcal{C}_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})
\end{tikzcd}$$

are strictly left s-filtering. Furthermore, the induced map on the quotients

$$l : \mathcal{C}_{\mathcal{S}}(\mathcal{E})/\mathcal{C}_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E}) \longrightarrow \mathcal{C}_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})/\mathcal{C}_{\mathcal{T}}(\mathcal{E})$$

is an equivalence of exact categories.

Proof. The inclusion functors are strictly left s-filtering by lemma 6.17. We claim that the map $r_{\mathcal{S}} : \mathcal{C}^0_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E}) \to \mathcal{C}^0_{\mathcal{S}}(\mathcal{E})$, which is right adjoint to the inclusion $\mathcal{C}^0_{\mathcal{S}}(\mathcal{E}) \subset \mathcal{C}^0_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})$, induces exact functors r_1, r_2 as in the diagram

$$\begin{tikzcd}
\mathcal{C}^0_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E}) \arrow[r] \arrow[d, "r_{\mathcal{S}}"'] & \mathcal{C}_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E}) \arrow[r] \arrow[dr, dashed, "r_1"'] & \mathcal{C}_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})/\mathcal{C}_{\mathcal{T}}(\mathcal{E}) \arrow[d, dashed, "r_2"] \\
\mathcal{C}^0_{\mathcal{S}}(\mathcal{E}) \arrow[r] & \mathcal{C}_{\mathcal{S}}(\mathcal{E}) \arrow[r] & \mathcal{C}_{\mathcal{S}}(\mathcal{E})/\mathcal{C}_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E})
\end{tikzcd}$$

such that r_2 is inverse to l. We show that r_1 is well defined. By remark 6.3 and the universal property of localizations, we only have to check that for $G \in \mathcal{C}^0_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})$ the map $r_{\mathcal{S}}(z_G) : r_{\mathcal{S}}G \to r_{\mathcal{S}}z.G$ is an isomorphism in $\mathcal{C}_{\mathcal{S}}(\mathcal{E})/\mathcal{C}_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E})$. The map $r_{\mathcal{S}}(z_G)$ fits into the commutative square

$$\begin{tikzcd} r_{\mathcal{S}}G \arrow[r, "r_{\mathcal{S}}(z_G)"] \arrow[d, "z_{r_{\mathcal{S}}G}"'] & r_{\mathcal{S}}z.G \arrow[d, "adj_{z.G}"] \\ z.r_{\mathcal{S}}G \arrow[r, "z.adj_G"'] \arrow[ur, dashed, "\rho_G"] & z.G. \end{tikzcd}$$

The map ρ_G is the unique map making the lower triangle commute. It exists by adjointness and the fact that $z.r_{\mathcal{S}}G$ is an object of $\mathcal{C}^0_{\mathcal{S}}(\mathcal{E})$. It is unique because $adj_{z.G}$ is an admissible monomorphism, hence a monic. The upper left triangle then commutes again because $adj_{z.G}$ is a monic. By definition $z_{r_{\mathcal{S}}G}$ is an isomorphism in $\mathcal{C}_{\mathcal{S}}(\mathcal{E})$, hence in $\mathcal{C}_{\mathcal{S}}(\mathcal{E})/\mathcal{C}_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E})$. So we are left with proving that ρ_G is an isomorphism in the latter quotient category. The map ρ_G is an admissible monomorphism (at least in the idempotent completion) since composed with $adj_{z.G}$ it is the admissible monomorphism $z.adj_G$. It therefore suffices to show that its quotient lies in the (idempotent completion) of $\mathcal{C}^0_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E})$. This will follow once we see that the diagram

$$\begin{tikzcd} r_{\mathcal{S}\cap\mathcal{T}}(z.r_{\mathcal{S}}G) \arrow[r, "r_{\mathcal{S}}(\rho_G)"] \arrow[d, tail, "adj"'] & r_{\mathcal{S}\cap\mathcal{T}}(r_{\mathcal{S}}z.G) \arrow[d, tail, "adj"'] \\ z.r_{\mathcal{S}}G \arrow[r, "\rho_G"] & r_{\mathcal{S}}z.G \end{tikzcd}$$

is (pointwise) cocartesian, since then the cokernel of ρ_G is isomorph to the cokernel of $r_{\mathcal{S}}(\rho_G)$ which certainly lies in (the idempotent completion) of $\mathcal{C}^0_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E})$. The last square evaluated in $x \in \mathbb{N}^k$ is equivalent to the square

$$\begin{tikzcd} colim_{t\in\mathcal{A}}G(t) \arrow[r] \arrow[d, tail] & colim_{t\in\mathcal{C}}G(t) \arrow[d, tail] \\ colim_{t\in\mathcal{B}}G(t) \arrow[r] & colim_{t\in\mathcal{D}}G(t) \end{tikzcd}$$

with

$\mathcal{A} = \mathcal{S} \cap (\mathcal{S}\cap\mathcal{T} + z)_{sat} \cap \{x+z\}_{sat}$ (after identification using 6.11),

$\mathcal{B} = \mathcal{S} \cap \{x+z\}_{sat}$,

$\mathcal{C} = (\mathcal{S}\cap\mathcal{T} + z)_{sat} \cap \{x+z\}_{sat}$ and

$\mathcal{D} = (\mathcal{S} + z)_{sat} \cap \{x+z\}_{sat}$.

Here $\mathcal{M}_{sat}$ is the saturated subposet of $\mathbb{N}^k$ generated by $\mathcal{M}$, i.e. the set of all elements $t \in \mathbb{N}^k$ such that there is a $m \in \mathcal{M}$ with $t \leq m$. Since G is an element of $\mathcal{C}^0_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})$, we can replace $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$

by $\mathcal{A} \cap (\mathcal{S} \cup \mathcal{T})$, $\mathcal{B} \cap (\mathcal{S} \cup \mathcal{T})$ etc. We have $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$ and $\mathcal{D} \cap (\mathcal{S} \cup \mathcal{T}) = \mathcal{B} \cup (\mathcal{C} \cap (\mathcal{S} \cup \mathcal{T}))$. By 6.6 the square is then cocartesian. Hence r_1 is well defined.

r_2 is well defined since $r_{\mathcal{S}}$ is exact and sends $\mathcal{C}^0_{\mathcal{T}}(\mathcal{E})$ into $\mathcal{C}^0_{\mathcal{S}\cap\mathcal{T}}(\mathcal{E})$.

r_2 is inverse to l since $r_2 \circ l = id$ and because the adjunction map $r_{\mathcal{S}}G \to G$ is an isomorphism in $\mathcal{C}_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})/\mathcal{C}_{\mathcal{T}}(\mathcal{E})$. It is an admissible monomorphism by lemma 6.13. Its cokernel lies in $\mathcal{C}^0_{\mathcal{T}}(\mathcal{E})$ because the diagram

$$\begin{tikzcd} r_{\mathcal{T}} r_{\mathcal{S}} G \arrow[r, tail] \arrow[d, tail] & r_{\mathcal{T}} G \arrow[d, tail] \\ r_{\mathcal{S}} G \arrow[r, tail] & G \end{tikzcd}$$

is cocartesian for $G \in \mathcal{C}^0_{\mathcal{S}\cup\mathcal{T}}(\mathcal{E})$ by 6.6. □

7 The functor $I\!K : Excat \to Sp^{\Sigma}$

In this chapter we will define a functor $I\!K : Excat \to Sp^{\Sigma}$ such that $\pi_i(I\!K(\mathcal{E})) = I\!K_i(\mathcal{E})$ are the Quillen K-groups for $i > 0$ and the negative K-groups as defined in section 5 by means of flasque resolutions for $i < 0$. Here the category Sp^{Σ} is the category of symmetric spectra as defined in [HSS]. Recall that this category has the same homotopy category as the usual category of spectra, but the advantage of having a strictly associative symmetric monoidal smash product. Our functor is compatible with products in the sense that a biexact pairing $\mathcal{A} \times \mathcal{B} \to \mathcal{E}$ induces a map of symmetric spectra $I\!K(\mathcal{A}) \wedge I\!K(\mathcal{B}) \to I\!K(\mathcal{E})$. If R is a ring then we define a symmetric spectrum $I\!K(R)$ stably equivalent to $I\!K(\mathcal{P}(R))$, $\mathcal{P}(R)$ being the split exact category of finitely generated projective modules, in a way that $I\!K(R)$ is a commutative ring spectrum whenever R is commutative. Moreover, every of the above spectrum will be a module over $I\!K(Z\!\!\!Z)$. Finally, we will show that applying $I\!K$ to the sequence of exact categories $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ in which $\mathcal{A}$ is left or right s-filtering in $\mathcal{U}$ gives a homotopy fibration of (non-connective) symmetric spectra extending proposition 5.1. At the end of the section we construct an exact category $C(f)$ for any exact functor f between exact categories such that its $I\!K$-theory spectrum is the homotopy cofiber of $I\!K(f)$.

For $\mathcal{E}$ an exact category, we chose a zero object $* \in \mathcal{E}$ called base point.

7.1 For $s \in I\!N$, $s > 0$ there is a functor L_s from exact categories to exact categories defined as follows. If $\mathcal{A}$ is an exact category, an object of $L_s\mathcal{A}$ is given by an s-tuple $l = (l_1, ..., l_s) \in I\!N^s$ and for every $0 \le i \le l$, $i \in I\!N^s$ an object $A_i \in \mathcal{A}$. Abbreviating we write $l(A_i)$ for this object. A morphism from $l(A_i)$ to $m(B_j)$ is given by a morphism $\phi_{j,i} : A_i \to B_j$ of $\mathcal{A}$ for every $0 \le (i,j) \le (l,m)$. Composition of morphisms is the usual composition of matrices, i.e. given $\phi = \{\phi_{j,i}\} : l(A_i) \to m(B_j)$ and $\psi = \{\psi_{k,j}\} : m(B_j) \to n(C_k)$, their composition is $(\psi \circ \phi)_{k,i} = \Sigma_{0 \le j \le m}\ \psi_{k,j} \circ \phi_{j,i}$. Moreover, we identify all objects $l(A_i)$ such that $A_i = *$ for all $0 \le i \le l$ with the basepoint of $L_s\mathcal{A}$. The categories $L_s\mathcal{A}$ are all equivalent to $\mathcal{A}$ and carry therefore a natural structure of an exact category induced by $\mathcal{A}$. We let $L_0\mathcal{A} = \mathcal{A}$. If we let $\mathcal{A}$ to be the two-object category, one object the basepoint, the other object having endomorphism ring R, then the definition of $L_s\mathcal{A}$ still makes sense and we write L_sR instead of $L_s\mathcal{A}$. The category L_sR is equivalent to the category of finitely generated free R-modules for $s > 0$ and L_sR is the two-object category above. The symmetric group Σ_n acts on $L_n\mathcal{A}$ by interchanging the coordinates. A biexact paring $\wedge : \mathcal{A} \times \mathcal{B} \to \mathcal{E}$ gives raise to a biexact $\Sigma_m \times \Sigma_n$-equivariant pairing

$$L_m(\mathcal{A}) \times L_n(\mathcal{B}) \longrightarrow L_{m+n}(\mathcal{E})$$

$$m(X_i) \wedge n(Y_j) \mapsto (m,n)(i,j \mapsto X_i \wedge Y_j).$$

7.2 We first define functors $\mathcal{C}_p : Excat \to Excat$ giving a p-fold non-connective delooping $K(\mathcal{C}_p\mathcal{E})$ of $K(\mathcal{E})$ analogue to the constructions of Pedersen and Weibel in [PW85]. We endow $I\!N$ with the usual structure of a (locally finite) poset given by the natural order. Elements of $\mathcal{N}_2 := I\!N \times I\!N$ are written as (a^+, a^-). Let $\mathcal{Z}$ be the saturated subposet of $\mathcal{N}_2$ whose elements satisfy $a^+ = 0$ or $a^- = 0$. The map $\mathcal{Z} \to Z\!\!Z :\ (a^+, a^-) \mapsto a^+ - a^-$ is a bijection which is not order preserving.

7.3 Definition. Let $\mathcal{E}$ be an exact category, we set

$$\mathcal{C}_p(\mathcal{E}) = \mathcal{C}_{\mathcal{Z}^p}(\mathcal{E})$$

(see section 6).

By the results of section 6, $\mathcal{C}_p$ is a functor from exact categories to exact categories. The symmetric group Σ_p acts on $\mathcal{C}_p(\mathcal{E})$ by the action on the right of $\mathcal{Z}^p$ interchanging its factors. A biexact pairing of exact categories $\wedge : \mathcal{A} \times \mathcal{B} \to \mathcal{E}$ induces a biexact pairing

$$\wedge : \mathcal{C}_p(\mathcal{A}) \times \mathcal{C}_q(\mathcal{B}) \longrightarrow \mathcal{C}_{p+q}(\mathcal{E})$$

by defining $(G \wedge H)(x, y) = G(x) \wedge G(y)$ for $x \in {\mathcal{N}_2}^p$ and $y \in {\mathcal{N}_2}^q$. It is not difficult to verify that this is well defined. Moreover, the last biexact pairing is $\Sigma_p \times \Sigma_q$ invariant.

7.4 Lemma. *All inclusion functors in the commutative square*

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{Z}^p}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{C}_{\{0\}\times I\!N\times\mathcal{Z}^p}(\mathcal{E}) \arrow[d] \\
\mathcal{C}_{I\!N\times\{0\}\times\mathcal{Z}^p}(\mathcal{E}) \arrow[r] & \mathcal{C}_{\mathcal{Z}^{p+1}}(\mathcal{E})
\end{tikzcd}$$

are strictly left s-filtering. Furthermore, the induced functors on the quotient categories are exact equivalence. The categories $\mathcal{C}_{\{0\}\times I\!N\times\mathcal{Z}^p}(\mathcal{E})$ and $\mathcal{C}_{I\!N\times\{0\}\mathcal{Z}^p}(\mathcal{E})$ are flasques. In particular the K-theory of $\mathcal{C}_p(\mathcal{E})$ is a p-fold delooping of the K-theory of $\tilde{\mathcal{E}}$. Moreover, there are maps $\mathcal{C}_p(\mathcal{E}) \to S_I^p(\mathcal{E})$ functoriel in $\mathcal{E}$ such that for all $n \in I\!N$ the map $S^n\mathcal{C}_p(\mathcal{E}) \to S^n S_I^p(\mathcal{E})$ is a K-theory equivalence.

Proof. The statement about the filtering properties and the equivalences of the quotients is proposition 6.18. The categories $\mathcal{C}_{\{0\}\times I\!N\mathcal{Z}^p}(\mathcal{E})$ and $\mathcal{C}_{I\!N\times\{0\}\mathcal{Z}^p}(\mathcal{E})$ are flasque by the usual Eilenberg swindle along the $I\!N$ component as in lemma 4.6. The delooping result follows by proposition 5.1.

For the last statement it suffices to show that there is a natural map $\mathcal{C}_{p+1}(\mathcal{E}) \to S_I\mathcal{C}_p(\mathcal{E})$ such that $S^n\mathcal{C}_{p+1}(\mathcal{E}) \to S^n S_I \mathcal{C}_p(\mathcal{E})$ is a K-theory equivalence for all $n \in I\!N$. The claim then follows by induction on p. First of all, the first part of the lemma and lemma 5.5 prove that the natural map

$\mathcal{C}_{p+1}(\mathcal{E}) \to \mathcal{C}_{p+1}(\mathcal{E})/\mathcal{C}_{\mathcal{Z}^p\times 0\times I\!N} \simeq \mathcal{C}_{\mathcal{Z}^p\times I\!N\times 0}(\mathcal{E})/\mathcal{C}_{\mathcal{Z}^p}(\mathcal{E})$ induce K-theory equivalences of their n-fold suspensions.

Second, the 2-action μ of $I\!N^{2p} \times I\!N$ on $\mathcal{C}^0_{\mathcal{Z}^p\times I\!N}(\mathcal{E})$ restricts to $\mu_{|I\!N^{2p}\times 0}$ and to $\mu_{|0\times I\!N}$. The action $\mu_{|0\times I\!N}$ induces an action on $\mu^{-1}_{|I\!N^{2p}\times 0}\mathcal{C}^0_{\mathcal{Z}^p\times I\!N}(\mathcal{E})$ and we have $\mathcal{C}_{\mathcal{Z}^p\times I\!N}(\mathcal{E}) = \mu^{-1}_{|0\times I\!N}\mu^{-1}_{|I\!N^{2p}\times 0}\mathcal{C}^0_{\mathcal{Z}^p\times I\!N}(\mathcal{E})$. Having this in mind, the functor $\mathcal{C}^0_{\mathcal{Z}^p\times I\!N}(\mathcal{E}) = \mathcal{F}^0_I\mathcal{C}^0_{\mathcal{Z}^p}(\mathcal{E}) \to \mathcal{F}^0_I\mathcal{C}_{\mathcal{Z}^p}(\mathcal{E})$ induces the following functor $\mu^{-1}_{|I\!N^{2p}\times 0}\mathcal{C}^0_{\mathcal{Z}^p\times I\!N}(\mathcal{E}) \to \mathcal{F}^0_I\mathcal{C}_{\mathcal{Z}^p}(\mathcal{E})$. Composing it with the localization map $\mathcal{F}^0_I \to \mathcal{F}_I$ induces the functor $\mathcal{C}_{\mathcal{Z}^p\times I\!N}(\mathcal{E}) \to \mathcal{F}_I\mathcal{C}_{\mathcal{Z}^p}(\mathcal{E})$. We have a commutative diagram of exact categories

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{Z}^p}\mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{C}_{\mathcal{Z}^p\times I\!N}(\mathcal{E}) \arrow[r] \arrow[d] & \mathcal{C}_{\mathcal{Z}^p\times I\!N}(\mathcal{E})/\mathcal{C}_{\mathcal{Z}^p}\mathcal{E} \arrow[d] \\
\mathcal{C}_p\mathcal{E} \arrow[r] & \mathcal{F}_I\mathcal{C}_p\mathcal{E} \arrow[r] & S_I\mathcal{C}_p\mathcal{E}
\end{tikzcd}$$

in which the left horizontal functors are left s-filtering and the right vertical functors are the corresponding quotient maps. By the flasqueness of the middle terms and lemma 5.5, the functor between the quotients induce a K-theory equivalence of their n-fold suspensions for all $n \in I\!N$. Combined with the first result above, this proves the statement. $\square$

7.5 The functor $I\!K$ to be constructed also uses Gillet and Grayson's G-construction ([GG87]) so recall the definitions and results needed later on. The following notations are taken from [K$\ddot{9}$4]. Let Ord be the category of totally ordered finite sets and let A be an object in Ord. Then $\gamma(A)$ is defined to be the disjoint union $\{L, R\} \sqcup A$, being ordered in such a way that A is an ordered subset of $\gamma(A)$ and that $L < a$ and $R < a$ for all $a \in A$. The elements L and R are not comparable. Let $\Gamma(A)$ be the category of arrows in $\gamma(A)$, an object being represented by $\frac{i}{j}$ with $j \leq i$. And call a sequence in $\gamma(A)$ exact if it has the form $\frac{i}{j} \to \frac{k}{j} \to \frac{k}{i}$. For an exact category $\mathcal{A}$ with chosen zero object $* = 0$ let $G^k(\mathcal{A})$ be the simplicial set with

$$G^k(\mathcal{A})(A) = Exact(\Gamma(A)^k,\ \mathcal{A})$$

where a functor is said to be exact if it sends objects of the form $(\frac{i_1}{j_1}, ..., \frac{i_k}{j_k})$ to $*$ whenever $i_l = j_l$ for some $1 \leq l \leq k$ and exact sequences to exact sequences for each variable separately. $G^k(\mathcal{A})$ is naturally homotopy equivalent to the loop space of Quillen's Q-construction for any $k > 0$ (see [GG87] and [K$\ddot{9}$4]). Remark that- as a limit over the empty set- $\Gamma(A)^0 = *$, the final object in cat. And therefore $G^0(\mathcal{A})(A)$ is the constant simplicial set $Ob\mathcal{A}$. We let Σ_n act on G^n by permuting the factors of $\Gamma(A)^n$ from the right. A biexact functor $\epsilon : \mathcal{A} \times \mathcal{B} \to \mathcal{E}$ gives raise to $\Sigma_m \times \Sigma_n$ equivariant maps

$$G^k(\mathcal{A}) \wedge G^l(\mathcal{B}) \to G^{k+l}(\mathcal{E})$$

by

$$A \wedge B \mapsto A \otimes B := [(\frac{i_1}{j_1}, ..., \frac{i_{m+n}}{j_{m+n}}) \mapsto \epsilon(\ A(\frac{i_1}{j_1}, ..., \frac{i_m}{j_m})\ ,\ B(\frac{i_{m+1}}{j_{m+1}}, ..., \frac{i_{m+n}}{j_{m+n}})\ )].$$

7.6 Definition. For an exact category $\mathcal{A}$ with base point $*$ we define a symmetric sequence $\mathbb{K}(\mathcal{A})$ by

$$\mathbb{K}(\mathcal{A})_n = G^n(\mathcal{C}_n\widetilde{(L_n}\mathcal{A})), \quad n \geq 0, \quad \text{and}$$

$$\mathbb{K}(\mathcal{A})_0 = Ob\mathcal{A}.$$

For R a ring we set

$$\mathbb{K}(R)_n = G^n(\mathcal{C}_n\widetilde{(L_n}R)), \quad n \geq 0, \quad \text{and}$$

$$\mathbb{K}(\mathcal{A})_0 = \{*, R\},$$

the two point set. We let Σ_n act on $\mathbb{K}(\mathcal{A})_n$ and on $\mathbb{K}(R)_n$ by simultaneously acting on L_n, $\mathcal{C}_n$ and G^n as described in 7.1, 7.2 and 7.5.

7.7 Lemma. *A biexact pairing $\wedge : \mathcal{A} \times \mathcal{B} \to \mathcal{E}$ induces by 7.1, 7.2 and 7.5 a pairing of symmetric sequences as defined in [HSS], i.e. $\Sigma_p \times \Sigma_q$-equivariant maps of simplicial sets*

$$\mathbb{K}(\mathcal{A})_p \wedge \mathbb{K}(\mathcal{B})_q \to \mathbb{K}(\mathcal{E})_{p+q}.$$

In particular there are $\Sigma_p \times \Sigma_q$-equivariant maps

$$\mathbb{K}(\mathbb{Z})_p \wedge \mathbb{K}(\mathcal{A})_q \to \mathbb{K}(\mathcal{A})_{p+q}$$

natural in $\mathcal{A}$ for exact functors. Furthermore, if R is a commutative ring, $\mathbb{K}(R)$ is a commutative monoid in the category of symmetric sequences. $\square$

7.8 We write $\mathcal{L}\mathcal{A}$ instead of $L_1\mathcal{A}$ (7.1). There is an additive functor $\gamma : \mathcal{L}(\mathbb{Z}[t, t^{-1}]) \to S_I\mathcal{L}(\mathbb{Z})$ determined by sending the object $\mathbb{Z}[t, t^{-1}]$ to $Z = (\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}t \subset \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \subset ...\ )$ and the multiplication by $t : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]$ to

$$\begin{tikzcd}
Z \arrow[d] & (\mathbb{Z} \arrow[r, tail] \arrow[d, "\times t"] & \mathbb{Z} \oplus \mathbb{Z}t \arrow[r, tail] \arrow[d, "\times t"] & \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \arrow[r, tail] \arrow[d, "\times t"] & ...\ ) \\
m_1^*Z & (\mathbb{Z} \oplus \mathbb{Z}t \arrow[r, tail] & \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \arrow[r, tail] & \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \oplus \mathbb{Z}t^3 \arrow[r, tail] & ...\ )
\end{tikzcd}$$

which is an isomorphism in $S_I\mathcal{L}(\mathbb{Z})$ since $Z \to m_1^*Z$ is an admissible monomorphism in $\mathcal{F}_I\mathcal{L}(\mathbb{Z})$ with cokernel the constant diagram $c(\mathbb{Z})$. The automorphism "multiplication by t" in $\mathbb{Z}[t, t^{-1}]$ yields an element in $K_1(\mathbb{Z}[t,t^{-1}])$ which corresponds to a map $\times t : S^1 \to G^1(\mathcal{L}(\mathbb{Z}[t,t^{-1}]))$. Now choose a map $\eta : S^1 \to I\!K(\mathbb{Z})_1$ making the diagram

$$\begin{tikzcd} S^1 \arrow[r, "\eta"] \arrow[d, "\times t"'] & I\!K(\mathbb{Z})_1 \arrow[d, "\wr"] \\ G^1(\mathcal{L}(\mathbb{Z}[t,t^{-1}])) \arrow[r, "\gamma"'] & G^1(S_I\mathcal{L}(\mathbb{Z})) \end{tikzcd}$$

commute up to homotopy. This is possible since the projection map $\widetilde{\mathcal{C}_1\mathcal{L}\mathbb{Z}} \to \widetilde{S_I\mathcal{L}(\mathbb{Z})}$ identifying $S_I\mathcal{L}(\mathbb{Z})$ with $(\mathcal{C}_{\mathbb{Z}}\mathcal{L}\mathbb{Z})/(\mathcal{C}_{I\!N\times 0}\mathcal{L}\mathbb{Z})$ is a K-theory equivalence by 5.1, 5.5 and 7.4. The choice of η induces a map of commutative monoids of symmetric sequences $\eta : S \to I\!K(\mathbb{Z})$ which makes $I\!K(\mathcal{A})$ into a symmetric spectrum as a module over the commutative symmetric ring spectrum $I\!K(\mathbb{Z})$.

7.9 Proposition. *For $\mathcal{E}$ an exact category, $I\!K(\mathcal{E})$ is an Ω-spectrum beyond the 0-th space.*

Proof. The corresponding statement for rings has been proven by Loday in [Lod76] and Karoubi in [Kar80b]. We have to show that the adjoint of $S^1 \wedge K(\widetilde{\mathcal{C}_p\mathcal{E}}) \xrightarrow{\eta\wedge id} K(\widetilde{\mathcal{C}_1\mathbb{Z}}) \wedge K(\widetilde{\mathcal{C}_p\mathcal{E}}) \xrightarrow{\wedge} K(\widetilde{\mathcal{C}_{p+1}\mathcal{E}})$ is a homotopy equivalence. Since we have natural homotopy equivalences (by 5.1, 5.5 and 7.4) $K(\widetilde{\mathcal{C}_p\mathcal{E}}) \simeq (\widetilde{S^p\mathcal{E}})$ it suffices to show that the adjoint of $S^1 \wedge G^1(\mathcal{E}) \xrightarrow{\eta\wedge id} G^1(S(\mathbb{Z})) \wedge G^1(\mathcal{E}) \xrightarrow{\wedge} G^2(S\mathcal{E})$ is a homotopy equivalence if $\mathcal{E}$ is idempotent complete.

Since $\mathcal{F}\mathcal{L}\mathbb{Z}$ is flasque, the functor S of the proof of lemma 4.6 and the natural equivalence $\varepsilon : 1 \oplus S \simeq S$ induce a contraction $h : I \wedge G^1(\mathcal{F}\mathcal{L}\mathbb{Z}) \to G^1(\mathcal{F}\mathcal{L}\mathbb{Z})$ of $G^1(\mathcal{F}\mathcal{L}\mathbb{Z})$ to its base point. Let $\rho : S^1 \to G^1(S\mathcal{L}\mathbb{Z})$ be the unique map making the diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d, "1\wedge c\mathbb{Z}"'] & I/S^0 = S^1 \arrow[r, "\rho"] & G^1(S\mathcal{L}\mathbb{Z}) \\ I \wedge G^1(\mathcal{F}\mathcal{L}\mathbb{Z}) \arrow[rr, "h"] & & G^1(\mathcal{F}\mathcal{L}\mathbb{Z}) \arrow[u] \end{tikzcd} \tag{7.10}$$

commute. Smashing 7.10 with $G^1(\mathcal{E})$ yields a commutative diagram

$$\begin{tikzcd} I \wedge G^1(\mathcal{E}) \arrow[r] \arrow[d, "1\wedge c\mathbb{Z}\wedge 1"'] & S^1 \wedge G^1(\mathcal{E}) \arrow[r, "\rho\wedge 1"] & G^1(S\mathcal{L}\mathbb{Z}) \wedge G^1(\mathcal{E}) \arrow[r, "\wedge"] & G^2(S\mathcal{E}) \\ I \wedge G^1(\mathcal{F}\mathcal{L}\mathbb{Z}) \wedge G^1(\mathcal{E}) \arrow[rr, "h\wedge 1"] & & G^1(\mathcal{F}\mathcal{L}\mathbb{Z}) \wedge G^1(\mathcal{E}). \arrow[u] & \end{tikzcd} \tag{7.11}$$

We write $H : I \wedge G^1(\mathcal{F}\mathcal{L}\mathbb{Z}) \wedge G^1(\mathcal{E}) \to G^2(S\mathcal{E})$ and $R : S^1 \wedge G^1(\mathcal{E}) \to G^2(S\mathcal{E})$ for the corresponding compositions of diagram 7.11.

There is a commutative diagram

$$\begin{tikzcd}
S^0 \wedge G(\mathcal{E}) \arrow[r, "c\mathbb{Z}\wedge id"] \arrow[d, "\mathbb{Z}\wedge id"', "\wr"] & G(\mathcal{FL}(\mathbb{Z})) \wedge G(\mathcal{E}) \arrow[r] \arrow[d, "\wedge"] & G^2(S\mathcal{E}) \arrow[d, equal] \\
G^2(\mathcal{E}) \arrow[r, "c"] & G^2(\mathcal{FE}) \arrow[r] & G^2(S\mathcal{E})
\end{tikzcd}$$

in which the the second row is a homotopy fibration by 5.1 with contractable total space. Since $G(\mathcal{FL}(\mathbb{Z}))\wedge G(\mathcal{E})$ is contractable, the left factor being so by flasqueness, all vertical maps are homotopy equivalences. It follows that the first row is a homotopy fibration.

The adjoints of H and R induce maps of homotopy fibrations

$$\begin{tikzcd}
G(\mathcal{E}) \arrow[r] \arrow[d, "adj(R)"'] & G(\mathcal{FL}(\mathbb{Z})) \wedge G(\mathcal{E}) \arrow[r] \arrow[d, "adj(H)"] & G^2(S\mathcal{E}) \arrow[d, equal] \\
\Omega G^2(S\mathcal{E}) \arrow[r] & PG^2(S\mathcal{E}) \arrow[r] & G^2(S\mathcal{E}).
\end{tikzcd}$$

The maps linking the total resp. base spaces are homotopy equivalences. Hence $adj(R)$ is a homotopy equivalence.

Therefore, we are left with proving that ρ as defined in 7.10 represents the same element as η in $K_1(S\mathcal{L}(\mathbb{Z}))$ which is the image of $[\times t]$ under $\gamma : K_1(\mathbb{Z}[t,t^{-1}]) \to K_1(S\mathcal{L}(\mathbb{Z}))$. This can be checked directly, and we omit the details. □

7.12 Theorem. *Let $\mathcal{A}$ be a left or right s-filtering subcategory of an exact category $\mathcal{U}$, then the sequence $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a homotopy fibration of symmetric spectra*

$$I\!K(\mathcal{A}) \to I\!K(\mathcal{U}) \to I\!K(\mathcal{U}/\mathcal{A}).$$

Proof. Using lemma 7.4 and lemma 5.5 we see that

$$I\!K(\mathcal{A})_n \to I\!K(\mathcal{U})_n \to I\!K(\mathcal{U}/\mathcal{A})_n$$

is a homotopy fibration of spaces for $n > 0$. Now proposition 7.9 and the definition of the closed model structure on Sp^{Σ} (see [HSS]) prove the theorem. □

7.13 Definition. For an exact category $\mathcal{E}$ we define

$$I\!K_i(\mathcal{E}) = \pi_i(I\!K(\mathcal{E})), \quad i \in \mathbb{Z}.$$

7.14 Remark. We have $I\!\!K_i(\mathcal{E}) = K_i(\mathcal{E})$ for $i > 0$, $I\!\!K_0(\mathcal{E}) = K_0(\tilde{\mathcal{E}})$ and $I\!\!K_{-i}(\mathcal{E}) = K_{-i}(\mathcal{E})$ for $i > 0$ as defined in 5.3.

7.15 Remark. It follows from theorem 7.12 that for $\mathcal{A}$ a left or right s-filtering subcategory of an exact category $\mathcal{U}$, there is a long exact sequence

$$\cdots \to I\!\!K_{i+1}(\mathcal{U}/\mathcal{A}) \to I\!\!K_i(\mathcal{A}) \to I\!\!K_i(\mathcal{U}) \to I\!\!K_i(\mathcal{U}/\mathcal{A}) \to I\!\!K_{i-1}(\mathcal{A}) \to I\!\!K_{i-1}(\mathcal{U}) \to \cdots, \ i \in \mathbb{Z}.$$

7.16 Remark. It follows also from 7.12 that we can calculate negative K-groups using any flasque resolution.

7.17 Relative $I\!\!K$-theory. In the same way as Wagoner did for rings ([Wag72]) we can construct the mapping cone of an exact functor between exact categories. Let $\mathcal{A} \to \mathcal{U}$ be strictly left s-filtering and write $g : \mathcal{U} \to \mathcal{U}/\mathcal{A}$ for the quotient map. If $f : \mathcal{L} \to \mathcal{U}/\mathcal{A}$ is an exact functor, let $\mathcal{W}$ be the category whose objects are triples (L, U, φ) with $\varphi : f(L) \xrightarrow{\sim} g(U)$ an $\mathcal{U}/\mathcal{A}$-isomorphism. A map from (L, U, φ) to (L', U', φ') is given by an $\mathcal{L}$-map $l : L \to L'$ and a $\mathcal{U}$-map $u : U \to U'$ such that $g(u)\varphi = \varphi' f(l)$. Then $\mathcal{W}$ is an exact category in which a sequence is exact iff its projections onto $\mathcal{U}$ and $\mathcal{L}$ are exact. We shall write $\mathcal{L} \times_{\mathcal{U}/\mathcal{A}} \mathcal{U}$ for $\mathcal{W}$ although it is not really the fiber product in *cat*. By construction, $\mathcal{A} \to \mathcal{W} : A \mapsto (0, A, 0) \in \mathcal{L} \times_{\mathcal{U}/\mathcal{A}} \mathcal{U}$ makes $\mathcal{A}$ into an exact subcategory of $\mathcal{W}$ and it is straight forward to verify that $\mathcal{A}$ is left s-filtering in $\mathcal{W}$ and the natural map $\mathcal{W}/\mathcal{A} \to \mathcal{L}$ is an equivalence of exact categories. Hence the commutative square of exact categories

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathcal{L} \arrow[r] & \mathcal{U}/\mathcal{A} \end{tikzcd}$$

gives rise to a homotopy cartesian square of $I\!\!K$-theory spectra by 7.12.

Given an exact functor $f : \mathcal{E} \to \mathcal{L}$, set $C(f) = S\mathcal{E} \times_{S\mathcal{L}} \mathcal{F}\mathcal{L}$ then the preceding argument identifies the commutative diagram of spectra

$$\begin{tikzcd} I\!\!K(C(f)) \arrow[r] \arrow[d] & I\!\!K(\mathcal{F}\mathcal{L}) \arrow[d] \\ I\!\!K(S\mathcal{E}) \arrow[r] & I\!\!K(S\mathcal{L}), \end{tikzcd}$$

in which the upper right corner is contractible, as homotopy cartesian. It follows that

$$I\!\!K(\mathcal{E}) \to I\!\!K(\mathcal{L}) \to I\!\!K(C(f))$$

is a homotopy fibration of non-connective spectra.

8 Additivity, filtered colimits and resolution

In this section we prove that Quillen's resolution ([Qui73] 4 corollary 1) and additivity theorems ([Qui73] 3 Corollary 1) continue to hold for negative K-theory and that negative K-theory commutes with filtered colimits.

8.1 Theorem. *(Additivity) Let $\mathcal{M}'$ and $\mathcal{M}$ be exact categories and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of exact functors from $\mathcal{M}'$ to $\mathcal{M}$. Then*

$$\mathbb{K}_i(F) = \mathbb{K}_i(F') + \mathbb{K}_i(F'') : \mathbb{K}_i(\mathcal{M}') \to \mathbb{K}_i(\mathcal{M}), \qquad i \in \mathbb{Z}.$$

Proof. For $i > 0$ this is [Qui73] 3 Corollary 1. The general case follows by observing that if $F',\ F,\ F'' : \mathcal{M}' \to \mathcal{M}$ satisfy the hypothesis of 8.1, so do $SF',\ SF,\ SF'' : S\mathcal{M}' \to S\mathcal{M}$ where "S" is one of the suspension functors defined in section 4. □

8.2 Theorem. *Given a filtered category $\mathcal{I}$ and a functor $\mathcal{C} : \mathcal{I} \to Excat$. Then the natural map $colim_{i\in\mathcal{I}}\mathbb{K}_n(\mathcal{C}_i) \to \mathbb{K}_n(colim_{i\in\mathcal{I}}\mathcal{C}_i)$ is an isomorphism for $n \in \mathbb{Z}$.*

Proof. It is immediate to verify that $colim_{i\in\mathcal{I}}\mathcal{C}_i$ is strictly s-filtering in $colim_{i\in\mathcal{I}}\mathcal{F}\mathcal{C}_i$, that the latter category is flasque and that $colim_{i\in\mathcal{I}}\mathcal{F}\mathcal{C}_i/colim_{i\in\mathcal{I}}\mathcal{C}_i \to colim_{i\in\mathcal{I}}(\mathcal{F}\mathcal{C}_i/\mathcal{C}_i)$ is an equivalence of exact categories by the universal properties of quotients and colimits. It follows by 7.12 that the map $colim_{i\in\mathcal{I}}S\mathcal{C}_i \to Scolim_{i\in\mathcal{I}}\mathcal{C}_i$ is a $\mathbb{K}$-equivalence, i.e. an equivalence for positive and negative K-groups. Iterating, we find that $colim_{i\in\mathcal{I}}S^n\mathcal{C}_i \to S^n colim_{i\in\mathcal{I}}\mathcal{C}_i$ is a $\mathbb{K}$-equivalence. The result is now a consequence of the fact that higher algebraic K-theory commutes with filtered colimits and that $\mathbb{K}_{-n}(\ )$ is naturally isomorph to $\mathbb{K}_1(S^{n+1}(\ ))$. □

8.3 Theorem. *(Resolution) Let $\mathcal{P}$ be an extension closed full exact subcategory of an exact category $\mathcal{M}$. Assume further that*

a) if $0 \to M' \to M \to M'' \to 0$ is exact in $\mathcal{M}$ and $M, M'' \in \mathcal{P}$, then $M' \in \mathcal{P}$ and that

b) for any object $M \in \mathcal{M}$ there is a finite resolution $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ with $P_i \in \mathcal{P}$

then $\mathbb{K}_i(\mathcal{P}) \to \mathbb{K}_i(\mathcal{M})$ is an isomorphism for $i \in \mathbb{Z}$.

Proof. For $i > 0$ this is Quillen's resolution theorem. Since by 8.2 $I\!K$-theory commutes with filtered colimits, it suffices to show the theorem under hypothesis a) and for the situation in which every $M \in \mathcal{M}$ admits a resolution $0 \to P_1 \to P_0 \to M \to 0$ with $P_i \in \mathcal{P}$ as in Quillen's proof of the case $i > 0$. We will prove that if $\mathcal{P} \subset \mathcal{M}$ satisfies the hypothesis of 8.3 with $n \leq 1$ then $\mathcal{C}_k(\mathcal{P}^{op})^{op} \subset \mathcal{C}_k(\mathcal{M}^{op})^{op}$ satisfies condition b) and a weaker version of a) which is sufficient to conclude the theorem. We show that $P_\infty(\mathcal{P}) \subset P_\infty(\mathcal{M})$ satisfies the hypothesis of 8.3 with $n \leq 1$. a) is clear since we take kernels pointwise. Now we show b). Let $M_* = (\, M_0 \overset{m_1}{\twoheadleftarrow} M_1 \overset{m_2}{\twoheadleftarrow} M_2 \overset{m_3}{\twoheadleftarrow} \cdots)$ be an object of $P_\infty(\mathcal{M})$. Let $Q_0 \overset{i_0}{\rightarrowtail} P_0 \overset{r_0}{\twoheadrightarrow} M_0$ be a resolution of M_0 by objects of $\mathcal{P}$ and let $P_0 \times_{M_0} M_1$ be the pullback of r_0 and m_1. By b) there is a resolution $Q \overset{i}{\rightarrowtail} P_1 \overset{p}{\twoheadrightarrow} P_0 \times_{M_0} M_1$. Let r_1 be the composition of p and $P_0 \times_{M_0} M_1 \to M_1$, let p_1 be the composition of p and $P_0 \times_{M_0} M_1 \to P_0$. The maps r_1 and p_1 are admissible epimorphisms. Let $Q_1 \overset{i_1}{\to} P_1$ be a kernel for r_1. Then Q_1 is as an extension of Q_0 and Q an object of $\mathcal{P}$. Let $q_1 : Q_1 \to Q_0$ be the kernel map induced be r_1 and m_1. The morphism q_1 is an admissible epimorphism with kernel Q. The $\mathcal{M}$-admissible epimorphism p_1 has by a) its kernel in $\mathcal{P}$ and is therefore a $\mathcal{P}$-admissible epimorphism. We can continue with P_1 and Q_1 instead of P_0 and Q_0, and we get an exact sequence $0 \to Q_* \to P_* \to M_* \to 0$ in $P_\infty(\mathcal{M})$. By a) the objects P_* and Q_* are objects of $P_\infty(\mathcal{P})$. So $I\!N(\mathcal{P}^{op})^{op} = P_\infty(\mathcal{P}) \subset P_\infty(\mathcal{M}) = I\!N(\mathcal{M}^{op})^{op}$ satisfies the hypothesis of 8.3 with $n \leq 1$. The same argument shows that $\mathcal{Z}(\mathcal{P}^{op})^{op} \subset \mathcal{Z}(\mathcal{M}^{op})^{op}$ satisfies the hypothesis of 8.3 with $n \leq 1$. Iterating (see lemma 6.10), we have that $\mathcal{Z}^i(\mathcal{P}^{op})^{op} \subset \mathcal{Z}^i(\mathcal{M}^{op})^{op}$ satisfy the hypothesis of 8.3 with $n \leq 1$. As a localization of $\mathcal{Z}^i((\ )^{op})^{op}$ (for its exact structure see 6.16), $\mathcal{C}_i(\mathcal{P}^{op})^{op} \subset \mathcal{C}_i(\mathcal{M}^{op})^{op}$ still satisfy condition b).

We don't know whether a) is also true, but we will use the weaker condition C2^{op} of theorem 12.1 of [Kel96]. It states that an extension closed fully exact subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ induces a fully faithful canonical functor between their bounded derived categories if for any exact sequence of $\mathcal{A}$, $0 \to A'' \to A' \to B \to 0$ with $B \in \mathcal{B}$, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B'' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A'' \arrow[r] & A' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

with $B', B'' \in \mathcal{B}$ and in which the first row is also exact. Instead of showing that $\mathcal{C}_i(\mathcal{P}^{op})^{op}$ is closed under kernels of surjections in $\mathcal{C}_i(\mathcal{M}^{op})^{op}$ we show that for every admissible epimorphism $r : M_* \to P_*$ in $\mathcal{C}_i(\mathcal{M}^{op})^{op}$ with target in $\mathcal{C}_i(\mathcal{P}^{op})^{op}$ there is an admissible epimorphism $q : Q_* \to M_*$ with $Q_* \in \mathcal{C}_i(\mathcal{P}^{op})^{op}$ such that the kernel of $r \circ q$ lies in $\mathcal{C}_i(\mathcal{P}^{op})^{op}$. This is sufficient for C2^{op} to hold. Given a $\mathcal{C}_i(\mathcal{M}^{op})^{op}$-admissible epimorphism $[r] : M_* \to P_*$ represented by $r : m.M_* \to P_*$. By the definition of the exact structure of $\mathcal{C}_i(\mathcal{M}^{op})^{op}$ there is a $\mathcal{C}_i^0(\mathcal{M}^{op})^{op}$-admissible epimorphism $p : M'_* \to N_*$

and a $\mathcal{C}_i^0(\mathcal{M}^{op})^{op}$-commutative square

$$\begin{tikzcd} m.M_* \arrow[r, "r"] \arrow[d, "\alpha"'] & P_* \arrow[d, "\beta"] \\ M'_* \arrow[r, "p"] & N_* \end{tikzcd}$$

with α and β isomorphisms in $\mathcal{C}_i(\mathcal{M}^{op})^{op}$. Let M''_* be a $\mathcal{C}_i^0(\mathcal{M}^{op})^{op}$-pullback of β along p, the induced maps being $\bar{\alpha} : M_* \to M''_*$, $\bar{\beta} : M''_* \to M'_*$ and $\bar{p} : M''_* \to P_*$. Then $\bar{p}$ is a $\mathcal{C}_i^0(\mathcal{M}^{op})^{op}$-admissible epimorphism and $\bar{\alpha}$ is a isomorphism in $\mathcal{C}_i(\mathcal{M}^{op})^{op}$ since α and $\bar{\beta}$ are. We have shown above that $\mathcal{C}_i^0(\mathcal{P}^{op})^{op} \subset \mathcal{C}_i^0(\mathcal{M}^{op})^{op}$ satisfy a) and b). That's why there is a $\mathcal{C}_i^0(\mathcal{M}^{op})^{op}$-admissible epimorphism $q : Q_* \to M''_*$ with $Q_* \in \mathcal{C}_i(\mathcal{P}^{op})^{op}$ and such that the kernel of $\bar{p} \circ q$ lies in $\mathcal{C}_i(\mathcal{P}^{op})^{op}$. Then the $\mathcal{C}_i(\mathcal{M}^{op})^{op}$-map $\bar{\alpha}^{-1} \circ q$ is an admissible epimorphism. The kernels of $r \circ (\bar{\alpha}^{-1} \circ q)$ and $\bar{p} \circ q$ are isomorphic and therefore in $\mathcal{C}_i(\mathcal{P}^{op})^{op}$. This shows C2^{op}. It follows that $\mathcal{D}^{\mathrm{b}}(\mathcal{C}_i(\mathcal{P}^{op})^{op}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{C}_i(\mathcal{M}^{op})^{op})$ is fully faithful. By b) for $\mathcal{C}_i(\mathcal{P}^{op})^{op} \subset \mathcal{C}_i(\mathcal{M}^{op})^{op}$ it is also full, hence an equivalence of derived categories. By Thomason's theorems [TT90] 1.9.8 and 1.11.7, the inclusions $\mathcal{C}_i^0(\mathcal{P}^{op})^{op} \subset \mathcal{C}_i^0(\mathcal{M}^{op})^{op}$ induce K-theory equivalences. The theorem follows by theorem 7.12 and the dual of lemma 7.4. $\square$

9 Negative K-theory of noetherian abelian categories

In this section we prove that the negative K-theory of noetherian abelian categories is zero. All noetherian abelian categories in this section are supposed to be small. Although not all categories we are considering in this section are small, we will work as if they were so, since we can replace them by small ones (see remark 9.11).

9.1 Definition. An object N of an exact category $\mathcal{E}$ is called noetherian if any ascending chain of subobjects of N eventually stops, i.e. given a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, tail, "i_0"] \arrow[drrr, tail] & A_1 \arrow[r, tail, "i_1"] \arrow[drr, tail] & A_2 \arrow[r, tail, "i_2"] \arrow[dr, tail] & \cdots \\ & & & N \end{tikzcd}$$

with all morphisms admissible monomorphisms, then there is a $k \in \mathbb{N}$ such that i_l is an isomorphism for $l \geq k$.

9.2 Definition. An exact category is called noetherian if all of its objects are noetherian.

Example. Let R be a noetherian ring then the category $\mathcal{M}_{fg}(R)$ of finitely generated left R-modules is a noetherian abelian category.

9.3 Lemma. *Every exact subcategory of a noetherian exact category is noetherian. Every quotient category of a noetherian abelian category by a Serre subcategory is also a noetherian abelian category.*

Proof. The first statement follows from the definition. The second statement is less obvious, we give a direct proof of it, but it follows also from [Pop73] lemma 5.8.3 and proposition 5.8.13. Let $\mathcal{U}$ be a noetherian abelian category and $\mathcal{A}$ be a Serre subcategory.

a) We first remark that a $\mathcal{U}$-map $f : Y \to Z$ which is a $\mathcal{U}/\mathcal{A}$-monomorphism has kernel in $\mathcal{A}$. This is because the $\mathcal{U}$-epimorphism $Y \to Im(f)$ is at the same time a $\mathcal{U}/\mathcal{A}$-monomorphism and a $\mathcal{U}/\mathcal{A}$-epimorphism, hence a $\mathcal{U}/\mathcal{A}$-isomorphism, so the kernel of f is in $\mathcal{A}$.

b) We claim that given two $\mathcal{U}$-monomorphism $i : Y \to X$ and $j : Z \to X$ and a fraction $fs^{-1} : Y \to Z$ such that $i = j \circ fs^{-1}$ in $\mathcal{U}/\mathcal{A}$, then there are $\mathcal{U}$-monomorphisms $k : Z' \to X$, $f' : Y \to Z'$ and $t : Z \to Z'$ with $i = k \circ f'$, $j = k \circ t$ and the cokernel of t lies in $\mathcal{A}$. We begin the proof. By calculus of fraction, we can suppose $i \circ s = j \circ f$. Let T be a pushout of s along f. The induced map $\bar{s} : Z \to T$

has kernel and cokernel in $\mathcal{A}$ since s has. Let $\bar{f} : Y \to T$ and $g : T \to X$ be the other induced maps. g is a $\mathcal{U}/\mathcal{A}$-monomorphism since j is and $\bar{s}$ is a $\mathcal{U}/\mathcal{A}$-isomorphism. Let $Z' = Im(g)$ and write $g = T \xrightarrow{p} Z' \xrightarrow{k} X$ with p surjective and k injective. Then by a), the map $p : T \to Z'$ has kernel in $\mathcal{A}$. Let $f' = p \circ \bar{f}$ and $t = p \circ \bar{s}$. The maps f' and t are monomorphisms since composed with the monomorphism $k : Z' \to X$ they are monomorphisms. The monomorphism t has cokernel in $\mathcal{A}$, since $\bar{s}$ and p have. This proves b).
a) and b) together imply that any chain of subobjects in $\mathcal{U}/\mathcal{A}$ of an object X is $\mathcal{U}/\mathcal{A}$-isomorphic to the image of a chain of subobjects in $\mathcal{U}$ of X. It follows that if X is noetherian, its image in $\mathcal{U}/\mathcal{A}$ is also noetherian. $\square$

The goal of this section is to prove the following theorem.

9.4 Theorem. *Let $\mathcal{A}$ be a noetherian abelian category, then $\mathbb{K}_{-i}(\mathcal{A}) = 0$ for $i > 0$.*

Together with the resolution theorem 8.3 this gives the well known fact

9.5 Corollary. *The negative K-theory of regular rings is zero.*

We first prove theorem 9.4 for $i = 1$.

9.6 Lemma. *Let $\mathcal{A}$ be a noetherian abelian category, then $\mathbb{K}_{-1}(\mathcal{A}) = 0$.*

Proof. By taking pointwise image and kernel, we see that every map $f : G \to H$ in $I_\infty \mathcal{A}$ has a kernel and an image. We have an exact sequence $ker(f) \to G \to Im(f)$. Since the maps $m_i^* Im(f) \to m_i^* H$ are monic in $I_\infty \mathcal{A}$, $i \in \mathbb{N}$, $Im(f) \to H$ is monic in $\mathcal{F}_I \mathcal{A}$. The functor $I_\infty \mathcal{A} \to \mathcal{F}_I \mathcal{A}$ preserves exact sequences, so every map in $\mathcal{F}_I \mathcal{A}$ is the composition of an admissible epimorphism and a monic. Hence every map in $\mathcal{F}_I \mathcal{A}$ has a kernel. Furthermore, since $\mathcal{A}$ is noetherian, the inclusion $c : \mathcal{A} \to \mathcal{F}_I \mathcal{A}$ is not only left filtering but also right filtering (factor $G \to cA$ through its image which by noetherianity is $\mathcal{F}_I \mathcal{A}$-isomorphic to an object of $\mathcal{A}$). It follows that $S_I \mathcal{A}$ is a localization of $\mathcal{F}_I \mathcal{A}$ by a calculus of fraction on both sides. Such localizations preserve in particular kernels, hence in $S_I \mathcal{A}$ every map has a kernel and so $S_I \mathcal{A}$ is idempotent complete. But we have $\mathbb{K}_{-1}(\mathcal{A}) = K_0(\widetilde{S_I \mathcal{A}}) = K_0(S_I \mathcal{A}) = 0$. $\square$

We will first show how to prove theorem 9.4 for $\mathcal{A} = \mathcal{M}_{fg}(R)$, the category of finitely generated modules over a noetherian ring R. The second half of the section will develop the necessary tools so that the proof for $\mathcal{A} = \mathcal{M}_{fg}(R)$ goes through for the case of arbitrary noetherian abelian categories.

Proof of 9.4 for $\mathcal{A} = \mathcal{M}_{fg}(R)$. The idea is the same as in the proof that the negative K-theory of regular noetherian rings vanish. We will show that $\mathbb{K}_{*-1}(\mathcal{M}_{fg}(R))$ is a direct factor of $\mathbb{K}_*(\mathcal{M}_{fg}(R[t,t^{-1}]))$. $R[t,t^{-1}]$ is also noetherian and by lemma 9.6 $\mathbb{K}_{-1}(\mathcal{M}_{fg}(R[t,t^{-1}])) = 0$, hence $\mathbb{K}_{-2}(\mathcal{M}_{fg}(R)) = 0$ as it is a direct factor of 0. Iteration shows the theorem.

For a ring R let $\mathcal{L}(R)$ be the split exact category of finitely generated free R modules. The claim follows from the existence of a commutative diagram

$$\begin{tikzcd}
\mathcal{L}(\mathbb{Z}[t,t^{-1}]) \times \mathcal{M}_{fg}(R) \arrow[rr, "\alpha"] \arrow[d, "\gamma\times 1"'] & & \mathcal{M}_{fg}(R[t,t^{-1}]) \arrow[d, "\varphi"] \\
S_I\mathcal{L}(\mathbb{Z}) \times \mathcal{M}_{fg}(R) \arrow[r, "\beta"] & S_I\mathcal{M}_{fg}(R) \arrow[r, "colim"] & \mathcal{M}(R)/\mathcal{M}_{fg}(R)
\end{tikzcd}$$

in which α is the biexact functor $L, M \mapsto L \otimes M$, β is the biexact functor

$$(L_0 \to L_1 \to \ldots), M \mapsto (L_0 \otimes M \to L_1 \otimes M \to \ldots),$$

colim is the exact functor induced by $colim_{\mathbb{N}} : \mathcal{F}_I\mathcal{M}_{fg}(R) \to \mathcal{M}(R)$ and φ is the exact functor induced by the natural inclusion of $\mathcal{M}_{fg}(R[t])$ into $\mathcal{M}(R)$. Here we use that for a finitely generated $R[t]$-module M multiplication by t has kernel and cokernel which are finitely generated as R-modules. The additive functor $\gamma : \mathcal{L}(\mathbb{Z}[t,t^{-1}]) \to S_I\mathcal{L}(\mathbb{Z})$ is determined by sending the object $\mathbb{Z}[t,t^{-1}]$ to $Z = (\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}t \subset \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \subset \ldots)$ and the multiplication by $t : \mathbb{Z}[t,t^{-1}] \to \mathbb{Z}[t,t^{-1}]$ to

$$\begin{tikzcd}
Z \arrow[d] & (\mathbb{Z} \arrow[r, tail] \arrow[d, "\times t"] & \mathbb{Z} \oplus \mathbb{Z}t \arrow[r, tail] \arrow[d, "\times t"] & \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \arrow[r, tail] \arrow[d, "\times t"] & \ldots) \\
m_1^*Z & (\mathbb{Z} \oplus \mathbb{Z}t \arrow[r, tail] & \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \arrow[r, tail] & \mathbb{Z} \oplus \mathbb{Z}t \oplus \mathbb{Z}t^2 \oplus \mathbb{Z}t^3 \arrow[r, tail] & \ldots)
\end{tikzcd}$$

which is an isomorphism in $S_I\mathcal{L}(\mathbb{Z})$ since $Z \to m_1^*Z$ is an admissible monomorphism in $\mathcal{F}_I\mathcal{L}(\mathbb{Z})$ with cokernel the constant diagram $c(\mathbb{Z})$. By 7.7 the commutative diagram induces a commutative diagram of symmetric spectra

$$\begin{tikzcd}
\mathbb{K}(\mathbb{Z}[t,t^{-1}]) \wedge \mathbb{K}(\mathcal{M}_{fg}(R)) \arrow[rr] \arrow[d] & & \mathbb{K}(\mathcal{M}_{fg}(R[t,t^{-1}])) \arrow[d] \\
\mathbb{K}(S_I\mathcal{L}(\mathbb{Z})) \wedge \mathbb{K}(\mathcal{M}_{fg}(R)) \arrow[r] & \mathbb{K}(S_I\mathcal{M}_{fg}(R)) \arrow[r] & \mathbb{K}(\mathcal{M}(R)/\mathcal{M}_{fg}(R)).
\end{tikzcd}$$

Choosing a map $\varepsilon : S^1 \to \mathbb{K}(\mathbb{Z}[t,t^{-1}])$ (which corresponds to a map $S^1 \to \mathbb{K}(\mathbb{Z}[t,t^{-1}])_1$) representing the automorphism "multiplication by t" in $K_1(\mathbb{Z}[t,t^{-1}])$ we complete the diagram on the left and obtain a commutative square

$$\begin{tikzcd}
S^1 \wedge \mathbb{K}(\mathcal{M}_{fg}(R)) \arrow[r] \arrow[d] & \mathbb{K}(\mathcal{M}_{fg}(R[t,t^{-1}])) \arrow[d] \\
\mathbb{K}(S_I\mathcal{M}_{fg}(R)) \arrow[r] & \mathbb{K}(\mathcal{M}(R)/\mathcal{M}_{fg}(R)).
\end{tikzcd}$$

By construction, the element $[\varepsilon] \in K_1(\mathbb{Z}[t,t^{-1}])$ goes to $[\eta] \in K_1(S\mathbb{Z})$ under the following map $K_1(\gamma) : K_1(\mathbb{Z}[t,t^{-1}]) \to K_1(S\mathbb{Z})$. Then by construction of the functor $\mathbb{K}$ in 7.8, the map of spectra $(\mathbb{K}(\eta) \circ \varepsilon) \wedge 1 : S^1 \wedge \mathbb{K}(\mathcal{M}_{fg}(R)) \to \mathbb{K}(S_I\mathcal{M}_{fg}(R))$ is a weak equivalence of spectra. Since by 3.12 $\mathcal{M}_{fg}(R)$ is a left s-filtering subcategory of the flasque exact category $\mathcal{M}(R)$, the map $\mathbb{K}(S_I\mathcal{M}_{fg}(R)) \to \mathbb{K}(\mathcal{M}(R)/\mathcal{M}_{fg}(R))$ is also a weak equivalence of symmetric spectra (7.12). Therefore, $S^1 \wedge \mathbb{K}(\mathcal{M}_{fg}(R))$ is a direct factor of $\mathbb{K}(\mathcal{M}_{fg}(R[t,t^{-1}]))$ in the homotopy category of spectra and $\mathbb{K}_{*-1}(\mathcal{M}_{fg}(R))$ is a direct factor of $\mathbb{K}_*(\mathcal{M}_{fg}(R[t,t^{-1}]))$. □

9.7 We proceed to give the details for the proof of theorem 9.4 in its full generality. Let $\mathcal{A}$ be a small abelian category, let $\mathcal{A}$-mod the category of additive functors from $\mathcal{A}^{op}$ to the category of abelian groups. $\mathcal{A}$-mod is a Grothendieck category, i.e. it is an abelian category in which all small colimits exist, filtered colimits are exact and which has a set of generators. There is a Yoneda embedding $\mathcal{A} \to \mathcal{A}$-mod sending A to $\hat{A} = \mathcal{A}(\ .\ , A) : B \mapsto \mathcal{A}(B, A)$ which is full and faithful. The $\hat{A}$, $A \in \mathcal{A}$ constitute a set of generators for $\mathcal{A}$-mod, i.e. if $f : X \to Y \in \mathcal{A}$-mod and $f \neq 0$ then there is a $A \in \mathcal{A}$ and a map $g : \hat{A} \to X$ such that $f \circ g \neq 0$. Let Lex$\mathcal{A}$ be the full subcategory of $\mathcal{A}$-mod consisting of the left exact functors. The Yoneda embedding factors through Lex$\mathcal{A}$. The category Lex$\mathcal{A}$ is also a Grothendieck category with the same set of generators as $\mathcal{A}$-mod. We describe now the exact structure of Lex$\mathcal{A}$. The inclusion Lex$\mathcal{A} \to \mathcal{A}$-mod has a left adjoint $L : \mathcal{A}-\text{mod} \to \text{Lex}\mathcal{A}$ which is the identity on Lex$\mathcal{A}$. A sequence $F \to G \to H$ of left exact functors is exact in Lex$\mathcal{A}$ iff $F \to G$ is a monomorphism in $\mathcal{A}$-mod and the map $L(G/F) \to H$ an isomorphism, G/F being the $\mathcal{A}$-mod-quotient of G by F. We see for instance that for an object A of Lex$\mathcal{A}$, the isomorphism classes of subobjects of A form a set, because this is obviously true for $\mathcal{A}$-mod. Further details may be found in [Pop73], [TT90] and [Kel90].

For later reference we state the following lemma which is known in the literature.

9.8 Lemma. *Let $\mathcal{A}$ be a small noetherian abelian category. Then*
i) the Yoneda embedding makes $\mathcal{A}$ into a Serre subcategory of Lex$\mathcal{A}$*,*
ii) every object of Lex$\mathcal{A}$ *is the filtered colimit of its noetherian subobjects,*
iii) $\mathcal{A}$ is the category of noetherian objects of Lex$\mathcal{A}$*, and*
iv) the Yoneda embedding makes $\mathcal{A}$ into a left s-filtering subcategory of Lex$\mathcal{A}$*.*

Proof. iii) is [Pop73] proposition 5.8.8 and Theorem 5.8.9, ii) is [Pop73] proposition 5.8.6, and i) is [Pop73] proposition 5.7.2 together with iii). We show iv). The category $\mathcal{A}$ is closed under taking subobjects and quotients, and the filtering conditions are satisfied because of i). It remains to show that surjections onto objects of $\mathcal{A}$ are special. Given a surjection $p: G \twoheadrightarrow A$ onto an object of $\mathcal{A}$. Let $\mathcal{N}$ be the filtered category of noetherian subobjects of G, maps being inclusions. By ii), $colim_{N\in\mathcal{N}} N \to G$ is an isomorphism. For $N \in \mathcal{N}$, we let N_A be the image of N in A under p. If $N \subset N'$ so is $N_A \subset N'_A$. Since filtered colimits are exact in Lex$\mathcal{A}$, the map $colim_{N\in\mathcal{N}} N_A \to A$ is an isomorphism. By iii), A is noetherian. It follows that there is a $N \in \mathcal{N}$ with $N_A \to A$ an isomorphism. Therefore, the inclusion of N in G surjects onto A. By iii), N_A is an object of $\mathcal{A}$ and we are done. $\square$

9.9 The category $\mathcal{A}[t]$. For an abelian category $\mathcal{B}$ let End$\mathcal{B}$ the category of endomorphisms of $\mathcal{B}$, objects are endomorphisms and morphisms are maps of objects in $\mathcal{B}$ commuting with the endomorphisms. Obviously, End$\mathcal{B}$ is an abelian category. For any $A \in \mathcal{A} \subset$ Lex$\mathcal{A}$ there is an object $A[t] \in$ End(Lex$\mathcal{A}$) constructed as follows. The underlying Lex$\mathcal{A}$-object is $A[t] = A \oplus At \oplus At^2 ...$, here At^i stands for a copy of A. The endomorphism of $A[t]$ is "multiplication by t", i.e. the map which sends At^i to At^{i+1} identifying them. We remark that for an End(Lex$\mathcal{A}$)-object $e: X \to X$ there is a canonical bijection Lex$\mathcal{A}(A, X) =$ End(Lex$\mathcal{A}$)$(A[t], e)$. Let $\mathcal{A}[t]$ be the full subcategory of End(Lex$\mathcal{A}$) of objects X for which there is a End(Lex$\mathcal{A}$)-surjection $A[t] \twoheadrightarrow X$.

9.10 Lemma. *$\mathcal{A}[t]$ is a noetherian abelian category, and if $e: M \to M$ is an object of $\mathcal{A}[t]$ with e nilpotent, then M is an object of $\mathcal{A}$.*

Proof. a) We show first that for every End(Lex$\mathcal{A}$)-subobject M of $A[t]$ there is a End(Lex$\mathcal{A}$)-surjection $B[t] \twoheadrightarrow M$. This is more or less a transcription of the corresponding theorem for noetherian rings into this context. Let $M_n = M \cap (A \oplus At \oplus At^2 ... \oplus At^n)$. It is an object of $\mathcal{A}$ since it is a subobject of the $\mathcal{A}$-object $A \oplus At \oplus At^2 ... \oplus At^n$ and $\mathcal{A}$ is by 9.8 a Serre subcategory of Lex$\mathcal{A}$. We remark that $M_{n-1} = ker(M_n \to At^n)$. Let $I_n = Im(M_n \to At^n)$ then we have a commutative diagram

$$\begin{tikzcd}
M_{n-1} \arrow[r, "t"] \arrow[d, tail] & M_n \arrow[r, "t"] \arrow[d, tail] & M_{n+1} \arrow[r, "t"] \arrow[d, tail] & \cdots \\
M_n \arrow[r, "t"] \arrow[d, two heads] & M_{n+1} \arrow[r, "t"] \arrow[d, two heads] & M_{n+2} \arrow[r, "t"] \arrow[d, two heads] & \cdots \\
I_n \arrow[r, tail] \arrow[d, tail] & I_{n+1} \arrow[r, tail] \arrow[d, tail] & I_{n+2} \arrow[r, tail] \arrow[d, tail] & \cdots \\
At^n \arrow[r, "t", "\sim"'] & At^{n+1} \arrow[r, "t", "\sim"'] & At^{n+2} \arrow[r, "t", "\sim"'] & \cdots
\end{tikzcd}$$

Therefore, the I_n, $n \in \mathbb{N}$, form an ascending chain of subobjects of A which by noetherianity of A becomes stationary, say $I_{n-1} \overset{\cong}{\rightarrowtail} I_n \overset{\cong}{\rightarrowtail} I_{n+1} \overset{\cong}{\rightarrowtail} \cdots$. It follows that the upper squares in the diagram are bicartesian squares. The inclusion of M_n into M induces a End(Lex$\mathcal{A}$)-map $M_n[t] \to M$. We claim that this map is a surjection. We prove by induction on k that $p_k : M_n \oplus M_n t \oplus M_n t^2 \oplus \cdots \oplus M_n t^k \to M_{n+k}$ is a surjection. The claim follows by taking the filtered colimit over k which preserves surjections since Lex$\mathcal{A}$ is a Grothendieck category. For $k = 0$ there is nothing to prove. The case $k = 1$ follows from the fact that the upper left square in the above diagram is bicartesian. Suppose p_k an epimorphism. We have a commutative diagram

$$\begin{tikzcd}
M_n \oplus M_n t \arrow[d, "p_1"'] & M_n \arrow[l, tail] \arrow[r, tail, "t^k"] \arrow[d, equal] & M_n \oplus M_n t \oplus M_n t^2 \oplus \cdots \oplus M_n t^k \arrow[d, "p_k"] \\
M_{n+1} & M_n \arrow[l, tail] \arrow[r, "t^k"] & M_{n+k}
\end{tikzcd}$$

in which the vertical maps are surjections by induction hypotheses. Therefore, the map of the pushouts of the two horizontal diagrams is also a surjection. But this map is nothing else than p_{k+1} using the the bicartesian squares of the first diagram.

b) We show that for A an object of $\mathcal{A}$, the object $A[t]$ is noetherian in End(Lex$\mathcal{A}$) in the following way. Let $M_O \subset M_1 \subset M_2 \subset ... \subset A[t]$ an ascending chain of subobjects of $A[t]$. Let $M = colim_i M_i$ this is still a subobject of $A[t]$ since End(Lex$\mathcal{A}$) is a Grothendieck category. By a), there is a surjection $p : B[t] \twoheadrightarrow M$, $B \in \mathcal{A}$. Let $B_i = B \cap p^{-1}(M_i)$. Then the B_i, $i \in \mathbb{N}$, form an ascending chain of subobjects of B. So there is a k with $B_k = B_{k+1} = ...$ Since $M = colim_i M_i$, we have $B = colim_i B_i$ and hence $B_k = B_{k+1} = ... = B$. Then p factors through M_k and we have $M_k = M_{k+1} = ... = M$.

c) Every object of $\mathcal{A}[t]$ is noetherian in End(Lex$\mathcal{A}$). Let M be an object of $\mathcal{A}[t]$. By definition there is an $A \in \mathcal{A}$ and a surjection $p : A[t] \to M$. Any ascending chain of subobjects of M gives by pulling back along p an ascending chain of subobjects of $A[t]$ which is stationary by b). But then the original ascending chain of subobjects of M is also stationary.

d) $\mathcal{A}[t]$ is an abelian subcategory of End(Lex$\mathcal{A}$). Let M be an object of $\mathcal{A}[t]$. Quotients of M are trivially in $\mathcal{A}[t]$. Let M' be a subobject of M and $p : A[t] \to M$ a surjection. $p^{-1}(M')$ is a subobject of $A[t]$ which surjects onto M'. By a) there is a surjection $B[t] \to p^{-1}(M')$. Composing with p gives a surjection of $B[t]$ onto M', so M' is an object of $\mathcal{A}[t]$. It follows that kernel, image and cokernel performed in End(Lex$\mathcal{A}$) of a map between two objects of $\mathcal{A}[t]$ are also in $\mathcal{A}[t]$. Hence $\mathcal{A}[t]$ is an abelian subcategory of End(Lex$\mathcal{A}$).

By c) and d) $\mathcal{A}[t]$ is a noetherian abelian category.

e) For every object $e : M \to M$ of $\mathcal{A}[t]$ the object $coker(e)$ is an object of $\mathcal{A}$ as it is a quotient of $A = coker(t : A[t] \to A[t])$ for a $\mathcal{A}[t]$-surjection $A[t] \to M$. If the $\mathcal{A}[t]$-object $e : M \to M$ is nilpotent then we have a finite filtration of M

$$0 = Im(e^n) \subset Im(e^{n-1}) \subset ...Im(e) \subset M$$

with quotients $Im(e^i)/Im(e^{i+1}) = coker(e : Im(e^i) \to Im(e^i))$ in $\mathcal{A}$ by the last sentence. It follows that M is an object of $\mathcal{A}$ by 9.8 i). □

Proof of 9.4 for $\mathcal{A}$ noetherian abelian. Let Nil be the full subcategory of $\mathcal{A}[t]$ of nilpotent endomorphisms. It is easily seen that Nil is a Serre subcategory of $\mathcal{A}[t]$. We define the category $\mathcal{A}[t, t^{-1}]$ to be the category $\mathcal{A}[t]/Nil$. By lemma 9.3, $\mathcal{A}[t, t^{-1}]$ is a noetherian abelian category. Now the proof of 9.4 for $\mathcal{A} = \mathcal{M}_{fg}(R)$ goes through if we replace $\mathcal{M}_{fg}(R)$ by $\mathcal{A}$, $\mathcal{M}(R)$ by Lex$\mathcal{A}$, $\mathcal{M}_{fg}(R[t])$ by $\mathcal{A}[t]$ and $\mathcal{M}_{fg}(R[t, t^{-1}])$ by $\mathcal{A}[t, t^{-1}]$ using Lemma 9.8 and Lemma 9.10 □

9.11 Remark. Since Lex$\mathcal{A}$ is not small, there are some problems in speaking of its K-theory and the K-theory of Lex$\mathcal{A}/\mathcal{A}$. We can go around this problem by choosing an abelian subcategory Lex$^\omega\mathcal{A}$ of Lex$\mathcal{A}$ which is equivalent to a small category containing $\mathcal{A}$ and which is stable under taking subobjects and quotients in Lex$\mathcal{A}$ and under taking countable sums of an object with itself. Then we work with this smaller category rather than with Lex$\mathcal{A}$. We construct Lex$^\omega\mathcal{A}$ for instance as follows. Let Lex$^0\mathcal{A} = \mathcal{A}$ which is a small category. Suppose we have already defined Lex$^i\mathcal{A}$ which is equivalent to a small category. Then Lex$^{i+1}\mathcal{A}$ is the full subcategory of Lex$\mathcal{A}$ on objects which are subobjects, quotients or countable sums of an object $A \in$ Lex$^i\mathcal{A}$. Since the isomorphism classes of subobjects of a fixed object of Lex$\mathcal{A}$ is a set, Lex$^{i+1}\mathcal{A}$ is also equivalent to a small category. We let Lex$^\omega\mathcal{A} = \bigcup_{i\geq 0}$ Lex$^i\mathcal{A}$. This category is by its construction an abelian category satisfying the above conditions. It is flasque since it contains with an object its countable sum with itself, and taking countable sums is an exact functor in Lex$\mathcal{A}$. It contains also all quotients of objects of the form $A[t]$ for $A \in \mathcal{A}$. Therefore, we can replace in the proof Lex$\mathcal{A}$ by a small category equivalent to Lex$^\omega\mathcal{A}$ without any further change.

9.12 Corollary. *An abelian category which is the filtered colimit of noetherian abelian categories has trivial negative $\mathbb{K}$-groups.*

Proof. This is 9.4 combined with 8.2. □

In view of the last corollary and of the fact that there is no failure of surjectivity in $K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B})$, for $\mathcal{B}$ a Serre subcategory of the abelian category $\mathcal{A}$, we conjecture the following.

9.13 Conjecture. *Let $\mathcal{A}$ be a small abelian category, then $\mathbb{K}_i(\mathcal{A}) = 0$ for $i < 0$.*

10 The localization as an exact sequence of derived categories

In this section we will show that if $\mathcal{A}$ is an idempotent complete right s-filtering subcategory of an exact category $\mathcal{U}$ then the sequence of bounded derived categories $\mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is an exact sequence of triangulated categories. This gives a second proof of the localization proposition 5.1. As a corollary we also obtain that if $\mathcal{A}$ is an idempotent complete exact category, then its bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is idempotent complete, too.

Let $\mathcal{E}$ be an exact category, then $\mathcal{D}^{\mathrm{b}}(\mathcal{E})$ is the category of bounded chain complexes localized with respect to chain maps whose cones are homotopy equivalent to an acyclic complex ([Kel96]), i.e. a complex assembled of short exact sequences of $\mathcal{E}$. A chain complex is zero in $\mathcal{D}^{\mathrm{b}}(\mathcal{E})$ iff it homotopy equivalent to an acyclic complex (loc.cit). We consider the category of bounded chain complexes $Ch^b(\mathcal{E})$ as an exact category in which a sequence is exact iff it is exact in each degree.

10.1 Theorem. *Let $\mathcal{A}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathcal{U}$. Then the sequence of bounded derived categories*

$$\mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$$

is an exact sequence of triangulated categories, i.e. $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{U})$ on objects zero in $\mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A}) = \mathcal{D}^{\mathrm{b}}(\mathcal{U})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})$.

Proof. a) We show first that $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is the kernel of $\mathcal{D}^{\mathrm{b}}(\mathcal{U}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$. The fact that every admissible monomorphism from an object of $\mathcal{A}$ to an object of $\mathcal{U}$ is special, has as consequence that the hypothesis C2 of [Kel96], theorem 12.1 is fulfilled. Therefore, $\mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U})$ is fully faithful. So it is sufficient to prove that a chain complex

$$U_* : \qquad 0 \longrightarrow U_0 \xrightarrow{d_0} U_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} U_n \longrightarrow 0$$

of $Ch^{\mathrm{b}}(\mathcal{U})$ which is zero in $\mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is isomorphic in $\mathcal{D}^{\mathrm{b}}(\mathcal{U})$ to a chain complex of $Ch^{\mathrm{b}}(\mathcal{A})$.

We first treat the case of U_* acyclic in $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$. We will construct a sequence of chain complexes and maps

$$(*) \qquad U_* = \qquad U_*^0 \xleftarrow{\sim} V_*^0 \xrightarrow{\simeq} U_*^1 \xleftarrow{\sim} V_*^1 \xrightarrow{\simeq} \cdots \qquad U_*^n \xleftarrow{\sim} V_*^n \xrightarrow{\simeq} 0$$

where $\rightarrowtail^{\sim}$ denotes an admissible monomorphism of chain complexes with cokernel in $Ch^{\mathrm{b}}(\mathcal{A})$ and $\xrightarrow{\simeq}$ a quasi isomorphism of $Ch^{\mathrm{b}}(\mathcal{U})$. Then U_* is isomorphic in $\mathcal{D}^{\mathrm{b}}(\mathcal{U})$ to an object of $Ch^{\mathrm{b}}(\mathcal{A})$ since $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is closed under extension in $\mathcal{D}^{\mathrm{b}}(\mathcal{U})$ as it is a full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{U})$.

Using the calculus of right fractions, 3.16 iv) and vii), we see that for a $\mathcal{U}$-map $f : X \to Y$ which becomes an admissible monomorphism in $\mathcal{U}/\mathcal{A}$, there are admissible monomorphisms in $\mathcal{U}$ with cokernel in $\mathcal{A}$, $i : X' \to X$ and $j : Y \to Y'$, such that $j \circ f \circ i : X' \to Y'$ is an admissible monomorphism in $\mathcal{U}$. We apply this argument to $d_0 : U_0 \to U_1$ which is a $\mathcal{U}/\mathcal{A}$-admissible monomorphism because U_* is $\mathcal{U}/\mathcal{A}$-acyclic. We therefore obtain $\mathcal{U}$-admissible monomorphisms $\alpha : V_0 \overset{\sim}{\rightarrowtail} U_0$ and $\beta : U_1 \overset{\sim}{\rightarrowtail} W_1$ such that $\beta \circ d_0 \circ \alpha$ is a $\mathcal{U}$-admissible monomorphism. Let V_*^0 be the chain complex

$$V_*^0 : \qquad 0 \longrightarrow V_0 \xrightarrow{d_0 \circ \alpha} U_1 \xrightarrow{d_1} \cdots \xrightarrow{d_0} U_n \longrightarrow 0.$$

The map

$$\begin{tikzcd}
U_* : & 0 \arrow[r] & U_0 \arrow[r, "d_0"] & U_1 \arrow[r, "d_1"] \arrow[d, equal] & \cdots \arrow[r, "d_0"] & U_n \arrow[r] \arrow[d, equal] & 0 \\
V_*^0 : \arrow[u, tail, "\wr"'] & 0 \arrow[r] & V_0 \arrow[r, "d_0 \circ \alpha"] \arrow[u, tail, "\wr"', "\alpha"] & U_1 \arrow[r, "d_1"] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] & 0
\end{tikzcd}$$

is an admissible monomorphism of chain complexes with cokernel in $Ch^{\mathrm{b}}(\mathcal{A})$. Let W_2 be the pushout of d_1 and β. d_2 and $0 : W_1 \to U_3$ induce a map $\gamma : W_2 \to U_3$. Let W_*^0 be the chain complex

$$W_*^0 : \qquad 0 \longrightarrow V_0 \overset{\beta \circ d_0 \circ \alpha}{\rightarrowtail} W_1 \xrightarrow{\bar{d_1}} W_2 \xrightarrow{\gamma} U_3 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-1}} U_n \longrightarrow 0.$$

The map

$$\begin{tikzcd}
V_*^0 : \arrow[d] & 0 \arrow[r] & V_0 \arrow[r, "d_0 \circ \alpha"] \arrow[d, equal] & U_1 \arrow[r, "d_1"] \arrow[d, tail, "\beta"', "\wr"] & U_2 \arrow[r, "d_2"] \arrow[d, tail, "\bar{\beta}"', "\wr"] & U_3 \arrow[r, "d_3"] \arrow[d, equal] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] \arrow[d, equal] & 0 \\
W_*^0 : & 0 \arrow[r] & V_0 \arrow[r, tail, "\beta \circ d_0 \circ \alpha"] & W_1 \arrow[r, "\bar{d_1}"] & W_2 \arrow[r, "\gamma"] & U_3 \arrow[r, "d_3"] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] & 0
\end{tikzcd}$$

is a quasi-isomorphism since it is an admissible monomorphism with contractible cokernel. Let U_*^1 be the chain complex

$$0 \longrightarrow W_1/V_0 \xrightarrow{\bar{d_1}} W_2 \xrightarrow{\gamma} U_3 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-1}} U_n \longrightarrow 0.$$

induced be W_*^0. Then the map

$$\begin{tikzcd}
W_*^0 : \arrow[d] & 0 \arrow[r] & V_0 \arrow[r, tail, "\beta \circ d_0 \circ \alpha"] \arrow[d] & W_1 \arrow[r, "\bar{d_1}"] \arrow[d, two heads] & W_2 \arrow[r, "\gamma"] \arrow[d, equal] & U_3 \arrow[r, "d_3"] \arrow[d, equal] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] \arrow[d, equal] & 0 \\
U_*^1 : & 0 \arrow[r] & 0 \arrow[r] & W_1/V_0 \arrow[r] & W_2 \arrow[r, "\gamma"] & U_3 \arrow[r, "d_3"] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] & 0
\end{tikzcd}$$

is a quasi-isomorphism since it is an admissible epimorphism with contractible kernel. Now U_*^1 is a shorter chain complex than U_* and we repeat the construction to get $(*)$.

We now treat the case of U_* contractible in $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$. We will construct a sequence of chain complexes as in $(*)$. Suppose $n \geq 2$. By lemma 3.16 and the contractability of U_* in $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ there is a $\mathcal{U}$-commutative diagram

$$\begin{tikzcd}
U_0 \arrow[r, "d_0"] & U_1 \arrow[d, "r"] \\
V_0 \arrow[u, tail, "\alpha", "\wr"'] \arrow[r, tail, "\sim", "\beta"'] & W_0 .
\end{tikzcd}$$

Let V_*^0 be the chain complex

$$V_*^0 : \qquad 0 \longrightarrow V_0 \xrightarrow{d_0 \circ \alpha} U_1 \xrightarrow{d_1} \cdots \xrightarrow{d_0} U_n \longrightarrow 0.$$

As in the acyclic case we have an admissible monomorphism of chain complexes $V_*^0 \rightarrowtail U_*$ with cokernel in $Ch^{\mathrm{b}}(\mathcal{A})$. Let W_1 be the pushout of $d_0 \circ \alpha$ and β as in the picture

$$\begin{tikzcd}
V_0 \arrow[r, "d_0 \circ \alpha"] \arrow[d, tail, "\beta", "\wr"'] & U_1 \arrow[d, tail, "\bar{\beta}"', "\wr"] \\
W_0 \arrow[r, "s"] & W_1 .
\end{tikzcd}$$

d_1 gives raise a map $\bar{d_1} : W_1 \to U_2$. as in the first case d_2 did. r induces a retraction $\bar{r}$ of s in $\mathcal{U}$. s is perhaps not admissible, but $(s, 0) : W_0 \to W_1 \oplus W_0$ certainly is. The composition of maps of chain complexes

$$\begin{tikzcd}
V_*^0 : \arrow[d] & 0 \arrow[r] & V_0 \arrow[r, "d_0 \circ \alpha"] \arrow[d, tail, "\beta", "\wr"'] & U_1 \arrow[r, "d_1"] \arrow[d, "(\bar{\beta}, 0)"'] & U_2 \arrow[r, "d_2"] \arrow[d, "(1,0)"'] & U_3 \arrow[r, "d_3"] \arrow[d, equal] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] \arrow[d, equal] & 0 \\
W_*^0 : \arrow[d] & 0 \arrow[r] & W_0 \arrow[r, tail, "(s,0)"'] \arrow[d] & W_1 \oplus W_0 \arrow[r, "\bar{d_1} \oplus 1"] \arrow[d, two heads] & U_2 \oplus W_0 \arrow[r, "(d_2, 0)"] \arrow[d, equal] & U_3 \arrow[r, "d_3"] \arrow[d, equal] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] \arrow[d, equal] & 0 \\
U_*^1 : & 0 \arrow[r] & 0 \arrow[r] & coker(s, 0) \arrow[r] & U_2 \oplus W_0 \arrow[r, "(d_2, 0)"] & U_3 \arrow[r, "d_3"] & \cdots \arrow[r, "d_{n-1}"] & U_n \arrow[r] & 0
\end{tikzcd}$$

is a quasi-isomorphism in $Ch^{\mathrm{b}}(\mathcal{U})$ since the first is an admissible monomorphism with contractible cokernel and the second an admissible epimorphism with contractible kernel. U_*^1 is a shorter complex than U_* and we can repeat the procedure. If $n = 1$, then d_0 is a $\mathcal{U}/\mathcal{A}$-isomorphism and by 3.16 vii) it is a weak equivalence. Using 3.16 iv) we find an admissible monomorphism $i : V \to U_0$ with cokernel in $\mathcal{A}$ such that $d_0 \circ i$ is also an admissible monomorphism with cokernel in $\mathcal{A}$. But then the map of complexes

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "1"] \arrow[d, tail, "i"', "\wr"] & V \arrow[r] \arrow[d, tail, "d_0 \circ i"', "\wr"] & 0 \\
0 \arrow[r] & U_0 \arrow[r, "d_0"] & U_1 \arrow[r] & 0
\end{tikzcd}$$

is an admissible monomorphism with contractible domain and cokernel in $Ch^{\mathrm{b}}(\mathcal{A})$.

We now treat the general case of U_* homotopy equivalent in $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ to a $Ch^{\mathrm{b}}(\mathcal{A})$-acyclic complex V_*. Using the calculus of right fractions with respect to admissible monomorphisms having their cokernel in $\mathcal{A}$ we see that $Ch^{\mathrm{b}}(\mathcal{U}) \to Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is essentially surjective and we can therefore assume $V_* \in Ch^{\mathrm{b}}(\mathcal{U})$. In fact, $Ch^{\mathrm{b}}(\mathcal{A})$ is right s-filtering in $Ch^{\mathrm{b}}(\mathcal{U})$ with quotient equivalent to $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$. This follows from 3.16. The $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$-chain homotopy equivalence $h : V_* \to U_*$ can be written as a fraction of two $Ch^{\mathrm{b}}(\mathcal{U})$ maps $V_* \xleftarrow{\sim} W_* \xrightarrow{f} U_*$. By the first case V_* is $Ch^{\mathrm{b}}(\mathcal{U})$-quasi-isomorph to a chain complex of $Ch^{\mathrm{b}}(\mathcal{A})$. It follows that W_* is also $Ch^{\mathrm{b}}(\mathcal{U})$-quasi-isomorphic to a chain complex of $Ch^{\mathrm{b}}(\mathcal{A})$. Since f is a $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ homotopy equivalence, the cone $C(f) \in Ch^{\mathrm{b}}(\mathcal{U})$ of f is $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$-contractible and by the second case $Ch^{\mathrm{b}}(\mathcal{U})$-quasi-isomorphic to a chain complex of $Ch^{\mathrm{b}}(\mathcal{A})$. Now, U_* is an extension in $\mathcal{D}^{\mathrm{b}}(\mathcal{U})$ of two complexes of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ and hence U_* is isomorph to a an object of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$.

b) We show that $\mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is equivalent to the cokernel of $\mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U})$ in the category of triangulated categories. As above, $Ch^{\mathrm{b}}(\mathcal{U}) \to Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is essentially surjective and every $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$-morphism of two $Ch^{\mathrm{b}}(\mathcal{U})$ objects is a right fraction of a $Ch^{\mathrm{b}}(\mathcal{U})$-morphism by a $Ch^{\mathrm{b}}(\mathcal{U})$-admissible monomorphism with cokernel in $Ch^{\mathrm{b}}(\mathcal{A})$. Therefore, $Ch^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is a localization of $Ch^{\mathrm{b}}(\mathcal{U})$ (up to equivalence of categories). Hence $\mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{U})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ are localizations of $Ch^{\mathrm{b}}(\mathcal{U})$ with respect to certain sets of morphisms in $Ch^{\mathrm{b}}(\mathcal{U})$. Since the map $\mathcal{D}^{\mathrm{b}}(\mathcal{U})/\mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ is well defined, it suffices to prove that a $Ch^{\mathrm{b}}(\mathcal{U})$-map $f : U_* \to V_*$ which is an isomorphism in $\mathcal{D}^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$ has its cone in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. But this was shown in a). □ □

10.2 Corollary. *Let $\mathcal{E}$ be an idempotent complete exact category. Then its bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathcal{E})$ is also idempotent complete.*

Proof. By the results of section 4 we can embed $\mathcal{E}$ as a right s-filtering subcategory of a flasque exact category $\mathcal{F}$. Then $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$ is flasque, too. Since in a triangulated category a morphism which admits a retraction has a cokernel, $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$ is idempotent complete. We see this by calculating in the idempotent completion of the underlying additive category of $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$. Let $p : A \to A$ be an idempotent in $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$ and let $T : \mathcal{D}^{\mathrm{b}}(\mathcal{F}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{F})$ be a functor such that $T \oplus id \simeq T$ which exists because $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$ is flasque. We have

$$TA \simeq T(Im(p) \oplus ker(p)) \simeq T(Im(p)) \oplus T(ker(p)) \simeq Im(p) \oplus T(Im(p)) \oplus T(ker(p)) \simeq Im(p) \oplus TA.$$

Since TA is an object of $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$, the morphism $TA \to TA \simeq Im(p) \oplus TA$ which identifies TA with the direct factor TA in $Im(p) \oplus TA$ admits a retraction, so $Im(p)$ must be isomorphic to an object of $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$. Hence $\mathcal{D}^{\mathrm{b}}(\mathcal{F})$ is idempotent complete. Then $\mathcal{D}^{\mathrm{b}}(\mathcal{E})$ is also idempotent complete since by 10.1 it is the kernel of $\mathcal{D}^{\mathrm{b}}(\mathcal{F}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{F}/\mathcal{E})$. □

Second proof of proposition 5.1. By Waldhausen's fibration theorem ([Wal85] theorem 1.6.4) there is a homotopy fibration of K-theory spaces of Waldhausen categories (categories with cofibrations and weak equivalences)

$$K(wCh^{\mathrm{b}}(\mathcal{U})^{v}) \to K(wCh^{\mathrm{b}}(\mathcal{U})) \to K(vCh^{\mathrm{b}}(\mathcal{U}))$$

in which w is the set of chain maps with cone homotopy equivalent in $\mathcal{U}$ to an acyclic complex of $\mathcal{U}$ and v is the set of chain maps with cone homotopy equivalent in $\mathcal{U}/\mathcal{A}$ to an acyclic complex of $\mathcal{U}/\mathcal{A}$. Cofibrations are degree wise admissible monomorphisms. Let a be the set of chain maps in $Ch^{\mathrm{b}}(\mathcal{A})$ with cone homotopy equivalent to an acyclic complex in $\mathcal{A}$. By theorem 10.1, the maps of complicial BiWaldhausen categories (for the terminology see [TT90])

$$aCh^{\mathrm{b}}(\mathcal{A}) \to wCh^{\mathrm{b}}(\mathcal{U})^{v}, \quad \text{and}$$

$$vCh^{\mathrm{b}}(\mathcal{U}) \to vCh^{\mathrm{b}}(\mathcal{U}/\mathcal{A})$$

induce equivalences of their bounded derived categories. Using Thomason's theorems [TT90] 1.9.8 and 1.11.7 the following maps of K-theory spaces are homotopy equivalences

$$K(\mathcal{A}) \to K(aCh^{\mathrm{b}}(\mathcal{A})) \to K(wCh^{\mathrm{b}}(\mathcal{U})^{v}),$$

$$K(\mathcal{U}) \to K(wCh^{\mathrm{b}}(\mathcal{U})), \quad \text{and}$$

$$K(\mathcal{U}/\mathcal{A}) \to K(vCh^{\mathrm{b}}(\mathcal{U}/\mathcal{A})) \leftarrow K(vCh^{\mathrm{b}}(\mathcal{U})).$$

□

11 Comparison with other definitions of K_{-i}

We will show that our definition of negative $I\!K$-theory coincides with the definition of Bass for rings and of Karoubi for split exact categories. We conjecture that our spectrum $I\!K(Vect(X))$ coincides with Thomason-Trobaugh's $K^B(X)$ when X is a quasi separated scheme which supports an ample family of line bundles.

11.1 Theorem. *Let A be a ring then $I\!K_i(A) = K_i(A)$ as defined by Quillen ($i \geq 0$) and by Bass ($i \leq 0$). If $\mathcal{A}$ is split exact then $I\!K_i(A) = K_i(A)$, $i \leq 0$ as defined by Karoubi ([Kar70] see also [Ped84]).*

Proof. The statement about non negative K-theories is incorporated in our definition of $I\!K_i$. Our negative $I\!K$-theory coincides with Karoubi's since it is independent of the chosen flasque resolution by theorem 7.12. In [Kar71] Karoubi showed that his negative K-groups coincide with Bass' groups (see also [Ped84]). □

11.2 Conjecture. *If X is a quasi separated scheme which supports an ample family of line bundles, then $I\!K_i(Vect(X)) = K_i^B(X)$ where $Vect(X)$ is the exact category of vector bundles on X and K_i^B is the K-theory functor as defined in [TT90].*

11.3 *Sketch of a proof:* For positive K-theory it is [TT90] corollary 3.9. By theorem 7.12, we can calculate negative $I\!K$-groups by means of an appropriately chosen flasque resolution. We would like to show that $K(S^nVect(X))$ is a natural retract of $K(Vect(X[t_1, t_1^{-1}, ...t_n, t_n^{-1}]))$. We only have to find an exact functor

$$Vect(X[t_1, t_1^{-1}]) \to S(Vect(X))$$

which fits into the commutative diagram

$$\begin{tikzcd} \mathbb{Z}[t_1, t_1^{-1}] \times Vect(X) \arrow[r] \arrow[d, "\otimes"] & S(\mathbb{Z}) \times Vect(X) \arrow[d, "\otimes"] \\ Vect(X[t_1, t_1^{-1}]) \arrow[r] & S(Vect(X)). \end{tikzcd}$$

Then the same argument as in the proof of theorem 9.4 shows that $K(SVect(X))$ is a direct factor of $K(Vect(X[t_1, t_1^{-1}]))$. Iterating it would follow that $I\!K(Vect(X))$ is a natural retract of $K^B(X)$ if X supports an ample family of line bundles. Thomason proved a Mayer-Vietoris principle for K^B. Therefore, as a natural retract of K^B, $I\!K(Vect(\quad))$ also satisfies Mayer-Vietoris. Since on affine schemes both definitions coincide (11.1), the Mayer-Vietoris principle would allow us to conclude the conjecture. We know that this strategy works for affine schemes, i.e. the exact functor to be constructed exists in this case. We don't know whether it exists in general. One difficulty is that even if $\mathcal{A} \subset \mathcal{U}$ is left or right s-filtering, $\tilde{\mathcal{A}} \subset \tilde{\mathcal{U}}$ is in general not.

12 Applications

Here we will give two applications basically of theorem 7.12. We will see that the long exact sequence of K-groups associated to a localization of a ring with respect to a denominator set of non zero divisors is a consequence of theorem 7.12 (compare [Car80]). As a second application we will construct a homology theory on finite simplicial complexes as in the spirit of [PW89].

12.1 Localization of rings. Let R be a ring with unit, $S \subset R$ a multiplicatively closed subset of elements satisfying the Øre condition

$$\forall s \in S,\ a \in R\ \exists t \in S,\ b \in R:\ ta = bs$$

and such that right multiplication with elements of S is injective. It follows that S is a left dominator set and that the ring $S^{-1}R$ can be obtained from R by a calculus of left fraction.

Example. Let $\alpha : A \to A$ be an endomorphism of rings, and let $A_\alpha[t]$ be the twisted polynomial ring which as an A module is $A[t]$ but where $ta = \alpha(a)t$ for $a \in A$. Then the multiplicative set $\{t^i |\ i \in \mathbb{N}\}$ satisfies the two conditions above. As in [Bad99] we let $A' = colim(A \xrightarrow{\alpha} A \xrightarrow{\alpha} A \xrightarrow{\alpha} \cdots)$ and $\alpha' : A' \to A'$ the endomorphism induced by α. Then α' is an automorphism and

$$[\{t^i |\ i \in \mathbb{N}\}^{-1}]A_\alpha[t] = [\{t^i |\ i \in \mathbb{N}\}^{-1}]A'_{\alpha'}[t] = A'_{\alpha'}[t, t^{-1}].$$

We remark that in general the localization map $A_\alpha[t] \to [\{t^i |\ i \in \mathbb{N}\}^{-1}]A_\alpha[t]$ is not injective.

Let $S_n \subset M_n(R)$ be the multiplicative set of $n \times n$ upper triangular matrices with diagonal entries in S. We consider R^n as a left R-module and write its elements as row vectors. Moreover right multiplication by $s \in S_n$ is then a morphism of left R-modules which is injective since right multiplication with elements of S is injective. We write $\mathcal{H}_S(R)$ for the full subcategory of left R-modules which are cokernels of the right multiplication by s for s in some S_n. It is easy to see that $\mathcal{H}_S(R)$ is extension closed in the category of left R-modules, we therefore consider it as an exact category. The main argument is that an extension of upper triangular matrices is an upper triangular matrix. Let $F_1(R)$ be the full category of left R-modules M for which there is an exact sequence

$$0 \to R^n \to M \to T \to 0$$

with $T \in \mathcal{H}_S(R)$. We claim that $F_1(R)$ is extension closed in the category of left R-modules. Let $0 \to M \xrightarrow{j} E \xrightarrow{p} N \to 0$ be an exact sequence of R-modules with M and N in $F_1(R)$. There are injective maps $i_M : R^m \to M$ and $i_N : R^n \to N$ with quotients T_M and T_N in $\mathcal{H}_S(R)$. Since R^n is a free R-module there is a lifting $i : R^n \to E$ with $p \circ i = i_N$. The map $(j \circ i_M, i) : R^m \oplus R^n \to E$ is injective as it is an extension of i_M and i_N. Its cokernel is an extension of T_M and T_N lying therefore in $\mathcal{H}_S(R)$. Hence, E is an object of $F_1(R)$. By the extension closure of $F_1(R)$ in the category of left R-modules, we consider $F_1(R)$ as en exact category.

12.2 Lemma. *$\mathcal{H}_S(R)$ is a right s-filtering subcategory of $F_1(R)$. The quotient category $F_1(R)/\mathcal{H}_S(R)$ is equivalent to $F(S^{-1}R)$, the category of finitely generated free left $S^{-1}R$-modules. Furthermore, $F(R) \subset F_1(R)$ is a $\mathbb{K}$-theory equivalence.*

Proof. i) We first prove that any diagram $R^n \xrightarrow{a} R^m \xleftarrow{s} R^m$ with $s \in S_m$ can be completed into a commutative diagram of left R-modules

$$\begin{tikzcd} R^n \arrow[r, dashed] \arrow[d, dashed, tail, "t"'] & R^m \arrow[d, "s"] \\ R^n \arrow[r, "a"'] & R^m \end{tikzcd}$$

with $t \in S_n$. The proof proceeds by induction on m. For $m = 1$ this is the Øre condition. Suppose the claim is proven for $1, ..., m-1$. The map $s : R^m \to R^m$ fits into a commutative diagram

$$\begin{tikzcd} R^{m-1} \arrow[r, tail, "j"] \arrow[d, "r"'] & R^m \arrow[r, two heads, "q"] \arrow[d, "s"'] & R \arrow[d, "r'"] \\ R^{m-1} \arrow[r, tail, "i"] & R^m \arrow[r, two heads, "p"] & R \end{tikzcd}$$

in which the horizontal lines are short exact sequences and $r \in S_{m-1}$ and $r' \in S$. By induction hypothesis there is a $t_1 \in S_n$ and a $b' : R^n \to R$ such that $r'b' = pat_1$. Since R^n is a free left R-module, there is a $b : R^n \to R^m$ with $qb = b'$. Moreover, since $p(at_1 - sb) = 0$ there exists a $u : R^n \to R^{m-1}$ such that $iu = at_1 - sb$. By induction hypothesis, there is a $v : R^n \to R^{m-1}$ and a $t_2 \in S_n$ with $rv = ut_2$. Let $c = jv + bt_2 : R^n \to R^m$. We then have $sc = at_1t_2$, and $t_1t_2 \in S_n$.

ii) - iv) will show that $\mathcal{H}_S(R)$ is right s-filtering in $F_1(R)$.

ii) We show that the inclusion $\mathcal{H}_S(R) \subset F_1(R)$ is right filtering. Given a map $g : M \to T$ with $M \in F_1(R)$ and $T \in \mathcal{H}_S(R)$. There is a monomorphism $j : R^m \to M$ with cokernel T_M in $\mathcal{H}_S(R)$, and there is an exact sequence $0 \to R^n \xrightarrow{s} R^n \xrightarrow{q} T \to 0$ with $s \in S_n$. Since R^m is free, there is a map $a : R^m \to R^n$ such that $qa = gj$. Applying i) to the diagram $R^m \xrightarrow{a} R^n \xleftarrow{s} R^n$, we find a $t \in R^m$ and a $b : R^m \to R^n$ such that $at = sb$. This shows that g factors through the quotient map $M \to coker(jt)$. Since $coker(jt)$ is an extension of $coker(j) = T_M \in \mathcal{H}_S(R)$ and $coker(t) \in \mathcal{H}_S(R)$ it lies in $\mathcal{H}_S(R)$.

iii) Admissible monomorphisms $i : T \rightarrowtail M$ with $T \in \mathcal{H}_S(R)$ and $M \in F_1(R)$ are special. Let p be the cokernel map $p : M \to coker(i)$. Since $coker(i)$ is an object of $F_1(R)$, there is an admissible monomorphism $j : R^n \to coker(i)$ with cokernel T' in $\mathcal{H}_S(R)$. As R^n is free, there is a map $j_M : R^n \to M$ with $pj_M = j$. j_M is an injective R-module homomorphism, because composed with p it is one. Its cokernel is an extension of T' and T and lies therefore in $\mathcal{H}_S(R)$. It follows that j_M is an admissible monomorphism, its cokernel map $q : M \to coker(j_M)$ an admissible epimorphism and qi is an admissible monomorphism.

iv) $\mathcal{H}_S(R)$ is by definition closed under extensions in $F_1(R)$. It is also closed under admissible subobjects and admissible quotients. Let $M \in F_1(R)$ be an admissible subobject or an admissible quotient of $T \in \mathcal{H}_S(R)$. There is an admissible monomorphism $i : R^n \to M$ with quotient in $\mathcal{H}_S(R)$. Then R^n is an admissible subobject of T, by composition if M itself is a admissible subobject, or by the lifting property if M is an admissible quotient of T. If $S^{-1}R$ is not zero, then $R^n = 0$, otherwise after tensoring with $S^{-1}R$ we would find $S^{-1}R^n$ an admissible subobject of $S^{-1}T = 0$ since the functor $S^{-1}R \otimes_R$ is exact for S a left dominator set. It follows that $M \in \mathcal{H}_S(R)$. If $S^{-1}R = 0$ we have $\mathcal{H}_S(R) = F_1(R)$.

v) We identify the quotient category $F_1(R)/\mathcal{H}_S(R)$ with $F(S^{-1}R)$ using the universal properties of quotients and localizations.

vi) $F(R) \subset F_1(R)$ is a $\mathbb{K}$-theory equivalence by resolution (theorem 8.3) applied to the inclusion of $\widetilde{F(R)}$ into the exact category generated by $\widetilde{F(R)}$ and $F_1(R)$ which is an exact subcategory of $\widetilde{F(R)}_1$ (see 3.4), every object of $\widetilde{F(R)}_1^{F(R)}$ is an extension of objects of $\widetilde{F(R)}$ and $F_1(R)$. □

12.3 Corollary. *There is a homotopy fibration of non-connective spectra*

$$\mathbb{K}(\mathcal{H}_S(R)) \to \mathbb{K}(F(R)) \to \mathbb{K}(F(S^{-1}R)).$$

Proof. This is theorem 7.12 together with lemma 12.2 and the resolution theorem 8.3. □

12.4 A homology theory on finite simplicial complexes. Let $\mathbb{R}_+ = [0, \infty)$, let $M \subset \mathbb{R}_+^n$ such that with $m \in M$ we have $tm \in M$ for $t \in \mathbb{R}_+$ and if $n \leq m$ then $n \in M$ as well. For $\mathcal{E}$ an exact category, the category $\mathcal{C}_M(\mathcal{E}) := \mathcal{C}_{M \cap \mathbb{N}^n}(\mathcal{E})$ is defined in section 6. To the spectrum $\mathbb{K}(\mathcal{E})$ there is associated a reduced homology theory $\tilde{H}_*(\,.\, ; \mathbb{K}(\mathcal{E})) = \pi_*(\,.\, \wedge \mathbb{K}(\mathcal{E}))$. We write $\dot{M}$ for the one point compactification of M.

12.5 Theorem. *Let M be as in 12.4 and let $\mathcal{E}$ be an exact category, then*

$$\mathbb{K}_*(\mathcal{C}_M(\mathcal{E})) = \tilde{H}_*(\dot{M}; \mathbb{K}(\mathcal{E})).$$

Proof. Let $\mathcal{P}_M$ be the set of saturated subsets V of M such that if $v, w \in V$ then $sv + tw \in M$ for all $s, t \in \mathbb{R}_+$. The set $\mathcal{P}_M$ is obviously finite and a poset by inclusion. The hypothesis on M are such that $colim_{V \in \mathcal{P}_M} V = \bigcup_{V \in \mathcal{P}_M} V = M$. Let $Sets_f$ be the category of finite sets with inclusions as morphisms. Every $V \in \mathcal{P}_M$ spans a linear vector space $< V >$ in $\mathbb{R}^n$. We have $V =< V > \cap \mathbb{R}^n_+$. The category $\mathcal{P}_M$ is a full subcategory of $Sets_f$ if we identify $V \in \mathcal{P}_M$ with the set of $W \in \mathcal{P}_M$ such that $dim < W >= 1$.

For $V \subset \mathbb{R}^n_+$ we let $L_V\mathcal{E}$ be the full subcategory of $L_n\mathcal{E}$ (see 7.1) on objects $l(A_i)$ with $l \in V$. If $V \subset W$ then $L_V\mathcal{E} \subset L_W\mathcal{E}$.

Since $\mathbb{K}(\mathcal{C}_{\mathbb{R}_+} L_{\mathbb{R}_+} \mathbb{Z})$ is contractible, there is a Sp^Σ-map $\sigma : I \to \mathbb{K}(\mathcal{C}_{\mathbb{R}_+}(L_{\mathbb{R}_+}\mathbb{Z}))$ making the following diagram commute

$$\begin{tikzcd} S^0 \arrow[r, "\eta"] \arrow[d] & \mathbb{K}(\mathbb{Z}) \arrow[d] \\ I \arrow[r, "\sigma"] & \mathbb{K}(\mathcal{C}_{\mathbb{R}_+} L_{\mathbb{R}_+} \mathbb{Z}). \end{tikzcd}$$

This induces a natural transformation of functors $Sets_f \to Sp^\Sigma$ which for a finite set p is given by $\sigma^{\wedge p} : I^{\wedge p} \to \mathbb{K}(\mathcal{C}_{\mathbb{R}_+} L_{\mathbb{R}_+} \mathbb{Z})^{\wedge p} \xrightarrow{\wedge} \mathbb{K}(\mathcal{C}_{\mathbb{R}^p_+} L_{\mathbb{R}^p_+} \mathbb{Z})$. The choice of a homeomorphism $\dot{\mathbb{R}}_+ \to I$ induces a natural transformation of functors $\mathcal{P}_M \to Sp^\Sigma$ which for $V \in \mathcal{P}_M$ is a map $\dot{V} \to \mathbb{K}(\mathcal{C}_V L_V \mathbb{Z})$ given by the inclusion of $\mathcal{P}_M$ into $Sets_f$. Smashing with $\mathbb{K}(\mathcal{E})$ yields a natural transformation of functors

$$\tau_V : \dot{V} \wedge \mathbb{K}(\mathcal{E}) \to \mathbb{K}(\mathcal{C}_V L_V \mathcal{E}).$$

This is a weak equivalence of spectra for every $V \in \mathcal{P}_M$, it is the identity for $V = 0$, otherwise both sides are contractible. Taking homotopy colimits, we obtain weak equivalences of spectra

$$colim_{V \in \mathcal{P}_M} \dot{V} \wedge \mathbb{K}(\mathcal{E}) \xleftarrow{\sim} hocolim_{V \in \mathcal{P}_M} \dot{V} \wedge \mathbb{K}(\mathcal{E}) \xrightarrow{\sim} hocolim_{V \in \mathcal{P}_M} \mathbb{K}(\mathcal{C}_V L_V \mathcal{E}) \xrightarrow{\sim} \mathbb{K}(\mathcal{C}_M L_M \mathcal{E}).$$

The map $hocolim_{V \in \mathcal{P}_M} \dot{V} \wedge \mathbb{K}(\mathcal{E}) \to colim_{V \in \mathcal{P}_M} \dot{V} \wedge \mathbb{K}(\mathcal{E})$ is a weak equivalence since for all saturated subposets $S \subset T \subset \mathcal{P}_M$ we have a cofibration $colim_{V \in S} \dot{V} \to colim_{V \in T} \dot{V}$. The middle map is a weak equivalence since τ is pointwise a weak equivalence. The map $hocolim_{V \in \mathcal{P}_M} \mathbb{K}(\mathcal{C}_V L_V \mathcal{E}) \to \mathbb{K}(\mathcal{C}_M L_M \mathcal{E})$ is a weak equivalence since both sides satisfy a Mayer-Vietoris principle in M (for the right hand term see 6.18 and 7.12). But we have $\dot{M} \wedge \mathbb{K}(\mathcal{E}) = colim_{V \in \mathcal{P}_M} \dot{V} \wedge \mathbb{K}(\mathcal{E})$ and $\mathbb{K}(\mathcal{C}_M L_M \mathcal{E}) \simeq \mathbb{K}(\mathcal{C}_M(\mathcal{E}))$.

$\square$

12.6 Let K be a finite simplicial complex, i.e. a finite set K_0 called points together with a subset K of all subsets of K_0 such that all subsets of a set of K are also in K. We associate a saturated subposet $\mathcal{O}(K) \subset {I\!\!R_+}^{K_0}$ with K by

$$\mathcal{O}(K) = \{x \in {I\!\!R_+}^{K_0} |\ \exists \sigma \in K, n_t \in I\!\!R_+,\ t \in \sigma :\ x = \sum_{t\in\sigma} n_t \delta_t\}.$$

This space is homeomorphic to the open cone of [PW89].

12.7 Theorem. *(compare theorem 3.1 of [PW89]). Let $\mathcal{E}$ be an exact category and K a finite simplicial complex, then*

$$I\!\!K_i(\mathcal{C}_{\mathcal{O}(K)}(\mathcal{E})) = \tilde{H}_{i-1}(K; I\!\!K(\mathcal{E})), \qquad i \in Z\!\!\!Z.$$

Proof. We remark that the one point compactification of $\mathcal{O}(K)$ is homeomorphic to the one fold unreduced topological suspension of K and that $M = \mathcal{O}(K)$ satisfies the hypothesis of 12.4. So the theorem follows from theorem 12.5. □

PART II

13 Complétion en groupe et construction plus

Dans cette section on construit pour une catégorie monoïdale symétrique $\mathcal{A}$ pour laquelle tous les morphismes sont des isomorphismes un groupe $G(\mathcal{A})$ tel que la construction plus appliquée a son classifiant a le même type d'homotopie que la composante connexe de la complétion en groupe du H-espace $B\mathcal{A}$. Ceci généralise la construction $BGl(R)^+$ bien connue en K-théorie algébrique.

13.1 Soit $\mathcal{A}$ une catégorie monoïdale symétrique unitaire telle que:

i) $\mathcal{A} = Iso\mathcal{A}$ et

ii) le foncteur $A\oplus\_ : \mathcal{A} \to \mathcal{A}$ soit fidèle pour chaque $A \in |\mathcal{A}|$ (comparer les hypothèses de la construction $\mathcal{S}^{-1}\mathcal{S}$ de Quillen, voir [Gra76]).

On va lui associer un groupe quasi-parfait $G(\mathcal{A})$ tel que $BG(\mathcal{A})^+$ ait le même type d'homotopie que $B(\mathcal{A}^{-1}\mathcal{A})_0$, la composante connexe de $B(\mathcal{A}^{-1}\mathcal{A})$.

Si $\mathcal{M}$ est un ensemble, notons par $\mathcal{P}_f(\mathcal{M})$ la catégorie des sous-ensembles finis de $\mathcal{M}$. On écrira $S \subset_f \mathcal{M}$ pour un objet de $\mathcal{P}_f(\mathcal{M})$. Les morphismes sont les inclusions. Cette catégorie est filtrante (filtered category) au sens de [Mac71]. En particulier la flèche naturelle de la colimite homotopique vers la colimite sur cette catégorie est une équivalence d'homotopie (voir [BK72]).

Pour un ensemble fini $S \subset_f |\mathcal{A}|$ on choisit un ordre c'est à dire une bijection $\sigma : \{1, 2, ..., |S|\} \to S$. Posons $G_S(\mathcal{A}) = aut_{\mathcal{A}}(\bigoplus S)$ où $\bigoplus S = (...((\sigma_1 \oplus \sigma_2) \oplus \sigma_3)... \oplus \sigma_{|S|})$. L'inclusion $S \subset T$ induit un morphisme canonique $G_S(\mathcal{A}) \to G_T(\mathcal{A})$. A l'ordre près c'est $\alpha \mapsto \alpha \oplus id_{\oplus T\setminus S}$. Qu'on obtienne de cette façon un foncteur $G_\bullet(\mathcal{A}) : \mathcal{P}_f(|\mathcal{A}|) \to Groups$ résulte de la théorie de cohérence pour les catégories monoïdales symétriques (voir [Mac71]). On définit alors:

$$G(\mathcal{A}) := colim_{S\subset_f |\mathcal{A}|} G_S(\mathcal{A}).$$

13.2 Remarque. Si $\mathcal{A} = Iso\mathcal{B}$ pour une catégorie additive $\mathcal{B}$, il n'est pas nécessaire de choisir un ordre car $G_S(\mathcal{A})$ peut s'écrire comme groupe des matrices inversibles $(\alpha_{i,j})_{i,j\in S}$ où $\alpha_{i,j} \in \mathcal{A}(j, i)$ et on a $G(Iso\mathcal{B}) = GL(\mathcal{B})$.

Pour des raisons de stabilisation on demande que $\mathcal{A}$ satisfasse aussi la condition suivante:

iii) pour $S \subset_f |\mathcal{A}|$ il existe un $A \in \mathcal{A}$ et $T \subset_f |\mathcal{A}|$ tels que $S \cap T = \emptyset$ et $A \oplus \bigoplus S \simeq \bigoplus T$.

En rajoutant des objets isomorphes aux objets dans $\mathcal{A}$ on peut toujours supposer iii) satisfait. Par exemple la catégorie permutative associée à une catégorie monoïdale symétrique vérifie iii). La condition iii) entrane par exemple que pour chaque $S \subset_f |\mathcal{A}|$ l'inclusion $G_S(\mathcal{A}) \subset G(\mathcal{A})$ se factorise en $G_S(\mathcal{A}) \subset aut(\bigoplus S \oplus \bigoplus S \oplus \bigoplus S) \subset G(\mathcal{A})$.

13.3 Lemme. *$G(\mathcal{A})$ est quasi-parfait.*

Démonstration. Soit $g, h \in G(\mathcal{A})$. Il existe un ensemble fini $S \subset_f |\mathcal{A}|$ tel que $g, h \in G_S(\mathcal{A})$. Dans $aut(\bigoplus S \oplus \bigoplus S \bigoplus S)$ on a

$$ghg^{-1}h^{-1} \oplus id_{\oplus S} \oplus id_{\oplus S} = [g \oplus g^{-1} \oplus id_{\oplus S}, h \oplus id_{\oplus S} \oplus h^{-1}].$$

Il suffit alors de voir que $g \oplus g^{-1}$ est dans le commutateur. Or
$(1_S \oplus g^{-1} \oplus 1_S) \circ (123) \circ (g \oplus 1_S \oplus 1_S) = (123)$ où (123) désigne la permutation cyclique $1 \mapsto 2 \mapsto 3 \mapsto 1$ et $(123) = (23)(12)(23)^{-1}(12)^{-1} \in [aut(\bigoplus S \oplus \bigoplus S \oplus \bigoplus S), aut(\bigoplus S \oplus \bigoplus S \oplus \bigoplus S)] \subset [G(\mathcal{A}), G(\mathcal{A})]$.
D'où le lemme. □

Pour comparer $BG(\mathcal{A})^+$ avec $B(\mathcal{A}^{-1}\mathcal{A})_0$ on va construire une flèche bien définie à homotopie près entre ces deux espaces qui sera un isomorphisme en homologie. Afin de pouvoir démontrer qu'elle est une équivalence d'homotopie il faut s'assurer que $BG(\mathcal{A})^+$ est un espace nilpotent (une application entre deux espaces connexes nilpotents de type d'un CW-complexe induisant un isomorphisme en homologie entière est une équivalence d'homotopie, voir [Ber82]).

13.4 Lemme. *Soit G un groupe quasi-parfait de commutateur (parfait) E tel que pour tous $g \in G$ et $e_1, ..., e_n \in E$ il existe un $e \in E$ tel que $ge_ig^{-1} = ee_ie^{-1}$ $i = 1, ..., n$. Alors BG^+ est un espace nilpotent.*

Démonstration. $\pi_1(BG^+)$ agit sur lui-même par conjugaison, mais $\pi_1(BG^+) = G/E$ est abélien donc l'action sur $\pi_1(BG^+)$ est nilpotente. Pour voir que $\pi_1(BG^+)$ agit de manière nilpotente sur $\pi_i(BG^+)$, $i \geq 2$, il suffit de le prouver pour son action sur les groupes d'homotopie de son revêtement universel, qui en l'occurrence est BE^+. Or le groupe $\pi_1(BG^+) = G/E$ agit sur $\pi_i(BE^+)$ par le procédé suivant: soit $[g] \in G/E$ représenté par $g \in G$, la conjugaison par g induit un morphisme de groupes $g : E \to E : e \mapsto geg^{-1}$ qui induit une application pointée $g^+ : BE^+ \to BE^+$ et donc un morphisme de groupes $\pi_i(BE^+) \to \pi_i(BE^+)$. BE^+ est un espace simplement connexe donc nilpotent. Un théorème de Hilton, Roitberg, Singer (voir [HRS78]) dit qu'un groupe opérant à homotopie pointée près par des équivalences d'homotopie sur un espace nilpotent agit de manière nilpotente sur l'homotopie de cet espace ssi il agit de manière nilpotente sur l'homologie. Or dans notre cas les hypothèses du lemme assurent que l'action de G sur $H_*(BE^+) = H_*(E)$ est triviale donc nilpotente. D'où le lemme. □

La condition de stabilité iii) entrane que $G(\mathcal{A})$ remplit les hypothèses du lemme. Avec le lemme suivant la démonstration que $BG(\mathcal{A})^+$ et $B(\mathcal{A}^{-1}\mathcal{A})_0$ ont le même type d'homotopie sera achevée.

13.5 Lemme. *Il existe des flèches naturelles*

$$G(\mathcal{A}) \xleftarrow{\sim} hocolim_{S\subset_f|\mathcal{A}|} G_S(\mathcal{A}) \to (\mathcal{A}^{-1}\mathcal{A})_0$$

dont la première est une équivalence d'homotopie et la deuxième un isomorphisme en homologie.

Démonstration. On choisit la construction de la colimite homotopique au niveau des catégories due à Thomason (voir [Tho79]). Un objet de $hocolim_{S\subset_f|\mathcal{A}|} G_S(\mathcal{A})$ est donc un $S \subset_f |\mathcal{A}|$. Un morphisme $S \to T$ est une inclusion $S \subset T$ plus un élément $g \in G_T(\mathcal{A})$. La composition des morphismes $g : S \to T$ et $h : T \to R$ est donnée par $hg : S \to R$ où g est considéré comme étant dans $G_R(\mathcal{A})$ via l'application $G_{T\subset R}(\mathcal{A}) : G_T(\mathcal{A}) \to G_R(\mathcal{A})$.

Le foncteur $hocolim_{S\subset_f|\mathcal{A}|} G_S(\mathcal{A}) \to G(\mathcal{A})$ envoie un objet sur l'unique objet de $G(\mathcal{A})$ et un morphisme sur le morphisme correspondant dans $G(\mathcal{A})$. Ceci est une équivalence d'homotopie car $\mathcal{P}_f(|\mathcal{A}|)$ est une catégorie filtrante au sens de [Mac71].

Le foncteur $hocolim_{S\subset_f|\mathcal{A}|} G_S(\mathcal{A}) \to (\mathcal{A}^{-1}\mathcal{A})_0$ envoie l'objet S sur l'objet $(\bigoplus S, \bigoplus S)$ et un morphisme $g : S \to T$ sur le morphisme $(\bigoplus T \setminus S, id, g)$.

On a le diagramme commutatif suivant où
$colim_{A\in\pi_0\mathcal{A}} H_*(aut_\mathcal{A}(A)) \to H_*((\mathcal{A}^{-1}\mathcal{A})_0)$ correspond à la localisation de
$H_*(\mathcal{A}) \to H_*(\mathcal{A}^{-1}\mathcal{A})$ par rapport au monoïde central $\pi_0\mathcal{A}$ et qui est d'après les travaux de Quillen (voir [Gra76]) un isomorphisme

$$\begin{tikzcd}
 & colim_{A\in\pi_0\mathcal{A}} H_*(aut_\mathcal{A}(A)) \arrow[dl] \arrow[d] \arrow[dr, "\sim"] & \\
H_*(G(\mathcal{A})) & H_*(hocolim_{S\subset_f|\mathcal{A}|} G_S(\mathcal{A})) \arrow[l, "\sim"'] \arrow[r] & H_*((\mathcal{A}^{-1}\mathcal{A})_0)
\end{tikzcd}$$

Ici $colim_{A\in\pi_0\mathcal{A}} H_*(aut_\mathcal{A}(A)) \to H_*(G(\mathcal{A})) = colim_{S\subset_f|\mathcal{A}|} H_*(G_S(\mathcal{A}))$ est induit par
$id : aut_\mathcal{A}(A) \to G_{\{A\}}(\mathcal{A}) \to G(\mathcal{A})$. La flèche inverse

$H_*(G(\mathcal{A})) = colim_{S \subset_f |\mathcal{A}|} H_*(G_S(\mathcal{A})) \to colim_{A \in \pi_0 \mathcal{A}} H_*(aut_{\mathcal{A}}(A))$ est induite par l'application "identité" $id : G_S(\mathcal{A}) \to aut_{\mathcal{A}}(\bigoplus S)$. On vérifie qu' elle est bien définie et bien l'inverse. D'où le lemme.

$\square$

14 Une démonstration topologique de 5.1 dans le cas additif filtrant

Pour illustrer l'utilité de la construction $G(\mathcal{A})$ de la section 13 on va donner une démonstration "élémentaire" du théorème de localisation de Pedersen et Weibel ([PW85] theorem 5.3, voir aussi [CP97]), c'est le cas additif filtrant de la proposition clé 5.1. Notre démonstration ne fait pas recours aux méthodes de Thomason ([Tho82], [TT90]) ni à celles de Waldhausen ([Wal85]). Elle utilise les méthode classique de la topologie algébrique.

On rappelle quelques notations. Soit $\mathcal{U}$ une catégorie additive (toute catégorie notée par une lettre calligraphique est supposée petite). Si U et V sont deux objets de $\mathcal{U}$, on appelle morphisme direct de U dans V la donnée de deux flèches $s : U \to V$ et $p : V \to U$ telles que $p \circ s = id_U$. Les morphismes directs sont les flèches d'une catégorie dont les objets sont les objets de $\mathcal{U}$. On notera un morphisme direct comme suit $(p, s) : \ U \twoheadrightarrow V$.

14.1 Définition. (voir [Kar70]) Soit $\mathcal{A}$ une sous-catégorie additive, pleine de $\mathcal{U}$ et soit U un objet de $\mathcal{U}$. Une $\mathcal{A}$-filtration de U est un ensemble I de morphismes directs $(p_i, s_i) : \ A_i \twoheadrightarrow U$, $i \in I$ de sources des objets de $\mathcal{A}$ vérifiant (A1)-(A3):

(A1) Si $(p_i, s_i) : \ A_i \twoheadrightarrow U$ et $(p_j, s_j) : \ A_j \twoheadrightarrow U$ sont deux morphismes directs de la filtration, il existe un troisième morphisme de la filtration $(p_k, s_k) : \ A_i \twoheadrightarrow U$, des morphisme directe $A_i \twoheadrightarrow A_k$ qui rendent commutatif le diagramme

$$\begin{tikzcd}
A_i \arrow[dr, "|" marking] \arrow[drr, "|" marking] & & \\
& A_k \arrow[r, "|" marking] & U \\
A_i \arrow[ur, "|" marking] \arrow[urr, "|" marking] & &
\end{tikzcd}$$

(A2) Pour chaque objet A de $\mathcal{A}$ et chaque flèche $f : A \to U$ il existe un morphisme direct de la filtration $(p_i, s_i) : \ A_i \twoheadrightarrow U$ tel que f se factorise à travers A_i, i.e. il existe $g : A \to A_i$ tel que $f = s_i \circ g$.

(A3) Pour chaque objet A de $\mathcal{A}$ et chaque flèche $f : U \to A$ il existe un morphisme direct de la filtration $(p_i, s_i) : A_i \longrightarrow\!\!\!\!\!\rightarrow U$ tel que f se factorise à travers A_i, i.e. il existe $g : A_i \to A$ tel que $f = g \circ p_i$.

On dit qu'un objet U de $\mathcal{U}$ est $\mathcal{A}$-filtré si U possède une $\mathcal{A}$-filtration. On dit qu'un objet U de $\mathcal{U}$ est stablement $\mathcal{A}$-filtré s'il existe un morphisme direct $(p, s) : U \longrightarrow\!\!\!\!\!\rightarrow V$ tel que V soit $\mathcal{A}$-filtré. On dit que la catégorie $\mathcal{U}$ est (stablement) $\mathcal{A}$-filtrée si tout objet de $\mathcal{U}$ est (stablement) $\mathcal{A}$-filtré.

Si $\mathcal{C}$ est une catégorie additive, on note $K(\mathcal{C})$ l'espace de la K-théorie de Quillen du pseudo-abélianisé $\tilde{\mathcal{C}}$ de (idempotent completion of) $\mathcal{C}$. On choisit la construction fonctorielle $K(\mathcal{C}) = B(Iso\tilde{\mathcal{C}})^{-1}(Iso\tilde{\mathcal{C}})$ (voir [Gra76]). Pour alléger la notation on écrira souvent $\mathcal{C}^{-1}\mathcal{C}$ au lieu de $(Iso\tilde{\mathcal{C}})^{-1}(Iso\tilde{\mathcal{C}})$. Par $K(\mathcal{C})_0$ on entend la composante connexe de l'unité du H-groupe $K(\mathcal{C})$, on a donc $K(\mathcal{C}) \simeq K_0(\mathcal{C}) \times K(\mathcal{C})_0$.

On remarque que si $\mathcal{U}$ est stablement $\mathcal{A}$-filtrée alors $\tilde{\mathcal{U}}$ est stablement $\tilde{\mathcal{A}}$-filtrée.

14.2 Définition. Soit $\mathcal{U}$ une catégorie additive et $\mathcal{A}$ une sous-catégorie pleine additive. La catégorie quotient $\mathcal{U}/\mathcal{A}$ est la catégorie dont les objets sont les objets de $\mathcal{U}$ et dont les morphismes sont les morphismes de $\mathcal{U}$ modulo ceux qui se factorisent par $\mathcal{A}$.

On remarque que la catégorie $\mathcal{U}/\mathcal{A}$ est additive.

14.3 Théorème. *(theorem 5.3 [PW85]) Soit $\mathcal{U}$ une catégorie additive stablement $\mathcal{A}$-filtrée par une sous-catégorie additive, pleine $\mathcal{A}$. Alors on a une fibration à homotopie près*

$$K(\mathcal{A}) \to K(\mathcal{U}) \to K(\mathcal{U}/\mathcal{A}).$$

14.4 L'idée est d'utiliser le fait que $K(\mathcal{A})_0 \simeq BGL(\mathcal{A})^+$, reliés par une suite d'équivalences d'homotopie naturelles. Ici $GL(\mathcal{A})$ est le groupe des matrices inversibles $(a_{ij})_{i,j \in |\mathcal{A}|}$ indexées sur les objets de $\mathcal{A}$. Leurs entrées $a_{ij} \in \mathcal{A}(j, i)$ vérifient $a_{ij} = 0$ pour presque tout $i \neq j$ et $a_{ii} = id_i$ pour presque tout i (voir section 13). On peut écrire $GL(\mathcal{A})$ comme $colim_{S \subset |\mathcal{A}| fini} GL_S(\mathcal{A})$ où $GL_S(\mathcal{A})$ est le groupe des matrices inversible $(a_{ij})_{i,j \in S}$. Des suites exactes de groupes (quasi-parfaits) donneront des fibrations au niveau de leurs espaces classifiants ce qui sera préservé en appliquant la construction plus.

Démonstration du théorème. Pour faciliter les démonstrations des lemmes suivants on va supposer que $\mathcal{U}$ est $\mathcal{A}$-filtrée. On remarque que la sous-catégorie pleine des objets filtrés d'une catégorie stablement filtrée est une sous-catégorie cofinale.

Le foncteur $\mathcal{U} \to \mathcal{U}/\mathcal{A}$ induit un morphisme de groupes $\varphi : Gl(\mathcal{U}) \to GL(\mathcal{U}/\mathcal{A})$. Le commutateur $E(\mathcal{U}/\mathcal{A})$ de $GL(\mathcal{U}/\mathcal{A})$ est engendré par les matrices élémentaires $e_{ij}(\alpha) = (a_{kl})_{k,l\in|\mathcal{U}/\mathcal{A}|}$, $i \neq j$ avec $a_{kk} = id_k$, $a_{kl} = 0$ si $(k,l) \neq (i,j)$ et $a_{ij} = \alpha \in \mathcal{U}/\mathcal{A}(j,i)$. $\mathcal{U}(j,i) \to \mathcal{U}/\mathcal{A}(j,i)$ est surjective. C'est pourquoi $E(\mathcal{U}/\mathcal{A}) \subseteq im\varphi$, en particulier $im\varphi$ est distingué dans $GL(\mathcal{U}/\mathcal{A})$ et

$$coker\varphi = coker[GL(\mathcal{U})/E(\mathcal{U}) \to GL(\mathcal{U}/\mathcal{A})/E(\mathcal{U}/\mathcal{A})] = coker[K_1(\mathcal{U}) \to K_1(\mathcal{U}/\mathcal{A})] = K_0$$

On a alors démontré le

14.5 Lemme.

$$Bker\varphi \to BGL(\mathcal{U}) \to Bim\varphi, \quad \text{et}$$

$$Bim\varphi \to BGL(\mathcal{U}/\mathcal{A}) \to BK_0$$

sont des fibrations.

14.6 Lemme. *$ker\varphi$ est quasi-parfait.*

Démonstration. Soit E le sous-groupe de $ker\varphi$ engendré par les matrices $e_{U,V}(\alpha) = (a_{kl})_{k,l\in|\mathcal{U}|}$, $U \neq V$ avec $a_{kk} = id_k$, $a_{kl} = 0$ si $(k,l) \neq (U,V)$ et $a_{U,V} = \alpha \in \mathcal{U}(V,U)$ se factorisant par $\mathcal{A}$. E est parfait car si $\alpha = \beta \circ \gamma$ où $\gamma : V \to A$ et $\beta : A \to U$ avec $A \in |\mathcal{A}|$ et $A \neq U, V$ on peut écrire $e_{U,V}(\alpha) = [e_{U,A}(\beta), e_{A,V}(\gamma)]$.

Il reste à voir que le commutateur de $ker\varphi$ est E. Comme d'habitude il suffit de voir que les matrices de la forme $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ où $a \in aut_{\mathcal{U}}(U)$ telles que $a-1$ et $a^{-1}-1$ se factorisent par $\mathcal{A}$ appartiennent à E. Puisque U est $\mathcal{A}$-filtré il existe un morphisme direct $(p,s) : A \longrightarrow\!\!\!\!\!\rightarrow U$ et des flèches $b, c : A \to A$ telles que $a - 1 = s \circ b \circ p$ et $a^{-1} - 1 = s \circ c \circ p$. En posant $q = p \circ s$ on trouve $q(a-1) = (a-1)q = a-1$ et $q(a^{-1}-1) = (a^{-1}-1)q = a^{-1}-1$. Les identités de Whitehead deviennent

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-q-a^{-1} \\ 0 & 1 \end{pmatrix}$$

où toute matrice à droite est dans E. D'où le lemme. $\square$

14.7 Lemme. *Soit $\mathcal{U}$ une catégorie additive $\mathcal{A}$-filtrée par une sous-catégorie pleine additive $\mathcal{A}$. Alors*

$$i^+ : BGL(\mathcal{A})^+ \to Bkerp^+$$

est une équivalence d'homotopie.

Démonstration. Elle consiste en deux parties 1) et 2). 1) Pour tout $\beta \in kerp$ il existe un $g \in kerp$ tel que $g\beta g^{-1} \in GL(\mathcal{A})$.

2) Pour $e_0, ..., e_n \in E$ et $z_0, ..., z_m \in GL(\mathcal{A})$ il existe $g \in E$ tel que $ge_ig^{-1} \in E(\mathcal{A})$ et $gz_jg^{-1} = z_j$.

Pourquoi le lemme s'ensuit-il de 1) et 2)? $\pi_1(i^+)$ s'identifie à $GL(\mathcal{A})/E(\mathcal{A}) \to kerp/E$. De 1) on déduit que $\pi_1(i^+)$ est surjectif. De 2) avec $n = m = 0$ $z_0 = e_0$ on déduit que $\pi_1(i^+)$ est injectif.
Pour voir l'isomorphie de $\pi_k(i^+)$ pour $k \geq 2$ il suffit que $BE(\mathcal{A})^+ \to BE^+$ soit une équivalence d'homotopie, puisque $BE(\mathcal{A})^+$ (BE^+) est le revêtement universel (à homotopie près) de $BGL(\mathcal{A})^+$ $(Bkerp^+)$. Les espaces en question sont simplement connexes, donc il suffit qu'on ait un isomorphisme $H_*(BE(\mathcal{A})^+) \to H_*(BE^+)$, c'est à dire un isomorphisme $H_*(i) : H_*(E(\mathcal{A})) \to H_*(E)$.

$H_n(i)$ est surjectif: soit $\sum_k n_k(e_1^k, ..., e_n^k)$ un représentant d'un élément dans $H_n(E)$. D'après 2) il existe un $g \in E$ tel que $ge_j^kg^{-1} \in E(\mathcal{A})$. Or $\sum_k n_k(ge_1^kg^{-1}, ..., ge_n^kg^{-1})$ est un cycle de $H_n(E(\mathcal{A}))$ homologue à $\sum_k n_k(e_1^k, ..., e_n^k)$ dans $H_n(E)$.

$H_n(i)$ est injectif: soit $z = \sum_k n_k(z_1^k, ..., z_n^k)$ un représentant d'une classe dans $H_n(E(\mathcal{A}))$ nulle dans $H_n(E)$, c'est à dire qu'il existe une chane $e = \sum_l n_l(e_1^l, ..., e_{n+1}^l)$, $e_j^l \in E$ telle que $\partial e = z$. D'après 2) il existe $g \in E$ tel que $ge_i^lg^{-1} \in E(\mathcal{A})$ et $gz_j^kg^{-1} = z_j^k$. Or $geg^{-1} = \sum_l n_l(ge_1^lg^{-1}, ..., ge_{n+1}^lg^{-1})$ est une chane du bar-complexe de $E(\mathcal{A})$ et $\partial geg^{-1} = g\partial eg^{-1} = gzg^{-1} = z$.

Il ne reste qu'à démontrer 1) et2).

Démonstration de 1): on peut supposer $\beta \in aut_{\mathcal{U}}(U)$. Puisque $\beta \in kerp$, $\beta - id_U$ factorise par $\mathcal{A}$. U est $\mathcal{A}$-filtré, c'est pourquoi on peut choisir $(p, s) : A \longrightarrow\!\!\!\!\!\!+\!\!\!\!\!\!\longrightarrow U$ tel que $\beta - id_U = s\alpha p$ pour un certain $\alpha : A \to A \in \mathcal{A}$. On pose

$$g = \begin{pmatrix} id_U - sp & s \\ p & 0 \end{pmatrix} \in aut_{\mathcal{U}}(U \oplus A), \text{ alors } g^2 = 1_{U\oplus A} \text{ et } g \in kerp \text{ et}$$

$$\begin{pmatrix} id_U & 0 \\ 0 & id_A + \alpha \end{pmatrix} = \begin{pmatrix} id_U - sp & s \\ p & 0 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & id_A \end{pmatrix} \begin{pmatrix} id_U - sp & s \\ p & 0 \end{pmatrix} \in GL(\mathcal{A}).$$

Démonstration de 2): il suffit de trouver un $g \in kerp$ car on peut toujours le remplacer par l'élément $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in E$ sans changer l'action sur e_i et z_j. Puisque E est engendré par les matrices élémentaires (voir la démonstration du lemme 14.6), on peut supposer les e_i élémentaires donc de la forme $e_i = e_{U_i,V_i}(\alpha_i)$, $U_i \neq V_i$ où les α_i se factorisent par $\mathcal{A}$. L'équation $e_{U,V}(\alpha) = [e_{U,A}(\beta), e_{A,V}(\gamma)]$ pour $\alpha = \beta \circ \gamma$ nous permet de supposer $e_i = e_{U_i,A_i}(\alpha_i)$ ou $e_{A_i,U_i}(\beta_i)$. En passant à la somme A des A_i et à la somme U des U_i on peut supposer $e_i = e_{U,A}(\alpha)$ ou $e_{A,U}(\beta_i)$. Une $\mathcal{A}$-filtration de U nous donne un morphisme direct $(p,s) : U \longrightarrow\!\!\!\!\!\!+\!\!\!\!\!\!\rightarrow V$ avec $B \in |\mathcal{A}|$ et $\alpha_i = s \circ a_i$, $\beta_i = b_i \circ p$ où $\alpha_i : A \to B$ et $\beta_i : B \to A$. Posons $g = \begin{pmatrix} 0 & 0 & p \\ 0 & id_A & 0 \\ s & 0 & id_U - sp \end{pmatrix} \in aut_{\mathcal{U}}(B \oplus A \oplus U)$. On trouve $g^2 = id_{B\oplus A\oplus U}$, $g \in kerp$ et

$$\begin{pmatrix} id_B & a_i & 0 \\ b_i & z_j & 0 \\ 0 & 0 & id_U \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & p \\ 0 & id_A & 0 \\ s & 0 & id_U - sp \end{pmatrix} \begin{pmatrix} id_B & 0 & 0 \\ 0 & z_j & \beta_i \\ 0 & \alpha_i & id_U \end{pmatrix} \begin{pmatrix} 0 & 0 & p \\ 0 & id_A & 0 \\ s & 0 & id_U - sp \end{pmatrix}$$

D'où $ge_ig^{-1} \in E(\mathcal{A})$, $gz_jg^{-1} = z_j$ et donc le lemme. □

14.8 Lemme.

$$Bker\varphi^+ \to BGL(\mathcal{U})^+ \to Bim\varphi^+, \quad \text{et}$$

$$Bim\varphi^+ \to BGL(\mathcal{U}/\mathcal{A})^+ \to BK_0$$

sont des fibrations à homotopie près.

Démonstration. Que $Bim\varphi^+ \to BGL(\mathcal{U}/\mathcal{A})^+ \to BK_0$ soit une fibration à homotopie près découle du fait qu'elle s'insère dans le diagramme

$$\begin{array}{ccccc} BE(\mathcal{U}/\mathcal{A})^+ & = & BE(\mathcal{U}/\mathcal{A})^+ & & \\ \downarrow & & \downarrow & & \\ Bim\varphi^+ & \to & BGL(\mathcal{U}/\mathcal{A})^+ & \to & BK_0 \\ \downarrow & & \downarrow & & \| \\ B(im\varphi/E(\mathcal{U}/\mathcal{A})) & \to & B(GL(\mathcal{U}/\mathcal{A})/E(\mathcal{U}/\mathcal{A})) & \to & BK_0 \end{array}$$

dans lequel toute les lignes horizontales ainsi que verticales sont des fibrations homotopiques sauf (peut-être) la ligne du milieu. Ceci force cette dernière à en être aussi une.

Démonstration que $Bker\varphi^+ \to BGL(\mathcal{U})^+ \to Bim\varphi^+$ est une fibration à homotopie près: soit F la fibre homotopique de $\varphi^+ : BGL(\mathcal{U})^+ \to Bim\varphi^+$. Puisque φ^+ est une application équivariante entre

H-espaces connexes et $\pi_1(B\varphi^+)$ est surjective d'après construction, F est un H-espace connexe, donc un H-groupe. On a le diagramme suivant

$$\begin{array}{ccccc} Bker\varphi & \longrightarrow & BGL(\mathcal{U}) & \longrightarrow & Bim\varphi \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & BGL(\mathcal{U})^+ & \longrightarrow & Bim\varphi^+ \end{array}$$

où les applications entre les bases et les espaces totaux respectivement sont des isomorphismes en homologie. Si les groupes fondamentaux des bases des deux fibrations agissent de façon triviale sur l'homologie de leurs fibres, le théorème de comparaison des suites spectrales nous permet de conclure que l'application entre les fibres est aussi un isomorphisme en homologie. Il s'ensuit que $Bker\varphi^+ \to F$ est un isomorphisme en homologie entre des H-espaces connexes donc une équivalence d'homotopie. Il reste à voir que les groupes fondamentaux agissent trivialement.

$GL(\mathcal{U})$ agit trivialement sur $H_*(kerp)$:soit maintenant $g \in GL_S(\mathcal{U})$ et b un cycle du bar-complexe de $kerp$. D'après le lemme 14.7 $H_*(GL(\mathcal{A})) \to H_*(kerp)$ est un isomorphisme c'est à dire b est homologue à un cycle a qui vient de $GL(\mathcal{A})$. Il existe une permutation $\pi \in GL(\mathcal{A})$ telle que a soit homologue à $\pi a \pi^{-1} \in GL_T(\mathcal{U})$ avec $S \cap T = \emptyset$. Or l'action de g sur $\pi a \pi^{-1} \in GL_T(\mathcal{U})$ est triviale, donc l'action de g sur b est homologiquement triviale.

Que l'action de $\pi_1(Bim\varphi^+)$ sur $H_*(F)$ soit triviale est un fait général sur les fibrations des H-espaces connexes (voir [Bro63]). D'où le lemme. □

14.9 Remarque. Le lemme ci-dessus résulte aussi d'un théorème de Berrick (voir [Ber82] theorem 6.4).

Puisque $BGL(\mathcal{C})^+$ et $B(\mathcal{C}^{-1}\mathcal{C})_0$ sont reliés par une suite d'équivalences d'homotopie naturelles, les deux lemmes précédents montrent:

14.10 Lemme. *Il existe un espace P et des fibrations homotopiques*

$$B(\mathcal{A}^{-1}\mathcal{A})_0 \to B(\mathcal{U}^{-1}\mathcal{U})_0 \to P, \quad \text{et}$$

$$P \to B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0 \to BK_0$$

avec $K_0 = coker(K_1(\mathcal{U}) \to K_1(\mathcal{U}/\mathcal{A}))$

14.11 Remarque. Par le théorème de cofinalité le lemme 14.10 est également vrai si $\mathcal{U}$ est stablement $\mathcal{A}$-filtrée car la sous-catégorie pleine des objets filtrés d'une catégorie stablement filtrée est une sous-catégorie cofinale.

14.12 Lemme. *Soit $\mathcal{U}$ une catégorie stablement $\mathcal{A}$-filtrée pour une sous-catégorie additive, pleine $\mathcal{A}$.*

i) *Si $\alpha : U \oplus B \to V \oplus C$ et $p_V \circ \alpha \circ i_U : U \to V$ sont des isomorphismes dans $\tilde{\mathcal{U}}$, alors $B \simeq C$ dans $\tilde{\mathcal{U}}$.*

ii) *Si $\varphi : U \to V \in \tilde{\mathcal{U}}$ est un isomorphisme dans $\mathcal{U}\tilde{/}\mathcal{A}$, alors il existe $A, B \in |\tilde{\mathcal{A}}|$, $W, \bar{U}, \bar{V} \in |\tilde{\mathcal{U}}|$ et des isomorphismes $u : U \oplus W \to \bar{U} \oplus A$ et $v : V \oplus W \to \bar{V} \oplus B$ dans $\tilde{\mathcal{U}}$ tels que $p_{\bar{V}} \circ v \circ (\varphi \oplus id_W) \circ u^{-1} \circ i_{\bar{U}}$ est aussi un isomorphisme dans $\tilde{\mathcal{U}}$.*

iii) *Si dans ii) $\varphi - \alpha$ est un isomorphisme dans $\tilde{\mathcal{U}}$ pour un α qui se factorise par $\mathcal{A}$, alors on peut choisir $A = B$ dans ii).*

Démonstration de i): On écrit $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ et $\alpha^{-1} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$. En particulier $d = p_V \circ \alpha \circ i_U$ et on calcule $id_C = (a - b \circ d^{-1} \circ c) \circ \bar{a}$ et $id_B = \bar{a} \circ (a - b \circ d^{-1} \circ c)$. Il s'ensuit que $\bar{a} : C \to B$ est un isomorphisme dans $\tilde{\mathcal{U}}$.

Démonstration de ii): Soit $\psi : V \to U \in \tilde{\mathcal{U}}$ un inverse de φ dans $\mathcal{U}\tilde{/}\mathcal{A}$. Sans restreindre la généralité on peut supposer U $\tilde{\mathcal{A}}$-filtré (remplacer U par $U \oplus W$ $\tilde{\mathcal{A}}$-filtré, V par $V \oplus W$, etc, W existe car U est stablement filtré et $\tilde{\mathcal{U}}$ est pseudo-abélien). Il existe alors un $A \in |\tilde{\mathcal{A}}|$, un morphisme direct $(q,s) :\ A \longrightarrow\!\!\!\!\!\!\rightarrow U$ et un $\alpha : A \to A$ tels que $\psi \circ \varphi = id_U + s \circ \alpha \circ q$. $s \circ q$ soit un projecteur dans U. Puisque $\tilde{\mathcal{U}}$ est pseudo-abélien $\bar{U} = ker(s \circ q)$ existe. On a $A \simeq im(s \circ q)$ et donc une décomposition $u : U \tilde{\to} \bar{U} \oplus A$ telle que $p_{\bar{U}} u \psi \circ \varphi u^{-1} i_{\bar{U}} = id_{\bar{U}}$. D'où un morphisme direct $(p_{\bar{U}} u \psi, \varphi u^{-1} i_{\bar{U}}) :\ \bar{U} \longrightarrow\!\!\!\!\!\!\rightarrow V$ On en déduit que $id_V - \varphi i_{\bar{U}} \circ p_{\bar{U}} \psi$ est un projecteur qui se factorise par $\tilde{\mathcal{A}}$ parce que $[\varphi u^{-1} i_{\bar{U}} \circ p_{\bar{U}} u \psi] = [\varphi \circ \psi] = id_V$ dans $\mathcal{U}\tilde{/}\mathcal{A}$. Comme ci-dessus on peut supposer V $\tilde{\mathcal{A}}$-filtré (remplacer U par $U \oplus W$, $\bar{U}$ par $\bar{U} \oplus W$, V par $V \oplus W$). Mais l'image d'un projecteur d'un objet filtré factorisant par $\tilde{\mathcal{A}}$ est isomorphe à un objet de $\tilde{\mathcal{A}}$. Posons $B = Im(id_V - \varphi u^{-1} i_{\bar{U}} \circ p_{\bar{U}} u \psi)$, $\bar{V} = Im(\varphi u^{-1} i_{\bar{U}} \circ p_{\bar{U}} u \psi)$ et $v : V \tilde{\to} \bar{V} \oplus B$. Par construction on a $p_{\bar{V}} \circ v \circ (\varphi \oplus id_W) \circ u^{-1} \circ i_{\bar{U}} : \bar{U} \tilde{\to} \bar{V}$ dans $\mathcal{U}$.

Démonstration de iii): Puisque $\varphi - \alpha : U \to V$ est un isomorphisme dans $\tilde{\mathcal{U}}$ on peut supposer $U = V$. On fait la somme directe de U avec un $W \in |\tilde{\mathcal{U}}|$ pour que U devienne $\tilde{\mathcal{A}}$-filtré. Il existe alors un $A \in |\tilde{\mathcal{A}}|$, un morphisme direct (q,s) : $A \twoheadrightarrow U$ et un $\beta : A \to A$ tels que $\varphi - \alpha = s \circ \beta \circ q$. Le morphisme $s \circ q$ est un projecteur dans U. Puisque $\tilde{\mathcal{U}}$ est pseudo-abélien $\bar{U} = ker(s \circ q)$ existe. On a $A \simeq im(s \circ q)$ et donc $u : U \tilde{\to} \bar{U} \oplus A$ et $p_{\bar{U}}\psi \circ \varphi i_{\bar{U}} = id_{\bar{U}}$ où $\psi = (\varphi - \alpha)^{-1}$ et $p_{\bar{U}}\psi \circ \varphi i_{\bar{U}} = id_{\bar{U}}$ Posons $\bar{V} = Im(\varphi i_{\bar{U}} \circ p_{\bar{U}}\psi)$. Puisque $id_U - \varphi i_{\bar{U}} \circ p_{\bar{U}}\psi = (\varphi - \alpha) i_A \circ p_A \psi$ on a un isomorphisme $(\varphi - \alpha) i_A : A \tilde{\to} ker(\varphi i_{\bar{U}} \circ p_{\bar{U}})$, donc il existe $v : U \tilde{\to} \bar{U} \oplus A$ tel que $p_{\bar{V}} \circ v \circ \varphi \circ u^{-1} \circ i_{\bar{U}} : \bar{U} \tilde{\to} \bar{V}$ soit un isomorphisme dans $\tilde{\mathcal{U}}$. D'où le lemme. □

14.13 Lemme. *Soit $\mathcal{U}$ stablement $\mathcal{A}$-filtrée. Alors*

$$K_0(\mathcal{A}) \to K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A})$$

est exacte.

Démonstration. Que la composée des deux flèches soit nulle est clair. Soit $[U] - [V] \in K_0(\mathcal{U})$ nul dans $K_0(\mathcal{U}/\mathcal{A})$. Il existe alors T tel que $U \oplus T \simeq V \oplus T$ dans $\mathcal{U}/\mathcal{A}$. D'après lemme 14.12 ii) il existe $W, \bar{U}, \bar{V} \in |\tilde{\mathcal{U}}|$ et $A, B \in |\tilde{\mathcal{A}}|$ tels que $W \oplus U \oplus T \simeq \bar{U} \oplus A$, $W \oplus V \oplus T \simeq \bar{V} \oplus B$ et $\bar{U} \simeq \bar{V}$. Dans $K_0(\mathcal{U})$ on a alors $[U] - [V] = [W \oplus U \oplus T] - [W \oplus V \oplus T] = [\bar{U} \oplus A] - [\bar{V} \oplus B] = [A] - [B]$ qui est dans l'image de $K_0(\mathcal{A}) \to K_0(\mathcal{U})$. D'où le lemme. □

Démonstration du théorème 14.3: Soit $N \subset K_0(\mathcal{C})$ un sous-ensemble, on notera $(\mathcal{C}^{-1}\mathcal{C})_N$ la sous-catégorie pleine de $\mathcal{C}^{-1}\mathcal{C}$ dont les objets sont les (U,V) tels que $[U] - [V] \in N$, c'est à dire $B(\mathcal{C}^{-1}\mathcal{C})_N$ est la réunion des composantes connexes de $K(\mathcal{C})$ correspondant à N.

$$B(\mathcal{A}^{-1}\mathcal{A}) \to B(\mathcal{U}^{-1}\mathcal{U}) \to B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})$$

est une fibration homotopique ssi

$$B(\mathcal{A}^{-1}\mathcal{A}) \longrightarrow B(\mathcal{U}^{-1}\mathcal{U})_{ker(K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}))} \longrightarrow B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0$$

l'est. Or ceci est vrai ssi pour chaque $\xi \in ker(K_0(\mathcal{U}) \to K_0(\mathcal{U}/\mathcal{A}))$

$$B(\mathcal{A}^{-1}\mathcal{A})_{\alpha^{-1}(\xi)} \to B(\mathcal{U}^{-1}\mathcal{U})_\xi \to B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0$$

est une fibration homotopique, où $\alpha : K_0(\mathcal{A}) \to K_0(\mathcal{U})$ et $\alpha^{-1}(\xi) \neq \emptyset$ par le lemme 14.13. Or pour tout $[A] - [B] \in \alpha^{-1}(\xi)$ on a un diagramme commutatif

$$\begin{array}{ccccc}
B(\mathcal{A}^{-1}\mathcal{A})_{\alpha^{-1}(\xi)} & \longrightarrow & B(\mathcal{U}^{-1}\mathcal{U})_\xi & \longrightarrow & B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0 \\
\downarrow_{\oplus(B,A)} & & \downarrow_{\oplus(B,A)} & & \downarrow_{\oplus(B,A)} \\
B(\mathcal{A}^{-1}\mathcal{A})_{ker(K_0(\mathcal{A}) \to K_0(\mathcal{U}))} & \longrightarrow & B(\mathcal{U}^{-1}\mathcal{U})_0 & \longrightarrow & B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0
\end{array}$$

où les flèches verticales sont des équivalences d'homotopie. On en conclut que le théorème 14.3 est démontré si on peut prouver le

14.14 Lemme.

$$B(\mathcal{A}^{-1}\mathcal{A})_{ker(K_0(\mathcal{A})\to K_0(\mathcal{U}))} \longrightarrow B(\mathcal{U}^{-1}\mathcal{U})_0 \longrightarrow B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0$$

est une fibration à homotopie près.

Démonstration. Soit F la fibre homotopique de $B(\mathcal{U}^{-1}\mathcal{U})_0 \to B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0$. Il faut voir que l'application canonique $\varrho : B(\mathcal{A}^{-1}\mathcal{A})_{ker} \to F$ avec $ker = ker(K_0(\mathcal{A}) \to K_0(\mathcal{U}))$ est une équivalence d'homotopie. Or, les deux espaces sont des H-groupes, il faut alors vérifier que $\pi_*(\varrho)$ est un isomorphisme. Que $\pi_i(\varrho)$ soit un isomorphisme pour $i \geq 1$ découle du lemme des cinq appliqué aux suites exactes longes des groupes d'homotopie associées aux fibrations et du diagramme commutatif suivant

$$\begin{array}{ccccc} & B(\mathcal{A}^{-1}\mathcal{A})_0 & \hookrightarrow \; B(\mathcal{A}^{-1}\mathcal{A})_{ker} \; \to & F \\ & \downarrow & & \downarrow \\ & B(\mathcal{U}^{-1}\mathcal{U})_0 & \longrightarrow & B(\mathcal{U}^{-1}\mathcal{U})_0 \\ & \downarrow & & \downarrow \\ coker \;\to & P & \longrightarrow & B(\mathcal{U}/\mathcal{A}^{-1}\mathcal{U}/\mathcal{A})_0 \end{array}$$

où $coker = coker(K_1(\mathcal{U}) \to K_1(\mathcal{U}/\mathcal{A})) = K_0$, et où les lignes verticales et la dernière ligne horizontale sont des fibrations homotopiques (voir lemme 14.10).

L'isomorphisme $\pi_0(F) \xrightarrow{\sim} coker(K_1(\mathcal{U}) \to K_1(\mathcal{U}/\mathcal{A})) = coker(GL(\mathcal{U}) \to GL(\mathcal{U}/\mathcal{A}))$ est donné par: pour $[x] \in \pi_0(F)$ on choisit un chemin σ reliant dans l'espace total x et le point base, la projection de σ dans la base donne un lacet dont la classe d'homotopie est l'image de $[x]$ par l'isomorphisme. Appliqué à notre situation le morphisme

$$\pi_0(\varrho) : \pi_0(B(\mathcal{A}^{-1}\mathcal{A})_{ker}) = ker(K_0(\mathcal{A}) \to K_0(\mathcal{U})) \to \pi_0(F) = coker(GL(\mathcal{U}) \to GL(\mathcal{U}/\mathcal{A}))$$

est donné par: soit $[A] - [B] \in ker(K_0(\mathcal{A}) \to K_0(\mathcal{U}))$, il existe $U \in |\tilde{\mathcal{U}}|$ et $\varphi : A \oplus U \xrightarrow{\sim} B \oplus U$ dans $\tilde{\mathcal{U}}$, on a $\pi_0(\varrho)([A] - [B]) = p_U \circ \varphi \circ i_U \in im(aut_{\mathcal{U}/\mathcal{A}}(U) \to coker(GL(\mathcal{U}) \to GL(\mathcal{U}/\mathcal{A})))$. Il reste à voir que ceci est un isomorphisme.

$\pi_0(\varrho)$ est surjectif: soit $\varphi \in coker(GL(\mathcal{U}) \to GL(\mathcal{U}/\mathcal{A}))$. Sans restreindre la généralité on peut supposer $\varphi \in im(aut_{\mathcal{U}/\mathcal{A}}(U) \to coker)$. D'après le lemme 14.12 ii) il existe $A, B \in |\tilde{\mathcal{A}}|$, $W, \bar{U}, \bar{V} \in |\tilde{\mathcal{U}}|$ et des isomorphismes $u : U \oplus W \to \bar{U} \oplus A$ et $v : V \oplus W \to \bar{V} \oplus B$ dans $\tilde{\mathcal{U}}$ tels que $p_{\bar{V}} \circ v \circ (\varphi \oplus id_W) \circ u^{-1} \circ i_{\bar{U}}$ est aussi un isomorphisme dans $\tilde{\mathcal{U}}$. C'est pourquoi $\pi_0(\varrho)([B] - [A]) = \varphi$.

$\pi_0(\varrho)$ est injectif: soit $[A]-[B] \in ker(K_0(\mathcal{A}) \to K_0(\mathcal{U}))$ et $\pi_0(\varrho)([A]-[B]) = p_U \circ \varphi \circ i_U$ comme dans la construction de $\pi_0(\varrho)$ ci-dessus, tel que $\pi_0(\varrho)([A]-[B]) = 0 \in coker$. Alors il existe un $\alpha : U \to U$ avec $p_U \circ \varphi \circ i_U - \alpha$ inversible. D'après le lemme 14.12 iii) il existe $C \in |\tilde{\mathcal{A}}|$, $W, \bar{U}, \bar{V} \in |\tilde{\mathcal{U}}|$ et des isomorphismes $u : U \oplus W \to \bar{U} \oplus C$ et $v : U \oplus W \to \bar{V} \oplus C$ dans $\tilde{\mathcal{U}}$ tels que $p_{\bar{V}} \circ v \circ (p_U \circ \varphi \circ i_U \oplus id_W) \circ u^{-1} \circ i_{\bar{U}}$ est aussi un isomorphisme dans $\tilde{\mathcal{U}}$. A l'aide du lemme 6i) et le fait que φ est un isomorphisme on déduit que $A \oplus C \simeq B \oplus C$ ce qu'il fallait démontrer. D'où le lemme et le théorème 14.3. □ □

14.15 Remarque. On remarque que dans la démonstration originale de [PW89] au lieu de la catégorie $\underline{\underline{Q}}$ on peut utiliser la catégorie $< \underline{\underline{A}}, \underline{\underline{U}} >$ (pour une définition voir [Gra76]). On a le foncteur "projection sur la deuxieme composante" $f : \underline{\underline{Q}} \to < \underline{\underline{A}}, \underline{\underline{U}} >$ qui envoie (A, U) sur U et un morphisme $(A, U) \to (B, V)$ donné par $\alpha : A \to E \oplus B \oplus F \in Mor\underline{\underline{A}}$ et $\varphi : F \oplus U \to V \in Mor\underline{\underline{U}}$ sur $\varphi : F \oplus U \to V$. Soit $\tilde{\underline{\underline{A}}} = < \underline{\underline{A}}, \underline{\underline{A}} >^{op}$ la catégorie de translation qui est contractiles car elle a un objet final. Le foncteur $\tilde{\underline{\underline{A}}} \to (X \downarrow f)$ donné par $A \mapsto (A, X, id : X \to X)$ est adjoint à gauche au foncteur $(X \downarrow f) \to \tilde{\underline{\underline{A}}}$ donné par $(B, V, e : E \oplus X \to V) \mapsto B \oplus E$. C'est pourquoi les $(X \downarrow f)$ sont contractiles. Pour démontrer le théorème 5.3 de [PW89] il suffit donc de voir que le foncteur canonique $\Sigma' : < \underline{\underline{A}}, \underline{\underline{U}} > \to \underline{\underline{S}}$ est une équivalence d'homotopie. Σ' est une équivalence d'homotopie car les catégories $(S \downarrow \Sigma')$ sont cofiltrées par le calcul des fractions comme dans lemme 3.16. Donc, on peut utiliser théorème A de Quillen sans détour.

PART III

15 Une KK-théorie pour les algèbres de Banach

Dans ce qui suit on va démontrer les deux théorèmes ci-dessous.

15.1 Soit $\mathbb{K}$ un foncteur de la catégorie des algèbres de Banach non-unitaires sur $\mathbb{C}$ vers la catégorie des spectres symétriques ([HSS]) tel que l'image de $\mathbb{K}$ soit toujours sémistable (en fait un Ω-spectre apart $\mathbb{K}_0$), tel que $\mathbb{K}(\mathbb{C})$ soit un spectre d'anneau commutatif, $\mathbb{K}(A)$ un $\mathbb{K}(\mathbb{C})$-module et tel que $\pi_*\mathbb{K}(A) = K_*(A)$, la K-théorie topologique de l'algèbre de Banach A. En outre $\mathbb{K}$ envoie les suites exactes courtes d'algèbres de Banach en des fibrations homotopiques de $\mathbb{K}(\mathbb{C})$-modules. Pour une construction d'un tel foncteur voir l'appendice.

La catégorie des $\mathbb{K}(\mathbb{C})$-modules porte une structure de catégorie de modèle fermé décrite dans [SS]. Si R est un spectre d'anneau commutatif les morphismes de X vers Y dans la catégorie homotopique des R-modules seront notés par $[X,Y]_R$. En outre, on note par S le spectre symétrique des sphères et par S^n sa n-ième suspension, $n \in \mathbb{Z}$, qui d'ailleurs est cofibrante. C'est pourquoi $R \wedge S^n$ est cofibrant dans R-mod et $[R \wedge S^n, X]_R = [S^n, X]_S =: \pi_n(X)$.

Puisque $\pi_*\mathbb{K}(\mathbb{C}) = K_*(\mathbb{C})$ est périodique de période deux, la périodicité étant induite par le produit cup avec un élément de $K_2(\mathbb{C})$, il en va de même pour les $\mathbb{K}(\mathbb{C})$-modules. On regarde les groupes d'homotopies d'un $\mathbb{K}(\mathbb{C})$-module comme module $\mathbb{Z}/2\mathbb{Z}$ gradué. Pour deux modules M_* et N_* $\mathbb{Z}/2\mathbb{Z}$ gradué on pose $Hom(M_*, N_*) = Hom(M_0, N_0) \times Hom(M_1, N_1)$ et
$Ext(M_*, N_*) = Ext(M_0, N_0) \times Ext(M_1, N_1)$.

15.2 Théorème. *(Théorème des coefficients universels) Soit X et Y deux $\mathbb{K}(\mathbb{C})$-modules. Il existe une suite exacte courte naturelle*

$$0 \to Ext(\pi_*(X), \pi_{*+1}(Y)) \xrightarrow{\delta} [X,Y]_{\mathbb{K}(\mathbb{C})} \xrightarrow{\pi_*} Hom(\pi_*(X), \pi_*(Y)) \to 0$$

15.3 On peut décrire la flèche $\delta^{-1} : ker\pi_* \to Ext(\pi_*(X), \pi_{*-1}(Y))$ de la façon explicite suivante: Soit X cofibrant et Y fibrant. Un élément de $[X,Y]_{\mathbb{K}(\mathbb{C})}$ peut alors être représenté par un morphisme de $\mathbb{K}(\mathbb{C})$-modules $f : X \to Y$. Notons par $F(f)$ la fibre homotopique de f. Alors on a une fibration homotopique

$$\Omega Y \to F(f) \to X$$

Si maintenant $\pi_*(f) = 0$, la suite exacte longue de groupes d'homotopies associée à la fibration homotopique se décompose en des suites exactes courtes

$$0 \to \pi_*(\Omega Y) \to \pi_* F(f) \to \pi_*(X) \to 0$$

D'où un élément de $Ext(\pi_*(X), \pi_{*+1}(Y))$.

15.4 Pour deux C^*-algèbres A, B Kasparov a introduit un groupe abélien $KK(A,B)$ et un produit associatif $KK(A,B) \otimes KK(B,C) \to KK(A,C)$ (pour les détails voir [Bla86] et [Cun87]). On a alors une catégorie additive dont les objets sont les C^*-algèbres et dont les morphismes de A vers B sont les éléments de $KK(A,B)$. D'après Cuntz ([Cun87]) $KK(A,B)$ peut être vu comme l'ensemble des morphismes de C^*-algèbres de qA vers $B \otimes \mathcal{K}$ modulo l'homotopie où $qA = ker(A * A \xrightarrow{id*id} A)$ et $\mathcal{K}$ est la C^*-algèbre des opérateurs compactes d'un espace d'Hilbert de dimension infinie. On a des morphismes $r : qA \xrightarrow{id*0} A$ et $i : B \to B \otimes \mathcal{K} : b \mapsto b \otimes p$ où p est un projecteur dont l'image est de dimension 1. Ces deux morphismes induisent des isomorphismes en K-théorie.

On dit qu'une paire de C^*-algèbres (A,B) vérifie le théorème des coefficients universels (TCU) si l'homomorphisme $K_* : KK(A,B) \to Hom(K_*(A), K_*(B)) : f \mapsto K_*(i)^{-1} \circ K_*(f) \circ K_*(r)^{-1}$ est surjectif et si l'homomorphisme $d^{-1} : kerK_* \to Ext(K_*(A), K_{*+1}(B))$ décrit ci-après est un isomorphisme. Si $f : A \to B$ est un morphisme de C^*-algèbres sa fibre homotopique $F(f)$ est définie comme suit: Soit $PB = ker(B^I \xrightarrow{\epsilon_0} B)$ et $\Omega B = ker(PB \xrightarrow{\epsilon_1} B)$ où ϵ_t est l'évaluation en t. La K-théorie de la C^*-algèbre PB est triviale, donc on a $K_*(\Omega B) = K_{*+1}(B)$. $F(f)$ est le pull-back du diagramme $(A \xrightarrow{f} B \xleftarrow{\epsilon_1} PB)$. On a donc une suite exacte courte $\Omega B \to F(f) \to A$. Appliqué à un $f : qA \to B \otimes \mathcal{K}$ tel que $K_*(f) = 0$ la suite exacte longue en K-théorie associée à la suite exacte courtes de C^*-algèbres décrite ci-dessus se décompose en des suites exactes courtes

$$0 \to K_*(\Omega B) \to K_*(F(f)) \to K_*(A) \to 0$$

. D'où un élément de $Ext(K_*(A), K_{*+1}(B))$. C'est l'image de f par d^{-1}.

15.5 Théorème. *Soit A et B deux C^*-algèbres. Il existe un morphisme fonctoriel en A et en B et compatible avec le produit de Kasparov*

$$\gamma : KK(A,B) \to [\mathbb{K}(A), \mathbb{K}(B)]_{\mathbb{K}(\mathbb{C})}$$

Si la paire (A,B) vérifie le théorème des coefficients universels alors γ est un isomorphisme.

Démonstration du théorème 15.5: Puisque i et r sont inversible dans la catégorie homotopique des $\mathbb{K}(\mathbb{C})$-modules l'application $\gamma : KK(A,B) \to [\mathbb{K}(A), \mathbb{K}(B)]_{\mathbb{K}(\mathbb{C})} : f \mapsto i^{-1} \circ f \circ r^{-1}$ a un sens. On vérifie qu'elle est bien définie et qu'elle respecte le produit. Supposons pour l'instant théorème 15.2 démontré les définitions de δ^{-1} et d^{-1} assurent que γ induit un morphisme de suites exactes courtes dans le cas où la paire (A, B) vérifie TCU. Avec le lemme des cinq on conclut le théorème 15.5. $\square$

15.6 Un $\mathbb{K}(\mathbb{C})$-module L est dit libre si $\pi_*(L)$ est un groupe abélien libre. Ceci est équivalent à dire que L est faiblement équivalent à un module de la forme $\bigvee_{J_0} \mathbb{K}(\mathbb{C}) \vee \bigvee_{J_1} S^1 \wedge \mathbb{K}(\mathbb{C})$. C'est parce que $\pi_*(\mathbb{K}(\mathbb{C}))$ est $\mathbb{Z}$ pour $* = 0$ et 0 pour $* = 1$.

15.7 Lemme. *Pour un $\mathbb{K}(\mathbb{C})$-module libre L, $\pi_* : [L, X]_{\mathbb{K}(\mathbb{C})} \to Hom(\pi_*(L), \pi_*(X))$ est un isomorphisme.*

Démonstration. On peut prendre X fibrant. Soit $L = \bigvee_{J_0} \mathbb{K}(\mathbb{C}) \vee \bigvee_{J_1} S^1 \wedge \mathbb{K}(\mathbb{C})$, en particulier L est cofibrant. Si on désigne par "$\sim$" la relation d'homotopie simpliciale pointée on trouve

$$\begin{aligned}
[L, X]_{\mathbb{K}(\mathbb{C})} &= \mathbb{K}(\mathbb{C})-\mathrm{mod}(L, X)/\sim \\
&= \Pi_{J_0}\mathbb{K}(\mathbb{C})-\mathrm{mod}(\mathbb{K}(\mathbb{C}), X)/\sim \;\times\; \Pi_{J_1}\mathbb{K}(\mathbb{C})-\mathrm{mod}(S^1 \wedge \mathbb{K}(\mathbb{C}), X)/\sim \\
&= \Pi_{J_0} Sp^{\Sigma}(S, X)/\sim \;\times\; \Pi_{J_1} Sp^{\Sigma}(S^1, X)/\sim \\
&= \Pi_{J_0}\pi_0(X) \;\times\; \Pi_{J_1}\pi_1(X) \\
&= Hom(\pi_0(L), \pi_0(X)) \times Hom(\pi_1(L), \pi_1(X))
\end{aligned}$$

parce que $\pi_*(L) = \bigoplus_{J_*} \mathbb{Z}$. On vérifie que l'isomorphisme ainsi obtenu est π_*. $\square$

Démonstration du théorème 15.2: On peut supposer X et Y fibrant et cofibrant en même temps. En particulier X est un Ω-spectre. Soit

$$L = \bigvee_{S \to X_0} \mathbb{K}(\mathbb{C}) \vee \bigvee_{S^1 \to X_0} S^1 \wedge \mathbb{K}(\mathbb{C}) \xrightarrow{p} X$$

le morphisme de $\mathbb{K}(\mathbb{C})$-modules évident. Et soit F la fibre homotopique de $L \to X$. D'après construction $\pi_*(L) \to \pi_*(X)$ est surjectif. C'est pourquoi la longue suite exacte de groupes d'homotopies associée à la fibration homotopique $F \xrightarrow{i} L \xrightarrow{p} X$ se décompose en des suites exactes courtes

$$0 \to \pi_*(F) \to \pi_*(L) \to \pi_*(X) \to 0.$$

En tant que sous-groupe du groupe abélien libre $\pi_*(L)$, $\pi_*(F)$ est aussi libre, donc F est un $\mathbb{K}(\mathbb{C})$-module libre. Toute fibration homotopique de $\mathbb{K}(\mathbb{C})$-modules est une cofibration homotopique. On a alors la suite exacte longue

$$\ldots \to [L, \Omega Y]_{\mathbb{K}(\mathbb{C})} \xrightarrow{i^{*+1}} [F, \Omega Y]_{\mathbb{K}(\mathbb{C})} \xrightarrow{\delta} [X, Y]_{\mathbb{K}(\mathbb{C})} \xrightarrow{p^*} [L, Y]_{\mathbb{K}(\mathbb{C})} \xrightarrow{i^*} [F, Y]_{\mathbb{K}(\mathbb{C})} \to \ldots$$

D'où la suite exacte courte

$$0 \longrightarrow \mathrm{coker} i^{*+1} \xrightarrow{\delta} [X,Y]_{\mathbb{K}(\mathbb{C})} \xrightarrow{p^*} \mathrm{ker} i^* \longrightarrow 0.$$

Le lemme 15.7 permet d'identifier cokeri^{*+1} avec $Ext(\pi_*(X), \pi_{*+1}(Y))$ et keri^* avec $Hom(\pi_*(X), \pi_*(Y))$. On vérifie que la description de δ^{-1} ainsi obtenu concide avec celle donnée plus haut. Ce qui achève la démonstration. □

Le théorme 15.5 suggère une définition d'une KK-théorie algébrique.

15.8 Définition. Soit A, B deux k-algèbres, on pose

$$KK_k(A,B) = [\mathbb{K}(A), \mathbb{K}(B)]_{\mathbb{K}(k)}.$$

Pour deux schémas quasi-séparés X, Y sur k on pose

$$KK_k^i(X,Y) = [S^i K^B(X), K^B(B)]_{\mathbb{K}(Y)}.$$

Par définition on a $KK_k^{i-j}(\mathrm{Spec}A, \mathrm{Spec}B) = KK_k(S^iA, S^jB)$ et $KK(S^i\mathbb{Z}, S^jA) = K_{i-j}(A)$. Puisque le spectre K^B de Thomason satisfait excision, il en est de même en les deux variables de KK_k.

15.9 Appendice

Pour les C^*-algèbre Bunke, Joachim et Stolz ont construit des spectres symétriques qui représentent la K-théorie topologique [BJS] vérifiant les condition dans 15.1. Une autre manière de procéder est de remplacer $\mathcal{C}_p\mathcal{A}$ par $S^p_{top}\mathcal{A}$ définit par Karoubi par exemple dans [Kar70] et de mettre une topologie sur les catégories $G^p(S^p_{top}\mathcal{A})$ dans l'esprit de [Pal96]. Ceci marche pour toutes les algèbres de Banach. Si A est une algèbre de Banach, on peut lui associer une algèbre simpliciale A_* avec $A_n = appl(\Delta_n, A)$. Ici on note par $appl(X,Y)$ l'ensemble des applications continues de X vers Y. Karoubi affirme que $|n \mapsto K(S^k_\delta A_n)|$ a le même type d'homotopie que $K(S^k_{top}A)$ où S^k_δ (resp. S^k_{top}) désigne la k-ième suspension algébrique (resp. topologique) d'un anneau discret (resp. de Banach). Ceci permettrait d'utiliser les constructions de la section 6.

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