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**Multizeta values: Lie algebras and periods on  $\mathfrak{M}_{0,n}$**

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**Valeurs multizêta : algèbres de Lie et périodes sur  $\mathfrak{M}_{0,n}$**

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## ABSTRACT

Multizeta values: Lie algebras and periods on  $\mathfrak{M}_{0,n}$

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This thesis is a study of algebraic and geometric relations between multizeta values. There are many such known sets of relations, coming from different theories, which are conjecturally equivalent to each other and which conjecturally describe all relations on multizeta values. This thesis was inspired by the conjectures of equivalence of these relations.

To study the algebraic relations, we begin by looking at the double shuffle Lie algebra associated to multizeta values, which encodes the double shuffle relations. In chapter 2 of this thesis, we prove a result which gives the dimension of the associated depth-graded pieces of the double shuffle Lie algebra in depths 1 and 2, thus verifying the conjecture that the double shuffle Lie algebra is isomorphic to the Grothendieck-Teichmüller Lie algebra in small depths.

Another conjecturally equivalent set of relations between multizeta values comes from their expression as periods on  $\mathfrak{M}_{0,n}$ , stemming originally from the work of Cartier and Kontsevich (among others). In chapters 3 and 4, we study these geometric relations. The results obtained from this study provide some evidence toward the conjecture that the associated formal period algebra is isomorphic to the formal zeta value algebra. The key ingredient in this study is the top dimensional de Rham cohomology of partially compactified moduli spaces of genus 0 curves with  $n$  marked points,  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ . In order to encode multizeta values in a formal period algebra, we give an explicit expression for a basis of  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ . The techniques used in this construction are generalized in chapter 4, in which we explicitly describe the bases of the cohomology of other partially compactified moduli spaces. This thesis concludes with a result which gives a new presentation of  $Pic(\overline{\mathfrak{M}}_{0,n})$ .

## RÉSUMÉ

Valeurs multizêta : algèbres de Lie et périodes sur  $\mathfrak{M}_{0,n}$

Sarah Carr

Leila Schneps, directrice de thèse

Cette thèse est une étude des relations algébriques et géométriques entre valeurs multizêta. Il y a de nombreux ensembles de telles relations, provenant de théories différentielles. Conjecturalement, ces ensembles sont équivalents et décrivent de plus toutes les relations entre valeurs multizêta. Cette thèse s'inspire de ces conjectures portant sur l'équivalence de ces relations.

Afin d'étudier les relations algébriques, on commence par regarder l'algèbre de Lie,  $\mathfrak{ds}$ , qui encode les relations de double mélange. Dans le chapitre 2, on démontre un résultat qui donne la dimension des parties graduées de  $\mathfrak{ds}$  associées à sa filtration par profondeur en profondeurs 1 et 2. On démontre donc que  $\mathfrak{ds}$  est isomorphe à l'algèbre de Lie  $\mathfrak{grt}$  dans les petites profondeurs.

Un autre ensemble de relations entre multizêtas, conjecturalement équivalent au système de double mélange, découle de leur expression comme périodes sur  $\mathfrak{M}_{0,n}$ , suivant les méthodes de Cartier et Kontsevich (parmi d'autres). Dans les chapitres 3 et 4, on étudie ces relations géométriques. Les résultats obtenus sont en accord avec la conjecture affirmant que l'algèbre formelle des périodes est isomorphe à l'algèbre formelle des multizêtas. L'ingrédient principal dans cette étude est la cohomologie de de Rham des espaces de modules de courbes en genre 0 avec  $n$  points marqués partiellement compactifiés,  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ . Afin d'encoder les valeurs multizêta dans l'algèbre formelle des périodes, on donne une expression explicite pour une base de  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ . Ces techniques sont généralisées dans le chapitre 4, dans lequel on décrit explicitement les bases de la cohomologie d'autres espaces de modules partiellement compactifiés. Dans la dernière partie, on fournit une nouvelle présentation de  $Pic(\overline{\mathfrak{M}}_{0,n})$ .

*This thesis is dedicated to my father, Bill Higgins, because he understands.*

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# Chapter 1

## Introduction

My research focuses on the study of multizeta values, real numbers defined by the iterated sums,

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 > n_2 > \dots > n_d > 0} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_d^{k_d}} \quad k_i \in \mathbb{Z}, (k_1 \geq 2).$$

Multizeta values are objects meriting much attention of late, and the multizeta value has acquired many nicknames in the process. We will refer to a multizeta value as a multiple zeta value, a multizeta, an *MZV*, a zeta value, or simply a zeta.

There is a myriad of conjectures and important recent results about multizeta values in various fields. One well-known number theoretic question is “Are multizeta values transcendental numbers?” Euler proved that  $\zeta(2n)$  is a rational multiple of  $\pi^{2n}$  and more recently R. Apéry and T. Rivoal showed that certain  $\zeta(2n + 1)$  are irrational. The conjecture underlying my research interests however is an even larger question, the understanding of which would prove the transcendence conjecture. The conjecture arises from number theoretic and geometric identities on multizeta values.

If one multiplies two multizeta values, one obtains a sum of multizeta values according to the double shuffle multiplication laws, shuffle and stuffle. The shuffle multiplication law comes from multizetas viewed as periods on the moduli space of genus 0 curves, while stuffle multiplication (already known to Euler) comes from the number theoretic expression of multizetas. We may then endow the vector space over  $\mathbb{Q}$  of multizetas with multiplication given by shuffle, and therefore multizetas form an algebra with a set of quadratic relations given by stuffle. We denote by  $\mathcal{Z}$ , the algebra generated by  $\mathbb{Q}$  and multizeta values, and call the set of multiplication relations the double shuffle (there is universal convention to consider  $1 = \zeta(\emptyset)$ , so that  $\mathbb{Q} \subset \mathcal{Z}$ ).

**Definition 1.1.** The *depth* of  $\zeta(k_1, \dots, k_d)$  is  $d$  and its *weight* is  $\sum_{i=1}^d k_i$ .

Both the shuffle and stuffle relations preserve the weight of an expression for multizetas. I emphasize “expression” since two expressions for a multizeta value may give the same number. Although the depth of an expression for a multizeta is easy to understand, it is not an invariant of a multizeta number. One example of this was known already to Euler, who proved that  $\zeta(3) = \zeta(2, 1)$ . The study of the weight and depth of multizetas leads to the main algebraic conjecture on multizetas.

**Conjecture 1.2.** A generating system of relations over  $\mathbb{Q}$  between multizeta values is essentially given by the shuffle and stuffle relations (for complete detail see definition 1.12). In particular,

there are no linear relations between multizetas of different weight, hence  $\mathcal{Z}$  forms a graded algebra.

This thesis is not an attempt to make progress toward this conjecture, which is extremely difficult because of the analytic nature of the transcendence problem. Rather, this thesis is an attempt to better understand its implications, in particular the combinatorial identities that arise from the known relations on multizeta values. Hence, we may define graded algebras that satisfy major families relations on multizetas and see what we can learn about multizetas from these algebras. This thesis is a study and comparison of two such algebras, the double shuffle Lie algebra and the period algebra of formal cell numbers.

In chapter 1, I give the main objects and state well-known theorems on which this study of multizetas is based. In this introduction, I will define the algebras and Lie algebras associated to multizeta values and explain how they are related and what the main conjectures are. These conjectures are there to provide the reader with a flavor of the questions that inspired this thesis. The conjectures presented in chapter 1 may be summarized as saying that all of the maps between the algebras (in upper case calligraphic font) and the maps between Lie algebras (in lower case Fraktur font) in the following commutative diagram are isomorphisms. Those shown are known to exist, except for the dotted arrows, whose conjectural definitions are known, but which are not proved to be well-defined, much less isomorphisms.

$$\begin{array}{ccccc}
 & & \mathfrak{nf}\mathfrak{z} & \longrightarrow & \widetilde{\mathfrak{nf}\mathfrak{z}} \\
 & & \uparrow & & \parallel \\
 \mathcal{FC} & \overset{\dots\dots}{\longleftarrow} & \mathcal{FZ} & & \mathfrak{d}\mathfrak{s}^\vee \overset{\dots\dots}{\longleftarrow} \mathfrak{grt}^\vee \\
 \downarrow & & \downarrow & & \downarrow \swarrow \\
 \mathcal{C} & \overset{\equiv}{=} & \mathcal{Z} & \longrightarrow & \mathfrak{nz}.
 \end{array}$$

Chapter 2 presents some evidence toward the well-known conjecture that the double shuffle Lie algebra,  $\mathfrak{d}\mathfrak{s}$ , is isomorphic to the Grothendieck-Teichmüller Lie algebra,  $\mathfrak{grt}$ . In chapter 2, I calculate the dimensions of the first two associated depth-graded parts of  $\mathfrak{d}\mathfrak{s}$ , confirming the conjecture that  $\mathfrak{d}\mathfrak{s} \simeq \mathfrak{grt}$  in small depths since the analogous dimensions were computed by Ihara [Ih2] for  $\mathfrak{grt}$ . (I note here that I found these dimensions in 2005. The result I present has since been published in [IKZ] using different methods.)

Chapters 3 and 4 deal with the algebra of periods and the study of the cohomology of  $\mathfrak{M}_{0,n}^\delta$ , the partially compactified moduli space of genus 0 curves with  $n$  marked points consisting of  $\mathfrak{M}_{0,n}$ , with the boundary divisors that bound the standard associahedron adjoined. Chapter 3 is an intact article which is joint work with F. Brown and L. Schneps. The main result presented in this article is the presentation of an explicit basis for the top dimensional de Rham cohomology of the partially compactified moduli space,  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ . This presentation allows us to compute the dimension of the cohomology using a recursive formula. In this article, we construct the algebra of periods on  $\mathfrak{M}_{0,n}^\delta$ , denoted  $\mathcal{C}$ , which is isomorphic to the algebra of multizeta values,  $\mathcal{Z}$  [Br]. This leads us to define an algebra of formal cell numbers,  $\mathcal{FC}$ , which encodes the known combinatorial relations coming from geometry on certain special generating periods called cell numbers. Since multizeta values are cell numbers, there is reason to believe that these geometric combinatorial relations describe all relations on multizeta values.

In chapter 4, I generalize the results of chapter 3 to calculate the top dimensional de Rham cohomology of some more general partial compactifications of  $\mathfrak{M}_{0,n}$  and give explicit dimensions for each cohomology. The investigation into the description of the cohomology also led to finding a new presentation of the Picard group,  $Pic(\overline{\mathfrak{M}}_{0,n})$ .

Although the chapters have disparate titles, they are intimately linked by the search for the connection between sets of relations and properties coming from the different geometric and number theoretic expressions for multizeta values. I outline these here because the goal of this thesis is to present results that emphasize the common properties of these different points of view. The double shuffle Lie algebra is an object which encodes both the number theoretic expression of multizeta values and the geometric expression of multizeta values as periods. In fact, it was recently shown with a clever manipulation of Cartier [Br], that the number theoretic identity, stuffle, may be seen as a period identity on multizeta values. This observation (among others) led naturally from our study of the Lie algebra of multizetas to the period algebra and we believe that all of the identities on multizeta values may be encoded as the identities which we derived from the geometry of moduli spaces. In this way, the study of the double shuffle Lie algebra is closely related to that of the period algebra.

## 1.1 Properties of multizeta values

In this section, I give the basic properties of multizeta values and the definitions from which this thesis is built.

To a sequence of positive integers,  $(k_1, \dots, k_d)$ , we associate a sequence in the non-commutative variables,  $x$  and  $y$ , by associating every  $k_i$  to the monomial  $x^{k_i-1}y$ . By concatenation, we then associate the sequence to the monomial,

$$(k_1, \dots, k_d) \sim x^{k_1-1}y \dots x^{k_d-1}y \quad (1.1.1)$$

whose degree is the same as the weight of the sequence of integers. Then we denote  $\zeta(k_1, \dots, k_d)$  by  $\zeta(x^{k_1-1}y \dots x^{k_d-1}y)$ .

**Definition 1.3.** Let  $\underline{k} = (k_1, \dots, k_d)$  be a sequence of positive integers and  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$  be corresponding a monomial in  $x$  and  $y$ . The sequences are **convergent** if  $k_1 \geq 2$ ,  $\epsilon_1 = 0$  and  $\epsilon_n = 1$ .

The  $x, y$  notation for a multizeta comes from its expression as an iterated integral. The following proposition is due to Kontsevich and is found in many texts about multiple zeta values, for example [Dr] and [IKZ].

**Proposition 1.4.** Let  $(k_1, k_2, \dots, k_r)$  be a sequence of positive integers and  $\underline{\omega}$  the corresponding word in  $x$  and  $y$ . We associate to  $\underline{\omega}$  a tuple of 0's and 1's,  $\underline{\epsilon}$ , by replacing each  $x$  and  $y$  in  $\underline{\omega}$  by 0 and 1 respectively to obtain the sequence,

$$(\epsilon_n, \dots, \epsilon_1) = (0, \dots, 0, 1, 0, \dots, 0, 1),$$

so that  $r$  is the number of 1's in the tuple, and  $\epsilon_1 = 1$ .

Then for an indeterminate  $z$ , we have

$$\sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \dots n_r^{k_r}} = (-1)^r \int_0^z \frac{dt_n}{t_n - \epsilon_n} \int_0^{t_n} \frac{dt_{n-1}}{t_{n-1} - \epsilon_{n-1}} \dots \int_0^{t_2} \frac{dt_1}{t_1 - \epsilon_1}. \quad (1.1.2)$$

When  $k_1 > 1$ , by setting  $z = 1$ , we have

$$\begin{aligned}\zeta(k_1, \dots, k_r) &= (-1)^r \int_0^1 \frac{dt_n}{t_n - \epsilon_n} \int_0^{t_n} \frac{dt_{n-1}}{t_{n-1} - \epsilon_{n-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - \epsilon_1} \\ &= (-1)^r \int_{0 < t_1 < t_2 < \cdots < t_n < 1} \frac{dt_1 dt_2 \cdots dt_n}{(t_n - \epsilon_n) \cdots (t_1 - \epsilon_1)}.\end{aligned}$$

*Proof.* We prove (1.1.2) by induction on  $n$ . For the base case  $n = 1$ ,  $k_1 = k_r = 1$ , so  $\epsilon_1 = 1$ . We have then that

$$\begin{aligned}\int_0^z \frac{dt_1}{(1 - t_1)} &= \int_0^z \left( \sum_{n=0}^{\infty} t_1^n \right) dt_1 \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n}.\end{aligned}$$

We now check the two base cases where  $n = 2$ , namely the tuple  $(0, 1)$  and the tuple  $(1, 1)$ , by repeated use of the series expansion  $1/(1 - t) = \sum_{i \geq 0} t^i$ . For the case  $(1, 1)$  we have

$$\int_0^z \frac{dt_2}{1 - t_2} \int_0^{t_2} \frac{dt_1}{1 - t_1} = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1 n_2}.$$

And for  $(0, 1)$  we have,

$$\int_0^z \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1 - t_1} = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Now assume (1.1.2) true for tuples of length  $n - 1$  and consider a tuple  $(\epsilon_n, \dots, \epsilon_1)$ . Assume first that  $\epsilon_n = 0$ . Then by the induction hypothesis the right hand side of (1.1.2) becomes

$$\begin{aligned}\int_0^z \frac{dt_n}{t_n} \sum_{n_1 > \cdots > n_r > 0} \frac{t_n^{n_1}}{n_1^{k_1-1} \cdots n_r^{k_r}} &= \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1-1} \cdots n_r^{k_r}} \int_0^z t_n^{n_1-1} dt_n \\ &= \sum_{n_1 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}}.\end{aligned}$$

To finish we only need to deal with the case where  $\epsilon_n = 1$ .

$$\begin{aligned}\int_0^z \frac{dt_n}{1 - t_n} \sum_{n_2 > \cdots > n_r > 0} \frac{t_n^{n_2}}{n_2^{k_2} \cdots n_r^{k_r}} &= \sum_{n_2 > \cdots > n_r > 0} \frac{1}{n_2^{k_2} \cdots n_r^{k_r}} \int_0^z \sum_{i \geq 0} t_n^{i+n_2} dt_n \\ &= \sum_{i \geq 0} \sum_{n_2 > \cdots > n_r > 0} \frac{1}{n_2^{k_2} \cdots n_r^{k_r}} \int_0^z t_n^{i+n_2} dt_n \\ &= \sum_{i \geq 0} \sum_{n_2 > \cdots > n_r > 0} \frac{1}{n_2^{k_2} \cdots n_r^{k_r}} \frac{z^{i+n_2+1}}{(i + n_2 + 1)} \\ &= \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}},\end{aligned}$$

where in the last line we set  $n_1 = i + n_2 + 1$  and by the hypothesis that  $\epsilon_1 = 1$ ,  $k_1 = 1$ .  $\square$

## 1.2 Quadratic relations on multizeta values

If one multiplies two multizeta values, one obtains the sum of multizeta values, but this expression is not unique. One expression was known already to Euler. In order to give these identities, we shall first define the shuffle and the stuffle products on sequences. We let  $\cdot$  denote the concatenation product of sequences.

**Definition 1.5.** For any two sequences of positive integers,  $\underline{a}$ ,  $\underline{b}$  the **stuffle product** of  $\underline{a}$  and  $\underline{b}$ , denoted  $st(\underline{a}, \underline{b})$  or  $\underline{a} * \underline{b}$ , is the formal sum obtained by the recursion:

1.  $st(\underline{a}, \emptyset) = st(\emptyset, \underline{a}) = \underline{a}$ ,
2.  $st(a_0 \cdot \underline{a}, b_0 \cdot \underline{b}) = a_0 \cdot st(\underline{a}, b_0 \cdot \underline{b}) + b_0 \cdot st(a_0 \cdot \underline{a}, \underline{b}) + (a_0 + b_0) \cdot st(\underline{a}, \underline{b})$ .

Morally, the stuffle product is obtained by taking permutations of  $\underline{a} \cdot \underline{b}$  such that the orders of both sequences are preserved and then adding adjacent pairs of elements, one from  $\underline{a}$  and one from  $\underline{b}$ , in all possible ways, in other words “stuffing” the elements of  $\underline{a}$  and  $\underline{b}$  into the same slot.

**Example 1.6.** The stuffle product  $(2, 1) * (3) = (2, 1, 3) + (2, 3, 1) + (3, 2, 1) + (2, 4) + (5, 1)$ .

**Definition 1.7.** Let  $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$  and  $\underline{\beta} = (\beta_1, \dots, \beta_l)$  be two sequences. The **shuffle product** of  $\underline{\alpha}$  and  $\underline{\beta}$ , denoted by  $sh(\underline{\alpha}, \underline{\beta})$ , or  $\underline{\alpha} \amalg \underline{\beta}$ , is the formal sum obtained by the recursive procedure:

1.  $sh(\underline{\alpha}, \emptyset) = sh(\emptyset, \underline{\alpha}) = \underline{\alpha}$ ,
2.  $sh(a_0 \cdot \underline{\alpha}, b_0 \cdot \underline{\beta}) = a_0 \cdot sh(\underline{\alpha}, b_0 \cdot \underline{\beta}) + b_0 \cdot sh(a_0 \cdot \underline{\alpha}, \underline{\beta})$ .

We will often rely on an equivalent definition of the shuffle product,

$$sh(\underline{\alpha}, \underline{\beta}) = \sum_{\sigma} \sigma(\underline{\alpha} \cdot \underline{\beta}),$$

where  $\sigma \in \mathfrak{S}_{k+l}$  runs over all permutations which preserve the orders of  $\underline{\alpha}$  and  $\underline{\beta}$ . For ease of notation, we write  $\underline{\gamma} \in sh(\underline{\alpha}, \underline{\beta})$  to mean that  $\underline{\gamma}$  is a term in the sum  $sh(\underline{\alpha}, \underline{\beta})$ .

**Example 1.8.**  $(0, 1) \amalg (0, 1) = 2(0, 1, 0, 1) + 4(0, 0, 1, 1)$ .

Both of the combinatorial products above are commutative. They were defined here in order to present the following two classical expressions for the product of multizetas.

**Proposition 1.9** (Euler). Let  $\underline{a}_1$  and  $\underline{a}_2$  be two convergent sequences of positive integers. Then,

$$\zeta(\underline{a}_1)\zeta(\underline{a}_2) = \sum_{\underline{a} \in st(\underline{a}_1, \underline{a}_2)} \zeta(\underline{a}) = \zeta(\underline{a}_1 * \underline{a}_2).$$

The iterated integral expression in proposition 1.1.2 for multizetas written in the  $x, y$  notation leads to the following alternative expression for the product of multizetas, also attributed to Kontsevich.

**Proposition 1.10.** Let  $\underline{\epsilon}_1$  and  $\underline{\epsilon}_2$  be two convergent sequences in the variables  $x$  and  $y$ . Then,

$$\zeta(\underline{\epsilon}_1)\zeta(\underline{\epsilon}_2) = \sum_{\underline{\epsilon} \in \underline{\epsilon}_1 \amalg \underline{\epsilon}_2} \zeta(\underline{\epsilon}) = \zeta(\underline{\epsilon}_1 \amalg \underline{\epsilon}_2).$$

The shuffle product on multizetas endows the vector space over  $\mathbb{Q}$  generated by multizeta values with the structure of a  $\mathbb{Q}$  algebra, while the stuffle product gives this algebra a set of relations. The system of relations on multizeta values given by the shuffle and the stuffle products is known as the system of double shuffle relations. If we restrict ourselves to double shuffle relations on multizeta values, we do not obtain what is conjectured to be a complete set of relations on multizetas.

An important system of relations on multizetas comes from regularization of non-convergent zeta values, a technique from physics to define a notion of cancelling divergences. Based on early work of Ecalle and Zagier (see [IKZ]), one may extend the double shuffle relations by allowing identities to be obtained from applying the double shuffle relations to divergent sums. The following important relation coming from regularization, known as **Hoffman's Relation**, is conjectured to complete the system of generating relations, along with double shuffle, on multizeta values.

**Proposition 1.11** (HO). *Let  $\underline{k}$  be a convergent sequence of positive integers and let  $\underline{\omega}$  be its corresponding sequence in  $x$  and  $y$  by the association (1.1.1). Then,*

$$\zeta\left(\sum_{l \in (1)*\underline{k}} l - \sum_{\lambda \in (y)\amalg \underline{\omega}} \lambda\right) = 0.$$

Note that although Hoffman's relation comes from regularization, it is a relation only on convergent zeta values, since each sum in the expression has only one non-convergent term,  $(1, \underline{k})$  and  $(y, \underline{\omega})$ , but these terms are equal and disappear in the difference.

The propositions 1.9, 1.10 and 1.11 are conjectured to be a generating set of relations on the algebra of multizeta values,  $\mathcal{Z}$ . This conjecture is one of the precise formulations of conjecture 1.2. As before, the analytic nature of this conjecture renders it out of reach with present techniques. Furthermore, even the algebraic structure of these three sets of relations is not fully understood. In order to study these relations, while avoiding the transcendence problem, we define the algebra of formal multizetas consisting of symbolic multizeta values and satisfying only these three sets of relations by definition.

**Definition 1.12.** *Let  $\mathcal{FZ}$  be the formal algebra generated by the symbols,*

$$W = \{\zeta^F(\omega); \text{ where } \omega \text{ is a convergent word in } x, y\},$$

*and containing the symbols*

$$T = \{\zeta^F(a); \text{ where } a \text{ is a convergent sequence of positive integers}\},$$

*with the following relations,*

1. *For every  $\zeta^F(\omega) \in W$  there is a unique  $\zeta^F(a) \in T$  such that  $\zeta^F(\omega) = \zeta^F(a)$  by the correspondence (1.1.1),*
2. *For all  $\zeta^F(\omega_1), \zeta^F(\omega_2) \in W$ ,  $\zeta^F(\omega_1)\zeta^F(\omega_2) = \zeta^F(\omega_1 \amalg \omega_2)$ ,*
3. *For all  $\zeta^F(a), \zeta^F(b) \in T$ ,  $\zeta^F(a)\zeta^F(b) = \zeta^F(a * b)$ ,*
4. *For all  $\omega \in W$  and  $a \in T$  such that  $\zeta^F(\omega) = \zeta^F(a)$  by relation 1,*

$$\zeta^F\left(\sum_{l \in (1)*(a)} l - \sum_{\lambda \in (y)\amalg (w)} \lambda\right) = 0.$$

The formal multizeta algebra is graded by weight, so we have

$$\mathcal{FZ} = \bigoplus_{n=0}^{\infty} \mathcal{FZ}_n,$$

where we set  $\mathcal{FZ}_0 = \mathbb{Q}$ ,  $\mathcal{FZ}_1 = 0$ . We also write  $\mathcal{FZ}_{>0} = \bigoplus_{n=1}^{\infty} \mathcal{FZ}_n$ .

**Definition 1.13.** We let  $\mathfrak{nf}_3$  denote the vector space obtained by quotienting  $\mathcal{FZ}$  by products,  $\zeta(2)$  and  $\mathbb{Q}$ :

$$\mathfrak{nf}_3 := \mathcal{FZ} / \langle \mathcal{FZ}_{>0}^2 \oplus \mathcal{FZ}_2 \oplus \mathcal{FZ}_0 \rangle.$$

We denote the elements of  $\mathfrak{nf}_3$  by  ${}_3(w)$  where  $w$  is a convergent word, or by  ${}_3(a)$  where  $a$  is a convergent sequence of integers.

In the following section, we introduce the double shuffle Lie algebra,  $\mathfrak{ds}$ , and relate it to a Lie coalgebra,  $\widetilde{\mathfrak{nf}}_3$ , which is conjecturally isomorphic to  $\mathfrak{nf}_3$ .

### 1.3 The Lie algebras, $\mathfrak{ds}$ and $\mathfrak{grt}$

The motivation for studying the double shuffle Lie algebra is its close relationship to multizeta values, which will be outlined in detail in the following section. I will summarize this relationship to motivate the results that are presented in this section.

**Definition 1.14.** Let  $\mathcal{Z}^2$  be the ideal in  $\mathcal{Z}$  generated by products of multizeta values. We define the  $\mathbb{Q}$ -vector space of *new zeta values* to be  $\mathfrak{n}_3 = \mathcal{Z} / \langle \mathcal{Z}^2 \oplus \mathbb{Q} \cdot \zeta(2) \oplus \mathbb{Q} \rangle$ . We denote an element of  $\mathfrak{n}_3$  by  $\bar{\zeta}(k_1, \dots, k_d)$  or by  $\bar{\zeta}(x^{k_1-1}y \cdots x^{k_d-1}y)$  where  $k_1, \dots, k_d$  is a convergent sequence of integers.

In G. Racinet's thesis, he constructs a subspace of the power series algebra,  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ , that is conjecturally isomorphic to dual space,  $\mathfrak{n}_3^\vee$ , and proves that this subspace,  $\mathfrak{ds}$ , is a Lie algebra for the Poisson bracket. We call this Lie algebra **the double shuffle Lie algebra**. This section is dedicated to defining the double shuffle Lie algebra.

In this chapter we work in the two noncommutative power series algebras,  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  and  $\mathbb{Q}\langle\langle y_i; 1 \leq i < \infty \rangle\rangle$ . For  $f$  a power series in one of these algebras, we denote by  $(f|w)$  the coefficient of the word  $w$  in  $f$ .

We associate an element in  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  to  $\mathbb{Q}\langle\langle y_i \rangle\rangle$  via the linear map,  $\pi_y$ , following [Ec] and [Ra]. It is closely linked to the alternative notation for a multizeta in the association (1.1.1).

**Definition 1.15.**

$$\begin{aligned} \pi_y : \mathbb{Q}\langle\langle x, y \rangle\rangle &\rightarrow \mathbb{Q}\langle\langle y_i \rangle\rangle \\ \tilde{\pi}_y(x^{k_1-1}y x^{k_2-1}y \cdots x^{k_n-1}y x^{k_{n+1}}) &= \begin{cases} 0 & k_{n+1} \neq 0 \\ y_{k_1} y_{k_2} \cdots y_{k_n} & k_{n+1} = 0 \end{cases} \\ \pi_y(f) &= \tilde{\pi}_y(f) + \sum_{n=2} (f|x^{n-1}y) \frac{(-1)^{n-1}}{n} y_1^n. \end{aligned}$$

The polynomial algebras  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  and  $\mathbb{Q}\langle\langle y_i \rangle\rangle$  may be equipped with the following coproducts defined on the generators,  $x, y, y_i$ , and extended multiplicatively:



**Definition 1.16.**

$$\begin{aligned}\Delta_{\text{III}} : \mathbb{Q}\langle\langle x, y \rangle\rangle &\rightarrow \mathbb{Q}\langle\langle x, y \rangle\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle\langle x, y \rangle\rangle \\ x &\mapsto x \otimes 1 + 1 \otimes x \\ y &\mapsto y \otimes 1 + 1 \otimes y \\ \Delta_* : \mathbb{Q}\langle\langle y_i \rangle\rangle &\rightarrow \mathbb{Q}\langle\langle y_i \rangle\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle\langle y_i \rangle\rangle \\ y_i &\mapsto \sum_{n+m=i} y_n \otimes y_m\end{aligned}$$

**Definition 1.17.** *The vector subspace,  $\mathfrak{d}\mathfrak{s} \subset \mathbb{Q}\langle\langle x, y \rangle\rangle$ , is generated by elements,  $f$ , that are primitive for  $\Delta_{\text{III}}$ , and such that  $\pi_{\mathcal{Y}}(f)$  is primitive for  $\Delta_*$ :*

$$\Delta_{\text{III}}(f) = f \otimes 1 + 1 \otimes f, \quad \Delta_*(\pi_{\mathcal{Y}}(f)) = \pi_{\mathcal{Y}}(f) \otimes 1 + 1 \otimes \pi_{\mathcal{Y}}(f).$$

**Definition 1.18.** *The Poisson bracket on elements of  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  is the Lie bracket given by*

$$\{f, g\} = [f, g] + D_f(g) - D_g(f)$$

where  $[f, g] = fg - gf$  and the  $D_f$  are derivations defined recursively by  $D_f(x) = 0$ ,  $D_f(y) = [y, f]$  and such that  $D_f(gh) = D_f(g)h + gD_f(h)$ .

The following result is one of the key ingredients in the understanding of the formal multiple zeta algebra (see section 4 for details).

**Theorem 1.19.** *[Ra] The double shuffle vector space,  $\mathfrak{d}\mathfrak{s}$ , forms a Lie algebra for the Poisson bracket.*

In the next section, we define a vector space  $\widetilde{\mathfrak{nf}}_3$  and prove that it is isomorphic to the dual space  $\mathfrak{d}\mathfrak{s}^{\vee}$  of  $\mathfrak{d}\mathfrak{s}$  (thereby proving in particular that  $\widetilde{\mathfrak{nf}}_3$  is a Lie coalgebra). The importance of the double shuffle Lie algebra in relation to multiple zeta values lies in the fact that the surjection from  $\widetilde{\mathfrak{nf}}_3$  (identified with  $\mathfrak{d}\mathfrak{s}^{\vee}$ ) to  $\mathfrak{n}_3$  given in the following proposition is conjectured to be an isomorphism.

**Proposition 1.20.** *We have a surjective,  $\mathbb{Q}$ -linear map from  $\mathfrak{d}\mathfrak{s}^{\vee}$  to  $\mathfrak{n}_3$ ,*

$$\mathfrak{d}\mathfrak{s}^{\vee} \rightarrow \mathfrak{n}_3.$$

The proof will be given in section 1.4.

The relationship between  $\mathfrak{d}\mathfrak{s}$ , multizetas, grt and mixed Tate motives, which will be outlined in the remainder of this introduction, is what led to our interest in studying multizeta values.

The double shuffle Lie algebra is graded by weight and each graded piece can be endowed with a filtration by depth. By definition 1.1, the depth of a monomial in  $x$  and  $y$  is the number of times  $y$  appears. We denote the depth filtration in the weight  $n$  part by

$$\mathfrak{d}\mathfrak{s} = F_n^1 \mathfrak{d}\mathfrak{s} \supset F_n^2 \mathfrak{d}\mathfrak{s} \supset \cdots \supset F_n^{n-1} \mathfrak{d}\mathfrak{s} \supset F_n^n(\mathfrak{d}\mathfrak{s}) = 0,$$

where  $F_n^i$  are generated by weight  $n$  polynomials whose terms all have depth greater than or equal to  $i$ . The Lie algebra  $\mathfrak{d}\mathfrak{s}$  is not graded by depth, since stuffle multiplication does not preserve depth. However, we may define an associated depth-graded object,

$$\bigoplus_{i \geq 1} F_n^i \mathfrak{d}\mathfrak{s} / F_n^{i+1} \mathfrak{d}\mathfrak{s}.$$

The dimensions of the vector spaces,  $F_n^i \mathfrak{d}\mathfrak{s} / F_n^{i+1} \mathfrak{d}\mathfrak{s}$ , are an essential feature of the structure of  $\mathfrak{d}\mathfrak{s}$ . This leads us to the main result in chapter 2 of this thesis:

**Theorem 1.21.** *The dimensions of the associated  $i$ th depth-graded parts of  $\mathfrak{d}\mathfrak{s}$  for  $i = 0, 1$  are*

$$\begin{aligned} \dim(F_n^1 \mathfrak{d}\mathfrak{s} / F_n^2 \mathfrak{d}\mathfrak{s}) &= \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \\ \dim(F_n^2 \mathfrak{d}\mathfrak{s} / F_n^3 \mathfrak{d}\mathfrak{s}) &= \begin{cases} 0 & n \text{ odd} \\ \lfloor \frac{n-2}{6} \rfloor & n \text{ even.} \end{cases} \end{aligned} \quad (1.3.1)$$

(This calculation was done before we knew that this result has been known by Zagier who published it in 1993 [Za1] and it was restated in [IKZ] and [GKZ].)

Here, I will explain the motivation for theorem 1.21. Y. Ihara defined the Lie algebra,  $\mathfrak{grt}$ , which is related to the Lie algebra of the braid group on 5 strands,  $\mathfrak{B}_5$  [Ih1].

**Definition 1.22.** *Let  $\mathfrak{B}_5$  be the Lie algebra with generators  $x_i, 1 \leq i \leq 5$  with the following relations:*

1.  $[x_i, x_j] = 0$  whenever  $1 < |i - j| < 4$ ,
2.  $[x_1, x_2] + [x_2, x_3] + [x_3, x_4] + [x_4, x_5] + [x_5, x_1] = 0$ .

**Definition 1.23.** *The Grothendieck-Teichmüller Lie algebra,  $\mathfrak{grt}$ , is the subspace of polynomials,  $\mathbb{Q} \oplus \langle f \in [\mathbb{L}[x, y], \mathbb{L}[x, y]] \rangle$ , such that the generators,  $f$ , satisfy the following 3 sets of relations:*

1.  $f(x, y) + f(y, x) = 0$ ,
2.  $f(x, y) + f(y, z) + f(z, x) = 0$ , where  $z = -x - y$ ,
3.  $\sum_{i \in \mathbb{Z}/5} f(x_i, x_{i+1}) = 0$ , for  $x_i \in \mathfrak{B}_5$ .

*This subspace is a Lie algebra for the Poisson bracket.*

One conjectures, and computations have verified in low weight, that  $\mathfrak{d}\mathfrak{s} \simeq \mathfrak{grt}$  and that the isomorphism is given simply by

$$\begin{aligned} f &: \mathfrak{d}\mathfrak{s} \rightarrow \mathfrak{grt} \\ f(x, y) &\mapsto f(x, -y). \end{aligned} \quad (1.3.2)$$

This conjecture remains remarkably elusive, although Ecalle claims to have shown that elements of  $\mathfrak{d}\mathfrak{s}$  satisfy the first relation of definition 1.23 and an unpublished and incomplete preprint of Deligne and Terasoma claims to have proven that map (1.3.2) gives an injection  $\mathfrak{grt} \hookrightarrow \mathfrak{d}\mathfrak{s}$ .

The algebras,  $\mathfrak{d}\mathfrak{s}$  and  $\mathfrak{grt}$ , encode two distinct, yet conjecturally equivalent, sets of relations on multizeta values. We have following theorem, due to Furusho [Fu], based on properties of the Drinfel'd associator,  $\Phi_{KZ}$  (see chapter 2). This theorem, analogous to proposition 1.20, underlines the relationship between  $\mathfrak{grt}$  and  $\mathfrak{d}\mathfrak{s}$ .

**Theorem 1.24.** *Let  $\mathfrak{grt}^\vee$  be the dual vector space to  $\mathfrak{grt}$ . Then there exists a canonical, surjective  $\mathbb{Q}$  linear map,*

$$\Psi_{DR} : \mathfrak{grt}^\vee \rightarrow \mathfrak{n}_3.$$

In [Ih2], Y. Ihara finds the following dimensions of the depth-graded pieces of  $\mathfrak{grt}$ :

**Theorem 1.25.** *The dimensions of the  $i$ th depth-graded parts of  $\mathfrak{grt}$  for  $i = 0, 1$  are*

$$\begin{aligned} \dim(F_n^1 \mathfrak{grt} / F_n^2 \mathfrak{grt}) &= \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \\ \dim(F_n^2 \mathfrak{grt} / F_n^3 \mathfrak{grt}) &= \begin{cases} 0 & n \text{ odd} \\ \lfloor \frac{n-2}{6} \rfloor & n \text{ even.} \end{cases} \end{aligned} \quad (1.3.3)$$

*Remark:* Theorem 1.21 was proved in 2005. Since then, a more general result has been published in [IKZ], where it is also shown that  $\dim(F_n^i \mathfrak{d}\mathfrak{s} / F_n^{i+1} \mathfrak{d}\mathfrak{s}) = 0$  whenever  $n$  and  $i$  have different parity. They also state without proof that for odd  $n$ ,

$$\dim(F_n^3 \mathfrak{d}\mathfrak{s} / F_n^4 \mathfrak{d}\mathfrak{s}) \geq \lfloor \frac{(n-3)^2 - 1}{48} \rfloor,$$

and they conjecture that  $\geq$  is an equality, a conjecture which has apparently been proven by Goncharov [Go1]. Our interpretation of this result is as follows. One conjectures that  $\mathfrak{d}\mathfrak{s}$  is a free Lie algebra with one depth 1 generator in each odd weight [Ih2]. We then believe that  $F_n^3 \mathfrak{d}\mathfrak{s} / F_n^4 \mathfrak{d}\mathfrak{s}$  is generated by images of  $\{\{f_i, f_j\}, f_k\} \bmod F_n^4 \mathfrak{d}\mathfrak{s}$ , where  $f_i, f_j, f_k$  are depth one elements,  $f_i, f_j, f_k \in \mathfrak{d}\mathfrak{s}$  are homogeneous of odd weight and  $i, j, k \geq 3$ . Furthermore, we believe that the only linear relations between these depth 3 elements come from bracketing a relation in depth 2 by a depth 1 element. Indeed, by counting the number of partitions,  $n = i + j + k$ , and subtracting off the number of depth 2 relations from theorem 1.21, we do indeed obtain the expected dimension  $\lfloor \frac{(n-3)^2 - 1}{48} \rfloor$ .

The situation in depth 4 (and higher) is much more complicated due to the presence of other generators that are not just Poisson brackets of depth 1 elements. For example, it is known by computation that the element,  $\{f_3, f_9\} - 3\{f_5, f_7\}$  is an element of depth 4, where  $f_i$  for  $i = 3, 5, 7, 9$  is the unique depth 1 element in weight  $i$  such that  $(f_i | x^{i-1}y) = 1$ .

## 1.4 The double shuffle Lie algebra and multizetas

In this section, we define a vector space,  $\widetilde{\mathfrak{nf}}_3$  which by work of Zagier, Ecalle, Le and Murakami, and finally Furusho, is known to surject onto  $\mathfrak{n}_3$ . We prove that  $\widetilde{\mathfrak{nf}}_3$  is isomorphic to the dual of  $\mathfrak{d}\mathfrak{s}$  defined in the previous section and conclude that  $\widetilde{\mathfrak{nf}}_3$  is a Lie coalgebra. The cobracket on  $\widetilde{\mathfrak{nf}}_3$  has been explicitly computed by Goncharov [Go]. Although none of the results in this section are new, they provide a framework for our work on  $\mathfrak{d}\mathfrak{s}$  in chapter 2.

**Definition 1.26.** *The Lie coalgebra,  $\widetilde{\mathfrak{nf}}_3$ , is the  $\mathbb{Q}$  vector space generated by symbols  $\mathfrak{z}^{\text{III}}(w)$  for all monomials  $w$  in  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  and symbols  $\mathfrak{z}^*(v)$  for all monomials  $v \in \mathbb{Q}\langle\langle x, y \rangle\rangle y$  (power series whose terms end in  $y$ ) modulo the following relations:*

1.  $\mathfrak{z}^{\text{III}}(w_1 \text{III} w_2) = 0$ ,
2.  $\mathfrak{z}^*(v_1 * v_2) = 0$ ,
3.  $\mathfrak{z}^{\text{III}}(1) = \mathfrak{z}^{\text{III}}(y) = \mathfrak{z}^{\text{III}}(x) = \mathfrak{z}^{\text{III}}(xy) = 0$ ,

4. If  $w = v$  is a word ending in  $y$  but not a power of  $y$ , then  $\mathfrak{z}^*(v) = \mathfrak{z}^{\text{III}}(w)$ , and  $\mathfrak{z}^*(y^n) = \frac{(-1)^{n-1}}{n} \mathfrak{z}^{\text{III}}(x^{n-1}y)$ .

**Proposition 1.27.**

$$\widetilde{\text{nf}}\mathfrak{z} \simeq \mathfrak{d}\mathfrak{s}^\vee.$$

*Proof.* Let  $f \in \mathfrak{d}\mathfrak{s} \subset \mathbb{Q}\langle\langle x, y \rangle\rangle$ . The relations in  $\mathfrak{d}\mathfrak{s}^\vee$  are given by the duals of the relations in  $\mathfrak{d}\mathfrak{s}$ , which are given by  $\mathfrak{d}\mathfrak{s}_1 = \mathfrak{d}\mathfrak{s}_2 = 0$  and relations (1.4.2) and (1.4.3) below, with

$$\pi_{\mathcal{Y}}(f) = \sum_{v \in \mathbb{Q}\langle\langle x, y \rangle\rangle \cdot y} (f|v)v + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (f|x^{n-1}y)y^n. \quad (1.4.1)$$

Let an element  $\mathfrak{z}^{\text{III}}(w) \in \mathfrak{d}\mathfrak{s}^\vee$ , be identified with the linear map,  $\mathfrak{d}\mathfrak{s} \rightarrow \mathbb{Q}$  given by  $\mathfrak{z}^{\text{III}}(w)(f) = (f|w)$  for all  $f \in \mathfrak{d}\mathfrak{s}$ . Let  $\mathfrak{z}^*(v) \in \mathfrak{d}\mathfrak{s}^\vee$  be identified with the linear map,  $\mathfrak{d}\mathfrak{s} \rightarrow \mathbb{Q}$  given by  $\mathfrak{z}^*(v)(f) = (\pi_{\mathcal{Y}}(f)|v)$  for all  $f \in \mathfrak{d}\mathfrak{s}$ . Note that in this proof, the symbols  $\mathfrak{z}^{\text{III}}$  and  $\mathfrak{z}^*$  refer to elements of  $\mathfrak{d}\mathfrak{s}^\vee$ , not to elements of  $\widetilde{\text{nf}}\mathfrak{z}$ , and the proof shows that they are equal. As usual,  $w$  always stands for an arbitrary word in  $x$  and  $y$ , and  $v$  for a word ending in  $y$ .

We will show that these linear maps,  $\mathfrak{z}^{\text{III}}(w) \in \mathfrak{d}\mathfrak{s}^\vee$ , satisfy the defining relations 1-4 of  $\widetilde{\text{nf}}\mathfrak{z}$  and no others. Indeed, the relations between the linear maps  $\mathfrak{z}^{\text{III}}(w)$  are exactly the duals of the relations in  $\mathfrak{d}\mathfrak{s}$ . Let us compute the dual relations of each of the four relations in  $\mathfrak{d}\mathfrak{s}$ .

First, we know that all  $f \in \mathfrak{d}\mathfrak{s}$  are primitive for  $\Delta_{\text{III}}$  which is equivalent to the condition:

$$\sum_{w \in w_1 \text{III} w_2} (f|w) = 0, \quad (1.4.2)$$

and hence  $\mathfrak{z}^{\text{III}}(w_1 \text{III} w_2) = 0$ . This is defining relation 1 of  $\widetilde{\text{nf}}\mathfrak{z}$ .

Next, we know that since for all  $f \in \mathfrak{d}\mathfrak{s}$ ,  $\pi_{\mathcal{Y}}(f)$  is primitive for  $\Delta_*$ , which is equivalent to the condition:

$$\sum_{v \in v_1 * v_2} (\pi_{\mathcal{Y}}(f)|v) = 0, \quad (1.4.3)$$

and hence  $\mathfrak{z}^*(v_1 * v_2) = 0$ . This is defining relation 2 of  $\widetilde{\text{nf}}\mathfrak{z}$ .

Finally, we know that  $\mathfrak{d}\mathfrak{s}_1 = \mathfrak{d}\mathfrak{s}_2 = 0$ . This immediately implies relation 3 of the definition of  $\widetilde{\text{nf}}\mathfrak{z}$ . Notice that  $ny^n = (y)_{\text{III}}(y^{n-1})$ , so that for  $f \in \mathfrak{d}\mathfrak{s}$ ,  $(f|y^n) = 0$ . It follows immediately that  $\mathfrak{z}^{\text{III}}(y^n) = 0$  for all  $n \geq 1$ . Similarly,  $\mathfrak{z}^{\text{III}}(x^n) = 0$  for all  $n \geq 1$ .

The last relation in  $\mathfrak{d}\mathfrak{s}$  is the defining formula (1.4.1). Therefore the coefficients of any word must be the same on both sides. If  $v$  is a word ending in  $x$ , this coefficient is 0 on both sides. If  $v$  is a word ending in  $y$  but not a power of  $y$ , the equality of the coefficients implies that  $\mathfrak{z}^*(v) = \mathfrak{z}^{\text{III}}(v)$ , which is the first part of defining relation 4 of  $\widetilde{\text{nf}}\mathfrak{z}$ . Finally, if  $v$  is a power of  $y$ , the equality of the coefficients shows that

$$\mathfrak{z}^*(y^n) = \mathfrak{z}^{\text{III}}(y^n) + \frac{(-1)^{n-1}}{n} \mathfrak{z}^{\text{III}}(x^{n-1}y)$$

since  $\mathfrak{z}^{\text{III}}(y^n) = 0$  as we showed above.

We have shown that the set of relations of  $\mathfrak{d}\mathfrak{s}^\vee$  is equal to the set of relations from definition 1.26, thus we have an isomorphism  $\mathfrak{d}\mathfrak{s}^\vee \simeq \widetilde{\text{nf}}\mathfrak{z}$ . □

Now we can prove proposition 1.20 from the previous section.

*Sketch of proof of proposition 1.20.* The proof of this proposition relies on proposition 1.27 proving that  $\partial\mathfrak{s}^\vee$  is isomorphic to the Lie coalgebra  $\widetilde{\mathfrak{nf}}_3$  defined in 1.26. The conclusion then follows from the regularization formula given by Furusho [Fu] (proposition 3.2.3), expressing non-convergent symbols  $\widetilde{\mathfrak{nf}}_3$  as explicit linear combinations of convergent symbols, thus giving an obvious surjection from  $\widetilde{\mathfrak{nf}}_3$  to  $\mathfrak{n}_3$  by mapping convergent symbols to the corresponding zeta values.  $\square$

Another proof of this proposition can be obtained by directly adapting Drinfel'd's and Furusho's proof of theorem 1.24 [Fu].

Now we are in a position to translate theorem 1.21 into the language of multizetas. As explained above,  $\widetilde{\mathfrak{nf}}_3$  surjects onto  $\mathfrak{n}_3$ . Theorem 1.21 implies that every depth 2 new zeta value in  $\mathfrak{n}_3$  of odd weight,  $\bar{\zeta}(a, b)$  where  $a + b$  is odd, is equal to a rational multiple of the depth 1 new zeta value,  $\bar{\zeta}(a + b)$ .

The proof of theorem 1.21 yields as a corollary the following formula for the coefficient of  $\bar{\zeta}(i, j)$  in terms of  $\bar{\zeta}(i + j)$  in  $\mathfrak{n}_3$ , which is actually the simplification of a result known to Euler, who gave the complete expression for  $\zeta(i, j)$  in  $\mathcal{Z}$ .

**Corollary 1.28.** *Assume that  $i + j$  is odd,  $i, j \geq 2$ . Then,*

$$\bar{\zeta}(i, j) = \frac{(-1)^{j-1} \binom{i+j}{j} - 1}{2} \bar{\zeta}(i + j).$$

## 1.5 The moduli space of genus 0 curves, $\mathfrak{M}_{0,n}$

Chapters 3 and 4 of this thesis are a study of multizeta values as periods on moduli space via the top dimensional de Rham cohomology,  $H^{n-3}(\mathfrak{M}_{0,n})$  (we drop the subscript,  $_{DR}$  for "de Rham", for the rest of the text). We begin this section by recalling the useful notations and properties of moduli space.

**Definition 1.29.** *The moduli space of genus 0 curves over  $\mathbb{C}$ ,  $\mathfrak{M}_{0,n}$ , is the space whose points are isomorphism classes of Riemann spheres with  $n$  distinct, ordered marked points modulo the action of  $\mathrm{PSL}_2(\mathbb{C})$  on the points.*

The action of  $\mathrm{PSL}_2$  is triply transitive, so we may denote a point in  $\mathfrak{M}_{0,n}$  by  $\overline{(z_1, \dots, z_n)}$  or by a well-chosen representative in its equivalence class,  $(0, t_1, \dots, t_\ell, 1, \infty)$ ,  $\ell = n - 3$ . In this way, we have the isomorphism,

$$\mathfrak{M}_{0,n} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^\ell \setminus \Delta, \quad (1.5.1)$$

where  $\Delta$  denotes the "fat" diagonal,  $\Delta = \{t_{i_1} = t_{i_j}; \text{ for all distinct } i_k, 1 \leq k \leq j\}$ .

The moduli space,  $\mathfrak{M}_{0,n}$  is not compact. A stable compactification,  $\overline{\mathfrak{M}}_{0,n}$  was defined by Deligne and Mumford [DM]. Adding boundary components to  $\mathfrak{M}_{0,n}$  corresponds to adding stable curves to  $\mathfrak{M}_{0,n}$ . These are genus 0 Riemann surfaces with nodes, such that each component has at least 3 marked or singular points. A visual interpretation of a point on the boundary  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$  is given in figure 1, where the simple closed loop on the left has been pinched to a geodesic of length 0 on the right.

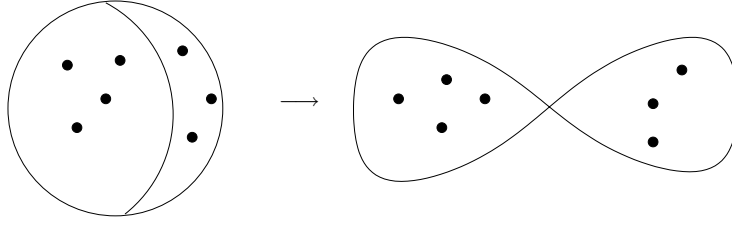


Figure 1. A point on the boundary of  $\mathfrak{M}_{0,n}$

The boundary divisors of  $\overline{\mathfrak{M}}_{0,n}$  are closed, irreducible codimension 1 subvarieties in  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$ . (Many authors use the term boundary divisor to denote  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$ , whereas we use the term for the irreducible components of  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$ .) In the association given in (1.5.1), they correspond to blowups of the regions in  $\Delta$ . We sometimes denote a boundary divisor by an equation,  $t_{i_1} = \cdots = t_{i_j}$ , which is understood to be the blowup in  $\overline{\mathfrak{M}}_{0,n}$  of that region in  $(\mathbb{P}^1)^{n-3}$ .

The boundary divisors may be combinatorially enumerated by specifying a partition of  $S = \{0, t_1, \dots, t_\ell, 1, \infty\}$  into two subsets,  $A$  and  $S \setminus A$ , with  $2 \leq |A| \leq n - 2$ . This is because any simple closed loop on the sphere with  $n$  marked points partitions the points of  $S$  into two subsets as in figure 1. We may alternatively denote by  $d_A$ , the boundary divisor in which the simple closed loop pinches the subset  $A \subset S$ , hence  $d_A = d_{S \setminus A}$ . During sections of this thesis where no confusion may arise, we may simply denote the boundary divisor by the set  $A$ .

**Definition 1.30.** We denote by  $\mathfrak{M}_{0,n}(\mathbb{R})$  the space of points,  $\{(0, t_1, \dots, t_\ell, 1, \infty); t_i \in \mathbb{R}\}$ .

While  $\mathfrak{M}_{0,n}$  is a connected manifold,  $\mathfrak{M}_{0,n}(\mathbb{R})$  is not connected. Each connected component in  $\mathfrak{M}_{0,n}(\mathbb{R})$  can be completely described by the real ordering of its marked points,  $t_{i_1} < \cdots < 0 < \cdots < 1 < \cdots < t_{i_{n-3}}$ .

**Definition 1.31.** A connected component of  $\mathfrak{M}_{0,n}(\mathbb{R})$  is called a *cell*. The cells in  $\mathfrak{M}_{0,n}(\mathbb{R})$  are also called *associahedra*. We denote a cell by the cyclic ordering corresponding to the real ordering of its marked points, where  $(s_1, s_2, \dots, s_n)$  denotes to the cell  $s_1 < s_2 < \cdots < s_n$  such that  $\{s_i, 1 \leq i \leq n\} = \{0, t_1, \dots, t_\ell, 1, \infty\}$ .

**Example 1.32.** Figure 1.1 depicts  $\mathfrak{M}_{0,5}(\mathbb{R})$ , where the lines are absent from the space, and the cells are the regions between the lines.

## 1.6 Periods on $\mathfrak{M}_{0,n}$ and the algebra, $\mathcal{C}$

The inspiration for chapters 3 and 4 of this thesis is a recent theorem of Francis Brown [Br] in which he proves that every period on  $\mathfrak{M}_{0,n}$  is a  $\mathbb{Q}$  linear combination of multiple zeta values. This led naturally to the question of whether the structure of the multiple zeta value algebra might not be more transparent or more symmetric by taking all periods as generators, and relations coming from the geometry of moduli spaces.

**Definition 1.33.** We define a *period* on  $\mathfrak{M}_{0,n}$  to be a convergent integral,  $\int_\gamma \omega$ , where  $\gamma$  is a cell in  $\mathfrak{M}_{0,n}(\mathbb{R})$  and  $\omega$  is a differential  $(n - 3)$ -form which is holomorphic on  $\mathfrak{M}_{0,n}$  and which has at most simple poles along the boundary divisors. We denote by  $\mathcal{C}$  the  $\mathbb{Q}$  algebra generated by periods on  $\mathfrak{M}_{0,n}$ .

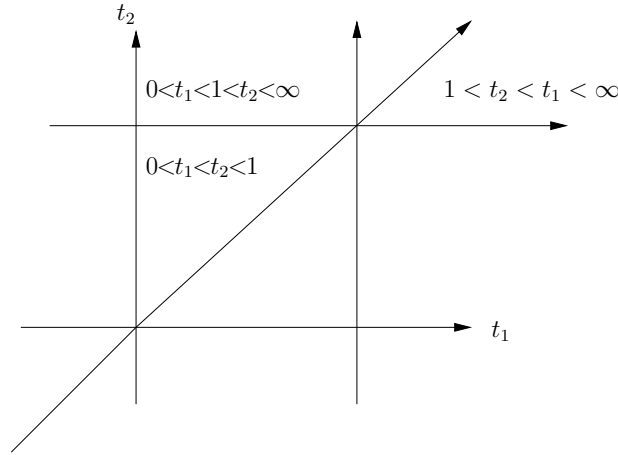


Figure 1.1:  $\mathfrak{M}_{0,5}(\mathbb{R})$

Up to a variable change corresponding to permuting the marked points, all periods may be written as integrals over the standard cell,  $\delta := 0 < t_1 < \dots < t_{n-3} < 1$ .

One of the main points of chapter 3 is that the combinatorial properties of periods can be expressed by using polygons. Let us now explain how polygons can be used to encode cells on  $\mathfrak{M}_{0,n}(\mathbb{R})$ , and also to encode certain differential forms on  $\mathfrak{M}_{0,n}$  called *cell forms*.

We may identify an oriented  $n$ -gon,  $\gamma$ , to a cell in  $\mathfrak{M}_{0,n}(\mathbb{R})$  by labelling the sides of the  $n$ -gon with the marked points. This  $n$ -gon is associated to the cell given by the clockwise cyclic ordering of the labelled edges of the polygon as in figure 3. Let  $Z = \{s_1, \dots, s_n\} = \{0, 1, \infty, t_1, \dots, t_\ell\}$  and let  $\gamma$  be a polygon decorated by  $Z$ , such that  $s_i$  is followed by  $s_{i+1}$  in the clockwise labelling of the edges and where  $i$  is taken modulo  $n$ . Then we denote  $\gamma$  by  $(s_1, \dots, s_n)$  and we have that  $\gamma = \sigma(s_1, \dots, s_n)$  where  $\sigma$  is any cyclic permutation in  $\mathfrak{S}_n$ .

Each component of the boundary of  $\gamma$  lies in some boundary divisor  $d_A \subset \overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$  such that  $A = \{s_i, s_{i+1}, \dots, s_{i+j}\}$  is a successive block in the cyclically ordered tuple,  $(s_1, \dots, s_n)$ .

**Example 1.34.** A polygon cyclically labelled  $(t_1, 0, t_3, 1, t_2, \infty) = \gamma$  is identified with the cell  $t_1 < 0 < t_3 < 1 < t_2 < \infty$  in  $\mathfrak{M}_{0,6}(\mathbb{R})$  as in figure 3.

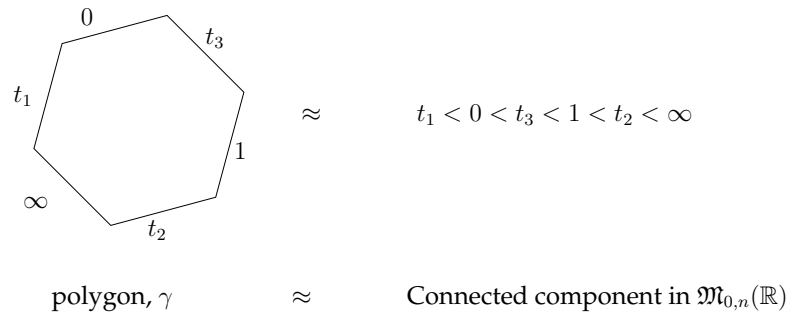


Figure 3. Polygon representation of a cell

For each cell in  $\mathfrak{M}_{0,n}(\mathbb{R})$ , there exists a unique differential  $\ell$ -form up to scalar multiple that is holomorphic on the interior and has simple poles on all of the divisors on



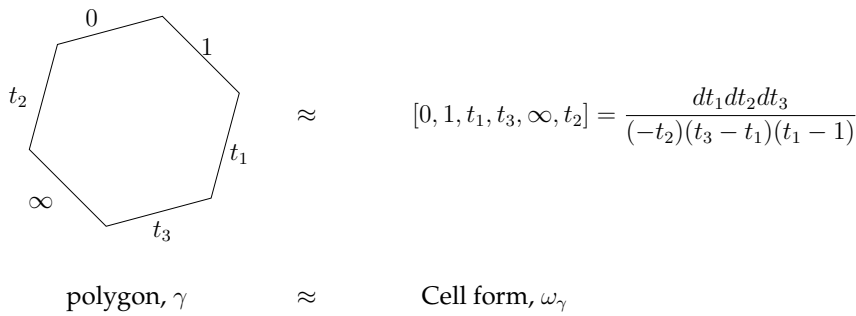


Figure 1.2: The polygon representation of a cell form

the boundary of that cell. We call such a form associated to the pole divisors of a cell a *cell form*.

**Definition 1.35.** Let  $\gamma$  be the cell,  $\gamma = (s_1, s_2, \dots, s_n)$ . The **cell form**,  $\omega_\gamma$ , associated to  $\gamma$  is defined as

$$\omega_\gamma = \frac{dt_1 \wedge \dots \wedge dt_{n-3}}{\prod (s_i - s_{i-1})},$$

where the  $s_i$  are the cyclically labelled sides of the polygon and where the side labelled  $\infty$  is left out of the product. This form is holomorphic on  $\mathfrak{M}_{0,n}$  and has simple poles along exactly those boundary divisors bounding  $\gamma$  and nowhere else on  $\overline{\mathfrak{M}}_{0,n}$ . We denote a cell form  $\omega_\gamma$  by the cyclic ordering  $[s_1, \dots, s_n]$ .

**Example 1.36.** The polygon cyclically labelled  $[0, 1, t_1, t_3, \infty, t_2]$  corresponds to the cell form in figure 1.2,  $\frac{dt_1 dt_2 dt_3}{(-t_2)(t_3 - t_1)(t_1 - 1)}$ .

We prove in chapter 3 that cell forms generate the de Rham cohomology group,  $H^\ell(\mathfrak{M}_{0,n})$ , so that every differential  $\ell$ -form can be written as a linear combination of these. We also explicitly determine a basis for the subspace  $H^\ell(\mathfrak{M}_{0,n}^\delta)$  of differential  $\ell$ -forms converging on the boundary divisors which bound the standard associahedron,  $\delta$ . To do this, we associate the integral  $\int_\gamma \omega_\beta$  to the polygon pair  $(\gamma, \beta)$  (even if this integral diverges). Because some linear combinations of cell forms, which individually diverge on  $\gamma$ , may actually converge on  $\gamma$ , the above results show that every convergent integral over  $\gamma$  can be expressed as a linear combination of pairs of polygons.

Using this association and Brown's theorem, we have defined (in a joint paper with F. Brown and L. Schneps, included as chapter 3) a formal algebra of periods, which is generated by polygon pairs, with relations coming from geometric properties of moduli spaces. The formal polygon pair algebra,  $\mathcal{FC}$ , generalizes the formal multizeta algebra and allows us to prove some results about periods and the cohomology of  $\mathfrak{M}_{0,n}$ . This gives a new approach to some conjectures about multizeta values and formal multizeta values. To begin with, in the theorem below, we use polygons to give a new basis for the top dimensional de Rham cohomology group,  $H^\ell(\mathfrak{M}_{0,n})$ , different from Arnol'd's well-known basis and more useful for the study of periods.

Following a theorem of Arnol'd (which is more precise, see chapter 4), we have the following characterization of the top dimensional de Rham cohomology group,  $H^\ell(\mathfrak{M}_{0,n})$  (top dimensional in the sense that  $H^m(\mathfrak{M}_{0,n}) = 0$  for all  $m > \ell$ ).

**Claim 1.37.**  $H^\ell(\mathfrak{M}_{0,n})$  is isomorphic to the vector space over  $\mathbb{Q}$  of differential forms which are holomorphic on  $\mathfrak{M}_{0,n}$  and which have at most simple poles along the boundary divisors,  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$ .



**Definitions 1.38.** Let  $\mathcal{P}_Z$  be the  $\mathbb{Q}$  vector space generated by oriented  $n$ -gons decorated by the marked points in  $\mathfrak{M}_{0,n}$ .

Let  $I_Z \subset \mathcal{P}_Z$  be the vector subspace generated by shuffle sums with respect to  $\infty$ , in other words polygon sums of the form

$$\sum_{W \in \text{Amb} B} [W, \infty],$$

where  $A, B$  is a partition of  $\{0, t_1, \dots, t_{n-3}, 1\}$ .

**Definition 1.39.** Let a cell form corresponding to a polygon in which 0 appears just to the left of 1 be called a 01-cell form.

**Theorem 1.40.**  $\mathcal{P}_Z/I_Z$  is isomorphic to  $H^\ell(\mathfrak{M}_{0,n})$  and a basis for  $H^\ell(\mathfrak{M}_{0,n})$  is given by the set of 01-cell forms,  $\{[0, 1, \sigma(\infty, t_1, \dots, t_\ell)], \sigma \in \mathfrak{S}_{n-2}\}$ .

Thus, each cohomology class contains a representative 01-cell form.

**Definition 1.41.** Let  $Z$  be the set denoting marked points on  $\mathfrak{M}_{0,n}$ ,  $Z = \{z_1, \dots, z_n\}$ . Let  $\rho$  be the set of partitions of  $Z$ , in which each set in the partition has cardinality greater than or equal to 2. We denote by  $D$  the disjoint union,  $\sqcup_{i \in \rho} D_i$  where each  $D_i$  is the (irreducible) boundary divisor in  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$  defined by the partition  $i$ . Likewise, if  $\gamma \sqcup \gamma^c$  is a partition of  $\rho$ , we denote by  $D_{\gamma^c} := \sqcup_{i \in \gamma^c} D_i$ . We denote by  $\mathfrak{M}_{0,n}^\gamma := \overline{\mathfrak{M}}_{0,n} \setminus D_{\gamma^c}$  and call  $\mathfrak{M}_{0,n}^\gamma$  a partial compactification of  $\mathfrak{M}_{0,n}$ .

So we have  $\mathfrak{M}_{0,n} \subset \mathfrak{M}_{0,n}^\gamma \subset \overline{\mathfrak{M}}_{0,n}$ . When no ambiguity can occur, to lighten the notation we may note  $D_\delta$  by  $\delta$ .

Based on a theorem of Grothendieck [Gr1], we use in chapter 3 and completely prove in chapter 4 that any period on  $\mathfrak{M}_{0,n}$  may be written as the integral of a linear combination of 01-forms which converges on  $D_\delta$ , the set of divisors each of which contains a face of the boundary of the standard associahedron,  $\delta$ , and such forms span the top dimensional de Rham cohomology of the partially compactified moduli space,  $H^\ell(\mathfrak{M}_{0,n}^\delta)$ .

Some 01-forms naturally converge on  $D_\delta$ . We define a *chord* on a cell form,  $\omega$ , to be a set of marked points of a consecutive subsequence on  $\omega$  of the length between 2 and  $\lfloor \frac{n}{2} \rfloor$ . The 01-forms which do not have any chords in common with the polygon  $\delta$  converge on the cell defined by the cyclic ordering  $\delta$ .

However, there are also some linear combinations of nonconvergent 01-forms which converge on  $D_\delta$ ; a basis for the space of these is the set of *insertion forms* defined in chapter 3.

With the above definitions, we can state one of the most important theorems in the article contained in chapter 3, which is a key ingredient in the definition of the algebra of periods (see next section). It gives a combinatorial construction of an explicit basis of  $H^\ell(\mathfrak{M}_{0,n}^\delta)$  and allows us to give a recursive formula for the dimension of this cohomology group.

**Theorem 1.42.** *The insertion forms and the convergent 01-cell forms form a basis for  $H^\ell(\mathfrak{M}_{0,n}^\delta)$ .*

The proof of this theorem is the heart of our recent work and is given in chapter 3. The goal is to attain an explicit combinatorial description of an algebra generated by “formal periods” in analogy with the formal multizeta value algebra,  $\mathcal{FZ}$ .

## 1.7 The algebra of formal periods, $\mathcal{FC}$

The period algebra,  $\mathcal{C}$ , has three known sets of relations coming from the following three important geometric properties of moduli spaces:

1. Invariance under the symmetric group action corresponding to a variable change,
2. Forms given by shuffles with respect to one point are identically 0,
3. Product map relations coming from the pullback of maps on moduli spaces (these are outlined in [BCS] and [Br]).

In the style of [KZ], who conjecture that only algebraic relations of certain geometric types exist between periods, we conjecture that these are the only relations on the periods on  $\mathfrak{M}_{0,n}$ . This is why our strategic approach to understanding the implications of this conjecture is to define a formal algebra on polygon pairs satisfying these and only these relations.

**Definition 1.43.** *The formal cell number algebra,  $\mathcal{FC}$ , is defined as the algebra generated by pairs of polygons,  $\mathcal{P}_Z \otimes_{\mathbb{Q}} \mathcal{P}_Z$ , decorated by the marked points in  $Z$  with the following sets of relations:*

1.  $(\gamma, \omega) = (\sigma(\gamma), \sigma(\omega)) \forall \sigma \in \mathfrak{S}_n$ ,
2. For any  $e \in Z$  and for any partition  $A, B$  of  $Z \setminus \{e\}$ ,

$$((e, A \amalg B), \omega) = (\gamma, (e, A \amalg B)) = 0,$$

3. For any partition,  $A, B$  of  $Z \setminus \{0, 1, \infty\}$ , and for any four polygons,  $\gamma_1$  and  $\omega_1$  decorated by  $A \cup \{0, 1, \infty\}$ ,  $\gamma_2$  and  $\omega_2$  decorated by  $B \cup \{0, 1, \infty\}$ , we have the product map relation,

$$(\gamma_1, \omega_1)(\gamma_2, \omega_2) = (\gamma_1 \amalg \gamma_2, \omega_1 \amalg \omega_2).$$

The first relation on  $\mathcal{FC}$  comes from variable changes on periods, the second relation from theorem 1.40, and the third from product maps on moduli spaces.

Using the definition of the period algebra and Brown's theorem, one shows easily that the algebra of periods,  $\mathcal{C}$ , is isomorphic to the algebra of multizeta values,  $\mathcal{Z}$  [Br]. This key remark is our main motivation for the definition of  $\mathcal{FC}$ , and leads naturally to the conjecture that  $\mathcal{FC}$  is isomorphic to the formal multizeta value algebra,  $\mathcal{FZ}$ . This conjecture seems likely because the algebra of formal cell numbers has shuffle multiplication and  $\mathcal{FC}$  encodes multizeta values, so it should also have stuffle. We have not yet been able to prove this, but computer calculations do support the hypothesis that  $\mathcal{FC}$  has the stuffle relation. Such calculations are given at the end of chapter 3. The relation between the different algebras is depicted in the following commutative diagram, where  $f$  is conjecturally an isomorphism:

$$\begin{array}{ccc} \mathcal{FC} & \overset{f}{\longleftrightarrow} & \mathcal{FZ} \\ \downarrow & & \downarrow \\ \mathcal{C} & \overset{\sim}{\longrightarrow} & \mathcal{Z}. \end{array}$$

## 1.8 Cohomology of partially compactified moduli spaces

Chapter 4 of this thesis extends the methods of chapter 3 to calculating the cohomology of  $\mathfrak{M}_{0,n}^\gamma$  for certain sets of divisors,  $\gamma$ , such that  $\mathfrak{M}_{0,n}^\gamma$  is affine. The search for criteria for affineness led to a new, combinatorial description of the Picard group,  $\text{Pic}(\mathfrak{M}_{0,n})$ , with a basis given by polygons.

In the first two sections of chapter 4, we recall, in a self-contained way, the proof of the following proposition, using the Leray theorem of spectral sequences and a theorem of Grothendieck on algebraic de Rham complexes.

**Proposition 1.44.** *If  $\mathfrak{M}_{0,n}^\gamma$  is an affine variety, then the top dimensional de Rham cohomology,  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  is isomorphic to the subspace of  $H^\ell(\mathfrak{M}_{0,n})$  of the classes of differential forms which have a representative that is holomorphic on  $\mathfrak{M}_{0,n}^\gamma$  and has at most logarithmic singularities along  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}^\gamma$ .*

The third section is dedicated to defining certain criteria for  $\mathfrak{M}_{0,n}^\gamma$  to be affine. The key observation of this section is that if  $\gamma$  is a subset of divisors that bound an associahedron, then  $\mathfrak{M}_{0,n}^\gamma$  is affine. In particular, using the geometry of  $\mathfrak{M}_{0,n}$ , we obtain the following proposition as a corollary to this observation.

**Proposition 1.45.** *If  $\gamma$  is a set containing only one divisor  $\{d_A\}$ , two divisors  $\{d_A, d_B\}$ , or the set of three divisors  $\{d_A, d_B, d_{A \cup B}\}$ , then  $\mathfrak{M}_{0,n}^\gamma$  is affine.*

In section 4 of chapter 4, we generalize the results of chapter 3 to find an explicit basis of polygons for  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  in the cases given in proposition 1.45. To do this, we exploit the residue map on polygons and cell forms.

As before, we denote by  $\mathcal{P}_{S_i}$  the  $\mathbb{Q}$ -vector space generated by polygons decorated by the marked points in a set  $S_i$ , and by  $I_{S_i} \subset \mathcal{P}_{S_i}$  the subspace of shuffles with respect to one element as in definition 1.38. We associate a divisor,  $d_{S_i}$  to a chord on a polygon,  $\omega^p = [s_{i_1}, \dots, s_{i_n}] \in \mathcal{P}_Z$  to be a partition of  $\omega^p$  into consecutive blocks,  $[s_{i_j}, \dots, s_{i_{j+k}}]$  and  $[s_{i_{j+k+1}}, \dots, s_{i_{j-1}}]$  such that  $k \geq 1$  as in the left-hand object in figure 5.

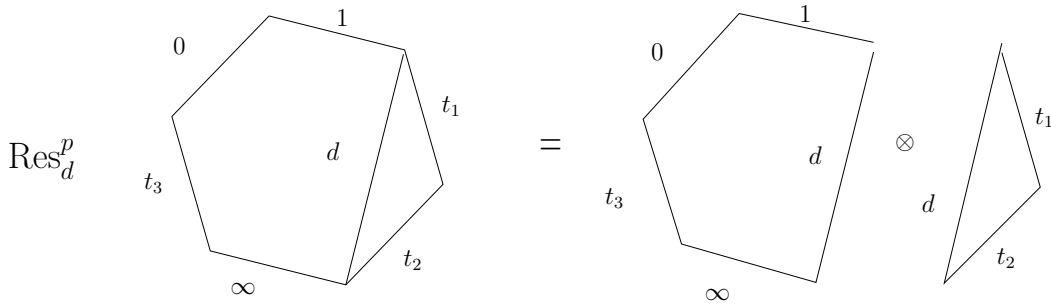


Figure 5. Residue chord of a polygon for the divisor,  $d = d_{t_1, t_2}$

Now, for every partition  $Z$  given by  $Z = S_1 \cup S_2$ , we define a residue map on polygons with respect to the divisor,  $d_{S_1} = d_{S_2} = d$ :

$$\text{Res}_d^p : \mathcal{P}_Z \rightarrow \mathcal{P}_{S_1 \cup \{d\}} \otimes_{\mathbb{Q}} \mathcal{P}_{S_2 \cup \{d\}},$$

which is simply the tensor product of the two polygons formed by cutting along the divisor  $d$ .

**Definition 1.46.** Let  $\omega^p$  be a polygon in  $\mathcal{P}_Z$ . If the partition  $S_1, S_2$  corresponds to a chord of  $\omega^p$ , then it cuts  $\omega^p$  into two subpolygons  $\omega_i^p$  ( $i = 1, 2$ ) whose edges are indexed by the set  $S_i$  and an edge labelled  $d$  corresponding to the chord  $d$ . We set

$$\text{Res}_d^p(\omega^p) = \begin{cases} \omega_1^p \otimes \omega_2^p & \text{if } d \text{ is a chord of } \omega^p \\ 0 & \text{if } d \text{ is not a chord of } \omega^p. \end{cases} \quad (1.8.1)$$

Let  $\pi : \mathcal{P}_Z \rightarrow H^\ell(\mathfrak{M}_{0,n})$  be the map from polygons to cell forms as in definition 1.35. In chapter 4, the following theorem is proved.

**Theorem 1.47.** Let  $\gamma = \{\gamma_1, \dots, \gamma_k\}$  be a set of boundary divisors of  $\overline{\mathfrak{M}}_{0,n}$  such that  $\mathfrak{M}_{0,n}^\gamma$  is affine. Then, the  $\mathbb{Q}$  vector space,  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  coincides with the differential forms in the intersection of vector spaces,

$$\bigcap_{i=1}^k \pi((\text{Res}_{\gamma_i}^p)^{-1}(I_{\gamma_i \cup \{d\}} \otimes \mathcal{P}_{Z \setminus \gamma_i \cup \{d\}})).$$

Furthermore, a basis for  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  can easily be deduced from a Lyndon basis of the polygons in  $I_{\gamma_i \cup \{d\}} \otimes \mathcal{P}_{Z \setminus \gamma_i \cup \{d\}}$  using insertion forms.

As a corollary to this theorem we display explicit bases for  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  for sets  $\gamma = \{d_A\}$ ,  $\{d_A, d_B\}$  and  $\{d_A, d_B, d_{A \cup B}\}$  and give a closed formula for the dimensions.

The search for criteria on  $\gamma$  such that  $\mathfrak{M}_{0,n}^\gamma$  is affine led us to investigate  $\text{Pic}(\overline{\mathfrak{M}}_{0,n})$ . If a divisor  $\gamma^c$  is ample in the Picard group, then  $\mathfrak{M}_{0,n}^{\gamma^c}$  is an affine space. Although we didn't succeed in proving that  $\gamma^c$  was affine, for certain  $\gamma$  that we were interested in (such as the pole divisors of a multizeta form), this search led to a new presentation of  $\text{Pic}(\overline{\mathfrak{M}}_{0,n})$  with a basis of polygons. The final section of this thesis is dedicated to the statement and proof of this result.

## Chapter 2

# Comparison and combinatorics of the Lie algebras, $\mathfrak{ds}$ , $\mathfrak{grt}$ and $\mathfrak{nfz}$

In this chapter we prove a dimension result on  $\mathfrak{ds}$  which provides evidence toward the conjecture stated in the introduction that  $\mathfrak{ds} \simeq \mathfrak{grt}$ . Some theorems and definitions given in the introduction and used in the chapter are restated for easy reference for the reader.

A multizeta value is a real number defined by the iterated sum,

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 > n_2 > \dots > n_d > 0} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_d^{k_d}},$$

where  $(k_1, \dots, k_d)$  is a sequence of positive integers such that  $k_1 \geq 2$ . We may call a multizeta value a multiple zeta value, a multizeta, an *MZV*, a zeta value or simply a zeta. The **depth** of  $\zeta(k_1, \dots, k_d)$  is  $d$  and its **weight** is  $\sum_{i=1}^d k_i$ . Let  $\mathcal{Z}$  denote the algebra over  $\mathbb{Q}$  generated by multizeta values, and let  $\mathcal{Z}_n$  denote the vector space over  $\mathbb{Q}$  generated by multizeta values of weight  $n$ .

Although  $\mathcal{Z}$  is simple to define, there remain many open questions about this algebra. The motivation for the results in this chapter stem from the following open problem about  $\mathcal{Z}$ . It is believed that all linear relations over  $\mathbb{Q}$  on multizeta values are generated by the double shuffle relations and Hoffman's relation, relations which preserve the weight of elements in  $\mathcal{Z}$ . This in turn would imply the well-known "direct sum conjecture":

**Conjecture.** *The algebra,  $\mathcal{Z}$ , is graded by weight and hence  $\mathcal{Z} := \bigoplus_{n=0}^{\infty} \mathcal{Z}_n$ .*

Note that this ambitious conjecture would imply the transcendence of every multizeta value, since the minimal polynomial of an algebraic multizeta value would yield a linear relation in different weights.

Of particular interest to us are the depth 1 generators of  $\mathcal{Z}$ ,  $\zeta(n)$ . The depth 1 generators in even weight are well understood and have long been known to be transcendental.

**Theorem 2.1** (Euler).

$$\zeta(2) = \frac{\pi^2}{6}$$
$$\zeta(2n) = \frac{2^{2n-1} |B_n| \pi^{2n}}{(2n)!},$$

where  $B_r$  is the Bernoulli number that is obtained by expanding the series,

$$\frac{y}{e^y - 1} = \sum_{r=0}^{\infty} B_r \frac{y^r}{r!}.$$

However, the depth 1 generators in odd weight are less well understood. They are conjectured to be transcendental numbers. R. Apéry [Ap] proved that  $\zeta(3)$  was irrational and T. Rivoal [BR] recently proved that there are infinitely many irrational  $\zeta(2n + 1)$ .

This chapter is not an attempt to tackle the question of irrationality of depth 1 zeta values, which seems very difficult because of the analytic nature of the problem. Yet, by working in the Lie algebra, we obtain results relating to the conjecture that the double shuffle Lie algebra is isomorphic to the free Lie algebra with one generator in each odd weight,

$$\mathfrak{f} = \mathbb{L}[x_{2n+1} : n \geq 1].$$

## 2.1 The double shuffle Lie algebra, $\mathfrak{ds}$

In this chapter, we work in the two noncommutative power series algebras,  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  and  $\mathbb{Q}\langle\langle y_i; 1 \leq i < \infty \rangle\rangle$ . For  $f$  a polynomial in one of these algebras, we denote by  $(f|w)$  the coefficient of the monomial  $w$  in  $f$ .

### 2.1.1 Shuffle on $\mathbb{L}[x, y]$

The power series algebra,  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ , may be graded in two ways, by weight and by depth according to the following definition.

**Definition 2.2.** The algebra  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  possesses a grading by the length of its monomials,  $\omega$ , which we call the **weight** and we denote the weight of  $\omega$  by  $w(\omega)$ . Similarly, we can define a grading on  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  by the **depth** of the monomial, which is the number of times  $y$  appears and we denote the depth of  $\omega$  by  $d(\omega)$ . The notation,  $V_n$ , where  $V$  is any vector space of polynomials, refers to its weight  $n$  graded part.

The algebra,  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ , may be equipped with the following coproduct to form a Hopf algebra,

$$\Delta_{\text{III}} : \mathbb{Q}\langle\langle x, y \rangle\rangle \rightarrow \mathbb{Q}\langle\langle x, y \rangle\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle\langle x, y \rangle\rangle \quad (2.1.1)$$

$$x \mapsto x \otimes 1 + 1 \otimes x \quad (2.1.2)$$

$$y \mapsto y \otimes 1 + 1 \otimes y. \quad (2.1.3)$$

**Definition 2.3.** A element  $f \in \mathbb{Q}\langle\langle x, y \rangle\rangle$  is **primitive** for the coproduct  $\Delta_{\text{III}}$ , if  $\Delta_{\text{III}}(f) = 1 \otimes f + f \otimes 1$ .

**Definition 2.4.** The Lie algebra,  $\mathbb{L}[x, y] \subset \mathbb{Q}\langle\langle x, y \rangle\rangle$ , is the subspace of polynomials generated by successive bracketings of  $x, y$  for the Lie bracket,  $[f, g] = fg - gf$ .

The Lie algebra,  $\mathbb{L}[x, y]$ , possesses the grading by weight and depth inherited from  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ . We denote by  $\mathbb{L}_n[x, y]$  the weight  $n$  graded part and by  $\mathbb{L}_n^i[x, y]$  the depth  $i$ , weight  $n$  graded part, so that

$$\begin{aligned} \mathbb{L}[x, y] &= \bigoplus_n \mathbb{L}_n[x, y] \\ \mathbb{L}_n[x, y] &= \bigoplus_{1 \leq i < n} \mathbb{L}_n^i[x, y]. \end{aligned}$$

The vector space,  $\mathbb{L}_n[x, y]$ , is of finite dimension for each  $n$  (we will recall the dimension formula in section 2.2).

Here we recall the definition of the shuffle product on monomials.

**Definition 2.5.** Let  $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$  and  $\underline{\beta} = (\beta_1, \dots, \beta_l)$  be two sequences. The **shuffle product** of  $\underline{\alpha}$  and  $\underline{\beta}$ , denoted by  $sh(\underline{\alpha}, \underline{\beta})$ , or  $\underline{\alpha \# \beta}$ , is the formal sum obtained by the recursive procedure:

1.  $sh(\underline{\alpha}, \emptyset) = sh(\emptyset, \underline{\alpha}) = \underline{\alpha}$ ,
2.  $sh(a_0 \cdot \alpha, b_0 \cdot \beta) = a_0 \cdot sh(\alpha, b_0 \cdot \beta) + b_0 \cdot sh(a_0 \cdot \alpha, \beta)$ .

The shuffle product on sequences  $\underline{\alpha}$  and  $\underline{\beta}$  is the sum over all of the permutations of  $\underline{\alpha} \cdot \underline{\beta}$  that preserve the orders of both sequences. For ease of notation, we write  $\underline{\gamma} \in sh(\underline{\alpha}, \underline{\beta})$  to mean that  $\underline{\gamma}$  is a term in the sum  $sh(\underline{\alpha}, \underline{\beta})$ .

**Proposition 2.6.** [Se][Re] For  $f \in \mathbb{Q}\langle\langle x, y \rangle\rangle$  the following conditions are equivalent:

1.  $f \in \mathbb{L}[x, y]$ ,
2.  $\Delta_{\text{III}}(f) = f \otimes 1 + 1 \otimes f$ ,
3. For any  $\underline{\omega}_1, \underline{\omega}_2$ , non-empty sequences in  $x$  and  $y$ ,

$$\sum_{\underline{\omega} \in sh(\underline{\omega}_1, \underline{\omega}_2)} (f|\underline{\omega}) = 0.$$

Let  $\mathcal{U}(\mathbb{L}[x, y])$  be the universal enveloping algebra of  $\mathbb{L}[x, y]$  and we denote by  $\cdot$  its product.

$$\mathcal{U}(\mathbb{L}[x, y]) = \bigoplus_{n=0}^{\infty} T^{\otimes n} / \langle f \otimes g - g \otimes f - [f, g] \mid f, g \in \mathbb{L}[x, y] \rangle, \quad (2.1.4)$$

where  $T^{\otimes n} = \bigotimes^n \mathbb{L}[x, y]$  is the  $n$ th tensor power of  $\mathbb{L}[x, y]$ . The universal enveloping algebra naturally possesses a Hopf algebra structure with coproduct,  $\Delta_{\mathbb{L}[x, y]}$ , which is the unique algebra morphism which is primitive for the elements in  $\mathbb{L}[x, y]$ . So we have the following corollary:

**Corollary 2.7.** [Se] We have the isomorphism of Hopf algebras,

$$\begin{aligned} (\mathbb{Q}\langle\langle x, y \rangle\rangle, \Delta_{\text{III}}, \cdot) &\simeq (\mathcal{U}(\mathbb{L}[x, y]), \Delta_{\mathbb{L}[x, y]}, \cdot) \\ x &\mapsto x \\ y &\mapsto y. \end{aligned}$$

*Proof.* The universal enveloping algebra on the free Lie algebra on  $n$  generators is isomorphic to the free polynomial algebra on  $n$  variables. By taking  $n = 2$  we have  $(\mathbb{Q}\langle\langle x, y \rangle\rangle, \cdot) \simeq (\mathcal{U}(\mathbb{L}[x, y]), \cdot)$ . By the theorem 2.6, the primitive elements for the coproduct,  $\Delta_{\text{III}}$  are exactly those in the Lie algebra,  $\mathbb{L}[x, y]$ .  $\square$

The weight grading that we give to  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  is the same as one gives to the grading in the universal enveloping algebra defined by the theorem of Poincaré-Birkhoff-Witt.

### 2.1.2 Stuffle on $\mathbb{L}[y_i]$

Here, we make analogous definitions and statements for the algebra  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ .

The power series algebra,  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ , possesses a grading given by the sum of the indices of the monomial which we call the **weight** of the monomial, i.e.  $w(y_{i_1} \cdots y_{i_r}) = \sum_{j=1}^r i_j$ . Similarly, we can define a grading on  $\mathbb{Q}\langle\langle y_i \rangle\rangle$  by the **depth** of the monomial, which is the length of the monomial, i.e.  $d(y_{i_1} \cdots y_{i_r}) = r$ .

The algebra,  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ , may be equipped with the following coproduct to form a Hopf algebra,

$$\Delta_* : \mathbb{Q}\langle\langle y_i \rangle\rangle \rightarrow \mathbb{Q}\langle\langle y_i \rangle\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle\langle y_i \rangle\rangle \quad (2.1.5)$$

$$y_i \mapsto \sum_{n+m=i} y_n \otimes y_m. \quad (2.1.6)$$

If  $\Delta_*(f) = 1 \otimes f + f \otimes 1$ , then  $f$  is primitive for  $\Delta_*$  as in definition 2.3.

**Definition 2.8.** *The Lie algebra,  $\mathbb{L}[y_i] \subset \mathbb{Q}\langle\langle y_i \rangle\rangle$ , is the subspace of polynomials generated by successive bracketings of  $y_i$  for the Lie bracket,  $[f, g] = fg - gf$ .*

The Lie algebra,  $\mathbb{L}[y_i]$ , possesses the grading by weight and depth inherited from  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ .

We define here the stuffle product of monomials in  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ , which is analogous to the stuffle product on sequences of positive integers given in the introduction.

**Definition 2.9.** *For any monomials in  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ ,  $\underline{a}$ ,  $\underline{b}$  the **stuffle product** of  $\underline{a}$  and  $\underline{b}$ , denoted  $st(\underline{a}, \underline{b})$  or  $\underline{a} * \underline{b}$ , is the formal sum obtained by the recursion:*

1.  $st(\underline{a}, \emptyset) = st(\emptyset, \underline{a}) = \underline{a}$ ,
2.  $st(y_i \cdot \underline{a}, y_j \cdot \underline{b}) = y_i \cdot st(\underline{a}, y_j \cdot \underline{b}) + y_j \cdot st(y_i \cdot \underline{a}, \underline{b}) + (y_{i+j}) \cdot st(\underline{a}, \underline{b})$ .

The following proposition due to J. Ecalle, gives us an easy method of determining whether  $f \in \mathbb{Q}\langle\langle y_i \rangle\rangle$  is in  $\mathbb{L}[y_i]$  and also gives the link between the stuffle relation and the coproduct,  $\Delta_*$ .

**Proposition 2.10.** *[Ec] For  $f \in \mathbb{Q}\langle\langle y_i \rangle\rangle$  the following conditions are equivalent:*

1.  $f \in \mathbb{L}[y_i]$ .
2.  $\Delta_*(f) = f \otimes 1 + 1 \otimes f$ .
3. For all  $\underline{\omega}_1, \underline{\omega}_2$ , non-empty sequences in  $\{y_i\}$ ,

$$\sum_{\underline{\omega} \in st(\underline{\omega}_1, \underline{\omega}_2)} (f|_{\underline{\omega}}) = 0.$$

We associate an element in  $\mathbb{Q}\langle\langle x, y \rangle\rangle$  to  $\mathbb{Q}\langle\langle y_i \rangle\rangle$  via the linear map,  $\pi_y$ , the corrected projection onto  $\mathbb{Q}\langle\langle y_i \rangle\rangle$ . It is closely linked to the alternative notation for a multizeta in the association,

$$\begin{aligned} x^{k_1-1} y \cdots x^{k_d-1} y &\sim k_1 k_2 \cdots k_d \\ \zeta(x^{k_1-1} y \cdots x^{k_d-1} y) &= \zeta(k_1 k_2 \cdots k_d). \end{aligned}$$



**Definition 2.11.** Let  $\pi_{\mathcal{Y}}$  be the  $\mathbb{Q}$  linear map defined by:

$$\pi_{\mathcal{Y}} : \mathbb{Q}\langle\langle x, y \rangle\rangle \rightarrow \mathbb{Q}\langle\langle y_i \rangle\rangle \quad (2.1.7)$$

$$\tilde{\pi}_{\mathcal{Y}}(x^{k_1-1}yx^{k_2-1}y \cdots x^{k_n-1}yx^{k_{n+1}}) = \begin{cases} 0 & k_{n+1} \neq 0 \\ y_{k_1}y_{k_2} \cdots y_{k_n} & k_{n+1} = 0 \end{cases} \quad (2.1.8)$$

$$\pi_{\mathcal{Y}}(f) = \tilde{\pi}_{\mathcal{Y}}(f) + \sum_{n=2} (f|x^{n-1}y) \frac{(-1)^{n-1}}{n} y_1^n. \quad (2.1.9)$$

**Example 2.12.** Let  $f = 2x^2y + x^3y + 4xy^2 - 8yxy + 4y^2x$ . Then,

$$\begin{aligned} \tilde{\pi}_{\mathcal{Y}}(f) &= 2y_3 + y_4 + 4y_2y_1 - 8y_1y_2, \\ \pi_{\mathcal{Y}}(f) &= 2y_3 + y_4 + 4y_2y_1 - 8y_1y_2 + \frac{2}{3}y_1^3 - \frac{1}{4}y_1^4. \end{aligned}$$

Note that  $\tilde{\pi}_{\mathcal{Y}}$  preserves the depth and the weight, but  $\pi_{\mathcal{Y}}$  only preserves the weight.

### 2.1.3 The double shuffle Lie algebra, $\mathfrak{ds}$

**Definition 2.13.** The vector subspace,  $\mathfrak{ds} \subset \mathbb{L}[x, y]$ , is generated by polynomials,  $f$ , that satisfy the following sets of relations,

1. The weight of any term in  $f$  is greater than or equal to 3,
2.  $f$  is primitive for  $\Delta_{\text{III}}$ :  $\Delta_{\text{III}}(f) = f \otimes 1 + 1 \otimes f$ ,
3.  $\pi_{\mathcal{Y}}(f)$  is primitive for  $\Delta_*$ :  $\Delta_*(\pi_{\mathcal{Y}}(f)) = \pi_{\mathcal{Y}}(f) \otimes 1 + 1 \otimes \pi_{\mathcal{Y}}(f)$ .

**Definition 2.14.** The Poisson bracket on elements of  $\mathbb{L}[x, y]$  is the Lie bracket given by

$$\{f, g\} = [f, g] + D_f(g) - D_g(f)$$

where  $[f, g] = fg - gf$  and the  $D_f$  are derivations defined recursively by  $D_f(x) = 0$ ,  $D_f(y) = [y, f]$ .

**Theorem 2.15.** [Ra] The double shuffle elements,  $\mathfrak{ds}$ , form a Lie algebra for the Poisson bracket.

The double shuffle Lie algebra is graded by weight because the double shuffle relations preserve the weight, and we denote each weight  $n$  graded part by  $\mathfrak{ds}_n$ . However,  $\mathfrak{ds}$  is not graded by depth because the stuffle forces relations between words of different depth, such as the classical relation,  $\zeta(2) * \zeta(2) = 2\zeta(2, 2) + \zeta(4)$ . In the proof of the main theorem 2.30, the relations between depth one and depth two elements given by stuffle are fully explained.

A useful way to calculate the action of the derivation,  $D_f(g)$ , is given in [Sc].

**Proposition 2.16.** [Sc] Let  $f, g \in \mathbb{L}[x, y]$ , such that the depth of  $g$  is  $d$ . Then  $D_f(g)$  is given by the sum over the Lie elements,  $\sum_{i=1}^d g_i(x, y, [y, f])$  where each  $g_i$  is gotten by substituting one  $y$  in  $g$  by  $[y, f]$ .

**Example 2.17.** Let  $g = [[[x, y], [x, [x, y]]], [y, x]]$ , so we have

$$\begin{aligned} D_f(g) &= [[[x, [y, f]], [x, [x, y]]], [y, x]] + [[[x, y], [x, [x, [y, f]]]], [y, x]] + \\ &\quad [[[x, y], [x, [x, y]]], [[y, f], x]]. \end{aligned}$$

## 2.2 Lyndon-Lie words

**Definition 2.18.** A Lyndon word is a monomial,  $\omega \in \mathbb{Q}\langle\langle x, y \rangle\rangle$ , such that all of the right factors of  $\omega$  are greater than  $\omega$  for the lexicographic ordering. In other words, if  $\omega = a_1 \cdots a_n$ ,  $a_i \in \{x, y\}$ , then  $a_1 \cdots a_n < a_i \cdots a_n \forall i > 1$ .

The simplest example of a Lyndon word is given in depth 1, where the only Lyndon word is  $x^n y$ .

Given a Lyndon word,  $\omega$ , we can construct an element of  $\mathbb{L}[x, y]$ , denoted  $[\omega]$ , by recursively bracketing in the following manner. Let  $\omega$  be written as  $\omega = u \cdot v$  such that  $v$  is the smallest, non-trivial right factor. Then we bracket  $[u, v]$ . We can repeat this procedure recursively on  $u$  and  $v$ , since  $u$  and  $v$  are Lyndon words. If  $v$  is the smallest right factor, it is smaller than all of its right factors. Furthermore,  $u$  is smaller than all of its right factors since if  $u = u_1 \cdot u_2$  where  $u_2 < u$ , then  $u_2 \cdot v < u \cdot v = \omega$  which is impossible since we supposed that  $\omega < u_2 \cdot v$ . So we can recursively bracket in the same way as the base step until we obtain an element of  $\mathbb{L}[x, y]$ .

**Definition 2.19.** A *Lyndon-Lie word* (or *Lyndon-Lie monomial*) is an element of  $\mathbb{L}[x, y]$  obtained by a bracketing a Lyndon word in the above recursive procedure.

**Theorem 2.20.** [Re] Lyndon-Lie words form a basis for the  $\mathbb{Q}$  vector space  $\mathbb{L}[x, y]$ . We call this basis the *Lyndon-Lie basis*.

**Theorem 2.21** (Witt dimension formula). [Se] Let  $\mathbb{L}[x_1, \dots, x_r]$  be the free Lie algebra on  $r$  generators. The dimension of the  $n$ th graded piece is given by

$$\dim(\mathbb{L}_n[x_1, \dots, x_r]) = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d},$$

where the Möbius function,  $\mu$ , is defined by

$$\mu(d) \begin{cases} 1 & d = 1 \\ (-1)^k & d = p_1 \dots p_k \text{ (} p_i \text{ distinct primes)} \\ 0 & d \text{ has a square factor.} \end{cases}$$

In particular,

$$\dim(\mathbb{L}_n[x, y]) = \frac{1}{n} \sum_{d|n} \mu(d) 2^{n/d}.$$

**Lemma 2.22.** [Re] Let  $f \in \mathbb{L}[x, y]$  and let  $\overleftarrow{\omega}$  be the word  $\omega$  written backwards. Then,  $(f|\omega) = (-1)^{n-1} (f|\overleftarrow{\omega})$ .

From theorem 2.20, we obtain the following corollary.

**Corollary 2.23.** For any  $n$ ,  $\mathbb{L}_n^1[x, y]$  has dimension 1 and its Lyndon-Lie basis is  $\{[x^{n-1}y]\}$ .

This corollary is immediate, since  $x^{n-1}y$  is the only Lyndon word in depth 1.

In this thesis, we use the notation  $C_a^b$  to denote the binomial coefficient,  $\binom{a}{b}$ .

**Lemma 2.24.** We have the following expression for the depth one basis element as a polynomial in  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ ,

$$[x^{n-1}y] = \sum_{i=0}^{n-1} (-1)^i C_{n-1}^i x^{n-1-i} y x^i.$$

*Proof.* We reason by induction. The smallest right factor of  $x^{n-1}y$  is  $x^{n-2}y$ . By applying the recursive procedure,  $[x^{n-1}y] = [x, [x, \dots, [x, y]]] = ad(x)^{n-1}(y)$ . The lemma is true for  $n = 1$  and we suppose it's true for  $n$ . Then,

$$\begin{aligned}
ad(x)^n(y) &= ad(x)\left(\sum_{i=0}^{n-1}(-1)^i C_{n-1}^i x^{n-1-i} y x^i\right) \\
&= \sum_{i=0}^{n-1}(-1)^i C_{n-1}^i x^{n-i} y x^i - \sum_{i=0}^{n-1}(-1)^i C_{n-1}^i x^{n-1-i} y x^{i+1} \\
&= \sum_{i=0}^{n-1}(-1)^i C_{n-1}^i x^{n-i} y x^i + \sum_{i=1}^n(-1)^i C_{n-1}^{i-1} x^{n-i} y x^i \\
&= C_{n-1}^0 x^n y + \sum_{i=1}^{n-1}(-1)^i (C_{n-1}^{i-1} + C_{n-1}^i) x^{n-i} y x^i + (-1)^n y x^n \\
&= \sum_{i=0}^n(-1)^i C_n^i x^{n-i} y x^i.
\end{aligned}$$

□

We obtain a similar corollary for the weight  $n$ , depth 2 graded parts.

**Corollary 2.25.** *The Lyndon-Lie basis for  $\mathbb{L}_n^2[x, y]$  is given by  $\{[x^r y x^s y] \mid r > s, r + s = n - 2\}$  and its dimension is  $\lfloor \frac{n-1}{2} \rfloor$ .*

*Proof.* A Lyndon word must end in a  $y$ , otherwise the right factor  $x$  would be smaller than the word. Also if  $r \leq s$  then the right factor  $x^s y$  would be smaller than the word. Then the dimension is just the number of ways to distribute  $y$  into the sequence  $x^{n-2}$  such that the number of  $x$  on the left is greater than the number of  $x$  on the right. □

## 2.3 Coefficients on monomials in $\mathfrak{d}\mathfrak{s}$

Let  $f \in \mathfrak{d}\mathfrak{s}$ . By the shuffle relation, we know that  $f \in \mathbb{L}[x, y]$ . To study the behavior in depths 1 and 2, we write  $f$  in two ways, one in terms of the Lyndon-Lie basis and one in terms of the basis of monomials in  $\mathbb{Q}\langle\langle x, y \rangle\rangle$ . Since being in  $\mathbb{L}[x, y]$  is equivalent to satisfying the shuffle relation, we only need to study how the stuffle relation behaves in  $\mathbb{L}[x, y]$ . The stuffle relation is seen by projecting onto terms ending in  $y$ . For this reason, we only label coefficients on monomials that end in  $y$  and to study depths 1 and 2, we only label coefficients in those depths.

$$\begin{aligned}
f &= A[x^{n-1}y] + \sum_{s=0}^{\lfloor \frac{n-3}{2} \rfloor} a_s [x^r y x^s y] + \dots \\
&= Ax^{n-1}y + b_0 x^{n-2}y^2 + b_1 x^{n-3}yxy + \dots + b_{n-2}yx^{n-2}y + \dots,
\end{aligned} \tag{2.3.1}$$

where the subscript on the  $b_i$  coefficients equals the number of  $x$  between the two  $y$ ,  $x^{n-2-i}yx^i y$ .

**Lemma 2.26.** *The coefficients of  $f$  satisfy the following relation:*

$$b_i = \sum_{j=0}^i (-1)^{i-j} a_j (C_{j+1}^{i-j} + C_j^{i-j-1}),$$

following the convention that  $C_c^d = 0$  whenever  $c < d$  or  $d < 0$  and taking  $a_j = 0$  whenever  $j > \frac{n-3}{2}$ .

*Proof.* For each  $j < \lfloor \frac{n-1}{2} \rfloor$  let  $L_j^n = [x^{n-2-j} y x^j y]$  be the basis element from corollary 2.25. So we have by the recursive procedure for bracketing Lyndon words,

$$L_j^n = [x, \dots [[x^{j+1} y], [x^j y]] \dots] \quad (2.3.2)$$

$$= [x, \dots [\sum_{k=0}^{j+1} (-1)^k C_{j+1}^k x^{j+1-k} y x^k, \sum_{l=0}^j (-1)^l C_j^l x^{j-l} y x^l] \dots]. \quad (2.3.3)$$

To isolate the coefficients on  $x^{n-2-i} y x^i y$ , thus determining their contribution to  $b_i$ , we only need to consider those terms coming from the inner most bracket product. (This follows from the fact that in the Lie word  $[x^{n-1} y]$ , there is only one term not ending in an  $x$  and its coefficient is 1.) There are two terms,  $x^{2j+1-i} y x^i y$  in this product, one coming from  $l = 0$ ,  $k = i - j$ , the other one coming from  $k = 0$ ,  $l = i - j - 1$ . These two contributions give a coefficient of

$$(-1)^{i-j} C_{j+1}^{i-j} - (-1)^{i-j-1} C_j^{i-j-1} = (-1)^{i-j} (C_{j+1}^{i-j} + C_j^{i-j-1}).$$

**Claim 2.27.** *These are the only terms in  $L_j^n$  that contribute to  $b_i$ .*

Since the terms we are looking for must end in a  $y$ , either  $l = 0$  or  $k = 0$ . If  $l = 0$ , then  $k = i - j$  because  $x^k x^{j-l} = x^k x^j = x^i$  and if  $k = 0$ , then  $l = i - j - 1$  because  $x^l x^{j+1-k} = x^l x^{j+1} = x^i$ .

To find the complete contribution to  $b_i$  from all of the  $L_j^n$ , we only need to sum from  $j = 0$  to  $i$ . This is seen by reapplying the argument from the above claim. Namely, if  $l = 0$  then there are at least  $j$   $x$ 's in front of the final  $y$ , so we won't find any terms,  $x^{2j+1-i} y x^i y$  when  $j > i$ . Similarly, if  $k = 0$ , then we have at least  $j + 1$   $x$ 's in front of the final  $y$ , so the coefficients on these terms do not contribute to  $b_i$  when  $j > i - 1$ .

In light of the above analysis, the complete expression for  $b_i$  in terms of the coefficients on the Lyndon Lie basis polynomials is

$$b_i = \sum_{j=0}^i (-1)^{i-j} a_j (C_{j+1}^{i-j} + C_j^{i-j-1}).$$

□

**Corollary 2.28.** *The coefficient  $b_0$  is given by*

$$b_0 = \frac{n-1}{2} (f|[x^{n-1} y]) = \frac{n-1}{2} A.$$

*Proof.* We use the convention that  $C_a^b = 0$  whenever  $a < b$  so that by lemma 2.26, we may write

$$\begin{aligned} \sum_{i=0}^{n-2} b_i &= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} a_j (-1)^{i-j} (C_{j+1}^{i-j} + C_j^{i-j-1}) \\ &= \sum_{j=0}^{n-2} a_j \left( \sum_{i=0}^{n-2} (-1)^{i-j} C_{j+1}^{i-j} + \sum_{i=0}^{n-2} (-1)^{i-j} C_j^{i-j-1} \right) \\ &= -a_0, \end{aligned}$$

where the last equality is gotten from the identity,  $\sum_{k=0}^j (-1)^k C_j^k = 0$ , which leaves a coefficient 0 on all terms except  $a_0$ .

For  $n$  odd, we have the  $\frac{n-1}{2}$  stuffle relations,  $b_{i-1} + b_{n-i-1} = -A$ . By summing over all such relations, we have  $\sum_{i=0}^{n-2} b_i = -\frac{n-1}{2}A$ . To finish, note that  $b_0 = a_0$  by lemma 2.26. So we have,

$$b_0 = a_0 = \frac{n-1}{2}A.$$

For  $n$  even, we have the  $\frac{n}{2}$  stuffle relations,  $b_{i-1} + b_{n-i-1} = -A$ . The last of the stuffle relations gives  $2b_{n/2-1} = -A$ ,  $b_{n/2-1} = -\frac{1}{2}A$ . By summing over all of the stuffle relations we obtain,  $\sum_{i=0}^{n-2} b_i = -\frac{n}{2}A - b_{n/2-1} = -\frac{n-1}{2}A$ . So we also have that

$$b_0 = a_0 = \frac{n-1}{2}A.$$

□

## 2.4 Statement of main theorem and generating polynomials

The purpose of this section is to shed some light, and give some evidence toward the conjecture that  $\text{grt} \simeq \mathfrak{ds}$ . In subsequent sections, we use combinatorial methods based on Lyndon-Lie theory from [Re] to prove a result parallel to work by Ihara on  $\text{grt}$  [Ih2]:

**Theorem 2.29.** *The dimensions of the  $i$ th depth-graded parts of  $\text{grt}$  for  $i = 0, 1$  are*

$$\begin{aligned} \dim(F_n^1 \text{grt} / F_n^2 \text{grt}) &= \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \\ \dim(F_n^2 \text{grt} / F_n^3 \text{grt}) &= \begin{cases} 0 & n \text{ odd} \\ \lfloor \frac{n-2}{6} \rfloor & n \text{ even.} \end{cases} \end{aligned} \tag{2.4.1}$$

First, we establish some key properties about the structure of the double shuffle Lie algebra.

We have a depth filtration of  $\mathfrak{ds}$ , as a vector subspace of  $\mathbb{L}[x, y]$ ,

$$\mathfrak{ds} = F^1 \mathfrak{ds} \supseteq F^2 \mathfrak{ds} \supseteq F^3 \mathfrak{ds} \supseteq \dots,$$

where each depth-filtered part is defined as

$$\begin{aligned} F^i \mathfrak{ds} &= \{f(x, y) \in \mathfrak{ds} \mid (f|_t) = 0, \forall \text{ terms } t \text{ of depth } < i\} \\ &= \bigoplus_{n=1}^{\infty} F_n^i \mathfrak{ds} \subset \bigoplus_{n=1}^{\infty} F_n^i \mathbb{L}[x, y]. \end{aligned} \tag{2.4.2}$$

In each weight  $n$  piece  $F_n^i \mathbb{L}[x, y]$  is finite dimensional so that  $F_n^i \mathfrak{d}\mathfrak{s}$  is finite dimensional.

We are interested in the depth grading given by  $F_n^i \mathfrak{d}\mathfrak{s} / F_n^{i+1} \mathfrak{d}\mathfrak{s} = gr_n^i(\mathfrak{d}\mathfrak{s})$ , which is of course also finite dimensional.

**Theorem 2.30.** *The dimensions of the depth-graded parts for depths 1 and 2 of  $\mathfrak{d}\mathfrak{s}$  are:*

$$(i) \quad \dim(F_n^1 \mathfrak{d}\mathfrak{s} / F_n^2 \mathfrak{d}\mathfrak{s}) = \begin{cases} 1 & n \geq 3, \text{ odd} \\ 0 & n = 1 \text{ or } n \text{ even,} \end{cases}$$

in other words, there exists at least one element  $f_n \in \mathfrak{d}\mathfrak{s}_n$  for all odd  $n \geq 3$  such that  $(f_n | x^{n-1}y) = 1$ .

Furthermore,

$$(ii) \quad \dim(F_n^2 \mathfrak{d}\mathfrak{s} / F_n^3 \mathfrak{d}\mathfrak{s}) = \begin{cases} 0 & n \text{ odd} \\ \lfloor \frac{n-2}{6} \rfloor & n \text{ even,} \end{cases}$$

and this space is generated by the images of  $\{f_i, f_j\}$  with  $i + j = n$ ,  $i, j \geq 3$  are odd, and where the  $f_i$  are the unique depth 1 elements from case (i).

The proof of theorem 2.30 is done in the sections 2.5 and 2.6. Before proceeding to the proof, we give some preliminary combinatorial lemmas necessary in the proof.

**Lemma 2.31.** *Let  $P$  be the rational function in the ring of power series on three commutative variables,  $\mathbb{Q}[[X, Y, T]]$ , given by*

$$P = \frac{1 + YT^2}{(1 - (XT + YT^2))(1 - (X + Y))}. \quad (2.4.3)$$

The coefficients of the monomials,  $X^{n-i}Y^i T^j$ , for the  $n, i, j$  listed below, in the series expansion of  $P$  are given by

$$(P | X^{n-i}Y^i T^j) = \begin{cases} C_n^i & i + 1 \leq j \leq n \text{ or } j = 0 \\ C_n^i - 1 & j = n + 1, i \text{ even} \\ C_n^i + 1 & j = n + 1, i \text{ odd} \\ C_n^i + 1 & j = i, i > 0 \text{ even} \\ C_n^i - 1 & j = i, i \text{ odd} \\ 0 & i = n, j \text{ odd and } j \leq n \\ 2 & i = n, j \text{ even and } 0 < j \leq n. \end{cases} \quad (2.4.4)$$

*Proof.* Let  $R$  denote the ring  $\mathbb{Q}\langle\langle x, y \rangle\rangle[[T]]$  of power series in which  $x$  and  $y$  do not commute with each other, but  $T$  commutes with both  $x$  and  $y$ . We define the power series in  $R$ :

$$F(x, y, T) := \frac{1}{1 - (xT + yT^2)} = \sum_{r=0}^{\infty} (xT + yT^2)^r,$$

$$G(x, y) := \frac{1}{1 - (x + y)} = \sum_{s=0}^{\infty} (x + y)^s.$$

Set  $H = FG$ . For any  $J \in R$ , let  $J^{ab}$  be its image in the commutative power series ring,  $\mathbb{Q}[[X, Y, T]]$  by the map which sends  $x$  to  $X$  and  $y$  to  $Y$ . In particular, the power series expansion of  $P$  is equal to  $(H + yT^2H)^{ab}$ .

*Case 1,  $i=0$ :* For  $0 \leq j \leq n$ , we have  $(P|X^n T^j) = C_n^i = 1$  because  $(F^{ab}|X^j T^j) = 1$  and  $(G^{ab}|X^{n-j}) = 1$ . For  $j = n + 1$ ,  $X^n T^{n+1}$  has coefficient  $0 = C_n^i - 1$  in  $P$  because the factor of  $T$  must come from  $F^{ab}$ , and in this expansion, since the number of  $T$  in a term in  $F^{ab}$  is equal to the sum of the powers of  $X$  and twice the powers of  $Y$ .

*Case 2,  $j=0$ :* The coefficients,  $X^{n-i} Y^i$ , come uniquely from  $G^{ab}$ , and by the binomial expansion, this coefficient is  $C_n^i$ .

*Case 3,  $i < n, i \leq j \leq n + 1$ :*

We say that a monomial in  $R$  is of "type  $(n - i, i)$ " if it is of degree  $n - i$  in  $x$  and degree  $i$  in  $y$ . Let

$$\epsilon = \begin{cases} -1 & j = i, i \text{ odd or } j = n + 1, i \text{ even} \\ +1 & j = i, i \text{ even or } j = n + 1, i \text{ odd} \\ 0 & i + 1 \leq j \leq n. \end{cases} \quad (2.4.5)$$

We say that a monomial in  $R$  is "divisible at  $j$ " if it can be written  $V \cdot WT^j$  where if  $\alpha$  is the degree in  $y$  in  $V$  and  $\beta$  the degree in  $x$  in  $V$ , then  $j = 2\alpha + \beta$ . We say in this case that  $V$  is of weight  $j$  in  $T$ . A monomial,  $V \cdot WT^j$ , is divisible at  $j$  if and only if  $VT^j$  is a monomial appearing in  $F$ .

The coefficient of every monomial in  $H$  is 1, hence the coefficient  $(P|X^{n-i} Y^i T^j)$  is equal to the number of monomials of type  $(n - i, i)$  in  $H$  which are divisible at  $j$  plus the number of monomials of type  $(n - i, i - 1)$  in  $H$  which are divisible at  $j - 2$ . We will partition the entire contribution to  $(P|X^{n-i} Y^i T^j)$  of monomials from  $H + yT^2 H$  into the three following sets:

$$\begin{aligned} & \#\{\text{type } (n - i, i), \text{ beginning with } x \text{ and divisible at } j\} + \\ & \#\{\text{type } (n - i, i), \text{ beginning with } y \text{ and divisible at } j\} + \\ & \#\{\text{type } (n - i, i - 1) \text{ and divisible at } j - 2\}. \end{aligned}$$

But multiplying the elements of the third set on the left by  $yT^2$  yields a bijection between the second and third sets so that

$$(P|X^{n-i} Y^i T^j) = \#\{xV \text{ divisible at } j\} + 2\#\{yV \text{ divisible at } j\}.$$

Note that the total number of monomials of type  $(n - i, i)$ , with  $0 < i < n$  and  $i \leq j \leq n + 1$  is just  $C_n^i$ , so we have,

$$C_n^i = \#\{xV \text{ divisible at } j\} + \#\{yV \text{ divisible at } j\} + \#\{V \text{ not divisible at } j\}.$$

Thus to show that  $(P|X^{n-i} Y^i T^j) = C_n^i + \epsilon$ , we only need to show that the cardinalities of the two following sets of words of type  $(n - i, i)$  with  $0 < i < n$  and  $i \leq j \leq n + 1$  satisfy:

$$\#\{yV \text{ divisible at } j\} = \#\{V \text{ not divisible at } j\} + \epsilon. \quad (2.4.6)$$

*Base case,  $n = 2, (i, j) = (1, 1), (1, 2), (1, 3)$ :* The words can be counted by hand. In the first case, there are no words starting with  $y$ , divisible at 1, and one word,  $yx$ , not divisible at 1. In the second case,  $\#\{y|x\} = 1 = \#\{xy\}$ . And in the third case, there is one word,  $yx$ , starting with  $y$  and divisible at 3 and no words which are not divisible at 3.

*Induction Case:* Now we assume the induction hypothesis that  $\#\{yV|W, \text{ type}(n - i, i - 1), \text{ divisible at } j - 1\} + \epsilon = \#\{V|yW, \text{ type}(n - i, i - 1), \text{ divisible at } j - 2\}$  (where

the  $\epsilon$  is on left hand side since the parity of the number  $y$  is different in this induction hypothesis).

We now partition the left hand set of (2.4.6) into the two subsets of monomials of the form:

$$\{yV \text{ divisible at } j\} = \{yV|xW\} \sqcup \{yV|yW\}, \quad (2.4.7)$$

Furthermore, the set of words that is not divisible at  $j$  is equal to the set of words divisible at  $j - 1$  of type  $(n - i, i)$  of the form:

$$\begin{aligned} \{V \text{ not divisible at } j\} &= \{V|yW, \text{ divisible at } j - 1\} \\ &= \{Vx|yW\} \sqcup \{Vy|yW\}. \end{aligned} \quad (2.4.8)$$

The first set of (2.4.7) is in bijection with the first set of (2.4.8) by permutation of the terms:

$$\{yV|xW, \text{ divisible at } j\} \leftrightarrow \{Vx|yW, \text{ divisible at } j - 1\}.$$

Furthermore, we have the following equalities relating the second sets of (2.4.7) and (2.4.8):

$$\begin{aligned} \#\{yV|yW\} &= \#\{V|yW, \text{ type } (n - i, i - 1), \text{ divisible at } j - 2\} \text{ (removal of leading } y) \\ &= \#\{yV|W, \text{ type } (n - i, i - 1), \text{ divisible at } j - 1\} + \epsilon \quad \text{(induction)} \\ &= \#\{Vy|yW\} + \epsilon \quad \text{(by permutation of the terms and removal of } y). \end{aligned}$$

This proves the lemma for  $i < n, i \leq j \leq n + 1$ .

*Case 4,  $i = n, 0 < j \leq n$ :* If  $j$  odd, there is no way to cut a word,  $y^\alpha|y^\beta T^j$  in such a way that  $2\alpha = j$ . So,  $(P|Y^i T^j) = 0$ . Likewise, if  $j$  is even, all words are divisible, therefore  $P$  has a coefficient of 1 coming from  $H^{ab}$  and a coefficient of 1 coming from  $(yT^2 H)^{ab}$ .  $\square$

**Remark.** *I would like to thank the reporter D. Zagier, for suggesting an alternative proof of this lemma, which is shorter and provides all of the coefficients of  $P$ .*

**Corollary 2.32.** *The coefficient of  $X^{n-i} Y^i T^j$  in the rational function,*

$$Q(X, Y, T) = \frac{(1 + YT^2)(1 + X)}{(1 - XT - YT^2)(1 - X - Y)} \quad (2.4.9)$$

is equal to

$$\begin{cases} C_n^i + C_{n-1}^i & i + 1 \leq j \leq n - 1 \\ C_n^i + C_{n-1}^i - 1 & j = n, i \text{ even}, i < n \\ C_n^i + C_{n-1}^i + 1 & j = n, i \text{ odd}, i < n \\ 2 & i = n, j \text{ even and } \leq n \\ 0 & i = n, j \text{ odd and } \leq n \end{cases} \quad (2.4.10)$$

*Proof.* The rational function,  $Q(X, Y, T) = P(X, Y, T) + XP(X, Y, T)$  where  $P$  is the same as in lemma 2.31. The coefficient,  $(Q|X^{n-i} Y^i T^j)$  is equal to  $(P|X^{n-i} Y^i T^{i+1}) + (P|X^{n-i-1} Y^i T^{i+1})$ .  $\square$

**Definition 2.33.** *Let  $\Lambda$  be the Pascal triangle obtained by the recurrence relation,  $\Lambda_0^0 = 1, \Lambda_1^0 = 2, \Lambda_1^1 = 1, \Lambda_n^k = 0$  for  $i < 0$  or  $k < 0$ , and  $\Lambda_n^k = \Lambda_{n-1}^{k-1} + \Lambda_{n-1}^k$ .*



We note here that  $\Lambda$  is the sum of two Pascal triangles with binomial coefficients interposed on top of one another, one with its tip on the first column of the second row of the other. We have then that  $\Lambda_n^k = C_n^k + C_{n-1}^k$ ,  $0 \leq k \leq n$ . In this expression,  $\Lambda_n^k$  is  $k$ th column of the  $n$ th row of  $\Lambda$ :

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & & 2 & 1 \\ & & & & 2 & 3 & 1 \\ & & 2 & 5 & 4 & 1 \\ 2 & 7 & 9 & 5 & 1 \end{array}$$

Now we associate commutative monomials to terms of  $\Lambda$  in two different ways. Let  $(\Lambda_D)_n^k = \Lambda_n^k X^k Y^{n-k} T^{k+2(n-k)}$  and  $(\Lambda_A)_n^k = T^k X^{n-k} Y^k$ :

$$\Lambda_D = \begin{array}{cccc} & & & 1 \\ & & & 2YT^2 & 1XT \\ & & & 2Y^2T^4 & 3XYT^3 & 1X^2T^2 \\ & & 2Y^3T^6 & 5XY^2T^5 & 4X^2YT^4 & 1X^3T^3 & \nearrow D_3 \\ 2Y^4T^8 & 7XY^3T^7 & 9X^2Y^2T^6 & 5X^3YT^5 & 1X^4T^4 & \nearrow D_4 \end{array}$$

The descending arrows,  $D_i$  represent the vectors of monomials with  $T^i$  formed from the corresponding descending diagonal of  $\Lambda_A$ .

$$\Lambda_D = \begin{array}{cccccc} & & & & & 1 \\ & & & & & 2X & 1Y \\ & & & & & 2X^2 & 3XY & 1Y^2 & \nearrow A_3 \\ & & & 2X^3 & 5X^2Y & 4XY^2 & 1Y^3 & \nearrow A_4 \\ & 2X^4 & 7X^3Y & 9X^2Y^2 & 5XY^3 & 1Y^4 \end{array}$$

The ascending arrows represent vectors,  $A_i$ , such that the power on  $X$  plus twice the power on  $Y$  equals the constant  $i$ .

We define the two rational functions,  $Q_1$  and  $Q_2$  such that  $Q = Q_1 Q_2$ :

$$Q_1 = \frac{1 + YT^2}{1 - (XT + YT^2)} = \sum_{i=0}^{\infty} (XT + YT^2)^i + YT^2 \sum_{i=0}^{\infty} (XT + YT^2)^i \quad (2.4.11)$$

$$Q_2 = \frac{1 + X}{1 - (X + Y)} = \sum_{i=0}^{\infty} (X + Y)^i + X \sum_{i=0}^{\infty} (X + Y)^i. \quad (2.4.12)$$

The expansion of  $Q_1$  gives us  $(Q_1|X^{n-i}Y^i T^{n+i}) = C_n^{n-i} + C_{n-1}^{n-i} = \Lambda_n^{n-i}$ , and so  $Q_1$  is the infinite sum of all of the terms of  $\Lambda_D$ . Likewise,  $Q_2$  is the infinite sum of all terms in  $\Lambda_A$ .

The arrows in the triangles  $\Lambda_D$  and  $\Lambda_A$  represent descending and ascending vectors,  $D_j$  and  $A_k$ , such that their scalar product is a monomial in  $Q_1Q_2$ , by adding the extra condition that we must add some zeroes at the beginning of the vectors,  $D_j$ , according to the following rule.

**Definition 2.34.** Let  $D_{j,z}$  be the vector of  $z$  zeroes concatenated by the descending diagonal vector  $D_j$  in  $\Lambda_D$  as in the above diagram. A closed formula for  $D_{j,z}$  is given by the formula:

$$\begin{aligned} D_{j,z} &= T^j \left( \overbrace{(0, 0, \dots, 0)}^z, \Lambda_{\lfloor \frac{j+1}{2} \rfloor}^{j \bmod 2} Y^{\lfloor \frac{j}{2} \rfloor} X^{j \bmod 2}, \dots, \Lambda_{j-2}^{j-4} Y^2 X^{j-4}, \Lambda_{j-1}^{j-2} Y X^{j-2}, \Lambda_j^j X^j \right) \\ &= T^j \left( 0^z \cdot \left( \Lambda_{\lfloor \frac{j+1}{2} \rfloor + l}^{j-2\lfloor \frac{j}{2} \rfloor + 2l} Y^{\lfloor \frac{j}{2} \rfloor - l} X^{j-2\lfloor \frac{j}{2} \rfloor + 2l} \right)_{l=0}^{\lfloor \frac{j}{2} \rfloor} \right), \end{aligned} \quad (2.4.13)$$

where  $T^j$  is distributed to all of the terms in the vector.

**Definition 2.35.** We define the vectors,  $A_k$  associated to  $\Lambda_A$  similarly:

$$\begin{aligned} A_k &= (\Lambda_k^0 X^k, \Lambda_{k-1}^1 X^{k-2} Y, \dots, \Lambda_{k-\lfloor \frac{k}{2} \rfloor}^{\lfloor \frac{k}{2} \rfloor} X^{k-2\lfloor \frac{k}{2} \rfloor} Y^{\lfloor \frac{k}{2} \rfloor}) \\ &= \left( \Lambda_{k-m}^m Y^m X^{k-2m} \right)_{m=0}^{\lfloor \frac{k}{2} \rfloor} \end{aligned} \quad (2.4.14)$$

**Definition 2.36.** The scalar product,  $A_k \cdot D_{j,z}$ , is defined as

$$\begin{aligned} \sum_{m=z}^{\lfloor \frac{k}{2} \rfloor} \Lambda_{k-m}^m Y^m X^{k-2m} \cdot \Lambda_{\lfloor \frac{j+1}{2} \rfloor + (m-z)}^{j-2\lfloor \frac{j}{2} \rfloor + 2(m-z)} Y^{\lfloor \frac{j}{2} \rfloor - (m-z)} X^{j-2\lfloor \frac{j}{2} \rfloor + 2(m-z)} T^j = \\ \left( \sum_{m=z}^{\lfloor \frac{k}{2} \rfloor} \Lambda_{k-m}^m \cdot \Lambda_{\lfloor \frac{j+1}{2} \rfloor + (m-z)}^{j-2\lfloor \frac{j}{2} \rfloor + 2(m-z)} \right) X^{k+j-2\lfloor \frac{j}{2} \rfloor - 2z} Y^{\lfloor \frac{j}{2} \rfloor + z} T^j. \end{aligned} \quad (2.4.15)$$

**Lemma 2.37.** The term  $a_{n-i,i,j} X^{n-i} Y^i T^j$  ( $i+1 \leq j \leq n$ ) in  $Q(X, Y, T)$  is equal to the scalar product  $A_{n+i-j} \cdot D_{j,i-\lfloor \frac{j}{2} \rfloor}$ , where  $a_{n-i,i,j}$  is given by corollary 2.32.

*Proof.* By the expression (2.4.15), we know that  $A_{n+i-j} \cdot D_{j,i-\lfloor \frac{j}{2} \rfloor}$  is a monomial of the right form. Now,  $A_{n+i-j} \cdot D_{j,i-\lfloor \frac{j}{2} \rfloor}$  is exactly the term,  $a_{n-i,i} X^{n-i} Y^i T^j$  in the product  $Q_1Q_2$ . But  $Q = Q_1Q_2$ , by (2.4.9) and this proves the lemma.  $\square$

## 2.5 grt $\simeq$ $\mathfrak{d}\mathfrak{s}$ in depth 1

The goal of this section is to prove that  $F_n^1 \mathfrak{d}\mathfrak{s}$  is non-empty for odd  $n$  and that  $F_n^1 \mathfrak{d}\mathfrak{s} = 0$  for even  $n$ . The proof of this part, for odd  $n$ , relies on sophisticated machinery due to Racinet, Furusho and their inspirations which include, among others, Drinfel'd, Ihara, Le and Murakami. The style of this proof is very different from the combinatorial nature of the rest of this chapter. One reason for this is that this result follows almost immediately from theirs, therefore a long combinatorial construction was unnecessary. We prove the existence, without explicit construction, of depth 1 elements of  $\mathfrak{d}\mathfrak{s}$  which are conjectured to be the generators of  $\mathfrak{d}\mathfrak{s}$  as a Lie algebra, and are therefore important objects in furthering the study of multizeta values.

*Proof of theorem 2.30 (i).* Let  $f \in \mathfrak{d}\mathfrak{s} \subset \mathbb{L}[x, y]$  and we write  $f$  in a Lyndon-Lie basis as in expression (2.3.1). Since  $F_n^1 \mathfrak{d}\mathfrak{s} \subset \mathbb{L}[x, y]$  and since  $\mathbb{L}[x, y]$  has only one depth 1 element in each weight  $n$ , namely  $[x^{n-1}y]$ , any two elements in  $F_n^1 \mathfrak{d}\mathfrak{s}$  are equivalent modulo  $F_n^2 \mathfrak{d}\mathfrak{s}$ . Therefore, we may assume that  $\dim F_n^1 \mathfrak{d}\mathfrak{s} / F_n^2 \mathfrak{d}\mathfrak{s} = 0$  or 1. Without loss of generality, we may assume that  $(f|[x^{n-1}y]) = 0$  or 1.

*Case 1,  $n$  is even:* The stuffle relation,  $(n-1) * (1) = (n-1, 1) + (1, n-1) + (n)$ , gives

$$0 = (f|x^{n-2}y^2) + (f|yx^{n-2}y) + (f|x^{n-1}y).$$

From lemma 2.22,  $(f|yx^{n-2}y) = 0$ . We have then that

$$(f|x^{n-2}y^2) = -(f|x^{n-1}y).$$

Let's assume that  $(f|x^{n-1}y) \neq 0$ . But by 2.28,

$$(f|x^{n-2}y^2) = \frac{n-1}{2}(f|x^{n-1}y),$$

which is a contradiction since  $n \neq -1$ . Therefore,  $(f|x^{n-1}y) = 0$ , and so  $F_n^1 \mathfrak{d}\mathfrak{s} / F_n^2 \mathfrak{d}\mathfrak{s} = 0$ .

*Case 2,  $n$  is odd:*

In order to treat this case, we introduce Drinfel'd's associator.

**Definition 2.38.** *The Drinfel'd associator,  $\Phi_{KZ}$ , is defined as*

$$\Phi_{KZ}(x, y) = \sum (-1)^{d(w)} \zeta^{sh}(w) w \in \mathbb{C}\langle\langle x, y \rangle\rangle, \quad (2.5.1)$$

where the sum is a power series over all of the monomials,  $w$ , in  $x$  and  $y$ . The coefficients,  $\zeta^{sh}(w)$ , are real numbers, called regularized zeta values, which have the following properties:

1. If  $w$  is a convergent word, then  $\zeta^{sh}(w) = \zeta(w)$ ,
2. For all non-convergent words,  $w$ , the  $\zeta^{sh}(w)$  are linear combinations of convergent multizeta values that satisfy the property that  $\zeta^{sh}(w_1)\zeta^{sh}(w_2) = \zeta^{sh}(w_1 \amalg w_2)$ .

An explicit expression for  $\zeta^{sh}(w)$  was calculated by Furusho [Fu] and is based on work of Le and Murakami [LM]. We set  $\Phi_{\text{III}}$  to be  $\Phi_{KZ}(x, -y)$ .

The group,  $DM$ , or "double mélange", which was defined by Racinet and is related to  $\mathfrak{d}\mathfrak{s}$ , plays a key role in this proof. The group,  $DM \subset \mathbb{C}\langle\langle x, y \rangle\rangle$ , is graded by weight and its weight  $n$  graded piece is denoted  $DM^n$ . The group law on  $DM$  is denoted by  $\otimes$ . For an explicit description of generators and relations on  $DM$ , see [Ra].

**Definition 2.39.** *Let  $DM_\lambda$  be set of power series,  $f \in DM$ , such that  $(f|xy) = \lambda$ .*

The series,  $\Phi_{\text{III}}(x, y)$  and  $\Phi_{\text{III}}(-x, -y)$ , are both elements of  $DM_{\zeta(2)}$ . Racinet proved that for all  $F \in DM_0$ , the weight  $n$  part of the power series,  $(\ln F)^n$ , satisfies the double shuffle relations given in 2.13.

**Theorem 2.40.** [Ra] *The set,  $DM_0$ , is a group that acts transitively on  $DM_{\zeta(2)}$ .*

As a consequence, there exists an element,  $F \in DM_0$  such that

$$\Phi_{\text{III}}(x, y) \otimes F = \Phi_{\text{III}}(-x, -y). \quad (2.5.2)$$

This  $F$  provides us with a depth 1 double shuffle element according to the following construction.

To see what happens in the Lie algebra,  $(\ln DM_0)$ , we take  $\ln$  of both sides of the equation to obtain,

$$\ln \Phi_{\text{III}}(x, y) \underset{C-H}{\star} \ln F = \ln \Phi_{\text{III}}(-x, -y), \quad (2.5.3)$$

where  $\underset{C-H}{\star}$  is the multiplication law by the Campbell-Baker-Hausdorff formula. The goal is to show that  $(\ln F|x^{n-1}y)$  cannot be 0 for odd  $n$ , in order to give a non-zero element in  $F_n^1 \mathfrak{ds}$  over  $\mathbb{Q}$ . To do this, we write the expansion of  $\ln \Phi_{\text{III}}$  and  $\ln F$  as a power series in a Lyndon-Lie basis of  $\mathbb{L}_{\mathbb{C}}[x, y]$ . To lighten the notation, we will write  $\ln F = f$ .

It is known that for any element in  $F \in DM_0$  and for any  $i$ ,  $(F|x^i) = (F|y^i) = 0$  and that for any  $i$ ,  $(\Phi_{\text{III}}|x^i) = (\Phi_{\text{III}}|y^i) = 0$  [Ra] [Dr]. Let  $\Phi_{\text{III}}(x, y) = 1 + \Phi_0(x, y)$ . By writing

$$\ln \Phi_{\text{III}} = \ln(1 + \Phi_0) = \Phi_0 - \frac{1}{2}\Phi_0^2 + \dots$$

we see that the contribution to the monomials,  $x^{n-1}y$ , in the Lie algebra comes uniquely from the first term,  $\Phi_0$ . And we have by a formula of Furusho [Fu],

$$\begin{aligned} \ln \Phi_{\text{III}}(x, y) &= \zeta(2)[x, y] + \zeta(3)[x, [x, y]] + \zeta(4)[x, [x, [x, y]]] + \dots \\ \ln \Phi_{\text{III}}(-x, -y) &= \zeta(2)[x, y] - \zeta(3)[x, [x, y]] + \zeta(4)[x, [x, [x, y]]] + \dots \end{aligned}$$

By the formula of Campbell-Baker-Hausdorff, (2.5.3) becomes

$$\begin{aligned} (\zeta(2)[x, y] + \zeta(3)[x, [x, y]] + \zeta(4)[x, [x, [x, y]]] + \dots) + f + \frac{1}{2}\{\ln \Phi_{\text{III}}, f\} + \dots = \\ \zeta(2)[x, y] - \zeta(3)[x, [x, y]] + \zeta(4)[x, [x, [x, y]]] + \dots \quad (2.5.4) \end{aligned}$$

We have the following important properties:  $(f|x) = (f|y) = 0 = (\ln \Phi_{\text{III}}|x) = (\ln \Phi_{\text{III}}|y)$ . Hence, both  $\ln \Phi_{\text{III}}$  and  $f$  are in  $[\mathbb{L}_{\mathbb{C}}[x, y], \mathbb{L}_{\mathbb{C}}[x, y]]$ . Because of this, the terms of bracketings of  $f$  and  $\ln \Phi_{\text{III}}$  do not contribute to the coefficient of  $[x^{n-1}y]$  since  $[x^{n-1}y] \notin [[\mathbb{L}_{\mathbb{C}}[x, y], \mathbb{L}_{\mathbb{C}}[x, y]], [\mathbb{L}_{\mathbb{C}}[x, y], \mathbb{L}_{\mathbb{C}}[x, y]]]$ .

By studying both sides of equation (2.5.4), we see that for even  $n$ ,  $(f|x^{n-1}y) = 0$  and for odd  $n$ ,  $(f|x^{n-1}y) = -2\zeta(n)$ . Let  $f_n$  be the homogeneous degree  $n$  graded part of  $f$ . In order to find a non-zero element in  $\mathfrak{ds}$ , we write

$$f_n = \sum a_j L_n^j, \quad a_j \in \mathbb{C}$$

where  $\{L_n^j\}$  is a Lyndon basis of  $\mathbb{L}[x, y]$ . Let  $V$  be the  $\mathbb{Q}$ -vector space generated by the complex numbers,  $\langle a_i \rangle$ , so we may choose the basis for  $V$  over  $\mathbb{Q}$ ,  $\{\zeta_1 = \zeta(n), \zeta_2, \dots, \zeta_r\}$ .

We then write,

$$f_n = \sum_{i=1}^r p_i(x, y) \zeta_i$$

where the  $p_i(x, y)$  are homogeneous polynomials of degree  $n$  with coefficients in  $\mathbb{Q}$ . Since  $f_n$  satisfies double shuffle, for any couple of words,  $u, v$ , we have

$$\begin{aligned} 0 &= \sum_{w \in sh/st(u,v)} (f_n|w) \\ &= \sum_{w \in sh/st(u,v)} \left( \sum_{i=1}^r p_i(x, y) \zeta_i | w \right) \\ &= \sum_{i=1}^r \left( \sum_{w \in sh/st(u,v)} (p_i(x, y)|w) \right) \zeta_i. \end{aligned}$$

The  $\zeta_i$  are linearly independent by hypothesis, so we see that each  $p_i(x, y)$  satisfies double shuffle. Therefore we have,  $(p_1|x^{n-1}y) = -2$  and  $p_1$  furnishes us with a nonzero element in  $F_n^1\partial\mathfrak{s}$ .  $\square$

**Remark.** It is tempting to take the log of the element,  $\Phi_{\text{III}}$ , in order to obtain a non-zero element of  $F_n^1\partial\mathfrak{s}$ , because the coefficients of  $\Phi_{\text{III}}$  are multizeta values, and multizeta values satisfy the double shuffle relations. There is still some work to do in this construction, but I will not outline it, since the proof fails at a certain point which I will explain here. In fact,  $\ln \Phi_{\text{III}} \in \widehat{\bigoplus}_{n \geq 2} (\mathfrak{n}_3 \otimes_{\mathbb{Q}} \partial\mathfrak{s}_n)$ , where  $\mathfrak{n}_3$  is the new zeta space from definition 1.14. This proof then fails in the last step of the correct proof above, because  $\mathfrak{n}_3$  is modded out by  $\mathbb{Q}$ , and since we do not know whether  $\zeta(n)$  is irrational, it might be identically 0 in  $\mathfrak{n}_3$ , and therefore we may not take it as a basis element in  $\mathfrak{n}_3$  over  $\mathbb{Q}$ .

## 2.6 $\text{grt} \simeq \partial\mathfrak{s}$ in depth 2

In this section, we study  $F_n^2\partial\mathfrak{s}/F_n^3\partial\mathfrak{s}$  by looking at the coefficients of polynomials in  $\partial\mathfrak{s}$  considered as a subspace of  $\mathbb{L}[x, y]$ . In section 2.4, we showed some combinatorial properties of depth 2 words. Now, we use these properties for, first of all, showing that for  $n$  odd, all of the coefficients on depth 2 monomials only depend on the coefficient of  $[x^{n-1}y]$  and deduce from that that for  $n$  odd,  $F_n^2\partial\mathfrak{s}/F_n^3\partial\mathfrak{s} = 0$ . Analogously, we prove that the dimension of  $F_n^2\partial\mathfrak{s}/F_n^3\partial\mathfrak{s}$  for  $n$  even is  $\lfloor \frac{n-2}{6} \rfloor$ , thereby making an important connection between the Lie algebras,  $\text{grt}$  and  $\partial\mathfrak{s}$ .

*Proof of 2.30 (ii).* Recall that we may write  $f \in \partial\mathfrak{s}$  as in equation (2.3.1),

$$\begin{aligned} f &= A[x^{n-1}y] + \sum_{s=0}^{\lfloor \frac{n-3}{2} \rfloor} a_s[x^r y x^s y] + \dots \\ &= Ax^{n-1}y + b_0x^{n-2}y^2 + b_1x^{n-3}yxy + \dots + b_{n-2}yx^{n-2}y + \dots \end{aligned}$$

*Case 1,  $n$  is odd:* We have the following  $\frac{n-1}{2}$  relations on the coefficients given by stuffle:

$$b_{i-1} + b_{n-1-i} + A = 0, \quad 1 \leq i \leq \frac{n-1}{2}. \quad (2.6.1)$$

By substituting the relation from lemma 2.26 between the  $b_i$  and the  $a_i$  into (2.6.1), we have the following system of relations on the coefficients,  $a_i$ , for each  $i$ ,  $0 \leq i \leq \frac{n-3}{2}$ :

$$\begin{aligned} \sum_{j=0}^i (-1)^{i-j} a_j \left( C_{j+1}^{i-j} + C_j^{i-j-1} \right) + \\ \sum_{k=0}^{n-2-i} (-1)^{n-2-i-k} a_k \left( C_{k+1}^{n-2-i-k} + C_k^{n-3-i-k} \right) = -A. \end{aligned} \quad (2.6.2)$$

The system given in (2.6.2) may be solved by finding a solution to the matrix equation,  $M \cdot (a_0, a_1, \dots, a_{\frac{n-3}{2}}) = -A(1, 1, \dots, 1)$ , where the matrix  $M$  is given by

$$\begin{aligned} M(i, j) &= (-1)^{i-j} \left( C_j^{i-j} + C_{j-1}^{i-j-1} \right) + \\ &\quad (-1)^{n-i-j} \left( C_j^{n-i-j} + C_{j-1}^{n-1-i-j} \right), \quad 1 \leq i, j \leq \frac{n-1}{2}, \end{aligned} \quad (2.6.3)$$

where a row  $i$  of  $M$  corresponds to the equation  $b_{i-1} + b_{n-1-i} = -A$ , so the  $j$ th entry in that row is equal to the coefficient of  $a_{j-1}$  in  $b_{i-1} + b_{n-i-1}$ .

**Remark.** Let  $M'$  be the block matrix in the upper left-hand corner of  $M$  for  $i, j \leq \frac{n-1}{3}$ . We have in this block that  $M'(i, j) = (-1)^{i-j} \Lambda_j^{2j-i}$  and the  $i$ th row of  $M'$  is equal, up to sign, to the vector of coefficients of  $D_i$ , such that the right-most term in the vector is on the diagonal, and where the sign is given by  $(-1)^{i-j}$ . For  $i \leq \frac{n-1}{3}$ ,  $j > \frac{n-1}{3}$  we have  $M(i, j) = (-1)^{n-i-j} \Lambda_j^{2j-n+i}$ . The other terms of  $M$  are the sum of these two types. This structure permits us to associate certain rows of  $M$  to vectors  $D_{i,z}$  and certain columns to rows of  $\Lambda$ .

**Example 2.41.** For  $n = 11$ , the matrix,  $M$ , is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ -2 & 1 & 0 & 0 & 9 \\ 0 & -3 & 1 & 2 & -16 \\ 0 & 2 & -4 & -6 & 14 \\ 0 & 0 & 3 & 4 & -5 \end{pmatrix}.$$

Let  $N$  be the  $\frac{n-1}{2}$  square invertible matrix such that the top of the  $(\frac{n-1}{2} - k)$ th column is equal to  $\overbrace{(0, \dots, 0)}^{2k} \cdot A_{\frac{n-3}{2}-3k}$ ,  $0 \leq k \leq \lfloor \frac{n-5}{6} \rfloor$  up to sign, and let the sign be given by  $\text{sgn}(N(i, j)) = (-1)^{i+j}$ . Let the other coefficients of  $N$  be those of the identity matrix.

Now we show that  $MN$  is a lower triangular matrix. From the corollary 2.37, above the diagonal, in the  $k$ th row and the  $i$ th column,  $i < \frac{n-1}{2} - k$ , we have

$$\begin{aligned} MN(i, \frac{n-1}{2} - k) &= (-1)^{i+\frac{n-1}{2}-k} D_{i, \lfloor \frac{i-1}{2} \rfloor - 2k} \cdot A_{\frac{n-3}{2}-3k} + M(i, \frac{n-1}{2} - k) \\ &= (-1)^{i+\frac{n-1}{2}-k} (Q|X^{\frac{n+1}{2}+k-i} Y^{i-1-2k} T^i) \\ &\quad + (-1)^{\frac{n+1}{2}-i+k} \left( C_{\frac{n-1}{2}-k}^{i-1-2k} + C_{\frac{n-3}{2}-k}^{i-1-2k} \right) \\ &= (-1)^{\frac{n-1}{2}-k+i} \left( C_{\frac{n-1}{2}-k}^{i-1-2k} + C_{\frac{n-3}{2}-k}^{i-1-2k} \right) \\ &\quad + (-1)^{\frac{n+1}{2}-i+k} \left( C_{\frac{n-1}{2}-k}^{i-1-2k} + C_{\frac{n-3}{2}-k}^{i-1-2k} \right) \\ &= 0. \end{aligned}$$

Therefore, we have that  $MN$  is lower triangular. We now need to show that the terms on the diagonal of  $MN$  are non-zero to show that  $M$  is invertible.

If  $k > \lfloor \frac{n-5}{6} \rfloor$ , then the  $(\frac{n-1}{2} - k)$ th diagonal term of  $MN$  is that of  $M$ , since  $N$  is the identity in this upper, right block. These terms are -1 or 1.

If  $0 \leq k \leq \lfloor \frac{n-5}{6} \rfloor$  and  $\frac{n-3}{2} - 3k$  is odd, the term on the diagonal,  $MN(\frac{n-1}{2} - k, \frac{n-1}{2} - k)$ , for  $\frac{n-1}{2} - k \geq \lfloor \frac{n}{3} \rfloor$ , is equal to

$$\begin{aligned} MN(\frac{n-1}{2} - k, \frac{n-1}{2} - k) &= D_{\frac{n-1}{2}-k, \lfloor \frac{n-3-2k}{4} \rfloor - 2k} \cdot A_{\frac{n-3}{2}-2k} + M(\frac{n-1}{2} - k, \frac{n-1}{2} - k) \\ &= (Q|X^{2k+1} Y^{\frac{n-3}{2}-3k} T^{\frac{n-1}{2}-k}) + C_{\frac{n-1}{2}-k}^0 - C_{\frac{n-1}{2}-k}^{2k+1} - C_{\frac{n-3}{2}-k}^{2k} \\ &= C_{\frac{n-1}{2}-k}^{\frac{n-3}{2}-3k} + C_{\frac{n-3}{2}-k}^{\frac{n-3}{2}-3k} + 1 + 1 - C_{\frac{n-1}{2}-k}^{\frac{n-3}{2}-3k} - C_{\frac{n-3}{2}-k}^{\frac{n-3}{2}-3k} \\ &= 2. \end{aligned}$$

When  $\frac{n-3}{2} - 3k$  is even, the length of the  $(\frac{n-1}{2} - k)$ th column of  $N$  ( $0^{2k}A_{\frac{n-3}{2}-3k}$ ) is  $\frac{n+1+2k}{4}$ . The  $(\frac{n-1}{2} - k)$ th row of  $M$  is equal to  $D_{\frac{n-1}{2}-k, \frac{n-3-2k}{4}} - D_{\frac{n-1}{2}+k+1, \frac{n+1+2k}{4}}$ . So the only term of  $D_{\frac{n-1}{2}+k+1, \frac{n+1+2k}{4}}$  that plays a role in the scalar product is the first and it is equal to  $-2$  when  $\frac{n-3}{2} - 3k$  is even. Furthermore, this term contributes to the scalar product by multiplication of the last term of  $A$ . which is 1 in this case.

So we have,

$$\begin{aligned}
MN\left(\frac{n-1}{2} - k, \frac{n-1}{2} - k\right) &= D_{\frac{n-1}{2}-k, \lfloor \frac{n-3-2k}{4} \rfloor - 2k} \cdot A_{\frac{n-3}{2}-2k} + M\left(\frac{n-1}{2} - k, \frac{n-1}{2} - k\right) - 2 \\
&= (Q|X^{2k+1}Y^{\frac{n-3}{2}-3k}T^{\frac{n-1}{2}-k}) + C_{\frac{n-1}{2}-k}^0 - C_{\frac{n-1}{2}-k}^{2k+1} - C_{\frac{n-3}{2}-k}^{2k} - 2 \\
&= C_{\frac{n-1}{2}-k}^{\frac{n-3}{2}-3k} + C_{\frac{n-3}{2}-k}^{\frac{n-3}{2}-3k} - 1 + 1 - C_{\frac{n-1}{2}-k}^{\frac{n-3}{2}-3k} - C_{\frac{n-3}{2}-k}^{\frac{n-3}{2}-3k} - 2. \\
&= -2.
\end{aligned}$$

So for  $n$  odd, we have that the matrix  $M$  is invertible, so that there exists a unique solution to the matrix equation, namely  $M^{-1} \cdot (-A, -A, \dots, -A) = (a_0, a_1, \dots, a_{\frac{n-3}{2}})$ , and the Lyndon-Lie basis elements are  $\mathbb{Q}$  linear combinations of  $A$ , showing that

$$F_n^2 \mathfrak{d}\mathfrak{s} / F_n^3 \mathfrak{d}\mathfrak{s} = 0.$$

*Case 2,  $n$  is even:* We have the following  $\frac{n-2}{2}$  relations on the coefficients given by stuffle:

$$b_{i-1} + b_{n-1-i} + A = 0, \quad 1 \leq i \leq \frac{n-2}{2}, \quad (2.6.4)$$

where we have now from part (i) that  $A = 0$ . By substituting the relation from lemma 2.26 between the  $b_i$  and the  $a_i$  into (2.6.4), we have the following system of relations on the coefficients,  $a_i$ , for each  $i$ ,  $0 \leq i \leq \frac{n-4}{2}$ :

$$\begin{aligned}
\sum_{j=0}^i (-1)^{i-j} a_j (C_{j+1}^{i-j} + C_j^{i-j-1}) + \\
\sum_{k=0}^{n-2-i} (-1)^{n-2-i-k} a_k (C_{k+1}^{n-2-i-k} + C_k^{n-3-i-k}) = 0. \quad (2.6.5)
\end{aligned}$$

The system given in (2.6.5) may be solved by finding solutions to the matrix equation,  $M \cdot (a_0, a_1, \dots, a_{\frac{n-4}{2}}) = (0, 0, \dots, 0)$ , in other words by finding the kernel of  $M$ . We will only find its dimension, the nullity of  $M$ .

In the even case, the matrix  $M$  is given by the same formula as in the odd case,

$$\begin{aligned}
M(i, j) &= (-1)^{i-j} (C_j^{i-j} + C_{j-1}^{i-j-1}) + \\
&(-1)^{n-i-j} (C_j^{n-i-j} + C_{j-1}^{n-1-i-j}), \quad 1 \leq i, j \leq \frac{n-2}{2}. \quad (2.6.6)
\end{aligned}$$

In the same way as the odd case, we construct a matrix  $N$  such that  $MN$  is lower triangular. The top of the  $(\frac{n-2}{2} - k)$ th column of  $N$  is  $(\overbrace{0, \dots, 0}^{2k+1}) \cdot A_{\frac{n-6}{2}-3k}$  for  $0 \leq k \leq \lfloor \frac{n-8}{6} \rfloor$

and equal to the identity matrix elsewhere up to sign, where  $\text{sgn}(N(i, j)) = (-1)^{i+j-1}$ , except on the diagonal, where the sign is positive. A similar calculation shows that  $MN$  is a lower triangular matrix.

To find the rank of this matrix, we calculate the terms on the diagonal. If  $\frac{n-6}{2} - 3k$  is odd, for  $\frac{n-2}{2} - k > \lfloor \frac{n}{3} \rfloor$ ,  $MN(\frac{n-2}{2} - k, \frac{n-2}{2} - k)$  is equal to

$$\begin{aligned} & -D_{\frac{n-2}{2}-k, \lfloor \frac{n-4-2k}{4} \rfloor - 2k-1} \cdot A_{\frac{n-6}{2}-3k} + M\left(\frac{n-2}{2} - k, \frac{n-2}{2} - k\right) \\ & = -(Q|X^{2+2k}Y^{\frac{n-6}{2}-3k}T^{\frac{n-2}{2}-k}) + 1 + C_{\frac{n-2}{2}-k}^{\frac{n-6}{2}-3k} + C_{\frac{n-4}{2}-k}^{\frac{n-6}{2}-3k} \\ & = -C_{\frac{n-2}{2}-k}^{\frac{n-6}{2}-3k} - C_{\frac{n-4}{2}-k}^{\frac{n-6}{2}-3k} - 1 + 1 + C_{\frac{n-2}{2}-k}^{\frac{n-6}{2}-3k} + C_{\frac{n-4}{2}-k}^{\frac{n-6}{2}-3k} \\ & = 0. \end{aligned}$$

Finally, for identical reasons as in the odd  $n$  case, if  $\frac{n-6}{2} - 3k$  is even  $MN(\frac{n-2}{2} - k, \frac{n-2}{2} - k)$  is equal to

$$\begin{aligned} & -D_{\frac{n-2}{2}-k, \lfloor \frac{n-4-2k}{4} \rfloor - 2k-1} \cdot A_{\frac{n-6}{2}-3k} - 2 + M\left(\frac{n-2}{2} - k, \frac{n-2}{2} - k\right) \\ & = -(Q|X^{2+2k}Y^{\frac{n-6}{2}-3k}T^{\frac{n-2}{2}-k}) - 2 + 1 + C_{\frac{n-2}{2}-k}^{\frac{n-6}{2}-3k} + C_{\frac{n-4}{2}-k}^{\frac{n-6}{2}-3k} \\ & = -C_{\frac{n-2}{2}-k}^{\frac{n-6}{2}-3k} - C_{\frac{n-4}{2}-k}^{\frac{n-6}{2}-3k} + 1 - 2 + 1 + C_{\frac{n-2}{2}-k}^{\frac{n-6}{2}-3k} + C_{\frac{n-4}{2}-k}^{\frac{n-6}{2}-3k} \\ & = 0. \end{aligned}$$

Now, because  $0 \leq k \leq \frac{n-8}{6}$  the nullity of  $M$  is equal to  $\lfloor \frac{n-2}{6} \rfloor$ . In other words,  $F_n^2 \mathfrak{d}\mathfrak{s} / F_n^3 \mathfrak{d}\mathfrak{s} \leq \lfloor \frac{n-2}{6} \rfloor$ , since there may be relations between the generators that come from other systems of equations besides the equations 2.6.1. In fact there are not any other relations, and we use the combinatorial properties of the Poisson bracket to justify this.

We will now verify that  $\lfloor \frac{n-2}{6} \rfloor$  is a lower bound for the dimension. From theorem 2.30 (i), let  $S = \{\{f_{2i+1}, f_{n-2i-1}\}, 1 \leq i \leq \lfloor \frac{n-4}{4} \rfloor\}$ , be the set of Poisson brackets of weight  $n$  generators of  $F_1^{2i+1} \mathfrak{d}\mathfrak{s}$ . Let  $D$  be the vector space generated by  $S$ .

We consider the image of  $D, \bar{D}$  in  $F_2^n \mathfrak{d}\mathfrak{s} / F_3^n \mathfrak{d}\mathfrak{s}$ . By work of Zagier, Ihara and Takao (unpublished, see [Sc]), we know that the nullity of this system of equations is equal to a number which turns out to be exactly the dimension of the space of period polynomials, which is itself equal to the dimension of the space of cusp forms of weight  $n$  on  $\text{SL}_2(\mathbb{Z})$  (denoted  $S_n(\text{SL}_2(\mathbb{Z}))$ ) [Sc]. Therefore, we have that

$$\begin{aligned} \dim(\bar{D}) & = |S| - \dim(S_n(\text{SL}_2(\mathbb{Z}))) \\ & = \lfloor \frac{n-4}{4} \rfloor - \begin{cases} \lfloor n/12 \rfloor - 1 & n \equiv 2 \pmod{12} \\ \lfloor n/12 \rfloor & \text{otherwise} \end{cases} \\ & = \lfloor \frac{n-2}{6} \rfloor. \end{aligned}$$

Since  $\bar{D} \subset F_2^n \mathfrak{d}\mathfrak{s} / F_3^n \mathfrak{d}\mathfrak{s}$ ,

$$\lfloor \frac{n-2}{6} \rfloor \leq \dim(F_2^n \mathfrak{d}\mathfrak{s} / F_3^n \mathfrak{d}\mathfrak{s}) \leq \lfloor \frac{n-2}{6} \rfloor,$$

and hence the theorem is proved.  $\square$



Recall the definition 1.14 of the new zeta value algebra,  $\mathfrak{n}_3$ , which is the quotient of the algebra of multizeta values by products. We showed in chapter 1 that  $\partial\mathfrak{s}^\vee$  surjects onto  $\mathfrak{n}_3$ . The proof of theorem 1.21 yields the following corollary which gives the expression of depth 2 multizeta value as a rational multiple of a depth 1 multizeta modulo products, thus recovering a (weaker version of a) result well-known to Euler.

**Corollary 2.42.** *Let  $\bar{\zeta}(i, j)$  be a new zeta value of depth 2, and odd weight  $n$  ( $i + j = n$  is odd). We have the following expression for  $\bar{\zeta}(i, j)$  in terms of  $\bar{\zeta}(i + j)$ :*

$$\begin{aligned}\bar{\zeta}(x^{i-1}yx^{j-1}y) &= \frac{(-1)^{j-1}C_n^j - 1}{2}\bar{\zeta}(x^{n-1}y) \\ \bar{\zeta}(i, j) &= \frac{(-1)^{j-1}C_n^j - 1}{2}\bar{\zeta}(n).\end{aligned}$$

*Proof.* For  $n$  odd, since the matrix  $M$  is invertible, there exists a unique solution to the equation,  $M \cdot (a_0, \dots, a_{\frac{n-3}{2}}) = -A(1, \dots, 1)$ . We propose  $a_i = A\frac{(-1)^i}{2}C_{n-i-1}^{i+1}$  and show that this is the solution. In this case we have,

$$\begin{aligned}M \cdot (a_0, \dots, a_{\frac{n-3}{2}}) &= \left( A \sum_{j=\lfloor \frac{i+1}{2} \rfloor}^i \frac{(-1)^{n-j}}{2} C_{n-j}^j (-1)^{i-j} \left( C_j^{i-j} + C_{j-1}^{i-j-1} \right) + \right. & (2.6.7) \\ &\left. A \sum_{k=\frac{n-1}{2}-\lfloor \frac{i-1}{2} \rfloor}^{\frac{n-1}{2}} \frac{(-1)^{n-k}}{2} C_{n-k}^k (-1)^{n-i-k} \left( C_k^{n-i-k} + C_{k-1}^{n-i-k-1} \right) \right)_{i=1}^{\frac{n-1}{2}}.\end{aligned}$$

The power series,  $P$ , from lemma 2.31, has the expression,

$$P = YT^2 \left( \sum_{j=0} (XT + YT^2)^j \right) \left( \sum_{k=0} (X + Y)^k \right) + \left( \sum_{j=0} (XT + YT^2)^j \right) \left( \sum_{k=0} (X + Y)^k \right).$$

Hence, the first term in equation (2.6.7) gives exactly  $A\frac{(-1)^{i-1}}{2}$  times the coefficient of  $X^{n-i}Y^i T^i$  in the expansion of  $P$ . By lemma 2.31, this term is equal to  $A\frac{-C_n^i-1}{2}$  for even  $i$  and equal to  $A\frac{C_n^i-1}{2}$  for odd  $i$ . Furthermore, this term is exactly the expression of  $b_{i-1}$ , by the constructions (2.6.2) and (2.6.3) of  $M$ . The second term gives  $A\frac{(-1)^i}{2}$  times the coefficient of  $X^i Y^{n-i} T^{n-i}$  in the expansion of  $P$  and is equal to  $A\frac{-C_n^i-1}{2}$  for odd  $i$  and equal to  $A\frac{C_n^i-1}{2}$  for even  $i$ . This second term is equal to  $b_{n-i-1}$ . In both even and odd  $i$  cases, this sum is equal to  $-A$ , so the expression  $a_i = A\frac{(-1)^i}{2}C_{n-i-1}^{i+1}$  is indeed the unique solution to the system.

Since the first term in equation (2.6.7) is equal to  $b_{i-1}$ , we have

$$b_{i-1} = (f|x^{n-i-1}yx^{i-1}y) = A\frac{(-1)^{i-1}C_n^i - 1}{2} = \frac{(-1)^{i-1}C_n^i - 1}{2}(f|x^{n-1}y).$$

By the definition 1.26 of the dual space,  $\partial\mathfrak{s}^\vee = \widetilde{\mathfrak{nf}}_3$ , this equation is  $\mathfrak{z}^{\text{III}}(x^{n-i-1}yx^{i-1}y) = \frac{(-1)^{i-1}C_n^i-1}{2}\mathfrak{z}^{\text{III}}(x^{n-1}y)$ . But  $\widetilde{\mathfrak{nf}}_3$  surjects onto  $\mathfrak{n}_3$ , by the map  $\mathfrak{z}^{\text{III}}(w) \mapsto \bar{\zeta}(w)$ , so this relation is true also in the new zeta space and we have the desired expression.  $\square$

**Remark.** Note that the preceding corollary does not work when  $i + j = n$  is even; a double zeta is not equal to a rational multiple of a single zeta in even weight in  $\widetilde{\mathfrak{nf}}_3 \simeq \mathfrak{ds}^\vee$ . This follows from the fact that  $F_n^1 \mathfrak{ds} / F_n^2 \mathfrak{ds} = 0$  for even  $n$  (theorem 2.30). In [IKZ], the authors prove in complete generality that  $F_n^d \mathfrak{ds} / F_n^{d+1} \mathfrak{ds} = 0$  whenever  $d$  and  $n$  have opposite parities.

## Chapter 3

# The algebra of cell zeta values

This chapter is an intact article entitled *The algebra of cell-zeta values*, [BCS], which is joint work with Francis Brown and Leila Schneps awaiting publication. In [BCS], we give an explicit basis of polygons for the de Rham cohomology space,  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ , and use this to present a new structure for the  $\mathbb{Q}$  algebra of multizeta values,  $\mathcal{Z}$ , by considering the algebra generated by all periods on  $\mathfrak{M}_{0,n}$ . Here, a *period* on  $\mathfrak{M}_{0,n}$  is considered to be the integral of a rational function over a simplex in  $\mathfrak{M}_{0,n}(\mathbb{R})$ , the real part of moduli space. In chapter 2, we presented Kontsevich's construction of multizeta values as integrals of rational functions over simplices,  $\delta := 0 < t_1 < \cdots < t_{n-3} < 1$ , which are simplices in  $\mathfrak{M}_{0,n}(\mathbb{R})$ , thus showing that multizetas are indeed periods.

This work was inspired by the recent theorem of F. Brown [Br] in which he proves that every period on  $\mathfrak{M}_{0,n}$  is a  $\mathbb{Q}$ -linear combination of multizeta values. Here, I give a brief and intuitive introduction to the development of the special periods that we take as generators of the period algebra, which are called cell-zeta values. The definitions, structure of the paper and background are given in the introduction of [BCS].

In [Br], product maps on moduli space are introduced,

$$f : \mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,s} \times \mathfrak{M}_{0,r}, \quad r + s = n + 3,$$

which are simply the products of two forgetful maps (see [BCS] section 3.2.3). He defines two particular product maps, the simplicial product map, which gives the shuffle relation on multizetas, and the cubical product map, which gives the stuffle relation on multizetas. At the end of section of 7.5 [Br], Brown comments that these two product maps are extreme cases of a range of intermediate product formulae. This paper is a study of the intermediate product formulae. These new product formulae yield relations on cell-zeta values, analogous to, but more general than, the double shuffle relations on multizeta values, and have the advantage that they reflect the geometry and symmetry of the  $\mathfrak{M}_{0,n}$ .

Recall the definition of  $\mathfrak{M}_{0,n}(\mathbb{R})$ . The connected components of  $\mathfrak{M}_{0,n}(\mathbb{R})$  are cells and are denoted by the real ordering of the marked points inside them. To any such cell in  $\mathfrak{M}_{0,n}(\mathbb{R})$  we may associate a differential form, called a *cell form*, which is the form that has a simple pole along each irreducible boundary divisor which contains a face of the boundary of the associahedron  $\delta$ .

The cells generate the top dimensional homology group and, by duality between the homology and the cohomology, the cell forms generate the top dimensional de Rham cohomology,  $H^{n-3}(\mathfrak{M}_{0,n})$ . Based on a theorem of Arnol'd [Ar], we found that a basis for  $H^{n-3}(\mathfrak{M}_{0,n})$  is given by 01-forms (proposition 3.47).

Our paper answers the following three questions that arose naturally from studying multizetas as periods.

**Question.** *What subspace of the cohomology is the space of differential forms that give periods, i.e. that converge on a cell?*

It is not useful to look at the whole cohomology group since the integral of any 01-form converges over some cell, but diverges on others, while certain linear combinations of divergent cell forms will actually converge on a cell. The periods on  $\mathfrak{M}_{0,n}$  are by definition convergent integrals of the forms,  $\omega \in H^{n-3}(\mathfrak{M}_{0,n})$ , over some cell  $\gamma$ . By a variable change, any period can be written as an integral over the standard cell,  $\delta$ . Therefore, to study of periods on  $\mathfrak{M}_{0,n}$  it is sufficient to study the forms in the cohomology that are convergent on  $\mathfrak{M}_{0,n}$  and on its set of boundary components  $\delta$  which bound the standard cell. In chapter 4, we prove that this subspace of convergent differential forms is isomorphic to  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$ , the cohomology of the partially compactified moduli space. In section 3.4.3 of chapter 3, we give an explicit basis for  $H^{n-3}(\mathfrak{M}_{0,n}^\delta)$  thereby answering this first question.

**Question.** *How does one use periods to study multizeta values?*

With the explicit basis given above, we now have a way to formally represent periods as linear combinations pairs of polygons of the form  $(\gamma, \omega)$ , where  $\gamma$  is a cell and  $\omega$  a cell form. The product of periods which are integrals of a cell form over a cell is given by the pullback formula of a product map, (proposition 3.21)

$$\int_{\delta_1} \omega_1 \int_{\delta_2} \omega_2 = \int_{\delta_1 \amalg \delta_2} \omega_1 \amalg \omega_2.$$

Therefore any period can be represented as a linear combination of pairs of polygons and these polygons form an algebra for the shuffle product,

$$(\delta_1, \omega_1)(\delta_2, \omega_2) = (\delta_1 \amalg \delta_2, \omega_1 \amalg \omega_2).$$

We denote the algebra of periods or *cell numbers* by  $\mathcal{C}$ . By [Br], all the periods on  $\mathfrak{M}_{0,n}$  are  $\mathbb{Q}$  linear combinations of multizeta values. Therefore, we have answered the second question,  $\mathcal{C} \simeq \mathcal{Z}$ , on the level of real numbers. However, on the combinatorial level of formal multizetas and formal periods, there is still much work to be done.

We know how to explicitly express all of the multizetas as polygon sums by Kontsevich's identity, but we cannot as of yet explicitly express all of the periods as multizetas (even though by [Br], we know such an expression exists).

This leads us to the question of finding a set of generating relations over  $\mathbb{Q}$  for  $\mathcal{C}$ . We conjecture that the answer to this question is that the algebra of cell numbers has only the relations coming from variable changes on periods, algebraic identities on differential forms and product maps. As usual, because conjectures of this analytic type seem very difficult to prove, as they would imply important results, such as the transcendence conjecture on multizeta values, we concentrate our study on the formal situation in which the only relations are decreed to be the known relations. This is the same principle as in chapter 1 where we defined the formal zeta value algebra which satisfies only shuffle, stuffle and regularization relations, and leads to the final main question addressed in [BCS].

**Question.** *How can we use the three known sets of relations on periods to study relations between multiple zeta values?*

In order to study this, in section 3.2.4, we define the *formal cell number algebra*,  $\mathcal{FC}$ , which satisfies exactly the three sets of period relations outlined above. Since  $\mathcal{FC}$  surjects onto  $\mathcal{C}$ , any identities that we can find on formal cell numbers are also true for multizeta values, hence the structure of  $\mathcal{FC}$  provides a new method for studying multizeta values. If the formal cell numbers provide an adequate structure for multiple zeta values, then we should have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{FC} & \overset{?}{\longleftrightarrow} & \mathcal{FZ} \\
 \downarrow & & \downarrow \\
 \mathcal{C} & \overset{\sim}{\longleftrightarrow} & \mathcal{Z}
 \end{array}$$

In the section 3.4 of [BCS], we discuss the implications of this hypothesis, and give examples of how and why it should be true.

# THE ALGEBRA OF CELL-ZETA VALUES

FRANCIS BROWN, SARAH CARR, LEILA SCHNEPS

*Abstract.* Traditionally, multiple zeta values are viewed as convergent nested series which can also be expressed as iterated integrals on the projective line minus three points  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ . They are known to satisfy two sets of quadratic relations known as the double shuffle relations, which are conjectured to generate all algebraic relations between them. They were subsequently interpreted as the periods of the (motivic) fundamental group of  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ . Recently, Goncharov and Manin introduced a new version of motivic multiple zeta values, in which they are interpreted as periods of mixed Tate motives attached to the moduli spaces  $\mathfrak{M}_{0,n}$  of genus zero curves with  $n$  marked points.

In this paper, we introduce *cell-forms* on  $\mathfrak{M}_{0,\ell+3}$ , which are differential  $\ell$ -forms diverging along the boundary of exactly one connected component (cell) of the real moduli space  $\mathfrak{M}_{0,\ell+3}(\mathbb{R})$ . We give a basis for the space of all forms convergent on a given cell  $X$  in terms of cell-forms and define *cell-zeta values* to be the real numbers obtained by integrating these forms over  $X$ . The cell-zeta values satisfy algebraic relations generalizing the double shuffle relations, coming from simple geometric operations on the moduli spaces, and the algebra of cell-zeta values is in fact equal to the algebra of multiple zeta values. We conjecture that this new combinatorial system of generators and relations gives a complete description of the algebra of multiple zeta values.

## 3.1 Introduction

Let  $n_1, \dots, n_r \in \mathbb{N}$  and suppose that  $n_r \geq 2$ . The multiple zeta values (MZV's)

$$\zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} \in \mathbb{R}, \quad (3.1.1)$$

were first defined by Euler, and have recently acquired much importance in their relation to mixed Tate motives. It is conjectured that the periods of all mixed Tate motives over  $\mathbb{Z}$  are expressible in terms of such numbers. By a remark due to Kontsevich, every multiple zeta value can be written as an iterated integral:

$$\int_{0 \leq t_1 \leq \dots \leq t_\ell \leq 1} \frac{dt_1 \dots dt_\ell}{(\varepsilon_1 - t_1) \dots (\varepsilon_\ell - t_\ell)}, \quad (3.1.2)$$

where  $\varepsilon_i \in \{0, 1\}$ , and  $\varepsilon_1 = 1$  and  $\varepsilon_\ell = 0$  to ensure convergence, and  $\ell = n_1 + \dots + n_r$ . The iterated integral (3.1.2) is a period of the motivic fundamental group of  $\mathfrak{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ , whose de Rham cohomology  $H^1(\mathfrak{M}_{0,4})$  is spanned by the forms  $\frac{dt}{t}$  and  $\frac{dt}{1-t}$  [De1, DG]. One proves that the multiple zeta values satisfy two sets of quadratic relations [Ch1, Ho], known as the regularised double shuffle relations, and it has been conjectured that these generate all algebraic relations between MZV's [Ca2, Wa]. This is the traditional point of view on multiple zeta values.

On the other hand, by a general construction due to Beilinson, one can view the iterated integral (3.1.2) as a period integral in the ordinary sense, but this time of the

$\ell$ -dimensional affine scheme

$$\mathfrak{M}_{0,n} \simeq (\mathfrak{M}_{0,4})^\ell \setminus \{\text{diagonals}\} = \{(t_1, \dots, t_\ell) : t_i \neq 0, 1, t_i \neq t_j\},$$

where  $n = \ell + 3$ . This is the moduli space of curves of genus 0 with  $n$  ordered marked points. Indeed, the open domain of integration  $X = \{0 < t_1 < \dots < t_\ell < 1\}$  is one of the connected components of the set of real points  $\mathfrak{M}_{0,n}(\mathbb{R})$ , and the integrand of (3.1.2) is a regular algebraic form in  $H^\ell(\mathfrak{M}_{0,n})$  which converges on  $X$ . Thus, the study of multiple zeta values leads naturally to the study of all periods on  $\mathfrak{M}_{0,n}$ , which was initiated by Goncharov and Manin [Br, GM]. These periods can be written

$$\int_X \omega, \quad \text{where } \omega \in H^\ell(\mathfrak{M}_{0,n}) \text{ has no poles along } \overline{X}. \quad (3.1.3)$$

The general philosophy of motives and their periods [KZ] indicates that one should study relations between all such integrals. This leads to the following problems:

1. Construct a good basis of all regular (logarithmic)  $\ell$ -forms  $\omega$  in  $H^\ell(\mathfrak{M}_{0,n})$  whose integral over the cell  $X$  converges.
2. Find all relations between the integrals  $\int_X \omega$  which arise from natural geometric considerations on the moduli spaces  $\mathfrak{M}_{0,n}$ .

In this paper, we give an explicit solution to (1), and a family of relations which conjecturally answers (2). Firstly, we give a complete description of the convergent part of the cohomology  $H^\ell(\mathfrak{M}_{0,n})$  in terms of the combinatorics of polygons. The corresponding integrals are much more general than (3.1.2), and the numbers one obtains are called *cell-zeta values*. For (2), it turns out that there are essentially two types of relations. The first arises from the dihedral subgroup of automorphisms of  $\mathfrak{M}_{0,n}$  which stabilise  $X$ , and the other is a quadratic relation, which we call the *modular shuffle product*, arising from a product of forgetful maps between moduli spaces. We conjecture that these two simple families of relations generate the complete set of relations for the periods of the moduli spaces  $\mathfrak{M}_{0,n}$ .

### 3.1.1 Main results

We give a brief presentation of the main objects introduced in this paper, and the results obtained using them.

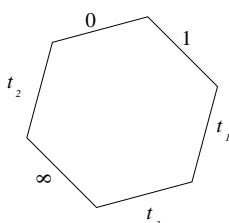
There is a stable compactification  $\overline{\mathfrak{M}}_{0,n}$  of  $\mathfrak{M}_{0,n}$ , such that  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$  is a smooth normal crossing divisor whose irreducible components correspond bijectively to partitions of the set of  $n$  marked points into two subsets of cardinal  $\geq 2$  [DM, Kn]. The real part  $\mathfrak{M}_{0,n}(\mathbb{R})$  of  $\mathfrak{M}_{0,n}$  is not connected, but has  $n!/2n$  connected components (open cells) corresponding to the different cyclic orders of the real points  $0, t_1, \dots, t_\ell, 1, \infty \in \mathbb{P}^1(\mathbb{R})$ , up to dihedral permutation [Dev1]. Thus, we can identify cells with  $n$ -sided polygons with edges labeled by  $\{0, t_1, \dots, t_\ell, 1, \infty\}$ . In the compactification  $\overline{\mathfrak{M}}_{0,n}(\mathbb{R})$ , the closed cells have the structure of associahedra or Stasheff polytopes; the boundary of a given cell is a union of irreducible divisors corresponding to partitions given by the chords in the associated polygon. The standard cell is the cell corresponding to the standard order we denote  $\delta$ , given by  $0 < t_1 < \dots < t_\ell < 1$ . We write  $\mathfrak{M}_{0,n}^\delta$  for the union of  $\mathfrak{M}_{0,n}$  with the boundary divisors of the standard cell. This is a smooth affine scheme introduced in [Br].

### Cell-forms.

A cell-form is a holomorphic differential  $\ell$ -form on  $\mathfrak{M}_{0,n}$  with logarithmic singularities along the boundary components of the stable compactification, having the property that its singular locus forms the boundary of a single cell in the real moduli space  $\mathfrak{M}_{0,n}(\mathbb{R})$ .

### Polygons.

Since a cell of  $\mathfrak{M}_{0,n}(\mathbb{R})$  is given by an ordering of  $\{0, t_1, \dots, t_\ell, 1, \infty\}$  up to dihedral permutation, we can identify it as above with an *unoriented*  $n$ -sided polygon with edges indexed by the set  $\{0, t_1, \dots, t_\ell, 1, \infty\}$ . Up to sign, the cell-form diverging on a given cell is obtained by taking the successive differences of the edges of the polygon (ignoring  $\infty$ ) as factors in the denominator:



$$\longleftrightarrow \pm \frac{dt_1 dt_2 dt_3}{(t_1 - 1)(t_3 - t_1)(-t_2)}$$

Let  $\mathcal{P}$  denote the  $\mathbb{Q}$ -vector space of *oriented*  $n$ -gons indexed by  $\{0, 1, t_1, \dots, t_\ell, 1, \infty\}$ . The orientation fixes the sign of the corresponding cell form, and this gives a map

$$\rho : \mathcal{P} \rightarrow H^\ell(\mathfrak{M}_{0,n}). \quad (3.1.4)$$

In section 3.4.1 we prove that this map is surjective and identify its kernel.

### 01-cell-forms.

These are the cell-forms corresponding to polygons in which 0 appears adjacent to 1. In theorem 3.16, we show that they form a basis of the cohomology  $H^\ell(\mathfrak{M}_{0,n})$ . In particular, the subspace of  $\mathcal{P}$  of polygons having 0 adjacent to 1 is isomorphic to  $H^\ell(\mathfrak{M}_{0,n})$  via (3.1.4).

### Insertion forms.

These very particular linear combinations of 01 cell-forms are constructed in section 3.3.3. We prove in theorem 3.45 and proposition 3.54 that they form a basis for the cohomology group  $H^\ell(\mathfrak{M}_{0,n}^\delta)$  of forms with no poles along the boundary of the standard cell of  $\mathfrak{M}_{0,n}(\mathbb{R})$ . These are precisely the forms whose integral (3.1.3) converges.

### Cell-zeta values.

These are real numbers obtained by integrating insertion forms over the standard cell as in (3.1.3). They are a generalization of multiple zeta values to a larger set of periods on  $\mathfrak{M}_{0,n}$ , such as

$$\int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1 dt_2 dt_3}{(1 - t_1)(t_3 - t_1)t_2}.$$

Note that unlike the multiple zeta values, this is not an iterated integral as in (3.1.2).



### Product maps.

Via the pullback, the maps  $f : \mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,r} \times \mathfrak{M}_{0,s}$  obtained by forgetting disjoint complementary subsets of the marked points  $t_1, \dots, t_\ell$  yield expressions for products of cell-zeta values on  $\mathfrak{M}_{0,r}$  and  $\mathfrak{M}_{0,s}$  as linear combinations of cell-zeta values on  $\mathfrak{M}_{0,n}$ :

$$\int_{X_1} \omega_1 \int_{X_2} \omega_2 = \int_{f^{-1}(X_1 \times X_2)} f^*(\omega_1 \wedge \omega_2). \quad (3.1.5)$$

There is a simple combinatorial algorithm to compute the multiplication law in terms of cell-forms. This generalizes the double shuffle multiplication laws for multiple zeta values, and is explained in section 3.2.3.

### Dihedral relations

These relations between cell-zeta values are given by

$$\int_X \omega = \int_X \sigma^*(\omega), \quad (3.1.6)$$

where  $\sigma$  is an automorphism of  $\mathfrak{M}_{0,n}$  which maps the standard cell to itself:  $\sigma(X) = X$ , and thus  $\sigma$  is a dihedral permutation of the marked points  $\{0, 1, t_1, \dots, t_\ell, \infty\}$ .

### The cell-zeta value algebra $\mathcal{C}$ .

The multiplication laws associated to product maps (3.1.5) make the space of all cell-zeta values on  $\mathfrak{M}_{0,n}$ ,  $n \geq 5$ , into a  $\mathbb{Q}$ -algebra which we denote by  $\mathcal{C}$ . By Brown's theorem [Br], which states essentially that all periods on  $\mathfrak{M}_{0,n}$  are linear combinations of multiple zeta values, together with Kontsevitch's expression (3.1.2) of multiple zeta values, we see that  $\mathcal{C}$  is equal to the algebra of multiple zeta values  $\mathcal{Z}$ .

### The formal cell-zeta value algebra $\mathcal{FC}$ .

By lifting the previous constructions to the level of polygons along the map (3.1.4), we define in section 3.2.4 an algebra of formal cell-zeta values which we denote by  $\mathcal{FC}$ . It is generated by the *insertion words*, which are formal sums of polygons corresponding to the insertion forms introduced above, subject to combinatorial versions of the product map relations (3.1.5) and the dihedral relations (3.1.6). This is analogous to the formal MZV algebra given by the double shuffle and Hoffmann relations.

The paper is organised as follows. In §2, we introduce cell forms, polygons and define the modular shuffle and dihedral relations. In §3, we define insertion words of polygons which are constructed out of Lyndon words, which may be of independent combinatorial interest. These are used to construct the insertion basis of convergent forms in §4. In §3.4.4, we give complete computations of this basis and the corresponding modular shuffle relations for  $\mathfrak{M}_{0,n}$ , where  $n = 5, 6, 7$ .

In the remainder of this introduction we sketch the connections between the formal cell-zeta value algebra and standard results and conjectures in the theory of multiple zeta values and mixed Tate motives.

### 3.1.2 Relation to mixed Tate motives and conjectures

Let  $\mathcal{MT}(\mathbb{Z})$  denote the category of mixed Tate motives which are unramified over  $\mathbb{Z}$  [DG]. Let  $\delta$  denote the standard cyclic structure on  $S = \{1, \dots, n\}$ , and let  $B_\delta$  denote the divisor which bounds the standard cell  $X_\delta$ . Let  $A_\delta$  denote the set of all remaining divisors on  $\overline{\mathfrak{M}}_{0,S} \setminus \mathfrak{M}_{0,S}$ , so that  $\mathfrak{M}_{0,S}^\delta = \mathfrak{M}_{0,S} \cup B_\delta$  ([Br]), and  $A_\delta = \overline{\mathfrak{M}}_{0,S} \setminus \mathfrak{M}_{0,S}^\delta$ . We write:

$$M_\delta = H^\ell(\overline{\mathfrak{M}}_{0,n} \setminus A_\delta, B_\delta \setminus (B_\delta \cap A_\delta)). \quad (3.1.7)$$

By a result due to Goncharov and Manin [GM],  $M_\delta$  defines an element in  $\mathcal{MT}(\mathbb{Z})$ , and therefore is equipped with an increasing weight filtration  $W$ . They show that  $\mathrm{gr}_\ell^W M_\delta$  is isomorphic to the de Rham cohomology  $H^\ell(\mathfrak{M}_{0,n}^\delta)$ , and that  $\mathrm{gr}_0^W M_\delta$  is isomorphic to the dual of the relative Betti homology  $H_\ell(\overline{\mathfrak{M}}_{0,n}, B_\delta)$ .

Let  $M$  be any element in  $\mathcal{MT}(\mathbb{Z})$ . A framing for  $M$  consists of an integer  $n$  and non-zero maps

$$v_0 \in \mathrm{Hom}(\mathbb{Q}(0), \mathrm{gr}_0^W M) \quad \text{and} \quad f_n \in \mathrm{Hom}(\mathrm{gr}_{-2n}^W M, \mathbb{Q}(n)). \quad (3.1.8)$$

Two framed motives  $(M, v_0, f_n)$  and  $(M', v'_0, f'_n)$  are said to be equivalent if there is a morphism  $\phi : M \rightarrow M'$  such that  $\phi \circ v_0 = v'_0$  and  $f_n \circ \phi = f'_n$ . This generates an equivalence relation whose equivalence classes are denoted  $[M, v_0, f_n]$ . Let  $\mathcal{M}(\mathbb{Z})$  denote the set of equivalence classes of framed mixed Tate motives which are unramified over  $\mathbb{Z}$ , as defined in [Go1]. It is a commutative, graded Hopf algebra.

To every convergent cohomology class  $\omega \in H^\ell(\mathfrak{M}_{0,n}^\delta)$ , we associate the following framed mixed Tate motive:

$$m(\omega) = [M_\delta, [X_\delta], \omega], \quad (3.1.9)$$

where  $[X_\delta]$  denotes the relative homology class of the standard cell. This defines a map  $\mathcal{FC} \rightarrow \mathcal{M}(\mathbb{Z})$ . The maximal period of  $m(\omega)$  is exactly the cell-zeta value

$$\int_{X_\delta} \omega.$$

**Proposition 3.1.** *The dihedral symmetry relation and modular shuffle relations are motivic. In other words,*

$$\begin{aligned} m(\sigma^*(\omega)) &= m(\omega), \\ m(\omega_1 \cdot \omega_2) &= m(\omega_1) \otimes m(\omega_2), \end{aligned}$$

for every dihedral symmetry  $\sigma$  of  $X_\delta$ , and for every modular shuffle product  $\omega_1 \cdot \omega_2$  of convergent forms  $\omega_1, \omega_2$  on  $\mathfrak{M}_{0,r}, \mathfrak{M}_{0,s}$  respectively.

The motivic nature of our constructions will be clear from the definitions. We therefore obtain a well-defined map  $m$  from the algebra of formal cell-zeta numbers  $\mathcal{FC}$  to  $\mathcal{M}(\mathbb{Z})$ .

**Conjecture 3.2.** *The map  $m : \mathcal{FC} \rightarrow \mathcal{M}(\mathbb{Z})$  is an isomorphism.*

Since the structure of  $\mathcal{M}(\mathbb{Z})$  is known, we are led to more precise conjectures on the structure of the formal cell-zeta algebra. To motivate this, let  $\mathcal{L} = \mathbb{Q}[e_3, e_5, \dots]$  denote the free Lie algebra generated by one element  $e_{2n+1}$  in each odd degree. Set

$$\mathfrak{F} = \mathbb{Q}[e_2] \oplus \mathcal{L}.$$

The underlying graded vector space is generated by, in increasing weight:

$$e_2 ; e_3 ; e_5 ; e_7 ; [e_3, e_5] ; e_9 ; [e_3, e_7] ; [e_3, [e_5, e_3]] , e_{11} ; [e_3, e_9] , [e_5, e_7] ; \dots$$

Let  $\mathcal{U}\mathfrak{F}$  denote the universal enveloping algebra of the Lie algebra  $\mathfrak{F}$ . Then it is known that  $\mathcal{M}(\mathbb{Z})$  is dual to  $\mathcal{U}\mathfrak{F}$ . From the explicit description of  $\mathfrak{F}$  given above, one can deduce that the graded dimensions  $d_k = \dim_{\mathbb{Q}} \text{gr}_k^W \mathcal{M}(\mathbb{Z})$  satisfy Zagier's recurrence relation

$$d_k = d_{k-2} + d_{k-3} , \quad (3.1.10)$$

with the initial conditions  $d_0 = 1, d_1 = 0, d_2 = 1$ .

**Conjecture 3.3.** *The dimension of the  $\mathbb{Q}$ -vector space of formal cell-zeta values on  $\mathfrak{M}_{0,n}$ , modulo all linear relations obtained from the dihedral and modular shuffle relations, is equal to  $d_\ell$ , where  $n = \ell + 3$ . Equivalently, the dual Lie algebra to the co-Lie algebra obtained by quotienting  $\mathcal{FC}$  by products is isomorphic to  $\mathfrak{F}$ .*

We verified this conjecture for  $\mathfrak{M}_{0,n}$  for  $n \leq 9$  by direct calculation (see §3.4.4). When  $n = 9$ , the dimension of the convergent cohomology  $H^6(\mathfrak{M}_{0,9}^\delta)$  is 1089, and after taking into account all linear relations coming from dihedral and modular shuffle products, this reduces to a vector space of dimension  $d_6 = 2$ .

To compare this picture with the classical picture of multiple zeta values, let  $\mathcal{FZ}$  denote the formal multi-zeta algebra. This is the quotient of the free  $\mathbb{Q}$ -algebra generated by formal symbols (3.1.2) modulo the regularised double shuffle relations. It has been conjectured that  $\mathcal{FZ}$  is isomorphic to  $\mathcal{M}(\mathbb{Z})$ , which leads to the second main conjecture.

**Conjecture 3.4.** *The formal algebras  $\mathcal{FC}$  and  $\mathcal{FZ}$  are isomorphic.*

Put more prosaically, this states that the formal ring of periods of  $\mathfrak{M}_{0,n}$  modulo dihedral and modular shuffle relations, is isomorphic to the formal ring of periods of the motivic fundamental group of  $\mathfrak{M}_{0,4}$  modulo the regularised double shuffle relations.

By (3.1.2), we have a natural linear map  $\mathcal{FZ} \rightarrow \mathcal{FC}$ . However, at present we cannot show that it is an algebra homomorphism. Indeed, although it is easy to deduce the regularised shuffle relation for the image of  $\mathcal{FZ}$  in  $\mathcal{FC}$  from the dihedral and modular shuffle relations, we are unable to deduce the regularised stuffle relations.

**Remark 3.5.** *The motivic nature of the regularised double shuffle relations proved to be somewhat difficult to establish [Go1, Go2, T]. It is interesting that the motivic nature of the dihedral and modular shuffle relations we define here is immediate.*

## 3.2 The cell-zeta value algebra associated to moduli spaces of curves

Let  $\mathfrak{M}_{0,n}$ ,  $n \geq 4$  denote the moduli space of genus zero curves (Riemann spheres) with  $n$  ordered marked points  $(z_1, \dots, z_n)$ . This space is described by the set of  $n$ -tuples of distinct points  $(z_1, \dots, z_n)$  modulo the equivalence relation given by the action of  $\text{PSL}_2$ . Because this action is triply transitive, there is a unique representative of each equivalence class such that  $z_1 = 0, z_{n-1} = 1, z_n = \infty$ . We define simplicial coordinates  $t_1, \dots, t_\ell$  on  $\mathfrak{M}_{0,n}$  by setting

$$t_1 = z_2 , \quad t_2 = z_3 , \quad \dots , \quad t_\ell = z_{n-2} , \quad (3.2.1)$$

where  $\ell = n - 3$  is the dimension of  $\mathfrak{M}_{0,n}(\mathbb{C})$ . This gives the familiar identification

$$\mathfrak{M}_{0,n} \cong \{(t_1, \dots, t_\ell) \in (\mathbb{P}^1 - \{0, 1, \infty\})^\ell \mid t_i \neq t_j \text{ for all } i \neq j\}. \quad (3.2.2)$$

### 3.2.1 Cell forms

**Definition 3.6.** Let  $S = \{1, \dots, n\}$ . A cyclic structure  $\gamma$  on  $S$  is a cyclic ordering of the elements of  $S$  or equivalently, an identification of the elements of  $S$  with the edges of an oriented  $n$ -gon modulo rotations. A dihedral structure  $\delta$  on  $S$  is an identification with the edges of an unoriented  $n$ -gon modulo dihedral symmetries.

We can write a cyclic structure as an ordered  $n$ -tuple  $\gamma = (\gamma(1), \gamma(2), \dots, \gamma(n))$  considered up to cyclic rotations.

**Definition 3.7.** Let  $(z_1, \dots, z_n) = (0, t_1, \dots, t_\ell, 1, \infty)$  be a representative of a point on  $\mathfrak{M}_{0,n}$  as above. Let  $\gamma$  be a cyclic structure on  $S$ , and let  $\sigma$  be the unique ordering of  $z_1, \dots, z_n$  compatible with  $\gamma$  such that  $\sigma(n) = n$ . The cell-form corresponding to  $\gamma$  is defined to be the differential  $\ell$ -form

$$\omega_\gamma = [z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}] = \frac{dt_1 \cdots dt_\ell}{(z_{\sigma(2)} - z_{\sigma(1)})(z_{\sigma(3)} - z_{\sigma(2)}) \cdots (z_{\sigma(n-1)} - z_{\sigma(n-2)})}. \quad (3.2.3)$$

In other words, by writing the terms of  $\omega_\gamma = [z_{\sigma(1)}, \dots, z_{\sigma(n)}]$  clockwise around a polygon, the denominator of a cell form is just the product of successive differences  $(z_{\sigma(i)} - z_{\sigma(i-1)})$  with the two factors containing  $\infty$  simply left out.

**Remark 3.8.** To every dihedral structure there correspond two opposite cyclic structures. If these are given by  $\gamma$  and  $\tau$ , then we have

$$\omega_\gamma = (-1)^n \omega_\tau. \quad (3.2.4)$$

**Example 3.9.** Let  $n = 7$ , and  $S = \{1, \dots, 7\}$ . Consider the cyclic structure  $\gamma$  on  $S$  given by the order 1635724. The unique ordering  $\sigma$  of  $S$  compatible with  $\gamma$  and having  $\sigma(n) = n$ , is the ordering 2416357, which can be depicted by writing the elements  $z_{\sigma(1)}, \dots, z_{\sigma(7)}$ , or  $0, 1, t_2, t_4, \infty, t_1, t_3$  clockwise around a circle:

$$\gamma = (z_{\sigma(1)}, \dots, z_{\sigma(7)}) = (t_1, t_3, 0, 1, t_2, t_4, \infty).$$

The corresponding cell-form on  $\mathfrak{M}_{0,7}$  is

$$\omega_\gamma = [t_1, t_3, 0, 1, t_2, t_4, \infty] = \frac{dt_1 dt_2 dt_3 dt_4}{(t_3 - t_1)(-t_3)(t_2 - 1)(t_4 - t_2)}.$$

The symmetric group  $\mathfrak{S}(S)$  acts on  $\mathfrak{M}_{0,n}$  by permutation of the marked points. It therefore acts both on the set of cyclic structures  $\gamma$ , and also on the ring of differential forms on  $\mathfrak{M}_{0,n}$ . These actions coincide for cell forms.

**Lemma 3.10.** For every cyclic structure  $\gamma$  on  $S$ , we have the formula:

$$\sigma^*(\omega_\gamma) = \omega_{\sigma(\gamma)} \quad \text{for all } \sigma \in \mathfrak{S}(S). \quad (3.2.5)$$

*Proof.* Consider the regular  $n$ -form on  $(\mathbb{P}^1)_*^S$  defined by the formula:

$$\tilde{\omega}_\gamma = \frac{dz_1 \wedge \dots \wedge dz_n}{(z_{\gamma(1)} - z_{\gamma(2)}) \dots (z_{\gamma(n)} - z_{\gamma(1)})}. \quad (3.2.6)$$

It clearly satisfies  $\sigma^*(\tilde{\omega}_\gamma) = \tilde{\omega}_{\sigma(\gamma)}$  for all  $\sigma \in D_\gamma$ . A simple calculation shows that  $\tilde{\omega}_\gamma$  is invariant under the action of  $\mathbb{P}\mathrm{SL}_2$  by Möbius transformations. Let  $\pi : (\mathbb{P}^1)_*^S \rightarrow \mathfrak{M}_{0,S}$  denote the projection map with fibres isomorphic to  $\mathbb{P}\mathrm{SL}_2$ . There is a unique (up to scalar multiple in  $\mathbb{Q}^\times$ ) non-zero invariant regular 3-form  $v$  on  $\mathbb{P}\mathrm{SL}_2(\mathbb{C})$  which is defined over  $\mathbb{Q}$ . Then, by renormalising  $v$  if necessary, we have  $\omega_\gamma \wedge v = \tilde{\omega}_\gamma$ . In fact,  $\omega_\gamma$  is the unique  $\ell$ -form on  $\mathfrak{M}_{0,S}$  satisfying this equation. We deduce that  $\sigma^*(\omega_\gamma) = \omega_{\sigma(\gamma)}$  for all  $\sigma \in D_\gamma$ .  $\square$

Each dihedral structure  $\eta$  on  $S$  corresponds to a unique connected component of the real locus  $\mathfrak{M}_{0,n}(\mathbb{R})$ , namely the component associated to the set of Riemann spheres with real marked points  $(z_1, \dots, z_n)$  whose real ordering is given by  $\eta$ . We denote this component by  $X_{S,\eta}$  or  $X_{n,\eta}$ . It is an algebraic manifold with corners with the combinatorial structure of a Stasheff polytope, so we often refer to it as a *cell*. A cyclic structure compatible with  $\eta$  corresponds to a choice of orientation of this cell. Let  $\delta$  once and for all denote the cyclic order corresponding to the ordering  $(1, 2, \dots, n)$ . We call  $X_{S,\delta} = X_{n,\delta}$  the *standard cell*. It is the set of points on  $\mathfrak{M}_{0,n}$  given by real marked points  $(0, t_1, \dots, t_\ell, 1, \infty)$  in that cyclic order; in simplicial coordinates it is given by the standard real simplex  $0 < t_1 < \dots < t_\ell < 1$ .

The distinguishing feature of cell-forms, from which they derive their name, is given in the following proposition.

**Proposition 3.11.** *Let  $\eta$  be a dihedral structure on  $S$ , and let  $\gamma$  be either of the two cyclic substructures of  $\eta$ . Then the cell form  $\omega_\gamma$  has simple poles along the boundary of the cell  $X_{S,\eta}$  and no poles anywhere else.*

*Proof.* Let  $D \subset \overline{\mathfrak{M}}_{0,S} \setminus \mathfrak{M}_{0,S}$  be a divisor given by a stable partition  $S = S_1 \amalg S_2$  (i.e., such that  $|S_i| \neq 1$  for  $i = 1, 2$ ). In [Br], the following notation was introduced:

$$\mathbb{I}_D(i, j) = \mathbb{I}(\{i, j\} \subset S_1) + \mathbb{I}(\{i, j\} \subset S_2),$$

where  $\mathbb{I}(A \subset B)$  is the indicator function which takes the value 1 if  $A$  is contained in  $B$  and 0 otherwise. Therefore  $\mathbb{I}_D(i, j) \in \{0, 1\}$ . Then we have

$$2 \operatorname{ord}_D(\omega_\gamma) = (\ell - 1) - \mathbb{I}_D(\gamma(1), \gamma(2)) - \mathbb{I}_D(\gamma(2), \gamma(3)) - \dots - \mathbb{I}_D(\gamma(n), \gamma(1)). \quad (3.2.7)$$

To prove this, observe that  $\omega_\gamma = f_\gamma \omega_0$ , where

$$f_\gamma = \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{(z_i - z_{i+2})}{(z_{\gamma(i)} - z_{\gamma(i+1)})},$$

and

$$\omega_0 = \frac{dt_1 \dots dt_\ell}{t_2(t_3 - t_1)(t_4 - t_2) \dots (t_\ell - t_{\ell-2})(1 - t_\ell)}$$

is the canonical volume form with no zeros or poles along the standard cell defined in [Br]. The proof of (3.2.7) follows on applying proposition 7.5 from [Br].

Now, (3.2.7) shows that  $\omega_\gamma$  has the worst singularities when the most possible  $\mathbb{I}_D(\gamma(i), \gamma(i+1))$  are equal to 1. This happens when only two of them are equal to zero, namely

$$S_1 = \{\gamma(1), \gamma(2), \dots, \gamma(k)\} \quad \text{and} \quad S_2 = \{\gamma(k+1), \gamma(k+2), \dots, \gamma(n)\}, \quad 2 \leq k \leq n-2.$$

In this case, (3.2.7) yields  $2\text{ord}_D\omega_\gamma = (\ell-1) - (n-2) = -2$ , so  $\text{ord}_D\omega_\gamma = -1$ . In all other cases we must therefore have  $\text{ord}_D\omega_\gamma \geq 0$ . Thus the singular locus of  $\omega_\gamma$  is precisely given by the set of divisors bounding the cell  $X_{S,\eta}$ .  $\square$

### 3.2.2 01 cell-forms and a basis of the cohomology of $\mathfrak{M}_{0,n}$

We first derive some useful identities between certain rational functions. Let  $S = \{1, \dots, n\}$  and let  $v_1, \dots, v_n$  denote coordinates on  $\mathbb{A}^n$ . For every cyclic structure  $\gamma$  on  $S$ , let  $\langle \gamma \rangle = \langle v_{\gamma(1)}, \dots, v_{\gamma(n)} \rangle$  denote the rational function

$$\frac{1}{(v_{\gamma(2)} - v_{\gamma(1)}) \cdots (v_{\gamma(n)} - v_{\gamma(n-1)})(v_{\gamma(1)} - v_{\gamma(n)})} \in \mathbb{Z}\left[v_i, \frac{1}{v_i - v_j}\right]. \quad (3.2.8)$$

We refer to such a function as a cell-function. We can extend its definition linearly to  $\mathbb{Q}$ -linear combinations of cyclic structures. Let  $X = \{x_1, \dots, x_n\}$  denote any alphabet on  $n$  symbols. Recall that the shuffle product [Re] is defined on linear combinations of words on  $X$  by the inductive formula

$$w \sqcup e = e \sqcup w \quad \text{and} \quad a w \sqcup a' w' = a(w \sqcup a' w') + a'(a w \sqcup w'), \quad (3.2.9)$$

where  $w, w'$  are any words in  $X$  and  $e$  denotes the empty or trivial word.

**Definition 3.12.** Let  $A, B \subset S$  and let  $A \cap B = C = \{c_1, \dots, c_r\}$ . Let  $\gamma_A$  be a cyclic order on  $A$  such that the elements  $c_1, \dots, c_r$  appear in their standard cyclic order, and let  $\gamma_B$  be a cyclic order on  $B$  with the same property. We write  $\gamma_A = (c_1, A_{1,2}, c_2, A_{2,3}, \dots, c_r, A_{r,1})$  and  $\gamma_B = (c_1, B_{1,2}, c_2, B_{2,3}, \dots, c_r, B_{r,1})$ , where the  $A_{i,i+1}$ , (resp. the  $B_{i,i+1}$ ) together with  $C$ , form a partition of  $A$  (resp.  $B$ ). We denote the shuffle product of the two cell-functions  $\langle \gamma_A \rangle$  and  $\langle \gamma_B \rangle$  with respect to  $c_1, \dots, c_r$  by

$$\langle \gamma_A \rangle \sqcup_{c_1, \dots, c_r} \langle \gamma_B \rangle$$

which is defined to be the sum of cell functions

$$\langle c_1, A_{1,2} \sqcup B_{1,2}, c_2, A_{2,3} \sqcup B_{2,3}, \dots, c_r, A_{r,1} \sqcup B_{r,1} \rangle. \quad (3.2.10)$$

The shuffle product of two cell-functions is related to their actual product by the following lemma.

**Proposition 3.13.** Let  $A, B \subset S$ , such that  $|A \cap B| \geq 2$ . Let  $\gamma_A, \gamma_B$  be cyclic structures on  $A, B$  such that the cyclic structures on  $A \cap B$  induced by  $\gamma_A$  and  $\gamma_B$  coincide. If  $\gamma_{A \cap B}$  denotes the induced cyclic structure on  $A \cap B$ , we have:

$$\frac{\langle \gamma_A \rangle \cdot \langle \gamma_B \rangle}{\langle \gamma_{A \cap B} \rangle} = \langle \gamma_A \rangle \sqcup_{\gamma_{A \cap B}} \langle \gamma_B \rangle. \quad (3.2.11)$$

*Proof.* Write the cell functions  $\langle \gamma_A \rangle$  and  $\langle \gamma_B \rangle$  as  $\langle a_{i_1}, P_1, a_{i_2}, P_2, \dots, a_{i_r}, P_r \rangle$  and  $\langle a_{i_1}, R_1, a_{i_2}, R_2, \dots, a_{i_r}, R_r \rangle$ , where  $P_i, R_i$  for  $1 \leq i \leq r$  are tuples of elements in  $S$ . Let  $\Delta_{ab} = (b - a)$ . We will first prove the result for  $r = 2$  and  $P_2, R_2 = \emptyset$ :

$$\Delta_{ab}\Delta_{ba}\langle a, p_1, \dots, p_{k_1}, b \rangle \langle a, r_1, \dots, r_{k_2}, b \rangle = \langle a, (p_1, \dots, p_{k_1})\sqcup(r_1, \dots, r_{k_2}), b \rangle. \quad (3.2.12)$$

We prove this case by induction on  $k_1 + k_2$ .

Trivially, for  $k_1 + k_2 = 0$  we have

$$\Delta_{ab}\Delta_{ba}\langle a, b \rangle \langle a, b \rangle = \langle a, b \rangle.$$

Now assume the induction hypothesis that

$$\begin{aligned} \Delta_{ab}\Delta_{ba}\langle a, p_2, \dots, p_{k_1}, b \rangle \langle a, r_1, \dots, r_{k_2}, b \rangle &= \langle a, ((p_2, \dots, p_{k_1})\sqcup(r_1, \dots, r_{k_2})), b \rangle \text{ and} \\ \Delta_{ab}\Delta_{ba}\langle a, p_1, \dots, p_{k_1}, b \rangle \langle a, r_2, \dots, r_{k_2}, b \rangle &= \langle a, ((p_1, \dots, p_{k_1})\sqcup(r_2, \dots, r_{k_2})), b \rangle. \end{aligned}$$

To lighten the notation, let  $p_2, \dots, p_{k_1} = \underline{p}$  and  $r_2, \dots, r_{k_2} = \underline{r}$ . By the shuffle recurrence formula (3.2.9) and the induction hypothesis:

$$\begin{aligned} \langle a, ((p_1, \underline{p})\sqcup(r_1, \underline{r})), b \rangle &= \langle a, p_1, ((\underline{p})\sqcup(r_1, \underline{r})), b \rangle + \langle a, r_1, ((p_1, \underline{p})\sqcup(\underline{r})), b \rangle \\ &= \frac{\Delta_{p_1 b} \langle p_1, ((\underline{p})\sqcup(r_1, \underline{r})), b \rangle}{\Delta_{ab}\Delta_{ap_1}} + \frac{\Delta_{r_1 b} \langle r_1, ((p_1, \underline{p})\sqcup(\underline{r})), b \rangle}{\Delta_{ab}\Delta_{ar_1}} \\ &= \frac{\Delta_{p_1 b} \Delta_{bp_1} \Delta_{p_1 b} \langle p_1, \underline{p}, b \rangle \langle p_1, r_1, \underline{r}, b \rangle}{\Delta_{ab}\Delta_{ap_1}} \\ &\quad + \frac{\Delta_{r_1 b} \Delta_{br_1} \Delta_{r_1 b} \langle r_1, p_1, \underline{p}, b \rangle \langle r_1, \underline{r}, b \rangle}{\Delta_{ab}\Delta_{ar_1}} \end{aligned}$$

Using identities such as  $\langle p_1, \underline{p}, b \rangle = \frac{\Delta_{ap_1}\Delta_{ba}}{\Delta_{bp_1}} \langle a, p_1, \underline{p}, b \rangle$ , this is

$$\begin{aligned} &\left[ \frac{\Delta_{p_1 b}^2 \Delta_{bp_1}}{\Delta_{ab}\Delta_{ap_1}} \frac{\Delta_{ap_1} \Delta_{ba}}{\Delta_{bp_1}} \frac{\Delta_{ba} \Delta_{ar_1}}{\Delta_{bp_1} \Delta_{p_1 r_1}} + \frac{\Delta_{r_1 b}^2 \Delta_{br_1}}{\Delta_{ab}\Delta_{ar_1}} \frac{\Delta_{ap_1} \Delta_{ba}}{\Delta_{r_1 p_1} \Delta_{br_1}} \frac{\Delta_{ba} \Delta_{ar_1}}{\Delta_{br_1}} \right] \langle a, p_1, \underline{p}, b \rangle \langle a, r_1, \underline{r}, b \rangle \\ &= \Delta_{ab} \left[ \frac{\Delta_{ar_1} \Delta_{bp_1}}{\Delta_{p_1 r_1}} + \frac{\Delta_{br_1} \Delta_{ap_1}}{\Delta_{r_1 p_1}} \right] \langle a, p_1, \underline{p}, b \rangle \langle a, r_1, \underline{r}, b \rangle = \Delta_{ab}\Delta_{ba}\langle a, p_1, \underline{p}, b \rangle \langle a, r_1, \underline{r}, b \rangle. \end{aligned}$$

The last equality is the Plücker relation  $\Delta_{ar_1}\Delta_{bp_1} - \Delta_{br_1}\Delta_{ap_1} = \Delta_{p_1 r_1}\Delta_{ba}$ . This proves the identity (3.2.12). Now, using the identity

$$\langle a_{i_1} P_1 a_{i_2} P_2 \dots a_{i_r} P_r \rangle = \Delta_{a_{i_2} a_{i_1}} \langle a_{i_1} P_1 a_{i_2} \rangle \times \Delta_{a_{i_3} a_{i_2}} \langle a_{i_2} P_2 a_{i_3} \rangle \times \dots \times \Delta_{a_{i_r} a_{i_{r-1}}} \langle a_{i_{r-1}} P_{r-1} a_{i_r} \rangle,$$

the general case follows from (3.2.12).  $\square$

**Corollary 3.14.** *Let  $X$  and  $Y$  be disjoint sequences of indeterminates and let  $e$  be an indeterminate not appearing in either  $X$  or  $Y$ . We have the following identity on cell functions:*

$$\langle (X, e) \sqcup (Y, e) \rangle = \langle X \sqcup Y, e \rangle = 0. \quad (3.2.13)$$



*Proof.* Write  $X = x_1, x_2, \dots, x_n$  and  $Y = y_1, y_2, \dots, y_m$ . By the recurrence formula for the shuffle product and proposition 3.13, we have

$$\begin{aligned}
\langle X \boxplus Y, e \rangle &= \langle x_1, (x_2, \dots, x_n \boxplus y_1, \dots, y_m), e \rangle + \langle y_1, (x_1, \dots, x_n \boxplus y_2, \dots, y_m), e \rangle \\
&= \langle X, e \rangle \langle x_1, Y, e \rangle (e - x_1)(x_1 - e) + \langle y_1, X, e \rangle \langle Y, e \rangle (y_1 - e)(e - y_1) \\
&= \frac{(e - x_1)(x_1 - e)}{(x_2 - x_1) \cdots (e - x_n)(x_1 - e) (y_1 - x_1)(y_2 - y_1) \cdots (e - y_m)(x_1 - e)} \\
&\quad + \frac{(y_1 - e)(e - y_1)}{(x_1 - y_1)(x_2 - x_1) \cdots (e - x_n)(y_1 - e) (y_2 - y_1) \cdots (e - y_m)(y_1 - e)} \\
&= \frac{(-1) + (-1)^2}{(x_2 - x_1) \cdots (e - x_n) (y_1 - x_1)(y_2 - y_1) \cdots (e - y_m)} = 0.
\end{aligned}$$

□

By specialization, we can formally extend the definition of a cell function to the case where some of the terms  $v_i$  are constant, or one of the  $v_i$  is infinite, by setting

$$\begin{aligned}
\langle v_1, \dots, v_{i-1}, \infty, v_{i+1}, \dots, v_n \rangle &= \lim_{x \rightarrow \infty} x^2 \langle v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n \rangle \\
&= \frac{1}{(v_2 - v_1) \cdots (v_{i-1} - v_{i-2})(v_{i+2} - v_{i+1}) \cdots (v_n - v_{n-1})(v_1 - v_n)}.
\end{aligned}$$

This is the rational function obtained by omitting all terms containing  $\infty$ . By taking the appropriate limit, it is clear that (3.2.11) and (3.2.13) are valid in this case too. In the case where  $\{v_1, \dots, v_n\} = \{0, 1, t_1, \dots, t_\ell, \infty\}$  we have the formula

$$[v_1, \dots, v_n] = \langle v_1, \dots, v_n \rangle dt_1 dt_2 \dots dt_\ell. \quad (3.2.14)$$

**Definition 3.15.** A 01 cyclic (or dihedral) structure is a cyclic (or dihedral) structure on  $S$  in which the numbers 1 and  $n - 1$  are consecutive. Since  $z_1 = 0$  and  $z_{n-1} = 1$ , a 01 cyclic (or dihedral) structure is a set of orderings of the set  $\{z_1, \dots, z_n\} = \{0, t_1, \dots, t_\ell, 1, \infty\}$ , in which the elements 0 and 1 are consecutive. In these terms, each dihedral structure can be written as an ordering  $(0, 1, \pi)$  where  $\pi$  is some ordering of  $\{t_1, \dots, t_\ell, \infty\}$ . To each such ordering we associate a cell-function  $\langle 0, 1, \pi \rangle$ , which is called a 01 cell-function.

Since 01 cell-functions corresponding to different  $\pi$  are clearly different, it follows that there exist exactly  $(n - 2)!$  distinct 01 cell-functions  $\langle 0, 1, \pi \rangle$ . To these correspond  $(n - 2)!$  distinct 01 cell-forms  $\omega_{(0,1,\pi)} = \langle 0, 1, \pi \rangle dt_1 \dots dt_\ell$ .

**Theorem 3.16.** The set of 01 cell-forms  $\omega_{(0,1,\pi)}$ , where  $\pi$  denotes any ordering of  $\{t_1, \dots, t_\ell, \infty\}$ , has cardinal  $(n - 2)!$  and forms a basis of  $H^\ell(\mathfrak{M}_{0,n}, \mathbb{Q})$ .

*Proof.* The proof is based on the following well-known result due to Arnol'd [Ar].

**Theorem 3.17.** A basis of  $H^\ell(\mathfrak{M}_{0,n}, \mathbb{Q})$  is given by the classes of the forms

$$\Omega(\underline{\varepsilon}) := \frac{dt_1 \dots dt_\ell}{(t_1 - \varepsilon_1) \cdots (t_\ell - \varepsilon_\ell)}, \quad \varepsilon_i \in E_i, \quad (3.2.15)$$

where  $E_1 = \{0, 1\}$  and  $E_i = \{0, 1, t_1, \dots, t_{i-1}\}$  for  $2 \leq i \leq \ell$ .



It suffices to prove that each element  $\Omega(\underline{\varepsilon})$  in (3.2.15) can be written as a linear combination of 01 cell-forms. We begin by expressing a given rational function  $\frac{1}{(t_1 - \varepsilon_1) \cdots (t_\ell - \varepsilon_\ell)}$  as a product of cell-functions and then apply proposition 3.13. To every  $t_i$ , we associate its *type*  $\tau(t_i) \in \{0, 1\}$  (which depends on  $\varepsilon_1, \dots, \varepsilon_\ell$ ) as follows. If  $\varepsilon_i = 0$  then  $\tau(t_i) = 0$ ; if  $\varepsilon_i = 1$ , then  $\tau(t_i) = 1$ , but if  $\varepsilon_i \neq 0, 1$  then  $\varepsilon_i = t_j$  for some  $j < i$ , and the type of  $t_i$  is defined to be equal to the type of  $t_j$ . Since the indices decrease, the type is well-defined.

We associate a cell-function  $F_i$  to each factor  $(t_i - \varepsilon_i)$  in the denominator of  $\Omega(\underline{\varepsilon})$  as follows:

$$F_i = \begin{cases} \langle 0, 1, t_i, \infty \rangle & \text{if } \varepsilon_i = 1 \\ -\langle 0, 1, \infty, t_i \rangle & \text{if } \varepsilon_i = 0 \\ \langle 0, 1, \varepsilon_i, t_i, \infty \rangle & \text{if } \varepsilon_i \neq 1 \text{ and the type } \tau(t_i) = 1 \\ -\langle 0, 1, \infty, t_i, \varepsilon_i \rangle & \text{if } \varepsilon_i \neq 0 \text{ and the type } \tau(t_i) = 0. \end{cases} \quad (3.2.16)$$

We have

$$\Omega(\underline{\varepsilon}) = \Delta \prod_{i=1}^{\ell} F_i,$$

where

$$\Delta = \prod_{j|\varepsilon_j \neq 0,1} (-1)^{\tau(\varepsilon_j)-1} (\varepsilon_j - \tau(\varepsilon_j))$$

is exactly the factor occurring when multiplying cell-functions as in proposition 3.13. This product can be expressed as a shuffle product, which is a sum of cell-functions. Furthermore each one corresponds to a cell beginning  $0, 1, \dots$  since this is the case for all of the  $F_i$ . The 01-cell forms thus span  $H^\ell(\mathfrak{M}_{0,n}, \mathbb{Q})$ . Since there are exactly  $(n-2)!$  of them, and since  $\dim H^\ell(\mathfrak{M}_{0,n}, \mathbb{Q}) = (n-2)!$ , they must form a basis.  $\square$

### 3.2.3 Pairs of polygons and multiplication

Let  $S = \{1, \dots, n\}$ , and let  $\mathcal{P}_S$  denote the  $\mathbb{Q}$ -vector space generated by the set of cyclic structures  $\gamma$  on  $S$ , modulo the relation  $\gamma = (-1)^n \overleftarrow{\gamma}$ , where  $\overleftarrow{\gamma}$  denotes the cyclic structure with the opposite orientation to  $\gamma$ .

#### Shuffles of polygons

Let  $T_1, T_2$  denote two subsets of  $Z = \{z_1, \dots, z_n\}$  satisfying:

$$T_1 \cup T_2 = Z \quad (3.2.17)$$

$$|T_1 \cap T_2| = 3. \quad (3.2.18)$$

Let  $E = \{z_{i_1}, z_{i_2}, z_{i_3}\}$  denote the set of three points common to  $T_1$  and  $T_2$ . Given two cyclic structures  $\gamma_1, \gamma_2$  on  $T_1, T_2$  respectively, the restriction  $\gamma_1|_E$  gives a cyclic order on  $E$ . Let  $\varepsilon = 1$  if this order is compatible with the standard order on  $\{1, \dots, n\}$ ,  $\varepsilon = -1$  otherwise. We define the *shuffle*  $\gamma_1 \sqcap \gamma_2$  of  $\gamma_1$  and  $\gamma_2$  relative to the three points of intersection of  $T_1$  and  $T_2$  by the formula

$$\varepsilon(\gamma_1 \sqcap \gamma_2) = \begin{cases} \sum_{\gamma|_{T_1}=\gamma_1, \gamma|_{T_2}=\gamma_2} \gamma & \text{if } \gamma_1|_E = \gamma_2|_E \\ \sum_{\gamma|_{T_1}=\gamma_1, \gamma|_{T_2}=\overleftarrow{\gamma_2}} (-1)^{|T_2|} \gamma & \text{if } \gamma_1|_E = \overleftarrow{\gamma_2}|_E. \end{cases} \quad (3.2.19)$$

The formal sum of polygons  $\gamma_1 \sqcup \gamma_2$  is well-defined and non-zero.

Assume  $\{z_1, \dots, z_n\} = \{0, 1, \infty, t_1, \dots, t_\ell\}$  with  $E = \{0, 1, \infty\}$ . Assume also that  $\gamma_1 = (0, A_{1,2}, 1, A_{2,3}, \infty, A_{3,1})$  where  $T_1$  is the disjoint union of  $A_{1,2}, A_{2,3}, A_{3,1}$ , and  $0, 1, \infty$ , and  $\gamma_2 = (0, B_{1,2}, 1, B_{2,3}, \infty, B_{3,1})$ , where  $T_2$  is the disjoint union of  $B_{1,2}, B_{2,3}, B_{3,1}$ , and  $0, 1, \infty$ . Then  $\gamma_1 \sqcup \gamma_2$  is the sum of cyclic structures

$$\gamma = (0, C_{1,2}, 1, C_{2,3}, \infty, C_{3,1}),$$

where each  $C_{i,j}$  is a shuffle of the ordered disjoint sets  $A_{i,j}$  and  $B_{i,j}$ .

**Example 3.18.** Let  $T_1 = \{0, 1, \infty, t_1, t_3\}$  and  $T_2 = \{0, 1, \infty, t_2, t_4\}$ . If  $\gamma_1$  and  $\gamma_2$  denote the cyclic orders  $(0, t_1, 1, t_3, \infty)$  and  $(0, 1, t_2, \infty, t_4)$ , then we have

$$\gamma_1 \sqcup \gamma_2 = (0, t_1, 1, t_2, t_3, \infty, t_4) + (0, t_1, 1, t_3, t_2, \infty, t_4).$$

We will often write, for example,  $(0, t_1, 1, t_2 \sqcup t_3, \infty, t_4)$  for the right-hand side.

### Multiplying pairs of polygons: the modular shuffle relation

We will now consider pairs of polygons  $(\gamma, \eta) \in \mathcal{P}_S \times \mathcal{P}_S$ . We can associate a geometric meaning to a pair of polygons as follows. The left-hand polygon  $\gamma$ , which we will write using round parentheses, for example  $(0, t_1, \dots, t_\ell, 1, \infty)$ , is associated to the real cell  $X_\gamma$  of the moduli space  $\mathfrak{M}_{0,n}$  associated to the cyclic structure. The right-hand polygon  $\eta$ , which we will write using square parentheses, for example  $[0, t_1, \dots, t_\ell, 1, \infty]$ , is associated to the cell-form  $\omega_\eta$  associated to the cyclic structure. The pair of polygons will be associated to the (possibly divergent) integral  $\int_{X_\gamma} \omega_\eta$ . In the following section we will investigate in detail the map from pairs of polygons to integrals.

**Definition 3.19.** Given sets  $T_1, T_2$  as above, the modular shuffle relation on the vector space  $\mathcal{P}_S \times \mathcal{P}_S$  is defined by

$$(\gamma_1, \eta_1) \sqcup (\gamma_2, \eta_2) = (\gamma_1 \sqcup \gamma_2, \eta_1 \sqcup \eta_2), \quad (3.2.20)$$

for pairs of polygons  $(\gamma_1, \eta_1) \sqcup (\gamma_2, \eta_2)$ , where  $\gamma_i$  and  $\eta_i$  are cyclic structures on  $T_i$  for  $i = 1, 2$ .

**Example 3.20.** The following product of two polygon pairs is given by

$$\begin{aligned} & ((0, t_1, 1, \infty, t_4), [0, \infty, t_1, t_4, 1]) ((0, t_2, 1, t_3, \infty), [0, t_3, t_2, \infty, 1]) \\ & = -((0, t_1 \sqcup t_2, 1, t_3, \infty, t_4), [0, t_3, t_2, \infty, t_1, t_4, 1]). \end{aligned}$$

Let us now give a geometric interpretation of (3.2.20) in terms of integrals of forms on moduli space. Recall that a *product map* between moduli spaces was defined in [Br] as follows. Let  $T_1, T_2$  denote two subsets of  $Z = \{z_1, \dots, z_n\}$  satisfying:

$$T_1 \cup T_2 = Z \quad (3.2.21)$$

$$|T_1 \cap T_2| = 3. \quad (3.2.22)$$

Then we can consider the product of forgetful maps:

$$f = f_{T_1} \times f_{T_2} : \mathfrak{M}_{0,n} \longrightarrow \mathfrak{M}_{0,T_1} \times \mathfrak{M}_{0,T_2}. \quad (3.2.23)$$

The map  $f$  is a birational embedding because

$$\dim \mathfrak{M}_{0,S} = |S| - 3 = |T_1| - 3 + |T_2| - 3 = \dim \mathfrak{M}_{0,T_1} \times \mathfrak{M}_{0,T_2} .$$

If  $f$  is a product map as above and  $z_i, z_j, z_k$  are the three common points of  $T_1$  and  $T_2$ , use an element  $\alpha \in \mathbb{P}\mathrm{SL}_2$  to map  $z_i$  to 0,  $z_j$  to 1 and  $z_k$  to  $\infty$ . Let  $t_1, \dots, t_\ell$  denote the images of  $z_1, \dots, z_n$  (excluding  $z_i, z_j, z_k$ ) under  $\alpha$ . Given the indices  $i, j$  and  $k$ , the product map is then determined by specifying a partition of  $\{t_1, \dots, t_\ell\}$  into  $S_1$  and  $S_2$ . We use the notation  $T_i = \{0, 1, \infty\} \cup S_i$  for  $i = 1, 2$ .

The multiplication formula (3.2.20) on pairs of polygons translates to a multiplication formula for integrals of cell-forms.

**Proposition 3.21.** *Let  $S = \{1, \dots, n\}$ , and let  $T_1$  and  $T_2$  be subsets of  $S$  as above, of orders  $r + 3$  and  $s + 3$  respectively. Let  $\omega_1$  (resp.  $\omega_2$ ) be a cell-form on  $\mathfrak{M}_{0,r}$  (resp. on  $\mathfrak{M}_{0,s}$ ), and let  $\gamma_1$  and  $\gamma_2$  denote cyclic orderings on  $T_1$  and  $T_2$ . Then the product rule for integrals is given by*

$$\int_{X_{\gamma_1}} \omega_1 \int_{X_{\gamma_2}} \omega_2 = \int_{X_{\gamma_1 \amalg \gamma_2}} \omega_1 \amalg \omega_2, \quad (3.2.24)$$

where  $\omega_1 \amalg \omega_2$  converges on the cell  $X_\gamma$  for each term  $\gamma$  in  $\gamma_1 \amalg \gamma_2$ .

*Proof.* The subsets  $T_1$  and  $T_2$  correspond to a product map

$$f : \mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,r} \times \mathfrak{M}_{0,s}.$$

The pullback formula gives a multiplication law on the pair of integrals:

$$\int_{X_{\gamma_1}} \omega_1 \int_{X_{\gamma_2}} \omega_2 = \int_{X_{\gamma_1} \times X_{\gamma_2}} \omega_1 \wedge \omega_2 = \int_{f^{-1}(X_{\gamma_1} \times X_{\gamma_2})} f^*(\omega_1 \wedge \omega_2). \quad (3.2.25)$$

The preimage  $f^{-1}(X_{\gamma_1} \times X_{\gamma_2})$  decomposes into a disjoint union of cells of  $\mathfrak{M}_{0,n}$ , which are precisely the cells given by cyclic orders of  $\gamma_1 \amalg \gamma_2$ . In other words,

$$f^{-1}(X_{\gamma_1} \times X_{\gamma_2}) = \sum_{\gamma \in \gamma_1 \amalg \gamma_2} X_\gamma,$$

where the sum denotes a disjoint union. Now we can assume without loss of generality that  $T_1 = \{0, 1, \infty, t_1, \dots, t_k\}$ ,  $T_2 = \{0, 1, \infty, t_{k+1}, \dots, t_\ell\}$  and that  $\delta_1, \delta_2$  are the cyclic structures on  $T_1, T_2$  corresponding to  $\omega_1, \omega_2$ , respectively, where  $\delta_1, \delta_2$  restrict to the standard cyclic order on  $0, 1, \infty$ . Then, in cell function notation,

$$f^*(\omega_1 \wedge \omega_2) = \langle \delta_1 \rangle \langle \delta_2 \rangle dt_1 \dots dt_\ell = \frac{\langle \delta_1 \amalg_{\{0,1,\infty\}} \delta_2 \rangle}{\langle 0, 1, \infty \rangle} dt_1 \dots dt_\ell = \omega_1 \amalg \omega_2,$$

by proposition 3.13. Since  $\omega_1$  and  $\omega_2$  converge on the closed cells  $\overline{X}_{\gamma_1}$  and  $\overline{X}_{\gamma_2}$  respectively,  $\omega_1 \wedge \omega_2$  has no poles on the contractible set  $\overline{X}_{\gamma_1} \times \overline{X}_{\gamma_2}$ , and therefore  $\omega_1 \amalg \omega_2 = f^*(\omega_1 \wedge \omega_2)$  has no poles on the closure of  $f^{-1}(X_{\gamma_1} \times X_{\gamma_2})$ . But  $\sum_{\gamma \in \gamma_1 \amalg \gamma_2} X_\gamma$  is a cellular decomposition of  $f^{-1}(X_{\gamma_1} \times X_{\gamma_2})$ , so, in particular,  $\omega_1 \amalg \omega_2$  can have no poles along the closure of each cell  $X_\gamma$ , where  $\gamma \in \gamma_1 \amalg \gamma_2$ .  $\square$

### $\mathfrak{S}(n)$ action on pairs of polygons

The symmetric group  $\mathfrak{S}(n)$  acts on a pair of polygons by permuting their labels in the obvious way, and this extends to the vector space  $\mathcal{P}_S \times \mathcal{P}_S$  by linearity. If  $\tau : \mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,n}$  is an element of  $\mathfrak{S}(n)$ , then the corresponding action on integrals is given by the pullback formula:

$$\int_{X_\gamma} \omega_\eta = \int_{\tau(X_\gamma)} \tau^*(\omega_\eta) = \int_{X_{\tau(\gamma)}} \omega_{\tau(\eta)}. \quad (3.2.26)$$

Note that, unlike for formal pairs of polygons, this formula only holds for linear combinations of cell-forms which are convergent, even if each individual cell-form is not convergent over the integration domain.

Suppose that  $\tau$  belongs to the dihedral group which preserves the dihedral structure underlying a cyclic structure  $\gamma$ . Let  $\epsilon = 1$  if  $\tau$  preserves  $\gamma$ , and  $\epsilon = -1$  if  $\tau$  reverses its orientation. We have the following *dihedral relation* between convergent integrals:

$$\int_{X_\gamma} \omega_\eta = (-1)^\epsilon \int_{X_\gamma} \tau^*(\omega_\eta) = (-1)^\epsilon \int_{X_\gamma} \omega_{\tau(\eta)}. \quad (3.2.27)$$

As above, this formula extends to linear combinations of integrals of cell-forms as long as the linear combination converges over the integration domain.

**Example 3.22.** *The form corresponding to  $\zeta(2, 1)$  on  $\mathfrak{M}_{0,6}$  is*

$$\frac{dt_1 dt_2 dt_3}{(1-t_1)(1-t_2)t_3} = [0, 1, t_1, t_2, \infty, t_3] + [0, 1, t_2, t_1, \infty, t_3],$$

*which gives  $\zeta(2, 1)$  after integrating over the standard cell. By applying the rotation (1,2,3,4,5,6), a dihedral rotation of the standard cell, to this form, one obtains*

$$\begin{aligned} [t_1, \infty, t_2, t_3, 0, 1] + [t_1, \infty, t_3, t_2, 0, 1] &= [0, 1, t_1, \infty, t_2, t_3] + [0, 1, t_1, \infty, t_3, t_2] \\ &= \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2 t_3}, \end{aligned}$$

*which gives  $\zeta(3)$  after integrating over the standard cell. Therefore, we have the following relation on linear combinations of pairs of polygons:*

$$\begin{aligned} &((0, t_1, t_2, t_3, 1, \infty), [0, 1, t_1, t_2, \infty, t_3] + [0, 1, t_2, t_1, \infty, t_3]) \\ &= ((0, t_1, t_2, t_3, 1, \infty), [0, 1, t_1, \infty, t_2, t_3] + [0, 1, t_1, \infty, t_3, t_2]) \end{aligned} \quad (3.2.28)$$

or

$$\zeta(2, 1) = \int_{X_{3,\delta}} \frac{dt_1 dt_2 dt_3}{t_3(1-t_2)(1-t_1)} = \int_{X_{3,\delta}} \frac{dt_1 dt_2 dt_3}{t_3 t_2 (1-t_1)} = \zeta(3).$$

### Standard pairs and the product map relations

A standard pair of polygons is a pair  $(\delta, \eta)$  where the left-hand polygon is the standard cyclic structure. Let  $S = \{1, \dots, n\}$ , and  $T_1 \cup T_2 = S$  with  $T_1 \cap T_2 = \{0, 1, \infty\}$  be as above, and let  $\gamma_1$  and  $\gamma_2$  be cyclic orders on  $T_1$  and  $T_2$ . In the present section we show

how for each such  $\gamma_1, \gamma_2$ , we can modify the modular shuffle relation to construct a multiplication law on standard pairs.

Let  $\delta_1$  and  $\delta_2$  denote the standard orders on  $T_1$  and  $T_2$ . Then there is a unique permutation  $\tau_i$  mapping  $\delta_i$  to  $\gamma_i$  such that  $\tau_i(0) = 0$ , for  $i = 1, 2$ . The multiplication law, denoted by the symbol  $\times$ , and called the *product map relation*, is defined by

$$\begin{aligned} (\delta_1, \omega_1) \times (\delta_2, \omega_2) &= (\gamma_1, \tau_1(\omega_1)) \mathfrak{III}(\gamma_2, \tau_2(\omega_2)) \\ &= (\gamma_1 \mathfrak{III} \gamma_2, \tau_1(\omega_1) \mathfrak{III} \tau_2(\omega_2)) \\ &= \sum_{\gamma \in \gamma_1 \mathfrak{III} \gamma_2} (\delta, \tau_\gamma^{-1}(\tau_1(\omega_1) \mathfrak{III} \tau_2(\omega_2))), \end{aligned} \quad (3.2.29)$$

where for each  $\gamma \in \gamma_1 \mathfrak{III} \gamma_2$ ,  $\tau_\gamma$  is the unique permutation such that  $\tau_\gamma(\delta) = \gamma$  and  $\tau_\gamma(0) = 0$ .

**Example 3.23.** Let  $S = \{0, 1, \infty, t_1, t_2, t_3, t_4\}$ ,  $T_1 = \{0, 1, \infty, t_1, t_4\}$  and  $T_2 = \{0, 1, \infty, t_2, t_3\}$ . Let the cyclic orders on  $T_1$  and  $T_2$  be given by  $\gamma_1 = (0, t_1, 1, \infty, t_4)$  and  $\gamma_2 = (0, t_2, 1, t_3, \infty)$ . Applying the product map relation to the pairs of polygons below yields

$$\begin{aligned} &((0, t_1, t_4, 1, \infty), [0, 1, t_1, \infty, t_4]) \times ((0, t_2, t_3, 1, \infty), [0, 1, t_2, \infty, t_3]) \\ &= ((0, t_1, 1, \infty, t_4), [0, \infty, t_1, t_4, 1]) \mathfrak{III}((0, t_2, 1, t_3, \infty), [0, t_3, t_2, \infty, 1]) \\ &= -((0, t_1, t_2, 1, t_3, \infty, t_4), [0, t_3, t_2, \infty, t_1, t_4, 1]) \\ &\quad - ((0, t_2, t_1, 1, t_3, \infty, t_4), [0, t_3, t_2, \infty, t_1, t_4, 1]) \\ &= ((0, t_1, t_2, t_3, t_4, 1, \infty), [0, t_3, \infty, t_1, 1, t_2, t_4]) + [0, t_3, \infty, t_2, 1, t_1, t_4]. \end{aligned} \quad (3.2.30)$$

In terms of integrals, this corresponds to the relation

$$\begin{aligned} \zeta(2)^2 &= \int_{X_{5,\delta}} \frac{dt_1 dt_4}{(1-t_1)t_4} \int_{X_{5,\delta}} \frac{dt_2 dt_3}{(1-t_2)t_3} \\ &= \int_{X_{7,\delta}} \frac{dt_1 dt_2 dt_3 dt_4}{t_4(t_4-t_2)(1-t_2)(1-t_1)t_3} + \frac{dt_1 dt_2 dt_3 dt_4}{t_4(t_4-t_1)(1-t_1)(1-t_2)t_3} \end{aligned} \quad (3.2.31)$$

We will show in §3.4.4 that the last two integrals evaluate to  $\frac{7}{10}\zeta(2)^2$  and  $\frac{3}{10}\zeta(2)^2$  respectively.

### 3.2.4 The algebra of cell-zeta values

**Definition 3.24.** Let  $\mathcal{C}$  denote the  $\mathbb{Q}$ -vector space generated by the integrals  $\int_{X_{n,\delta}} \omega$ , where  $X_{n,\delta}$  denotes the standard cell of  $\mathfrak{M}_{0,n}$  for  $n \geq 5$  and  $\omega$  is a holomorphic  $\ell$ -form on  $\mathfrak{M}_{0,n}$  with logarithmic singularities at infinity (thus a linear combination of 01 cell-forms) which converges on  $X_{n,\delta}$ . We call these numbers cell-zeta values. The existence of product map multiplication laws in proposition 3.21 imply that  $\mathcal{C}$  is in fact a  $\mathbb{Q}$ -algebra.

**Theorem 3.25.** The  $\mathbb{Q}$ -algebra  $\mathcal{C}$  of cell-zeta values is isomorphic to the  $\mathbb{Q}$ -algebra  $\mathcal{Z}$  of multizeta values.

*Proof.* Multizeta values are real numbers which can all be expressed as integrals  $\int_{X_{n,\delta}} \omega$  where  $\omega$  is an  $\ell$ -form of the form

$$\omega = (-1)^d \prod_{i=1}^{\ell} \frac{dt_i}{t_i - \epsilon_i}, \quad (3.2.32)$$

where  $\epsilon_1 = 0$ ,  $\epsilon_i \in \{0, 1\}$  for  $2 \leq i \leq \ell - 1$ ,  $\epsilon_\ell = 1$ , and  $d$  denotes the number of  $i$  such that  $\epsilon_i = 1$ . Since each such form converges on  $X_{n,\delta}$ , the multizeta algebra  $\mathcal{Z}$  is a subalgebra of  $\mathcal{C}$ . The converse is a consequence of the following theorem due to F. Brown [Br].

**Theorem 3.26.** *If  $\omega$  is a holomorphic  $\ell$ -form on  $\mathfrak{M}_{0,n}$  with logarithmic singularities at infinity and convergent on  $X_{n,\delta}$ , then  $\int_{X_{n,\delta}} \omega$  is  $\mathbb{Q}$ -linear combination of multizeta values.*

Thus,  $\mathcal{C}$  is also a subalgebra of  $\mathcal{Z}$ , proving the equality. □

The structure of the multizeta algebra, or rather, of the formal version of it given by quotienting the algebra of symbols formally representing integrals of the form (3.2.32) by the main known relations between these forms (shuffle and stuffle), has been much studied of late. The present article provides a different approach to the study of this algebra, by turning instead to the study of a formal version of  $\mathcal{C}$ .

**Definition 3.27.** *Let  $|S| \geq 5$ . The formal algebra of cell-zeta values  $\mathcal{FC}$  is defined as follows. Let  $\mathcal{A}$  be the vector space of formal linear combinations of standard pairs of polygons in  $\mathcal{P}_S \times \mathcal{P}_S$*

$$\sum_i a_i(\delta, \omega_i)$$

such that the associated  $\ell$ -form  $\sum_i a_i \omega_i$  converges on the standard cell  $X_{n,\delta}$ . Let  $\mathcal{FC}$  denote the quotient of  $\mathcal{A}$  by the following three families of relations.

- *Product map relations.* These relations were defined in section 3.2.3. For every choice of subsets  $T_1, T_2$  of  $S = \{1, \dots, n\}$  such that  $T_1 \cup T_2 = S$  and  $|T_1 \cap T_2| = 3$ , and every choice of cyclic orders  $\gamma_1, \gamma_2$  on  $T_1, T_2$ , formula (3.2.29) gives a multiplication law expressing the product of any two standard pairs of polygons of sizes  $|T_1|$  and  $|T_2|$  as a linear combination of standard pairs of polygons of size  $n$ .
- *Dihedral relations.* For  $\sigma$  in the dihedral group associated to  $\delta$ , i.e.  $\sigma(\delta) = \pm\delta$ , there is a dihedral relation  $(\delta, \omega) = (\sigma(\delta), \sigma(\omega))$ .
- *Shuffles with respect to one element.* The linear combinations of pairs of polygons

$$(\delta, (A, e) \amalg (B, e))$$

where  $A$  and  $B$  are disjoint of length  $n - 1$  are zero, as in (3.2.13).

With the goal of approaching the combinatorial conjectures given in the introduction, the purpose of the next chapters is to give an explicit combinatorial description of a set of generators for  $\mathcal{FC}$ . We do this in two steps. First we define the notion of a linear combination of polygons convergent with respect to a chord of the standard polygon  $\delta$ , and thence, the notion of a linear combination of polygon convergent with respect to the standard polygon. We exhibit an explicit basis, the basis of *Lyndon insertion words and shuffles* for the subspace of such linear combinations. In the subsequent chapter, we deduce from this a set of generators for the formal cell-zeta value algebra  $\mathcal{FC}$  and also, as a corollary, a basis for the cohomology  $H^\ell(\mathfrak{M}_{0,n}^\delta)$ , where  $\mathfrak{M}_{0,n}^\delta$  denotes the union of  $\mathfrak{M}_{0,n}$  with the boundary components containing the boundary of the standard cell.

### 3.3 Polygons and convergence

In the present chapter, we define the notions of bad chord of a polygon (a generalization of the notion of a divisor on the boundary of the standard cell of  $\mathfrak{M}_{0,n}$  along which the differential form diverges), residue of a polygon along a bad chord, convergence of linear combinations of polygons along bad chords, and finally, convergence of linear combinations of polygons with respect to the standard polygon  $\delta$ . The main theorem exhibits an explicit basis for the space of linear combinations of polygons convergent with respect to the standard polygon, consisting of linear combinations called *Lyndon insertion words and shuffles*.

#### 3.3.1 Bad chords and polygon convergence

For any finite set  $R$ , let  $\mathcal{P}_R$  denote the  $\mathbb{Q}$  vector space of *polygons on  $R$* , i.e. cyclic structures on  $R$ , identified with planar polygons with edges indexed by  $R$ .

Let  $\mathcal{V}$  denote the free polynomial shuffle algebra on the alphabet of positive integers, and let  $V$  be the quotient of  $\mathcal{V}$  by the relations  $w = 0$  if  $w$  is a word in which any letter appears more than once (these relations imply that  $w \sqcup w' = 0$  if  $w$  and  $w'$  are not disjoint). The Lyndon basis for  $\mathcal{V}$  is given by Lyndon words and shuffles of Lyndon words. The elements of this basis which do not map to zero remain linearly independent in  $V$ , whose basis consists of Lyndon words with distinct letters – such a word is Lyndon if and only if the smallest character appears on the left – and shuffles of disjoint Lyndon words with distinct letters. Throughout this chapter, we work in  $V$ , so that when we refer to a ‘word’, we automatically mean a word with distinct letters, and shuffles of such words are zero unless the words are disjoint. Let  $V_S$  be the subspace of  $V$  spanned by the  $n!$  words of length  $n$  with distinct letters in the characters of  $S = \{1, \dots, n\}$ . Then the Lyndon basis for  $V_S$  is given by Lyndon words of degree  $n$  and shuffles of disjoint Lyndon words the union of whose letters is equal to  $S$ .

The vector space  $\mathcal{P}_S$  is generated by  $n$ -polygons with edges indexed by  $S$ . If we consider  $(n + 1)$ -polygons with edges indexed by  $S \cup \{d\}$ , we have a natural isomorphism

$$V_S \xrightarrow{\sim} \mathcal{P}_{S \cup \{d\}} \quad (3.3.1)$$

given by writing each cyclic structure on  $S \cup \{d\}$  as a word on the letters of  $S$  followed by the letter  $d$ . Let  $I_S \subset \mathcal{P}_{S \cup \{d\}}$  be the set of shuffles of polygons  $(A \sqcup B, d)$  where  $A \cup B = S$  and  $A \cap B = \emptyset$ . Then under the isomorphism above,  $I_S$  is identified with the subspace of  $V_S$  generated by the part of the Lyndon basis consisting of shuffles. By a slight abuse of notation, we use the same notation  $I_S$  for both the subspace of  $\mathcal{P}_{S \cup \{d\}}$  and that of  $V_S$ .

**Definition 3.28.** Let  $D = S_1 \cup S_2$  denote a stable partition of  $S$  (partition into two disjoint subsets of cardinal  $\geq 2$ ). Let  $\gamma$  be a polygon on  $S$ . We say that the partition  $D$  corresponds to a chord of  $\gamma$  if the polygon  $\gamma$  admits a chord which cuts  $\gamma$  into two pieces indexed by  $S_1$  and  $S_2$ . Let a block of  $\gamma$  be a subsequence of consecutive elements of  $\gamma$  for the cyclic order, of length at least two and at most  $n - 2$ . Thus, a chord divides  $\gamma$  into two blocks, and  $\chi(\gamma)$  indexes the set of stable partitions which are compatible with  $\gamma$ , in the sense that they can be realized as chords of  $\gamma$ , i.e. in the sense that the subsets  $S_1$  and  $S_2$  are blocks of  $\gamma$ .



**Definition 3.29.** Let  $\gamma, \eta$  denote two polygons on  $S$ . We say that  $\eta$  is convergent relative to  $\gamma$  if there are no stable partitions of  $S$  compatible with both  $\gamma$  and  $\eta$ :

$$\chi(\gamma) \cap \chi(\eta) = \emptyset. \quad (3.3.2)$$

In other words, there exists no block of  $\gamma$  having the same underlying set as a block of  $\eta$ . If  $\eta$  is a polygon on  $S$ , then a block of  $\eta$  is said to be a consecutive block if its underlying set corresponds to a block of the polygon with the standard cyclic order  $\delta$ . The polygon  $\eta$  is said to be convergent if it has no consecutive blocks at all, i.e., if it is convergent relative to  $\delta$ . Similarly, a polygon  $\eta \in \mathcal{P}_{S \cup \{d\}}$  is said to be convergent if it has no chords partitioning  $S \cup \{d\}$  into disjoint subsets  $S_1 \cup S_2$  such that  $S_1$  is a consecutive subset of  $S = \{1, \dots, n\}$ .

**Definition 3.30.** We now adapt the definition of convergence for polygons in  $\mathcal{P}_{S \cup \{d\}}$  to the corresponding words in  $V_S$ . A convergent word in the alphabet  $S$  is a word having no subword which forms a consecutive block. In other words, if  $w = a_{i_1} a_{i_2} \cdots a_{i_r}$ , then  $w$  is convergent if it has no subword  $a_{i_j} a_{i_{j+1}} \cdots a_{i_k}$  such that the underlying set  $\{a_{i_j}, a_{i_{j+1}}, \dots, a_{i_k}\} = \{i, i+1, \dots, i+r\} \subset \{1, \dots, n\}$ . A convergent word is in fact the image in  $V_S$  of a convergent polygon in  $\mathcal{P}_{S \cup \{d\}}$  under the isomorphism (3.3.1).

**Example 3.31.** When  $1 \leq n \leq 4$  there are no convergent polygons in  $\mathcal{P}_S$ . For  $n = 5$ , there is only one convergent polygon up to sign, given by  $\gamma = (13524)$ . The other convergent cyclic structure (14253) is just the cyclic structure (13524) written backwards. When  $n = 6$ , there are three convergent polygons up to sign:

$$(135264), \quad (152463), \quad (142635).$$

There are 23 convergent polygons for  $n = 7$ . Note that when  $n = 8$ , the dihedral structure  $\eta = (24136857)$  is not convergent even though no neighbouring numbers are adjacent, because  $\{1, 2, 3, 4\}$  forms a consecutive block for both  $\eta$  and  $\delta$ .

**Remark 3.32.** The enumeration of permutations satisfying the single condition that no two adjacent elements in  $\gamma$  should be consecutive (the case  $k = 2$ ) is known as the dinner table problem and is a classic problem in enumerative combinatorics. The more general problem of convergent words (arbitrary  $k$ ) seems not to have been studied previously. The problems coincide for  $n \leq 7$ , but the counterexample for  $n = 8$  above shows that the problems are not equivalent for  $n \geq 8$ .

### 3.3.2 Residues of polygons along chords

For every stable partition  $D$  of  $S$  given by  $S = S_1 \cup S_2$ , we define a residue map on polygons

$$\text{Res}_D^p : \mathcal{P}_S \longrightarrow \mathcal{P}_{S_1 \cup \{d\}} \otimes_{\mathbb{Q}} \mathcal{P}_{S_2 \cup \{d\}}$$

as follows. Let  $\eta$  be a polygon in  $\mathcal{P}_S$ . If the partition  $D$  corresponds to a chord of  $\eta$ , then it cuts  $\eta$  into two subpolygons  $\eta_i$  ( $i = 1, 2$ ) whose edges are indexed by the set  $S_i$  and an edge labelled  $d$  corresponding to the chord  $D$ . We set

$$\text{Res}_D^p(\eta) = \begin{cases} \eta_1 \otimes \eta_2 & \text{if } D \text{ is a chord of } \eta \\ 0 & \text{if } D \text{ is not a chord of } \eta. \end{cases} \quad (3.3.3)$$



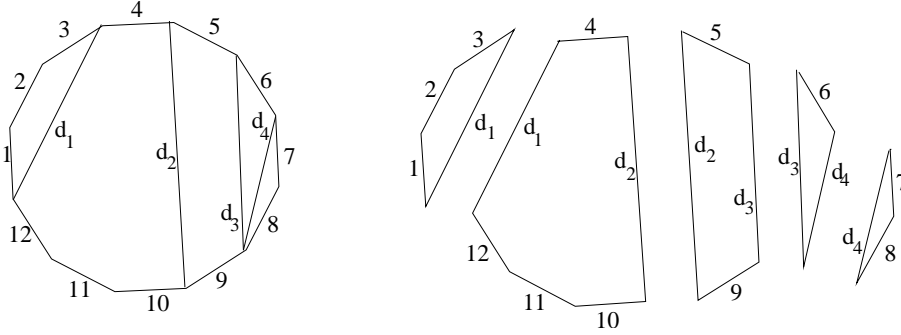
More generally, we can define the residue for several disjoint chords simultaneously. Let  $S = S_1 \cup \dots \cup S_{r+1}$  be a partition of  $S$  into  $r + 1$  disjoint subsets with  $r \geq 2$ . For  $1 \leq i \leq r$ , let  $D_i$  be the partition of  $S$  into the two subsets  $(S_1 \cup \dots \cup S_i) \cup (S_{i+1} \cup \dots \cup S_{r+1})$ . For any polygon  $\eta \in \mathcal{P}_S$ , we say that  $\eta$  admits the chords  $D_1, \dots, D_r$  if there exist  $r$  chords of  $\eta$ , disjoint except possibly for endpoints, partitioning the edges of  $\eta$  into the sets  $S_1, \dots, S_{r+1}$ . If  $\eta$  admits the chords  $D_1, \dots, D_r$ , then these chords cut  $\eta$  into  $r + 1$  subpolygons  $\eta_1, \dots, \eta_{r+1}$ . Let  $T_i$  denote the set indexing the edges of  $\eta_i$ , so that each  $T_i$  is a union of  $S_i$  and elements of the set  $\{d_1, \dots, d_r\}$  of indices of the chords. The composed residue map

$$\text{Res}_{D_1, \dots, D_r}^p : \mathcal{P}_S \rightarrow \mathcal{P}_{T_1} \otimes \dots \otimes \mathcal{P}_{T_r}$$

is defined as follows:

$$\text{Res}_{D_1, \dots, D_r}^p(\eta) = \begin{cases} \eta_1 \otimes \dots \otimes \eta_{r+1} & \text{if } \eta \text{ admits } D_1, \dots, D_r \text{ as chords} \\ 0 & \text{if } \eta \text{ does not admit } D_1, \dots, D_r \end{cases} \quad (3.3.4)$$

**Example 3.33.** In this example,  $n = 12$  and the partition of  $S$  given by  $D_1, D_2, D_3$  and  $D_4$  is  $S_1 = \{1, 2, 3\}, S_2 = \{4, 10, 11, 12\}, S_3 = \{5, 9\}, S_4 = \{6\}, S_5 = \{7, 8\}$ .



We have  $T_1 = S_1 \cup \{d_1\}, T_2 = S_2 \cup \{d_1, d_2\}, T_3 = S_3 \cup \{d_2, d_3\}, T_4 = S_4 \cup \{d_3, d_4\}, T_5 = S_5 \cup \{d_4\}$ . The composed residue map  $\text{Res}_{D_1, D_2, D_3, D_4}^p$  maps the standard polygon  $\delta = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$  to the tensor product of the five subpolygons shown in the figure.

The definition of the residue allows us to extend the definition of convergence of a polygon to linear combinations of polygons.

**Definition 3.34.** Let  $E$  be a partition of  $S \cup \{d\}$  into two subsets, one of which,  $T$ , is a consecutive subset of  $S$ . Let  $\eta = \sum_i a_i \eta_i$  be a linear combination of polygons. We say that  $E$  is a bad chord for  $\eta$  if it is a bad chord for any of the  $\eta_i$ . The linear combination  $\eta$  converges along  $E$  (or along  $T$ ) if the residue

$$\text{Res}_E^p(\eta) \in I_T \otimes \mathcal{P}_{S \setminus T \cup \{d\} \cup \{e\}}, \quad (3.3.5)$$

where we recall that  $I_T \subset \mathcal{P}_{T \cup \{e\}}$  is the subspace spanned by shuffles  $(A \amalg B, e)$  where  $A$  and  $B$  are disjoint words the union of whose letters is equal to  $T$ . A linear combination  $\eta$  is convergent if it converges along all of its bad chords.

The goal of the following section is to define a set of particular linear combinations of polygons, the Lyndon insertion words and shuffles, which are convergent, and show that they are linearly independent. In the section after that, we will prove that this set forms a basis for the convergent subspace of  $\mathcal{P}_{S \cup \{d\}}$ .

### 3.3.3 The Lyndon insertion subspace

Let a  $1n$ -word be a word of length  $n$  in the distinct letters of  $S = \{1, \dots, n\}$  in which the letter 1 appears just to the left of the letter  $n$ , and let  $W_S \subset V_S \simeq \mathcal{P}_{S \cup \{d\}}$  denote the subspace generated by these words. The space  $W_S$  is of dimension  $(n-1)!$ . The following lemma will show that  $V_S = W_S \oplus I_S$ , where  $I_S$  is the subspace of shuffles as before.

**Lemma 3.35.** *Fix two elements  $a_1$  and  $a_2$  of  $S = \{1, \dots, n\}$ . Let*

$$\tau = \sum_i c_i \eta_i,$$

where the  $\eta_i$  run over the words of length  $n$  in  $V_S$  such that  $a_1$  appears just to the left of  $a_2$ . Then  $\tau \in I_S$  if and only if  $c_i = 0$  for all  $i$ .

*Proof.* The assumption  $\tau \in I_S$  means that we can write  $\tau = \sum_i c_i u_i \text{III} v_i$ . Considering this in the space  $\mathcal{P}_{S \cup \{d\}}$  isomorphic to  $V_S$ , it is a sum of cyclic structures  $\sum_i c_i (u_i, d) \text{III} (v_i, d)$  shuffled with respect to the point  $d$ . Choose any bijection

$$\rho : \{1, \dots, n, d\} \rightarrow \{0, 1, \infty, t_1, \dots, t_{n-2}\}$$

that maps  $a_1$  to 0 and  $a_2$  to 1. Define a linear map from  $\mathcal{P}_{S \cup \{d\}}$  to  $H^{n-2}(\mathfrak{M}_{0, n+1})$  by first renumbering the indices  $(1, \dots, n, d)$  of each polygon  $\eta \in \mathcal{P}_{S \cup \{d\}}$  as  $(0, 1, \infty, t_1, \dots, t_{n-2})$  via  $\rho$ , then mapping the renumbered polygon to the corresponding cell-form (same cyclic order). By hypothesis,  $\tau = \sum_i c_i \eta_i$  maps to a sum  $\omega_\tau = \sum_i c_i \omega_{\eta_i}$  of 01 cell forms. Since  $\tau$  is a shuffle with respect to one point, we know by (3.2.13) that  $\omega_\tau = 0$ . But the 01 cell-forms  $\omega_{\eta_i}$  are linearly independent by theorem 3.16. Therefore each  $c_i = 0$ .  $\square$

Recall that the shuffles of disjoint Lyndon words form a basis for  $I_S$ ; we call them *Lyndon shuffles*. A *convergent Lyndon shuffle* is a shuffle of convergent Lyndon words.

**Definition 3.36.** *We will recursively define the set  $\mathcal{L}_S$  of Lyndon insertion shuffles in  $I_S$ . If  $S = \{1\}$ , then  $\mathcal{L}_S = 0$ . If  $S = \{1, 2\}$  then  $\mathcal{L}_S = \{1 \text{III} 2\}$ . In general, if  $D$  is any (lexicographically ordered) alphabet on  $m$  letters and  $S = \{1, \dots, m\}$ , we define  $\mathcal{L}_D$  to be the image of  $\mathcal{L}_S$  under the bijection  $S \rightarrow D$  corresponding to the ordering of  $D$ .*

Assume now that  $S = \{1, \dots, n\}$  with  $n > 2$ , and that we have constructed all of the sets  $\mathcal{L}_{\{1, \dots, i\}}$  with  $i < n$ . Let us construct  $\mathcal{L}_S$ . The elements of these set are constructed by taking convergent Lyndon shuffles on a smaller alphabet, and making insertions into every letter except for the leftmost letter of each Lyndon word in the shuffle, according to the following explicit procedure. Let  $T = \{a_1, \dots, a_k\}$  be an alphabet with  $3 \leq k \leq n$ , with the lexicographical ordering  $a_1 < \dots < a_k$ , and choose a convergent Lyndon shuffle  $\gamma$  of length  $k$  in the letters of  $T$ . Write  $\gamma$  as a shuffle of  $s > 1$  convergent Lyndon words:

$$\gamma = (a_{i_1} \cdots a_{i_{k_1}}) \text{III} (a_{i_{k_1+1}} \cdots a_{i_{k_2}}) \text{III} \cdots \text{III} (a_{i_{k_{s-1}+1}} \cdots a_{i_{k_s}})$$

where  $k_1 + \dots + k_s = k$ . Choose integers  $v_1, \dots, v_k \geq 1$  such that  $\sum_i v_i = n$  and such that for each of the indices  $l = 1, i_{k_1+1}, \dots, i_{k_{s-1}+1}$  of the leftmost characters of the  $s$  convergent Lyndon words in  $\gamma$ , we have  $v_l = 1$ . For  $1 \leq i \leq k$ , let  $D_i$  denote an alphabet  $\{b_1^i, \dots, b_{v_i}^i\}$ . When  $v_i = 1$ , insert  $b_1^i$  into the place of the letter  $a_i$  in  $\gamma$ ; when  $v_i > 1$ , choose any element  $V_i$  from  $\mathcal{L}_{D_i}$ , and insert this  $V_i$  into the place of the letter  $a_i$ .

The result is a sum of words in the alphabet  $\cup D_i$ . Note that this alphabet is of cardinal  $n$  and equipped with a natural lexicographical ordering given by the ordering  $D_1, \dots, D_k$  and the orderings within each alphabet  $D_i$ . We can therefore renumber this alphabet as  $1, \dots, n$ . Since it is a sum of shuffles, the renumbered element lies in  $I_S$ , and we call it a Lyndon insertion shuffle on  $S$ . The original convergent Lyndon shuffle  $\gamma$  on  $T$  is called the framing; together with the integers  $v_i$ , we call this the fixed structure of the insertion shuffle. We define  $\mathcal{L}_S$  to be the set of all Lyndon insertion shuffles on  $S$ . Note in particular that when  $k = n$ , so  $v_i = 1$  for  $1 \leq i \leq k$ , there are no non-trivial insertions, and the corresponding elements of  $\mathcal{L}_S$  are the convergent Lyndon shuffles.

**Example 3.37.** We have

$$\begin{aligned}\mathcal{L}_{\{1,2\}} &= \{1\mathfrak{m}2\} \\ \mathcal{L}_{\{1,2,3\}} &= \{1\mathfrak{m}2\mathfrak{m}3, 2\mathfrak{m}13\} \\ \mathcal{L}_{\{1,2,3,4\}} &= \{1\mathfrak{m}2\mathfrak{m}3\mathfrak{m}4, 13\mathfrak{m}2\mathfrak{m}4, 14\mathfrak{m}2\mathfrak{m}3, 24\mathfrak{m}1\mathfrak{m}3, \\ &\quad 3\mathfrak{m}142, 13\mathfrak{m}24, 1(3\mathfrak{m}4)\mathfrak{m}2\}\end{aligned}$$

The last element of  $\mathcal{L}_{\{1,2,3,4\}}$  is obtained by taking  $T = \{1, 2, 3\}$  and  $\gamma = 13\mathfrak{m}2$ . We can only insert in the place of the character 3 since 1 and 2 are leftmost letters of the Lyndon words in  $13\mathfrak{m}2$ . As for what can be inserted in the place of 3, the only possible choices are  $k = 1$ ,  $v_1 = 2$ ,  $D_1 = \{b_1, b_2\}$ , and  $V_1 = b_1\mathfrak{m}b_2$ , the unique element of  $\mathcal{L}_{D_1}$ . The natural ordering on the alphabet  $\{T \setminus 3\} \cup D_1$  is given by  $(1, 2, b_1, b_2)$  since  $b_1\mathfrak{m}b_2$  is inserted in the place of 3, so we renumber  $b_1$  as 3 and  $b_2$  as 4, obtaining the new element  $1(3\mathfrak{m}4)\mathfrak{m}2$ .

For  $n = 5$ ,  $\mathcal{L}_{\{1,2,3,4,5\}}$  has 34 elements. Of these, 25 are convergent Lyndon shuffles which we do not list. The remaining nine elements are obtained by insertions into the smaller convergent Lyndon shuffles: they are given by

$$\left\{ \begin{array}{ll} 2\mathfrak{m}1(4\mathfrak{m}35), 2\mathfrak{m}1(3\mathfrak{m}4\mathfrak{m}5) & \text{insertions into } 2\mathfrak{m}13 \\ 3\mathfrak{m}1(4\mathfrak{m}5)2, 4\mathfrak{m}15(2\mathfrak{m}3) & \text{insertions into } 3\mathfrak{m}142 \\ 13\mathfrak{m}2(4\mathfrak{m}5), 1(3\mathfrak{m}4)\mathfrak{m}25 & \text{insertions into } 13\mathfrak{m}24 \\ 1(3\mathfrak{m}4)\mathfrak{m}2\mathfrak{m}5 & \text{insertion into } 13\mathfrak{m}2\mathfrak{m}4 \\ 1(4\mathfrak{m}5)\mathfrak{m}2\mathfrak{m}3 & \text{insertion into } 14\mathfrak{m}2\mathfrak{m}3 \\ 2(4\mathfrak{m}5)\mathfrak{m}1\mathfrak{m}3 & \text{insertion into } 24\mathfrak{m}1\mathfrak{m}3. \end{array} \right.$$

**Definition 3.38.** We now define a complementary set, the set  $\mathcal{W}_S$  of Lyndon insertion words. Let a special convergent word  $w \in V_S$  denote a convergent word of length  $n$  in  $S$  such that in the lexicographical ordering  $(1, \dots, n, d)$ , the polygon (cyclic structure)  $\eta = (w, d)$  satisfies  $\chi(\delta) \cap \chi(\eta) = \emptyset$ ; in other words, the polygon  $\eta$  has no chords in common with the standard polygon. This condition is a little stronger than asking  $w$  to be a convergent word (for instance,  $13524$  is a convergent word but not a special convergent word, since  $13524d$  has a bad chord  $\{2, 3, 4, 5\}$ ). The first elements of  $\mathcal{W}_S$  are given by the special convergent  $1n$ -words. The remaining elements of  $\mathcal{W}_S$  are the Lyndon insertion words constructed as follows. Take a special convergent word  $w'$  in a smaller alphabet  $T = \{a_1, \dots, a_k\}$  with  $k < n$  such that  $a_1$  appears just to the left of  $a_{k-1}$ , and choose positive integers  $v_1, \dots, v_k$  such that  $v_1 = v_k = 1$  and  $\sum_i v_i = n$ . As above, we let  $D_i = \{b_1^i, \dots, b_{v_i}^i\}$  for  $1 \leq i \leq k$ , and choose an element  $D_i$  of  $\mathcal{L}_{D_i}$  for each  $i$  such that  $v_i > 1$ . For  $i$  such that  $v_i = 1$ , insert  $b_1^i$  in the place of  $a_i$  in  $w'$ , and for  $i$  such that  $v_i > 1$  insert  $D_i$  in the place of  $a_i$ . We obtain a sum of words  $w''$  in the letters  $\cup D_i$ . This alphabet has a natural lexicographic ordering  $D_1, \dots, D_k$  as above, so we can renumber its letters from 1 to  $n$ , which transforms  $w''$  into a sum of words  $w \in V_S$  called a Lyndon insertion word. Note that by

construction, the result is still a sum of  $1n$ -words. The set  $\mathcal{W}_S$  consists of the special convergent words and the Lyndon insertion words.

**Remark 3.39.** It follows from lemma 3.35 that the intersection of the subspace  $\langle \mathcal{W}_S \rangle$  in  $V_S$  with the subspace  $I_S$  of shuffles is equal to zero.

**Example 3.40.** We have

$$\begin{aligned}\mathcal{W}_{\{1,2\}} &= \emptyset, \quad \mathcal{W}_{\{1,2,3\}} = \emptyset, \quad \mathcal{W}_{\{1,2,3,4\}} = \{3142\}, \\ \mathcal{W}_{\{1,2,3,4,5\}} &= \{24153, 31524, (3\text{m}4)152, (415(2\text{m}3))\}\end{aligned}$$

The last two elements of  $\mathcal{W}_{\{1,2,3,4,5\}}$  are obtained by taking  $v_1 = 1, v_2 = 1, v_3 = 2, v_4 = 1$  and  $v_1 = 1, v_2 = 2, v_3 = 1, v_4 = 1$  and creating the corresponding Lyndon insertion word with respect to 3142.

**Theorem 3.41.** The set  $\mathcal{W}_S \cup \mathcal{L}_S$  of Lyndon insertion words and shuffles is linearly independent.

*Proof.* We will prove the result by induction on  $n$ . Since  $\mathcal{L}_S \subset I_S$  and we saw by lemma 3.35 that the space generated by  $\mathcal{W}_S$  has zero intersection with  $I_S$ , we only have to show that both  $\mathcal{W}_S$  and  $\mathcal{L}_S$  are linearly independent sets. We begin with  $\mathcal{L}_S$ . Since  $\mathcal{L}_{\{1,2\}}$  contains a single element, we may assume that  $n > 2$ .

Let  $W = A_1\text{m}\cdots\text{m}A_r$  be a Lyndon shuffle, with  $r > 1$ . We define its *fixed structure* as follows. Replace every maximal consecutive block (not contained in any larger consecutive block) in each  $A_i$  by a single letter. Then  $W$  becomes a convergent Lyndon shuffle  $W'$  in a smaller alphabet  $T'$  on  $k$  letters, which is equipped with an inherited lexicographical ordering. If  $T = \{1, \dots, k\}$ , then under the order-respecting bijection  $T' \rightarrow T$ ,  $W'$  is mapped to a convergent Lyndon shuffle  $V$  in  $T$ , called the framing of  $W$ . The fixed structure is given by the framing together with the set of integers  $\{v_i \mid 1 \leq i \leq k\}$  defined by  $v_i = 1$  if that letter in  $T$  does not correspond to a maximal block, and  $v_i$  is the length of the maximal block if it does. Thus we have  $v_1 + \cdots + v_k = n$ . We can extend this definition to the fixed structure of a Lyndon insertion shuffle, since by definition this is a linear combination of Lyndon shuffles all having the same fixed structure, and we recover the framing and fixed structure of the insertion shuffle given in the definition.

**Example 3.42.** If  $W$  is the Lyndon shuffle  $1546\text{m}237$ , we replace the consecutive blocks 23 and 546 by letters  $b_1$  and  $b_2$ , obtaining the convergent shuffle  $W' = 1b_2\text{m}b_17$  in the alphabet  $T' = \{1, b_1, b_2, 7\}$ ; renumbering this as 1, 2, 3, 4 we obtain  $V = 13\text{m}24 \in \mathcal{L}_{\{1,2,3,4\}}$ . The fixed structure is given by  $13\text{m}24$  and integers  $v_1 = 1, v_2 = 2, v_3 = 3, v_4 = 1$ .

The Lyndon insertion shuffles  $(1, (3\text{m}4))\text{m}(2, 5)$  and  $(1, 3)\text{m}(2, (4\text{m}5))$  have the same framing  $13\text{m}24$ , but since  $(v_1, v_2, v_3, v_4) = (1, 1, 2, 1)$  for the first one and  $(1, 1, 1, 2)$  for the second, they do not have the same fixed structure. The Lyndon insertion shuffles  $(1, (5)\text{m}(3, 4, 6))\text{m}(2, 7)$  and  $(1, (3, 5)\text{m}(4, 6))\text{m}(2, 7)$  have the same associated framing  $13\text{m}24$  and the same integers  $(v_1, v_2, v_3, v_4) = (1, 1, 4, 1)$ . so they have the same fixed structure.

For any fixed structure, given by a convergent Lyndon shuffle  $\gamma$  on an alphabet  $T$  of length  $k$  and associated integers  $v_1, \dots, v_k$  with  $v_1 + \cdots + v_k = n$ , let  $L(\gamma, v_1, \dots, v_k)$  be the subspace of  $V_S$  spanned by Lyndon shuffles with that fixed structure. Since Lyndon shuffles are linearly independent, we have

$$V_S = \bigoplus L(\gamma, v_1, \dots, v_k)$$

Now, as we saw above, a Lyndon insertion shuffle is a linear combination of Lyndon shuffles all having the same fixed structure, so every element of  $\mathcal{W}_S \cup \mathcal{L}_S$  lies in exactly one subspace  $L(\gamma, v_1, \dots, v_k)$ . Thus, to prove that the elements of  $\mathcal{L}_S$  are linearly independent, it is only necessary to prove the linear independence of Lyndon insertion shuffles with the same fixed structure. If all of the  $v_i = 1$ , then the fixed structure is just a single convergent Lyndon shuffle on  $S$ , and these are linearly independent. So let  $(\gamma, v_1, \dots, v_k)$  be a fixed structure with not all of the  $v_i$  equal to 1, and let  $\omega = \sum_q c_q \omega_q$  be a linear combination of Lyndon insertion shuffles of fixed structure  $\gamma, v_1, \dots, v_n$ .

Break up the tuple  $(1, \dots, n)$  into  $k$  successive tuples

$$B_1 = (1, \dots, v_1), B_2 = (v_1 + 1, \dots, v_1 + v_2), \dots, B_k = (v_1 + \dots + v_{k-1} + 1, \dots, n).$$

Let  $i_1, \dots, i_m$  be the indices such that  $B_{i_1}, \dots, B_{i_m}$  are the tuples of length greater than 1. These tuples correspond to the insertions in the Lyndon insertion shuffles of type  $(\gamma, v_1, \dots, v_k)$ . For  $1 \leq j \leq m$ , let  $T_j = \{B_{i_j}\} \cup \{d_j\}$ . This element  $d_j$  is the index of the chord  $D_j$  corresponding to the consecutive subset  $B_{i_j}$ , which is a chord of the standard polygon and also of every term of  $\omega$ . The chords  $D_1, \dots, D_r$  are disjoint and cut each term of  $\omega$  into  $m + 1$  subpolygons,  $m$  of which are indexed by  $T_j$ , and the last one of which is indexed by  $T' = S \setminus \{B_{i_1} \cup \dots \cup B_{i_m}\} \cup \{d_1, \dots, d_m\}$ . Thus we can take the composed residue map

$$\text{Res}_{D_1, \dots, D_m}^p(\omega) \in \mathcal{P}_{T_1} \otimes \dots \otimes \mathcal{P}_{T_m} \otimes \mathcal{P}_{T'}.$$

Let us compute this residue.

The alphabet  $T'$  is of length  $k$  and has a natural ordering corresponding to a bijection  $\{1, \dots, k\} \rightarrow T'$ . Let  $\gamma'$  be the image of  $\gamma$  under this bijection, i.e. the framing. Let  $P_1^q, \dots, P_m^q$  be the insertions corresponding to the  $m$  tuples  $B_{i_1}, \dots, B_{i_m}$  in each term of  $\omega = \sum_q c_q \omega_q$ . Each  $P_j^q$  lies in  $\mathcal{L}_{B_{i_j}}$ . The image of the composed residue map is then

$$\text{Res}_{D_1, \dots, D_m}^p(\omega) = \sum_q c_q (P_1^q, d_1) \otimes \dots \otimes (P_m^q, d_m) \otimes \gamma'. \quad (3.3.6)$$

Now assume that  $\omega = \sum_q c_q \omega_q = 0$ . Then

$$\sum_q c_q (P_1^q, d_1) \otimes \dots \otimes (P_m^q, d_m) \otimes \gamma' = 0,$$

and since  $\gamma'$  is fixed, we have

$$\sum_q c_q (P_1^q, d_1) \otimes \dots \otimes (P_m^q, d_m) = 0.$$

But for  $1 \leq j \leq m$ , the  $P_j^q$  lie in  $\mathcal{L}_{B_{i_j}}$  and thus, by the induction hypothesis, the distinct  $P_j^q$  for fixed  $j$  and varying  $q$  are linearly independent. Since  $d_i$  is the largest element in the lexicographic alphabet  $T_i$ , the sums  $(P_j^q, d_j)$  are also linearly independent for fixed  $j$  and varying  $q$ , because if  $\sum_q d_q (P_j^q, d_j) = 0$  then  $\sum_q d_q P_j^q = 0$  simply by erasing  $d_j$ . The tensor products are therefore also linearly independent, so we must have  $c_q = 0$  for all  $q$ . This proves that  $\mathcal{L}_S$  is a linearly independent set.

We now prove that  $\mathcal{W}_S$  is a linearly independent set. For this, we construct the framing and fixed structure of a of length  $n$  in  $V_S$  just as above, by replacing consecutive

blocks with single letters, obtaining a word in a smaller alphabet  $T'$  and a set of integers corresponding to the lengths of the consecutive blocks. For instance, replacing the consecutive blocks 12 and 354 in the word 12735486 by letters  $b_1$  and  $b_2$  gives a convergent word  $b_17b_286$  in the alphabet  $(b_1, b_2, 6, 7, 8)$ ; renumbering this as  $(1, 2, 3, 4, 5)$  gives the framing as 14253 and the associated integers as  $v_1 = 2, v_2 = 3, v_3 = 1, v_4 = 1, v_5 = 1$ . For every fixed structure of this type, now given as a convergent word  $\gamma$  of length  $k < n$  together with integers  $v_1, \dots, v_k$ , we let  $W(\gamma, v_1, \dots, v_k)$  denote the subspace of  $V_S$  generated by words with the fixed structure  $(\gamma, v_1, \dots, v_k)$ . Since the words of length  $n$  form a basis for  $V_S$ , we again have  $V_S = \bigoplus W(\gamma, v_1, \dots, v_k)$ . Therefore, to show that  $\mathcal{W}_S$  is a linearly independent set, we only need to show that the set of Lyndon insertion words with a given fixed structure is a linearly independent set. So assume that we have some linear combination  $\sum_q c_q w_q = 0$ , where the  $w_q$  are all Lyndon insertion words of given fixed structure  $(\gamma, v_1, \dots, v_k)$ . If  $k = n$ , then these insertion words are just words, so they are linearly independent and  $c_q = 0$  for all  $q$ . So assume that at least one  $v_i > 1$ . We proceed very much as above. Breaking up the tuple  $(1, \dots, n)$  into tuples  $B_1, \dots, B_k$  as above, and letting  $D_1, \dots, D_m, T_j$  and  $T'$  denote the same things, we compute the composed residue of  $\sum_q c_q w_q$  and obtain (3.3.6). Then because all of the insertions  $P_i^q$  lie in  $\mathcal{L}_{B_{i_j}}$  and we know that these sets are linearly independent, we find as above that  $c_q = 0$  for all  $q$ .  $\square$

### 3.3.4 Convergent linear combinations of polygons

**Definition 3.43.** Let  $S = \{1, \dots, n\}$ . Let  $J_S$  be the subspace of  $\mathcal{P}_{S \cup \{d\}}$  spanned by  $\mathcal{L}_S$  and let  $K_S$  be the subspace of  $\mathcal{P}_{S \cup \{d\}}$  spanned by  $\mathcal{W}_S$ .

We prove the main convergence results in two separate theorems, concerning the subspaces  $I_S$  and  $W_S$  of  $V_S \simeq \mathcal{P}_{S \cup \{d\}}$  respectively.

**Theorem 3.44.** If  $\omega \in I_S \subset \mathcal{P}_{S \cup \{d\}}$  is convergent, then  $\omega \in J_S$ .

*Proof.* One direction of this theorem is easy. We only need to show that any Lyndon insertion shuffle is convergent. If it is a shuffle of convergent Lyndon words, then there are no consecutive blocks in any of the words. Therefore if the letters of any consecutive subset  $T$  of  $S$  appear as a block in any term of  $\omega$ , it must be because they appeared in more than one of the convergent words which are shuffled together. So these letters appear as a shuffle, so the residue lies in  $I_T \otimes \mathcal{P}_{S \setminus T \cup \{d\}}$ . Now, if we are dealing with a Lyndon insertion shuffle with non-trivial insertions, then there are two kinds of bad chords: those corresponding to these insertions, and those corresponding to consecutive subsets of the insertion sets. By definition, the insertions themselves lie in  $\mathcal{L}_T \subset I_T$ , and their expressions are equal to the  $I_T$  factors of the residue, so  $\omega$  converges along all the bad chords corresponding to insertions. For the subchords of these, their letters appear shuffled inside the insertions, so the previous argument holds.

Write  $\omega = \sum_i c_i \omega_i$  where each  $\omega_i = (A_1^i \text{III} \cdots \text{III} A_{r_i}^i, d)$  is a Lyndon shuffle,  $r_i > 1$ . Assume that  $\omega$  converges along all of its bad chords. As above, a consecutive block appearing in any  $A_j^i$  is maximal if the same block does not appear in any other factor inside a bigger consecutive block. Factors may appear which contain more than one consecutive block, but the maximal blocks are disjoint.



We prove the result by induction on the length of the alphabet  $S = \{1, \dots, n\}$ . The smallest case is  $n = 3$ , since for  $n = 2$ , the polygons are triangles and have no chords. For  $n = 3$ , let

$$\omega = a_1(12\mathbb{m}3, d) + a_2(13\mathbb{m}2, d) + a_3(1\mathbb{m}2\mathbb{m}3, d) + a_4(23\mathbb{m}1, d).$$

The only non-trivial bad chords are  $D = \{1, 2\}$ ,  $E = \{2, 3\}$ . We have

$$\begin{aligned} \text{Res}_D^p(\omega) &= a_1(1, 2, e) \otimes (e\mathbb{m}3, d) + a_2(1\mathbb{m}2, e) \otimes (e, 3, d) \\ &\quad + a_3(1\mathbb{m}2, e) \otimes (e\mathbb{m}3, d) + a_4(1\mathbb{m}2, e) \otimes (e, 3, d). \end{aligned}$$

For this to converge means that the left-hand parts of the two right-hand tensor factors  $(e, 3, d)$  and  $(e\mathbb{m}3, d)$  must lie in  $I_{\{1,2\}}$ . This implies that  $a_1 = 0$ . For the other residue, we have

$$\begin{aligned} \text{Res}_E^p(\omega) &= a_1(2\mathbb{m}3, e) \otimes (1, e, d) + a_2(2\mathbb{m}3, e) \otimes (1, e, d) \\ &\quad + a_3(2\mathbb{m}3, e) \otimes (1\mathbb{m}e, d) + a_4(2, 3, e) \otimes (1\mathbb{m}e, d). \end{aligned}$$

This gives  $a_4 = 0$ . Therefore convergent  $\omega$  is a linear combination of  $13\mathbb{m}2$  and  $1\mathbb{m}2\mathbb{m}3$ , which are the basis elements of  $\mathcal{L}_{\{1,2,3\}}$ .

The induction hypothesis is that for every alphabet  $S' = \{1, \dots, i\}$  with  $i < n$ , if  $\omega \in V_{S'}$  is convergent, then  $\omega \in K_{S'}$ .

Now let  $S = \{1, \dots, n\}$  and assume that  $\omega \in V_S$  is convergent. If no consecutive block appears in any  $A_j^i$ , then  $\omega$  is a linear combination of convergent Lyndon words, so it is in  $J_S$ . Assume some consecutive blocks do appear, and consider a maximal consecutive block  $T$ , which corresponds to a bad chord  $E$ . Exactly as in the proof of the lemma, we decompose  $\omega = \gamma_1 + \gamma_2$  where  $\gamma_k$  is the sum  $\sum_{i \in I_k} c_i \omega_i$ , with  $I_1$  the set of indices  $i$  for which  $T$  appears as a block in some  $A_j^i$ , which by reordering we may assume to be  $A_1^i$ , and  $I_2$  is the set of indices for which  $T$  does not appear as a block in any  $A_j^i$ . As in the lemma, we see immediately that  $\text{Res}_E^p(\gamma_2) \in I_T \otimes \mathcal{P}_{S \setminus T \cup \{e\} \cup \{d\}}$ , so  $\gamma_2$  converges along  $E$ . Since we are assuming that  $\omega$  is convergent, also  $\gamma_1$  must converge, so we have

$$\text{Res}_E^p(\gamma_1) \in I_T \otimes \mathcal{P}_{S \setminus T \cup \{d\} \cup \{e\}}.$$

For each  $i \in I_1$ , write  $A_1^i = B_1^i Y^i C_1^i$ , where  $Y^i$  consists of the letters of  $T$  in some order, and  $B_1^i$  is Lyndon and non-empty. Then

$$\text{Res}_E^p(\gamma_1) = \sum_{i \in I_1} c_i (Y^i, e) \otimes (B^i e C^i \mathbb{m} A_2^i \mathbb{m} \dots \mathbb{m} A_{r_i}^i, d). \quad (3.3.7)$$

Putting an equivalence relation on  $I_1$  as in the proof of the lemma, so that  $i \sim i'$  if the right-hand factors of (3.3.7) are equal, and letting  $[i]$  denote the equivalence classes for this relation, we write the residue as

$$\text{Res}_E^p(\gamma_1) = \sum_{[i] \subset I_1} \left( \sum_{i \in [i]} c_i (Y^i, e) \right) \otimes (B^{[i]} e C^{[i]} \mathbb{m} A_2^{[i]} \mathbb{m} \dots \mathbb{m} A_{r_{[i]}}^{[i]}, d). \quad (3.3.8)$$

Since the right-hand factors in the sum over  $[i]$  are distinct Lyndon shuffles, they are linearly independent and therefore we find that

$$(S_{[i]}, e) = \sum_{i \in [i] \subset I_1} c_i (Y^i, e) \in I_T$$

for each  $[i] \subset I_1$ .

We now show that  $(S_{[i]}, e)$  is not merely in  $I_T$ , but in  $J_T$ . To see this, it is enough to show that  $(S_{[i]}, e)$  converges on every subchord of  $T$  (consecutive subset inside the set  $T$ ), and apply the induction hypothesis. So let  $E'$  be a subchord of  $E$ , corresponding to a consecutive block  $T'$  strictly contained in  $T$ .

We now decompose the set of indices  $I_1$  into two subsets  $I_3$  and  $I_4$ , where  $I_3$  contains the indices  $i \in I_1$  such that  $T'$  appears as a consecutive block inside the block  $T$  appearing in  $A_1^i$ , and  $I_4$  contains the indices  $i \in I_1$  such that the letters of  $T'$  do not appear consecutively inside the block  $T$ . Similarly, we partition  $I_2$ , the set of indices in the sum  $\omega = \sum_i c_i \omega_i$  for which  $T$  does not appear as a block in  $A_1^i$ , into two sets  $I_5$  and  $I_6$ , where  $I_5$  contains the indices  $i \in I_2$  such that  $T'$  appears as a block in some  $A_j^i$  which we may assume to be  $A_1^i$ , and  $I_6$  contains the indices  $i \in I_2$  of the terms in which  $T'$  does not appear as a block in any  $A_j^i$ . We have corresponding decompositions  $\gamma_1 = \gamma_3 + \gamma_4$ ,  $\gamma_2 = \gamma_5 + \gamma_6$ . As before,  $T'$  must appear as a shuffle in  $\gamma_6$ , so  $\gamma_6$  converges along  $E'$ . As for  $\gamma_4$ , since  $T'$  does not appear as a block or a shuffle, the residue along  $E'$  is 0. Since by assumption  $\omega$  converges along  $E'$ , we know that  $\gamma_3 + \gamma_5$  converges along  $E'$ . Let us show that in fact both  $\gamma_3$  and  $\gamma_5$  converge along  $E'$ .

Write  $A_1^i = R^i Z^i S^i$  for every  $i \in I_3 \cup I_5$ , where  $Z^i$  is a word in the letters of  $T'$ . Note that  $R^i$  is Lyndon, and non-empty because  $T'$  cannot appear as a block to the left of any  $A_j^i$  by the lemma. Then for  $k = 3, 5$ , we have

$$\text{Res}_{E'}^p(\gamma_k) = \sum_{i \in I_k} c_i(Z^i, e') \otimes (R^i e' S^i \text{III} A_2^i \text{III} \cdots \text{III} A_{r_i}^i). \quad (3.3.9)$$

For  $k = 3, 5$ , put the equivalence relation on  $I_k$  for which  $i \sim i'$  if the right-hand factors of (3.3.8) are equal, and let  $\langle i \rangle$  denote the equivalence classes for this relation. Note that because for  $i \in I_3$ ,  $T'$  appears as a block of  $T$ , we have  $B^i \subset R^i$  and  $C^i \subset S^i$ , in the sense that in fact  $B^i$  is the left-hand part of  $R^i$  and  $C^i$  is the right-hand part of  $S^i$ . Therefore in particular, the new equivalence relation is strictly finer than the old, i.e. the equivalence class  $[i]$  breaks up into a finite union of equivalence classes  $\langle i \rangle$ . The residue can now be written

$$\text{Res}_{E'}^p(\gamma_k) = \sum_{\langle i \rangle \subset I_k} \left( \sum_{i \in \langle i \rangle} c_i(Z^i, e') \right) \otimes (R^{\langle i \rangle} e' S^{\langle i \rangle} \text{III} A_2^{\langle i \rangle} \text{III} \cdots \text{III} A_{r_{\langle i \rangle}}^{\langle i \rangle}). \quad (3.3.10)$$

Then since the right-hand factors for each  $k$  are distinct Lyndon shuffles, they are linearly independent, and furthermore, none of these factors for  $\gamma_3$  can ever occur in  $\gamma_5$  for the following reason: the Lyndon words  $R^i e' S^i$  appearing for  $k = 3$  all have the letters of  $T \setminus T'$  grouped around  $e'$ , whereas none of the Lyndon words  $R^i e' S^i$  have this property. Therefore all the right-hand factors from the residues of  $\gamma_3$  and  $\gamma_5$  are linearly independent, so we find that all the left-hand factors

$$\sum_{i \in \langle i \rangle \subset I_k} (Z^i, e') \in I_{T'}, \quad (3.3.11)$$

so that both  $\gamma_3$  and  $\gamma_5$  converge along  $E'$ . In particular, this means that both  $\gamma_1$  and  $\gamma_2$  converge along  $E'$ .

Now, let us compute the composed residue map  $\text{Res}_{E, E'}^p(\gamma_1)$ . First, for each  $i \in I_3$ , write  $Y^i = U^i Z^i V^i$  where  $Z^i$  is a word in the letters of  $T'$ , so that  $R^i = B^i U^i$ ,  $S^i = U^i C^i$ ,



and  $A_1^i = B^i U^i Z^i V^i C^i$ . Then by (3.3.9), we have

$$\begin{aligned} \text{Res}_E^p(\gamma_1) &= \sum_{[i] \in I_3} \left( \sum_{i \in [i]} c_i(U^i Z^i V^i, e) \right) \otimes (B_1^{[i]} e C_1^{[i]} \text{III} A_2^{[i]} \text{III} \cdots \text{III} A_{r[i]}^{[i]}, d) + \\ &\quad \sum_{[i] \in I_4} \left( \sum_{i \in [i]} c_i(Y^i, e) \right) \otimes (B_1^{[i]} e C_1^{[i]} \text{III} A_2^{[i]} \text{III} \cdots \text{III} A_{r[i]}^{[i]}, d). \end{aligned}$$

The terms for  $i \in I_4$  converge along  $T'$ , so they vanish when taking the composed residue, and we find

$$\text{Res}_{E, E'}^p(\gamma_1) = \sum_{[i] \in I_3} \left( \sum_{i \in [i]} c_i(Z^i, e') \otimes (U^i e' V^i, e) \right) \otimes (B_1^{[i]} e C_1^{[i]} \text{III} A_2^{[i]} \text{III} \cdots \text{III} A_{r[i]}^{[i]}, d).$$

Since for each  $[i] \subset I_3$ , the right-hand factors are as usual distinct and linearly independent, this means that for each  $[i] \subset I_3$ ,

$$\text{Res}_{E'}^p(S_{[i]}, e) = \sum_{i \in [i]} c_i(Z^i, e') \otimes (U^i e' V^i, e) \in \mathcal{P}_{T' \cup \{e'\}} \otimes \mathcal{P}_{T \setminus T' \cup \{e'\} \cup \{e\}}.$$

Now, breaking  $[i]$  up into separate equivalence classes  $\langle i \rangle$ , we have that  $U^i$  and  $V^i$  are identical for all  $i$  in one subclass  $\langle i \rangle$  since  $B^i$  and  $C^i$  are already identical for all  $i \in [i]$ . So for each  $[i] \subset I_3$ , we can write

$$\text{Res}_{E'}^p(S_{[i]}, e) = \sum_{\langle i \rangle \subset [i]} \sum_{i \in \langle i \rangle} c_i(Z^i, e') \otimes (U^{\langle i \rangle} e' V^{\langle i \rangle}, e),$$

where the right-hand factors are all distinct words. Then (3.3.11) shows that this sum lies in  $I_{T'} \otimes \mathcal{P}_{T \setminus T' \cup \{e'\} \cup \{e\}}$ , so in fact  $(S_{[i]}, e)$  converges along  $E'$ . For  $[i] \subset I_4$ , we already saw that  $\text{Res}_{E'}^p(S_{[i]}, e) = 0$ , so  $(S_{[i]}, e)$  converges along  $E'$  for all  $[i] \subset I_1$ . Since this holds for all chords  $E'$  corresponding to consecutive subblocks  $T'$  of  $T$ , we see that each  $(S_{[i]}, e)$  is convergent along all its bad chords, and thus, by the induction hypothesis,  $(S_{[i]}, e) \in J_T$ . Now we can write  $\omega = \gamma_1 + \gamma_2$  with

$$\gamma_1 = \sum_{[i] \in I_1} c_{[i]} B^{[i]}(S_{[i]}) C^{[i]} \text{III} A_2^{[i]} \text{III} \cdots \text{III} A_{r_i}^{[i]}$$

with  $S_{[i]} \in J_T$ . This means that the maximal block  $T$ , which appeared only in  $\gamma_1$ , has been replaced by an insertion in the sense of the definition of Lyndon insertion shuffles.

To conclude the proof of the theorem, we successively replace each of the maximal blocks in  $\omega$  by insertion terms in the same way. Insertions are by definition convergent and contain no blocks, so as we proceed to substitute insertions for the maximal blocks one by one, blocks which were previously not maximal may become maximal; however the order in which the blocks are substituted by insertions is of no importance as long as only maximal blocks are treated at each step. The final result displays  $\omega$  as a linear combination of convergent Lyndon shuffles and Lyndon insertion shuffles, so  $\omega \in J_S$ .  $\square$

**Theorem 3.45.** *Let  $\eta \in W_S \subset \mathcal{P}_{S \cup \{d\}}$ . Then  $\eta$  is convergent if and only if  $\eta \in K_S = \langle \mathcal{W}_S \rangle$ .*

*Proof.* The proof that  $\omega \in K_S$  is convergent is exactly as at the beginning of the proof of the previous theorem. So let  $\omega \in W_S$ , write

$$\omega = \sum_i a_i \eta_i$$

where each  $\eta_i$  is a  $1n$ -polygon (a  $1n$ -word concatenated with  $d$ ), and assume  $\omega$  is convergent. The only possible bad chords for  $\omega$  are the consecutive blocks appearing in the  $\eta_i$ . Let  $T$  be a subset of  $S$  corresponding to a maximal consecutive block.

**Lemma 3.46.** *No maximal consecutive block having non-trivial intersection with  $\{1, n\}$  can appear in any of the  $1n$ -words  $\eta_i$  of  $\omega$ .*

*Proof.* If  $T$  is a maximal block containing both 1 and  $n$ , then  $T = \{1, \dots, n\}$  which does not correspond to a chord. Let  $T$  be a maximal consecutive block appearing in  $\omega$  which contains 1 but not  $n$ , say  $T = \{1, \dots, m\}$ . If  $T$  appears as a consecutive block in some  $\eta_i$ , we may write  $\eta_i = (K^i, Z^i, 1, n, H^i, d)$  where  $Z^i$  is an ordering of  $\{2, \dots, m\}$ . Then

$$\text{Res}_E^p\left(\sum_i a_i \eta_i\right) = \sum_i a_i(Z^i, 1, e) \otimes (K^i, e, n, H^i, d).$$

The assumption that  $\omega$  converges along  $E$  means that this residue lies in  $I_T \otimes \mathcal{P}_{S \setminus T \cup \{e, d\}}$ . So for constant words  $K, H$  (i.e. constant right-hand tensor factor), we must have

$$\sum_{i|K^i=K, H^i=H} a_i(Z^i, 1, e) \in I_T, \quad (3.3.12)$$

in other words, a sum of words  $\sum_i a_i(Z^i 1)$  must be a shuffle. But this is impossible by a Lyndon basis argument. Using a backwards Lyndon basis in which all Lyndon words are as usual but written right to left, the words ending in 1 generate the degree 1 part of the algebra and are linearly independent from the shuffles, which generate the part of degree  $\geq 2$ . So we must have  $a_i = 0$  for all  $i$ .

Now let  $T = \{m, \dots, n\}$ . We write  $\eta_i = (K^i, 1, n, Z^i, H^i, d)$  where  $(n, Z^i)$  is an ordering of  $T$ , and we have

$$\text{Res}_E^p\left(\sum_i a_i \eta_i\right) = \sum_i a_i(n, Z^i, e) \otimes (K^i, 1, e, H^i, d).$$

Convergence implies that

$$\sum_{i|K^i=K, H^i=H} a_i(n, Z^i, e) \in I_T, \quad (3.3.13)$$

Using a Lyndon basis in which the lexicographical ordering is the backwards order  $n < \dots < 1$ , the  $nZ^i$  are all Lyndon words, so as above, they cannot sum to a shuffle.  $\square$

Now we can complete the proof of the theorem. It runs almost exactly as the proof of the previous theorem. Let  $\omega = \sum_i a_i \eta_i$  be a sum of  $1n$ -words which converges and consider a maximal consecutive block  $T \subset \{2, \dots, n-1\}$ . Let  $I_1$  be the set of indices  $i$  such that  $\eta_i$  contains the block  $T$  and  $I_2$  the other indices. For  $i \in I_1$ , write  $\eta_i = (K^i, Z^i, H^i, d)$  where  $Z^i$  is an ordering of  $T$ . Then

$$\text{Res}_T^p(\omega) = \sum_{i \in I_1} a_i(Z^i, e) \otimes (K^i, e, H^i, d).$$

Let  $i \sim i'$  be the equivalence relation on  $I_1$  given by  $K^i = K^{i'}$  and  $H^i = H^{i'}$ . Then

$$\text{Res}_T^p(\omega) = \sum_{[i] \in I_1} \left( \sum_{i \in [i]} a_i(Z^i, e) \right) \otimes (K^{[i]}, e, H^{[i]}, d),$$

so by the convergence assumption, we have

$$S_{[i]} = \sum_{i \in [i]} a_i(Z^i, e) \in I_T$$

for each  $[i] \subset I_1$ . Therefore we can write  $\omega$  with the insertion  $S_{[i]}$  as

$$\omega = \sum_{[i] \subset I_1} a_i(K^{[i]}, S_{[i]}, H^{[i]}, d) + \sum_{i \in I_2} a_i \eta_i,$$

and the maximal block  $T$  no longer appears in  $\omega$ . We prove that  $S_{[i]} \in J_T$  exactly as in the proof of the previous theorem: considering a maximal consecutive block  $T' \subset T$  occurring in a factor of  $S_{[i]}$ , one shows that  $S_{[i]}$  converges along  $T'$  if and only if  $\omega$  converges along  $T'$ . Since  $\omega$  does converge by assumption,  $S_{[i]}$  also converges, and since this holds for all consecutive blocks  $T' \subset T$ ,  $S_{[i]}$  converges on all its subdivisors and therefore  $S_{[i]} \in J_S = \langle \mathcal{L}_S \rangle$ . Finally, one deals with the disjoint maximal blocks appearing in  $\omega$  one at a time until no blocks at all remain.  $\square$

### 3.4 Explicit generators for $\mathcal{FC}$ and $H^\ell(\mathfrak{M}_{0,n}^\delta)$

In this chapter, we show that the map from polygons to cell-forms is surjective, and compute its kernel. From this and the previous chapter, we will conclude that the pairs  $(\delta, \omega)$ , where  $\omega$  runs through the set  $\mathcal{W}_S$  of Lyndon insertion words for  $n \geq 5$  form a generating set for the formal cell-zeta algebra  $\mathcal{FC}$ . In the final section, we show that the images of the elements of  $\mathcal{W}_S$  in the cohomology  $H^\ell(\mathfrak{M}_{0,n})$  yield an explicit basis for the convergent cohomology  $H^\ell(\mathfrak{M}_{0,n}^\delta)$ , and discuss its dimension.

#### 3.4.1 From polygons to cell-forms

Let  $S = \{1, \dots, n\}$ . The bijection  $\rho : S \cup \{d\} \rightarrow \{0, t_1, \dots, t_{\ell+1}, 1, \infty\}$  given by associating the elements  $1, \dots, n, d$  to  $0, t_1, \dots, t_{\ell+1}, 1, \infty$  respectively, induces a map  $f$  from polygons to cell-forms:

$$\eta = (\sigma(1), \dots, \sigma(n), d) \xrightarrow{f} \omega_\eta = [\rho(\sigma(1)), \dots, \rho(\sigma(n)), \infty].$$

The map  $f$  extends by linearity to a map from  $\mathcal{P}_{S \cup \{d\}}$  to the cohomology group  $H^{n-2}(\mathfrak{M}_{0,n+1})$ . The purpose of this section is to prove that  $f$  is a surjection, and to determine its kernel.

Recall that  $I_S \subset \mathcal{P}_{S \cup \{d\}}$  denotes the subvector space of  $\mathcal{P}_{S \cup \{d\}}$  spanned by the *shuffles with respect to the element  $d$* , namely by the linear combinations of polygons

$$(S_1 \boxplus S_2, d)$$

for all partitions  $S_1 \amalg S_2$  of  $S$ .

**Proposition 3.47.** *Let  $S = \{1, \dots, n\}$ . Then the cell-form map*

$$f : \mathcal{P}_{S \cup \{d\}} \longrightarrow H^{n-2}(\mathfrak{M}_{0,n+1})$$

*is surjective with kernel equal to the subspace  $I_S$ .*

*Proof.* The surjectivity is an immediate consequence of the fact that 01 cell-forms form a basis of  $H^{n-2}(\mathfrak{M}_{0,n+1})$  (theorem 3.16), since all such cell-forms are the images under  $f$  of polygons having the edge labelled 1 next to the one labelled  $n$ .

Now,  $I_S$  lies in the kernel of  $f$  by the corollary to proposition 3.13. So it only remains to show that the kernel of  $f$  is equal to  $I_S$ . But this is a consequence of counting the dimensions of both sides. By theorem 3.16, we know that the dimension of  $H^{n-2}(\mathfrak{M}_{0,n+1})$  is equal to  $(n-1)!$ . As for the dimension of  $\mathcal{P}_{S \cup \{d\}}/I_S$ , recall from the beginning of chapter 3 that  $\mathcal{P}_{S \cup \{d\}} \simeq V_S$ , which can be identified with the graded  $n$  part of the quotient of the polynomial algebra on  $S$  by the relation  $w = 0$  for all words  $w$  containing repeated letters. Thus  $V_S$  is the vector space spanned by words on  $n$  distinct letters, so it is of dimension  $n!$ . But instead of taking a basis of words, we can take the Lyndon basis of Lyndon words (words with distinct characters whose smallest character is on the left) and shuffles of Lyndon words. The subspace  $I_S$  is exactly generated by the shuffles, so the dimension of the quotient is given by the number of Lyndon words on  $S$ , namely  $(n-1)!$ . Therefore  $\mathcal{P}_{S \cup \{d\}}/I_S \simeq H^{n-2}(\mathfrak{M}_{0,n+1})$ .  $\square$

**Remark 3.48.** *The above proof has an interesting consequence. Since the map from polygons to differential forms does not depend on the role of  $d$ , the kernel cannot depend on  $d$ , and any other element of  $S \cup \{d\}$  could play the same role. Therefore  $I_S$ , which is defined as the space generated by shuffles with respect to the element  $d$ , is equal to the space generated by shuffles of elements of  $S \cup \{d\}$  with respect to any element of  $S$ ; it is simply the subspace generated by shuffles with respect to one element of  $S \cup \{d\}$ .*

**Corollary 3.49.** *Let  $W_S \subset \mathcal{P}_{S \cup \{d\}}$  be the subset of polygons corresponding to  $1n$ -words (concatenated with  $d$ ). Then*

$$f : W_S \simeq H^{n-2}(\mathfrak{M}_{0,n+1}).$$

*Proof.* The proof follows from the fact that  $\mathcal{P}_{S \cup \{d\}} = W_S \oplus I_S$ .  $\square$

### 3.4.2 Generators for $\mathcal{FC}$

By definition,  $\mathcal{FC}$  is generated by all linear combinations of pairs of polygons  $\sum_i a_i(\delta, \omega_i)$  whose associated differential form converges on the standard cell, but modulo the relation (among others) that shuffles are equal to zero. In other words, since  $\mathcal{P}_{S \cup \{d\}} = W_S \oplus I_S$ , we can redefine  $\mathcal{FC}$  to be generated by linear combinations  $\sum_i a_i(\delta, \omega_i)$  such that  $\sum_i a_i \omega_i \in W_S$  and such that the associated differential form converges on the standard cell.

The following proposition states that the notion of the residue of a polygon and the residue of the corresponding cell-form coincide. In order to state it, we must recall that one can define the map

$$\rho : \mathcal{P}_S \longrightarrow \Omega^\ell(\mathfrak{M}_{0,S}),$$

from polygons labelled by  $S$  to cell forms in a coordinate-free way (one can do this directly from equation (3.2.6)). In §1, this map was defined in explicit coordinates by fixing any three marked points at 0, 1 and  $\infty$ . This essence of lemma 3.10 is that  $\rho$  is independent of the choice of three marked points, and is thus coordinate-free.

**Proposition 3.50.** *Let  $S = \{1, \dots, n\}$  and let  $D$  be a stable partition  $S_1 \cup S_2$  of  $S$  corresponding to a boundary divisor of  $\mathfrak{M}_{0,n}$ , with  $|S_1| = r$  and  $|S_2| = s$ . Let  $\rho$  denote the usual map from polygons to cell-forms. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{P}_S & \xrightarrow{\rho} & H^\ell(\mathfrak{M}_{0,n}) \\ \text{Res}_D^p \downarrow & & \downarrow \text{Res}_D \\ \mathcal{P}_{S_1 \cup \{d\}} \otimes \mathcal{P}_{S_2 \cup \{d\}} & \xrightarrow{\rho \otimes \rho} & H^{r-2}(\mathfrak{M}_{0,r+1}) \otimes H^{s-2}(\mathfrak{M}_{0,s+1}). \end{array}$$

In other words, the usual residue of differential forms corresponds to the combinatorial residue of polygons.

*Proof.* Let  $\eta \in \mathcal{P}_S$  be a polygon, and let  $\omega_\eta$  be the associated cell-form. If  $D$  is not compatible with  $\omega_\eta$ , then  $\omega_\eta$  has no pole on  $D$  by proposition 3.11, so  $\text{Res}_D(\omega) = 0$ .

We shall work in explicit coordinates, bearing in mind that this does not affect the answer, by the remarks above. Therefore assume that  $\eta$  is the polygon numbered with the standard cyclic order on  $\{1, \dots, n\}$ , and that  $D$  is compatible with  $\eta$ . The corresponding cell-form is given in simplicial coordinates by  $[0, t_1, \dots, t_\ell, 1, \infty]$ . By applying a cyclic rotation, we can assume that  $D$  corresponds to the partition

$$S_1 = \{1, 2, 3, \dots, k+1\} \text{ and } S_2 = \{k+2, \dots, n-1, n\}$$

for some  $1 \leq k \leq \ell$ . In simplicial coordinates,  $D$  corresponds to the blow-up of the cycle  $0 = t_1 = \dots = t_k$ . We compute the residue of  $\omega_\eta$  along  $D$  by applying the variable change  $t_1 = x_1 \dots x_\ell, \dots, t_{\ell-1} = x_{\ell-1} x_\ell, t_\ell = x_\ell$  to the form  $\omega_\eta = [0, t_1, \dots, t_\ell, 1, \infty]$ . The standard cell  $X_\eta$  is given by  $\{0 < x_1, \dots, x_\ell < 1\}$ . In these coordinates, the divisor  $D$  is given by  $\{x_k = 0\}$ , and the form  $\omega_\eta$  becomes

$$\omega_\eta = \frac{dx_1 \dots dx_\ell}{x_1(1-x_1) \dots x_\ell(1-x_\ell)}. \quad (3.4.1)$$

The residue of  $\omega_\eta$  along  $x_k = 0$  is given by

$$\frac{dx_1 \dots dx_{k-1}}{x_1(1-x_1) \dots x_{k-1}(1-x_{k-1})} \otimes \frac{dx_{k+1} \dots dx_\ell}{x_{k+1}(1-x_{k+1}) \dots x_\ell(1-x_\ell)}. \quad (3.4.2)$$

Changing back to simplicial coordinates via  $x_1 = a_1/a_2, \dots, x_{k-2} = a_{k-2}/a_{k-1}, x_{k-1} = a_{k-1}$ , and  $x_\ell = b_\ell, x_{\ell-1} = b_{\ell-1}/b_\ell, \dots, x_{k+1} = b_k/b_{k+1}$  defines simplicial coordinates on  $D \cong \mathfrak{M}_{0,r+1} \times \mathfrak{M}_{0,s+1}$ . The standard cells induced by  $\eta$  are  $(0, a_1, \dots, a_{k-1}, 1, \infty)$  on  $\mathfrak{M}_{0,r+1}$  and  $(0, b_k, \dots, b_\ell, 1, \infty)$  on  $\mathfrak{M}_{0,s+1}$ . If we compute (3.4.2) in these new coordinates, it gives precisely

$$[0, a_1, \dots, a_{k-1}, 1, \infty] \otimes [0, b_k, \dots, b_\ell, 1, \infty],$$

which is the tensor product of the cell forms corresponding to the standard cyclic orders  $\eta_1, \eta_2$  on  $S_1 \cup \{d\}$  and  $S_2 \cup \{d\}$  induced by  $\eta$ . Therefore  $\rho(\text{Res}_D^p \eta) = \text{Res}_D \omega_\eta$ .

To conclude the proof of the proposition, it is enough to notice that applying  $\sigma \in \mathfrak{S}(n)$  to the formula  $\text{Res}_D \omega_\eta = \omega_{\eta_1} \otimes \omega_{\eta_2}$  yields

$$\text{Res}_{\sigma(D)} \sigma^*(\omega_\eta) = \text{Res}_{\sigma(D)} \omega_{\sigma(\eta)} = \sigma^*(\omega_{\eta_1}) \otimes \sigma^*(\omega_{\eta_2}) = \omega_{\sigma(\eta_1)} \otimes \omega_{\sigma(\eta_2)}.$$

Here,  $\sigma(\eta_i)$  is the cyclic order induced by  $\sigma(\eta)$  on the set  $\sigma(S_1) \cup \{\sigma(d)\}$ , where  $\sigma(d)$  corresponds to the partition  $S = \sigma(S_1) \cup \sigma(S_2)$ . Thus  $\rho(\text{Res}_{\sigma(D)}^p \sigma(\eta)) = \text{Res}_{\sigma(D)} \omega_{\sigma(\eta)}$  for all  $\sigma \in \mathfrak{S}(n)$ , which proves that  $\rho(\text{Res}_D^p \gamma) = \text{Res}_D \omega_\gamma$  for all cyclic structures  $\gamma \in \mathcal{P}_S$ , and all divisors  $D$ .  $\square$

**Corollary 3.51.** *A linear combination  $\eta = \sum_i a_i \eta_i \in W_S \subset \mathcal{P}_{S \cup \{d\}}$  converges with respect to the standard polygon if and only if its associated form  $\omega_\eta$  converges on the standard cell.*

*Proof.* We first show that

$$\text{Res}_D^p(\eta) \in I_{S_1} \otimes \mathcal{P}_{S_2 \cup \{d\}} + \mathcal{P}_{S_1 \cup \{d\}} \otimes I_{S_2}, \quad (3.4.3)$$

if and only if  $\omega_\eta$  converges along the corresponding divisor  $D$  in the boundary of the standard cell. If (3.4.3) holds, then by proposition 3.47 together with the previous proposition,  $\text{Res}_D(\omega_\eta) = 0$ . Conversely, if  $\text{Res}_D(\omega_\eta) = 0$  for a divisor  $D$  in the boundary of the standard cell, then by the previous proposition,  $\text{Res}_D^p(\eta) \in \text{Ker}(\rho \otimes \rho)$ , which is exactly equal to  $I_{S_1} \otimes \mathcal{P}_{S_2 \cup \{d\}} + \mathcal{P}_{S_1 \cup \{d\}} \otimes I_{S_2}$ .

We now show that (3.4.3) is equivalent to the convergence of  $\eta$ . But since  $\eta \in W_S$ , the argument of lemma 3.46 implies that (3.4.3) holds automatically for any  $D$  which intersects  $\{1, n\}$  non-trivially. If  $D$  intersects  $\{1, n\}$  trivially, then we can assume that  $\{1, n\} \subset S_2$ . In that case, the fact that  $W_{S_2} \cap I_{S_2} = 0$  (lemma 3.35) implies that (3.4.3) is equivalent to the apparently stronger condition

$$\text{Res}_D^p(\eta) \in I_{S_1} \otimes \mathcal{P}_{S_2 \cup \{d\}},$$

and thus  $\eta$  converges along  $S_1$  in the sense of definition (3.3.5). This holds for all divisors  $D$  and thus completes the proof of the corollary.  $\square$

**Corollary 3.52.** *The Lyndon insertion words of  $\mathcal{W}_S$  form a generating set for  $\mathcal{FC}$ . Furthermore,  $\mathcal{FC}$  is defined by subjecting this generating set to only two sets of relations:*

- *dihedral relations*
- *product map relations*

### 3.4.3 The insertion basis for $H^\ell(\mathfrak{M}_{0,n}^\delta)$

**Definition 3.53.** *Let an insertion form be the sum of 01-cell forms obtained by renumbering the Lyndon insertion words of  $\mathcal{W}_S$  via  $(1, \dots, n, d) \rightarrow (0, t_1, \dots, t_{\ell+1}, 1, \infty)$ .*

**Proposition 3.54.** *The insertion forms form a basis for  $H^{n-2}(\mathfrak{M}_{0,n+1}^\delta)$ .*

This is an immediate corollary of all the preceding results.

It is interesting to attempt to determine the dimension of the spaces  $H^\ell(\mathfrak{M}_{0,n}^\delta)$ . The most important numbers needed to compute these are the numbers  $c_0(n)$  of special convergent words (convergent 01 cell-forms) on  $\mathfrak{M}_{0,n}$ . We have  $c_0(4) = 0$ ,  $c_0(5) = 1$ ,  $c_0(6) = 2$ ,  $c_0(7) = 11$ ,  $c_0(8) = 64$ ,  $c_0(9) = 461$ .

**Proposition 3.55.** *Set  $I_1 = 1$ , and let  $I_r$  denote the cardinal of the set  $\mathcal{L}_{\{1, \dots, r\}}$  for  $r \geq 2$ . The dimensions  $\dim H^\ell(\mathfrak{M}_{0,n}^\delta)$  are given by*

$$d_n = \sum_{r=5}^n \sum_{i_1 + \dots + i_{r-3} = n-3} I_{i_1} \dots I_{i_{r-3}} c_0(r), \quad (3.4.4)$$

where the inner sum is over all partitions of  $(n-3)$  into  $(r-3)$  strictly positive integers.

We have  $I_1 = I_2 = 1, I_3 = 2, I_4 = 7$ . The formula gives

$$\begin{cases} d_5 = I_1^2 c_0(5) = 1, \\ d_6 = I_1 I_2 c_0(5) + I_2 I_1 c_0(5) + I_1^3 c_0(6) = 1 + 1 + 2 = 4, \\ d_7 = I_1 I_3 c_0(5) + I_2^2 c_0(5) + I_3 I_1 c_0(5) + I_1^2 I_2 c_0(6) + I_1 I_2 I_1 c_0(6) + I_2 I_1^2 c_0(6) + c_0(7) \\ = 5c_0(5) + 3c_0(6) + c_0(7) = 5 + 6 + 11 = 22. \end{cases}$$

These expressions give the dimensions as sums of positive terms. A very different formula for  $\dim H^\ell(\mathfrak{M}_{0,n}^\delta)$  is given in the appendix using point-counting methods.

### 3.4.4 The insertion basis for $\mathfrak{M}_{0,n}, 5 \leq n \leq 9$

In this section we list the insertion bases in low weights. In the case  $\mathfrak{M}_{0,5}$ , there is a single convergent cell form:

$$\omega = [0, 1, t_1, \infty, t_2]. \quad (3.4.5)$$

The corresponding period integral is the cell-zeta value:

$$\zeta(\omega) = \int_{(0, t_1, t_2, 1, \infty)} [0, 1, t_1, \infty, t_2] = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1 dt_2}{(1-t_1)t_2} = \zeta(2).$$

Here we use the notation of round brackets for cells in the moduli space  $\mathfrak{M}_{0,n}$  introduced in section 3.2.3: the cell  $(0, t_1, t_2, 1, \infty)$  is the same as the cell  $X_{5,\delta}$  corresponding to the standard dihedral order on the set  $\{0, t_1, t_2, 1, \infty\}$ . Since  $C_0(5)$  is 1-dimensional, the space of periods in weight 2, namely the weight 2 graded part  $\mathcal{C}_2$  of the algebra of cell-zeta values  $\mathcal{C}$  of section 3.2.4, is just the 1-dimensional space spanned by  $\int_{X_{5,\delta}} \omega = \zeta(2)$ .

#### The case $\mathfrak{M}_{0,6}$

The space  $C(6)$  is four-dimensional, generated by two 01-convergent cell-forms (the first row in the table below) and two forms (the second row in the table below) which come from inserting  $\mathcal{L}_{1,2} = \{1\mathfrak{m}2\}$  and  $\mathcal{L}_{2,3} = \{2\mathfrak{m}3\}$  into the unique convergent 01 cell form on  $\mathfrak{M}_{0,5}$  (3.4.5). The position of the point  $\infty$  plays a special role. It gives rise to another grading, corresponding to the two columns in the table below, since  $\infty$  can only occur in two positions.

$C_0(6)$	$\omega_{1,1} = [0, 1, t_2, \infty, t_1, t_3]$	$\omega_{1,2} = [0, 1, t_1, t_3, \infty, t_2]$
$C_1(6)$	$\omega_{2,1} = [0, 1, t_1, \infty, t_2\mathfrak{m}t_3]$	$\omega_{2,2} = [0, 1, t_1\mathfrak{m}t_2, \infty, t_3]$

We therefore have four generators in weight 3. There are no product relations on  $\mathfrak{M}_{0,6}$ , so in order to compute the space of cell-zeta values, we need only compute the action of the dihedral group on the four differential forms. In particular, the order 6 cyclic generator  $0 \mapsto t_1 \mapsto t_2 \mapsto t_3 \mapsto 1 \mapsto \infty \mapsto 0$  sends

$$\omega_{1,1} \mapsto -\omega_{2,1} - \omega_{2,2}, \quad \omega_{1,2} \mapsto \omega_{1,1}, \quad \omega_{2,1} \mapsto -\omega_{1,2} - \omega_{2,1}, \quad \omega_{2,2} \mapsto \omega_{2,1}.$$

Thus, letting  $X$  denote the standard cell  $X_{6,\delta} = (0, t_1, t_2, t_3, 1, \infty)$ , we have  $\int_X \omega_{1,1} = \int_X \omega_{1,2}$ ,  $\int_X \omega_{2,1} = \int_X \omega_{2,2}$  and  $2 \int_X \omega_{2,2} = \int_X \omega_{1,2}$ , so in fact the periods form a single



orbit under the action of the cyclic group of order 6 on  $H^\ell(\mathfrak{M}_{0,S}^\delta)$ . We deduce that the space of periods of weight 3 is of dimension 1, generated for instance by  $\int \omega_{2,1}$ . Since  $\omega_{2,1}$  is the standard form for  $\zeta(3)$ , we have

$$\begin{aligned}\zeta(0, 1, t_2, \infty, t_1, t_3) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_2)(t_1-t_3)t_3} = 2\zeta(3), \\ \zeta(0, 1, t_1, t_3, \infty, t_2) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_1)(t_1-t_3)t_2} = 2\zeta(3), \\ \zeta(0, 1, t_1, \infty, t_2 \text{m} t_3) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_1)t_2 t_3} = \zeta(3), \\ \zeta(0, 1, t_1 \text{m} t_2, \infty, t_3) &= \int_X \frac{dt_1 dt_2 dt_3}{(1-t_1)(1-t_2)t_3} = \zeta(3),\end{aligned}$$

Note that  $\omega_{2,2}$  is the standard form usually associated to  $\zeta(2, 1)$ , so that we have recovered the well-known identity  $\zeta(2, 1) = \zeta(3)$ , which is normally obtained using stuffle, shuffle and Hoffmann relations on multizetas.

### The case $\mathfrak{M}_{0,7}$

The insertion basis is listed in the following table. It consists of 22 forms, eleven of which lie in  $C_0(7)$ , six of which come from making one insertion into a convergent 01 cell-form from  $C_0(6)$  (using  $\mathcal{L}_{1,2} = \{1\text{m}2\}$  and  $\mathcal{L}_{2,3} = \{2\text{m}3\}$ ), and five of which come from making two insertions into the unique convergent 01 cell-form from  $C_0(5)$  (which also uses  $\mathcal{L}_{1,2,3} = \{1\text{m}2\text{m}3, 2\text{m}13\}$  and  $\mathcal{L}_{2,3,4} = \{2\text{m}3\text{m}4, 3\text{m}24\}$ ).

$C_0(7)$	$[0, 1, t_2, \infty, t_3, t_1, t_4]$ $[0, 1, t_2, \infty, t_4, t_1, t_3]$ $[0, 1, t_3, \infty, t_1, t_4, t_2]$	$[0, 1, t_1, t_3, \infty, t_2, t_4]$ $[0, 1, t_1, t_3, \infty, t_4, t_2]$ $[0, 1, t_2, t_4, \infty, t_1, t_3]$ $[0, 1, t_3, t_1, \infty, t_2, t_4]$ $[0, 1, t_3, t_1, \infty, t_4, t_2]$	$[0, 1, t_1, t_4, t_2, \infty, t_3]$ $[0, 1, t_2, t_4, t_1, \infty, t_3]$ $[0, 1, t_3, t_1, t_4, \infty, t_2]$
$C_1(7)$	$[0, 1, t_2, \infty, t_1, t_3 \text{m} t_4]$ $[0, 1, t_3, \infty, t_1 \text{m} t_2, t_4]$	$[0, 1, t_1, t_4, \infty, t_2 \text{m} t_3]$ $[0, 1, t_2 \text{m} t_3, \infty, t_1, t_4]$	$[0, 1, t_1 \text{m} t_2, t_4, \infty, t_3]$ $[0, 1, t_1, t_3 \text{m} t_4, \infty, t_2]$
$C_2(7)$	$[0, 1, t_1, \infty, t_3 \text{m} (t_2, t_4)]$ $[0, 1, t_1, \infty, t_2 \text{m} t_3 \text{m} t_4]$	$[0, 1, t_1 \text{m} t_2, \infty, t_3 \text{m} t_4]$	$[0, 1, t_2 \text{m} (t_1, t_3), \infty, t_4]$ $[0, 1, t_1 \text{m} t_2 \text{m} t_3, \infty, t_4]$

The standard multizeta forms can be decomposed into sums of insertion forms as follows:

$$\begin{aligned}\frac{dt_1 dt_2 dt_3 dt_4}{(1-t_1)t_2 t_3 t_4} &= [0, 1, t_1, \infty, t_2 \text{m} t_3 \text{m} t_4] \\ \frac{dt_1 dt_2 dt_3 dt_4}{(1-t_1)(1-t_2)t_3 t_4} &= [0, 1, t_1 \text{m} t_2, \infty, t_3 \text{m} t_4] \\ \frac{dt_1 dt_2 dt_3 dt_4}{(1-t_1)t_2(1-t_3)t_4} &= [0, 1, t_1, t_3, \infty, t_2, t_4] + [0, 1, t_1, t_3, \infty, t_4, t_2] + \\ &\quad [0, 1, t_3, t_1, \infty, t_2, t_4] + [0, 1, t_3, t_1, \infty, t_4, t_2] \\ \frac{dt_1 dt_2 dt_3 dt_4}{(1-t_1)(1-t_2)(1-t_3)t_4} &= [0, 1, t_1 \text{m} t_2 \text{m} t_3, \infty, t_4]\end{aligned}\tag{3.4.6}$$



In general, the standard multizeta form having factors  $(1-t_{i_1}), \dots, (1-t_{i_r})$  (with  $i_1 = 1$ ) and  $t_{j_1}, \dots, t_{j_s}$  (with  $j_s = n$ ) in the denominator is equal to the shuffle form:

$$[0, 1, t_{i_1} \amalg \dots \amalg t_{i_r}, \infty, t_{j_1} \amalg \dots \amalg t_{j_s}],$$

so to decompose it into insertion forms it is simply necessary to decompose the shuffles  $t_{i_1} \amalg \dots \amalg t_{i_r}$  and  $t_{j_1} \amalg \dots \amalg t_{j_s}$  into linear combinations of Lyndon insertion shuffles.

Computer computation confirms that the space of periods on  $\mathfrak{M}_{0,7}$  is of dimension 1 and is generated by  $\zeta(2)^2$ . Indeed, up to dihedral equivalence, there are six product maps on  $\mathfrak{M}_{0,7}$ , given by

$$\left\{ \begin{array}{l} (0, t_1, t_2, t_3, t_4, 1, \infty) \mapsto (0, t_1, t_2, 1, \infty) \times (0, t_3, t_4, 1, \infty) \\ (0, t_1, t_2, 1, t_3, t_4, \infty) \mapsto (0, t_1, t_2, 1, \infty) \times (0, 1, t_3, t_4, \infty) \\ (0, t_1, t_2, 1, t_3, \infty, t_4) \mapsto (0, t_1, t_2, 1, \infty) \times (0, 1, t_3, \infty, t_4) \\ (0, t_1, t_2, 1, t_3, \infty, t_4) \mapsto (0, t_1, 1, t_3, \infty) \times (0, t_2, 1, \infty, t_4) \\ (0, t_1, t_2, t_3, 1, t_4, \infty) \mapsto (0, t_1, t_2, 1, \infty) \times (0, t_3, 1, t_4, \infty) \\ (0, t_1, t_2, 1, t_3, t_4, \infty) \mapsto (0, t_1, 1, t_3, \infty) \times (0, t_2, 1, t_4, \infty) \end{array} \right. \quad (3.4.7)$$

Following the algorithm from section 3.2.3, we have six associated relations between the integrals of the 22 cell-forms. Then, explicitly computing the dihedral action on the forms yields a further set of linear equations, and it is a simple matter to solve the entire system of equations to recover the 1-dimensional solution. It also provides the value of each integral of an insertion form as a rational multiple of any given one; for instance all the values can be computed as rational multiples of  $\zeta(2)^2$ . In particular, we easily recover the usual identities

$$\zeta(4) = \frac{2}{5}\zeta(2)^2, \quad \zeta(3, 1) = \frac{1}{10}\zeta(2)^2, \quad \zeta(2, 2) = \frac{3}{10}\zeta(2)^2, \quad \zeta(2, 1, 1) = \frac{2}{5}\zeta(2)^2.$$

### The cases $\mathfrak{M}_{0,8}$ and $\mathfrak{M}_{0,9}$

There are 64 convergent 01 cell-forms in on  $\mathfrak{M}_{0,8}$ , and the dimension of  $H^5(\mathfrak{M}_{0,8}^\delta)$  is 144. The remaining 80 forms are obtained by Lyndon insertion shuffles as follows:

- 44 forms obtained by making the four insertions:

$$(t_1 \amalg t_2, t_3, t_4, t_5), (t_1, t_2 \amalg t_3, t_4, t_5), (t_1, t_2, t_3 \amalg t_4, t_5), (t_1, t_2, t_3, t_4 \amalg t_5)$$

into the eleven 01 cell-forms of  $\mathfrak{M}_{0,7}$

- 12 forms obtained by the six insertion possibilities:

$$(t_1 \amalg t_2 \amalg t_3, t_4, t_5), (t_2 \amalg t_1 t_3, t_4, t_5), (t_1, t_2 \amalg t_3 \amalg t_4, t_5), (t_1, t_3 \amalg t_2 t_4, t_5), \\ (t_1, t_2, t_3 \amalg t_4 \amalg t_5), (t_1, t_2, t_4 \amalg t_3 t_5)$$

into the two 01 cell-forms of  $\mathfrak{M}_{0,6}$

- 6 forms obtained by the three insertion possibilities:

$$(t_1 \amalg t_2, t_3 \amalg t_4, t_5), (t_1 \amalg t_2, t_3, t_4 \amalg t_5), (t_1, t_2 \amalg t_3, t_4 \amalg t_5)$$

into the two 01 cell-forms of  $\mathfrak{M}_{0,6}$

- 4 forms obtained by the four insertions:

$$(t_1 \mathfrak{m} t_2 \mathfrak{m} t_3, t_4 \mathfrak{m} t_5), (t_2 \mathfrak{m} t_1 t_3, t_4 \mathfrak{m} t_5), (t_1 \mathfrak{m} t_2, t_3 \mathfrak{m} t_4 \mathfrak{m} t_5), (t_1 \mathfrak{m} t_2, t_4 \mathfrak{m} t_3 t_5)$$

into the single 01 cell-form of  $\mathfrak{M}_{0,5}$

- 14 forms obtained by the fourteen insertions:

$$(t_1 t_3 \mathfrak{m} t_2 t_4, t_5), (t_3 \mathfrak{m} t_1 t_4 t_2, t_5), (t_1 t_3 \mathfrak{m} t_2 \mathfrak{m} t_4, t_5), (t_1 t_4 \mathfrak{m} t_2 \mathfrak{m} t_3, t_5), (t_2 t_4 \mathfrak{m} t_1 \mathfrak{m} t_3, t_5), \\ (t_2 \mathfrak{m} t_1 (t_3 \mathfrak{m} t_4), t_5), (t_1 \mathfrak{m} t_2 \mathfrak{m} t_3 \mathfrak{m} t_4, t_5), (t_1, t_2 t_4 \mathfrak{m} t_3 t_5), (t_1, t_4 \mathfrak{m} t_2 t_5 t_3), (t_1, t_2 t_4 \mathfrak{m} t_3 \mathfrak{m} t_5), \\ (t_1, t_2 t_5 \mathfrak{m} t_3 \mathfrak{m} t_4), (t_1, t_3 t_5 \mathfrak{m} t_2 \mathfrak{m} t_4), (t_1, t_3 \mathfrak{m} t_2 (t_4 \mathfrak{m} t_5)), (t_1, t_2 \mathfrak{m} t_3 \mathfrak{m} t_4 \mathfrak{m} t_5)$$

into the single 01 cell-form of  $\mathfrak{M}_{0,5}$ .

The case of  $\mathfrak{M}_{0,9}$  is too large to give explicitly. There are 461 convergent 01 cell-forms, and  $\dim H^6(\mathfrak{M}_{0,9}^\delta) = 1089$ . An interesting phenomenon occurs first in the case  $\mathfrak{M}_{0,9}$ ; namely, this is the first value of  $n$  for which convergent (but not 01) cell-forms do not generate the cohomology. The 1463 convergent cell-forms for  $\mathfrak{M}_{0,9}$  generate a subspace of dimension 1088.

For  $5 \leq n \leq 9$ , computer computations have confirmed the main conjecture, namely: for  $n \leq 9$ , the weight  $n-3$  part  $\mathcal{FC}_{n-3}$  of the formal cell-zeta algebra  $\mathcal{FC}$  is of dimension  $d_{n-3}$ , where  $d_n$  is given by the Zagier formula  $d_n = d_{n-2} + d_{n-3}$  with  $d_0 = 1, d_1 = 0, d_2 = 1$ .

## Chapter 4

# Cohomology of $(\mathfrak{M}_{0,n}^\gamma)$

**Definition 4.1.** The  $k$ th de Rham cohomology group of a smooth manifold,  $X$ , is defined to be the group of closed differential  $k$  forms on  $X$  (those whose exterior derivative is 0) modulo exact ones (those which are the exterior derivative of a  $k - 1$  form).

In the first section of this chapter, we use the theory of spectral sequences of a fibration to review the proof of the following well-known dimension result on cohomology groups of genus 0 moduli space, a complete proof of which is difficult (or impossible) to find in the literature.

**Theorem 4.2.** For  $n \geq 3$ , the dimension of  $H^{n-3}(\mathfrak{M}_{0,n}, \mathbb{Q})$  is  $(n - 2)!$  and the dimension of  $H^k(\mathfrak{M}_{0,n}, \mathbb{Q})$  is 0 whenever  $k > n - 3$ .

This result was used and reproved (though not as explicitly as we do in the following sections) in a well-known theorem by Arnol'd. For each cohomology group,  $H^k(\mathfrak{M}_{0,n}, \mathbb{Q})$ , Arnol'd explicitly exhibits a set,  $B_n^k$  of differential forms whose classes form a basis of  $H^k(\mathfrak{M}_{0,n}, \mathbb{Q})$  and which has the astonishing property that the ring,  $A$ , generated by  $B_n^1$  contains  $B_n^k$  for all  $k$  and  $A$  is isomorphic to  $H^*(\mathfrak{M}_{0,n}, \mathbb{Q})$ . For the remainder of the text, we will denote  $H^k(\mathfrak{M}_{0,n}, \mathbb{Q})$  simply by  $H^k(\mathfrak{M}_{0,n})$ .

**Definition 4.3.** We denote by  $\omega_{t_i, t_j}$  the differential forms defined by :

$$\omega_{t_i, t_j} = \frac{dt_j - dt_i}{t_i - t_j}, \quad 1 \leq i < j \leq n - 3 \quad (4.0.1)$$

$$\omega_{0, t_j} = \frac{dt_j}{t_j} \quad (4.0.2)$$

$$\omega_{1, t_j} = \frac{dt_j}{1 - t_j}. \quad (4.0.3)$$

We call the ring generated by these forms **Arnol'd's Ring**,  $A$ .

**Theorem 4.4** (Arnol'd). Let  $i_1, \dots, i_k$  be distinct integers in the interval  $[1, n]$ . The elements of Arnol'd's ring of the form,

$$\bigwedge_{l=1}^k \omega_{z, t_{i_l}}, \quad z = t_j, 0 \text{ or } 1$$

(where  $j < i_l$ ) form a basis of  $H^k(\mathfrak{M}_{0,n})$ . In particular, a basis of  $H^{n-3}(\mathfrak{M}_{0,n})$  is given by

$$\bigwedge_{j=1}^{n-3} \frac{dt_j}{t_j - z} : z = 0, 1 \text{ or } t_i, i < j.$$

Because Arnol'd's theorem is a key ingredient in our work, in section 4.1 of this chapter, we recall a self-contained proof of the theorem 4.2. Here, we recall some definitions and properties of divisors on  $\overline{\mathfrak{M}}_{0,n}$  used throughout the rest of the text.

**Definition 4.5.** Let  $Z$  be the set denoting marked points on  $\mathfrak{M}_{0,n}$ ,  $Z = \{z_1, \dots, z_n\}$  and let  $\rho$  be the set of all partitions of  $Z$ , in which each set in the partition has cardinality greater than or equal to 2.

**Definition 4.6.** Let  $\{z_{i_1}, \dots, z_{i_k}\} = K \subset Z$  be a subset of the marked points. Then the divisor which is obtained as the exceptional divisor by blowing up along  $z_{i_1} = \dots = z_{i_k}$  in  $\mathbb{P}_1^{n-3}$  is denoted by  $d_K$ .

Recall from page 13 that  $d_A = d_{Z \setminus A}$  in  $\overline{\mathfrak{M}}_{0,n}$ .

**Definition 4.7.** We denote by  $D$  the disjoint union,  $\sqcup_{i \in \rho} \{d_i\}$  where each  $\{d_i\}$  is a singleton whose single element is the (irreducible) boundary divisor in  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$  defined by the partition  $i$  as in the definition 4.6. Likewise, if  $\gamma \sqcup \gamma^c$  is a partition of  $\rho$ , we denote by  $D_{\gamma^c} := \sqcup_{i \in \gamma^c} \{d_i\}$ . We denote by  $\mathfrak{M}_{0,n}^\gamma := \overline{\mathfrak{M}}_{0,n} \setminus D_{\gamma^c}$  and call  $\mathfrak{M}_{0,n}^\gamma$  a partial compactification of  $\mathfrak{M}_{0,n}$ .

So we have  $\mathfrak{M}_{0,n} \subset \mathfrak{M}_{0,n}^\gamma \subset \overline{\mathfrak{M}}_{0,n}$ .

We remark here that the results outlined in this chapter are combinatorial ones obtained by considering an irreducible boundary component as the set of its defining marked points, therefore by a slight abuse of notation, and when no ambiguity can arise, we will denote  $D_\gamma$  simply by  $\gamma$ .

In the second section of this chapter we show an analog of Arnol'd's theorem for partial compactifications. In particular, we show that the cohomology rings of the partial compactifications are the subrings of Arnol'd's ring converging on the partial compactification.

In the third section, we will recall Brown's proof that  $\mathfrak{M}_{0,n}^\delta$  is an affine variety when  $D_\delta$  is the set of divisors each of which contains a face of the boundary of an associahedron in  $\mathfrak{M}_{0,n}(\mathbb{R})$  (as in chapter 3), and deduce that any subset,  $D_\gamma \subseteq D_\delta$  of boundary divisors also has the property that  $\mathfrak{M}_{0,n}^\gamma$  is affine (for example when  $|\gamma| = 1, 2$ ).

In the fourth section, for some of the families  $D_\gamma$  of boundary divisors from section 4.3, we display explicit bases of the top dimensional cohomology groups of  $\mathfrak{M}_{0,n}^\gamma$ . For  $\gamma = \delta$ , the union of the boundary divisors of the standard cell, this computation was done in chapter 3 where we defined the basis of *insertion forms*. In chapter 4, we generalize the method of insertion forms.

In the last section, we study a combinatorial presentation of the Picard group of divisors on  $\overline{\mathfrak{M}}_{0,n}$  based on work of S. Keel and A. Gibney. We extend their techniques of calculating a basis to calculating an explicit expression of any boundary divisor in terms of these bases by using polygon techniques.

## 4.1 Spectral sequences of a fibration

The goal of this section is to recall the proof theorem 4.2 by induction, using only the Leray theorem of the cohomology of a spectral sequence.

*Proof (of theorem 4.2).* We begin the proof by justifying the base case,  $n = 3$ , in which case  $\mathfrak{M}_{0,3}$  is a point. The dimension of  $H^0(\mathfrak{M}_{0,3})$  is thus  $1! = 1$  and  $H^k(\mathfrak{M}_{0,3}) = 0$  for all  $k > 0$ .

So we have two induction hypotheses,  $\dim(H^{n-4}(\mathfrak{M}_{0,n-1})) = (n-3)!$  and for  $k > n-4$ ,  $H^k(\mathfrak{M}_{0,n-1}) = 0$ .

Before detailing the proof, let us recall some notation and some results on spectral sequences.

Let

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_k = X \quad (4.1.1)$$

be a filtration of the topological space  $X$ . And let

$$C_q(X_{-1}) \subset \cdots \subset C_q(X_k)$$

be the formal groups of  $\mathbb{Z}$  linear combinations of  $q$  dimensional oriented simplices in the subspaces  $(X_i)$ . Such filtrations exist for nice topological spaces, such as CW complexes. There exist linear maps,  $\partial_q : C_q(X_i) \rightarrow C_{q-1}(X_i)$  which send a simplex to its boundary in  $X_i$ . Since the simplices are oriented,  $\partial_{q-1} \circ \partial_q = 0$ .

Let

$$E_0^{i,q-i} := C_q(X_i)/C_q(X_{i-1}).$$

Then  $\partial_q$  induces an exact sequence,  $(E_0^{i,q-i}, d_0^{i,q-i})$ .

Let  $Z_r^{i,q-i} \subset E_0^{i,q-i}$  be the subgroup of elements,  $\alpha$ , such that the coset of  $\alpha$  contains a representative  $a$  such that  $\partial_q(a) \in C_{q-1}(X_{i-r})$ . From this definition, we see that  $Z_0^{i,q-i} = E_0^{i,q-i}$ .

By the filtration, we have  $Z_k^{i,q-i} \subset Z_{k-1}^{i,q-i}$ . If  $r$  is sufficiently large, we obtain a stable group,  $Z_\infty^{i,q-i}$  whose elements  $\alpha$  contain a coset  $a$  such that  $\partial_q(a) = 0$ .

Let  $B_r^{i,q-i} \subset E_0^{i,q-i}$  be the subset of elements  $\alpha$  whose coset contains an element  $a$  such that there exists an element  $b \in C_{q+1}(X_{i+r-1})$  such that  $a = \partial_{q+1}(b)$ .

By the filtration, we have that  $B_k^{i,q-i} \subset B_{k+1}^{i,q-i}$ . Furthermore, if  $r$  is large enough, we obtain a stable group,  $B_\infty^{i,q-i}$  which contains all of the elements  $\alpha = \bar{a}$  such that  $a$  is the boundary of some simplex in  $C_q(X)$ . Since  $\partial_{q-1} \circ \partial_q = 0$ , we have that  $B_\infty^{i,q-i} \subset Z_\infty^{i,q-i}$ .

Let  $E_r^{i,q-i} := Z_r^{i,q-i}/B_r^{i,q-i}$ . The differential  $d_0^{i,q-i}$  induces a complex,

$$d_r^{i,q-i} : E_r^{i,q-i} \rightarrow E_r^{i-r,q-i+r-1}.$$

Let

$$E_r := \bigoplus_{i,q} E_r^{i,q-i},$$

which is a complex for the differential  $d_r := \bigoplus d_r^{i,q-i} : E_r \rightarrow E_r$ .

**Definition 4.8.** The complex  $(E_r, d_r)$  is a spectral sequence for the given filtration of  $X$ .

**Theorem 4.9.**  $E_{r+1}$  is the homology group of  $E_r$  with respect to the differential  $d_r$ , in particular,

$$E_{r+1}^{p,q} \simeq \ker(d_r^{p,q})/\text{Im}(d_r^{p-r,q+r-1}).$$

The proof of this theorem can be found in many textbooks, such as [FFG].

In our studies, we are concerned with the stabilizing groups,

$$E_\infty^{i,q-i} = Z_\infty^{i,q-i}/B_\infty^{i,q-i}.$$

Now let us consider the fibration

$$\mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,n-1} \quad (4.1.2)$$

$$(0, t_1, \dots, t_{n-3}, 1, \infty) \mapsto (0, t_1, \dots, t_{n-4}, 1, \infty), \quad (4.1.3)$$

with fiber equal to  $\mathbb{P}^1(\mathbb{C}) \setminus \{0, t_1, \dots, t_{n-4}, 1, \infty\}$  over  $(0, t_1, \dots, t_{n-4}, 1, \infty)$ .

We write this fibration in the classical notation as

$$F \hookrightarrow E \rightarrow B \quad (4.1.4)$$

where  $F$  is the fiber, the projective line minus  $n - 1$  points,  $B$  the base,  $\mathfrak{M}_{0, n-1}$ , and  $E$  is  $\mathfrak{M}_{0, n}$ .

**Lemma 4.10.** *The homology groups of  $F$  as  $\mathbb{R}$  vector spaces are given by*

$$H_q(F, \mathbb{R}) \simeq \begin{cases} \mathbb{R} & q = 0 \\ \mathbb{R}^{n-2} & q = 1 \\ 0 & q > 1 \end{cases} \quad (4.1.5)$$

This simple lemma may be deduced by using a long-exact Mayer-Vietoris sequence.

Any differentiable manifold has the homotopy type of a CW complex [Mi]. In particular,  $\mathfrak{M}_{0, n}$  is a CW complex and so there exists a filtration as in (4.1.1) on  $\mathfrak{M}_{0, n}$ ,

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \dots \subseteq X_k = \mathfrak{M}_{0, n},$$

where  $X_i$  denotes the  $i$ th skeleton of  $\mathfrak{M}_{0, n}$ .

We have a bundle of groups over  $\mathfrak{M}_{0, n-1}$  given by the family,  $\{H_1(F_b) : b \in B\}$ , and the associated family of homomorphisms,

$$h[\lambda] : H_1(F_{b_0}) \rightarrow H_1(F_{b_1}),$$

for all paths  $\lambda$  from  $b_0$  to  $b_1$  on  $B$ . The homomorphism,  $h[\lambda]$  comes from lifting the path  $\lambda$  to  $\mathfrak{M}_{0, n}$ . The choice of a lift of  $b_0$  to  $F_{b_0} \subset \mathfrak{M}_{0, n}$  determines the lift of  $\lambda$  uniquely. Therefore the endpoint of this lift, a lift of  $b_1$ , is uniquely defined, giving a map from  $F_{b_0}$  to  $F_{b_1}$ . This map induces a map on the fundamental groups and hence passes to the  $H_1$ . The  $h[\lambda]$  satisfy

$$h[\text{Id}] = \text{Id} \quad (4.1.6)$$

$$h[\lambda] = h[\lambda'] \text{ whenever } \lambda \text{ is homotopic to } \lambda' \quad (4.1.7)$$

$$\lambda : b_0 \rightarrow b_1, \mu : b_1 \rightarrow b_2, \text{ then } h[\mu \circ \lambda] = h[\mu] \circ h[\lambda]. \quad (4.1.8)$$

**Definition 4.11.** *A fibration is simple if for all  $\lambda_1, \lambda_2 : b_0 \rightarrow b_1$ ,  $h[\lambda_1] = h[\lambda_2]$ .*

**Claim 4.12.** *The fibration (4.1.2) is simple.*

*Proof.* Given two points  $b_0 \neq b_1$  on  $B$ , we may fix a path  $\lambda_0$  from  $b_0$  to  $b_1$ , and every path on  $B$  from  $b_0$  to  $b_1$  is homotopic to a loop starting at  $b_0$  composed with  $\lambda_0$ . Thus, by definition 4.11 we only need to show simplicity for loops  $\lambda$  on  $B$  based at a point  $b$ , in other words that  $(h[\lambda] : H_1(F_b) \rightarrow H_1(F_b)) = \text{Id}$ . We saw above that loops on  $B$ , and in fact homotopy classes of loops on  $B$ , act on  $\pi_1(F)$ ; in other words there is a group action of  $\pi_1(B)$  on  $\pi_1(F)$ . This action can be explicitly computed as follows.

Recall the definition of the Artin braid group,  $B_{n-1}$ , generated by the fundamental braids  $\sigma_i, i = 1, \dots, n - 2$ , subject to the relations  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  and  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for all  $|i - j| \geq 2$ . The full mapping class group  $\Gamma_{0, [n-1]}$  is defined to be the quotient of  $B_{n-1}$  by the two relations,  $(\sigma_1 \cdots \sigma_{n-2})^{n-1} = 1$  and  $\sigma_1 \cdots \sigma_{n-2} \cdot \sigma_{n-2} \cdots \sigma_1 = 1$ . There is a surjection  $B_{n-1} \twoheadrightarrow \mathfrak{S}_{n-1}$  given by mapping  $\sigma_i \mapsto (i, i + 1)$ , which factors through  $\Gamma_{0, [n-1]}$ .

Let the free group,  $\pi_1(F)$ , be generated by the loops,  $x_1, \dots, x_{n-1}$  around the marked points of  $F$ , whose product equals 1. The group  $\Gamma_{0,[n-1]}$  acts on  $\pi_1(F)$  via

$$\sigma_i(x_j) = \begin{cases} x_j & j < i, j > i + 1 \\ x_{j+1} & j = i \\ x_j^{-1} x_{j-1} x_j & j = i + 1. \end{cases} \quad (4.1.9)$$

The fundamental group of  $B = \mathfrak{M}_{0,n-1}$ , known as the pure mapping class group,  $\Gamma_{0,n-1}$ , is the kernel of the surjection,  $\Gamma_{0,[n-1]} \twoheadrightarrow \mathfrak{S}_{n-1}$ . It is generated by the elements  $x_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ . It acts on  $\pi_1(F)$  by restriction of the action (4.1.9), and it is easy to see that each generator,  $x_{i,j}$  maps each  $x_k$  to a conjugate of  $x_k$ . Thus,  $\pi_1(B)$  passes to the trivial action on  $\pi_1(F)^{ab} = H_1(F)$ .  $\square$

**Theorem 4.13** (Leray). *Given a simple fibration,*

$$F \hookrightarrow E \rightarrow B,$$

*there exists a cohomology spectral sequence,  $\{E_r^{p,q}, d_r\}$  such that*

$$E_2^{p,q} \simeq H^p(B, H^q(F))$$

*and converging to  $H_*(E)$  in other words,*

$$\bigoplus_{r+s=n} E_\infty^{r,s} \simeq H^n(E).$$

An introductory proof of this famous theorem can be found for example in [FFG]. This theorem is the major ingredient in our proof of the dimension result.

**Lemma 4.14.** *For the fibration (4.1.2) we have  $E_2^{p,q} \simeq H^p(B) \otimes H^q(F)$ , and if  $q > 1$  then  $E_2^{p,q} = 0$ .*

*Proof.*  $H_q(F)$  is a finite dimensional real vector space,  $\mathbb{R}^k$ , so its dual,  $H^q(F) \simeq \mathbb{R}^k$ . By the Leray theorem,

$$E_2^{p,q} \simeq H^p(B, H^q(F)) \simeq H^p(B, \mathbb{R}^k) \simeq H^p(B) \otimes \mathbb{R}^k \simeq H^p(B) \otimes H^q(F). \quad (4.1.10)$$

This holds because the action of  $\pi_1(B)$  on  $\mathbb{R}^k \simeq H_q(F)$  is trivial as we saw in the proof of claim 4.12. If  $q > 1$  then  $E_2^{p,q} = 0$  by lemma 4.10.  $\square$

**Lemma 4.15.** *If  $E_2^{p,q} \neq 0$ , then  $0 \leq p \leq n - 4$  and  $0 \leq q \leq 1$ . Therefore,  $E_2^{p,q} \neq 0$  implies that  $p + q \leq n - 3$ .*

*Proof.* By 4.14,  $E_2^{p,q} = H^p(B) \otimes H^q(F)$ ; the left hand factor is 0 whenever  $p > \dim(B) = n - 4$  by the induction hypothesis and the right hand one is 0 whenever  $q > 1$  by lemma 4.10.  $\square$

We can now conclude the induction proof of the vanishing statement of theorem 4.2.

**Corollary 4.16.**  *$H^k(E) = 0$  whenever  $k > n - 3$ .*

*Proof.* If  $k > n - 3$ , then  $E_2^{p,q} = 0$  for  $p + q = k$ . Recall by theorem 4.9 that  $E_{r+1}^{p,q}$  is the homology group of  $E_r^{p,q}$ . Therefore, if  $E_2^{p,q} = 0$ , then  $E_\infty^{p,q} = 0$ . So by Leray,  $\bigoplus_{p+q=k} E_\infty^{p,q} = H^k(E) = 0$ .  $\square$

**Lemma 4.17.** For the given fibration,  $\mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,n-1}$ , we have  $E_2^{n-4,1} = E_\infty^{n-4,1}$ .

*Proof.* To prove this, we show that  $E_2^{n-4,1} = E_3^{n-4,1}$ , therefore by theorem 4.9, the sequence stabilizes at  $E_2^{n-4,1} = E_\infty^{n-4,1}$ .

We have

$$E_3^{n-4,1} \simeq \ker(d_2^{n-4,1})/\text{Im}(d_2^{n-2,0}).$$

The kernel of

$$d_2^{n-4,1} : E_2^{n-4,1} \rightarrow E_2^{n-6,2}$$

is all of  $E_2^{n-4,1}$  since the image,  $E_2^{n-6,2} = 0$  by 4.15 for  $q = 2 > 1$ .

Likewise, the image of  $d_2^{n-2,0} : E_2^{n-2,0} \rightarrow E_2^{n-4,1}$  is 0 since  $E_2^{n-2,0} = 0$  by 4.15 for  $p = n - 2 > n - 3$ .

This proves the lemma.  $\square$

The previous sequence of lemmas and claims allows us to deduce the following key proposition of this section.

**Proposition 4.18.** The cohomology group,

$$H^{n-3}(E) \simeq H^{n-4}(B) \otimes H^1(F).$$

*Proof.* By the Leray theorem, we have that

$$H^{n-3}(E) = \bigoplus_{p+q=n-3} E_\infty^{p,q} \quad (4.1.11)$$

$$= E_\infty^{n-4,1} \oplus E_\infty^{n-2,0} \text{ (by lemma 4.15)} \quad (4.1.12)$$

$$= E_\infty^{n-4,1} \text{ (also by lemma 4.15)} \quad (4.1.13)$$

$$= H^{n-4}(B) \otimes H^1(F). \quad (4.1.14)$$

The last equality follows from the previous lemma 4.17 and lemma 4.14, since

$$E_2^{n-4,1} = E_\infty^{n-4,1} = H^{n-4}(B) \otimes H^1(F).$$

$\square$

By the induction hypothesis,  $\dim(H^{n-4}(\mathfrak{M}_{0,n-1})) = (n-3)!$  and by lemma 4.10,  $\dim(H^1(F)) = n-2$ . Therefore as a corollary to proposition 4.18, we obtain the claim made in the main theorem 4.2,

$$\dim(H^{n-3}(\mathfrak{M}_{0,n})) = (n-2)(n-3)! = (n-2)!.$$

## 4.2 Cohomology of partial compactifications, $\mathfrak{M}_{0,n}^\gamma$

In this section, we prove an analog of Arnol'd's theorem for the cohomology of the subspaces  $\mathfrak{M}_{0,n}^\gamma \subset \overline{\mathfrak{M}}_{0,n}$ , as defined in definition (4.7), in the case where  $\mathfrak{M}_{0,n}^\gamma$  is an affine variety.

Firstly, it is shown that for certain sets of divisors,  $\gamma$ , those such that  $\mathfrak{M}_{0,n}^\gamma$  is an affine variety, we have a natural injection

$$H^{n-3}(\mathfrak{M}_{0,n}^\gamma) \hookrightarrow H^{n-3}(\mathfrak{M}_{0,n}).$$

Then we give a theorem that shows how to explicitly calculate  $H^{n-3}(\mathfrak{M}_{0,n}^\gamma)$  as a subspace of Arnol'd's ring of theorem 4.4 of differential  $n-3$  forms.



**Proposition 4.19.** *Let  $\gamma$  be such that  $\mathfrak{M}_{0,n}^\gamma$  is an affine variety. Then the top dimensional cohomology  $H^{n-3}(\mathfrak{M}_{0,n}^\gamma)$  is isomorphic to a subspace of  $H^{n-3}(\mathfrak{M}_{0,n})$ .*

*Proof.* The heart of justifying this proposition is the following important result of Grothendieck.

**Theorem 4.20.** [Gr1] *Let  $X$  be an affine algebraic scheme over  $\mathbb{C}$ , assume that  $X$  is regular (i.e. “non-singular”). Then the complex cohomology,  $H^\bullet(X, \mathbb{C})$  can be calculated as the cohomology of the algebraic de Rham complex, (i.e. the complex of differential forms on  $X$  which are “rational and everywhere defined”).*

The Deligne-Mumford compactification,  $\overline{\mathfrak{M}}_{0,n}$ , is a smooth manifold and the set of divisors we remove,  $D \setminus \gamma$ , is a closed subset of  $\overline{\mathfrak{M}}_{0,n}$ , so  $\mathfrak{M}_{0,n}^\gamma$  is a smooth manifold.

A  $k$  form,  $\omega$ , on  $X = \mathfrak{M}_{0,n}^\gamma$  will be denoted *algebraic* if it is rational and everywhere defined, in other words, it is global and holomorphic on  $X$  and there are rational functions,  $f_{i_1, \dots, i_k}(t_1, \dots, t_{n-3})$  such that

$$\omega = \sum f_{i_1, \dots, i_k} dt_{i_1} \wedge \cdots \wedge dt_{i_k}. \quad (4.2.1)$$

Such a form must be meromorphic on  $\overline{\mathfrak{M}}_{0,n}$  because it will have poles of finite order on the boundary of  $X$  which is given by blowing up at coalescing marked points.

What Grothendieck’s theorem says is that the cohomology group of classes of algebraic forms, in which two elements are in the same class if they differ by an exact form,  $d\alpha$  where  $\alpha$  is algebraic, is isomorphic to the usual de Rham cohomology group. Therefore, in the following arguments, we may assume that a cohomology class in  $H^{n-3}(X)$  is an equivalence class of algebraic  $n - 3$  forms.

Let

$$\Phi : \Omega^k(\mathfrak{M}_{0,n}^\gamma) \rightarrow \Omega^k(\mathfrak{M}_{0,n})$$

denote the restriction map applied to an algebraic  $k$  form. Let  $d\alpha$  denote an exact  $k$  form on  $\mathfrak{M}_{0,n}^\gamma$ . Then in particular  $\alpha$  is algebraic, and its restriction,  $\Phi(\alpha)$  is of Grothendieck type on  $\mathfrak{M}_{0,n}$ . Thus  $\Phi(d\alpha) = d\Phi(\alpha)$ . Thus  $\Phi$  descends to a  $\mathbb{Q}$ -linear map on cohomology,

$$\phi : H^{n-3}(\mathfrak{M}_{0,n}^\gamma) \rightarrow H^{n-3}(\mathfrak{M}_{0,n})$$

which sends a class in the cohomology,  $\bar{\omega}$ , to its restriction on  $\mathfrak{M}_{0,n}$ .

We will now justify that this map is injective. Let  $\omega$  be an  $n - 3$  form such that  $\bar{\omega} \neq 0$  is in the kernel of  $\phi$ . Then the restriction of  $\Phi(\omega)$  is an exact form,  $d\alpha$ , on  $\mathfrak{M}_{0,n}$ , for  $\alpha$  a meromorphic  $n - 4$  form on  $\overline{\mathfrak{M}}_{0,n}$ , holomorphic on  $\mathfrak{M}_{0,n}$ . We claim  $\alpha$  must also be holomorphic on  $\mathfrak{M}_{0,n}^\gamma$ . For if it weren’t, we can suppose that  $\alpha$  has a pole on  $\mathfrak{M}_{0,n}^\gamma$  (in particular on some boundary divisor  $\gamma_i$  in  $\gamma$ ) of order  $m > 0$ . Then since  $\alpha$  has the form (4.2.1),  $d\alpha = \Phi(\omega)$  would have a pole of order greater than or equal to  $m$  on  $\gamma$ . But  $\Phi(\omega)$  has the exact expression, (4.2.1), as  $\omega$ , which by assumption is holomorphic on  $\mathfrak{M}_{0,n}^\gamma$ . Thus  $\alpha$  is holomorphic on  $\mathfrak{M}_{0,n}^\gamma$  and  $\omega$  is exact. This proves injectivity.

Since  $\phi$  is injective, we can consider the injection map as an inclusion

$$H^{n-3}(\mathfrak{M}_{0,n}^\gamma) \hookrightarrow H^{n-3}(\mathfrak{M}_{0,n}).$$

□

**Proposition 4.21.** *Assume that  $\mathfrak{M}_{0,n}^\gamma$  is an affine variety. A basis for  $H^{n-3}(\mathfrak{M}_{0,n}^\gamma)$  is given by the classes of the  $n - 3$  forms in the basis of Arnol’ d’s ring from theorem 4.4 which do not have a pole on  $D_\gamma$ . We call such forms “convergent on  $\gamma$ ” or “holomorphic on  $\gamma$ ”.*

*Proof.* Let  $A_i$  be the sub-vector space of Arnol'd's ring  $A$  generated by  $i$  forms. By Arnol'd's theorem,  $A_{n-3} \simeq H^{n-3}(\mathfrak{M}_{0,n})$ . Let  $A^\gamma$  be the subspace of  $A_{n-3}$  of differential forms convergent on  $\gamma$ .

We have a map

$$\rho : A^\gamma \rightarrow H^{n-3}(\mathfrak{M}_{0,n}^\gamma),$$

given by associating a form to its cohomology class. This map is injective because as we saw above, if  $\omega_1 - \omega_2 = d\alpha$ , an exact form on  $\mathfrak{M}_{0,n}^\gamma$ , then  $d\alpha$  is an exact form on  $\mathfrak{M}_{0,n}$ . This shows that  $\rho$  is injective.

By Grothendieck's theorem, each cohomology class in  $H^{n-3}(\mathfrak{M}_{0,n}^\gamma)$  contains a representative which is algebraic, holomorphic on  $\gamma$  and thus an element of  $A^\gamma$ . We can therefore further conclude that  $\rho$  is surjective.

Hence,  $H^{n-3}(\mathfrak{M}_{0,n}^\gamma) \simeq A^\gamma$  as vector spaces, so a basis for  $A^\gamma$  yields a basis for  $H^{n-3}(\mathfrak{M}_{0,n}^\gamma)$ . □

### 4.3 Some affine subvarieties of $\overline{\mathfrak{M}}_{0,n}$

In this section, we prove that certain partial compactifications of  $\mathfrak{M}_{0,n}$  contained in  $\overline{\mathfrak{M}}_{0,n}$  are affine varieties, that is, we justify that the addition of some subsets of divisors to  $\mathfrak{M}_{0,n}$  gives an affine space. The partial compactifications we refer to are according to definition 4.7. We first recall some important definitions and properties of divisors.

**Definition 4.22.** *A prime divisor on  $\overline{\mathfrak{M}}_{0,n}$  is an irreducible subvariety of  $\overline{\mathfrak{M}}_{0,n}$  of codimension 1.*

**Definition 4.23.** *A Weil divisor on  $\overline{\mathfrak{M}}_{0,n}$  is a formal finite linear combination over  $\mathbb{Q}$  of prime divisors.*

In this thesis, we refer to irreducible boundary divisors in  $\overline{\mathfrak{M}}_{0,n} \setminus \mathfrak{M}_{0,n}$  simply as divisors as in definition 4.7.

Every divisor,  $d_K$  contains a face of the boundary of some associahedron  $(z_{i_1}, \dots, z_{i_n})$  in  $\overline{\mathfrak{M}}_{0,n}(\mathbb{R})$  where the elements of  $K$  are in a consecutive block in any order. We can picture the divisor as a chord along that associahedron as in chapter 3, page 64. For example, we can picture the divisor  $d_K, K = \{t_1, t_3\}$  in  $\overline{\mathfrak{M}}_{0,6}$  as the chord in the polygon in figure 8.

**Definition 4.24.** *Let  $d_J$  and  $d_K$  be two divisors satisfying the following three conditions: firstly  $d_K$  and  $d_J$  each contain a face of the boundary of a single associahedron, secondly as chords of the polygon representing that associahedron,  $d_J$  and  $d_K$  cross inside the polygon, and finally  $2 \leq |J \cup K| \leq n - 2$ . We call  $d_{J \cup K}$  an **intersection-divisor** of  $d_J$  and  $d_K$ .*

An intersection-divisor corresponds to the chord adjoining adjacent endpoints as in figure 8. An intersection-divisor is not the intersection of divisors, because the intersection of divisors is of codimension  $\geq 2$ .

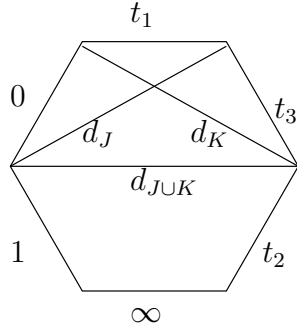


Figure 8: An intersection-divisor

Note that according to this definition, two equivalent divisors may have up to four associated intersection-divisors, depending on whether one chooses  $J$  or  $Z \setminus J$ ,  $K$  or  $Z \setminus K$  as the labelling for the divisors.

We have the following result by F. Brown allowing us to deduce the cohomology of certain partially compactified moduli spaces.

**Theorem 4.25.** [Br] *Let  $\delta$  be set of boundary divisors which contain faces on the boundary of the associahedron  $(z_{i_1}, \dots, z_{i_n})$ ,*

$$\delta = \{d_K : K = \{z_{i_j}, \dots, z_{i_{j+k}}\}\}, \quad (4.3.1)$$

where  $K$  is a consecutive set of marked points along the associahedron.

Then the partially compactified moduli space,  $\mathfrak{M}_{0,n}^\delta$ , is an affine variety.

*Sketch of proof.* Without loss of generality, we may assume that  $(i_1, \dots, i_n) = (1, \dots, n)$ . The proof for arbitrary dihedral orderings can be repeated by replacing  $j$  with  $i_j$  everywhere.

In F. Brown's thesis, he considers the ring of  $\mathbb{P}\mathrm{SL}_2$ -invariant regular functions on  $\mathfrak{M}_{0,n}$ ,  $\{u_{ij}, \frac{1}{u_{ij}} \mid i, j, i+1, j+1 \text{ distinct modulo } n\}$ , defined by the cross ratio

$$u_{ij} : (z_1, \dots, z_n) \mapsto \frac{(z_i - z_{j+1})(z_{i+1} - z_j)}{(z_i - z_j)(z_{i+1} - z_{j+1})}.$$

These functions can be labeled by chords on the polygon whose vertices are labeled in the standard cyclic order. The function  $u_{ij}$  corresponds (naturally) to the chord between the vertices  $i$  and  $j$ . This ring of functions has the following defining relation.

For any four distinct vertices,  $\{i, j, k, l\}$  with the imposed dihedral order of the polygon, define the sets,

$$A = \{\{p, q\} : i \leq p < j, k \leq q < l\} \quad (4.3.2)$$

$$B = \{\{p, q\} : j \leq p < k, l \leq q < i\}. \quad (4.3.3)$$

These are pairs of chords which "cross completely", namely every chord in  $A$  intersects every chord in  $B$  and vice versa. Let  $u_A = \prod_{a \in A} u_a$  and  $u_B = \prod_{b \in B} u_b$ . Then,

$$u_A + u_B = 1, \quad (4.3.4)$$

for any two sets of completely crossing chords.

Let  $J$  be the ideal generated by the relations (4.3.4). It is then shown that

$$\mathfrak{M}_{0,n}^\delta = \mathrm{Spec}(\mathbb{Z}[u_{ij}]/J).$$

□

In order to extend F. Brown's theorem to more general partial compactifications, we use the following classical algebro-geometric construction.

**Proposition 4.26.** *Let  $X$  be an affine variety,  $X = \text{Spec}(R)$ , and let  $D$  be a Cartier divisor on  $X$ . Then  $X \setminus D$  is an affine variety.*

*Proof.* A Cartier divisor,  $D$ , is associated to a line bundle  $\mathcal{L}(D)$  which is an invertible sheaf over  $X$ . Since  $X$  is affine, every invertible sheaf is ample, so there exists an  $n > 0$  such that  $\mathcal{L}(nD) = (\mathcal{L}(D))^{\otimes n}$  is very ample. Then,  $(\mathcal{L}(D))^{\otimes n}$  is given by a section  $f \in \Gamma(X, \mathcal{O}_X(nD))$  where  $f$  vanishes exactly on  $D$  and is non-zero on  $X \setminus D$ . We have then that  $X \setminus D = \text{Spec}(R[\frac{1}{f}])$  is affine.  $\square$

From definitions 4.7 and 4.23, the boundary divisors,  $D_\gamma$ , are Weil divisors. Since  $\overline{\mathfrak{M}}_{0,n}$  is smooth, Weil and Cartier divisors coincide, and hence we may consider  $D_\gamma$  as a Cartier divisor.

**Corollary 4.27.** *The partial compactifications,  $\mathfrak{M}_{0,n}^\gamma$ , are affine varieties for the following sets of boundary divisors  $\gamma \subset D$ :*

1. Any  $\gamma$  containing only one boundary divisor,  $|\gamma| = 1$ ,
2. Any  $\gamma$  containing any 2 boundary divisors,
3. Any  $\gamma$  containing 3 boundary divisors such that if two divisors intersect (as chords), then the third divisor is an intersection-divisor of the two.

*Proof.* All of these sets,  $\gamma$ , contain divisors which contain a face on the boundary of an associahedron in  $\overline{\mathfrak{M}}_{0,n}$ , so we apply proposition 4.26.

For part (1), let  $\gamma = \{d_K\}$  where  $K = \{z_{i_1}, \dots, z_{i_k}\}$  for  $k < n - 1$ . Then  $d_K$  contains the face of the boundary of any associahedron enumerated by  $\Delta = (z_{i_1}, \dots, z_{i_k}, \dots)$ . Let  $\delta$  denote the set of boundary divisors which contain a face of  $\Delta$  as in (4.3.1), so  $\mathfrak{M}_{0,n}^\delta$  is affine. By successive removal of all the divisors, we recursively obtain affine varieties, the final one being  $\mathfrak{M}_{0,n}^\gamma$ .

For part (2), let  $\gamma = \{d_P, d_Q\}$  where  $P = \{z_{i_1}, \dots, z_{i_p}\}$ ,  $Q = \{z_{j_1}, \dots, z_{j_q}\}$  and without loss of generality  $P \cap Q = \{z_{i_{k+1}}, \dots, z_{i_p}\} = \{z_{j_1}, \dots, z_{j_{p-k}}\}$  ( $P \cap Q$  may be empty). Then  $d_P, d_Q$  are divisors containing the face of the boundary of any associahedron enumerated by

$$\Delta = (z_{i_1}, \dots, z_{i_p}, z_{j_{p-k+1}}, \dots, z_{j_q}, \dots).$$

As in part (1), by recursive removal of divisors from  $\mathfrak{M}_{0,n}^\delta$ , we have that  $\mathfrak{M}_{0,n}^\gamma$  is affine.

Finally for part (3), assume first that  $\gamma$  contains any two divisors which intersect (as chords) and a third which is an intersection-divisor of the two. This means that we can find sets,  $P$  and  $Q$  as in part (2) such that  $\gamma = \{d_P, d_Q, d_{P \cup Q}\}$ . Then all of these divisors are on the boundary of the associahedron,  $(z_{i_1}, \dots, z_{i_p}, z_{j_{p-k+1}}, \dots, z_{j_q}, \dots)$ . By applying the proposition, we have that  $\mathfrak{M}_{0,n}^\gamma$  is affine. If  $\gamma$  contains three divisors defined by  $R, P, Q$  which are disjoint subsets of  $Z$ , we can construct  $d_P, d_Q$  and  $d_R$  as in part (2).

All of the varieties,  $\mathfrak{M}_{0,n}^\gamma$  for  $\gamma$  from the cases (1)-(3), are therefore affine.  $\square$

The previous corollary can be extended to many other partial compactifications, but we only treat these three in detail here.

#### 4.4 Explicit bases for $H^{n-3}(\mathfrak{M}_{0,n}^\gamma)$

In this section, I generalize a result of the previous chapter, namely I use the methods introduced there to calculate the top dimensional cohomology of the subspaces  $\mathfrak{M}_{0,n}^\gamma \subset \overline{\mathfrak{M}}_{0,n}$  for certain small subsets  $\gamma \subset \delta$ , where  $\delta$  denotes as usual the union of the divisors which contain the face of the boundary of an associahedron. We let  $\ell = n - 3$  and  $Z = \{z_1, \dots, z_n\} = \{0, 1, \infty, t_1, \dots, t_\ell\}$ .

Let us recall some definitions and results from chapter 3 that will be useful throughout this and subsequent sections.

**Definition 4.28.** [Re] Let  $\{x_1, \dots, x_n\}$  be a set of non-commutative variables with a lexicographic ordering and let  $P_n$  be the  $\mathbb{Q}$ -vector space generated by monomials of degree  $n$  such that every variable appears exactly once. The Lyndon basis for  $P_n$  is given by the set  $\{A_1 \# \dots \# A_k\}$  where the  $A_i$  form a partition of the variables and the first letter of every  $A_i$  is the smallest letter appearing in  $A_i$  for the imposed lexicographic ordering. We say that  $A_1 \# \dots \# A_k$  is a Lyndon shuffle of degree  $k$ .

The Lyndon basis is an alternative basis to the standard basis of permutations of the  $n$  variables. There are  $(n - 1)!$  degree 1 Lyndon elements, since these are all monomials which start with the smallest letter. Let  $I_n \subset P_n$  denote vector subspace of Lyndon shuffles of degree  $\geq 2$ , whose dimension is  $n! - (n - 1)! = (n - 1)(n - 1)!$ .

**Definitions 4.29.** Given a divisor,  $d_K$ ,  $K = \{t_{i_1}, \dots, t_{i_r}\}$ , we define  $\mathcal{P}_{d_K}$  to be  $\mathbb{Q}$ -vector space of polygons with sides decorated by the marked points in  $K$ . The subspace  $I_{d_K} \subset \mathcal{P}_{d_K}$  is generated by shuffle sums with respect to one point (as in definition 1.38).

We may often denote the vector spaces,  $\mathcal{P}_{d_K}$  and  $I_{d_K}$  simply by  $\mathcal{P}_K$  and  $I_K$ . We will often also note  $\mathcal{P}_{d_{K \cup \{e\}}}$  and  $I_{d_{K \cup \{e\}}}$  by  $\mathcal{P}_{d_K \cup \{e\}}$  and by  $I_{d_K \cup \{e\}}$ . Note that  $I_n$ , where  $n$  is an integer, is different from  $I_K$ , where  $K$  is a set.

Let  $\pi$  be the map that sends a polygon to its associated cell form. The  $\text{Res}_d$  map sends a cell form to its residue along a divisor  $d$  while the  $\text{Res}_d^p$  map sends a polygon to the tensor product of the polygons cut the chord,  $e$ , as in definitions (3.4.2) and (3.3.3). The  $\text{Res}_d$  and  $\text{Res}_d^p$  maps are related by the identity,

$$\text{Res}_d(\pi(\omega^p)) = \pi(\text{Res}_d^p(\omega^p)),$$

for any polygon  $\omega^p$ .

Recall corollary 3.51 that identifies the kernel of the residue map on a divisor  $d$ ,

$$\ker(\text{Res}_d) = \pi^{-1}(I_{d \cup \{e\}} \otimes \mathcal{P}_{Z \setminus d \cup \{e\}}).$$

**Theorem 4.30.** Let  $\gamma = \{\gamma_1, \dots, \gamma_k\}$  be a set of boundary divisors in  $\overline{\mathfrak{M}}_{0,n}$  such that  $\mathfrak{M}_{0,n}^\gamma$  is affine. Then, the  $\mathbb{Q}$ -vector space  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  coincides with the intersection of vector spaces,

$$\bigcap_{i=1}^k \pi((\text{Res}_{\gamma_i}^p)^{-1}(I_{\gamma_i \cup \{e\}} \otimes \mathcal{P}_{Z \setminus \gamma_i \cup \{e\}})).$$

Furthermore, a basis for  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  can easily be deduced from a Lyndon basis of the polygons in  $I_{\gamma_i \cup \{d\}} \otimes \mathcal{P}_{Z \setminus \gamma_i \cup \{d\}}$  using insertion forms.

*Proof.* From theorem 4.19, we have an injection

$$H^\ell(\mathfrak{M}_{0,n}^\gamma) \hookrightarrow H^\ell(\mathfrak{M}_{0,n}).$$

By theorem 3.16 a basis for  $H^\ell(\mathfrak{M}_{0,n})$  is given by 01-forms.

By applying proposition 4.21, we obtain a basis for  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  by taking the subspace of 01-forms which converge on  $\gamma$ . A form is convergent on  $\mathfrak{M}_{0,n}^\gamma$  if and only if it is convergent on all of the divisors,  $\gamma_i \in \gamma$ , since by the hypothesis, it is convergent on the interior,  $\mathfrak{M}_{0,n}$ .

A cell form,  $\omega$ , is convergent on  $\gamma_i$  if and only if its residue on  $\gamma_i$  is 0, in other words if and only if

$$\omega \in \ker(\text{Res}_{\gamma_i}). \quad (4.4.1)$$

We rely on two important combinatorial properties of 01-cyclic structures. Not only do 01-forms form a basis for the cohomology, but also 01-polygons form a basis for the  $\mathbb{Q}$  vector space which is freely generated by 01-cyclic structures. Therefore, each  $\ell$  form,  $\omega$ , has a unique lifting,  $\omega^p$  to a linear combination of 01-polygons. By proposition 3.51, the condition (4.4.1) can be restated as

$$\omega^p \in (\text{Res}_{\gamma_i}^p)^{-1}(I_{\gamma_i \cup \{e\}} \otimes \mathcal{P}_{Z \setminus \gamma_i \cup \{e\}}). \quad (4.4.2)$$

If a form is convergent on all  $\gamma_i$  it must be in the intersection of the spaces spanned by the spaces (4.4.2).  $\square$

In the examples that follow, we exploit this theorem and the methods of chapter 3 of insertion forms to calculate bases of cohomologies for some natural  $\mathfrak{M}_{0,n}^\gamma$ . Recall from definition 3.36 that an insertion form is a cell form coming from a linear combination of polygons such that the polygon residue map maps them to  $I_d \otimes \mathcal{P}_{d'}$  for some divisors  $d$  and  $d'$ .

### Case 1: $|\gamma| = 1$

Firstly, we treat the smallest and most natural case of a partial compactification, namely that obtained by removing all boundary divisors except one from  $\overline{\mathfrak{M}}_{0,n}$ . It was shown in corollary 4.27 that if  $|\gamma| = 1$ ,  $\mathfrak{M}_{0,n}^\gamma$  is an affine space.

Let  $\gamma = \{d_R\}$  for  $R = \{z_{i_1}, \dots, z_{i_r}\}$  and let  $\omega$  be a differential  $\ell$ -form written in the 01-basis, where  $\ell = n - 3$  as in chapter 3. In writing 01-cell forms, it is useful to choose an appropriate equivalence class representative modulo  $\mathbb{P}\text{SL}_2$ . So without loss of generality, we may assume that  $R = \{t_{i_1}, \dots, t_{i_r}\}$ , where one of the  $t_{i_j}$  may be  $\infty$ .

From theorem 4.30,  $\omega$  converges if and only if

$$\text{Res}_{\gamma}^p(\omega^p) \in I_{\gamma \cup \{e\}} \otimes \mathcal{P}_{Z \setminus \gamma \cup \{e\}}.$$

The 01-gons that have 0 residue along this divisor are those that don't contain the block  $t_{i_1}, \dots, t_{i_r}$ ; let this set of 01-gons be denoted  $\mathcal{W}_{\gamma_0}^p$ .

To calculate the dimension of the cohomology, we count the number of fixed structures containing this block. There are  $(n - 1 - r)!$  such fixed structures and  $r!$  ways of ordering the elements in the block. So the number of 01-cell forms that map identically to 0 by  $\text{Res}_{\gamma}^p$  is  $(n - 2)! - (n - r - 1)!r!$ . The projection from these polygons to 01-forms are in the basis of  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  along with the insertion forms.

The insertion forms for the divisor,  $\gamma$  are linear combinations of 01-forms which map to  $I_{\gamma \cup \{d\}} \otimes \mathcal{P}_{\gamma \cup \{d\}}$  and don't map identically to 0. These forms are the images of  $\pi$  of formal sums of  $n$ -gons,

$$P = [0, 1, Z_1, R_1 \sqcup R_2, Z_2],$$

where  $Z_1 \cup Z_2 = \{t_1, \dots, t_\ell\} \setminus \{t_{i_1}, \dots, t_{i_r}\}$  and  $R_1 \cup R_2 = \{t_{i_1}, \dots, t_{i_r}\}$ . Let  $\mathcal{W}_{\gamma \sqcup}^p$  be the set of such polygons. The image of  $\pi(\mathcal{W}_{\gamma \sqcup}^p)$  forms a linearly independent set of  $\ell$ -forms in the cohomology by an argument used in chapter 3, theorem 3.41. The 01-forms form a basis, therefore, we only need to worry about linear dependance for any fixed  $Z_1$  and  $Z_2$ . But the Lyndon shuffles form a basis for the polynomial algebra, so for any fixed  $Z_1$  and  $Z_2$  the forms  $[0, 1, Z_1, R_1 \sqcup R_2, Z_2]$  are linearly independent.

To count the dimension of polygons that map to  $I_{\gamma \cup \{e\}} \otimes \mathcal{P}_{\gamma \cup \{e\}}$  is simple. There are  $(r-1)!$  degree 1 Lyndon generators in the shuffle algebra, therefore there are  $r! - (r-1)! = (r-1)(r-1)!$  shuffle generators in  $I_{\gamma \cup \{e\}}$ . There are  $(n-r-1)!$  fixed structures, so we conclude that

$$\dim(H^\ell(\mathfrak{M}_{0,n}^\gamma)) = (n-2)! - (n-r-1)!r! + (n-r-1)!(r-1)!(r-1) \quad (4.4.3)$$

$$= (n-2)! - (n-r-1)!(r-1)!. \quad (4.4.4)$$

A basis for  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  is given by  $\pi(\mathcal{W}_{\gamma_0}^p \sqcup \mathcal{W}_{\gamma \sqcup}^p)$ .

**Examples 4.31.** (1)  $n = 6$ ,  $\gamma$  contains the boundary divisor corresponding to  $t_1 = t_2$ .

Then  $r = 2$  so we conclude that the dimension of  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  is 18. A basis for the cohomology is given by the 12 01-forms,  $[0, 1, \{t_1, t_2, t_3, \infty\}]$  such that  $t_1$  is not next to  $t_2$ , together with the 6 shuffle sums

$$[0, 1, t_1 \sqcup t_2, t_3, \infty], [0, 1, t_1 \sqcup t_2, \infty, t_3], [0, 1, t_3, t_1 \sqcup t_2, \infty], \\ [0, 1, \infty, t_1 \sqcup t_2, t_3], [0, 1, t_3, \infty, t_1 \sqcup t_2], [0, 1, \infty, t_3, t_1 \sqcup t_2].$$

(2)  $n = 6$ ,  $r = 3$ ,  $\gamma$  consists of the boundary divisor corresponding to  $t_1 = t_2 = t_3$ .

The dimension is 20, and the basis elements are given by the 6 forms,  $[0, 1, t_i, \infty, t_j, t_k]$ , the 6 forms  $[0, 1, t_i, t_j, \infty, t_k]$ , the 4 Lyndon shuffles,

$$[0, 1, t_1 \sqcup (t_2, t_3), \infty], [0, 1, (t_1, t_2) \sqcup t_3, \infty], [0, 1, (t_1, t_3) \sqcup t_2, \infty], [0, 1, t_1 \sqcup t_2 \sqcup t_3, \infty],$$

and finally the 4 Lyndon shuffles,

$$[0, 1, \infty, t_1 \sqcup (t_2, t_3)], [0, 1, \infty, (t_1, t_2) \sqcup t_3], [0, 1, \infty, (t_1, t_3) \sqcup t_2], [0, 1, \infty, t_1 \sqcup t_2 \sqcup t_3].$$

## Case 2: $|\gamma| = 2$ and the divisors are disjoint

In this case, we are considering two divisors that do not cross as chords of any polygon. Let these divisors be given by the equalities of the marked points in the sets  $R = \{z_{i_1}, \dots, z_{i_r}\}$ ,  $S = \{z_{j_s}, \dots, z_{j_s}\}$ ,  $R \cap S = \emptyset$ . (Recall that by corollary 4.27 we know that  $\mathfrak{M}_{0,n}^\gamma$  is affine.)

In this case a basis for the cohomology is given by sets of 01-forms whose associated polygon either maps identically to zero or to  $I_{\gamma_i \cup \{e\}} \otimes \mathcal{P}_{Z \setminus \gamma_i \cup \{e\}}$  for the corresponding  $\text{Res}_{\gamma_i}^p$  maps,  $i = 1, 2$ . As in case 1, the forms mapping identically to zero are all forms whose associated polygon contains no consecutive block of  $R$  or of  $S$ . The other forms are insertions of Lyndon shuffles of degree two or higher of  $R$  (resp.  $S$ , resp. both) into 01-forms on  $Z \setminus R$  (resp.  $Z \setminus S$ , resp.  $Z \setminus (R \cup S)$ ).



Here, we count the dimension and give a small example of an explicit basis. In the following formula we count the dimension of  $\mathfrak{M}_{0,n}^\gamma$  by methods similar to case 1. The first line counts the 01-polygons whose polygon residue is identically 0 for both  $d_R$  and  $d_S$  (of lengths  $r$  and  $s$ ), where the last term counts their overlap. The second (resp. third) line counts the insertions that land in  $I_{R \cup \{d\}}$  (resp.  $I_{S \cup \{d\}}$ ) for  $\text{Res}_R^p$  (resp.  $\text{Res}_S^p$ ) and map to 0 for  $\text{Res}_S^p$  (resp.  $\text{Res}_R^p$ ). The last line counts the number of terms that land in  $I_{R \cup \{d\}}$  and  $I_{S \cup \{d\}}$  for the respective residue maps.

$$(n-2)! - (n-r-1)!r! - (n-s-1)!s! + (n-r-s)!r!s! \quad (4.4.5)$$

$$+ (n-r-1)!(r-1)!(r-1) - (n-r-s)!(r-1)!(r-1)s! \quad (4.4.6)$$

$$+ (n-s-1)!(s-1)!(s-1) - (n-r-s)!(s-1)!(s-1)r! \quad (4.4.7)$$

$$+ (n-r-s)!(r-1)!(r-1)(s-1)!(s-1). \quad (4.4.8)$$

**Example 4.32.** (1) Let  $n = 6$ ,  $R = \{t_1, t_2\}$  and  $S = \{t_3, \infty\}$ . The dimension of the cohomology is 14 by the formula. There are 8 01-cyclic structures such that  $t_1$  is not next to  $t_2$  and  $t_3$  is not next to  $\infty$ ,

$$[0, 1, t_1, t_3, t_2, \infty], [0, 1, t_1, \infty, t_2, t_3], [0, 1, t_2, t_3, t_1, \infty], [0, 1, t_2, \infty, t_1, t_3], \\ [0, 1, t_3, t_1, \infty, t_2], [0, 1, t_3, t_2, \infty, t_1], [0, 1, \infty, t_1, t_3, t_2], [0, 1, \infty, t_2, t_3, t_1].$$

Then we add the 6 insertion elements to form the basis,

$$[0, 1, \infty, t_1 \text{mt} t_2, t_3], [0, 1, t_3, t_1 \text{mt} t_2, \infty], [0, 1, t_1, t_3 \text{m} \infty, t_2], [0, 1, t_2, t_3 \text{m} \infty, t_1] \\ [0, 1, t_3 \text{m} \infty, t_1 \text{mt} t_2], [0, 1, t_1 \text{mt} t_2, t_3 \text{m} \infty].$$

### Case 3: $|\gamma| = 3$ and contains two divisors that intersect as chords and their intersection-divisor

Let  $d_1$  and  $d_2$  be any divisors that intersect (as chords) as in definition 4.24 and consider them now as chords of a polygon. A chord between adjacent endpoints of  $d_1$  and  $d_2$  may represent an intersection-divisor if it cuts the polygon into two sections, each with at least two edges. For  $n \geq 5$  and for any  $d_1, d_2$  that intersect (as chords), there exists at least one well defined intersection-divisor, since the four possible intersection chords form partitions of the edges of the polygon. Therefore we can find sets  $R, S \subset Z$  such that  $d_1 = d_R, d_2 = d_S$ , and  $3 \leq |R \cup S| \leq n-2$  so  $d_{R \cup S}$  is a well defined intersection-divisor. Let  $R$  and  $S$  be such sets and let  $\gamma = \{d_R, d_S, d_{R \cup S}\}$ .

The ideas used in the description of the cohomology of  $\mathfrak{M}_{0,n}^\gamma$  are similar to the ones used in chapter 3 for finding the cohomology of  $\mathfrak{M}_{0,n}^\delta$ . In fact, they provide a sort of base case for studying the origin of insertion forms, since this cohomology space consists of forms which converge on many divisors at the same time, some of which overlap. The method of constructing this space consists simply of finding elements of the vector space of polygons decorated with the marked points in  $\mathfrak{M}_{0,n}$  and categorizing the polygons according to their image by the residue maps along the divisors in  $\gamma$ . We construct vector spaces of polygons that map to 0 or to  $I \otimes \mathcal{P}$  for the  $\text{Res}^p$  maps. According to corollary 3.51, this classification gives all of the differential forms convergent on the divisors in  $\gamma$  and on  $\mathfrak{M}_{0,n}$ .

To construct the combinatorial polygon sets that describe the cohomology, we consider the subsets of marked points that define the boundary components we are looking for.



Let  $R \subset Z$  be the set  $\{z_{i_1}, \dots, z_{i_r}\}$  and let  $S \subset Z$  be  $\{z_{j_1}, \dots, z_{j_s}\}$ . Since the intersection of  $R$  and  $S$  is supposed to be non-empty, we may assume that  $z_{i_1} = z_{j_1}, \dots, z_{i_k} = z_{j_k}$ .

Let the lexicographic order on  $R$  be  $z_{i_{k+1}} < \dots < z_{i_r} < z_{i_1} < z_{i_2} < \dots < z_{i_k}$ , and on  $S$  be  $z_{j_1} < \dots < z_{j_s}$ . In this way, we have that the elements of  $R$  are less than those of  $S$ .

For any set,  $P$ , with a given lexicographic order, define  $\mathcal{L}^i(P)$  to be the set of Lyndon shuffles on  $P$  of degree  $i$  for the shuffle product.

In the following paragraphs, we construct the sets,  $\mathcal{W}_0, \mathcal{W}_{d_R}, \mathcal{W}_{d_S}, \mathcal{W}_{d_{R \cup S}}, \mathcal{W}_{d_{R \cup S}, S}$ , that describe the cohomology of  $\mathfrak{M}_{0,n}^\gamma$ .

Let  $\mathcal{W}_0^p$  be the set of 01  $n$ -gons,  $\{\omega^p\}$ , such that  $\text{Res}_{d_R}^p(\omega^p) = 0, \text{Res}_{d_S}^p(\omega^p) = 0$  and  $\text{Res}_{d_{R \cup S}}^p(\omega^p) = 0$ , in other words all generating 01 polygons with no chord,  $e$ , that cuts  $\omega$  into a polygon in  $\mathcal{P}_{K \cup \{e\}} \otimes \mathcal{P}_{\overline{K} \cup \{e\}}$  where  $K = R, S$  or  $R \cup S$  and  $\overline{K} = \{z_1, \dots, z_n\} \setminus K$ .

The number of elements in  $\mathcal{W}_0^p$  is given by the following formula,

$$|\mathcal{W}_0^p| = (n-2)! - \left( (n-1-s-r+k)!(r+s-k)! \right) \quad (4.4.9)$$

$$+ r!((n-1-r)! - (n-1-s-r+k)!(s-k+1)!) \quad (4.4.10)$$

$$+ s!((n-1-s)! - (n-1-s-r+k)!(r-k+1)!). \quad (4.4.11)$$

The first term subtracted off counts all of the elements which do not map to 0 for  $\text{Res}_{d_{R \cup S}}^p$ , i.e. those polygons which can be cut by a chord in such a way that one side contains only elements labelled by  $R \cup S$ . The second (resp. third) term counts the polygons which do not map to 0 for the  $\text{Res}_{d_R}^p$  (resp.  $\text{Res}_{d_S}^p$ ) map and subtracts off the intersection with the first term.

**Example 4.33.** Let  $n = 6, R = \{t_1, t_2\}, S = \{t_2, t_3\}$ . Then  $\mathcal{W}_0^p$  contains the 4 cell forms,

$$[0, 1, t_1, t_3, \infty, t_2], [0, 1, t_3, t_1, \infty, t_2], [0, 1, t_2, \infty, t_1, t_3], [0, 1, t_2, \infty, t_3, t_1].$$

Let  $\mathcal{W}_{d_R}^p$  be the set of 01  $n$ -gons,  $\{\omega^p\}$ , such that

$$\begin{aligned} \text{Res}_{d_r}^p(\omega^p) &\in I_{R \cup \{e\}} \otimes \mathcal{P}_{Z \setminus R \cup \{e\}} \\ &\neq 0, \end{aligned}$$

and such that  $\text{Res}_{d_S}^p(\omega^p) = 0$  and  $\text{Res}_{d_{R \cup S}}^p(\omega^p) = 0$ . This set contains all shuffle sums of elements of  $\mathcal{L}^i(R)$ ,  $i \geq 2$ , inserted into polygons decorated by  $\{z_1, \dots, z_n, e\} \setminus \{z_{i_1}, \dots, z_{i_r}\}$  and such that the elements of  $S \cup \{e\} \setminus R$  are never consecutive. The cardinality of  $\mathcal{W}_{d_R}^p$  is  $(r-1)!(r-1)((n-r-1)! - (s-k+1)!(n-r-s+k-1)!)$ .

Let  $\mathcal{W}_{d_S}^p$  be defined similarly, so it has cardinality is  $(s-1)!(s-1)((n-s-1)! - (r-k+1)!(n-r-s+k-1)!)$ .

**Example 4.34.** Let  $n, R$  and  $S$  be as in example 4.33. Then  $\mathcal{W}_{d_R}^p$  is the set containing the two elements,

$$[0, 1, (t_1 \text{III} t_2), \infty, t_3], [0, 1, t_3, \infty, (t_1 \text{III} t_2)],$$

and  $\mathcal{W}_{d_S}^p$  contains the two elements,

$$[0, 1, (t_3 \text{III} t_2), \infty, t_1], [0, 1, t_1, \infty, (t_3 \text{III} t_2)].$$

We define  $\mathcal{W}_{d_{R \cup S}}^p$  to be the set of insertions of elements  $(A_1 \text{III} \dots \text{III} A_j) \in \mathcal{L}^i(R \cup S)$ ,  $i \geq 2$ , such that the shuffle factors,  $A_k$ , do not contain blocks that equal  $R$  or  $S$  into the fixed 01 structures on  $Z \cup \{e\} \setminus (R \cup S)$  in the place of  $e$ . One could say that these

are the standard type of shuffle elements that only have one level of insertion. These elements map to  $I_{R \cup S \cup \{e\}} \otimes \mathcal{P}_{Z \cup \{e\} \setminus (R \cup S)}$  for  $\text{Res}_{d_{R \cup S}}^p$  and can map either to 0 or to  $I \otimes \mathcal{P}$  for  $\text{Res}_{d_R}^p, \text{Res}_{d_S}^p$ . There are  $(r + s - k - 1)!(r + s - k - 1)$  degree two or higher Lyndon shuffle elements on  $R \cup S$ . Any shuffle term with a consecutive block of  $R$  begins with  $z_{i_{k+1}}$  by the lexicographic order we imposed on  $R \cup S$ , so the number of shuffles that have such a consecutive block in a factor is  $(r - 1)!(s - k)!(s - k)$ . The number of shuffles where  $S$  is a consecutive block in one of the factors is  $((r - k)!(r - k) - (r - k)!)s! + (r - k)!(s - 1)!$ . The first term in this expression counts the number of shuffle sums where  $S$  is in a successive block and in which there are other terms from  $R \setminus S$  in that factor. By the lexicographic order, the elements of  $S$  are never the first letters of those factors, therefore for each fixed structure, we have  $s!$  such shuffles. The second term counts the number of shuffles in which the letters in  $S$  form their own word. By the lexicographic ordering, we have  $(r - k)!(s - 1)!$  such shuffles. Then we may insert all such elements into the  $(n - 2 - (r + s - k) + 1)!$  fixed 01-structures on  $Z \cup \{e\} \setminus (R \cup S)$ . The cardinality of  $\mathcal{W}_{d_{R \cup S}}^p$  is therefore

$$(n - 1 - r - s + k)!((r + s - k - 1)!(r + s - k - 1) - (r - 1)!(s - k)!(s - k) - (((r - k)!(r - k) - (r - k)!)s! + (r - k)!(s - 1)!)).$$

**Example 4.35.** *Let  $n, R$  and  $S$  be as in example 4.33. Then there are the four elements in  $\mathcal{W}_{d_{R \cup S}'}^p$*

$$[0, 1, \infty, (t_1 \text{mt}_2 \text{mt}_3)], [0, 1, \infty, (t_1 t_3 \text{mt}_2)], [0, 1, (t_1 \text{mt}_2 \text{mt}_3), \infty], [0, 1, (t_1 t_3 \text{mt}_2), \infty].$$

The final set in the combinatorial description of this cohomology follows closely the methods of chapter 3, where we construct a second level of insertion sets by inserting elements in  $\mathcal{L}^{\geq 2}(S)$  into the place of  $e$  in those elements of  $\mathcal{L}^{\geq 2}(R \cup \{e\} \setminus S)$  in which a consecutive block of  $e$  is not equal to a shuffle factor. In order to respect the given lexicographic order on  $R \cup S$ ,  $e$  is greater than all elements in  $R \setminus S$ . Then, to obtain an element in the cohomology, insert these into the place of  $e$  in 01-gons decorated by  $Z \cup \{e\} \setminus (R \cup S)$ . Let  $\mathcal{W}_{d_{R \cup S}, S}^p$  be the set obtained by this insertion process. The cardinality of  $\mathcal{W}_{d_{R \cup S}, S}^p$  is  $(n - 1 - r - s + k)!(s - 1)!(s - 1)((r - k)!(r - k) - (r - k)!)$ , where the first factor is the number of 01 fixed structures on  $Z \cup \{e\} \setminus (R \cup S)$ , the second, the number of degree two Lyndon shuffles on  $S$  and the third, the number of fixed structures which are Lyndon shuffles on  $R \cup \{e\} \setminus S$  such that  $e$  is not its own shuffle factor.

**Examples 4.36.** *In the extended example, there are no level two insertion elements in  $\mathcal{W}_{d_{R \cup S}, S}^p$ . Take  $n = 7, R = \{t_1, t_2, t_3\}, S = \{t_3, t_4\}$ . Then there are four level two insertion elements in  $\mathcal{W}_{d_{R \cup S}, S}^p$  given by*

$$[0, 1, \infty, ((t_1, (t_3 \text{mt}_4)) \text{mt}_2)], [0, 1, \infty, (t_1 \text{mt}(t_2, (t_3 \text{mt}_4)))], \\ [0, 1, ((t_1, (t_3 \text{mt}_4)) \text{mt}_2), \infty], [0, 1, (t_1 \text{mt}(t_2, (t_3 \text{mt}_4))), \infty].$$

In the above examples, we constructed sets of polygons. To pass to elements of the cohomology, let  $\mathcal{W}_i$  be the image of  $\mathcal{W}_i^p$  for the map from polygons to cell forms for  $i = 0, d_R, d_S, d_{R \cup S}$  and  $d_{R \cup S}, S$ .

**Proposition 4.37.** *The cell forms in the disjoint union,*

$$\mathcal{W}_\gamma = \mathcal{W}_0 \cup \mathcal{W}_{d_R} \cup \mathcal{W}_{d_S} \cup \mathcal{W}_{d_{R \cup S}} \cup \mathcal{W}_{d_{R \cup S}, S} \quad (4.4.12)$$

form a basis for the  $\mathbb{Q}$  vector space of differential  $\ell$  forms,  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  and its dimension is given by

$$(n-2)! + (n-1-(r+s-k))!((r-1)!(s-k)! + (s-1)!(r-k)! - (r+s-k-1)!) \\ - (n-1-r)!(r-1)! - (n-1-s)!(s-1)!.$$

*Proof.* Because 01-polygons are in bijection with the 01-cell form basis for  $H^\ell(\mathfrak{M}_{0,n})$  by theorem 3.16, we may and will prove this theorem combinatorially on the polygons  $\mathcal{W}_\gamma^p$ .

First, we justify that the given sets are indeed disjoint. We can write the elements of the sets as sums of 01  $n$ -gons of the form,

$$\omega = \sum_{\sigma} [0, 1, \sigma(z_1, \dots, z_{n-2})]$$

where  $\sigma \in \mathfrak{S}_n$ . The elements of  $\mathcal{W}_0^p$  are sums of length one and contain no permutations of the marked points where the elements of  $R$ ,  $S$  or  $R \cup S$  are consecutive. The other sets contain sums of 01  $n$ -gons whose terms all contain consecutive elements from at least one of  $R$ ,  $S$  or  $R \cup S$ , so  $\mathcal{W}_0^p$  is disjoint from the union of the other four sets. Similarly, by construction,  $\mathcal{W}_{d_R}^p$  (resp.  $\mathcal{W}_{d_S}^p$ ) contains only sums of 01  $n$ -gons whose terms contain consecutive elements of  $R$  (resp.  $S$ ), but neither consecutive elements from  $S$  (resp.  $R$ ) nor  $R \cup S$ . Therefore,  $\mathcal{W}_{d_R}^p$  and  $\mathcal{W}_{d_S}^p$  are disjoint from the other sets. Finally, the sets  $\mathcal{W}_{d_{R \cup S}}^p$  and  $\mathcal{W}_{d_{R \cup S, S}}^p$  are disjoint from each other, since they are sums whose terms are insertions of Lyndon shuffles on  $r+s-k$  elements into 01  $n$ -gons. Since Lyndon shuffles form a basis for the shuffle algebra on  $n$  distinct letters, they are distinct sums. The Lyndon shuffles in  $\mathcal{W}_{d_{R \cup S}}^p$  do not contain shuffle factors with consecutive elements in either  $R$  or  $S$ , but those in  $\mathcal{W}_{d_{R \cup S, S}}^p$  contain only sums of Lyndon shuffles whose shuffle factors contain consecutive sequences in  $S$ . Therefore  $\mathcal{W}_{d_{R \cup S}}^p$  and  $\mathcal{W}_{d_{R \cup S, S}}^p$  are also disjoint sets. The map from 01-polygons to cell forms is injective, so the corresponding sets of cell forms,  $\mathcal{W}_i$ , are disjoint as well.

By theorem 4.30, we need to justify that  $\mathcal{W}_\gamma^p$  is a basis for

$$\bigcap_{K=R, S, R \cup S} \pi((\text{Res}_{d_K}^p)^{-1}(I_K \otimes \mathcal{P}_{Z \cup \{e\} \setminus K})). \quad (4.4.13)$$

Recall the definition of a framing for a divisor,  $d_K$  on page 66. If  $\text{Res}_K^p(\omega^p) \in \mathcal{P}_K \otimes f$ , such that  $f$  is a 01-polygon, we define the framing of  $\omega^p$  with respect to  $\overline{K}$  to be the right hand factor,  $f$ , of its image. Depending on the context, the framing may also be an element of a basis for  $\mathcal{P}_{\overline{K}}$  which is the sum of 01-gons. Since the right hand factors are assumed to be basis elements for  $\mathcal{P}_{\overline{K}}$ ,  $\text{Res}_K^p(\omega^p) = 0$  if and only if  $\text{Res}_K^p(\omega^p) = 0$  for each framing on the letters  $Z \cup \{e\} \setminus K$ .

Note: Polygons decorated by a set,  $K \cup \{e\}$ , are isomorphic to the noncommutative polynomial algebra in  $K$ , where  $e$  can be just considered as an ‘‘marker’’, i.e. the side that follows it in the clockwise ordering corresponds to the first letter of the corresponding monomial.

**Claim 4.38.** *The sets, (4.4.12), are linearly independent.*

*Proof.* To prove linear independence, we recall theorem 3.16. Since the 01-forms form a basis for  $H^\ell(\mathfrak{M}_{0,n})$ , and  $\mathcal{W}_0$  contains 01-forms, we only need to show linear independence for the other four sets. Let  $\omega^p$  be a linear combination of forms from  $\mathcal{W}_\gamma^p \setminus \mathcal{W}_0^p$ . The vector space of 01-gons can be written as the direct sum,  $V_0 \oplus V_{R \cup S} \oplus V_S \oplus V_R$ , where

$V_0$  is generated by 01-gons with no consecutive blocks of  $R, S$  or  $R \cup S$ ;  $V_R$  (resp.  $V_S$ ) is the vector space generated by 01-gons with consecutive blocks of  $R$  (resp.  $S$ ), but no consecutive blocks of  $R \cup S$  and no consecutive blocks of  $S$  (resp.  $R$ );  $V_{R \cup S}$  is generated by 01-gons with consecutive blocks of  $R \cup S$ .

By the hypothesis,  $\omega^p \in V_{R \cup S} \oplus V_S \oplus V_R$ , so write

$$\omega^p = \sum a_i^R v_i^R + \sum a_i^S v_i^S + \sum a_i^{R,S} v_i^{R,S},$$

where the  $v_i^K \in \mathcal{W}_K^p$ ,  $K = d_R, d_S$  and the last sum has terms from  $\mathcal{W}_{d_{R \cup S}}^p$  and  $\mathcal{W}_{d_{R \cup S}, S}^p$ . We only need to show linear independence for each individual sum, so we assume that  $\sum a_i^R v_i^R = \sum a_i^S v_i^S = \sum a_i^{R,S} v_i^{R,S} = 0$  and show that this implies that all of the coefficients,  $a_i^\bullet$ , must be 0.

Each  $v_i^R$  is a sum of 01-gons with a framing (the same in each term) by  $Z \cup \{e\} \setminus R$ . Then we can rewrite the first sum separating out terms with the same framing  $f_i$ ,

$$\sum_{f_i} \sum_j a_{f_i,j}^R v_{f_i,j}^R = 0,$$

so that

$$\begin{aligned} \text{Res}_{d_R}^p \left( \sum_{f_i} \sum_j a_{f_i,j}^R v_{f_i,j}^R \right) &= \sum_{f_i} \sum_j a_{f_i,j}^R \text{Res}_{d_R}^p (v_{f_i,j}^R) \\ &= \sum_{f_i} \sum_j a_{f_i,j}^R (P_{f_i,j}^R \otimes f_{i,j}) \\ &= 0, \end{aligned}$$

where  $P^R \in \mathcal{P}_{R \cup \{e\}}$ ,  $f_{i,j} \in \mathcal{P}_{Z \cup \{e\} \setminus R}$ . The fixed structures,  $f_{i,j}$ , are 01-polygons and are linearly independent, thus the sum only equals 0 if  $\sum_j a_{f_i,j}^R P_{f_i,j}^R = 0$  for each  $f_i$ . But for each  $j$ ,  $P_{f_i,j}^R$  is an element of the Lyndon basis for the polynomial algebra in  $R$ . Hence these are also linearly independent and  $\sum a_i^R v_i^R = 0$  if and only if all  $a_i^R$  are zero, proving the claim for  $\mathcal{W}_{d_R}^p$ .

The proof for  $\mathcal{W}_{d_S}^p$  is identical.

Next, we look at the sum,  $\sum a_i^{R,S} v_i^{R,S}$  and as in the previous two cases, we can write this term as  $\sum_{f_i} \sum_j a_{f_i,j}^{R,S} v_{f_i,j}^{R,S}$ , where here the  $f_i$  are fixed structures on  $Z \cup \{e\} \setminus (R \cup S)$ . Then we only need to prove linear independence for each  $f_i$ ,  $\sum_j a_{f_i,j}^{R,S} P_{f_i,j}^{R,S}$ . If the polynomials  $P^{R,S}$  come from  $\mathcal{W}_{d_{R \cup S}}^p$ , they are linearly independent for the reasons above. So we only need to look at level two insertion elements,  $P_{f_i,j}^{R,S} \in \mathcal{W}_{d_{R \cup S}, S}^p$ . Without loss of generality, assume that all of the terms,  $P_{f_i,j}^{R,S}$  are in  $\mathcal{W}_{d_{R \cup S}, S}^p$ . Now as above, we can break the sum of the  $P^{R,S}$  into fixed structures,  $g_i$ , on  $R \setminus S$ , and apply the residue map,

$$\begin{aligned} \text{Res}_{d_S}^p \left( \sum_{g_i} \sum_j a_{f_i,g_i,j}^{R,S} P_{f_i,g_i,j}^{R,S} \right) &= \sum_{g_i} \sum_j a_{f_i,g_i,j}^{R,S} \text{Res}(P_{f_i,g_i,j}^{R,S}) \\ &= \sum_{g_i} \sum_j a_{f_i,g_i,j}^{R,S} P_{f_i,g_i,j}^S \otimes g_i \\ &= 0. \end{aligned}$$

By definition the  $g_i$  are Lyndon shuffles in  $R \setminus S$  and are therefore linearly independent. And we proceed as before: the  $P^S$  are Lyndon shuffles in  $S$ , so the sum is zero only if all  $a_{f_i,j}^{R,S}$  are zero.  $\square$

**Claim 4.39.** *The set  $\mathcal{W}_\gamma$  spans the set of forms convergent on  $\gamma$ .*

*Proof.* From the previous claim, we may extend the set of linearly independent elements in  $\mathcal{W}_\gamma^p$  to a basis,  $\mathcal{B}$ , of 01-polygons. Then we show that if an element written in this basis is in the intersection (4.4.13), then the coefficient on the basis elements  $\mathcal{B} \setminus \mathcal{W}_\gamma^p$  must be 0. Therefore the set,  $\pi(\mathcal{W}_\gamma^p) = \mathcal{W}_\gamma$ , spans the space of convergent cell forms on  $\gamma$ .

The standard basis of 01-gons is  $B = \{[0, 1, \sigma(Z \setminus \{0, 1\})]; \sigma \in \mathfrak{S}_{n-2}\}$ . As in claim 4.38, we may write the space of 01-gons as

$$V = V_{R \cup S} \oplus V_0 \oplus V_S \oplus V_R.$$

We now construct an alternative basis to  $B$ . The polygons in  $\mathcal{W}_0^p$  span  $V_0$  since they are elements of  $B$ . Furthermore, they are in the intersection of the preimage of the three residue maps by definition 4.4. Let  $\mathcal{W}_0^p = \mathcal{B}_0$ .

As a basis for  $V_S$ , we take instead of permutations of  $S$ , the Lyndon basis for  $S$  and insert into the framings given by  $Z \cup \{e\} \setminus S$ . Let  $V_S = V_{S^1} \oplus V_{S \geq 2}$ . A basis for  $V_{S \geq 2}$  is given by  $\mathcal{W}_{d_S}^p$  and that for  $V_{S^1}$  is given by all insertions of degree 1 Lyndon elements in  $S$ . As before,  $\mathcal{W}_{d_S}^p$  is in the preimage of the three residue maps. Let  $\mathcal{W}_{d_S}^p = \mathcal{B}_{S \geq 2}$  and let the basis for  $V_{S^1} = \mathcal{B}_{S^1}$ .

By the same argument, we construct  $\mathcal{B}_{R^1}$  and  $\mathcal{B}_{R \geq 2}$  as bases for  $V_{R^1}$  and  $V_{R \geq 2}$ .

We take as a basis for  $V_{R \cup S}$ , insertions of Lyndon shuffles and write  $V_{R \cup S}$  as the direct sum of the two vector spaces,  $V_{(R \cup S)^1} \oplus V_{(R \cup S) \geq 2}$  with respective bases,  $\mathcal{B}_{(R \cup S)^1}$  and  $\mathcal{B}_{(R \cup S) \geq 2}$  as previously.

We can write an alternative to basis for  $V_{(R \cup S) \geq 2}$  by taking inserting Lyndon shuffles of  $S$  into consecutive blocks of  $S$  which appear in a shuffle factor, call this basis  $\mathcal{B}'_{R \cup S}$ .

**Example 4.40.** *Consider the subset of marked points in  $\overline{\mathfrak{M}}_{0,9}$ ,*

$$Z = \{0, 1, \infty, t_1, t_2, t_3, t_4, t_5, t_6\}$$

and let  $R = \{t_1, t_2, t_3\}$ ,  $S = \{t_3, t_4, t_5\}$ . In the usual basis for  $\mathcal{L}^{\geq 2}(R \cup S)$ , we have the elements,

$$B_1 = \{(t_1, t_3, t_4, t_5 \mathfrak{m} t_2), (t_1, t_3, t_5, t_4 \mathfrak{m} t_2)(t_1, t_4, t_3, t_5 \mathfrak{m} t_2), (t_1, t_4, t_5, t_3 \mathfrak{m} t_2), \\ (t_1, t_5, t_3, t_4 \mathfrak{m} t_2), (t_1, t_5, t_3, t_4 \mathfrak{m} t_2)\}.$$

We take the alternative basis in which the elements,

$$B_2 = \{(t_1, t_3, t_4, t_5 \mathfrak{m} t_2), (t_1, t_3, t_5, t_4 \mathfrak{m} t_2)(t_1, (t_3 \mathfrak{m} t_4, t_5) \mathfrak{m} t_2), (t_1, (t_3, t_4 \mathfrak{m} t_5) \mathfrak{m} t_2), \\ (t_1, (t_3, t_5 \mathfrak{m} t_4) \mathfrak{m} t_2), (t_1, (t_3 \mathfrak{m} t_4 \mathfrak{m} t_5) \mathfrak{m} t_2)\}$$

appear as insertions. To construct an element of  $\mathcal{B}'_{R \cup S}$ , we insert elements of  $B_2$  into  $e$  in framings of 01-polygons on  $\{0, 1, \infty, t_6, e\}$ .

The subspace spanned by  $\mathcal{B}'_{R \cup S}$  can be written as

$$W_{R^1} \oplus W_{S^1} \oplus W_{S \geq 2} \oplus W_{(R, S) \geq 2}.$$

The space  $W_{R^1}$  is spanned by the elements which are insertions of shuffles in which  $R$  appears as a block in one factor, likewise for  $W_{S^1}$ .  $W_{S \geq 2}$  is spanned by insertions of shuffles in  $S$  of degree  $\geq 2$  into one factor as in example 4.40. And  $W_{(R, S) \geq 2}$  is spanned by those elements in which neither  $S$  nor  $R$  appears as a block in any shuffle factor of

$R \cup S$ . The sets  $\mathcal{W}_{d_{RUS},S}$  and  $\mathcal{W}_{d_{RUS}}$  are subsets of  $B'_{RUS}$  and are respectively bases for  $W_{S \geq 2}$  and  $W_{(R,S) \geq 2}$ . As before let  $\mathcal{W}_{d_{RUS},S} = \mathcal{B}_{(RUS),S \geq 2}$  and  $\mathcal{W}_{d_{RUS}} = \mathcal{B}_{(RUS) \geq 2}$ . We let  $\mathcal{B}_{(RUS),R^1}$  and  $\mathcal{B}_{(RUS),S^1}$  be the bases for  $W_{R^1}$  and  $W_{S^1}$  respectively. Then  $\mathcal{B}_{(RUS),S \geq 2}$ ,  $\mathcal{B}_{(RUS) \geq 2}$ ,  $\mathcal{B}_{(RUS),R^1}$  and  $\mathcal{B}_{(RUS),S^1}$  form a partition of  $B'_{RUS}$

So now we have that

$$\begin{aligned} \mathcal{B} = & \mathcal{B}_0 \cup \mathcal{B}_{S \geq 2} \cup \mathcal{B}_{S^1} \cup \mathcal{B}_{R \geq 2} \cup \mathcal{B}_{R^1} \cup \mathcal{B}_{(RUS)^1} \cup \mathcal{B}_{(RUS) \geq 2} \\ & \cup \mathcal{B}_{(RUS),S \geq 2} \cup \mathcal{B}_{(RUS),R^1} \cup \mathcal{B}_{(RUS),S^1} \end{aligned}$$

is a basis for the 01-gons where

$$\mathcal{W}_\gamma = \mathcal{B}_0 \cup \mathcal{B}_{S \geq 2} \cup \mathcal{B}_{R \geq 2} \cup \mathcal{B}_{(RUS),S \geq 2} \cup \mathcal{B}_{(RUS) \geq 2}.$$

We can now justify that if  $\omega^p$  is in the intersection, (4.4.13), then  $\omega^p$  is in the space spanned by  $\mathcal{W}_\gamma$ .

The elements in the bases for the subspaces,  $V_0, V_S, V_R$  and  $V_{RUS}$  all have unique framings for a well-chosen basis for  $\mathcal{P}_{\overline{K}}$ ,  $K = R, S, R \cup S$ , namely the basis coming from the construction of the  $\mathcal{B}$  sets. Let  $\omega^p$  be in the intersection (4.4.13), we can write  $\omega^p$  in the basis  $\mathcal{B}$  as

$$\omega^p = \omega_{000} + \omega_{001} + \omega_{010} + \omega_{011} + \omega_{100} + \omega_{101} + \omega_{110} + \omega_{111},$$

where  $\omega_{000}$  are terms that are in the kernel of all three residue maps,  $\omega_{001}$  is in the kernel of  $\text{Res}_{d_R}^p$  and  $\text{Res}_{d_S}^p$ , but  $0 \neq \text{Res}_{d_{RUS}}^p(\omega_{001}) \in I_{RUS} \otimes \mathcal{P}_{\overline{RUS}}$ , and so on. This decomposition is unique.

First, we verify that the term  $\omega_{000} \in \mathcal{W}_\gamma^p$ . Since  $\text{Res}_{d_{RUS}}^p(\omega_{000}) = 0$ , the coefficient on the elements  $\mathcal{B}_K$  ( $K = (R \cup S)^\bullet, \bullet$ ) must be 0, since for each framing, the elements of  $\mathcal{B}_K$  form a basis for their image in  $I_{RUS}$ . Since  $\text{Res}_{d_S}^p(\omega_{000}) = 0$  and  $\text{Res}_{d_R}^p(\omega_{000}) = 0$ , then the coefficient on  $\omega_{000}$  on  $\mathcal{B}_K$ ,  $K = R^\bullet, S^\bullet$  is also 0 for the same reason. Therefore  $\omega_{000} \in \langle \mathcal{B}_0 \rangle$ .

Similarly,  $\omega_{010} \in \langle \mathcal{B}_{S \geq 2} \rangle$  and  $\omega_{100} \in \langle \mathcal{B}_{R \geq 2} \rangle$ .

The term  $\omega_{110}$  must be identically 0. For the same reasons as above, it must lie in the space,  $\langle \mathcal{B}_{S \geq 2} \cup \mathcal{B}_{S^1} \cup \mathcal{B}_{R \geq 2} \cup \mathcal{B}_{R^1} \rangle$ . The framing for blocks of  $R$  consists of permutations of  $Z \cup \{e\} \setminus R$  where the elements of  $S \cup \{e\} \setminus R$  are not in a consecutive block since these elements are in the spaces generated by the  $\mathcal{B}_{(RUS)^\bullet, \bullet}$  bases since  $R \cap S$  is non-empty. Therefore if an element maps not to 0, and to  $I_R \cup \mathcal{P}_{\overline{R}}$  for  $\text{Res}_{d_R}^p$ , it must map to 0 for  $\text{Res}_{d_S}^p$  map.

Now, we look at the last four terms,  $\omega_{R,S} = \omega_{001} + \omega_{101} + \omega_{011} + \omega_{111}$ . Since

$$0 \neq \text{Res}_{d_{RUS}}^p(\omega_{R,S}) \in I_{RUS} \otimes \mathcal{P}_{\overline{RUS}},$$

then the coefficients on  $\omega_{R,S}$  on the elements of  $\mathcal{B}_{(RUS)^1}$  are 0.

We have that  $\omega_{111} + \omega_{101}$  maps by  $\text{Res}_{d_R}^p$  to  $\mathcal{P}_R \otimes \mathcal{P}_{\overline{R}}$  which we can write as

$$(\mathcal{L}^1(R) \otimes \mathcal{P}_{\overline{R}}) \oplus (I_R \otimes \mathcal{P}_{\overline{R}}).$$

The residue map,

$$\begin{aligned} \text{Res}_{d_R}^p : & \langle \mathcal{B}_{(RUS),R^1} \rangle \oplus \langle \mathcal{B}_{(RUS) \geq 2} \cup \mathcal{B}_{(RUS),S \geq 2} \cup \mathcal{B}_{(RUS),S^1} \rangle \\ & \rightarrow (\mathcal{L}^1(R) \otimes \mathcal{P}_{\overline{R}}) \oplus (I_R \otimes \mathcal{P}_{\overline{R}}) \end{aligned}$$



is equal to the direct sum of the residue maps,

$$\mathcal{R}^1 \oplus \mathcal{R}^{\geq 2} : (\langle \mathcal{B}_{(RUS),R^1} \rightarrow \mathcal{L}^1(R) \otimes \mathcal{P}_{\overline{R}} \rangle \oplus \\ \langle \mathcal{B}_{(RUS),\geq 2} \cup \mathcal{B}_{(RUS),S^{\geq 2}} \cup \mathcal{B}_{(RUS),S^1} \rangle \rightarrow I_R \otimes \mathcal{P}_{\overline{R}}).$$

We have such a decomposition of the residue map because the polygons in  $\mathcal{B}_{(RUS),R^1}$  (for any framing) only have consecutive blocks in  $R$  which are of degree 1, so they map to  $\mathcal{L}^1(R) \otimes \mathcal{P}$ . Furthermore, for any framing in which  $R$  is not a consecutive block, it is a degree two or higher shuffle, and therefore maps to  $I_R \otimes \mathcal{P}$ . Since we are searching for elements that map to  $I_R$  as a left-hand factor,  $\langle \mathcal{B}_{(RUS),R^1} \rangle$  cannot be in the intersection (4.4.13) and therefore  $\omega_{101} + \omega_{111} \in \langle \mathcal{B}_{(RUS),\geq 2} \cup \mathcal{B}_{(RUS),S^{\geq 2}} \cup \mathcal{B}_{(RUS),S^1} \rangle$ .

We may repeat the same proof as above, substituting  $S$  for  $R$ , which shows that  $\langle \mathcal{B}_{(RUS),S^1} \rangle$  cannot be in the intersection (4.4.13) and therefore  $\omega_{011} + \omega_{111} \in \langle \mathcal{B}_{(RUS),\geq 2} \cup \mathcal{B}_{(RUS),S^{\geq 2}} \rangle$ .

We have now shown that if  $\omega^p$  is in the intersection (4.4.13), then  $\omega^p$  is in the space spanned by  $\mathcal{W}_\gamma^p$ . The map from 01-polygons to  $H^\ell(\mathfrak{M}_{0,n})$  is bijective, therefore by theorem 4.30,  $\mathcal{W}_\gamma$  spans  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$ .  $\square$

We have proven that  $\mathcal{W}_\gamma$  is a set of 01-forms which are linearly independent and span  $H^\ell(\mathfrak{M}_{0,n}^\gamma)$  and therefore form a basis.  $\square$

## 4.5 The non-adjacent bases of $Pic(\overline{\mathfrak{M}}_{0,n})$

The following result emerged from the search for sets of divisors,  $\gamma$ , that satisfy the criterion that  $\mathfrak{M}_{0,n}^\gamma$  be affine. If  $D$  is an ample divisor, then  $\overline{\mathfrak{M}}_{0,n} \setminus D = \mathfrak{M}_{0,n}^\gamma$  is an affine space. Given some “natural set”,  $\gamma$  (we considered for example sets  $\gamma$  which are in the support of a multizeta form), we searched for explicit divisors having support equal to  $\gamma^c$ , the complement of  $\gamma$ . We then attempted to prove, using a methods of A. Gibney and S. Keel, that these are ample in the Picard group. As we will outline in this section, their methods our similar to ours in that they describe the Picard group as generated by polygon divisors. We have not yet succeeded in proving ampleness for the desired sets,  $\gamma$ , however the search led to a new presentation of  $Pic(\overline{\mathfrak{M}}_{0,n})$  with a very simple form which we will prove in this section.

This section may stand alone from the rest of the text, hence we recall some definitions for the ease of the reader.

**Definition 4.41.** *Let  $X$  be a smooth manifold, and let  $Div(X)$  be the group formally generated by Weil divisors on  $X$ . The Picard group,  $Pic(X)$ , is the quotient of  $Div(X)$  by the principal divisors.*

We have the following characterization/definition of the Picard group of Weil divisors on  $\overline{\mathfrak{M}}_{0,n}$ .

**Theorem 4.42.** *[Ke] The Picard group,  $Pic(\overline{\mathfrak{M}}_{0,n})$ , is isomorphic to  $Div(\overline{\mathfrak{M}}_{0,n}) / \sim$ , where  $\sim$  denotes numerical equivalence of divisors.*

Any simple closed loop on a stable curve in the Deligne-Mumford stable compactification of  $\mathfrak{M}_{0,n}$  partitions the points of  $Z$  into two subsets as in figure 9. Pinching this loop to a point yields a nodal topological surface. The stable curves of this type are obtained by putting all possible complex structures on this topological surface. A single

boundary component parametrizes these stable curves for a given pinched loop. We denote by  $d_A$  the boundary divisor in which the loop pinches the subset  $A \subset Z$ , hence  $d_A = d_{Z \setminus A}$ . We denote the set of irreducible boundary divisors on  $\overline{\mathfrak{M}}_{0,n}$  by  $D^n$ . This set has cardinality  $2^{n-1} - 1 - n$ .

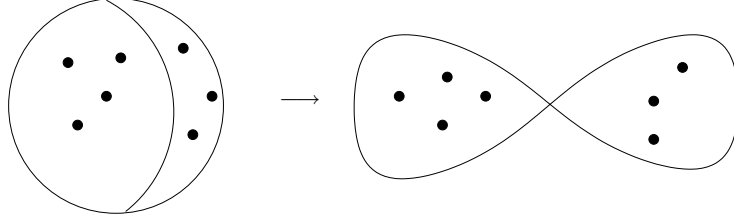


Figure 9: A point on a boundary divisor in  $\overline{\mathfrak{M}}_{0,n}$

**Example 4.43.** The set of boundary divisors,  $D^4$ , on  $\overline{\mathfrak{M}}_{0,4}$  contains the three divisors,  $d_{z_1, z_2}$ ,  $d_{z_1, z_3}$  and  $d_{z_1, z_4}$ .

The set of boundary divisors,  $D^5$ , on  $\overline{\mathfrak{M}}_{0,5}$  contains the 10 divisors,  $d_A$ , where  $A \subset Z$  has cardinality 2.

**Theorem 4.44.** [Ke] A presentation of the Picard group,  $Pic(\overline{\mathfrak{M}}_{0,n})$ , is given by taking the classes,  $\delta_A$  of the boundary divisors,  $d_A \in D^n$  as generators, subject to the following relations: for any four distinct elements,  $z_i, z_j, z_k, z_l$  in  $Z$ ,

$$\sum_{\substack{z_i, z_j \in A \\ z_k, z_l \notin A}} \delta_A = \sum_{\substack{z_i, z_k \in A \\ z_j, z_l \notin A}} \delta_A = \sum_{\substack{z_i, z_l \in A \\ z_j, z_k \notin A}} \delta_A.$$

The following theorem specifies a basis for  $Pic(\overline{\mathfrak{M}}_{0,n})$  and also yields an expression for its dimension.

**Theorem 4.45.** [Gi, 2008] Let  $[z_{i_1}, \dots, z_{i_n}]$  denote a cyclic ordering of the marked points, considered as labelling the consecutive edges of an  $n$ -gon. Then a basis for  $Pic(\overline{\mathfrak{M}}_{0,n})$  is given by the divisors defined by nonempty subsets of marked points on the  $n$ -gon which do not form an adjacent set of vertices on the  $n$ -gon. We call this set of divisors the **non-adjacent basis**.

**Remark.** The following combinatorial formula for the dimension follows immediately from counting the elements of the non-adjacent basis:

$$\dim(Pic(\overline{\mathfrak{M}}_{0,n})) = 2^{n-1} - 1 - n - \binom{n}{2} + n = 2^{n-1} - 1 - \binom{n}{2}. \quad (4.5.1)$$

This dimension was found by S. Keel in [Ke] as the dimension of the first Chow group of  $\overline{\mathfrak{M}}_{0,n}$ .

**Example 4.46.** Consider the standard ordering,  $(z_1, z_2, z_3, z_4, z_5)$ . Then the non-adjacent basis for  $Pic(\overline{\mathfrak{M}}_{0,5})$  for this ordering is given by the five divisors

$$d_{\{z_1, z_3\}}, d_{\{z_1, z_4\}}, d_{\{z_2, z_4\}}, d_{\{z_2, z_5\}}, d_{\{z_3, z_5\}}.$$



## 4.6 A new presentation of $Pic(\overline{\mathfrak{M}}_{0,n})$

In this section, we give a simple expression of each boundary divisor in  $Pic(\overline{\mathfrak{M}}_{0,n})$  in terms of any non-adjacent basis. This yields a new and very simple presentation for  $Pic(\overline{\mathfrak{M}}_{0,n})$  with a minimal set of relations.

Before stating the theorem, we introduce some notation. Given an  $n$ -gon decorated by marked points in the cyclic order,  $(z_{i_1}, \dots, z_{i_n})$ , a divisor in the basis of the Picard group can be described as an ordered list of disjoint subsets,  $B_1, \dots, B_N$ , where each  $B_i$  is a set of adjacent points on the  $n$ -gon but no pair  $(B_J, B_{J+1})$  is a set of adjacent points, and the divisor is given by the blowup at the equality of the marked points in  $\cup_1^N B_i$ . Each pair,  $B_J, B_{J+1} \bmod N$ , defines a non-empty gap between them which we denote  $G_J$ . Specifically, let  $B_J = \{z_{i_j}, \dots, z_{i_{j+k}}\}$ ,  $B_{J+1} = \{z_{i_p}, \dots, z_{i_{p+q}}\}$ . Then  $G_J = \{z_{i_{j+k+1}}, \dots, z_{i_{p-1}}\}$ . In this way, we can write a basis divisor as  $(B_1, G_1, \dots, B_N, G_N)$ .

**Theorem 4.47.** *Let  $\delta$  denote a dihedral ordering  $(z_{i_1}, \dots, z_{i_n})$  on the points  $z_1, \dots, z_n$ . Then  $Pic(\overline{\mathfrak{M}}_{0,n})$  is generated by the set of boundary divisors of  $\overline{\mathfrak{M}}_{0,n}$  (denoted by subsets of  $\{z_1, \dots, z_n\}$  of cardinality between 2 and  $n - 2$ ), subject to the relations*

$$\delta_I = \sum_{J \in \mathcal{J}} \delta_J - \sum_{K \in \mathcal{K}} \delta_K, \quad (4.6.1)$$

where  $I$  denotes a consecutive subset of points for the ordering  $\delta$ ,  $\mathcal{J}$  denotes the set of non-adjacent subsets

$$J = B_1 \cup \dots \cup B_j$$

of  $\{z_1, \dots, z_n\}$  such that  $I$  is equal to a "segment" of even length,

$$B_i, G_i, \dots, B_k, G_k \text{ or } G_i, B_{i+1}, G_{i+1}, \dots, G_k, B_{k+1}$$

of  $(B_1, G_1, \dots, B_N, G_N)$ , and  $\mathcal{K}$  denotes the set of non-adjacent subsets  $K = B_1 \cup \dots \cup B_j$  such that  $I$  is equal to a "segment" of odd length,

$$B_i, G_i, \dots, B_k, G_k, B_{k+1} \text{ or } G_i, B_{i+1}, G_{i+1}, \dots, B_k, G_k$$

of  $(B_1, G_1, \dots, B_N, G_N)$ .

The beauty of the theorem is more easily seen by rephrasing it as: the coefficients of any divisor in the basis of the Picard group given by a cyclic ordering can be calculated by the parity of the defining blocks of the divisor. The precise statement of the theorem does not do justice to its simplicity, as illustrated in the following example.

**Example 4.48.** *We have the following expression for the divisor,  $\delta_{1,2,3}$ , in the basis of  $Pic(\overline{\mathfrak{M}}_{0,6})$  given by the cyclic ordering  $(1, 2, 3, 4, 5, 6) = (z_1, z_2, z_3, z_4, z_5, z_6)$ :*

$$\delta_{1,2,3} = -\delta_{1,3} + \delta_{1,4} + \delta_{3,6} - \delta_{4,6} + \delta_{1,2,4} - \delta_{1,3,5} + \delta_{1,4,5}.$$

*Proof.* We will do this proof by induction.

Let  $\Delta_\delta$  be set of divisors which is a basis for  $Pic(\overline{\mathfrak{M}}_{0,n})$  with respect to the cyclic order  $\delta$  by theorem 4.45. We denote by  $\delta_{B_1 \dots B_N} = (B_1, G_1, \dots, B_N, G_N)$  an element of  $\Delta_\delta$ . Let  $I$  be a consecutive subset for the cyclic order  $\delta$ , and  $\delta_I$  the corresponding boundary divisor in the Picard group. Then we may restate the theorem as follows. One can express  $\delta_I$  as a linear combination of elements of  $\Delta_\delta$ :

$$\delta_I = \sum C_{B_1, \dots, B_{J_k}}^I(B_1, G_1, \dots, B_{J_k}, G_{J_k}), \quad (4.6.2)$$

and the coefficients are given by

$$C_{B_1, \dots, B_N}^I = \begin{cases} 1 & I = \bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \\ -1 & I = (\bigcup_{p=1}^j B_{i+p} \cup G_{i+p}) \cup B_{i+j+1} \\ 0 & \text{otherwise,} \end{cases} \quad (4.6.3)$$

where for  $1 < p < j$ ,  $i + p$  is taken modulo  $n$ .

The following theorem gives the base case for the induction.

**Theorem 4.49.** *[Gi] The coefficient,  $C_{B_1, B_2}^I$  in the Picard basis with respect to  $\gamma$  of the basis divisor  $(B_1, G_1, B_2, G_2)$  is given by*

$$C_{B_1, B_2}^I = \begin{cases} 1 & I = B_i \cup G_j \text{ for any } i, j \\ -1 & I = B_i \text{ or } I = G_j \text{ for any } i, j \\ 0 & \text{otherwise.} \end{cases} \quad (4.6.4)$$

Furthermore, by recursion on  $N$ , this formula allows one to calculate  $C_{B_1, \dots, B_N}^I$  for any basis element  $(B_1, \dots, G_N) \in \Delta_\delta$ .

To calculate the coefficient recursively for  $N = 3$ , we use the following artful technique due to A. Gibney. Let  $B_1 B_2 = B_1 \cup B_2$ ,  $G_1 G_2 = G_1 \cup G_2$  and  $G_3 B_1 = G_3 \cup B_1$ . Consider another basis of the Picard group containing  $(B_1 B_2, G_1 G_2, B_3, G_3)$ . Then the coefficient of  $\delta_I$  on  $(B_1 B_2, G_1 G_2, B_3, G_3)$ ,  $C_{B_1 B_2, B_3}^I$ , is equal to the coefficient of

$$\sum C_{B_1, \dots, B_{J_k}}^I(B_1, G_1, \dots, B_{J_k}, G_{J_k}) \text{ on } (B_1 B_2, G_1 G_2, B_3, G_3), \quad (4.6.5)$$

by the expression (4.6.2) in this new basis. By theorem 4.49, the the only non-zero terms in the expression (4.6.5) are the four terms in which the basis element in the basis with respect to  $\delta$  can be written as a union of the sets,  $B_1 B_2, G_1 G_2, B_3$  and  $G_3$ :

$$C_{B_1, B_2, B_3}^I, C_{B_1, B_2}^I, C_{G_1, G_2}^I \text{ and } C_{G_3 B_1, B_2}^I.$$

Hence we have

$$\begin{aligned} C_{B_1 B_2, B_3}^I &= C_{B_1, B_2, B_3}^I C_{B_1 B_2, B_3}^{B_1, B_2, B_3} + C_{B_1, B_2}^I C_{B_1 B_2, B_3}^{B_1, B_2} + C_{G_1, G_2}^I C_{B_1 B_2, B_3}^{G_1, G_2} \\ &\quad + C_{G_3 B_1, B_2}^I C_{B_1 B_2, B_3}^{G_3 B_1, B_2} \\ &= C_{B_1, B_2, B_3}^I - C_{B_1, B_2}^I - C_{G_1, G_2}^I + C_{G_3 B_1, B_2}^I \\ C_{B_1, B_2, B_3}^I &= C_{B_1 B_2, B_3}^I + C_{B_1, B_2}^I + C_{G_1, G_2}^I - C_{G_3 B_1, B_2}^I. \end{aligned} \quad (4.6.6)$$

**Example 4.50.** In  $\overline{\mathfrak{M}}_{0,6}$ , take a basis for the Picard group defined by the standard cyclic order,  $(z_1, z_2, z_3, z_4, z_5, z_6)$ . In this example, we write the divisor,  $\delta_I$ ,  $I = \{z_2, z_3, z_4\}$ , in this basis. By theorem 4.49 we can calculate most of the coefficients directly, since all but one of the basis elements can be written as the partition into four sets,  $(B_1, G_1, B_2, G_2)$ .

$$\delta_I = \delta_{z_1, z_4} - \delta_{z_1, z_5} - \delta_{z_2, z_4} + \delta_{z_2, z_5} + \delta_{z_1, z_3, z_4} + \delta_{z_1, z_4, z_6} + C_{\{z_1\}, \{z_3\}, \{z_5\}}^I \delta_{z_1, z_3, z_5}. \quad (4.6.7)$$

We can now apply the recursion step in (4.6.6). The first term,  $C_{\{z_1, z_3\}, \{z_4\}}^I = 0$  by theorem 4.49. Likewise,  $C_{\{z_1\}, \{z_3\}}^I = 0$ ,  $C_{\{z_2\}, \{z_4\}}^I = -1$  and  $C_{\{z_1, z_6\}, \{z_3\}}^I = 0$ , so  $C_{\{z_1\}, \{z_3\}, \{z_5\}}^I = -1$ .

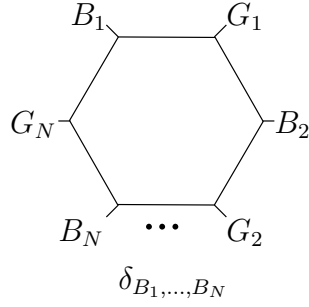


Figure 4.1: Polygon representing a basis divisor

We can generalize this recursive procedure to prove (4.6.3), which is equivalent to the formula (4.6.1) in the statement of the theorem.

It will be useful to consider a visual interpretation of a basis divisor  $(B_1, \dots, G_N)$ , which pictures the divisor as an  $N$ -gon with the sets  $B_i, G_j$  on the vertices as in figure 4.1.

We have an equivalent restatement of theorem using the pictorial representation in figure 4.1 of a basis divisor: The coefficient of  $\delta_I$  on the basis divisor  $\delta_{B_1, \dots, B_N}$  is  $(-1)^m$  if  $I$  is the union of  $m$  of the vertices on the polygon representation of the basis divisor  $\delta_{B_1, \dots, B_N}$ .

Using this pictorial interpretation, we prove (4.6.3) by induction on  $N$ , the number of blocks of consecutive elements on  $\delta$  that define the basis divisor.

Statement 4.6.1 is true for  $N = 2$  by theorem 4.49.

Assume statement 4.6.1 is true for  $N - 1$ . Let  $(B_1, \dots, G_N)$  be a basis element in  $\Delta_\delta$ .

We denote by  $B_1 \cdots B_{N-1} = B_1 \cup \cdots \cup B_{N-1}$ ,  $G_1 \cdots G_{N-1} = G_1 \cup \cdots \cup G_{N-1}$ ,  $G_N B_1 = G_N \cup B_1$ . Let  $\delta'$  denote the dihedral ordering of  $\{z_1, \dots, z_n\}$  given by

$$\delta' = (B_1, \dots, B_{N-1}, G_1, \dots, G_{N-1}, B_N, G_N),$$

where the ordering on the points inside each set  $B_i, G_i$  is that inherited from  $\delta$ . By theorem 4.45, the ordering  $\delta'$  determines a basis  $\Delta_{\delta'}$  of  $Pic(\overline{\mathcal{M}}_{0,n})$ , and the divisor  $\delta_{B_1 \cdots B_{N-1}, B_N}$ , which we denote by  $d = (B_1 \cdots B_{N-1}, G_1 \cdots G_{N-1}, B_N, G_N)$  is in the set  $\Delta_{\delta'}$ .

By expression (4.6.5), the coefficient of  $\delta_I$  on  $d$  in the  $\Delta_{\delta'}$  basis is equal to the coefficient of

$$\sum C_{B_1, \dots, B_{J_k}}^I(B_1, G_1, \dots, B_{J_k}, G_{J_k}) \text{ on } d.$$

Just as in expression (4.6.6) for  $N = 3$ , the only divisors in  $\Delta_\delta$  which have a non-zero coefficient on  $d$  are

$$\begin{cases} \delta_{B_1 \cup \dots \cup B_N} = (B_1, G_1, \dots, B_N, G_N) \\ \delta_{B_1 \cup \dots \cup B_{N-1}} = (B_1, G_1, \dots, B_{N-1}, G_{N-1} B_N G_N) \\ \delta_{G_1 \cup \dots \cup G_{N-1}} = (G_1, B_2, \dots, G_{N-1}, B_N G_N B_1) \\ \delta_{G_N \cup B_1 \cup B_2 \cup \dots \cup B_{N-1}} = (G_N B_1, G_1, B_2, \dots, G_{N-2}, B_{N-1}, G_{N-1} B_N). \end{cases} \quad (4.6.8)$$

By (4.6.4), these four coefficients are respectively 1,  $-1$ ,  $-1$  and 1. Thus, we obtain

$$C_{B_1, \dots, B_N}^I = C_{B_1 \cdots B_{N-1}, B_N}^I + C_{B_1, \dots, B_{N-1}}^I + C_{G_1, \dots, G_{N-1}}^I - C_{G_N B_1, B_2, \dots, B_{N-1}}^I \quad (4.6.9)$$

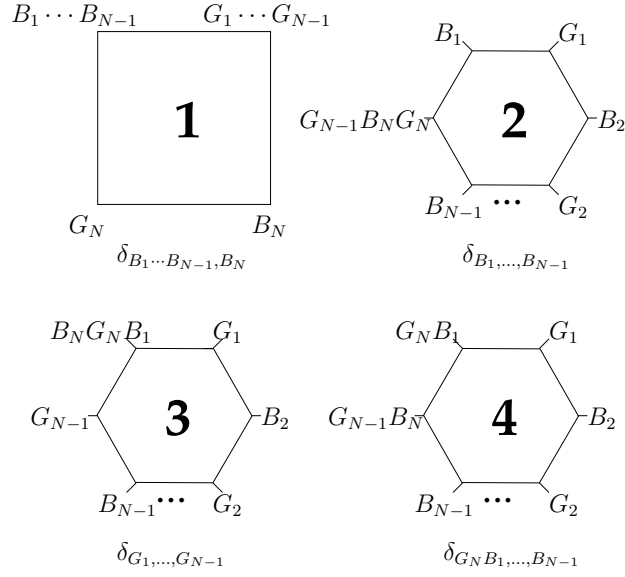


Figure 4.2: Polygons corresponding to  $T_1, T_2, T_3, T_4$

which we rewrite more concisely as

$$C = T_1 + T_2 + T_3 - T_4. \quad (4.6.10)$$

Each of the basis elements,  $\delta_{B_1 \dots B_{N-1}, B_N}, \delta_{B_1, \dots, B_{N-1}}, \delta_{G_1, \dots, G_{N-1}}, \delta_{G_N B_1, \dots, B_N}$  is a divisor defined by less than  $N$  blocks, so by the induction hypothesis,  $T_1 = (-1)^{m_1}$  if  $I$  is the union of  $m_1$  vertices in polygon 1 in figure 4.2,  $T_2 = (-1)^{m_2}$  if  $I$  is the union of  $m_2$  vertices in polygon 2. Likewise, polygon 3 gives  $T_3$  and polygon 4 gives  $T_4$ .

To prove (4.6.3), we need the induction step in each of the following cases.

*Case 1.*  $I$  is not the union of any collection of the sets  $B_i, G_j$ . Then by the expression (4.6.9) and the induction hypothesis, the coefficient of  $\delta_I$  on  $(B_1, \dots, G_N)$  is 0.

*Case 2.*  $I$  is the union of an even number of consecutive subsets:

$$I = \bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \text{ or } I = \bigcup_{p=1}^j G_{i+p} \cup B_{i+p+1}$$

and these sets are not "on the boundary", in other words  $i \geq 1$  and  $i + j \leq N - 2$ . By theorem 4.49,  $T_1 = 0$ . By the induction hypothesis,  $T_2 = T_3 = T_4 = 1$ . Then  $\delta_{B_1, \dots, B_N}^I = 1 + 1 - 1 = 1$  verifying statement 4.6.3 of the theorem. This case covers all divisors  $\delta_I$  such that  $I$  is the union of an even number of subsets from  $\{B_2, G_2, \dots, B_{N-2}, G_{N-2}\}$  or  $\{G_2, B_3, \dots, G_{N-2}, B_{N-1}\}$ .

*Case 3.*  $I$  is the union of an odd number of non-boundary consecutive subsets:

$$I = \left( \bigcup_{p=1}^j B_{i+p} \cup G_{i+p} \right) \cup B_{i+j+1} \text{ or } I = \left( \bigcup_{p=1}^j G_{i+p} \cup B_{i+p+1} \right) \cup G_{i+j+1},$$

$i \geq 1, i + j + 1 \leq N - 2$ . Then by the same arguments as in case 2,  $T_1 = 0, T_2 = T_3 = T_4 = -1$  so that  $\delta_{B_1, \dots, B_N}^I = -1 - 1 + 1 = -1$  verifying statement (4.6.3) of the theorem. This case covers all  $\delta_I$  where  $I$  is the union of an odd number of subsets from  $\{B_2, G_2, \dots, G_{N-3}, B_{N-2}\}$  or  $\{G_2, B_3, \dots, B_{N-2}, G_{N-2}\}$ .

For the following 12 boundary cases, we calculate  $T_1$  by theorem 4.49 and  $T_2, T_3, T_4$  are gotten from the induction hypothesis. The results of the calculation are summarized in the following table and can be deduced from the parity of loops around the vertices in the polygons in figure 4.2. Recall that the last column,  $C = T_1 + T_2 + T_3 - T_4$  denotes  $C_{B_1, \dots, B_N}^I$ . To prove the theorem, it suffices to calculate that  $C = 1$  when  $I$  is the union of an even number of subsets,  $B_i, G_j$ , and that  $C = -1$  when  $I$  is the union of an odd number of such subsets.

Case	$I$	$T_1$	$T_2$	$T_3$	$T_4$	$C$
4	$G_i \cup B_{i+1} \cup \dots \cup B_N \cup G_N \cup B_1, 2 \leq i \leq N-1$	0	1	1	1	1
5	$B_i \cup G_i \cup \dots \cup G_N \cup B_1, 2 \leq i \leq N-1$	0	-1	-1	-1	-1
6	$G_N$ or $B_N$	-1	0	0	0	-1
7	$B_N \cup G_N$	1	0	0	0	1
8	$G_i \cup B_{i+1} \cup \dots \cup B_N \cup G_N, 1 \leq i \leq N-1$	0	-1	0	0	-1
9	$B_i \cup G_i \cup \dots \cup B_N \cup G_N, 2 \leq i \leq N-1$	0	1	0	0	1
10	$B_i \cup \dots \cup B_N, 2 \leq i \leq N-1$	0	0	0	1	-1
11	$G_i \cup \dots \cup B_N, 1 \leq i \leq N-1$	0	0	0	-1	1
12	$G_i \cup \dots \cup G_{N-1}, 1 \leq i \leq N-1$	0	0	-1	0	-1
13	$B_i \cup G_i \cup \dots \cup G_{N-1}, 2 \leq i \leq N-1$	0	0	1	0	1
14	$G_i \cup \dots \cup B_{N-1}, 2 \leq i \leq N-2$	0	1	1	1	1
15	$B_i \cup \dots \cup B_{N-1}, 2 \leq i \leq N-1$	0	-1	-1	-1	-1

The fifteen cases above cover all possible consecutive subsets  $I$  and all verify the statement of the theorem. □

# Chapter 5

## Index of Notations and Definitions by Chapter

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