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Traces de Markov spéciales et formule de Gomi pour les groupes de réflexion

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Special Markov Traces and Gomi's Formula for Reflection Groups

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dirigée par Jean Michel

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Contents

Introduction

The main objects of interest in my thesis are the Markov traces on Hecke algebras and the weight formula discovered by Yasushi Gomi [Gom06] in 2006.

Gomi's "special" Markov traces are the natural generalization of the HOMFLY-PT trace [Jon87] used to construct the famous 2-variable knot invariant. The achievement of Gomi is that he found a beautiful formula for special Markov traces whose weights include Lusztig's Fourier transform matrix coming from the character theory of finite groups of Lie type. Moreover, Gomi's formula allowed him to give a construction of Markov traces for all finite Coxeter groups. Previous works of Meinolf Geck, Sofia Lambropoulou [GL97] and of Meinolf Geck, Lacrimioara Iancu, Gunter Malle [GIM00] have shown that there are infinitely many Markov traces satisfying Markov conditions for standard parabolic subgroups in the cases B_n, D_n and $G(e,1,n)$. Special Markov traces are unique and can be characterised by some additional Markov conditions.

My first goal was to extend Gomi's results to a much larger class of finite reflection groups, the so-called spetsial complex reflection groups. Works of George Lusztig and Gunter Malle [Lus94], Gunter Malle [Mal95] and recent work of Michel Broué, Gunter Malle and Jean Michel have shown that there exists a Fourier transform matrix even in the non-rational case, where there are no such objects as algebraic groups. Using Malle's formulae I show that Gomi's weight formula holds in the case of groups $G(e,1,n)$ and $G(e,e,n)$, leaving only a finite number of exceptional groups to consider. This work can be regarded as another evidence of existence of Spetses, mysterious objects which in the case of complex reflection groups play the role of algebraic groups in the case of Weyl groups.

Recently, after the discovery of a new invariant of oriented links, triply-graded Khovanov-Rozansky link homology [KR08], [Kho07], the situation becomes even more interesting in the case of finite Coxeter groups. Mikhail Khovanov shows that their link invariant can be constructed using Hochschild homology of Soergel bimodules. His construction makes sense for all finite Coxeter groups and gives a new way to obtain Markov traces on Hecke algebras. We call them Khovanov-Rozansky traces. These Markov traces have the same parameters as Gomi's special Markov traces. My sci-

ence advisor Jean Michel naturally conjectured that Khovanov-Rozansky traces should coincide with Gomi's traces for all finite Coxeter groups. One year later Ben Webster and Geordie Williamson [WW09] have proved this conjecture in the case of Weyl groups using Lusztig's theory of character sheaves. After that I gave a positive answer in the case of dihedral groups using the theory of Soergel bimodules and a direct calculation of character values on Kazhdan-Lusztig basis elements. I should point out that a useful idea to help calculate values of the Khovanov-Rozansky trace in the dihedral case was given to me by Raphael Rouquier in private communication. Hence only two cases of the groups H_3 and H_4 remain to be considered.
The thesis has the following structure:

- In the first two chapters we recall the basic definitions and facts about complex reflection groups, Coxeter groups, braid groups and Hecke algebras.

- Third chapter is dedicated to the special Markov traces, or the traces constructed by Gomi, and their generalisation for complex reflection groups $G(e,1,n)$ and $G(e,e,n)$. In section (3.1) we formulate Jones-Ocneanu theorem (3.1.3) which was the starting point of the study of Markov traces. Section (3.2) is dedicated to the notions of Markov trace, Markov property and the definition of special Markov traces of non-exceptional types. In section (3.3), following Geck, Iancu and Malle, we give definition (3.3.1) of Markov trace in the case of $G(e,1,n)$ as well as the formula for its weights in theorem (3.3.5). Section (3.4) contains the definition of the special Markov trace for the group $G(e,e,n)$. In all sections special Markov traces are constructed either as the unique Markov traces satisfying additional Markov properties regarding "non-standard" parabolic subgroups or as restrictions of some specialized Markov traces to Hecke subalgebras.

- Fourth chapter is pretty much independent of the first three chapters. It contains all necessary definitions and facts to construct the Khovanov-Rozansky trace. In theorem (4.4.9) I prove that the Khovanov-Rozansky trace is actually a Markov trace with the given parameter z. The proof there is a sort of compilation of the results in the articles [KR08], [Kho07] and some hints given to me by Raphael Rouquier during his visit to Paris.

- Fifth chapter contains the history of appearance and properties of Lusztig's Fourier transform matrix, generic and fake degrees. Section (5.1) is dedicated to Lusztig's Fourier transform matrix in the context of algebraic groups. In sections (5.2), (5.3) and (5.4) we give, following Lusztig and Malle, definitions and properties of Lusztig's Fourier transform matrix, generic and fake degrees for non-rational finite Coxeter groups and the groups $G(e,1,n)$ and $G(e,e,n)$. It should be pointed out that the Fourier matrix is not completely defined in the case of $G(e, e, n)$. At the moment we have only several properties that the conjectural Fourier matrix should satisfy. They are listed in conjecture (5.4.34).

- Chapter six contains Gomi's theorem (6.1.3) giving a construction of Markov traces for all finite Coxeter groups and the formula for the weigths of the special Markov traces of non-exceptional types. It can be briefly formulated as follows:

Theorem 0.0.1 *Let W be a finite Coxeter group and $\mathcal{H}_W$ the corresponding Hecke algebra with the parameters q and -1. Let τ_g be the trace on $\mathcal{H}_W$ whose weights ω^χ are given by the formula:*

$$\omega^\chi = \left(\frac{1-q}{1+r}\right)^n \sum_{\mu \in \mathrm{Irr} W} S_{\chi,\mu} P_\mu(q,r), \quad \text{for } \chi \in \mathrm{Irr} W \tag{0.0.2}$$

where S is the Fourier transform matrix and $P_\mu(q,r)$ is the Molien series. Then τ_g is a Markov trace. It is equal to the special Markov trace when W is of non-exceptional type.

In sections (6.2) and (6.3) we extend Gomi's result to the case of the groups $G(e, 1, n)$ and $G(e, e, n)$. These results support Broué, Malle, Michel's conjecture that for spetsial complex reflection groups there should be a similar theory to the theory of algebraic groups in the rational case.

- Chapter seven is devoted to the proof of Jean Michel's conjecture (7.0.1) that Gomi's traces coincide with Khovanov-Rozansky traces. Section (7.1) contains the plan of Webster-Williamson proof in the case of Weyl groups. Section (7.2) is dedicated to the proof of the conjecture in the case of dihedral groups.

Chapter 1

Complex Reflection Groups

1.1 Definition of Complex Reflection Groups

Definition 1.1.1 *Let k be a subfield of $\mathbb{C}$ and V be a k-vector space of dimension n. An element $s \in \mathrm{End}(V)$ is called* pseudo-reflection *if* $\mathrm{rank}(\mathrm{Id}_V - s) = 1$*. When s is of finite order it has a unique eigenvalue $\zeta \neq 1$, root of unity, and it is called* reflection.

Remark 1.1.2 *The term "reflection" comes from the case where $k = \mathbb{R}, \zeta = -1$ and s is a hyperplane reflection.*

By definition every reflection s has the form $s(x) = x - \overline{r}(x)r$ where $\overline{r} \in V^*$ is a linear form whose kernel is the hyperplane $H_s = \mathrm{Ker}(s - \mathrm{Id}_V)$ and r is an eigenvector corresponding to the eigenvalue ζ. The hyperplane H_s is called the *reflecting hyperplane* of the reflection s; the vector r and the form $\overline{r}$ are called *root* and *coroot* corresponding to s. They are defined up to a scalar.

Let W be a finite subgroup of $\mathrm{GL}(V)$. We denote by $\mathrm{Ref}(W)$ the set of reflections in W.

Definition 1.1.3 *The group W is called* (finite) complex reflection group *if it is generated by* $\mathrm{Ref}(W)$*. In that case V is called the* reflection representation *or the* natural representation *of W.*

Remark 1.1.4

- *In case $k = \mathbb{Q}$ we obtain the notion of* finite rational reflection group *which coincides with the notion of* Weyl *group.*
- *In case $k = \mathbb{R}$ we obtain the notion of* finite real reflection group *which coincides with the notion of* finite Coxeter group *which will be introduced later.*

Definition 1.1.5 *A* parabolic subgroup *of W is the subgroup of elements of W which act trivially on a fixed subspace of V.*

The following result obtained by Steinberg implies that a parabolic subgroup of a complex reflection group is also a complex reflection group.

Theorem 1.1.6 (Steinberg) *Let V' be a subspace of V and $W_{V'}$ be the parabolic subgroup of W which acts trivially on V'. Then $W_{V'}$ is generated by those reflections of W whose reflecting hyperplane contains V' and thus is a complex reflection group.*

Definition 1.1.7 *A finite complex reflection group is called* irreducible *if the corresponding reflection representation is irreducible.*

Maschke's theorem tells us that V can be decomposed into a direct sum $V_1 \oplus \ldots \oplus V_r$ of irreducible representations V_i of W. Denote by W_i the parabolic subgroups of W acting trivially on the subspaces $\oplus_{j \neq i} V_j$. Then the following proposition is an easy consequence of Steinberg theorem:

Proposition 1.1.8 *For $1 \leq i \leq r$ we have*

- *The group W_i is a complex reflection group with the reflection representation V_i.*
- *The action of W_i on V_i is irreducible. Thus W_i is an irreducible complex reflection group.*
- $W = W_1 \times \ldots \times W_r$.

Remark 1.1.9 *The above proposition implies that in order to classify all complex reflection groups we just need to classify the irreducible ones.*

1.2 Polynomial Invariants and Finite Complex Reflection Groups

The aim of this subsection is to give the classic characterization of finite complex reflection groups.

Let V be a k-vector space of dimension n and W be a finite subgroup of $\mathrm{GL}(V)$. Let $S(V)$ be the symmetric algebra on V (i.e. tensor algebra of V factorized by the ideal generated by all differences $v \otimes u - u \otimes v$ for all $u, v \in V$). For any choice of basis $(x_1, x_2, \ldots, x_n)$ of V one can identify $S(V)$ with the graded ring of polynomials $k[x_1, x_2, \ldots, x_n]$ (here we set $\deg x_i = 1$). The action $v \mapsto w(v)$ of W on V naturally extends to the action of W on $S(V)$. We will see that W is a finite complex reflection group if and only if the corresponding algebra of W-invariants $S(V)^W$ is also an algebra of polynomials. The first important result is as follows:

Theorem 1.2.1 *(Hilbert-Noether) $S(V)^W$ is a finitely generated algebra over k.*

Since the action of W preserves the degrees of elements of $S(V)$ one can choose a homogeneous system of generators $(f_1, f_2, \ldots, f_r)$ of $S(V)^W$. Every generator x_i of $S(V)$ satisfies the equation $\prod_{w \in W}(X - wx_i)$ thus the algebra $S(V)$ is of finite rank as $S(V)^W$-module. Taking into account that the transcendence degree of $S(V)$ is n we can choose n algebraically independent elements among f_i's. Thus (after renumbering of f_i's) we get an inclusion of the polynomial algebra $k[f_1, f_2, \ldots, f_n]$ into $S(V)^W$. Now we can formulate Shephard-Todd-Chevalley characterization theorem:

Theorem 1.2.2 *Let $(f_1, f_2, \ldots, f_n)$ be algebraically independent elements of $S(V)^W$ of homogeneous degrees $(d_1, d_2, \ldots, d_n)$ such that their product $d_1 d_2 \ldots d_n$ is minimal. Then $|W| \leq d_1 d_2 \ldots d_n$ and the following statements are equivalent:*

- $|W| = d_1 d_2 \ldots d_n$.
- *W is generated by* $\mathrm{Ref}(W)$ *and therefore is a* finite complex reflection group.
- $S(V)^W = k[f_1, f_2, \ldots, f_n]$.

Remark 1.2.3 *Let I be the ideal of $S(V)$ generated by the homogeneous elements of $S(V)^W$ of positive degree. It can be proven that if W is a complex reflection group then the representation $S(V)/I$ of W is a graded version of the regular representation of G and there is an isomorphism of G-modules $S(V) \cong (S(V)/I) \otimes_k S(V)^W$.*

Remark 1.2.4 *From now on we will consider W as finite complex reflection group.*

1.3 Groups $\mathbf{G(de, e, n)}$

We will now introduce an important family of complex reflection groups, the so-called *infinite series* which is one of the main subjects in this thesis.

For an integer l denote by ζ_l an l-th primitive root of 1 and by μ_l the set of all l-th roots of 1 in $\mathbb{C}$. Let d, e, n be positive integers and V be an n-dimensional vector space over the field $k = \mathbb{Q}(\zeta_{de}) \subset \mathbb{C}$. The group $G(de, e, n) \subset \mathrm{GL}(V)$ is defined as a set of monomial matrices (matrices with just one non-zero entry in every row and every column) whose coefficients lie in μ_{de} while their product lies in μ_d. If D is the subgroup of diagonal matrices of $G(de, e, n)$ then $|D| = (de)^n/e$ and $G(de, e, n) \cong D \rtimes S_n$ where S_n is the symmetric group.

Denote by s_i the matrices of permutation $(i, i+1)$, by s_1' the matrix $\begin{pmatrix} 0 & \zeta_e^{-1} & 0 \\ \zeta_e & 0 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$ and by t the matrix $\mathrm{diag}(\zeta_d, 1, \ldots, 1)$. All these matrices are reflections. One can easily see that:

- $G(e,e,n)$ is generated by $s_1', s_1, \ldots, s_{n-1}$.
- $G(d,1,n)$ is generated by $t, s_1, \ldots, s_{n-1}$.
- Two cases mentioned above are the only cases when groups $G(de,e,n)$ are generated by n reflections. In all the other cases $G(de,e,n)$ can be generated only by $n+1$ reflections.

If $x_1, x_2, \ldots, x_n$ is a basis of V then algebraically independent generators of the ring of invariants $S(V)^{G(de,e,n)}$ are $f_i := \sum_{j_1<j_2<\ldots<j_i} (x_{j_1} \ldots x_{j_i})^{de}$ for $i = 1, \ldots, n-1$ and $f_n = (x_1 x_2 \ldots x_n)^d$. Indeed, it is clear that these are invariant polynomials. They are algebraically independent since x_i^{de} are the roots of the polynomial

$$X^n - f_1 X^{n-1} + \ldots + (-1)^{n-1} f_{n-1} X + (-1)^n f_n^e$$

and thus $x_1, x_2, \ldots, x_n$ are algebraic over $f_1, f_2, \ldots, f_n$. To see that f_i's generate the whole $S(V)^{G(de,e,n)}$ we just need to verify that the product of their degrees $(de, 2de, \ldots, (n-1)de, nd)$ is equal to the order of $G(de,e,n)$. This is true since $|G(de,e,n)| = |D||S_n| = d(de)^{n-1}n!$.

1.4 Coxeter Groups

In this section we give all necessary definitions concerned with Coxeter groups.

Let W be a group generated by a set S of involutions. For any $s, t \in S$ denote by $m_{st} \in \mathbb{Z} \cup \{\infty\}$ the order of the product st in W. If m_{st} is finite, the word $\underbrace{ststs\ldots}_{m_{st} \text{ terms}}$ is denoted by $\Delta_{s,t}$.

Definition 1.4.1 *(W,S) is a* Coxeter system *if*

$$\Big\langle\ S \mid s^2 = 1, \Delta_{s,t} = \Delta_{t,s} \quad \forall s,t \in S \Big\rangle$$

is a presentation of W.

Remark 1.4.2 *In that case W is called a* Coxeter group *and elements of S are called* Coxeter generators.

Definition 1.4.3 *Let $w \in W$. Denote by $l(w)$ the length of w with respect to the Coxeter generators. We say that $w = s_1 \ldots s_k$ with $s_i \in S$ is a* reduced word *or a* reduced expression *of w if $k = l(w)$.*

Later we will need the notion of the *Bruhat order* $\leq$ on W.

Definition 1.4.4 *Let $x, y \in W$. We say that $x \leq y$ if there is a reduced expression $y = s_1 \ldots s_k$ such that omitting some of s_i's gives a product that represents x.*

Definition-Theorem 1.4.5 *For $I \subset S$ denote by W_I the subgroup of W generated by I. Then (W_I, I) is a Coxeter system and the group W_I is called a* parabolic subgroup *of W.*

Proof. Cf [Bou68, Ch 4, § 1, Th 2] □
Now we introduce the notion of the *geometric representation*:

Definition-Theorem 1.4.6 *Let (W, S) be a Coxeter system. Suppose that W is finite and $n = |S| < \infty$. Define an action of W on $V = \mathbb{R}^n$ with the basis $\{e_s\}_{s \in S}$ as follows:*

$$s(e_t) = e_t + 2cos(\pi/m_{st})e_s, \qquad for\ s, t \in S.$$

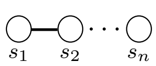
Then this is a well-defined action and we have an inclusion $W \hookrightarrow \mathrm{GL}(V)$. This inclusion is called the geometric representation *of W.*

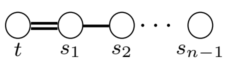
Proof. Cf [Bou68, Ch 4, § 4, Th 1] and [Bou68, Ch 4, § 4, Cor 2] □
An easy consequence of above theorem shows that a finite Coxeter group is a finite real reflection group (cf. (1.1.4)).

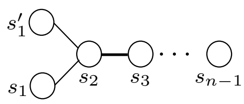
Remark 1.4.7 *The inverse is also true: For a finite real reflection group W one can find a set $S \subset W$ of involutions such that (W, S) is a Coxeter system.*

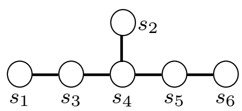
Let (W, S) be a Coxeter system. The presentation of the Coxeter group W is encoded by its *Coxeter graph* or a *Coxeter-Dynkin diagram* which is constructed as follows: The vertices are in bijection with the set S and there is an edge between s and t if $m_{st} \geq 3$. This edge is marked by m_{st}. There is also a convention: if $m_{st} = 3$ then the corresponding edge is just a simple line; if $m_{st} = 4$ it is double, if $m_{st} = 5$ it is a triple-line, etc.

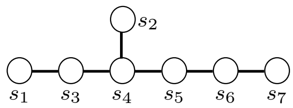
Proposition 1.4.8 *Let W be a finite Coxeter group and V its geometric representation. Let Γ be the Coxeter graph of W. Then V is an irreducible representation of W (and thus W is irreducible as a real reflection group) if and only if W and Γ are as follows:*

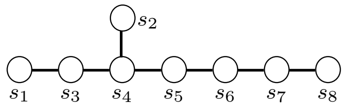
- *Type A_n: $W = G(1, 1, n+1)$* 

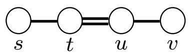
- *Type B_n: $W = G(2, 1, n)$* 

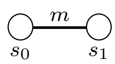
- *Type D_n: $W = G(2, 2, n)$* 

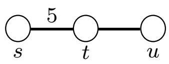
- *Type E_6: $W = G_{35}$* 

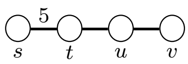
- *Type E_7: $W = G_{36}$* 

- *Type E_8: $W = G_{37}$* 

- *Type F_4: $W = G_{28}$* 

- *Type $I_2(m)$: $W = G(m, m, 2)$* 

- *Type H_3: $W = G_{23}$* 

- *Type H_4: $W = G_{30}$* 

Remark 1.4.9 *Coxeter groups of type $I_2(m)$ are called* dihedral groups*. Groups of types A_n, B_n, D_n and $I_2(m)$ are called* non-exceptional *Coxeter groups.*

Remark 1.4.10 *This classification is a part of the classification of irreducible complex reflection groups which will be briefly described in the next section.*

1.5 Classification of Finite Complex Reflection Groups

As we have pointed out in (1.1.9) in order to classify finite complex reflection groups it is sufficient to classify the irreducible ones. This classification has been made by Shephard and Todd (cf. [ST54]).

Definition 1.5.1 *Let W, W' be complex reflection groups and V, V' be their reflection representations over subfields k and k' of $\mathbb{C}$. We say that W is* isomorphic *to W' and write $W \simeq W'$ or $(V, W) \simeq (V', W')$ if there exists an isomorphism of $\mathbb{C}$-vector spaces $f : V \otimes_k \mathbb{C} \xrightarrow{\sim} V' \otimes_{k'} \mathbb{C}$ which induces a group isomorphism of W and W' regarded as subgroups of $\mathrm{GL}(V \otimes_k \mathbb{C})$ and $\mathrm{GL}(V' \otimes_{k'} \mathbb{C})$.*

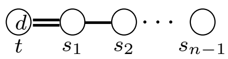
Theorem 1.5.2 (Shephard-Todd) *Let W be an irreducible complex reflection group and V its reflection representation. Then one of the following assertions is true:*

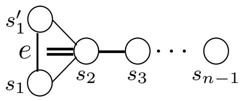
- *There exist integers d, e, n, with $de \geq 2, n \geq 1$ such that $(V, W) \simeq (V_n, G(de, e, n))$ where V_n is the matrix representation of $G(de, e, n)$ described in section (1.3).*

- *There exists an integer* n *such that* $(V, W) \simeq (\mathbb{Q}^{n-1}, S_n)$.
- W *is isomorphic to one of the* 34 *exceptional groups* $G_4, G_5, \dots, G_{37}$.

Remark 1.5.3 *It is not hard to see that if two finite Coxeter groups have the same Coxeter graphs then they are isomorphic as finite real reflection groups. Thus the classification (1.4.8) is a part of Shephard-Todd classification.*

One can extend the notion of Coxeter-Dynkin diagram and encode the presentations with generators and relations of all complex reflection groups by certain diagrams $\mathcal{D}$ "à la Coxeter-Dynkin" (cf. [BMR98, Appendix 2], [BM04, Chapter 2] and [Bes08, Th. 0.6]). This will give also the presentations of the associated braid groups (cf. (2.1.12)). We will provide some of these diagrams.

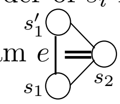
- $W = G(d, 1, n), d \geq 2$ $\quad \mathcal{D}:$ 

- $W = G(e, e, n), e \geq 2, n \geq 3$ $\quad \mathcal{D}:$ 

Here vertices and edges carry the same meaning as those in the Coxeter-Dynkin diagrams; 

means that the order of the generator t is d, 

means that the order of s_i is 2.

The diagram 

corresponds to the presentation

$$\begin{aligned} & s_1^2 = s_1'^2 = s_2^2 = 1, \\ & s_1 s_2 s_1 = s_2 s_1 s_2, s_1' s_2 s_1' = s_2 s_1' s_2, s_2 s_1' s_1 s_2 s_1' s_1 = s_1' s_1 s_2 s_1' s_1 s_2, \\ & \underbrace{s_1 s_1' s_1 \dots}_{e \text{ terms}} = \underbrace{s_1' s_1 s_1' \dots}_{e \text{ terms}}. \end{aligned} \tag{1.5.4}$$

Now assume that W is a complex reflection group $G(d, 1, n)$, or $G(e, e, n)$ or a Coxeter group. There is the following connection between diagrams and the parabolic subgroups of W in the sense of (1.1.5) and (1.4.5):

Proposition 1.5.5 *Let* $\mathcal{D}$ *be the diagram of* W.

- *Every full subdiagram* $\mathcal{D}'$ *of* $\mathcal{D}$ *gives a presentation of the corresponding subgroup* $W(\mathcal{D}')$ *of* W *generated by the reflections corresponding to the vertices of* $\mathcal{D}'$. *The subgroup* $W(\mathcal{D}')$ *is a parabolic subgroup of* W *in the sense of (1.1.5).*
- *If* $P_1 \subset \dots \subset P_k$ *is a chain of parabolic subgroups of* W, *there exists* $g \in W$ *and a chain* $\mathcal{D}_1 \subset \dots \subset \mathcal{D}_k$ *of full subdiagrams of* $\mathcal{D}$ *such that*

$$(P_1, \dots, P_k) = {}^g(W(\mathcal{D}_1), \dots, W(\mathcal{D}_k)).$$

Remark 1.5.6

- *The assumption about the group W in the above proposition is made for simplicity. Look [BMR98, Fact 1.7] for the general case.*
- *Proposition (1.5.5) shows that the definition (1.4.5) of parabolic subgroups agrees with the definition (1.1.5) when we look at a finite Coxeter group as a real reflection group.*

Remark 1.5.7 *As we will see in the next section the above diagrams can be used to provide presentations of braid groups and Hecke algebras corresponding to complex reflection groups.*

Chapter 2

Braid Groups and Hecke Algebras

Most results in this chapter are classic and can be found for example in [BMR98] , [MM10] and [GP00].

2.1 Braid Groups Associated with Complex Reflection Groups

Let W be a complex reflection group and V be its reflection representation over some $k \subset \mathbb{C}$. In this section we work with the "complexification" of the k-vector space V, namely with $V \otimes_k \mathbb{C}$ and denote it by the same letter V.

Notation 2.1.1 *Denote by $\mathcal{A}_W$ the set $\{H_s\}_{s \in \mathrm{Ref}(W)}$ of reflecting hyperplanes of W. For $H \in \mathcal{A}_W$ denote by W_H the pointwise stabilizer of H in W.*

Proposition 2.1.2 *For any $H \in \mathcal{A}_W$ the group W_H is cyclic and generated by an element s_H such that $det_V(s_H) = \zeta_{e_H}$ where $e_H = |W_H|$.*

Definition 2.1.3 *The reflection s_H in the above proposition is called a* distinguished *reflection in W.*

The group W has a natural action $H \mapsto w(H)$ on the set $\mathcal{A}_W$. If we fix $\ell \in W \backslash \mathcal{A}_W$ then $e_H, H \in \ell$ depends only on ℓ since $W_{w(H)} = wW_Hw^{-1}$. We will denote it by e_ℓ.

Definition 2.1.4 *Let $\mathcal{M} = V - \bigcup_{H \in \mathcal{A}_W} H$. Fix any $x_0 \in \mathcal{M}$. The fundamental group $P(W) = \pi_1(\mathcal{M}, x_0)$ is called* pure braid group *corresponding to W. Let $\overline{x}_0$ be the W-orbit of x_0. The fundamental group $B(W) = \pi_1(W \backslash \mathcal{M}, \overline{x}_0)$ is called* braid group *corresponding to W.*

Remark 2.1.5 *This definition is clearly independent of the choice of* x_0 *since both* $\mathcal{M}$ *and* $W\backslash\mathcal{M}$ *are connected topological spaces.*

According to Steinberg's theorem (1.1.6) the surjection $\mathcal{M} \twoheadrightarrow W\backslash\mathcal{M}$ is a covering map and we get the following exact sequence:

$$1 \longrightarrow P(W) \longrightarrow B(W) \longrightarrow W^{op} \longrightarrow 1.$$

Definition 2.1.6 (Generator of the monodromy) *Let* $H \in \mathcal{A}_W$, s_H *be the corresponding distinguished reflection and* $L_H := \mathrm{Im}(s_H - \mathrm{Id}_V)$. *Then* $s_H(x) = \zeta_{e_H}\mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)$ *where* pr_H *is the projection on* H *along* L_H *and* $\mathrm{pr}_H^{\perp}$ *is the projection on* L_H *along* H.

Define a path in $\mathcal{M}$ *from* x_0 *to* $s_H(x_0)$ *as follows:*

- *For a point* $x \in \mathcal{M}$ *define a path* $\sigma_{H,x}(t) = \zeta_{e_H}^t \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)$ *in* V.
- *For any path* $\gamma \subset \mathcal{M}$ *from* x_0 *to* x *define a path* $\sigma_{H,\gamma} = s_H(\gamma^{-1})\sigma_{H,x}\gamma$.
- *Choose the path* γ *and the point* x *"close enough" to* H *in such a way that* $\sigma_{H,\gamma} \subset \mathcal{M}$.

The image $s_{H,\gamma} \in B(W)$ *of* $\sigma_{H,\gamma}$ *via the surjection* $\mathcal{M} \twoheadrightarrow W\backslash\mathcal{M}$ *is called* generator of the monodromy around H.

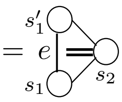
Remark 2.1.7 *It follows immediately from the definition that the image of* $s_{H,\gamma}$ *via the surjection* $B(W) \twoheadrightarrow W^{op}$ *is* s_H.

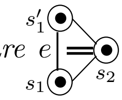
Remark 2.1.8 *It is not hard too see that for two different paths* γ *and* γ' *the generators* $s_{H,\gamma}$ *and* $s_{H,\gamma'}$ *are conjugated in* $B(W)$.

Definition 2.1.9 *Let* s *be a distinguished reflection in* W *with reflection hyperplane* H. *An* s-generator *of the monodromy is a generator of the monodromy* $\mathbf{s}$ *around the image of* H *in* $W\backslash\mathcal{M}$ *such that its image via the surjection* $B(W) \twoheadrightarrow W^{op}$ *is* s.

In order to formulate an important theorem about the structure of braid groups we introduce the following notation:

Notation 2.1.10 *Let* $\mathcal{D}$ *be a diagram for a complex reflection group* W *(cf. (1.5)). We denote by* $\mathcal{D}_{br}^{op}$ *the* opposite braid *diagramm associated with* $\mathcal{D}$*: the set of vertices of* $\mathcal{D}$ *subject to all opposite relations (words in reverse order) of* $\mathcal{D}$ *except the orders of the vertices of* $\mathcal{D}$. *These relations are called* braid relations.

Example 2.1.11 *For the diagram* $\mathcal{D} =$ 

for the complex reflection group $G(e, e, 3)$ *the diagram* $\mathcal{D}_{br}^{op}$ *is represented by the picture* 

and corresponds to the following presentation (cf. (1.5.4)):

$$\mathbf{s}_1\mathbf{s}_2\mathbf{s}_1 = \mathbf{s}_2\mathbf{s}_1\mathbf{s}_2, \mathbf{s}_1'\mathbf{s}_2\mathbf{s}_1' = \mathbf{s}_2\mathbf{s}_1'\mathbf{s}_2, \mathbf{s}_2\mathbf{s}_1\mathbf{s}_1'\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1' = \mathbf{s}_1\mathbf{s}_1'\mathbf{s}_2\mathbf{s}_1\mathbf{s}_1'\mathbf{s}_2,$$
$$\underbrace{\mathbf{s}_1\mathbf{s}_1'\mathbf{s}_1\ldots}_{e \text{ terms}} = \underbrace{\mathbf{s}_1'\mathbf{s}_1\mathbf{s}_1'\ldots}_{e \text{ terms}}.$$

Theorem 2.1.12 (Presentation of Braid Groups) *Let* W *be an irreducible complex reflection group. Let* $\mathcal{N}(\mathcal{D})$ *be the set of vertices of the diagram* $\mathcal{D}$ *for* W *identified with a set of reflections of* W*. For each* $s \in \mathcal{N}(\mathcal{D})$ *there exists an s-generator of the monodromy* $\mathbf{s} \in B(W)$ *such that the set* $\{\mathbf{s}\}_{s\in\mathcal{N}(\mathcal{D})}$*, together with the braid relations of* $\mathcal{D}_{br}^{op}$*, is a presentation of* $B(W)$*.*

The above theorem (cf. [BMR98, Th. 2.27]) shows that in the case when W is a real reflection group the definition of its braid group agrees with the definition of the braid group corresponding to W regarded as a Coxeter group, which is as follows:

Definition 2.1.13 *Let* (W, S) *be a Coxeter system. The group* $B(W)$ *with the presentation*

$$B(W) = \left\langle\ \{\sigma_s\}_{s\in S} \mid\ \underbrace{\sigma_s\sigma_t\sigma_s\ldots}_{m_{st} \text{ terms}} = \underbrace{\sigma_t\sigma_s\sigma_t\ldots}_{m_{st} \text{ terms}} \quad s, t \in S \right\rangle$$

is the braid group *corresponding to* W*.*

2.2 Hecke Algebra

2.2.1 Hecke Algebra: Classical Situation

Definition 2.2.1 *Let* G *be a finite group,* $H \leq G$ *a subgroup and let* ψ *be a* k*-character afforded by* kHe *for some idempotent* $e \in kH$*. The* Hecke algebra $\mathcal{H}(G, H, \psi)$ *is the subalgebra* $ekGe$ *of* kG*.*

Remark 2.2.2 *Since* kG *is semisimple and the Hecke algebra* $ekGe$ *is isomorphic to the opposite ring* $(\mathrm{End}_{kG}kGe)^{op}$ *where* $\mathrm{End}_{kG}kGe$ *is viewed as an algebra of left operators on* kGe *it is also* semisimple*.*

Remark 2.2.3 *The Hecke algebra* $\mathcal{H}(G, H, \psi)$ *is independent of the left ideal* kHe *affording* ψ*. Indeed, if* $kHe \cong kHe'$ *for another idempotent* e'*, then clearly* $kGe \cong kGe'$ *and by the preceeding remark* $ekGe \cong e'kGe'$*.*

Next theorem [Isa76,] establishes a connection between irreducible characters of the group G and irreducible characters of the Hecke algebra $\mathcal{H}(G, H, \psi)$.

Theorem 2.2.4 *Let $\mathcal{H} = \mathcal{H}(G, H, \psi)$ be the Hecke algebra associated with G, H, e and ψ as above. The following statements hold:*

(i) Let $\phi \in \mathrm{Irr} G$. Then the restriction $\phi_{\mathcal{H}} \neq 0$ if and only if the multiplicity $(\phi, \mathrm{Ind}_H^G \psi) \neq 0$.

(ii) The map $\phi \mapsto \phi_{\mathcal{H}}$ is a bijection from the set of irreducible characters ϕ of G such that $(\phi, \mathrm{Ind}_H^G \psi) \neq 0$, to the set of all irreducible characters of the semisimple k-algebra $\mathcal{H}$.

(iii) If φ is an irreducible character of $\mathcal{H}$ corresponding to $\phi \in \mathrm{Irr} G$ according to part (ii), then $\deg \varphi = (\phi, \mathrm{Ind}_H^G \psi)$.

We will be interested in Hecke algebras in a special case. Namely, let G be a finite group with BN-pair, W the Weyl group of G, and S the set of distinguished generators of W. Let

$$G = \coprod_{w \in W} BwB$$

be the *Bruhat decomposition* of G with respect to a Borel subgroup B. The pair (W, S) is a Coxeter system thus the group W has the presentation

$$\Big\langle\ S \mid s^2 = 1, \Delta_{s,t} = \Delta_{t,s} \quad \forall \forall s, t \in S \Big\rangle$$

where $m_{s,t}$ is the order of the product st (cf. (1.4.1)).

Iwahori and Matsumoto have shown that the Hecke algebra $\mathcal{H}(G, B) := \mathcal{H}(G, B, \mathbb{1}_B)$ has a presentation, as an algebra, similar to the presentation of the group W. This allows to establish a bijection between the irreducible characters of $\mathcal{H}(G, B)$ and the irreducible characters of W.

To formulate their theorem we need some definitions. First of all, recall that $\mathcal{H}(G, B) = ekGe$, where $e = |B|^{-1} \sum_{b \in B} b$ is the idempotent in kB such that the left ideal kGe affords the permutation representation $k(G/B)$. It has the *standard basis* consisting of the elements

$$a_D = |B|^{-1} \sum_{x \in D} x, \quad D \in B \backslash G / B.$$

Since there is a bijection map $w \leftrightarrow BwB$ from W to the double cosets $B \backslash G / B$ we shall label the standard basis elements by the elements of W, and denote them by $\{a_w\}_{w \in W}$.

For each $w \in W$, we define its *index*

$$\mathrm{ind} w = |B : {}^{w}B \cap B| = |BwB/B|,$$

here ${}^{w}B$ stands for ${}^{\dot{w}}B$, where $\dot{w}$ is a coset representative in N corresponding to the element $w \in N/T$. In particular, the *index parameters* of G are defined to be the numbers

$$q_s = \text{ind} s, \quad s \in S.$$

Now we can formulate

Theorem 2.2.5 (Iwahori-Matsumoto)

- *The Hecke algebra $\mathcal{H}(G,B)$ is generated by elements $\{a_s\}_{s\in S}$. These generators satisfy the quadratic relations*

  $$a_s^2 = (q_s - 1)a_s + q_s 1, \ s \in S,$$

  and the homogeneous relations

  $$\underbrace{a_s a_t a_s \cdots}_{m_{st} \text{ terms}} = \underbrace{a_t a_s a_t \cdots}_{m_{st} \text{ terms}}$$

- *The generators and relations given above define a presentation of the Hecke algebra $\mathcal{H}(G,B)$.*

Remark 2.2.6 *The above results can be applied in the following case: $G = \mathbf{G}^F, B = \mathbf{B}^F$ where $\mathbf{G}$ is a connected affine algebraic group over $\overline{\mathbb{F}}_q$, split over $\mathbb{F}_q$, with the corresponding Frobenius map F, and $\mathbf{B}$ is a rational Borel subgroup of $\mathbf{G}$. (cf. (5.1.3) and (5.1.5))*

Remark 2.2.7 *One can show using the properties of the BN-pair that if $s,t \in S$ are conjugate in W then $q_s = q_t$.*

Remark 2.2.8 *Theorem (2.2.5) makes it natural to investigate the so-called* Iwahori-Hecke algebras*, replacing numbers q_s by some parameters. We give the main definitions and properties of these algebras in next subsections.*

2.2.2 Iwahori-Hecke Algebras

Let (W,S) be a Coxeter system. Let $\mathbf{q} = \{q_s, q'_s\}_{s\in S}$ be a set of variables such that $q_s = q_t$ and $q'_s = q'_t$ if s and t are conjugate in W and $\mathcal{A} = \mathbb{Z}[\mathbf{q}^{\pm 1}]$.

Definition 2.2.9 *The* Iwahori-Hecke algebra $\mathcal{H}_{\{q_s,q'_s\}_{s\in S}}(W,\mathcal{A})$ *over the ring $\mathcal{A}$ with the parameters $\{q_s,q'_s\}_{s\in S} \subset \mathcal{A}$ is a unitary $\mathcal{A}$-algebra with the following presentation by generators and relations:*

$$\Big\langle \{\mathbf{T}_s\}_{s\in S} \mid \underbrace{\mathbf{T}_s\mathbf{T}_t\mathbf{T}_s\ldots}_{m_{st} \text{ terms}} = \underbrace{\mathbf{T}_t\mathbf{T}_s\mathbf{T}_t\ldots}_{m_{st} \text{ terms}}, \ (\mathbf{T}_s - q_s)(\mathbf{T}_s - q'_s) = 0 \quad s,t \in S \Big\rangle$$

Remark 2.2.10 *One can see that $\mathcal{H}_{\{q_s,q'_s\}_{s\in S}}(W,\mathcal{A})$ is just a quotient of the braid group algebra $\mathcal{A}B(W)$ by the ideal generated by the elements $(\sigma_s - q_s)(\sigma_s - q'_s)$ (cf. (2.1.13)).*

Remark 2.2.11 *It is obvious that Iwahori-Hecke algebra becomes the group algebra $\mathbb{Z}W$ via the specialization $q_s \mapsto 1, q'_s \mapsto -1$. Thus it can be regarded as a deformation of the group algebra $\mathbb{Z}W$.*

The following proposition can be found in [GP00, Th. 4.4.6]:

Proposition 2.2.12 (Alternative definition) Iwahori-Hecke algebra $\mathcal{H}_{\{q_s,q'_s\}_{s\in S}}(W,\mathcal{A})$ *is a free $\mathcal{A}$-module with the basis $\{\mathbf{T}_w\}_{w\in W}$ and the following multiplication rule:*

$$\forall s\in S \quad \mathbf{T}_s\mathbf{T}_w = \begin{cases} \mathbf{T}_{sw} & , \ l(sw) > l(w) \\ (q_s+q'_s)\mathbf{T}_w - q_sq'_s\mathbf{T}_{sw} & , \ l(sw) < l(w), \end{cases}$$

$\mathbf{T}_w = \mathbf{T}_{s_1}\dots\mathbf{T}_{s_k}$ where $w = s_1\dots s_k$ is any reduced expression of w (cf. (1.4.3)).

Proof. Cf [GP00, Cor 4.4.5] □

Definition 2.2.13 *Let $\mathcal{B}$ be any ring and $\mathcal{A}\xrightarrow{\kappa}\mathcal{B}$ be a ring morphism making $\mathcal{B}$ into an $\mathcal{A}$ algebra. We call the tensor product $\mathcal{B}\otimes_{\mathcal{A}} \mathcal{H}_{\{q_s,q'_s\}_{s\in S}}(W,\mathcal{A})$* Iwahori-Hecke algebra *(or just* Hecke algebra*) over $\mathcal{B}$ with the parameters $\{\kappa(q_s),\kappa(q'_s)\}_{s\in S}\subset\mathcal{B}$.*

An important example of an Iwahori-Hecke algebra is as follows

Example 2.2.14 *Let v be a variable, $\mathcal{B}=\mathbb{Z}[v^{\pm1}]$ and $\kappa:\mathcal{A}\longrightarrow\mathcal{B}$ is given by $q_s\mapsto v^{-2}, q'_s\mapsto -1$. We denote by $\mathcal{H}'_W$ the* Iwahori-Hecke algebra over $\mathbb{Z}[v^{\pm1}]$ *with the parameters $(v^{-2},-1)$.*

Proposition (2.2.12) shows that $\mathcal{H}'_W$ is a free $\mathcal{B}$-module with the basis $\{\mathbf{T}_w\}_{w\in W}$. Later we will need the notion of an alternative K-basis of $\mathcal{H}'_W$ defined as follows

Definition-Theorem 2.2.15 *Consider in the Hecke algebra $\mathcal{H}'_W$ the elements $\widetilde{\mathbf{T}}_w = v^{l(w)}\mathbf{T}_w$. By [KL79] there is exactly one involution $d:\mathcal{H}\longrightarrow\mathcal{H}$ with $d(v)=v^{-1}$ and $d(\mathbf{T}_w)=(\mathbf{T}_{w^{-1}})^{-1}$ and for $w\in W$ there is a unique $C'_w\in\mathcal{H}'_W$ with $d(C'_w)=C'_w$ and*

$$C'_w \in \widetilde{\mathbf{T}}_w + \sum_y v\mathbb{Z}[v]\widetilde{\mathbf{T}}_y.$$

These elements form the so-called Kazhdan-Lusztig basis *of $\mathcal{H}'_W$.*

Remark 2.2.16 *The Iwahori-Hecke algebra turns out to be a* geneneric Hecke algebra *corresponding to real reflection groups, whose definition we provide in the next subsection.*

2.2.3 Generic Hecke Algebra

Let W be a complex reflection group.

Definition 2.2.17 *Let*

$$\mathbf{u} = (u_{\ell,j})_{(\ell \in W\backslash\mathcal{A}_W)(0\leq j\leq e_\ell-1)}$$

be a set of $\sum_{\ell\in W\backslash\mathcal{A}_W} e_\ell$ *variables and* $\mathbb{Z}[\mathbf{u}^{\pm 1}]$ *the ring of Laurent polynomials in these variables. Then the* generic Hecke algebra $\mathcal{H}_\mathbf{u}(W)$ *is the* $\mathbb{Z}[\mathbf{u}^{\pm 1}]$-*algebra* $\mathbb{Z}[\mathbf{u}^{\pm 1}]B(W)/J$, *where* $B(W)$ *is the braid group associated to* W *(cf. (2.1.4)) and* J *is the ideal of* $\mathbb{Z}[\mathbf{u}^{\pm 1}]B(W)$ *generated by the elements of the form*

$$\prod_{j=0}^{e_\ell-1}(s_{H,\gamma} - u_{\ell,j}),$$

where $s_{H,\gamma}$ *is any generator of the monodromy around a hyperplane* $H \in \ell, \ell \in W\backslash\mathcal{A}_W$ *(cf. (2.1.6)).*

Remark 2.2.18 *It follows from (2.1.8) that in the above definition we do not need to consider all possible paths* γ *for a fixed* ℓ *and* $H \in \ell$, *but just one of them since the products* $\prod_{j=0}^{e_\ell-1}(s_{H,\gamma} - u_{\ell,j})$ *are conjugate in* $\mathbb{Z}[\mathbf{u}^{\pm 1}]B(W)$.

Now assume that W is an irreducible complex reflection group. Let $\mathcal{D}$ be the diagram of W (cf. (1.5)) and let $\mathbf{s}$ be an s-generator of the monodromy corresponding to a vertex $s \in \mathcal{N}(\mathcal{D})$ (cf. (2.1.12)). We set $u_{s,j} := u_{\ell,j}, e_s := e_\ell$ for $0 \leq j \leq e_\ell - 1$, where ℓ is the W-orbit of the reflecting hyperplane of s. Denote by $\mathbf{T}_s$ the image of $\mathbf{s}$ under the surjection

$$\mathbb{Z}[\mathbf{u}^{\pm 1}]B(W) \twoheadrightarrow \mathcal{H}_\mathbf{u}(W). \tag{2.2.19}$$

The following proposition is an immediate consequence of theorem (2.1.12):

Proposition 2.2.20 (Presentation of Generic Hecke Algebras) *Let* W *be an irreducible complex reflection group. The generic Hecke algebra* $\mathcal{H}_\mathbf{u}(W)$ *is isomorphic to the* $\mathbb{Z}[\mathbf{u}^{\pm 1}]$-*algebra generated by the elements* $(\mathbf{T}_s)_{s\in\mathcal{N}(\mathcal{D})}$ *with the following defining relations:*

- *the elements* $\mathbf{T}_s$ *satisfy the braid relations defined by* $\mathcal{D}_{br}^{op}$,
- *we have* $(\mathbf{T}_s - u_{s,0})(\mathbf{T}_s - u_{s,1})\ldots(\mathbf{T}_s - u_{s,e_s-1}) = 0$

Let I be a subspace of the reflection representation V of W and W_I the corresponding parabolic subgroup of W (cf. (1.1.5)). Denote by $\mathcal{A}_I$ the set of reflecting hyperplanes of W_I and by

$$\mathbf{t} = (t_{\ell,j})_{(\ell\in W_I\backslash\mathcal{A}_I)(0\leq j\leq e_\ell-1)}$$

a set of $\sum_{\ell\in W_I\backslash\mathcal{A}_S} e_\ell$ variables.

Let $a : W_I\backslash\mathcal{A}_I \longrightarrow W\backslash\mathcal{A}_W$ be the map which sends a W_I-orbit of hyperplanes onto the corresponding W-orbit. We have a morphism

$$\mathbb{Z}[\mathbf{t}^{\pm1}] \longrightarrow \mathbb{Z}[\mathbf{u}^{\pm1}], \quad t_{\ell,j} \mapsto u_{a(\ell),j}$$

and the injections (cf. (2.1.4) and [BMR98, (2.29)]):

$$B(W_I) \hookrightarrow \pi_1(W_I\backslash\mathcal{M}, \overline{x}_0) \hookrightarrow B(W)$$

which induce an inclusion

$$\mathcal{H}_{\mathbf{t}}(W_I) \otimes_{\mathbb{Z}[\mathbf{t}^{\pm1}]} \mathbb{Z}[\mathbf{u}^{\pm1}] \hookrightarrow \mathcal{H}_{\mathbf{u}}(W) .$$

Definition 2.2.21 • *The image of the above inclusion is called the* parabolic Hecke subalgebra *of $\mathcal{H}_{\mathbf{u}}(W)$ associated with W_I.*

- *Let W be as in (1.5.5), J be a subset of $\mathcal{N}(\mathcal{D})$ and $\mathcal{D}_J$ the corresponding full subdiagram of $\mathcal{D}$. Then the parabolic Hecke subalgebra of $\mathcal{H}_{\mathbf{u}}(W)$ associated with $W(\mathcal{D}_J)$ is called* the parabolic Hecke subalgebra *of $\mathcal{H}_{\mathbf{u}}(W)$ corresponding to J.*

Definition 2.2.22 *Let $\mathbb{Z}[\mathbf{u}^{\pm1}] \xrightarrow{\kappa} \mathcal{B}$ be a ring morphism making $\mathcal{B}$ an $\mathbb{Z}[\mathbf{u}^{\pm1}]$ algebra. By analogy with (2.2.13) we call the tensor product $\mathcal{B} \otimes_{\mathbb{Z}[\mathbf{u}^{\pm1}]} \mathcal{H}_{\mathbf{u}}(W)$ the* Hecke algebra *of W over $\mathcal{B}$ with the parameters $\kappa(u_{\ell,j})_{(\ell\in W\backslash\mathcal{A}_W)(0\le j\le e_\ell-1)}$.*

2.2.4 Structure of Hecke Algebras

In this section we formulate without proof some well-known structural properties of generic Hecke algebras. For more information and references one can look at [MM10].

The first important property is:

Theorem 2.2.23 *Let W be an irreducible complex reflection group different from $G_{17}, G_{18}, G_{19}, G_{29}, G_{31}, G_{32}, G_{33}$ and G_{34} (cf. (1.5.2)). Then the generic Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$ is a free $\mathbb{Z}[\mathbf{u}^{\pm1}]$-module of rank $|W|$.*

Proof. Cf [MM10, Tables 8,9] □

In order to formulate other properties of generic Hecke algebras we need the following definitions and theorems which can be found for example in [GP00].

Definition 2.2.24 *K-algebra $\mathcal{H}$ of finite dimension over a field K is called*

- split semisimple *if $\mathcal{H}$ is isomorphic as K-algebra to a sum $\oplus_i \mathrm{M}_{n_i}(K)$ of matrix algebras.*

- separable *if* $\mathcal{H} \otimes_K \overline{K}$ *is isomorphic as* $\overline{K}$*-algebra to a sum* $\oplus_i \mathrm{M}_{n_i}(\overline{K})$ *of matrix algebras, where* $\overline{K}$ *is the algebraic closure of* K*.*

The numbers n_i *are called* numerical invariants *of* $\mathcal{H}$*.*

Definition 2.2.25 *Assume that* $\mathcal{A}$ *is an integral domain and* $\mathcal{H}$ *is an* $\mathcal{A}$*-algebra finitely generated as an* $\mathcal{A}$*-module. Let* K *be an extension of* $\mathrm{Frac}(\mathcal{A})$*. The field* K *is called a* splitting field *of* $\mathcal{H}$ *if* $\mathcal{H} \otimes_{\mathcal{A}} K$ *is split semisimple.*

Theorem 2.2.26 (Tits deformation theorem) *Let* $f : \mathcal{A} \longrightarrow k$ *be a ring morphism where* $\mathcal{A}$ *is an integral domain and* k *is a field. Denote* $\mathrm{Frac}(\mathcal{A})$ *by* K*. Consider an* $\mathcal{A}$*-algebra* $\mathcal{H}$ *which is a free finite-dimensional* $\mathcal{A}$*-module.*

(1) Suppose that $\mathcal{H} \otimes_{\mathcal{A}} K$ *and* $\mathcal{H} \otimes_{\mathcal{A}} k$ *are separable. Then they have the same numerical invariants.*

(2) Let $\overline{\mathcal{A}}$ *be the integral closure of* $\mathcal{A}$ *in* $\overline{K}$ *and* $\overline{k}$ *the algebraic closure of* k*. Then* f *can be extended to* $\overline{f} : \overline{\mathcal{A}} \longrightarrow \overline{k}$*. And under the assumption (1) we have:*

  - *For any irreducible character* χ *of* $\mathcal{H} \otimes_{\mathcal{A}} \overline{K}$ *and any* $a \in \mathcal{H}$ *the value* $\chi(a)$ *is in* $\overline{\mathcal{A}}$*.*
  - $\overline{f} \circ \chi$ *is an irreducible character of* $\mathcal{H} \otimes_{\mathcal{A}} \overline{k}$*.*

Definition 2.2.27 *Let* $\mathcal{H}$ *be an* $\mathcal{A}$*-algebra which is a free* n*-dimensional* $\mathcal{A}$*-module.*

- *An* $\mathcal{A}$*-linear map* $\tau : \mathcal{H} \longrightarrow \mathcal{A}$ *is called a* central function *or a* trace *if* $\tau(hh') = \tau(h'h)$ *for any* $h, h' \in \mathcal{H}$*.*

- *Let* $(h_1, \ldots, h_n)$ *be some* $\mathcal{A}$*-basis of* $\mathcal{H}$*. The algebra* $\mathcal{H}$ *is called* symmetric *if there is a trace* τ *and an* $\mathcal{A}$*-basis* $(h'_1, \ldots, h'_n)$ *of* $\mathcal{H}$ *such that* $\tau(h_i h'_j) = \delta_{i,j}$*. In this case* τ *is called a* symmetrizing trace *for* $\mathcal{H}$*.*

Example 2.2.28 *The classical example of a symmetric algebra is a group algebra* kW *with the symmetrizing trace* τ_0 *given by* $\tau_0(w) = \delta_{w,1}$*. This trace is called* the canonical trace *on* kW*.*

The following theorem can be found in [GP00, (7.2)]:

Theorem 2.2.29 *Let* $\mathcal{H}$ *be a symmetric* $\mathcal{A}$*-algebra with a symmetrizing trace* τ *such that* $\tau(1) = 1$*. Assume that* $\mathcal{A}$ *is an integral domain and denote* $\mathrm{Frac}(\mathcal{A})$ *by* K*. Suppose that* $\mathcal{H} \otimes_{\mathcal{A}} K$ *is split semisimple. Then* τ *can be decomposed as follows:*

$$\tau = \sum_M \frac{1}{S_{\chi_M}} \chi_M,$$

where M runs over all isomorphism classes of simple $\mathcal{H} \otimes_{\mathcal{A}} K$-modules and χ_M are the corresponding characters. The elements $S_{\chi_M} \in K$ are integral over $\mathcal{A}$ and called the Schur elements *attached to M.*

It is not hard to see that the reflection representation V of W can be realized over the field $K_W \subset \mathbb{C}$ generated by the traces of the elements of W on V. It is a theorem of Benard and Bessis (cf. [Bes97, Th. 0.2]) that *all* representations of W can be realized over K_W.

Let $\mathcal{O}$ be the ring of integers of K_W and let $\widetilde{\mathcal{A}} = \mathcal{O}[\mathbf{v}^{\pm 1}]$ with

$$\mathbf{v} = (v_{\ell,j})_{(\ell \in W \setminus \mathcal{A}_W)(0 \leq j \leq e_\ell - 1)} \quad \text{such that } v_{\ell,j}^{|\mu(K_W)|} = \zeta_{e_\ell}^{-j} u_{\ell,j},$$

where $\mu(K_W)$ is the group of roots of unity in K_W.

Gunter Malle proves the following theorem in [Mal99, Th. 5.2]:

Theorem 2.2.30 (Malle) *Let W be any complex reflection group. Then the field $K_W(\mathbf{v})$ is a splitting field for the generic Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$.*

Remark 2.2.31 *One has stronger versions of the above theorem for certain classes of complex reflection groups:*

- *If $W = G(d, 1, n)$ (cf. (1.3)) then Ariki and Koike prove that the field $\mathbb{Q}(\mathbf{u})$ is a splitting field for the generic Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$ (cf. [AK94]).*
- *If $W = G(e, e, n)$ then Ariki proves that the field $\mathbb{Q}(\zeta_e, \mathbf{u})$ is a splitting field for the generic Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$ (cf. [Ari95]).*

Consider the specialization

$$f : \mathbb{Z}[\mathbf{u}^{\pm 1}] \longrightarrow K_W \text{ given by } u_{\ell,j} \mapsto \zeta_{e_\ell}^{j}. \tag{2.2.32}$$

Then proposition (2.2.20) implies that $\mathcal{H}_{\mathbf{u}}(W) \otimes_{\mathbb{Z}[\mathbf{u}^{\pm 1}]} K_W$ is isomorphic to the group algebra $K_W W$. Theorems (2.2.23), (2.2.30) and Tits deformation theorem applied to the $\mathbb{Z}[\mathbf{u}^{\pm 1}]$-module $\mathcal{H}_{\mathbf{u}}(W)$ show that the $K_W(\mathbf{v})$-algebras $\mathcal{H}_{\mathbf{u}}(W) \otimes_{\mathbb{Z}[\mathbf{u}^{\pm 1}]} K_W(\mathbf{v})$ and $K_W(\mathbf{v})W$ are isomorphic. This gives us a bijection between the irreducible characters of W and $\mathcal{H}_{\mathbf{u}}(W) \otimes_{\mathbb{Z}[\mathbf{u}^{\pm 1}]} K_W(\mathbf{v})$:

$$\mathrm{Irr} W \longrightarrow \mathrm{Irr} \mathcal{H}_{\mathbf{u}}(W) : \chi \mapsto \chi^{(q)}. \tag{2.2.33}$$

Remark 2.2.34 *Tits deformation theorem implies that the inverse $\chi^{(q)} \mapsto \chi$ of the above bijection is given by some extension to $\widetilde{\mathcal{A}}$ of the specialization (2.2.32).*

The next important result following from [MM10] is:

Proposition 2.2.35 *Let W be an irreducible complex reflection group different from $G_{17}, G_{18}, G_{19}, G_{29}, G_{31}, G_{32}, G_{33}$ and G_{34} (cf. (1.5.2)). Then the generic Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$ is a symmetric algebra with a symmetrizing trace τ which specializes to the canonical trace on $K_W W$ via (2.2.32).*

Remark 2.2.36 *In the case when W is a Coxeter group and $\mathcal{H}_{\mathbf{u}}(W)$ is the Iwahori-Hecke algebra $\mathcal{H}_{\{q_s,q'_s\}_{s\in S}}(W, \mathbb{Z}[\mathbf{q}^{\pm 1}])$ a symmetrizing trace τ can be constructed using the alternative definition of Iwahori-Hecke algebra (2.2.12): $\tau(\mathbf{T}_w) := \delta_{w,1}$.*

Since K_W is an algebraic number field we have $\mathrm{Frac}(\widetilde{\mathcal{A}}) = K_W(\mathbf{v})$ and theorem (2.2.29) applied to the symmetric algebra $\mathcal{H}_{\mathbf{u}}(W) \otimes_{\mathbb{Z}[\mathbf{u}^{\pm 1}]} \widetilde{\mathcal{A}}$ implies that the symmetrizing trace τ can be decomposed into the sum

$$\tau = \sum_{\chi \in \mathrm{Irr}\mathcal{H}_{\mathbf{u}}(W)} \frac{1}{S_\chi} \chi$$

where the Schur elements S_χ are integral over $\widetilde{\mathcal{A}}$ and thus lie in $\widetilde{\mathcal{A}}$ since it is integrally closed.

Remark 2.2.37 *In the case when W is a rational reflection group (Weyl group) and $\mathcal{H}_{\mathbf{u}}(W)$ is the Iwahori-Hecke algebra with the parameters $(q, -1)$ we have $K_W = \mathbb{Q}, \mathbb{Z}[\mathbf{u}^{\pm 1}] = \mathbb{Z}[q^{\pm 1}]$. Thus the corresponding Schur elements lie in $\widetilde{\mathcal{A}} = \mathbb{Z}[\sqrt{q}^{\pm 1}]$.*

In the next chapter we will introduce a generalization of the notion of a symmetrizing trace on a Hecke algebra: special Markov traces.

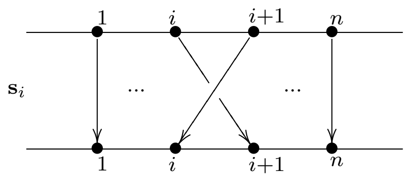
Chapter 3

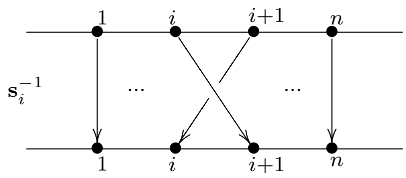
Special Markov Traces

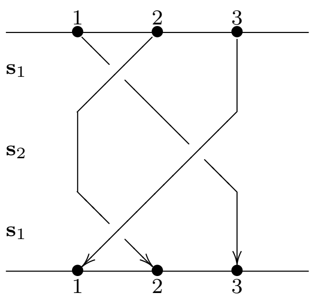
3.1 Symmetric Group Case: Links

In this introductory section we follow Geck and Pfeiffer [GP00, (4.5)]. Let n be a positive integer. An oriented *n-link* is an embedding of n copies of the interval $[0, 1] \subset \mathbb{R}$ into $\mathbb{R}^3$ such that 0 and 1 are mapped to the same point (the orientation is induced by the natural ordering of $[0, 1]$). A *1-link* is also called a *knot*. We are interested in knots and links modulo isotopy of $\mathbb{R}^3$. The problem of classifying oriented links modulo isotopy can be translated into a purely algebraic problem about braid groups. This is based on the following facts.

Let $B(S_n)$ be the braid group corresponding to the symmetric group S_n. Denote by $\mathbf{s}_i$ the generators of $B(S_n)$ corresponding to the standard generators $s_i = (i, i + 1), 1 \leq i \leq n - 1$. The generators of $B(S_n)$ can be represented by oriented diagrams as indicated below. Writing any $\mathfrak{g} \in B(S_n)$ as a product of the generators and their inverses, we also obtain a diagram for $\mathfrak{g}$ by concatenating the diagrams for the generators.

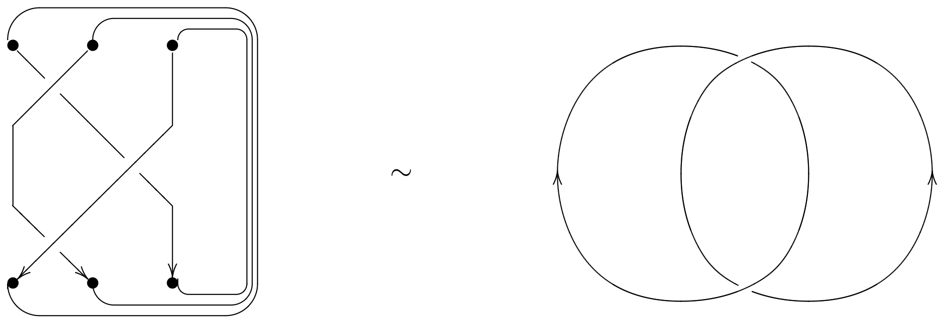






$\mathfrak{g} = \mathbf{s}_1\mathbf{s}_2\mathbf{s}_1 \in B(S_3)$

"Closing" such a diagram by joining the end points we obtain the plane projection of an oriented link in $\mathbb{R}^3$. We call it the *closure* of the corresponding element of the braid group.



closure of $\mathfrak{g}$

Theorem 3.1.1 (Alexander) *Every oriented link is isotopic in $\mathbb{R}^3$ to the closure of an element of $B(S_n)$.*

This theorem induces the question: Can the fact that two links in $\mathbb{R}^3$ are isotopic be expressed in terms of braid groups ? The answer is *positive* . For this purpose, we consider the infinite disjoint union

$$B_\infty = \coprod_{n\geq 1} B(S_n).$$

Given $\mathfrak{g}, \mathfrak{g}' \in B_\infty$, we write $\mathfrak{g} \sim \mathfrak{g}'$ if one of the following relations is satisfied:

(i) We have $\mathfrak{g}, \mathfrak{g}' \in B(S_n)$ and $\mathfrak{g}' = \mathbf{x}^{-1}\mathfrak{g}\mathbf{x}$ for some $\mathbf{x} \in B(S_n)$.

(ii) We have $\mathfrak{g} \in B(S_n), \mathfrak{g}' \in B(S_{n+1})$ and $\mathfrak{g}' = \mathfrak{g}\mathbf{s}_n$ or $\mathfrak{g}' = \mathfrak{g}\mathbf{s}_n^{-1}$.

The above relations are called the *Markov relations* due to the following theorem [Tra98]:

Theorem 3.1.2 (Markov) *Two elements of B_∞ are equivalent under the equivalence relation generated by $\sim$ if and only if the corresponding links obtained by closure are isotopic in $\mathbb{R}^3$.*

Thus to define an invariant of oriented links is the same as to define a map on B_∞ which takes equal values on elements $\mathfrak{g}, \mathfrak{g}' \in B_\infty$ satisfying (i) or (ii). In order to solve the latter problem, the key idea is to look for maps which factor through the corresponding Iwahori-Hecke algebras for suitable values of parameters.

Let $L = \mathbb{Q}(u, v)$ where u, v are variables. For any $n \geq 1$ we consider the Iwahori-Hecke algebra $\mathcal{H}_{u,v}(S_n, L)$ associated with S_n over the field L, with the parameters $q_{s_i} = u, q'_{s_i} = v$ for $1 \leq i \leq n-1$ (cf. (2.2.9)). We will regard $\mathcal{H}_{u,v}(S_n, L)$ as a parabolic Hecke subalgebra of $\mathcal{H}_{u,v}(S_{n+1}, L)$ (cf. (2.2.21)). The following theorem allows us to construct a two-variable invariant of oriented knots and links, the so-called *HOMFLY-PT polynomial*.

Theorem 3.1.3 (Jones-Ocneanu) *There is a unique family of L-linear maps $\tau_n : \mathcal{H}_{u,v}(S_n, L) \to L \quad (n \geq 1)$ such that the following conditions hold:*

(M1) $\tau_1(1) = 1$

(M2) $\tau_n(\mathbf{h}\mathbf{h}') = \tau_n(\mathbf{h}'\mathbf{h})$ *for all* $n \geq 1$ *and* $\mathbf{h}, \mathbf{h}' \in \mathcal{H}_{u,v}(S_n, L)$.

(M3) $\tau_{n+1}(\mathbf{h}\mathbf{T}_{s_n}) = \tau_{n+1}(\mathbf{h}\mathbf{T}_{s_n}^{-1}) = \tau_n(\mathbf{h})$ *for all* $n \geq 1$ *and* $\mathbf{h} \in \mathcal{H}_{u,v}(S_n, L)$.

Moreover, we have $\tau_{n+1}(\mathbf{h}) = \frac{1+uv}{u+v}\tau_n(\mathbf{h})$ *for all* $n \geq 1$ *and* $\mathbf{h} \in \mathcal{H}_{u,v}(S_n, L)$.

Remark 3.1.4 *Let $P = -uv$ and $Q = u + v$. Let $b \in B(S_n)$ and b^* its image under the surjection (2.2.19). Then Jones-Ocneanu theorem easily implies that $\tau_n(b^*) \in \mathbb{Z}[P^{\pm 1}, Q^{\pm 1}]$.*

To construct a HOMFLY-PT polynomial consider an oriented link $\mathbf{L}$. By theorem (3.1.1) it is isotopic to the closure of some $\mathfrak{g} \in B(S_n)$ for $n \geq 1$ (in fact, there is an algorithm due to Vogel [Vog90] allowing to find such $\mathfrak{g}$ using circles of Seifert). We set

$$X_{\mathbf{L}}(P, Q) := \tau_n(\mathfrak{g}^*) \in \mathbb{Z}[P^{\pm 1}, Q^{\pm 1}] \quad \text{(cf. (3.1.4))}$$

where τ_n is as in theorem (3.1.3); $\mathfrak{g}^*$ denotes the image of $\mathfrak{g}$ under the surjection $L(B(S_n)) \twoheadrightarrow \mathcal{H}_{u,v}(S_n, L), \mathbf{s}_i \mapsto \mathbf{T}_{s_i}$ (cf. (2.2.19)). The properties $(M2)$ and $(M3)$ make sure that $\tau_n(\mathfrak{g}^*)$ does not depend on the choice of $\mathfrak{g}$.

If we make the change of variables $P = t^2$ and $Q = tx$, we can identify the above invariant with the *HOMFLY-PT polynomial* $P_{\mathbf{L}}(t, x)$ in [Jon87, (6.2)]. Furthermore, the *Jones polynomial* $J_{\mathbf{L}}(t)$ (classic knot invariant) is obtained by setting $P = t^2, Q = t^{1/2}(t-1)$. Finally, setting $P = 1, Q = t^{1/2} - t^{-1/2}$ we obtain the classical *Alexander polynomial* $A_{\mathbf{L}}(t)$.

3.2 Coxeter Group Case

3.2.1 Markov Trace

Let (W, S) be a finite Coxeter system and $\mathcal{H}_S$ the corresponding Iwahori-Hecke algebra over a field K with generators $\{\mathbf{T}_s\}_{s \in S}$ and parameters $\{q_s, -1\}_{s \in S} \subset K$ (cf. (2.2.13)). For any $I \subset S$ we have a canonical embedding of the Iwahori-Hecke algebras $\mathcal{H}_I \subset \mathcal{H}_S$ (cf. (2.2.21)).

In order to reformulate Jones-Ocneanu theorem in a more convenient way for us, we need the following definition introduced by Gomi [Gom06, 3.3].

Definition 3.2.1 *Let $\tau : \mathcal{H}_S \to K$ be a K-linear function and $z_s \in K$ for $s \in S$. Then τ is called a* Markov trace *(with the parameters $\{z_s\}_{s\in S}$) if the following conditions are satisfied:*

(M1) $\tau(1) = 1$,

(M2) $\tau(ab) = \tau(ba)$ *for* $a, b \in \mathcal{H}_S$,

(M3) $\tau(a\mathbf{T}_s) = z_s\tau(a)$ *for any* $s \in S$ *and* $a \in \mathcal{H}_{S\setminus\{s\}}$.

We call the conditions above the Markov property. *If all the parameters z_s are equal to some $z \in K$ then we just say that τ is a* Markov trace with the parameter z.

3.2.2 Type A

We pass to the notations corresponding to the classification of finite Coxeter groups.

- Let $(W_{A_{n-1}}, S_{A_{n-1}})$ be a Coxeter system of type A_{n-1} (cf. (1.4.8)).
- Let q, z be variables and $K = \mathbb{Q}(q, z)$. The Iwahori-Hecke algebra of $W_{A_{n-1}} = S_n$ over K with the parameters $q_s = q, q'_s = -1$ is denoted by $\mathcal{H}_{S_{A_{n-1}}}$ (cf. (2.2.13)).
- As in (3.2.1) for any $I \subset S_{A_{n-1}}$ we denote by $\mathcal{H}_I$ the corresponding Hecke subalgebra.

We set $q = -uv^{-1}, z = -\frac{u+v}{v(1+uv)}$. Then Jones-Ocneanu theorem (3.1.3) is equivalent to the following theorem:

Theorem 3.2.2 (Jones-Ocneanu) *For any $n \geq 1$ there exists a unique Markov trace $\tau_{A_{n-1}} : \mathcal{H}_{S_{A_{n-1}}} \to K$ with the parameter z.*

Proof. $((\mathbf{3.2.2}) \Rightarrow (\mathbf{3.1.3}))$: Let $(\mathbf{T}_{s_i})_{1\leq i\leq n-1}$ be the standard generators of $\mathcal{H}_{S_{A_{n-1}}}$. Then the elements $\mathbf{T}'_{s_i} := -v\mathbf{T}_{s_i}, 1 \leq i \leq n-1$ satisfy the braid relations of type A_{n-1} (cf. (2.1.10), (1.4.8) type A_{n-1}) and

$$(\mathbf{T}'_{s_i} - u)(\mathbf{T}'_{s_i} - v) = 0 \quad \text{for} \quad 1 \leq i \leq n-1$$

and thus can be regarded as standard generators of the Iwahori-Hecke algebra $\mathcal{H}_{u,v}(S_n, L) \cong \mathcal{H}_{S_{A_{n-1}}} \otimes_K L$ with the parameters u, v, where $L = \mathbb{Q}(u, v)$.

Suppose that for any $n \geq 1$ there exists a trace $\tau_{A_{n-1}}$ on $\mathcal{H}_{S_{A_{n-1}}}$ as in (3.2.2). Then $\tau_{A_{n-1}} \otimes_K \mathrm{Id}$ is a trace on $\mathcal{H}_{u,v}(S_n, L)$. Define a family of traces

$$\tau_n := \left(\frac{1+uv}{u+v}\right)^{n-1} \tau_{A_{n-1}} \otimes_K \mathrm{Id}. \tag{3.2.3}$$

Then this family satisfies the conditions of theorem (3.1.3) with the standard basis $(\mathbf{T}'_{s_i})_{1\le i\le n-1}$. Indeed, the properties (M1) and (M2) follow immediately and we just need to check (M3):

$$\tau_{n+1}(\mathbf{h}\mathbf{T}'_{s_n}) = -v\left(\frac{1+uv}{u+v}\right)^n \tau_{A_n}(\mathbf{h}\mathbf{T}_{s_n}) = \left(\frac{1+uv}{u+v}\right)^{n-1} \tau_{A_n}(\mathbf{h}) = \tau_n(\mathbf{h})$$

$$\begin{aligned}\tau_{n+1}(\mathbf{h}\mathbf{T}'^{-1}_{s_n}) &= (u+v)u^{-1}v^{-1}\tau_{n+1}(\mathbf{h}) - u^{-1}v^{-1}\tau_{n+1}(\mathbf{h}\mathbf{T}'_{s_n})\\ &= (u+v)u^{-1}v^{-1}\left(\frac{1+uv}{u+v}\right)^n \tau_{A_n}(\mathbf{h}) - u^{-1}v^{-1}\tau_n(\mathbf{h})\\ &= (u+v)u^{-1}v^{-1}\left(\frac{1+uv}{u+v}\right)\tau_n(\mathbf{h}) - u^{-1}v^{-1}\tau_n(\mathbf{h}) = \tau_n(\mathbf{h})\end{aligned}$$

The uniqueness of such a family will follow from the second part of the proof. $((\mathbf{3.1.3}) \Rightarrow (\mathbf{3.2.2}))$: We suppose now that there exists a family of traces $\tau_n : \mathcal{H}_{u,v}(S_n, L) \to L \quad (n \ge 1)$ as in (3.1.3) with respect to the standard basis $(\mathbf{T}'_{s_i})_{1\le i\le n-1}$. We set

$$\tau_{A_{n-1}} := \left(\frac{u+v}{1+uv}\right)^{n-1} \tau_n$$

and consider its restriction on $\mathcal{H}_{S_{A_{n-1}}}$. Then $\tau_{A_{n-1}}$ is a Markov trace with the parameter z. Indeed, properties (M1) and (M2) of (3.2.1) are obvious. Uniqueness of the traces $\tau_{A_{n-1}}$ follows from the first part of the proof (cf. formula (3.2.3)) and the uniqueness of the family satisfying the conditions of theorem (3.1.3). To check property (M3) we use induction on n.

If $n=1$ then $\mathcal{H}_{S_{A_{n-1}}} = K\cdot \mathbf{1}$ and property (M3) is trivial.

Suppose we have checked property (M3) for the traces τ_{A_k} for all $k \le n-2$. We will check it for $\tau_{A_{n-1}}$. If $s = s_{n-1}$ then the equality $\tau_{A_{n-1}}(\mathbf{h}\mathbf{T}_{s_{n-1}}) = z\tau_{A_{n-1}}(\mathbf{h})$ is a consequence of

$$\begin{aligned}\tau_{A_{n-1}}(\mathbf{h}\mathbf{T}_{s_{n-1}}) &= -v^{-1}\left(\frac{u+v}{1+uv}\right)^{n-1} \tau_n(\mathbf{h}\mathbf{T}'_{s_{n-1}})\\ &= -v^{-1}\left(\frac{u+v}{1+uv}\right)^{n-1} \tau_{n-1}(\mathbf{h}) = -v^{-1}\left(\frac{u+v}{1+uv}\right)^{n} \tau_n(\mathbf{h}) = z\tau_{A_{n-1}}(\mathbf{h})\end{aligned}$$

Suppose that $s = s_i \quad (1 \le i \le n-2)$. The set of generators $\{\mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{n-1}}\} \setminus \{\mathbf{T}_{s_i}\}$ can be decomposed into the disjoint union of two mutually commuting sets $\mathbf{J}_1 = \{\mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{i-1}}\}$ and $\mathbf{J}_2 = \{\mathbf{T}_{s_{i+1}}, \ldots, \mathbf{T}_{s_{n-1}}\}$. By linearity it is enough to consider the case when $\mathbf{h} \in \mathcal{H}_{S_{A_{n-1}}\setminus\{s_i\}}$ is just a product of some elements and their inverses from $\mathbf{J}_1 \bigsqcup \mathbf{J}_2$. Hence $\mathbf{h} = \mathbf{h}_1\mathbf{h}_2$ where $\mathbf{h}_i \in \langle \mathbf{J}_i\rangle$. The subalgebra $\langle \mathbf{J}_2\rangle$ is just the Iwahori-Hecke algebra of type A_{n-i-1}. Thus its basis consists of reduced products of $\mathbf{T}_{s_j}, i < j$. Since $S_n = S_{n-1}s_{n-1}S_{n-1}$, we can write

$\mathbf{h}_2 = \mathbf{h}_3\mathbf{T}^{\varepsilon}_{s_{n-1}}\mathbf{h}_4$, where $\mathbf{h}_3, \mathbf{h}_4 \in \mathbf{J}_2 \setminus \{\mathbf{T}_{s_{n-1}}\}$ and $\varepsilon \in \{0,1\}$. If $\varepsilon = 0$ then $\mathbf{h} = \mathbf{h}_1\mathbf{h}_3\mathbf{h}_4$ and both $\mathbf{h}$ and $\mathbf{h}\mathbf{T}_{s_i}$ lie in the parabolic subalgebra $\mathcal{H}_{S_{A_{n-2}}}$ of $\mathcal{H}_{S_{A_{n-1}}}$ generated by $\{\mathbf{T}_{s_1}, \dots, \mathbf{T}_{s_{n-2}}\}$ thus we can use the induction:

$$\tau_{A_{n-1}}(\mathbf{h}\mathbf{T}_{s_i}) = \tau_{A_{n-2}}(\mathbf{h}\mathbf{T}_{s_i}) = z\tau_{A_{n-2}}(\mathbf{h}) = z\tau_{A_{n-1}}(\mathbf{h}).$$

In the case $\varepsilon = 1$ we have $\mathbf{h} = \mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4$ and the equalities

$$\begin{aligned}\tau_{A_{n-1}}(\mathbf{h}\mathbf{T}_{s_i}) &= \tau_{A_{n-1}}(\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4\mathbf{T}_{s_i}) = \tau_{A_{n-1}}(\mathbf{h}_4\mathbf{T}_{s_i}\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}})\\ &= z\tau_{A_{n-1}}(\mathbf{h}_4\mathbf{T}_{s_i}\mathbf{h}_1\mathbf{h}_3) = z\tau_{A_{n-2}}(\mathbf{h}_4\mathbf{T}_{s_i}\mathbf{h}_1\mathbf{h}_3) = z\tau_{A_{n-2}}(\mathbf{h}_1\mathbf{h}_3\mathbf{h}_4\mathbf{T}_{s_i})\\ &= z^2\tau_{A_{n-2}}(\mathbf{h}_1\mathbf{h}_3\mathbf{h}_4) = z^2\tau_{A_{n-2}}(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3) = z^2\tau_{A_{n-1}}(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3)\\ &= z\tau_{A_{n-1}}(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}) = z\tau_{A_{n-1}}(\mathbf{h}).\end{aligned}$$

make the step in the induction. □

The algebra $\mathcal{H}_{S_{A_{n-1}}}$ is split semisimple over K and its irreducible characters are in canonical bijection with those of S_n via the specialization $q \mapsto 1$ (cf. (2.2.31) and (2.2.34)).

Following e.g. Zelevinsky [Zel81, §6] we see that the irreducible characters of the symmetric group S_n are parametrized by the partitions of n: to a partition $\alpha = (0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_k)$ with $|\alpha| := \sum_{i=1}^{k} \alpha_i = n$ corresponds the unique common irreducible constituent $\{\alpha\}$ of the characters $\mathrm{Ind}_{S_\alpha}^{S_n}\mathbb{1}$ and $\mathrm{Ind}_{S_{\alpha^t}}^{S_n}\varepsilon$ where α^t is the transposed partitition of α, $S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times \ldots \times S_{\alpha_k}$ and ε is the sign character.

For an irreducible character $\{\alpha\}$ of S_n denote by $\{\alpha\}^{(q)}$ that of $\mathcal{H}_{S_{A_{n-1}}}$ via the bijection (2.2.34). Then we have

$$\tau_{A_{n-1}} = \sum_{\{\alpha\}\in \mathrm{Irr} S_n} \omega^{\{\alpha\}}\{\alpha\}^{(q)} \text{ with } \omega^{\{\alpha\}} \in K.$$

Elements $\omega^{\{\alpha\}}$ are called the *weights* of $\tau_{A_{n-1}}$. We need the following

Definition 3.2.4 *Let $\alpha = (0 \le \alpha_1 \le \ldots \le \alpha_k)$ be a partition of n.*

- *The* Young diagram *assigned to α is the set*

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i \le k, j \le \alpha_{k-i+1}\}.$$

- *We identify α with its Young diagram. For $x = (i,j) \in \alpha$ define its* hook length *$h(x)$ as the cardinality of the set $\{(i',j') \in \alpha \mid i' = i, j' \ge j$ or $i' \ge i, j' = j\}$.*

- *For $x = (i,j) \in \alpha$ define its* content *by $c(x) = j - i$.*

Ocneanu has given a formula for the weight $\omega^{\{\alpha\}}$ (cf. [Jon87, (5.4)]).

Theorem 3.2.5 *Let $\alpha = (0 \leq \alpha_1 \leq \ldots \leq \alpha_k)$ be a partition of n. Let $\tau_{A_{n-1}}$ be the Markov trace of type A_{n-1} with the parameter z. Then its weight $\omega^{\{\alpha\}}$ corresponding to the irreducible character $\{\alpha\}$ is as follows*

$$\omega^{\{\alpha\}} = q^{n(\alpha)} \prod_{x \in \alpha} \frac{1 - q + (1 - q^{c(x)})z}{1 - q^{h(x)}}$$

where $n(\alpha) = \sum_{i=1}^{k}(k-i)\alpha_i$, $h(x)$ and $c(x)$ are as in (3.2.4).

Remark 3.2.6 *The above theorem is a particular case ($e = 1$) of the theorem (3.3.5) due to Geck, Iancu and Malle.*

3.2.3 Type B

We use the following notations:

- Let Q, q, y, z be variables and $K = \mathbb{Q}(Q, q, y, z)$.
- Denote by (W_{B_n}, S_{B_n}) the Coxeter system of type B_n; with $S_{B_n} = \{t, s_1, \ldots, s_{n-1}\}$.
- For $n \geq 1$ denote by $\mathcal{H}_{S_{B_n}}^{Q,q}$ the Iwahori-Hecke algebra of type B_n over the field K with the parameters $q_t = Q, q'_t = -1, q_s = q, q'_s = -1$ for $s \in S_{B_n} \setminus \{t\}$.
- As in (3.2.1) for any $I \subset S_{B_n}$ we denote by $\mathcal{H}_I^{Q,q}$ the corresponding Hecke subalgebra.
- The standard generators of $\mathcal{H}_{S_{B_n}}^{Q,q}$ are denoted by $\mathbf{T}_s, s \in S_{B_n}$.

In [GL97, Th. 4.3] Geck and Lambropoulou state a theorem whose particular case is the following:

Theorem 3.2.7 *For any $n \geq 1$ there exists a unique trace $\tau_{B_n}^{y,z} : \mathcal{H}_{S_{B_n}}^{Q,q} \to K$ which satisfies the following properties:*

- $\tau_{B_n}^{y,z}(1) = 1$.
- $\tau_{B_n}^{y,z}(\mathbf{h}\mathbf{T}_{s_k}) = z\tau_{B_n}^{y,z}(\mathbf{h})$ *where $1 \leq k \leq n-1$ and $\mathbf{h}$ lies in the parabolic Hecke subalgebra of $\mathcal{H}_{S_{B_n}}^{Q,q}$ generated by the elements $\{\mathbf{T}_t, \mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{k-1}}\}$.*
- $\tau_{B_n}^{y,z}(\mathbf{T}'_0\mathbf{T}'_1 \ldots \mathbf{T}'_k) = y^{k+1}$ *for $0 \leq k \leq n-1$, where $\mathbf{T}'_i = \mathbf{T}_{s_i}\mathbf{T}_{s_{i-1}} \ldots \mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}^{-1} \ldots \mathbf{T}_{s_{i-1}}^{-1}\mathbf{T}_{s_i}^{-1}$ for $0 \leq i \leq n-1$.*

The above theorem implies the following:

Theorem 3.2.8 *For any $n \geq 1$ there exists a unique Markov trace $\tau_{B_n}^{y,z} : \mathcal{H}_{S_{B_n}}^{Q,q} \to K$ with the parameters*

$$z_t = y \text{ and } z_{s_1} = \ldots = z_{s_{n-1}} = z,$$

which satisfies the following additional properties:

$$\tau_{B_n}^{y,z}(\mathbf{T}_0'\mathbf{T}_1' \ldots \mathbf{T}_k') = y^{k+1} \text{ for } 0 \leq k \leq n-1,$$

where $\mathbf{T}_i' = \mathbf{T}_{s_i}\mathbf{T}_{s_{i-1}} \ldots \mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}^{-1} \ldots \mathbf{T}_{s_{i-1}}^{-1}\mathbf{T}_{s_i}^{-1}$ *for* $0 \leq i \leq n-1$.

Proof. Proof that $\tau_{B_n}^{y,z}$ in theorem (3.2.7) is a Markov trace in the sense of (3.2.1) is contained in the proof of lemma (3.3.2), properties $(M3)$ and $(M3')$ for $e = 2, \xi = z, \eta_1 = y, u_0 = Q$ and $u_1 = -1$. □

Definition 3.2.9 *The trace function $\tau_{B_n}^{y,z}$ on $\mathcal{H}_{S_{B_n}}^{Q,q}$ is called the* Markov trace of type B_n *with the parameters y, z.*

Definition 3.2.10 *The trace $\tau_{B_n} := \tau_{B_n}^{z,z}$ is called the* special Markov trace *of type B_n (with the parameter z) on the Iwahori-Hecke algebra $\mathcal{H}_{S_{B_n}}^{q,q}$ (the Iwahori-Hecke algebra of type B_n over the field $\mathbb{Q}(q, z)$ with the parameters $q_s = q, q_s' = -1$).*

We have the following

Proposition 3.2.11 *The special Markov trace τ_{B_n} can be characterized as the unique Markov trace on $\mathcal{H}_{S_{B_n}}^{q,q}$ with the parameter z satisfying the additional* twisted *Markov property:*

$$\tau_{B_n}(a\mathbf{T}_k') = z\tau_{B_n}(a) \text{ for } 1 \leq k \leq n-1 \text{ and } a \in \mathcal{H}_{S_{B_n} \setminus \{s_k\}}^{q,q},$$

where $\mathbf{T}_k' = \mathbf{T}_{s_k}\mathbf{T}_{s_{k-1}} \ldots \mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}^{-1} \ldots \mathbf{T}_{s_{k-1}}^{-1}\mathbf{T}_{s_k}^{-1}$

Proof. The existence and uniqueness of such trace τ will follow from lemma (3.3.2) (with $e = 2, \xi = \eta_1 = z, u_0 = q$ and $u_1 = -1$). Applying the twisted Markov property several times we have

$$\tau(\mathbf{T}_0'\mathbf{T}_1' \ldots \mathbf{T}_k') = z\tau(\mathbf{T}_0'\mathbf{T}_1' \ldots \mathbf{T}_{k-1}') = \ldots = z^k\tau(\mathbf{T}_0') = z^{k+1}.$$

Thus, τ satisfies the conditions of theorem (3.2.8) with $y = z, Q = q$ and hence coincides with τ_{B_n}. □

Remark 3.2.12 *When we consider $\mathcal{H}_{S_{A_{n-1}}}$ as a parabolic subalgebra of $\mathcal{H}_{S_{B_n}}^{q,q}$, the restriction of the special Markov trace τ_{B_n} to $\mathcal{H}_{S_{A_{n-1}}}$ coincides with the Markov trace $\tau_{A_{n-1}}$ (cf. 3.2.2). We call* special *the unique Markov trace $\tau_{A_{n-1}}$.*

Remark 3.2.13 *By analogy with the case A_{n-1} the trace τ_{B_n} can be decomposed into a sum of irreducible characters of $\mathcal{H}_{S_{B_n}}^{q,q}$. We do not give here the formula since it is a particular case of a general decomposition formula which will be given in (3.3.5).*

3.2.4 Type D

In this subsection we present the construction of the special Markov trace of type D_n. We follow [Gom06, 3.4], [GL97, 4.7] and use the following notations:

- Let q, y, z be variables and $K = \mathbb{Q}(q, y, z)$.
- Denote by (W_{D_n}, S_{D_n}) the Coxeter system of type D_n; with $S_{D_n} = \{s_1', s_1, \ldots, s_{n-1}\}$.
- For $n \geq 1$ denote by $\mathcal{H}_{S_{D_n}}$ the Iwahori-Hecke algebra of type D_n over the field K with the parameters $q_s = q, q_s' = -1$.
- As in (3.2.1) for any $I \subset S_{D_n}$ we denote by $\mathcal{H}_I$ the corresponding Hecke subalgebra.

The Coxeter group W_{D_n} can be regarded as the subgroup of W_{B_n} generated by the elements $\{t^{-1}s_1t, s_1, \ldots, s_{n-1}\}$. And the Iwahori-Hecke algebra $\mathcal{H}_{S_{D_n}}$ of type D_n can be regarded as a subalgebra of the specialization $\mathcal{H}^{1,q}_{S_{B_n}}$ via $Q \mapsto 1$ of the Iwahori-Hecke algebra $\mathcal{H}^{Q,q}_{S_{B_n}}$ of type B_n(cf. (3.2.3)). The generators of $\mathcal{H}_{S_{D_n}}$ are as follows

$$\{\mathbf{T}_{s_1'}, \mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{n-1}}\} \text{ where } \mathbf{T}_{s_1'} := \mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t.$$

Let $\tau^{y,z}_{B_n}$ be the Markov trace on $\mathcal{H}^{1,q}_{S_{B_n}}$ with the parameters y, z (cf. (3.2.9)). Consider the restriction of $\tau^{y,z}_{B_n}$ to $\mathcal{H}_{S_{D_n}}$. It is obvious that $\tau^{y,z}_{B_n}|_{\mathcal{H}_{S_{D_n}}}$ satisfies the conditions (M1) and (M2) of (3.2.1). If $s \in \{s_2, \ldots, s_{n-1}\}$ then the equations of (M3) are satisfied because of the Markov property of $\tau^{y,z}_{B_n}$. When $s = s_1'$ or $s = s_1$, the equations of (M3) are reduced to the equality $\tau^{y,z}_{B_n}(\mathbf{T}_{s_1'}\mathbf{T}_{s_1}) = z^2$ (here the argument is similar to that in (3.2.2), induction on n and the fact that $S_n = S_{n-1}s_{n-1}S_{n-1}$). We have

$$\tau^{y,z}_{B_n}(\mathbf{T}_{s_1'}\mathbf{T}_{s_1}) = \tau^{y,z}_{B_n}(\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}) = \tau^{y,z}_{B_n}(\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}).$$

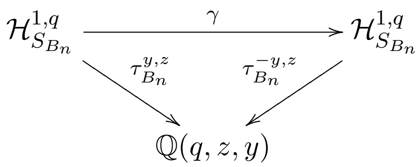
Since $\mathbf{T}_{s_1} = q\mathbf{T}_{s_1}^{-1} + (q-1)$ we get

$$\begin{aligned}\tau^{y,z}_{B_n}(\mathbf{T}_{s_1'}\mathbf{T}_{s_1}) &= q\tau^{y,z}_{B_n}(\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}^{-1}) + (q-1)\tau^{y,z}_{B_n}(\mathbf{T}_{s_1}) \\ &= q\tau^{y,z}_{B_n}(\mathbf{T}_0'\mathbf{T}_1') + (q-1)z = qy^2 + (q-1)z.\end{aligned}$$

Thus in order to get a Markov trace with the parameter z we need to specialize $y^2 \mapsto zq^{-1}(z-q+1)$.

Lemma 3.2.14 *For any $w \in W_{D_n} \leq W_{B_n}$ let $\mathbf{T}_w$ be the corresponding element of the standard K-basis of $\mathcal{H}^{1,q}_{S_{B_n}}$ (cf. (2.2.12)). Then $\tau^{y,z}_{B_n}(\mathbf{T}_w) \in \mathbb{Q}(q, z, y^2)$,*

Proof. Consider the following diagram:



where γ is the automorphism of the Iwahori-Hecke algebra $\mathcal{H}_{S_{B_n}}$ sending $\mathbf{T}_t$ to $-\mathbf{T}_t$ and $\mathbf{T}_{s_i}$ to $\mathbf{T}_{s_i}$. Theorem (3.2.8) implies that the above diagram is commutative. If we consider the element $\mathbf{T}_w$ as a word in generators $\mathbf{T}_t, \mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{n-1}}$ then the generator $\mathbf{T}_t$ appears an even number of times in it and hence γ sends $\mathbf{T}_w$ to itself. Thus $\tau^{y,z}_{B_n}(\mathbf{T}_w) = \tau^{-y,z}_{B_n}(\mathbf{T}_w)$ and $\tau^{y,z}_{B_n}(\mathbf{T}_w) \in \mathbb{Q}(q,z,y^2)$. □
The above lemma implies that we do not need to specify the specialization of y, just that of y^2.

Definition 3.2.15 *We define the* special Markov trace $\tau_{D_n} : \mathcal{H}_{S_{D_n}} \to \mathbb{Q}(q,z)$ of type D_n *as the specialization of* $\tau^{y,z}_{B_n}|_{\mathcal{H}_{S_{D_n}}}$ *via* $y^2 \mapsto zq^{-1}(z-q+1)$.

Remark 3.2.16 *By analogy with the cases A_{n-1} and B_n the trace τ_{D_n} can be decomposed into a sum of irreducible characters of $\mathcal{H}_{S_{D_n}}$. We do not give here the formula since it is a particular case of a general decomposition formula which will be given in the case $G(e,e,n)$.*

3.2.5 Type $\mathbf{I_2(m)}$

Kihara has shown that the notion of special Markov trace (cf. (3.2.10),(3.2.15)) can be extended to the dihedral case. We use the following notations:

- Let q, z be variables and $K = \mathbb{Q}(q,z)$.
- Denote by $(I_2(m), \{s_0, s_1\})$ the Coxeter system of type $I_2(m)$.
- For $m \geq 1$ denote by $\mathcal{H}_m$ the Iwahori-Hecke algebra of type $I_2(m)$ over the field K with the parameters $q_{s_0} = q_{s_1} = q, q'_{s_0} = q'_{s_1} = -1$.
- The standard generators of $\mathcal{H}_m$ are denoted by $\mathbf{T}_0$ and $\mathbf{T}_1$.

In [Kih04] Kihara proves the following:

Theorem 3.2.17 *There exists a unique trace function τ_m on $\mathcal{H}_m$ which satisfies the following equations:*

(1) $\tau_m(1) = 1$,

(2) $\tau_m(\mathbf{T}_0) = \tau_m(\mathbf{T}_1) = z$,

(3) $\tau_m(\underbrace{\mathbf{T}_0\mathbf{T}_1\dots}_{i+1}\underbrace{\dots\mathbf{T}_0^{-1}\mathbf{T}_1^{-1}}_{i-1}) = z^2$ *for* $1 \le i \le \left[\frac{m-1}{2}\right]$.

The above theorem can be reformulated in the following way:

Theorem 3.2.18 *There exists a unique Markov trace* $\tau_m : \mathcal{H}_m \to K$ *with the parameter* z *satisfying the additional* twisted *Markov property:*

$$\tau_m(a\mathbf{T}_i') = z\tau_m(a) \text{ for } 1 \le i \le \left[\frac{m-1}{2}\right] \text{ and } a \in \mathcal{H}_{\{s_0\}},$$

where $\mathbf{T}_i' = \mathbf{T}_1\mathbf{T}_0\dots\mathbf{T}_{i-1}\mathbf{T}_i\mathbf{T}_{i+1}^{-1}\dots\mathbf{T}_0^{-1}\mathbf{T}_1^{-1}$ *and all indices are taken modulo* 2.

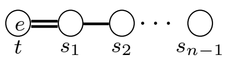
Proof. Proof that this theorem is equivalent to theorem (3.2.17) follows from the simple fact that $\mathcal{H}_{\{s_0\}} = K \oplus K \cdot \mathbf{T}_0$ as a K-vector space. □

Definition 3.2.19 *The trace function* τ_m *on* $\mathcal{H}_m$ *is called the* special Markov trace of type $I_2(m)$.

Remark 3.2.20 *By analogy with the cases* A_{n-1}, B_n *and* D_n *the trace* τ_m *can be decomposed into a sum of irreducible characters of* $\mathcal{H}_m$*. We do not give here the formula since it is a particular case of a general decomposition formula which will be given in the case* $G(e,e,n)$.

3.3 Geck-Iancu-Malle Definition in the Case $\mathbf{G(e,1,n)}$

Here we present the generalization of the notion of Markov trace in the case of $W_n^{(e)} := G(e,1,n)$ (cf. (1.3)). It was constructed by Lambropoulou in [Lam99, §4] and its weight formula has been given by Geck, Iancu and Malle in [GIM00, Th. 1.3]. Recall, that the diagram of $W_n^{(e)}$ is



(cf. (1.5)).

Thus $W_n^{(e)}$ has a presentation with the set of generators $S_n^{(e)} = \{t, s_1, s_2, \dots, s_{n-1}\}$ and the following defining relations:

(1) $t^e = 1$,

(2) $s_i^2 = 1$ for $1 \le i \le n-1$,

(3) $ts_1ts_1 = s_1ts_1t$,

(4) $ts_i = s_it$ for $2 \le i \le n-1$,

(5) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n-2$,

(6) $s_i s_j = s_j s_i$ if $|i-j| \geq 2$.

The corresponding generic Hecke algebra $\mathcal{H}_n^{(e)}(q, -1, u_0, u_1, \ldots, u_{e-1})$ (or symply $\mathcal{H}_n^{(e)}$) is an algebra over the ring $\mathcal{A} := \mathbb{Z}[q^{\pm 1}, u_0, u_1, \ldots, u_{e-1}]$ where $q, u_0, u_1, \ldots, u_{e-1}$ are indeterminates (cf. (2.2.17)). By proposition (2.2.20) the algebra $\mathcal{H}_n^{(e)}$ has a presentation with generators $\mathbf{T}_t, \mathbf{T}_{s_1}, \mathbf{T}_{s_2}, \ldots, \mathbf{T}_{s_{n-1}}$ and the defining relations:

(1) $(\mathbf{T}_t - u_0)(\mathbf{T}_t - u_1) \ldots (\mathbf{T}_t - u_{e-1}) = 0$,

(2) $\mathbf{T}_{s_i}^2 = q\mathbf{1} + (q-1)\mathbf{T}_{s_i}$ for $1 \leq i \leq n-1$,

(3) $\mathbf{T}_t \mathbf{T}_{s_1} \mathbf{T}_t \mathbf{T}_{s_1} = \mathbf{T}_{s_1} \mathbf{T}_t \mathbf{T}_{s_1} \mathbf{T}_t$,

(4) $\mathbf{T}_t \mathbf{T}_{s_i} = \mathbf{T}_{s_i} \mathbf{T}_t$ for $2 \leq i \leq n-1$,

(5) $\mathbf{T}_{s_i} \mathbf{T}_{s_{i+1}} \mathbf{T}_{s_i} = \mathbf{T}_{s_{i+1}} \mathbf{T}_{s_i} \mathbf{T}_{s_{i+1}}$ for $1 \leq i \leq n-2$,

(6) $\mathbf{T}_{s_i} \mathbf{T}_{s_j} = \mathbf{T}_{s_j} \mathbf{T}_{s_i}$ if $|i-j| \geq 2$.

where $\mathbf{1}$ denotes the identity element of $\mathcal{H}_n^{(e)}$.

Let $K = \mathbb{Q}(q, u_0, u_1, \ldots, u_{e-1})$ be the field of fractions of $\mathcal{A}$. We will use the same notation $\mathcal{H}_n^{(e)}$ for the algebra obtained by extending scalars from $\mathcal{A}$ to K. One can see that Jones-Ocneanu and Geck-Lambropoulou constructions of Markov traces of type A_n and B_n were defined inductively, using the chain of subgroups $W_0^{(e)} \subset W_1^{(e)} \subset \ldots \subset W_n^{(e)}$ ($e \in \{1, 2\}$) and the corresponding chain of subalgebras $\mathcal{H}_0^{(e)} \subset \mathcal{H}_1^{(e)} \subset \cdots \subset \mathcal{H}_n^{(e)}$ ($e \in \{1, 2\}$) , where $W_m^{(e)}$ (respectively $\mathcal{H}_m^{(e)}$) is generated by $t, s_1, s_2, \ldots, s_{m-1}$ (respectively by $\mathbf{T}_t, \mathbf{T}_{s_1}, \mathbf{T}_{s_2}, \ldots, \mathbf{T}_{s_{m-1}}$) for $1 \leq m \leq n$. Here we set $W_0^{(e)} = \{1\}$ and $\mathcal{H}_0^{(e)} = \{K \cdot \mathbf{1}\}$ by convention. Thus it was quite natural to give a similar definition with "inductive" properties:

Definition 3.3.1 *Let $\xi, \eta_1, \ldots, \eta_{e-1} \in K$ and $\tau : \mathcal{H}_n^{(e)} \to K$ be a K-linear trace function. We say that τ is a* Markov trace *with parameters $\xi, \eta_1, \ldots, \eta_{e-1}$ if*

(1) $\tau(\mathbf{1}) = 1$,

(2) $\tau(\mathbf{h}\mathbf{T}_{s_m}) = \xi \tau(\mathbf{h})$ *for* $1 \leq m \leq n-1$ *and* $\mathbf{h} \in \mathcal{H}_m^{(e)}$,

(3) $\tau(\mathbf{h}(\mathbf{T}'_m)^k) = \eta_k \tau(\mathbf{h})$ *for* $0 \leq m \leq n-1$, $1 \leq k \leq e-1$ *and* $\mathbf{h} \in \mathcal{H}_m^{(e)}$,

where $\mathbf{T}'_m = \mathbf{T}_{s_m} \mathbf{T}_{s_{m-1}} \cdots \mathbf{T}_{s_1} \mathbf{T}_t \mathbf{T}_{s_1}^{-1} \cdots \mathbf{T}_{s_{m-1}}^{-1} \mathbf{T}_{s_m}$.

By [Lam99, Th. 6], given $\xi, \eta_1, \dots, \eta_{e-1} \in K$, a Markov trace exists and is uniquely determined by the above conditions.

To show that the above definition is a natural generalization of the Markov traces of type A_{n-1} (cf. (3.2.2)) and B_n (cf. (3.2.9)) we prove the following simple lemma:

Lemma 3.3.2 (Markov trace via Gomi) *The trace $\tau : \mathcal{H}_n^{(e)} \to K$ in the definition (3.3.1) can be characterized as the unique K-linear function on $\mathcal{H}_n^{(e)}$ with the following properties:*

(M1) $\tau(1) = 1$,

(M2) $\tau(\mathbf{h}\mathbf{h}') = \tau(\mathbf{h}'\mathbf{h})$ *for* $\mathbf{h}, \mathbf{h}' \in \mathcal{H}_n^{(e)}$,

(M3) $\tau(\mathbf{h}\mathbf{T}_s) = \xi\tau(\mathbf{h})$ *for* $s \in \{s_1, \dots, s_{n-1}\}$ *and* $\mathbf{h} \in \mathcal{H}_{S_n^{(e)} \setminus \{s\}}$, *the Hecke subalgebra of* $\mathcal{H}_n^{(e)}$ *generated by all* $\mathbf{T}$*'s except* $\mathbf{T}_s$ *(cf. (3.2.1)).*

(M3') $\tau(\mathbf{h}\mathbf{T}_t^k) = \eta_k\tau(\mathbf{h})$ *for* $1 \le k \le e-1$ *and* $\mathbf{h} \in \mathcal{H}_{S_n^{(e)} \setminus \{t\}}$.

And the additional property:

(M4) $\tau(\mathbf{h}(\mathbf{T}'_m)^k) = \eta_k\tau(\mathbf{h})$ *for* $1 \le m \le n-1$, $1 \le k \le e-1$ *and* $\mathbf{h} \in \mathcal{H}_{S_n^{(e)} \setminus \{s_m\}}$

Proof. In order to prove the lemma we just need to prove the following statement: The properties $(1), (2)$ and (3) of (3.3.1) imply the properties $(M3), (M3')$ and $(M4)$.

Property (M3). We use induction on n. If $n = 1$ then $\mathcal{H}_n^{(e)} = K[\mathbf{T}_t]$ and property (M3) is trivial. Suppose we have proved our statement for all $k \le n-1$. We will now prove it for n. If $s = s_{n-1}$ then the equality $\tau(\mathbf{h}\mathbf{T}_s) = \xi\tau(\mathbf{h})$ holds by (2). Suppose that $s = s_i \quad (1 \le i \le n-2)$. The set of generators $\{\mathbf{T}_t, \mathbf{T}_{s_1}, \dots, \mathbf{T}_{s_{n-1}}\} \setminus \{\mathbf{T}_{s_i}\}$ can be decomposed into the disjoint union of two mutually commuting sets $\mathbf{J}_1 = \{\mathbf{T}_t, \mathbf{T}_{s_1}, \dots, \mathbf{T}_{s_{i-1}}\}$ and $\mathbf{J}_2 = \{\mathbf{T}_{s_{i+1}}, \dots, \mathbf{T}_{s_{n-1}}\}$. By linearity we can suppose that $\mathbf{h} \in \mathcal{H}_{S_n^{(e)} \setminus \{s_i\}}$ is just a product of some elements and their inverses from $\mathbf{J}_1 \bigsqcup \mathbf{J}_2$. Hence $\mathbf{h} = \mathbf{h}_1\mathbf{h}_2$ where $\mathbf{h}_i \in \langle \mathbf{J}_i \rangle$. The subalgebra $\langle \mathbf{J}_2 \rangle$ is just the Iwahori-Hecke algebra of type A_{n-i-1}. Thus its basis consists of reduced products of $\mathbf{T}_{s_j}, i < j$. And without loss of generality we can write $\mathbf{h}_2 = \mathbf{h}_3\mathbf{T}_{s_{n-1}}^{\varepsilon}\mathbf{h}_4$, where $\mathbf{h}_3, \mathbf{h}_4 \in \langle \mathbf{J}_2 \setminus \{\mathbf{T}_{s_{n-1}}\} \rangle$ and $\varepsilon \in \{0, 1\}$. If $\varepsilon = 0$ then $\mathbf{h} = \mathbf{h}_1\mathbf{h}_3\mathbf{h}_4$ and both $\mathbf{h}$ and $\mathbf{h}\mathbf{T}_{s_i}$ lie in the parabolic subalgebra $\mathcal{H}_{n-1}^{(e)}$ of $\mathcal{H}_n^{(e)}$ generated by $\{\mathbf{T}_t, \mathbf{T}_{s_1}, \dots, \mathbf{T}_{s_{n-2}}\}$ thus we can use the induction. In the case $\varepsilon = 1$ we have $\mathbf{h} = \mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4$ and the equalities

$$\tau(\mathbf{h}\mathbf{T}_{s_i}) = \tau(\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4\mathbf{T}_{s_i}) = \tau(\mathbf{h}_4\mathbf{T}_{s_i}\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}) = \xi\tau(\mathbf{h}_4\mathbf{T}_{s_i}\mathbf{h}_1\mathbf{h}_3) =$$
$$\xi\tau(\mathbf{h}_1\mathbf{h}_3\mathbf{h}_4\mathbf{T}_{s_i}) = \xi^2\tau(\mathbf{h}_1\mathbf{h}_3\mathbf{h}_4) = \xi^2\tau(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3) = \xi\tau(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}) = \xi\tau(\mathbf{h}).$$

make the step in induction.

Property (M3'). The proof is very similar to that in the case of (M3). We use induction on n. If $n = 1$ then $\mathcal{H}_n^{(e)} = K[\mathbf{T}_t]$ and we just have to verify the equality $\tau(\mathbf{T}_t^k) = \eta_k$ which is a consequence of property (3) in (3.3.1) for $m = 0$. Suppose we have checked the property (M3') for $n - 1$. We will check it for n. Since $\mathbf{h}$ is an element of $\mathcal{H}_{S_n^{(e)} \setminus \{t\}}$ which is just the Iwahori-Hecke algebra of type A_{n-1} we can assume that $\mathbf{h} = \mathbf{h}_1 \mathbf{T}_{s_{n-1}}^{\varepsilon} \mathbf{h}_2$, where $\mathbf{h}_1, \mathbf{h}_2$ lie in the parabolic subalgebra of $\mathcal{H}_{S_n^{(e)}}$ generated by the elements $\{\mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{n-2}}\}$ and $\varepsilon \in \{0, 1\}$. If $\varepsilon = 0$ then $\mathbf{h} = \mathbf{h}_1\mathbf{h}_2$ and both $\mathbf{h}$ and $\mathbf{h}\mathbf{T}_t^k$ lie in the parabolic subalgebra $\mathcal{H}_{n-1}^{(e)}$ of $\mathcal{H}_n^{(e)}$ generated by $\{\mathbf{T}_t, \mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{n-2}}\}$ thus we can use the induction. In the case $\varepsilon = 1$ we have $\mathbf{h} = \mathbf{h}_1 \mathbf{T}_{s_{n-1}} \mathbf{h}_2$ and the equalities

$$\begin{aligned} \tau(\mathbf{h}\mathbf{T}_t^k) = \tau(\mathbf{h}_1\mathbf{T}_{s_{n-1}}\mathbf{h}_2\mathbf{T}_t^k) = \tau(\mathbf{h}_2\mathbf{T}_t^k\mathbf{h}_1\mathbf{T}_{s_{n-1}}) = \\ \xi\tau(\mathbf{h}_2\mathbf{T}_t^k\mathbf{h}_1) = \xi\tau(\mathbf{h}_1\mathbf{h}_2\mathbf{T}_t^k) \overset{\text{induction}}{=} \xi\eta_k\tau(\mathbf{h}_1\mathbf{h}_2) = \\ \xi\eta_k\tau(\mathbf{h}_2\mathbf{h}_1) = \eta_k\tau(\mathbf{h}_2\mathbf{h}_1\mathbf{T}_{s_{n-1}}) = \eta_k\tau(\mathbf{h}). \end{aligned}$$

make the step in induction.

Property (M4). The proof is very similar to that in the cases of (M3) and (M3'). We use induction on n again. If $n = m + 1$ then property (M4) follows from property (3) of (3.3.1). Suppose we have proved our statement for $n - 1$. We will now prove it for n. Since $\mathbf{h} \in \mathcal{H}_{S_n^{(e)} \setminus \{s_m\}}$ and the sets $\mathbf{J}_1 = \{\mathbf{T}_t, \mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{m-1}}\}$ and $\mathbf{J}_2 = \{\mathbf{T}_{s_{m+1}}, \ldots, \mathbf{T}_{s_{n-1}}\}$ mutually commute we can suppose that $\mathbf{h} = \mathbf{h}_1\mathbf{h}_2$ where $\mathbf{h}_i \in \langle \mathbf{J}_i \rangle$. The subalgebra $\langle \mathbf{J}_2 \rangle$ is just the Iwahori-Hecke algebra of type A_{n-m-1}. Thus its basis consists of reduced products of $\mathbf{T}_{s_j}, m < j$. And without loss of generality we can write $\mathbf{h}_2 = \mathbf{h}_3\mathbf{T}_{s_{n-1}}^{\varepsilon}\mathbf{h}_4$, where $\mathbf{h}_3, \mathbf{h}_4 \in \langle \mathbf{J}_2 \setminus \{\mathbf{T}_{s_{n-1}}\} \rangle$ and $\varepsilon \in \{0, 1\}$. If $\varepsilon = 0$ then $\mathbf{h} = \mathbf{h}_1\mathbf{h}_3\mathbf{h}_4$ and both $\mathbf{h}$ and $\mathbf{h}(\mathbf{T}'_m)^k$ lie in the parabolic subalgebra $\mathcal{H}_{n-1}^{(e)}$ of $\mathcal{H}_n^{(e)}$ generated by $\{\mathbf{T}_t, \mathbf{T}_{s_1}, \ldots, \mathbf{T}_{s_{n-2}}\}$ thus we can use the induction. In the case $\varepsilon = 1$ we have $\mathbf{h} = \mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4$ and the equalities

$$\begin{aligned} \tau(\mathbf{h}(\mathbf{T}'_m)^k) &= \tau(\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4(\mathbf{T}'_m)^k) \\ &= \tau(\mathbf{h}_4(\mathbf{T}'_m)^k\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}) \overset{\text{(M3)}}{=} \xi\tau(\mathbf{h}_4(\mathbf{T}'_m)^k\mathbf{h}_1\mathbf{h}_3) \\ &= \xi\tau(\mathbf{h}_1\mathbf{h}_3\mathbf{h}_4(\mathbf{T}'_m)^k) \overset{\text{induction}}{=} \xi\eta_k\tau(\mathbf{h}_1\mathbf{h}_3\mathbf{h}_4) = \xi\eta_k\tau(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3) \\ &= \eta_k\tau(\mathbf{h}_4\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}) = \eta_k\tau(\mathbf{h}_1\mathbf{h}_3\mathbf{T}_{s_{n-1}}\mathbf{h}_4) = \eta_k\tau(\mathbf{h}). \end{aligned}$$

make the step in induction. □

By (2.2.31) the algebra $\mathcal{H}_n^{(e)}$ is split semisimple over K and its irreducible characters are in bijection with those of $W_n^{(e)}$.

The group $W_n^{(e)}$ is obviously isomorphic to the wreath product $C_e \wr S_n$ where C_e is the cyclic group of order e. Thus the classification of representations of wreath products by S_n can be applied (cf. [Zel81, §7]).

Definition 3.3.3 *Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$ be a e-tuple of partititions $\alpha_i = (0 \le \alpha_{i,1} \le \alpha_{i,2} \le \ldots \le \alpha_{i,m_i})$ such that $\sum_{i=0}^{e-1} |\alpha_i| = n$. Then $\boldsymbol{\alpha}$ is called* e-partition *of n.*

Proposition 3.3.4 *The set of irreducible characters of $W_n^{(e)}$ is parametrized by e-partitions of n as follows:*

$$\boldsymbol{\alpha} \leftrightarrow \chi_{\boldsymbol{\alpha}} := \mathrm{Ind}_{W_{\boldsymbol{\alpha}}^{(e)}}^{W_n^{(e)}} \left((\{\alpha_0\} \otimes \zeta_0) \boxtimes (\{\alpha_1\} \otimes \zeta_1) \boxtimes \ldots \boxtimes (\{\alpha_{e-1}\} \otimes \zeta_{e-1}) \right).$$

Here $W_{\boldsymbol{\alpha}}^{(e)} = W_{|\alpha_0|}^{(e)} \times \ldots \times W_{|\alpha_{e-1}|}^{(e)}$; characters $\{\alpha_i\}$ are regarded as characters of $W_{|\alpha_i|}^{(e)}$ provided by surjections $W_{|\alpha_i|}^{(e)} \twoheadrightarrow S_{|\alpha_i|}$; linear characters ζ_i of $W_{|\alpha_i|}^{(e)}$ are given as follows:

$$\zeta_i(S_{|\alpha_i|}) = 1 \text{ and } \zeta_i \text{ sends the generator of each } C_e \text{ to } \zeta^i,$$

where ζ is an e-th primitive root of 1. The operations $\otimes$ and $\boxtimes$ are inner and external tensor products of characters.

Denote by $\Lambda_n^{(e)}$ the set of e-partitions of n. Thus we can write:

$$\mathrm{Irr}(\mathcal{H}_n^{(e)}) = \left\{ \chi_{\boldsymbol{\alpha}}{}^{(q)} | \boldsymbol{\alpha} \in \Lambda_n^{(e)} \right\} \qquad \text{(cf. (2.2.33)).}$$

Since $\mathcal{H}_n^{(e)}$ is a symmetric algebra there exist unique elements $\omega_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1}) \in K$ such that

$$\tau = \sum_{\boldsymbol{\alpha} \in \Lambda_n^{(e)}} \omega_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1}) \chi_{\boldsymbol{\alpha}}{}^{(q)}.$$

They are called the *weights* of τ.

In [GIM00, 1.2 and 1.3] the authors have given an explicit combinatorial formulae for the weights of the Markov trace on $\mathcal{H}_n^{(e)}$, proving at the same time the existence of a Markov trace for any given choice of $\xi, \eta_1, \ldots, \eta_{e-1} \in K$. To state their main result we need to introduce some notations.

For any e-tuple $\underline{S} = (S_0, S_1, \ldots, S_{e-1})$ of finite subsets $S_k \subset \mathbb{Z}_{\ge 0}$ with $|S_0| = m+1$, $|S_1| = \ldots = |S_{e-1}| = m$ define

$$D_{\underline{S}}(q; u_0, \ldots, u_{e-1})$$

$$:= \frac{(-1)^{\binom{e}{2}\binom{m}{2}+ne} \cdot \prod\limits_{0 \le k \le l \le e-1} \prod\limits_{\substack{(\lambda,\lambda') \in S_k \times S_l \\ \lambda' < \lambda \text{ if } k=l}} (q^{\lambda} u_k - q^{\lambda'} u_l) \cdot \prod\limits_{k=0}^{e-1} u_k^n}{q^{f(m,e)} \cdot \prod\limits_{k,l=0}^{e-1} \prod\limits_{\substack{\lambda \in S_k \\ \lambda \ge 1}} \prod\limits_{h=1}^{\lambda} (q^h u_k - u_l) \cdot \prod\limits_{0 \le k < l \le e-1} (u_k - u_l)^m},$$

where $f(m,e) = \binom{e(m-1)+1}{2} + \binom{e(m-2)+1}{2} + \ldots$.

As in [Mal95, 2.13], to any e-partition $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$ of n we associate an e-tuple $S_{\boldsymbol{\alpha}} = (S_0, S_1, \ldots, S_{e-1})$ of finite subsets $S_k \subset \mathbb{Z}_{\geq 0}$ with $|S_0| - 1 = |S_1| = \ldots = |S_{e-1}|$, as follows. Writing $\alpha_0 = (0 \leq \alpha_{0,1} \leq \ldots \leq \alpha_{0,m+1})$ and $\alpha_k = (0 \leq \alpha_{k,1} \leq \cdots \leq \alpha_{k,m})$ for $1 \leq k \leq e-1$ for some large enough m, we set

$$\begin{aligned} S_0 &:= \{\alpha_{0,i} + i - 1 \mid 1 \leq i \leq m+1\}, \\ S_k &:= \{\alpha_{k,i} + i - 1 \mid 1 \leq i \leq m\} \quad (1 \leq k \leq e-1). \end{aligned}$$

Finally we define $D_{\boldsymbol{\alpha}} := D_{S_{\boldsymbol{\alpha}}}(q; u_0 q^{-1}, u_1, \ldots, u_{e-1}) \in K$. As pointed out in the remark following [Mal95, 2.19], $D_{\boldsymbol{\alpha}}$ does not depend on the choice of m and thus is well-defined.

Theorem 3.3.5 (Geck, Iancu, Malle) *Let $\xi, \eta_1, \ldots, \eta_{e-1} \in K$. For any $\boldsymbol{\alpha} \in \Lambda_n^{(e)}$ we define*

$$R_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1}) = \prod_{k=0}^{e-1} \prod_{x \in \alpha_k} \Bigg((-1)^{e-1} \prod_{\substack{l=0 \\ l \neq k}} u_l^{-1} \Big[\xi (1 - q^{c(x)}) \prod_{\substack{l=0 \\ l \neq k}}^{e-1} (u_k q^{c(x)} - u_l) + \\ + (1-q) \Big(\sum_{i=1}^{e-1} (u_k^{e-i-1} q^{(e-i)c(x)} \sum_{j=1}^{i} (-1)^{i-j} \eta_j \sigma_{i-j}) + (-1)^{e-1} \prod_{\substack{l=0 \\ l \neq k}}^{e-1} u_l \Big) \Big] \Bigg), \tag{3.3.6}$$

where σ_i denotes the i-th elementary symmetric function in $u_0, \ldots, u_{e-1}$ (by convention $\sigma_0 = 1$) and $c(x)$ denotes the content of $x \in \alpha_k$. Then

$$\tau_{\xi, \eta_1, \ldots, \eta_{e-1}} := \sum_{\boldsymbol{\alpha} \in \Lambda_n^{(e)}} D_{\boldsymbol{\alpha}} R_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1}) \chi_{\boldsymbol{\alpha}}^{(q)} \tag{3.3.7}$$

is a Markov trace on $\mathcal{H}_n^{(e)}$ with parameters $\xi, \eta_1, \ldots, \eta_{e-1}$.

Remark 3.3.8 *If we pose $e = 1, \xi = z, u_0 = 1$ then $\mathbf{T}_t = 1$ and η's will dissappear. Lemma (3.3.2) implies that the Markov trace τ is the Markov trace $\tau_{A_{n-1}}$ (cf. (3.2.2)). And theorem (3.3.5) implies Ocneanu's formula (3.2.5).*

Remark 3.3.9 *If we pose $e = 2, \xi = z, \eta_1 = y, u_0 = Q, u_1 = -1$ then lemma (3.3.2) implies that the Markov trace τ is the Markov trace $\tau_{B_n}^{y,z}$ (cf. (3.2.2)). And theorem (3.3.5) provides a formula for its weights.*

The special Markov traces $\tau_{A_{n-1}}$ and τ_{B_n} can be considered as particular cases of the following trace:

Definition 3.3.10 *Consider the Markov trace with parameters $\xi = z, \eta_i = zq^{i-1} (1 \leq i \leq e)$:*

$$\tau_e := \tau_{z,z,zq,\ldots,zq^{e-1}} : \mathcal{H} \to \mathbb{C}(q,z)$$

where $\mathcal{H}$ is the spetsial Hecke algebra obtained from the generic Hecke algebra $\mathcal{H}_n^{(e)}(q,-1,u_0,u_1,\ldots,u_{e-1})$ via the specialization

$$u_0 \mapsto q, \quad u_i \mapsto \zeta^i \quad (1 \le i \le e-1). \tag{3.3.11}$$

The trace τ_e is called the special *Markov trace of the group $W_n^{(e)}$.*

Lemma (3.3.2) immediately implies the following

Corollary 3.3.12 *The special Markov trace $\tau_e : \mathcal{H} \to \mathbb{C}(q,z)$ can be characterized as the unique trace function on $\mathcal{H}$ with the following properties:*

- $\tau_e(1) = 1$,
- $\tau_e(\mathbf{h}\mathbf{T}_s) = z\tau_e(\mathbf{h})$ *for $s \in \{s_1,\ldots,s_{n-1}\}$ and $\mathbf{h} \in \mathcal{H}_{S_n^{(e)}\setminus\{s\}}$, the Hecke subalgebra of $\mathcal{H}$ generated by all $\mathbf{T}$'s except $\mathbf{T}_s$ (cf. (3.2.1)).*
- $\tau_e(\mathbf{h}\mathbf{T}_t^k) = zq^{k-1}\tau_e(\mathbf{h})$ *for $1 \le k \le e-1$ and $\mathbf{h} \in \mathcal{H}_{S_n^{(e)}\setminus\{t\}}$.*

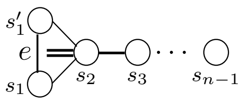
And the additional property:

- $\tau_e(\mathbf{h}(\mathbf{T}'_m)^k) = zq^{k-1}\tau_e(\mathbf{h})$ *for $1 \le m \le n-1$, $1 \le k \le e-1$ and $\mathbf{h} \in \mathcal{H}_{S_n^{(e)}\setminus\{s_m\}}$.*

3.4 Case $\mathbf{G(e,e,n)}$

3.4.1 Special Markov Trace for the Group $\mathbf{G(e,e,n)}$.

In this subsection we present a construction of the special Markov trace of the group $\widetilde{W}_n^{(e)} := G(e,e,n)$ (cf. (1.3)). It is a direct generalization of the construction done by Gomi in the case of D_n (cf. (3.2.4)). The group $\widetilde{W}_n^{(e)}$ is generated by the set $\widetilde{S}_n^{(e)} := \{s'_1, s_1, s_2, \ldots, s_{n-1}\}$ and has the following diagram:



(cf. (1.5)).

We will identify it with a subgroup of $W_n^{(e)}$ via

$$s'_1 \mapsto t^{-1}s_1t, \; s_i \mapsto s_i \text{ for } 1 \le i \le n-1 \text{ (cf. (1.5))}.$$

This is a normal subgroup of index e of $W_n^{(e)}$ since it is the kernel of the linear character $\chi_{(\varnothing,(n),\varnothing,\ldots,\varnothing)}$ (cf. (3.3.4)).

The corresponding Hecke algebra $\widetilde{\mathcal{H}}_n^{(e)}$ over the field $\mathbb{C}(q)$ can be regarded as a subalgebra of the Hecke algebra $\mathcal{H}_n^{(e)}(q,\zeta) :=$

$\mathcal{H}_n^{(e)}(q,-1,1,\zeta,\ldots,\zeta^{e-1})\otimes_{\mathbb{Q}(\zeta)}\mathbb{C}$ where $\mathcal{H}_n^{(e)}(q,-1,1,\zeta,\ldots,\zeta^{e-1})$ is the specialization via

$$u_j \mapsto \zeta^j, \quad 0 \le j \le e-1, \zeta = exp(2\pi i/e)$$

of the Hecke algebra $\mathcal{H}_n^{(e)}(q,-1,u_0,u_1,\ldots,u_{e-1})$ of the group $W_n^{(e)}$ (cf. (3.3)). The generators of $\widetilde{\mathcal{H}}_n^{(e)}$ are as follows

$$\{\mathbf{T}_{s_1'},\mathbf{T}_{s_1},\ldots,\mathbf{T}_{s_{n-1}}\} \text{ where } \mathbf{T}_{s_1'} := \mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t.$$

Let z be a variable. By theorem (3.3.5) there exists the Markov trace $\overline{\tau}$ in the sense of (3.3.1) with the parameters

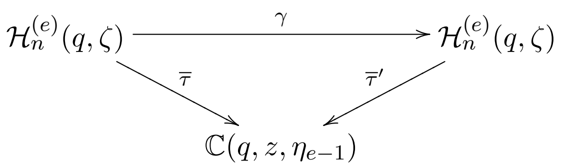
$$\xi = z, \eta_{e-1} = (zq^{-1}(z-q+1)^{e-1})^{\frac{1}{e}}, \eta_i = \eta_{e-1}^{e-i}(z-q+1)^{i+1-e} \qquad (3.4.1)$$

on the Hecke algebra $\mathcal{H}_n^{(e)}(q,\zeta)$. Denote by $\widetilde{\tau}_e$ the restriction of $\overline{\tau}$ from $\mathcal{H}_n^{(e)}(q,\zeta)$ to $\widetilde{\mathcal{H}}_n^{(e)}$.

Lemma 3.4.2 *We have*

$$\widetilde{\tau}_e(\widetilde{\mathcal{H}}_n^{(e)}) \subset \mathbb{C}(q,z).$$

Proof. Here we use the same argument as we did in (3.2.14). Consider the following diagram:



where γ is the automorphism of the Hecke algebra $\mathcal{H}_n^{(e)}(q,\zeta)$ sending $\mathbf{T}_t$ to $\zeta\mathbf{T}_t$ and $\mathbf{T}_{s_i}$ to $\mathbf{T}_{s_i}$. The trace $\overline{\tau}'$ is the Markov trace in the sense of (3.3.1) with the parameters

$$\xi = z, \eta_{e-1} = \zeta(zq^{-1}(z-q+1)^{e-1})^{\frac{1}{e}}, \eta_i = \eta_{e-1}^{e-i}(z-q+1)^{i+1-e}$$

on the Hecke algebra $\mathcal{H}_n^{(e)}(q,\zeta)$. Theorem (3.3.5) implies that the above diagram is commutative. Consider any element $\mathbf{h} \in \widetilde{\mathcal{H}}_n^{(e)} \subset \mathcal{H}_n^{(e)}(q,\zeta)$. By $\mathbb{C}(q)$-linearity we can think that $\mathbf{h}$ is just a product of the generators $\mathbf{T}_{s_1'},\mathbf{T}_{s_1},\ldots,\mathbf{T}_{s_{n-1}}$. Since $\mathbf{T}_{s_1'} = \mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t$, the total power of the generator $\mathbf{T}_t$ in this product is divisible by e. Hence γ sends $\mathbf{h}$ to itself. Thus $\overline{\tau}(\mathbf{h}) = \overline{\tau}'(\mathbf{h})$ and $\widetilde{\tau}_e(\mathbf{h}) = \overline{\tau}(\mathbf{h}) \in \mathbb{C}(q,z,\eta_{e-1}^e) = \mathbb{C}(q,z)$. $\square$

The following lemma shows that the trace $\widetilde{\tau}_e$ is a generalization of the special Markov trace of type D_n (cf. (3.2.15)):

Lemma 3.4.3 (Markov trace via Gomi) *The trace* $\widetilde{\tau}_e : \widetilde{\mathcal{H}}_n^{(e)} \to \mathbb{C}(q,z)$ *has the following properties:*

(M1) $\widetilde{\tau}_e(1) = 1$,

(M2) $\widetilde{\tau}_e(\mathbf{h}\mathbf{T}_s) = z\widetilde{\tau}_e(\mathbf{h})$ *for* $s \in \widetilde{S}_n^{(e)}$ *and* $\mathbf{h} \in \mathcal{H}_{\widetilde{S}_n^{(e)}\setminus\{s\}}$*, the Hecke subalgebra of* $\widetilde{\mathcal{H}}_n^{(e)}$ *generated by all* $\mathbf{T}$*'s except* $\mathbf{T}_s$ *(cf. (3.2.1)).*

Proof. Since $\widetilde{\tau}_e$ is a restriction of $\overline{\tau}$ property (M1) as well as property (M2) when $s = s_m, m > 1$ are inherited from properties (M1) and (M3) of $\overline{\tau}$ (cf. (3.3.2)). Thus in order to check property (M2) for any $s \in \widetilde{S}_n^{(e)}$ we just need to consider two cases:

Case $\mathbf{s = s_1' = t^{-1}s_1t}$. Here as in the proof of lemma (3.3.2) we proceed by induction on n. For $n = 2$ we need to prove that for $\mathbf{h} \in \langle \mathbf{T}_{s_1} \rangle$ we have $\widetilde{\tau}_e(\mathbf{h}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t) = z\widetilde{\tau}_e(\mathbf{h})$. Since $\widetilde{\tau}_e(\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t) = \widetilde{\tau}_e(\mathbf{T}_{s_1}) = z$ and $\mathbf{T}_{s_1}^2 = (q-1)\mathbf{T}_{s_1} + q$ we only need to check the equality $\widetilde{\tau}_e(\mathbf{T}_{s_1}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t) = z^2$. We use property (M4) of $\overline{\tau}$ in (3.3.2) for $k = e-1, m = 1$ and $\mathbf{h} = \mathbf{T}_t$:

$$\overline{\tau}(\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t^{e-1}\mathbf{T}_{s_1}^{-1}) = \eta_{e-1}\overline{\tau}(\mathbf{T}_t).$$

We have

$$\begin{aligned}\widetilde{\tau}_e(\mathbf{T}_{s_1}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t) =& \overline{\tau}(\mathbf{T}_{s_1}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t) = \overline{\tau}(\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t^{-1}(q\mathbf{T}_{s_1}^{-1} + q - 1)) = \\ & q\overline{\tau}(\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t^{e-1}\mathbf{T}_{s_1}^{-1}) + (q-1)\overline{\tau}(\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t^{-1}) = \\ & q\eta_{e-1}\overline{\tau}(\mathbf{T}_t) + (q-1)z = q\eta_{e-1}\eta_1 + (q-1)z = \\ & z(z-q+1) + (q-1)z = z^2.\end{aligned}$$

Suppose we have proved our statement for all $n' \leq n-1$. We need to prove it for n. Since the parabolic Hecke subalgebra $\mathcal{H}_{\widetilde{S}_n^{(e)}\setminus\{s\}}$ is of type A_{n-1} we do the step of the induction exactly the same way we did it in (3.3.2) property (M3').

Case $\mathbf{s = s_1}$. This case follows from the previous one. Indeed, using the relations (cf. (3.3))

$$\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1} = \mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t, \quad \mathbf{T}_t\mathbf{T}_{s_i} = \mathbf{T}_{s_i}\mathbf{T}_t \text{ for } 2 \leq i \leq n-1$$

we get $\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{h}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}^{-1} \in \mathcal{H}_{\widetilde{S}_n^{(e)}\setminus\{s_1'\}}$ for $\mathbf{h} \in \mathcal{H}_{\widetilde{S}_n^{(e)}\setminus\{s_1\}}$ and

$$\begin{aligned}\widetilde{\tau}_e(\mathbf{h}\mathbf{T}_{s_1}) = \overline{\tau}(\mathbf{h}\mathbf{T}_{s_1}) =& \overline{\tau}(\mathbf{T}_t\mathbf{h}\mathbf{T}_t^{-1}\mathbf{T}_t\mathbf{T}_{s_1}\mathbf{T}_t^{-1}) = \overline{\tau}(\mathbf{T}_t\mathbf{h}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}^{-1}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{T}_{s_1}) = \\ & \overline{\tau}((\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{h}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}^{-1})\mathbf{T}_t^{-1}\mathbf{T}_{s_1}\mathbf{T}_t) = \\ & z\overline{\tau}(\mathbf{T}_{s_1}\mathbf{T}_t\mathbf{h}\mathbf{T}_t^{-1}\mathbf{T}_{s_1}^{-1}) = z\overline{\tau}(\mathbf{h}) \qquad \square\end{aligned}$$

Definition 3.4.4 *The trace* $\widetilde{\tau}_e : \widetilde{\mathcal{H}}_n^{(e)} \to \mathbb{C}(q,z)$ *is called the* special *Markov trace for the group* $\widetilde{W}_n^{(e)}$.

In the next subsection we recall the Clifford theory and use it to classify irreducible characters of $\widetilde{W}_n^{(e)}$ using the classification of irreducible characters of $W_n^{(e)}$.

3.4.2 Clifford Theory and Irreducible Characters of $\mathbf{G(e,e,n)}$.

Definition 3.4.5 *Let H be a normal subgroup of a finite group G. For $\pi \in \mathrm{Irr}H$ define its* inertia group $I_G(\pi) := \{g \in G | \forall h \in H : \pi(h) = \pi(g^{-1}hg)\}$.

We need the following two theorems (cf. [Isa76, Th. 6.2 and 6.11]):

Theorem 3.4.6 (Clifford) *Let $H \trianglelefteq G$, $\chi \in \mathrm{Irr}G$ and $\pi \in \mathrm{Irr}H$ such that $\langle \pi, \chi_H \rangle_H \neq 0$, $T = I_G(\pi)$ then*

$$\chi_H = e \sum_{g \in G/T} \pi^g$$

where $e = \langle \chi_H, \pi \rangle_H$ and $\pi^g(h) = \pi(g^{-1}hg)$.

Theorem 3.4.7 *Again $H \trianglelefteq G$, $\pi \in \mathrm{Irr}H$, $T = I_G(\pi)$. Let*

$$A = \{\psi \in \mathrm{Irr}T | \langle \psi_H, \pi \rangle_H \neq 0\}, \quad B = \{\chi \in \mathrm{Irr}G | \langle \chi_H, \pi \rangle_H \neq 0\}.$$

(a) If $\psi \in A$ then $\mathrm{Ind}_T^G \psi$ is irreducible.

(b) The map $\psi \mapsto \mathrm{Ind}_T^G \psi$ is a bijection of A onto B.

(c) If $\mathrm{Ind}_H^G \psi = \chi$ with $\psi \in A$ then ψ is the unique irreducible constituent of χ_T which lies in A and $\langle \psi_H, \pi \rangle_H = \langle \chi_H, \pi \rangle_H$.

Now we apply the above theorems to the following example:

Example 3.4.8 *Suppose $G = H \rtimes \langle t \rangle$ where $t^e = 1$. For $\pi \in \mathrm{Irr}H$ its inertia group $T = I_G(\pi) = H \rtimes \langle t^p \rangle$ for some $p \geq 0$.*

Let E be a $\mathbb{C}$-vector space affording the irreducible representation corresponding to π. Since $t^p \in T$ there exists an isomorphism $f_0 : E \to E$ such that $t^{-p}ht^p f_0(v) = f_0(hv)$ for any $v \in E$.

Irreducibility of π implies that there exists $\lambda \in \mathbb{C}$ such that $f := \lambda f_0$ satisfy $f^{e/p} = \mathrm{Id}_E$. Defining the action of t^p on E by $t^p * v := f^{-1}(v)$ we get an action of T on E. Thus there exists $\psi \in \mathrm{Irr}T$ such that $\psi_H = \chi$. We will denote it by $\overline{\chi}$.

Comparing degrees of characters we have $\mathrm{Ind}_H^T \chi = \sum_{j=1}^{e/p} \overline{\chi} \otimes \xi^j$ where $\xi \in \mathrm{Irr}T$ is a linear character with $\xi(H) = 1, \xi(t^p) = exp(2ip\pi/e)$. And thus $\mathrm{Ind}_H^G \chi = \sum_{j=1}^{e/p} \mathrm{Ind}_T^G[\overline{\chi} \otimes \xi^j]$ where each character $\mathrm{Ind}_T^G[\overline{\chi} \otimes \xi^j]$ is irreducible by theorem (3.4.7)(a). Finally , $\mathrm{Ind}_H^G \chi = \sum_{j=0}^{e/p-1} \mathrm{Ind}_T^G[\overline{\chi}] \otimes \epsilon^j$ where $\epsilon \in \mathrm{Irr}G$ is a linear character with $\xi(H) = 1, \xi(t) = exp(2i\pi/e)$; also we have that e/p is the minimal number k such that $\mathrm{Ind}_T^G[\overline{\chi}] = \mathrm{Ind}_T^G[\overline{\chi}] \otimes \epsilon^k$.

Remark 3.4.9 *Note that the number e/p can be characterized as the minimal number $k > 0$ such that $\mathrm{Ind}_T^G[\overline{\chi}] = \mathrm{Ind}_T^G[\overline{\chi}] \otimes \epsilon^k$.*

If we consider any $\phi \in \mathrm{Irr}G$ *and* $\chi \in \mathrm{Irr}H$ *such that* $\langle \phi_H, \chi\rangle_H \neq 0$ *then* $\phi = \mathrm{Ind}_T^G(\overline{\chi})$ *for some extension* $\overline{\chi}$ *of* χ *to* T *(cf. Th. (3.4.7)(b)) where* $T = I_G(\chi) = H \rtimes \langle t^p\rangle$ *is known since the minimal number* k *such that* $\phi = \phi \otimes \epsilon^k$ *is known (cf. Rem. (3.4.9)). By Clifford theorem we have* $\phi_H = \langle \phi_H, \chi\rangle_H \sum_{g\in G/T} \chi^g = \sum_{j=0}^{p-1} \chi^{t^j}$ *since* $\langle \phi_H, \chi\rangle_H = \langle \overline{\chi}, \chi\rangle_H = 1$.

Since $W_n^{(e)} = \widetilde{W}_n^{(e)} \rtimes \langle t\rangle$ we can apply the results from the above example to classify the irreducible characters of $\widetilde{W}_n^{(e)}$. We will use the following notations:

- For an e-partition $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{e-1})$ of n (cf. (3.3.3)) we denote by $\pi(\boldsymbol{\alpha})$ its cyclic permutation $(\alpha_{e-1}, \alpha_0, \ldots, \alpha_{e-2})$ and by $s(\boldsymbol{\alpha})$ the cardinality of the set $\{0 \leq i \leq e-1 | \pi^i(\boldsymbol{\alpha}) = \boldsymbol{\alpha}\}$.

- Let $\chi_{\boldsymbol{\alpha}}$ be the irreducible character of $W_n^{(e)}$ corresponding to an e-partition $\boldsymbol{\alpha}$ (cf. (3.3.4)). We denote by $\chi_{(\boldsymbol{\alpha},1)}, \chi_{(\boldsymbol{\alpha},\zeta^{e/s(\boldsymbol{\alpha})})}, \ldots, \chi_{(\boldsymbol{\alpha},(\zeta^{e/s(\boldsymbol{\alpha})})^{(s(\boldsymbol{\alpha})-1)})}$ the irreducible characters of $\widetilde{W}_n^{(e)}$ which lie in the restriction of $\chi_{\boldsymbol{\alpha}}$ to $\widetilde{W}_n^{(e)}$.

- Let $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}$. We denote by $\boldsymbol{\alpha}(\mu)$ an e-partition (which is defined up to a cyclic permutation) such that $\chi_{\boldsymbol{\alpha}(\mu)}$ lies in $\mathrm{Ind}_{\widetilde{W}_n^{(e)}}^{W_n^{(e)}}\mu$.

3.4.3 Weights of the Special Markov Trace for the Group $\mathbf{G(e,e,n)}$.

By (2.2.31) the Hecke algebra $\widetilde{\mathcal{H}}_n^{(e)}$ is split semisimple over $\mathbb{C}(q)$ and its irreducible characters are in bijection with those of $\widetilde{W}_n^{(e)}$. We can write:

$$\mathrm{Irr}(\mathcal{H}_n^{(e)}) = \left\{\mu^{(q)} | \mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}\right\} \qquad \text{(cf. (2.2.33))}.$$

By (2.2.35) the Hecke algebra $\widetilde{\mathcal{H}}_n^{(e)}$ is a symmetric algebra. Lemma (3.4.2) and [GP00, Ex. (7.4)(b)] imply that there exist unique elements $\omega'_\mu \in \mathbb{C}(q,z)$ such that

$$\widetilde{\tau}_e = \sum_{\mu\in\mathrm{Irr}\widetilde{W}_n^{(e)}} \omega'_\mu \mu^{(q)}. \tag{3.4.10}$$

They are called the *weights* of $\widetilde{\tau}_e$. The following lemma gives us an expression for the weights of $\widetilde{\tau}_e$:

Lemma 3.4.11 *Let* $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}$. *Then*

$$\omega'_\mu = \frac{1}{s(\boldsymbol{\alpha}(\mu))} \sum_{i=0}^{e-1} \omega_{\pi^i(\boldsymbol{\alpha}(\mu))}$$

where $\omega_{\boldsymbol{\alpha}}$ *is the weight of the trace* $\overline{\tau}$ *corresponding to the irreducible character* $\chi_{\boldsymbol{\alpha}}$ *of* $W_n^{(e)}$.

Proof. Consider the decomposition of the trace $\overline{\tau}$:

$$\overline{\tau} = \sum_{\boldsymbol{\alpha} \in \Lambda_n^{(e)}} \omega_{\boldsymbol{\alpha}} \chi_{\boldsymbol{\alpha}}{}^{(q)}.$$

Results from the example (3.4.8), Tits deformation theorem (2.2.26) and the fact that the specializations of $\widetilde{\mathcal{H}}_n^{(e)} \leq \mathcal{H}_n^{(e)}(q, \zeta)$ via $q \mapsto 1$ are $\mathbb{C}\widetilde{W}_n^{(e)} \leq \mathbb{C}W_n^{(e)}$ imply:

$$\widetilde{\tau}_e = \overline{\tau}|_{\widetilde{\mathcal{H}}_n^{(e)}} = \sum_{\boldsymbol{\alpha} \in \Lambda_n^{(e)}} \omega_{\boldsymbol{\alpha}} \chi_{\boldsymbol{\alpha}}{}^{(q)}|_{\widetilde{\mathcal{H}}_n^{(e)}} = \sum_{\boldsymbol{\alpha} \in \Lambda_n^{(e)}} \omega_{\boldsymbol{\alpha}} \sum_{i=0}^{s(\boldsymbol{\alpha})-1} \chi^{(q)}_{(\boldsymbol{\alpha}, \zeta^{ei/s(\boldsymbol{\alpha})})} =$$

$$\sum_{\mu \in \mathrm{Irr} \widetilde{W}_n^{(e)}} \left(\sum_{i=0}^{e/s(\boldsymbol{\alpha}(\mu))-1} \omega_{\pi^i(\boldsymbol{\alpha}(\mu))} \right) \mu^{(q)} = \sum_{\mu \in \mathrm{Irr} \widetilde{W}_n^{(e)}} \frac{1}{s(\boldsymbol{\alpha}(\mu))} \left(\sum_{i=0}^{e-1} \omega_{\pi^i(\boldsymbol{\alpha}(\mu))} \right) \mu^{(q)}$$

By [GP00, Ex. (7.4)(b)] we have that the decomposition of the trace $\widetilde{\tau}_e$ in a sum of irreducible characters of $\widetilde{\mathcal{H}}_n^{(e)}$ is unique. This finishes the proof. □

Remark 3.4.12 *The formula in the above lemma can also be obtained using [Mal95, Lemma 5.11] and ideas from the proof of [Mal95, Satz 5.13].*

Chapter 4

Khovanov-Rozansky Trace

In this chapter we introduce the notion of the so-called Khovanov-Rozansky trace which is a natural generalization of the HOMFLY-PT polynomial, a two-variable knot invariant. In chapter 7 we will see that it coincides with the Gomi trace defined in chapter 6 for all finite Coxeter groups except, possibly, H_3 and H_4. Here we will often follow the article of Mikhail Khovanov [Kho07].

4.1 Hochschild Homology

Let k be a field, R be a commutative graded k-algebra and M be a graded R-*bimodule*. Thus M is a left and a right R-module, $um = mu$ for any $u \in k, m \in M$ and the two actions are compatible in the sense that $a(mb) = (am)b$ for any $a, b \in R$ and $m \in M$. The bimodule M can also be regarded as a left R^e-module, where $R^e = R \otimes_k R$ is the *enveloping algebra* of R. One can define a functor $*_R$ of R-coinvariants from the category of graded R-bimodules to the category of graded R-modules in the following way: $M_R := R \otimes_{R^e} M$. The R-coinvariants functor is right exact and its i-th derived functor takes M to $\mathrm{Tor}_i^{R^e}(R, M)$.

Definition 4.1.1 *The graded R-module* $\mathrm{Tor}_i^{R^e}(R, M)$ *is denoted by* $\mathrm{HH}_i(R, M)$ *and called the* i-th Hochschild homology group *of M. The* Hochschild homology *of M is the direct sum*

$$\mathrm{HH}(R, M) := \bigoplus_{i \geq 0} \mathrm{HH}_i(R, M).$$

To compute Hochschild homology one can take any resolution of the R-bimodule R by projective R-bimodules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow R \longrightarrow 0,$$

tensor it with M, removing the right term M_R:

$$\cdots \longrightarrow P_2 \otimes_{R^e} M \longrightarrow P_1 \otimes_{R^e} M \longrightarrow P_0 \otimes_{R^e} M \longrightarrow 0.$$

The homology of this complex is isomorphic to the Hochschild homology of M.

Any k-algebra R has the standard *bar resolution* $C_*^{\text{bar}}(R)$ by free R-bimodules $C_n^{\text{bar}}(R) := R^{\otimes n+2}, n \geq 0$ with boundary map b':

$$b'_n : C_n^{\text{bar}}(R) \to C_{n-1}^{\text{bar}}(R) :$$
$$b'_n(r_0, r_1, \ldots, r_{n+1}) = \sum_{i=0}^{n} (-1)^i (r_0, \ldots, r_i r_{i+1}, \ldots, r_{n+1}).$$

From now on we will consider the case when $k = \mathbb{C}$ and $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ the algebra of complex polynomials in n variables. We regard R as a graded $\mathbb{C}$-algebra with $\deg x_i = 2$.

We will use the following

Notation 4.1.2 *Let M be any graded R-module and M_i $(i \in \mathbb{Z})$ its homogeneous components. Then for $k \in \mathbb{Z}$ we denote by $M(k)$ the* "shifted" *graded R-modules defined by $M(k)_i := M_{k+i}$.*

Replacing R by R^e we get the notion of *shifted* graded R-bimodules.

The polynomial algebra R admits a "Koszul" resolution (which is much smaller then the "bar" one) by free $R^e = R \otimes R$-modules given by the tensor product over R^e of the complexes

$$0 \longrightarrow R^e(-2) \xrightarrow{\times \mu_i} R^e \longrightarrow 0,$$

for $1 \leq i \leq n$. Here $\times \mu_i$ means the multiplication by $\mu_i := 1 \otimes x_i - x_i \otimes 1$. We use a shift to make the multiplication preserve the polynomial degree.

4.2 The Category of Soergel Bimodules

Let (W, S) be a Coxeter system with $|S| = n < \infty$; let V be a complex vector space with basis $e_1, \ldots, e_n$ such that $W \hookrightarrow \mathrm{GL}(V)$ is the geometric representation of the Coxeter group W. The geometric representation gives us an action of W on V which can be naturally extended to the action of W on the symmetric algebra $S(V^*)$, which can also be regarded as a ring $R = \mathbb{C}[x_1, x_2, ..., x_n]$ of polynomials in n variables.

Now we will introduce the notion of *Soergel bimodules*, after Wolfgang Soergel, who introduced them and explained their importance for the infinite-dimensional representation theory of simple Lie algebras and closely related Kazhdan-Lusztig theory (cf. [Soe92] and [Soe95]). For any simple generator $s \in S$ we define an R-bimodule $B_s := R \otimes_{R^s} R$, where R^s is the ring of polynomials invariant by s, R acts on B_s by right and left multiplication.

Definition 4.2.1 *We define the* Soergel category $\mathcal{R}$ *as the full subcategory of the category of graded R-bimodules whose objects are*

$(B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_k})\,(l)$ and all sums of their direct summands. (Here $k \in \mathbb{Z}^+$,$l \in \mathbb{Z}$ and $s_i \in S$ for $1 \leq i \leq k$). The objects of the Soergel category are called Soergel bimodules.

Let $\langle\mathcal{R}\rangle$ be the split Grothendieck group of the Soergel category. Tensoring $\otimes_R$ defines a ring structure on $\langle\mathcal{R}\rangle$. Soergel has proven the following "categorification" theorem:

Theorem 4.2.2 (Soergel) *Let $\mathcal{H}'_W$ be the Iwahori-Hecke algebra of W with the parameters $q_s = v^{-2}, q'_s = -1$ (cf. (2.2.14)). Then there is exactly one ring homomorphism*

$$\mathcal{E} : \mathcal{H}'_W \to \langle\mathcal{R}\rangle$$

such that we have $\mathcal{E}(v) = \langle R(1)\rangle$ and $\mathcal{E}(T_s + 1) = \langle B_s\rangle\,, \forall s \in S$.

There is a famous Soergel conjecture concerning the above homomorphism $\mathcal{E}$:

Conjecture 4.2.3 (Soergel) *Let C'_x be a Kazhdan-Lusztig basis element corresponding to $x \in W$ (cf. (2.2.15)). Then there is an indecomposable graded R-bimodule B_x which is finitely generated as a left and as a right R-module such that*

$$\mathcal{E}(C'_x) = \langle B_x\rangle\,.$$

Remark 4.2.4 *In the case when W is a Weyl group the conjecture above is proven in [Soe92]. The case when W is a dihedral group is done in [Soe07, Th. 4.2].*

Since we will be particularly interested in the dihedral case we give more details here.

Notation 4.2.5 *Given any finite dimensional representation V of the group W we consider for any $x \in W$ the graph*

$$\mathrm{Gr}(x) = \{(x\lambda, \lambda)\} \subset V \times V$$

and form for any finite subset $A \subset W$ the Zariski closed subset in $V \times V$

$$\mathrm{Gr}(A) = \bigcup_{x \in A} \mathrm{Gr}(x).$$

Notation 4.2.6 *Recall that the ring of complex polynomials R can be regarded as the $\mathbb{C}$-algebra of all regular functions on V. If we identify $R^e = R \otimes R$ with the regular functions on $V \times V$ via the rule $(f \otimes g)(\lambda, \mu) = f(\lambda)g(\mu)$ then the ring of regular functions on $Gr(A)$ is a quotient of R^e and inherits the structure of graded R-bimodule. It's not hard to see that this*

R-bimodule is finitely generated as left and as right R-module. Following the notations of Soergel we denote it by

$$R(A) = R(\mathrm{Gr}(A)).$$

Let $\leq$ be the Bruhat order on W. For $A = \{y | y \leq x\}$ we put $R(A) = R_{\leq x}$.

Example 4.2.7 *With the notations above its not hard to see that $R(\leq 1) = R$ and $R(\leq s) = B_s$.*

Now we can formulate the property of the above homomorphism $\mathcal{E}$ in case of dihedral groups ($W = I_2(m)$):

Theorem 4.2.8 (Soergel) *For any element $x \in W$ we have*

$$\mathcal{E}(C'_x) = \langle R_{\leq x}(l(x)) \rangle ,$$

where $(l(x))$ means the shift in the graduation by the length of x.

4.3 Rouquier Complex

Raphael Rouquier has found a connection between the braid group and Soergel bimodules $B_s, s \in S$.

We assign to a generator σ_s of the braid group B a cochain complex $F(\sigma_s)$ of Soergel bimodules

$$F(\sigma_s): \qquad 0 \longrightarrow B_s \xrightarrow{m} R \longrightarrow 0,$$

where $m(a \otimes b) = ab$ for any $a, b \in R$; the bimodule B_s is placed in cohomological degree 0.

To the braid group element σ_s^{-1} we assign a cochain complex $F(\sigma_s^{-1})$ of Soergel bimodules

$$F(\sigma_s^{-1}): \qquad 0 \longrightarrow R \xrightarrow{\eta} B_s(2) \longrightarrow 0,$$

here the R-bimodule morphism η is defined by $\eta(1) := 1 \otimes a_s + a_s \otimes 1$, where $a_s := e_s^*$ is dual to the basis vector e_s; the bimodule R is placed in cohomological degree -1.

To a braid word

$$\sigma = \sigma_{s_1}^{\epsilon_1} \sigma_{s_2}^{\epsilon_2} \cdots \sigma_{s_k}^{\epsilon_k}, \quad \epsilon_i = \pm 1, s_i \in S \text{ for } 1 \leq i \leq k$$

we assign the tensor product over R of the above complexes and denote it by $F(\sigma)$:

$$F(\sigma) = F(\sigma_{s_1}^{\epsilon_1}) \otimes_R F(\sigma_{s_2}^{\epsilon_2}) \otimes_R \cdots \otimes_R F(\sigma_{s_k}^{\epsilon_k}). \qquad (4.3.1)$$

This complex is an element of the category $\mathcal{B}(R)$ of cochain complexes of graded R-bimodules up to chain homotopies. Rouquier [Rou06] has proved the following important theorem:

Proposition 4.3.2 (Rouquier) *If the braid words σ and $\overline{\sigma}$ represent the same element of the braid group then the complexes $F(\sigma)$ and $F(\overline{\sigma})$ are isomorphic in $\mathcal{B}(R)$.*

In particular, the tensor product $F(\sigma_i) \otimes_R F(\sigma_i^{-1})$ is chain homotopy equivalent to the complex $0 \longrightarrow R \longrightarrow 0$ (where R is placed in cohomological degree 0).

Remark 4.3.3 *From now on we will refer to $F(\sigma)$ as* Rouquier complexes*.*

4.4 Khovanov-Rozansky Trace

Now we pass to Khovanov's construction via Rouquier's complex of Khovanov-Rozansky link homology.

Again, let $\sigma = \sigma_{s_1}^{\epsilon_1} \sigma_{s_2}^{\epsilon_2} \cdots \sigma_{s_k}^{\epsilon_k}$ $(\epsilon_i = \pm 1, s_i \in S$ for $1 \leq i \leq k)$ be a braid word. The Rouquier complex $F(\sigma)$:

$$\cdots \xrightarrow{\partial} F^j(\sigma) \xrightarrow{\partial} F^{j+1}(\sigma) \xrightarrow{\partial} \cdots$$

has $k+1$ nontrivial terms. Each term $F^j(\sigma)$ is a direct sum of graded R-bimodules which are shifts of tensor products of B_{s_i}'s and thus $F^j(\sigma)$ is a Soergel bimodule.

The Hochschild homology $\mathrm{HH}(R, F^j(\sigma))$ of a bimodule $F^j(\sigma)$ is a bigraded $\mathbb{C}$-vector space. Taking Hochschild homology of each term we obtain a complex $\mathrm{HH}(F(\sigma))$ of bigraded vector spaces

$$\ldots \xrightarrow{\mathrm{HH}(\partial)} \mathrm{HH}(R, F^j(\sigma)) \xrightarrow{\mathrm{HH}(\partial)} \mathrm{HH}(R, F^{j+1}(\sigma)) \xrightarrow{\mathrm{HH}(\partial)} \ldots$$

Definition 4.4.1 *The homology of the above complex is denoted by* $\mathrm{HHH}(\sigma)$ *and called* Khovanov-Rozansky homology*.*

It is easy to see that $\mathrm{HHH}(\sigma)$ is a triply-graded $\mathbb{C}$-vector space. In the classical case when $W = S_n$ is a symmetric group Khovanov and Rozansky have proved the following theorem:

Theorem 4.4.2 *Up to an overall shift in the grading,* $\mathrm{HHH}(\sigma)$ *is an invariant of oriented links and, up to isomorphism, depends only on the closure of σ. The homology theory is isomorphic to the reduced homology $\overline{H}(\sigma)$ which is defined in [KR08].*

The theorem implies that the Euler characteristic of $\mathrm{HHH}(\sigma)$ in the classical case of the symmetric group is the HOMFLYPT link polynomial [FYH$^+$85], [PT87]. By introducing a fractional $\frac{1}{2}\mathbb{Z}$ triple grading and a suitable shift as in Wu [Wu08], the grading indeterminacy can be renormalized away. This fact makes it natural to investigate the Euler characteristic of $\mathrm{HHH}(\sigma)$ in the case of an arbitrary finite Coxeter group W.

We start with the definition of a map $\langle \cdot \rangle_V$ from the braid group B to the field $K = \mathbb{C}(v, r)$ of rational functions in v and r:

$$\begin{aligned} \langle\sigma\rangle_V &:= \left(\frac{1-v^2}{1+r}\right)^n \cdot \sum_i (-1)^i \mathrm{dimgr}_{v,t} \mathrm{HH}(R, F^i(\sigma)) \\ &= \left(\frac{1-v^2}{1+r}\right)^n \cdot \sum_{i,j,k} (-1)^i \dim_{\mathbb{C}} \mathrm{HH}_j(R, F^i(\sigma))_k \cdot v^k \cdot t^j, \end{aligned} \tag{4.4.3}$$

where $t = rv^{-2}$, the index i corresponds to the cohomological grading of the Rouquier complex, j - to the Hochschild homology grading and k - to the polynomial grading of R-modules $\mathrm{HH}_j(R, F^i(\sigma))$.

Remark 4.4.4 *The map above is nothing else but the graded Euler characteristic (multiplied by normalizing coefficient) of the complex* $\mathrm{HH}(F(\sigma))$.

Proposition 4.4.5 *The map* $\langle \cdot \rangle_V$ *is well-defined , i.e. is independent of the choice of the braid word representing* σ.

Proof. This proposition is a direct consequence of the Rouquier theorem. □

By linearity the above map can be extended to a K-linear morphism from the group algebra KB to K which we denote by the same symbol $\langle \cdot \rangle_V$.

Proposition 4.4.6 *The above morphism has the following properties:*

(a) (Trace property) For any $\sigma_1, \sigma_2 \in KB$ *we have* $\langle\sigma_1\sigma_2\rangle_V = \langle\sigma_2\sigma_1\rangle_V$.

(b) (Normalizing property) $\langle 1\rangle_V = 1$.

(c) (Hecke property) For any $\sigma \in KB$ *and* $s \in S$ *we have*
$\langle\sigma\sigma_s\rangle_V = v^2 \langle\sigma\sigma_s^{-1}\rangle_V + (v^2 - 1) \langle\sigma\rangle_V$

Proof. (a) Here we need to compare Euler characteristics of the complexes $\mathrm{HH}(F(\sigma_1) \otimes_R F(\sigma_2))$ and $\mathrm{HH}(F(\sigma_2) \otimes_R F(\sigma_1))$. The definition of a tensor product of complexes allows us to reduce this to the comparison of Hochschild homologies $\mathrm{HH}_i(R, M \otimes_R N)$ and $\mathrm{HH}_i(R, N \otimes_R M)$ where M, N are tensor products over R of some $B_s, s \in S$.

Let $\mathcal{A}$ be the category of all graded R-bimodules. We will work in $D(\mathcal{A})$, the derived category of $\mathcal{A}$. By definition $\mathrm{HH}_i(R, M \otimes_R N) = \mathrm{H}_i(\widetilde{R} \otimes_{R^e} (M \otimes_R N))$ where $\widetilde{R} \simeq R$ is a free R^e-resolution of R. We have a series of isomorphisms in $D(\mathcal{A})$:

$$\widetilde{R} \otimes_{R^e} (M \otimes_R N) \simeq \widetilde{R} \otimes_{R^e} M \widetilde{\otimes_R} N \simeq R \otimes_{R^e} M \widetilde{\otimes_R} N,$$

where $\widetilde{M \otimes_R N}$ is a free R^e-resolution of $M \otimes_R N$. Let us precise its structure. Denote by $\widetilde{M} \simeq M$ and $\widetilde{N} \simeq N$ free R^e-resolutions of M and N. Then $M \otimes_R N \simeq M \otimes_R \widetilde{N}$ since M is a free right R-module (as a tensor product of B_s which are free right R-modules). But $M \otimes_R \widetilde{N} \simeq \widetilde{M} \otimes_R \widetilde{N}$ and we get a free R^e-resolution $\widetilde{M} \otimes_R \widetilde{N}$ of $M \otimes_R N$.

Similarly, $\widetilde{N} \otimes_R \widetilde{M}$ is a free R^e-resolution of $N \otimes_R M$ and $\mathrm{HH}_i(R, N \otimes_R M) = \mathrm{H}_i(R \otimes_{R^e} (\widetilde{M} \otimes_R \widetilde{N}))$. Thus in order to prove the equality it is sufficient to prove that

$$R \otimes_{R^e} (\widetilde{M} \otimes_R \widetilde{N}) \simeq R \otimes_{R^e} (\widetilde{N} \otimes_R \widetilde{M}),$$

which is not hard to do. The morphism is simply $r \otimes_{R^e} a \otimes_R b \mapsto r \otimes_{R^e} b \otimes_R a$ for $r \in R, a \in \widetilde{M}_i, b \in \widetilde{N}_j$.
(b) In this case $F(1) = 0 \to R \to 0$ and we have to calculate $\mathrm{HH}_i(R, R)$. In order to do that we take a Koszul resolution $\widetilde{R}$ of R. Recall that it is a tensor product over R^e of complexes

$$0 \longrightarrow R^e(-2) \xrightarrow{\times \mu_i} R^e \longrightarrow 0,$$

for $1 \leq i \leq n$, where $\times \mu_i$ is the multiplication by $\mu_i := 1 \otimes x_i - x_i \otimes 1$. By definition $\mathrm{HH}_i(R, R) = \mathrm{H}_i(R \otimes_{R^e} \widetilde{R}) = R \otimes_{R^e} \widetilde{R}_i$ since all boundary operators of the complex $R \otimes_{R^e} \widetilde{R}$ are zero. This gives us the formula

$$\sum_{i,k} \dim_{\mathbb{C}} \mathrm{HH}_i(R, R)_k \cdot v^k \cdot t^i = (1 + tv^2)^n \mathrm{dimgr}_v R = \left(\frac{1 + tv^2}{1 - v^2}\right)^n.$$

But this is the inverse of the normalizing coefficient. Thus $\langle 1 \rangle_V = 1$.
(c) We have

$$F^i(\sigma\sigma_s) = F^i(\sigma) \otimes_R B_s \oplus F^{i-1}(\sigma)$$

and

$$F^i(\sigma\sigma_s^{-1}) = F^{i+1}(\sigma) \oplus F^i(\sigma) \otimes_R B_s(2).$$

We will use the following notation

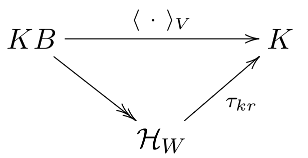
Notation 4.4.7 *For a finitely generated graded R-bimodule M we denote by $\spadesuit_{v,t}(M)$ the bigraded dimension of its Hochschild homology $\sum_{j,k} \dim_{\mathbb{C}} \mathrm{HH}_j(R, M)_k \cdot v^k \cdot t^j$.*

From the definition of $\langle \sigma\sigma_s \rangle_V$ we have

$$
\begin{aligned}
\left(\frac{1-v^2}{1+tv^2}\right)^n & \langle \sigma\sigma_s \rangle_V \\
&= \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma\sigma_s)) \\
&= \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma) \otimes_R B_s) + \sum_i (-1)^i \spadesuit_{v,t}(F^{i-1}(\sigma)) \\
&= v^2 \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma) \otimes_R B_s(2)) - \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma)) \\
&= v^2 \sum_i (-1)^i \spadesuit_{v,t}(F^{i+1}(\sigma)) + v^2 \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma) \otimes_R B_s(2)) \\
&\quad + v^2 \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma)) - \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma)) \\
&= v^2 \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma\sigma_s^{-1})) + (v^2-1) \sum_i (-1)^i \spadesuit_{v,t}(F^i(\sigma)) \\
&= v^2 \left(\frac{1-v^2}{1+tv^2}\right)^n \left\langle \sigma\sigma_s^{-1} \right\rangle_V + (v^2-1) \left(\frac{1-v^2}{1+tv^2}\right)^n \langle \sigma \rangle_V
\end{aligned}
$$

□

The above proposition allows us to factorize the morphism $\langle \cdot \rangle_V$ through the Iwahori-Hecke algebra $\mathcal{H}_W$ with the parameters $q_s = v^2, q'_s = -1$, which is a natural quotient of KB:



Definition 4.4.8 *The right diagonal map τ_{kr} in the above diagram is a trace such that $\tau_{kr}(1) = 1$ and is called the* Khovanov-Rozansky trace*.*

Theorem 4.4.9 *The trace τ_{kr} is a Markov trace with the parameter $z = \frac{r(v^2-1)}{r+1}$.*

Proof. The idea of the proof is based on the proof of [Kho07, Th. 1] where Koszul complexes are used. We will use the following notations.

Notation 4.4.10 *We pose*

$$
\begin{array}{lll}
R_x^n := \mathbb{C}[x_1,\dots,x_n] & R_y^n := \mathbb{C}[y_1,\dots,y_n] & R_z^n := \mathbb{C}[z_1,\dots,z_n] \\
R_{x,y}^n := R_x^n \otimes_{\mathbb{C}} R_y^n & R_{x,z}^n := R_x^n \otimes_{\mathbb{C}} R_z^n & R_{y,z}^n := R_y^n \otimes_{\mathbb{C}} R_z^n \\
R_{x,y,z}^n := R_x^n \otimes_{\mathbb{C}} R_y^n \otimes_{\mathbb{C}} R_z^n. & &
\end{array}
$$

The action of W on R_y^n, R_z^n is exactly the same as on $R = R_x^n$ and is directly induced on $R_{x,y}^n, R_{x,z}^n, R_{y,z}^n, R_{x,y,z}^n$.

Notation 4.4.11 *Let A be a graded ring and $a_1, \dots a_k$ some homogeneous elements in A. We denote by $(a_1, \dots, a_k)_A$ the corresponding* Koszul complex *which is defined as a tensor product over A of small complexes*

$$0 \to A(-\deg a_i) \xrightarrow{\times a_i} A \to 0$$

where $\times a_i$ means the multiplication by a_i.

We will also need two simple facts:

Fact 4.4.12 *Let $(a_1, \dots, a_k)_A$ be a Koszul complex and $\lambda \in A$ be an element such that λa_i is homogeneous of the same degree as a_j for some $i \neq j$. Then we have the following isomorphism of complexes:*

$$(a_1, \dots, a_i, \dots, a_j \dots, a_k)_A \simeq (a_1, \dots, a_i, \dots, a_j + \lambda a_i \dots, a_k)_A.$$

Fact 4.4.13 *Let A, B be some $\mathbb{C}$-algebras, $K^\bullet$ some complex of $A \otimes_{\mathbb{C}} B$-modules and $L^\bullet$ a complex of A-modules. Then there is an obvious isomorphism of complexes*

$$K^\bullet \otimes_A L^\bullet \simeq K^\bullet \otimes_{A \otimes B} (L^\bullet \otimes_{\mathbb{C}} B)$$

In order to prove the theorem it is enough to show that $\tau_{kr}(b\sigma_s) = z\tau_{kr}(b)$ for any $s \in S, b \in \mathcal{H}_{S \setminus \{s\}}$ since all the other properties follow from the proposition above. Linearity of τ_{kr} allows us to assume that b is a product of simple generators $\sigma_t, t \neq s$ of the braid group.

Recall that by definition

$$\begin{aligned}
\left(\tfrac{1-v^2}{1+tv^2}\right)^n \tau_{kr}(b\sigma_s) &= \textstyle\sum_i (-1)^i \spadesuit_{v,t}(F^i(b\sigma_s)) \\
&= \textstyle\sum_i (-1)^i \spadesuit_{v,t}(F^i(b) \otimes_R B_s) \\
&\quad - \textstyle\sum_i (-1)^i \spadesuit_{v,t}(F^i(b)) \\
\left(\tfrac{1-v^2}{1+tv^2}\right)^n z\tau_{kr}(b) &= \tfrac{tv^2(v^2-1)}{tv^2+1} \textstyle\sum_i (-1)^i \spadesuit_{v,t}(F^i(b)).
\end{aligned}$$

Thus it is enough to prove the following equality:

$$(tv^2 + 1)\spadesuit_{v,t}(\Theta_b \otimes_R B_s) = (tv^4 + 1)\spadesuit_{v,t}(\Theta_b),$$

where the bimodule Θ_b is a tensor product over R of some B_t's ($t \neq s$) since every $F^i(b)$ is a direct sum of such bimodules.

Let us look at our vector space V. Since $\dim_{\mathbb{C}} V = n > n - 1 = |S \setminus \{s\}|$ there exists a vector $v \in V^*$ invariant under the action of any $s' \in S, s' \neq s$. We can assume that $v = x_n$ and $x_1, \dots, x_{n-1} \in V^s$. Let $u_x = x_n - w_x$ ($w_x \in V^{*s}$) be an element of V^* dual to the root of s (vectors $u_y = y_n - w_y$ and $u_z = z_n - w_z$ will be the corresponding copies of u_x in R_y^n and R_z^n).

Lemma 4.4.14 *There is an isomorphism of $R^n_{x,y}$-modules*

$$\Theta_b \simeq \Theta'_b \otimes_{\mathbb{C}} \mathbb{C}[x_n],$$

where Θ'_b is some $R^{n-1}_{x,y}$-module, y_n acts on $\mathbb{C}[x_n]$ by multiplication by x_n.

Proof. Easy consequence from the fact that Θ_b is a tensor product of some B_t's. □
Let us prove the following

Proposition 4.4.15 *Let $\widetilde{\Theta'_b}$ be a free $R^{n-1}_{x,y}$-resolution of Θ'_b. Then the following complex $K_{b,s}$ is a free $R^n_{x,y,z}$-resolution of $\Theta_b \otimes_{R^n_y} B_s$:*

$$\left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R^n_{x,y,z}}.$$

Proof. Indeed, the complex above is isomorphic to

$$\begin{aligned} & \left[\left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n]\right) \otimes_{R^n_{x,y}} (x_n - y_n)_{R^n_{x,y}}\right] \\ & \qquad \otimes_{R^n_{x,y}} (y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R^n_{x,y,z}} \\ \simeq\; & \left[\widetilde{\Theta'_b} \otimes_{R^{n-1}_{x,y}} (x_n - y_n)_{R^n_{x,y}}\right] \\ & \qquad \otimes_{R^n_y} (y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R^n_{y,z}} \end{aligned}$$

Since the complex $\widetilde{\Theta'_b} \otimes_{R^{n-1}_{x,y}} (x_n - y_n)_{R^n_{x,y}}$ is isomorphic to $\widetilde{\Theta'_b} \otimes_{\mathbb{C}} (x_n - y_n)_{\mathbb{C}[x_n, y_n]}$, it is a free $R^n_{x,y}$-resolution of Θ_b. At the same time the complex $(y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2)_{R^n_{y,z}}$ considered as a complex of R^n_y-modules is homotopy equivalent to the free R^n_y-module B_s. This proves that their tensor product over R^n_y is a free $R^n_{x,y,z}$-resolution of $\Theta_b \otimes_{R^n_y} B_s$. □
In order to calculate $\mathrm{HH}(R, \Theta_b \otimes_R B_s)$ we take the tensor product over $R^n_{x,z}$ of $K_{b,s}$ and a free $R^n_{x,z}$-resolution of R represented by the Koszul complex $(x_1 - z_1, \ldots, x_n - z_n)_{R^n_{x,z}}$. This tensor product is equal to

$$K_{b,s} \otimes_{R^n_{x,y,z}} (x_1 - z_1, \ldots, x_n - z_n)_{R^n_{x,y,z}}.$$

And this is equal to

$$\begin{aligned} & \left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \\ & \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, u_y^2 - u_z^2, x_1 - z_1, \ldots, x_n - z_n)_{R^n_{x,y,z}} \\ \simeq\; & \left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \\ & \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, 0, x_1 - z_1, \ldots, x_n - z_n)_{R^n_{x,y,z}} \end{aligned}$$

where the element $u_y^2 - u_z^2 = (y_n - z_n - w_y + w_z)(y_n + z_n - w_y - w_z)$ of homogeneous degree 4 was "killed" by elements $x_n - y_n, x_n - z_n, y_1 - z_1 \ldots, , y_{n-1} - z_{n-1}$. Denoting the homology of the complex

$$\begin{aligned} & \left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \\ & \quad \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, x_1 - z_1, \ldots, x_n - z_n)_{R^n_{x,y,z}} \end{aligned}$$

by $\mathrm{H}^{\bullet}_{b,s}$ (which is double-graded) we have:

$$\spadesuit_{v,t}(\Theta_b \otimes_R B_s) = (1 + tv^4)\mathrm{dimgr}_{v,t}\mathrm{H}^{\bullet}_{b,s}.$$

The proof that

$$\spadesuit_{v,t}(\Theta_b) = (1 + tv^2)\mathrm{dimgr}_{v,t}\mathrm{H}^{\bullet}_{b,s}$$

is completely analogous. First we make a formal note that the complex $L_{b,s}$:

$$\left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, y_n - z_n)_{R^n_{x,y,z}}$$

is a free $R^n_{x,y,z}$-resolution of Θ_b. Then we take the tensor product over $R^n_{x,z}$ of $L_{b,s}$ and the Koszul complex $(x_1 - z_1, \dots, x_n - z_n)_{R^n_{x,z}}$. And we get

$$\begin{aligned} & \left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \\ & \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, y_n - z_n, x_1 - z_1, \dots, x_n - z_n)_{R^n_{x,y,z}} \\ \simeq \; & \left(\widetilde{\Theta'_b} \otimes_{\mathbb{C}} \mathbb{C}[x_n, y_n] \otimes_{\mathbb{C}} R^n_z\right) \\ & \otimes_{R^n_{x,y,z}} (x_n - y_n, y_1 - z_1, \dots, y_{n-1} - z_{n-1}, 0, x_1 - z_1, \dots, x_n - z_n)_{R^n_{x,y,z}}, \end{aligned}$$

where the element $y_n - z_n$ of homogeneous degree 2 was "killed" by elements $x_n - y_n$ and $x_n - z_n$. This ends the proof. □

Chapter 5

Fourier Transform

5.1 Case of Weyl Groups

Our goal in this section is to give a brief history of appearance of the Fourier transform matrix in the rational case.

5.1.1 Characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$

Let $\mathbf{G}$ be a connected affine algebraic group over $\overline{\mathbb{F}}_q$, split over $\mathbb{F}_q$; thus $\mathbf{G}$ contains a split maximal torus $\mathbf{T}_1$ and the $\mathbb{F}_q$-structure is given by a Frobenius map F which acts trivially on the Weyl group $W = N_{\mathbf{G}}(\mathbf{T}_1)/\mathbf{T}_1$.

The starting point in the study of irreducible representations of the finite group $\mathbf{G}^F$ over the field $\overline{\mathbb{Q}}_l$ was their construction. The obvious way to construct such representations is to start with some subgroup H of $\mathbf{G}^F$ (for example, $H = \mathbf{L}^F$, where $\mathbf{L}$ is a rational Levi subgroup of $\mathbf{G}$) and build representations of $\mathbf{G}^F$ using the usual induction functor $\mathrm{Ind}_H^{\mathbf{G}^F}$. Actually the representations thus obtained would not have the right properties, in particular, their decomposition into irreducible representations will be very difficult to work with. The right construction is to use the Harish-Chandra induction functor $R_{\mathbf{L}}^{\mathbf{G}}$.

Definition 5.1.1 *Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ a rational Levi subgroup of $\mathbf{P}$, so that we have a rational Levi decomposition $\mathbf{P} = \mathbf{LU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$. The functor from the category of left $\mathbf{L}^F$-modules the category of left $\mathbf{G}^F$-modules defined by*

$$R_{\mathbf{L}}^{\mathbf{G}} : E \mapsto \overline{\mathbb{Q}}_l \left[\mathbf{G}^F/\mathbf{U}^F\right] \otimes_{\overline{\mathbb{Q}}_l[\mathbf{L}^F]} E$$

where $\mathbf{G}^F$ acts on $\overline{\mathbb{Q}}_l \left[\mathbf{G}^F/\mathbf{U}^F\right]$ by left translations and $\mathbf{L}^F$ by right translations is called Harish-Chandra induction*.*

Remark 5.1.2 *The parabolic subgroup* $\mathbf{P}$ *does not appear in the notation since the Mackey formula allows to prove that* $R_{\mathbf{L}}^{\mathbf{G}}$ *does not depend on the parabolic subgroup used in its construction. (cf. [DM91, (6.1) , (5.1)])*

Remark 5.1.3 *As you can see the functor* $R_{\mathbf{L}}^{\mathbf{G}}$ *is only defined when* $\mathbf{L}$ *is a rational Levi subgroup of a* rational *parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$*.*

Trying to extend the construction of the functor $R_{\mathbf{L}}^{\mathbf{G}}$ to the case where $\mathbf{L}$ is a rational Levi subgroup of $\mathbf{G}$ but not the Levi subgroup of any rational parabolic subgroup of $\mathbf{G}$, Deligne and Lusztig came to the idea of using l-adic cohomology of certain varieties.

It is a very deep result that to any variety $\mathbf{X}$ over $\overline{\mathbb{F}}_q$ one can associate canonically l-adic cohomology groups with compact support $H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$, with properties analogous to those familiar from algebraic topology. These are finite dimensional $\overline{\mathbb{Q}}_l$-vector spaces, zero for $i < 0$ and for large i. They are functorial in the sense that a finite map $f : \mathbf{X} \to \mathbf{X}'$ induces a $\overline{\mathbb{Q}}_l$-linear map $f^* : H_c^i(\mathbf{X}', \overline{\mathbb{Q}}_l) \to H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$. Definition and properties of l-adic cohomology can be found in (S.G.A 4,5).

l-adic cohomology can be used to construct representations of finite groups as follows: Assume that G is a finite group acting on a variety $\mathbf{X}$. Then $g \in G$ acts on $H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$ by $(g^*)^{-1}$ and this is a representation of G. We can also form the alternating sum $H_c^*(\mathbf{X}) := \sum_i (-1)^i H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_l)$. This gives a *virtual* representation of G over $\overline{\mathbb{Q}}_l$, whose trace can be computed by Lefschetz's fixed-point formula.

Definition 5.1.4 *Let* $\mathbf{P}$ *be a parabolic subgroup of* $\mathbf{G}$ *and* $\mathbf{L}$ *a rational Levi subgroup of* $\mathbf{P}$*, so that we have a Levi decomposition* $\mathbf{P} = \mathbf{L}\mathbf{U}$*. Let* $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ *be the functor from the category of left* $\mathbf{L}^F$*-modules to the category of left* $\mathbf{G}^F$*-modules defined by*

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : E \mapsto H_c^*(\mathcal{L}^{-1}(\mathbf{U})) \otimes_{\overline{\mathbb{Q}}_l[\mathbf{L}^F]} E$$

where $\mathcal{L} : \mathbf{G} \to \mathbf{G}$ *is the* Lang map $x \mapsto x^{-1F}x$ *and the action of* $(g, l) \in \mathbf{G} \times (\mathbf{L}^F)^{opp}$ *is induced by that on* $\mathcal{L}^{-1}(\mathbf{U})$ *given by* $x \mapsto gxl$*. The functor* $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ *is called the* Deligne-Lusztig induction*.*

Remark 5.1.5 *In general it is not known whether* $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ *depends on the choice of a parabolic subgroup* $\mathbf{P}$ *or not (latest results on this topic can be found in [BM11]). But when* $\mathbf{L}$ *is a torus the Mackey formula holds for the Deligne-Lusztig induction and it can be proven that* $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ *does not depend on the parabolic subgroup used in its construction. (cf. [DM91, Th. (11.13) and remark after it]). Thus we can use the notation* $R_{\mathbf{L}}^{\mathbf{G}}$ *in that case.*

Remark 5.1.6 *When* $\mathbf{P}$ *is rational then* $\mathbf{U}$ *is also and* $H_c^*(\mathcal{L}^{-1}(\mathbf{U})) \simeq \overline{\mathbb{Q}}_l\left[\mathbf{G}^F/\mathbf{U}^F\right]$ *as* $\mathbf{G} \times (\mathbf{L}^F)^{opp}$*-modules (cf. [DM91, note after (11.1)]), thus Deligne-Lusztig induction is indeed a generalization of Harish-Chandra induction.*

Certain characters associated to representations constructed using Deligne-Lustig induction proved to be very useful in studying the irreducible representations of $\mathbf{G}^F$.

Definition 5.1.7 *When $\mathbf{L} = \mathbf{T}$ is a rational maximal torus, $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ is called a* Deligne-Lusztig character*.*

Recall that we are in a situation when $\mathbf{G}$ contains a rational Borel subgroup $\mathbf{B}$ and a split maximal torus $\mathbf{T}_1$ in it. This allows us to parametrize the $\mathbf{G}^F$-classes of rational tori by the conjugacy classes of W: For a class of $w \in W$, we choose $x \in \mathbf{G}$ such that $\mathcal{L}(x) = \dot{w}$ (we can do it since $\mathcal{L}$ is surjective) where $\dot{w}$ is any representative of w in $N_{\mathbf{G}}(\mathbf{T}_1)$, and we define $\mathbf{T}_w = x\mathbf{T}_1 x^{-1}$. This is a rational maximal torus and it is not hard to see that T_w is well-defined up to $\mathbf{G}^F$-conjugacy.

Proposition 5.1.8 *The $\mathbf{G}^F$-conjugacy class of T_w depends only on the W-conjugacy class of $w \in W$, and this gives a bijection between the $\mathbf{G}^F$-conjugacy classes of rational tori and the conjugacy classes of W.*

Proof. Cf. [DM91, (3.22)] □

5.1.2 Lusztig Series

Here we present a brief sketch of how characters $R_{\mathbf{T}}^{\mathbf{G}}$ lead to a classification of irreducible characters of $\mathbf{G}^F$. We start with the following important definition:

Definition 5.1.9 *Let $\mathbf{T}$ and $\mathbf{T}'$ be two rational tori, and let θ and θ' be characters respectively of $\mathbf{T}^F$ and $\mathbf{T}'^F$. We say that the pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are* geometrically conjugate *if there exists $g \in \mathbf{G}$ such that $\mathbf{T} = {}^g\mathbf{T}'$ and such that for any n satisfying $g \in \mathbf{G}^{F^n}$ we have*

$$\theta \circ N_{F^n/F} = \theta' \circ N_{F^n/F} \circ \mathrm{ad}(g)$$

where $N_{F^n/F} : \mathbf{T} \to \mathbf{T}$ defined by $\tau \mapsto \tau \cdot {}^F\tau \dots {}^{F^{n-1}}\tau$, is the so-called norm *on a torus.*

Remark 5.1.10 *If $g \in \mathbf{G}^F$ in the definition of the geometric conjugacy then we just get the usual* $\mathbf{G}^F$-conjugacy *of two pairs.*

Definition (5.1.9) is justified by the following proposition which gives a decomposition of the set of irreducible characters of $\mathbf{G}^F$ into disjoint subsets.

Proposition 5.1.11

- *For any $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ there exists a rational maximal torus $\mathbf{T}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ such that $\left\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\theta) \right\rangle_{\mathbf{G}^F} \neq 0$.*

- *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be two rational tori, and let* θ *and* θ' *be characters respectively of* $\mathbf{T}^F$ *and* $\mathbf{T}'^F$. *If the virtual representations* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *and* $R_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ *have a common irreducible constituent, then the pairs* $(\mathbf{T}, \theta)$ *and* $(\mathbf{T}', \theta')$ *are geometrically conjugate.*

Proof. Cf. [DM91, (13.1), (13.3)] □
Now, in order to parametrize the geometric conjugacy classes of pairs $(\mathbf{T}, \theta)$ we need to introduce the notion of the dual of a reductive group.

Definition 5.1.12 *Two connected reductive algebraic groups* $\mathbf{G}$ *and* $\mathbf{G}^*$ *are said to be* dual *to each other if there exists a maximal torus* $\mathbf{T}$ *of* $\mathbf{G}$ *(resp.* $\mathbf{T}^*$ *of* $\mathbf{G}^*$*) and an isomorphism from* $X(\mathbf{T})$ *to* $Y(\mathbf{T}^*)$ *which sends the roots of* $\mathbf{G}$ *relative to* $\mathbf{T}$ *to the coroots of* $\mathbf{G}^*$ *relative to* $\mathbf{T}^*$. *If in addition* $\mathbf{G}$ *and* $\mathbf{G}^*$ *are defined over* $\mathbb{F}_q$ *with respective Frobenius maps* F *and* F^*, *and if* $\mathbf{T}$ *and* $\mathbf{T}^*$ *are rational and the isomorphism above is compatible with the actions of* F *and* F^*, *we say that the pair* $(\mathbf{G}, F)$ *is* dual *to the pair* $(\mathbf{G}^*, F^*)$.

Remark 5.1.13 *In particular, two tori* $\mathbf{T}$ *and* $\mathbf{T}^*$ *are said to be dual to each other if we have an isomorphism* $X(\mathbf{T}) \xrightarrow{\sim} Y(\mathbf{T}^*)$. *Further we will usually say that two groups are dual to each other, assuming that we have chosen corresponding dual tori.*

The following proposition gives a parametrization of geometric conjugacy classes.

Proposition 5.1.14 *Assume that* $(\mathbf{G}, F)$ *and* $(\mathbf{G}^*, F^*)$ *are dual. Geometric conjugacy classes of pairs* $(\mathbf{T}', \theta')$ *in* $\mathbf{G}$ *are in one-to-one correspondence with* F^**-stable* $\mathbf{G}^*$*-conjugacy classes of semi-simple elements of* $\mathbf{G}^*$.

Proof. Cf. [DM91, (13.12)] □

Remark 5.1.15 *It is obvious that the character* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *depends only on the* $\mathbf{G}^F$*-conjugacy class of the pair* $(\mathbf{T}, \theta)$. *The* $\mathbf{G}^F$*-conjugacy class of the pair* $(\mathbf{T}', \theta')$ *is in one-to-one correspondence with the* $\mathbf{G}^{*F^*}$*-conjugacy class of a pair* $(\mathbf{T}'^*, s)$ *where* s *is a semi-simple element of* $\mathbf{G}^{*F^*}$ *and* $\mathbf{T}'^*$ *is some rational maximal torus of* $\mathbf{G}^*$ *containing* s *(cf. [DM91, (13.13)]). This allows us to use the notation* $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ *for* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.

The next definition is the first step towards the classification of irreducible characters of $\mathbf{G}^F$.

Definition 5.1.16 *The* Lusztig series $\mathcal{E}(\mathbf{G}^F, (s))$ *associated to the* $\mathbf{G}^*$*-conjugacy class* (s) *of a semi-simple element* $s \in \mathbf{G}^{*F^*}$ *is the set of irreducible characters of* $\mathbf{G}^F$ *which occur is some* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, *where* $(\mathbf{T}, \theta)$ *is in the geometric conjugacy class associated by (5.1.14) to* (s).

5.1.3 Unipotent Characters and Lusztig's Theorem

A particularly important Lusztig series, which is a kind of "prototype" for the other ones, is the series associated to the identity element of $\mathbf{G}^*$.

Definition 5.1.17 *The elements of $\mathcal{E}(\mathbf{G}^F, 1)$ (ie, the irreducible components of $R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}})$) are called* unipotent characters.

We have an inclusion $\mathrm{Irr}(W) \subset \mathcal{E}(\mathbf{G}^F, 1)$ $(\chi \mapsto \rho_\chi)$ defined as follows.

Consider the Hecke algebra $\mathcal{H}(W) = \mathrm{End}_{\mathbf{G}^F}\left(R_{\mathbf{T}_1}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_1})\right) = \mathrm{End}_{\mathbf{G}^F}\left(\overline{\mathbb{Q}}_l\left[\mathbf{G}^F/\mathbf{B}^F\right]\right) = \mathcal{H}(\mathbf{G}^F, \mathbf{B}^F)$ (cf. (2.2.6)). The characters of $\mathcal{H}(W)$ are in bijection with the characters of W, we will denote them by the same letters and write $\mathrm{Irr}W$ instead of $\mathrm{Irr}\mathcal{H}(W)$. We have a classical equality of characters $\overline{\mathbb{Q}}_l\left[\mathbf{G}^F/\mathbf{B}^F\right] = \sum_{\chi \in \mathrm{Irr}W} \chi \otimes \rho_\chi$ where ρ_χ are some different irreducible characters of $\mathbf{G}^F$ (the last equality follows from a decomposition of the $\mathcal{H}(W) \times \overline{\mathbb{Q}}_l[\mathbf{G}^F]$-module $\overline{\mathbb{Q}}_l\left[\mathbf{G}^F/\mathbf{B}^F\right]$ into the sum of simple modules). Characters ρ_χ are unipotent since the equality above implies the decomposition of the character $R_{\mathbf{T}_1}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_1})$ into the sum of irreducible characters: $R_{\mathbf{T}_1}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_1}) = \sum_{\chi \in \mathrm{Irr}W} \chi(1)\rho_\chi$.

Now we will give a statement of the main result [Lus84, Main theorem (4.23)] of Lusztig's classification of characters of finite groups of Lie type for algebraic groups with connected centre. We will present a simplified form of his result, assuming that the algebraic group is split. (This implies that in Lusztig's notations in his theorem (4.23) the automorphism γ is always the identity).

Theorem 5.1.18 (Lusztig) *Let $\mathbf{G}$ be a connected affine algebraic group defined over $\mathbb{F}_q$, split over $\mathbb{F}_q$, with connected centre, and $\mathbf{G}^*$ be the dual group of $\mathbf{G}$;*

- *For a semi-simple element $s \in \mathbf{G}^{*F^*}$, there is a bijection*

$$\mathcal{E}(\mathbf{G}^F, (s)) \leftrightarrow \mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, 1)$$

  and this bijection can be chosen such that, extended by linearity to virtual characters, it sends $R_{\mathbf{T}^*}^{\mathbf{G}}(s)$ to $\varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}^*}(s)} R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(s)}(\mathrm{Id}_{\mathbf{T}^*})$ for any rational maximal torus $\mathbf{T}^*$ of $C_{\mathbf{G}^*}(s)$.

- *There is a set which depends only on W*

$$\mathbf{X}(W) = \coprod_{\mathcal{F}} \mathfrak{M}(\mathcal{G}_{\mathcal{F}})$$

  a disjoint union over disjoint families (two-sided cells) $\mathcal{F}$ of W of the sets $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ of isomorphism classes of finite-dimensional irreducible $\mathcal{G}_{\mathcal{F}}$-equivariant vector bundles over the finite groups $\mathcal{G}_{\mathcal{F}}$ (cf. [Lus84,

(4.2), (4.3)-(4.13), (4.21.1)]);
and a bijection

$$\mathcal{E}(\mathbf{G}^F, 1) \leftrightarrow \mathbf{X}(W), \quad (\rho \leftrightarrow \mathrm{x}_\rho)$$

such that for any $\rho \in \mathcal{E}(\mathbf{G}^F, 1)$ *and any* $\chi \in \mathrm{Irr}(W) \subset \mathcal{E}(\mathbf{G}^F, 1)$ *we have*

$$\langle \rho, R_\chi \rangle_{\mathbf{G}^F} = \Delta(\mathrm{x}_\rho)\, \{\mathrm{x}_\rho, \mathrm{x}_\chi\}$$

where W *is the Weyl group of* $\mathbf{G}$ *with respect to the split torus* $\mathbf{T}_1$*;*
$\varepsilon_{\mathbf{G}} = (-1)^{(\mathbb{F}_q-\text{rank of } \mathbf{G})}$ *(cf. [DM91, remark after (8.11)]);*
the class functions R_χ *are defined by the following formula:*

$$R_\chi = |W|^{-1} \sum_{w \in W} \chi(w) R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id})$$

the map $\Delta : \mathbf{X}(W) \to \{\pm 1\}$ *is almost always* 1 *(cf. [Lus84, (4.14)]);*
the pairing $\{\ ,\ \} : \mathbf{X}(W) \times \mathbf{X}(W) \to \overline{\mathbb{Q}}_l$ *is defined by Lusztig in [Lus84, (4.14.3), (4.21.2)]. We will say more about this pairing later in the subsection (5.2.2).*

Remark 5.1.19 *The decomposition of the characters* R_χ *provides the decomposition of Deligne-Lusztig characters* $R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id})$ *since* $R^{\mathbf{G}}_{\mathbf{T}_w}(\mathrm{Id}) = \sum_{\chi \in \mathrm{Irr}(W)} \chi(w^{-1}) R_\chi$*, which in turn gives the decomposition of any Deligne-Lusztig character* $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ *by the first part of Lusztig's theorem.*

5.1.4 Lusztig Fourier Transform Matrix

Definition-Theorem 5.1.20 *Let* S *be a matrix whose lines and columns are indexed by the set* $\mathbf{X}(W)$*, such that* $S_{\mathrm{x},\mathrm{y}} = \{\mathrm{x}, \mathrm{y}\}$ *for any* $\mathrm{x}, \mathrm{y} \in \mathbf{X}(W)$*. Then* S *is a symmetric, unitary matrix, independent of* q*. It is called the* Lusztig Fourier transform *matrix (or simply* Fourier matrix*).*

Remark 5.1.21 *The matrix* S *is block diagonal with "small" diagonal blocks indexed by two-sided cells of* W*. The set of indices of the block corresponding to a cell* $\mathcal{F}$ *is the set* $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ *(also called* Lusztig family*) and its entries are given by a formula which uses only the character theory of* $\mathcal{G}_{\mathcal{F}}$ *and its subgroups.*

Remark 5.1.22 *Lusztig has completed the picture in [Lus85] by introducing class functions on* $\mathbf{G}^F$*, the characteristic functions of* character sheaves *which complete the set* $\{R_\chi | \chi \in \mathrm{Irr}(W)\}$ *to a basis of the subspace spanned by the unipotent characters. There is a function* R_{x} *attached to each* $x \in \mathbf{X}(W)$ *and the matrix* S *is the matrix of scalar products* $\langle R_{\mathrm{x}}, \rho_{\mathrm{x}'} \rangle_{\mathbf{G}^F}$ *for* $\mathrm{x}, \mathrm{x}' \in \mathbf{X}(W)$*.*

5.1.5 Character Degrees and Fourier Matrix

In this section we give one important property of the Fourier matrix relating fake degrees to generic degrees. We start with necessary definitions.

For a given Weyl group W and q any power of a prime number there is $\mathbf{G}$ connected affine algebraic group over $\overline{\mathbb{F}}_q$ with connected centre and split maximal torus $\mathbf{T}_1$ such that $W = N_{\mathbf{G}}(\mathbf{T}_1)/\mathbf{T}_1$. Thus, to any $\chi \in \mathrm{Irr}(W)$ we can associate a virtual character $R_\chi : \mathbf{G}^F \to \overline{\mathbb{Q}}_l$. One can show that $R_\chi(1) \in \mathbb{Z}$ and there is a polynomial $f_\chi \in \mathbb{Z}[\mathbf{q}]$ in a variable $\mathbf{q}$ such that $f_\chi(q) = R_\chi(1)$ for any q power of a prime.

Definition 5.1.23 *The polynomial $f_\chi \in \mathbb{Z}[\mathbf{q}]$ is called the* fake degree *of χ.*

Consider the complex vector space $V = X(\mathbf{T}_1) \otimes \mathbb{C}$ then we have an induced action of W on the symmetric algebra $S(V)$ and by (1.2.3) the *coinvariant* algebra $S_W := S(V)/I$, where I is the graded ideal generated by the W-invariants $S(V)^W$ of positive degree, is a graded version of the regular representation of W. This gives an alternative definition of the fake degree:

Definition 5.1.24 *For any $\chi \in \mathrm{Irr}(W)$ the* fake degree $f_\chi := \langle S_W, \chi \rangle_W$ *is the graded multiplicity of χ in the coinvariant algebra S_W.*

Remark 5.1.25 *On the other hand one can show that if $\mathrm{x} \in \mathbf{X}(W)$ and $\mathrm{x} \notin \mathrm{Irr}(W)$ then $R_{\mathrm{x}}(1)$ is always zero (always means: for any algebraic group as above). Thus we can think that the fake degrees $f_{\mathrm{x}}(\mathbf{q})$ of the corresponding unipotent characters are zero.*

Now we pass to the definition of the generic degrees. Again for a given W and q we have an algebraic group as above. Let $\mathbf{B}$ be a rational Borel subgroup containing the split torus $\mathbf{T}_1$. Then the pair $(\mathbf{B}^F, N_{\mathbf{G}}(\mathbf{T}_1)^F)$ is a (B, N)-pair with the corresponding Weyl group W (cf. [DM91, (1.3)]). Thus the Hecke algebra $\mathcal{H}(W) = \mathrm{End}_{\mathbf{G}^F}\left(R^{\mathbf{G}}_{\mathbf{T}_1}(\mathrm{Id})\right)$ has a natural basis $\{T_w\}_{w \in W}$ which comes from the Bruhat decomposition (cf. (2.2.6)).

The function $\tau : T_w \mapsto \mathrm{Trace}\left(T_w | R^{\mathbf{G}}_{\mathbf{T}_1}(\mathrm{Id})\right)$ is a symmetrizing trace on $\mathcal{H}(W)$. We have $\tau(T_w) = 0$ if $w \neq 1$, $\tau(T_1) = \dim R^{\mathbf{G}}_{\mathbf{T}_1}(\mathrm{Id})$. Since $\mathcal{H}(W)$ splits over $\mathbb{Z}[\sqrt{q}, \sqrt{q}^{-1}]$, one can attach a Schur element $S_\chi \in \mathbb{Z}[\sqrt{q}, \sqrt{q}^{-1}]$ to every $\chi \in \mathrm{Irr}\mathcal{H}(W)$ (cf. (2.2.29) and (2.2.37)). We have $\rho_\chi(1) = \dim R^{\mathbf{G}}_{\mathbf{T}_1}(\mathrm{Id})/S_\chi$ and the fact that $\rho_{\mathrm{Id}} = 1$ implies that $S_{\mathrm{Id}} = \dim R^{\mathbf{G}}_{\mathbf{T}_1}(\mathrm{Id})$.

One can prove that for every $\chi \in \mathrm{Irr}(W)$ there is a polynomial $g_\chi(\mathbf{q})$ such that $g_\chi(q) = S_{\mathrm{Id}}/S_\chi = \deg \rho_\chi$ for any q power of a prime.

Definition 5.1.26 *The polynomial g_χ is called the* generic degree *of χ.*

Remark 5.1.27 *The definition of the Fourier matrix S implies that for any algebraic group $\mathbf{G}$ and a power q as above, S transforms the vector*

$(R_{\mathrm{x}}(1))_{\mathbf{X}(W)} = (f_{\mathrm{x}}(q))_{\mathbf{X}(W)}$ *to the vector* $(\rho_{\mathrm{x}}(1))_{\mathbf{X}(W)}$ *of degrees of unipotent characters. Since the matrix* S *depends only on* W *we obtain that* generic degrees of unipotent characters *can be defined as polynomials* $g_{\mathrm{x}}(\mathbf{q})$ *such that* $g_{\mathrm{x}}(q) = \rho_{\mathrm{x}}(1)$ *for any power of a prime* q.

Thus we can state:

Proposition 5.1.28 *The Fourier matrix* S *transforms the vector of fake degrees, completed by zero, to the vector of generic degrees of unipotent characters. In particular, its top left (square) part transforms fake degrees to generic degrees.*

Remark 5.1.29 *It is possible to determine combinatorially the Fourier matrix starting from* W*, using the above property and a few others, like symmetry of* S *and* $S^{-1} = S$*, etc. This allows to generalize the notion of Fourier matrix to finite Coxeter groups and to certain classes of complex reflection groups. We will talk about it in the next sections.*

5.2 Case of Non-Rational Coxeter Groups

As we have seen in the previous section the set $\mathcal{E}(\mathbf{G}^F, 1) \simeq \mathbf{X}(W)$ of unipotent characters of a group $\mathbf{G}^F$ and the corresponding Fourier matrix depend only on the Weyl group associated to $\mathbf{G}$. In the course of his classification Lusztig observed that similar sets and matrices can also be attached to some non-rational finite Coxeter groups, just as if there was a "fake algebraic group" whose Weyl group is non-rational. In this section we will briefly describe his constructions.

5.2.1 Unipotent Characters and their Degrees

Let W be a finite Coxeter group. We start with the construction of a finite set $\mathcal{U}(W)$ and a function Deg which associates to each $\rho \in \mathcal{U}(W)$ a polynomial $\mathrm{Deg}(\rho) \in \mathbb{R}[q]$, which in the case of a Weyl group is equal to the generic degree $g_{\mathrm{x}_\rho}(q)$ of a unipotent representation ρ (cf Remark (5.1.27)).

First of all Lusztig postulates that if W and W' are two finite Coxeter groups then $\mathcal{U}(W \times W') = \mathcal{U}(W) \times \mathcal{U}(W')$ and $\mathrm{Deg}(\rho, \rho') = \mathrm{Deg}(\rho)\mathrm{Deg}(\rho')$ for $\rho \in \mathcal{U}(W)$ and $\rho' \in \mathcal{U}(W')$ [Lus93, Postulate (2.3)]. In the case of Weyl groups these bijections are given by external tensor product of unipotent representations.

Since the case where W is a Weyl group is understood, we see using the above postulate that it is enough to define $\mathcal{U}(W)$ and Deg in the case when W is irreducible and non-rational, hence a dihedral group $I_2(p)$ of order $2p$ ($p = 5$ or $p \geq 7$) or a Coxeter group of type H_3 or H_4 (cf. (1.4.8)).

Notation 5.2.1 *By analogy with the rational case we will use the word* representations *for the elements of* $\mathcal{U}(W)$.

Lusztig starts with the case where W is one of the groups $H_2 = I_2(10), H_3$ or H_4. First he constructs a set $\mathcal{U}(W, \varnothing, 1)$ as the set of irreducible representations of the Hecke algebra $\mathcal{H}_{q,-1}$ whose generic degrees are known. After this using his postulates [Lus93, (2.2)] and [Lus93, (2.4)-(2.7)] (which hold in the rational case) he extends the set $\mathcal{U}(W, \varnothing, 1)$ constructing new representations and their degrees. Finally , using his postulate [Lus93, (2.8)] about the sum $\sum_{\rho\in\mathcal{U}(W)} \mathrm{Deg}(\rho)^2$ he shows that the set of constructed representations exhausts $\mathcal{U}(W)$.

The second case is the case of dihedral groups. Let W be a dihedral group $I_2(p)$ where $p \geq 3$. Define an integer $k \geq 1$ by $p = 2k + 1$ if p is odd and by $p = 2k + 2$ if p is even. By methods similar to those used for the groups H_2,H_3 and H_4 Lusztig obtains that, if $p = 2k + 1$, then $\mathcal{U}(W)$ consists of two objects $1, \sigma$ and of k^2 other objects denoted ρ_i $(1 \leq i \leq k)$ and $\rho_{i,j}, \rho'_{i,j}$ $(1 \leq i < j \leq k)$; if $p = 2k+2$, then $\mathcal{U}(W)$ consists of two objects $1, \sigma$ and k^2+k+2 objects denoted ρ_i, ρ'_i $(1 \leq i \leq k)$, $\rho_{i,j}, \rho'_{i,j}$ $(1 \leq i < j \leq k)$ and ϵ', ϵ''. The values of Deg are given as follows.

$$
\begin{aligned}
&\mathrm{Deg}(1) = 1, \mathrm{Deg}(\sigma) = q^p,\\
&\mathrm{Deg}(\rho_i) = \frac{(1-\xi^i)(1-\xi^{-i})}{p}\frac{q(1-q^2)(1-q^p)}{(1-q)^2(q-\xi^i)(q-\xi^{-i})},\\
&\mathrm{Deg}(\rho'_i) = \frac{(1+\xi^i)(1+\xi^{-i})}{p}\frac{q(1-q^2)(1-q^p)}{(1+q)^2(q-\xi^i)(q-\xi^{-i})},\\
&\mathrm{Deg}(\rho_{i,j}) = \frac{\xi^i+\xi^{-i}-\xi^j-\xi^{-j}}{p}\frac{q(1-q^2)(1-q^p)}{(q-\xi^i)(q-\xi^{-i})(q-\xi^j)(q-\xi^{-j})},\\
&\mathrm{Deg}(\rho'_{i,j}) = \mathrm{Deg}(\rho_{i,j}),\\
&\mathrm{Deg}(\epsilon') = \mathrm{Deg}(\epsilon'') = \frac{2}{p}\frac{q(1-q^2)(1-q^p)}{(1-q^2)^2}.
\end{aligned}
$$

where $\xi = e^{2\pi i/p}$.

Remark 5.2.2 *During his construction Lusztig obtains an analogue of the inclusion* $\mathrm{Irr}(W) \subset \mathcal{E}(\mathbf{G}^F, 1)$ *(cf. 5.1.3) identifying IrrW with* $\mathcal{U}(W, \varnothing, 1) \subset \mathcal{U}(W)$.

Now we pass to the construction of the Fourier transform matrix.

5.2.2 Fusion Datum and Fourier Matrix

Every modular tensor category [BK01, Chapter 3] provides data known as a *fusion datum*. A fusion datum in turn contains a Fourier Matrix corresponding to it. The following definition can be found in [Lus94, (1.1)].

Definition 5.2.3 (Fusion datum) *Let X be a finite set with a given element x_0 and with given commuting involutions $^\# : X \to X$ and $^\flat : X \to X$ such that $x_0^\# = x_0^\flat = x_0$. Set $\mathbf{S} = (s_{x,y})_{x,y\in X}$ be a matrix with complex* *entries and let* $\mathbf{t} = (t_x)_{x\in X}$ *be a collection of non-zero complex numbers indexed by* X. *We say that* $(X, x_0, {}^{\#}, {}^{\flat}, \mathbf{S}, \mathbf{t})$ *is a* fusion datum *if the conditions (a)-(g) below are satisfied.*

(a) $s_{x,y} = s_{x^{\#},y^{\flat}} = s_{x^{\flat},y^{\#}} = s_{y^{\#},x^{\#}} = s_{y^{\flat},x^{\flat}} = \overline{s}_{y,x}$, *for all* $x, y \in X$*;*

(b) $t_{x^{\flat}} = t_{x^{\#}} = t_x^{-1} = \overline{t}_x$, *for all* $x \in X$*;*

(c) $\sum_{z\in X} s_{x,z}s_{z,y} = \delta_{x,y}$, *for all* $x, y \in X$*;*

(d) $s_{x,x_0} \in \mathbb{R}_{>0}$, *for all* $x \in X$*;*

(e) $t_{x_0} = 1$*;*

(f) $\sum_{u\in X} \frac{s_{x,u}s_{y,u}s_{z,u}}{s_{x_0,u}} \in \mathbb{N}$, *for all* $x, y, z \in X$*;*

(g) $\sum_{x,y\in X} s_{u,x^{\flat}} s_{x,y^{\flat}} s_{y,z^{\flat}} t_x^{-1}t_y^{-1}t_z^{-1} = \delta_{u,z}$, *for all* $u, z \in X$.

The matrix $\mathbf{S}$ *is called the* Fourier transform *corresponding to the fusion datum.*

Now we explain how a Fusion datum appears in the case of Weyl groups. Recall that in the rational case we have a decomposition $\mathbf{X}(W) = \coprod_{\mathcal{F}} \mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ where $\mathfrak{M}(\mathcal{G}_{\mathcal{F}})$ are isomorphism classes of finite-dimensional irreducible $\mathcal{G}_{\mathcal{F}}$-equivariant vector bundles over some finite groups $\mathcal{G}_{\mathcal{F}}$ and we have a pairing $\{\ ,\ \} : \mathbf{X}(W) \times \mathbf{X}(W) \to \overline{\mathbb{Q}}_l$ which provides the decomposition of the characters R_χ in irreducible characters.

Let $\mathcal{G}$ be a finite group. Consider the modular tensor category of finite-dimensional $\mathcal{G}$-equivariant vector bundles over $\mathcal{G}$ where $\mathcal{G}$ acts on itself by conjugation. It is easy to see that the set $\mathfrak{M}(\mathcal{G})$ of isomorphism classes of irreducible $\mathcal{G}$-equivariant vector bundles is in one-to-one correspondence with the set of pairs (a, σ) where $a \in \mathcal{G}$ and σ is the character of an irreducible representation of the centralizer $C_{\mathcal{G}}(a)$, modulo the equivalence given by $(a, \sigma) \sim (gag^{-1}, \sigma^g)$ where $g \in G$.

Let $x_0 \in \mathfrak{M}(\mathcal{G})$ be the orbit of $(1, \mathbb{1}) \in \mathfrak{M}(\mathcal{G})$. Let ${}^{\#} : \mathfrak{M}(\mathcal{G}) \to \mathfrak{M}(\mathcal{G})$ and ${}^{\flat} : \mathfrak{M}(\mathcal{G}) \to \mathfrak{M}(\mathcal{G})$ be given by $(a, \sigma) = (a, \overline{\sigma})$ (here $\overline{\sigma}$ is the composition of σ and any automorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q})$ sending roots of unity to their inverses) and $(a, \sigma)^{\flat} = (a^{-1}, \sigma)$.

Let $\mathbf{S} = (\{m, m'\})$ be the matrix indexed by elements $m, m' \in \mathfrak{M}(\mathcal{G})$ given by

$$\{m, m'\} = \sum_{g\in\mathcal{G};aga'g^{-1}=ga'g^{-1}a} |C_{\mathcal{G}}(a)|^{-1}|C_{\mathcal{G}}(a')|^{-1}|\sigma'(g^{-1}a^{-1}g)\sigma(ga'g^{-1})$$

where (a, σ) is any representative of m and (a', σ') is any representative of m'.

Let $\mathbf{t} = (t_m)$ be defined by $t_m = \sigma(a)/\sigma(1)$ for any $m \in \mathfrak{M}(\mathcal{G})$ and any representative (a, σ) of m.

Proposition 5.2.4 *$X = \mathfrak{M}(\mathcal{G})$ together with $x_0, {}^{\#}, {}^{\flat}, \mathbf{S}, \mathbf{t}$ as above is a fusion datum.*

Proof. Cf. [Lus94, Prop. 1.6] □
Now we will precise Remark (5.1.21). Namely, to any two-sided cell $\mathcal{F}$ one can associate a finite group $\mathcal{G}_{\mathcal{F}}$ and the entries of the block corresponding to $\mathcal{F}$ of Lusztig's Fourier matrix are given by the Fourier matrix corresponding to the fusion datum $(\mathfrak{M}(\mathcal{G}_{\mathcal{F}}), x_0, {}^{\#}, {}^{\flat}, \mathbf{S}, \mathbf{t})$ described above.

The notion of fusion datum proved to be very convenient for the construction of a Fourier transform matrix even in the non-rational case. Here we present the construction of Lusztig in the case of dihedral groups $I_2(p)$ where $p \geq 3$.

In this case there are only three two-sided cells: the cell containing the trivial character, the cell containing the character $w \mapsto (-1)^{l(w)}$ and the cell containing all other characters. The blocks corresponding to the first two cells are trivial. We only need to define the block of the Fourier matrix corresponding to the "big" cell. It will be given by the Fourier matrix corresponding to the following fusion datum.

Let p be an integer ≥ 3. Let X' be the set consisting of all ordered pairs (i, j) of integers such that either

(a) $0 < i < j < p$ and $i + j < p$, or

(b) $0 = i < j < \frac{p}{2}$.

Let X'' be the set consisting of the two elements $(0, p/2)', (0, p/2)''$, if p is even, and let $X'' = \varnothing$, if p is odd. Let X be the disjoint union $X' \sqcup X''$.

Let $x_0 = (0, 1) \in X'$ and $x \mapsto x^{\#} = x^{\flat}$ be the involution of X defined by $(i, j)^{\#} = (i, j)^{\flat} = (i, p - j)$ if $(i, j) \in X'$ and $i > 0$; $x^{\#} = x^{\flat} = x$ for all other $x \in X$. Let $\xi = e^{2\pi i/p}$. Let $\mathbf{S} = (\{x, x'\})_{x,x' \in X}$ be the matrix defined by

$$\{(i, j), (k, l)\} = \frac{\xi^{kj-li} + \xi^{-kj+li} - \xi^{lj-ki} - \xi^{-lj+ki}}{p}$$

if $(i, j) \in X'$ and $(k, l) \in X'$;

$$\begin{aligned} &\{(i, j), (0, p/2)'\} = \{(i, j), (0, p/2)''\} \\ &= \{(0, p/2)', (i, j)\} = \{(0, p/2)'', (i, j)\} = \frac{(-1)^i - (-1)^j}{p} \end{aligned}$$

if $(i, j) \in X'$ and p is even;

$$\begin{aligned} \{(0, p/2)', (0, p/2)'\} = \{(0, p/2)'', (0, p/2)''\} &= \frac{1 - (-1)^{p/2} + p}{2p}, \\ \{(0, p/2)', (0, p/2)''\} = \{(0, p/2)'', (0, p/2)'\} &= \frac{1 - (-1)^{p/2} - p}{2p}. \end{aligned}$$

if p is even.

Let $\mathbf{t} = (t_x)_{x \in X}$ be defined by $t_{(i,j)} = \xi^{-ij}$ if $(i,j) \in X'$ and by $t_x = 1$ if $x \in X''$. Then we have the following proposition [Lus94, Prop. 3.2]:

Proposition 5.2.5 $(X, x_0, {}^{\#}, {}^{\flat}, \mathbf{S}, \mathbf{t})$ *just defined is a fusion datum. It is called a* dihedral fusion datum.

In order to construct the Fourier matrix of $I_2(p)$ we define the following bijection between the set $\mathcal{U}(I_2(p)) - \{1, \sigma\}$ (Lusztig family corresponding to the "big" cell) and the set X which appears in the dihedral fusion datum corresponding to p.

- Case $p = 2k+1$: $\begin{cases} \rho_i \leftrightarrow (0,i) \text{ for } 1 \le i \le k, \\ \rho_{i,j} \leftrightarrow (i,j) \text{ for } 1 \le i < j \le k, \\ \rho'_{i,j} \leftrightarrow (i, p-j) \text{ for } 1 \le i < j \le k. \end{cases}$

- Case $p = 2k+2$: $\begin{cases} \rho_i \leftrightarrow (0,i) \text{ for } 1 \le i \le k, \\ \rho'_i \leftrightarrow (i,k+1) \text{ for } 1 \le i \le k, \\ \rho_{i,j} \leftrightarrow (i,j) \text{ for } 1 \le i < j \le k, \\ \rho'_{i,j} \leftrightarrow (i, p-j) \text{ for } 1 \le i < j \le k, \\ \epsilon' \leftrightarrow (0, p/2)', \\ \epsilon'' \leftrightarrow (0, p/2)''. \end{cases}$

Definition 5.2.6 *Define a block-diagonal matrix S indexed by the set $\mathcal{U}(I_2(p))$ with three blocks: two one-dimensional trivial blocks corresponding to the representations* 1 *and σ and one "big" block indexed by the set $\mathcal{U}(I_2(p)) - \{1, \sigma\}$, entries of S in this block are obtained via the above bijection from the entries of the Fourier matrix* **S** *corresponding to the dihedral fusion datum. Then the matrix S is called the* Fourier matrix *of $I_2(p)$.*

Remark 5.2.7 *The third property of the Fourier matrix corresponding to a fusion datum (cf. (c) in (5.2.3)) implies that the Fourier matrix of $I_2(p)$ is equal to its inverse. (Cf. 5.1.29).*

Proposition 5.2.8 *The Fourier matrix just constructed transforms the vector* $(\mathrm{Deg}(\rho))_{\mathcal{U}(I_2(p))}$ *formed by the degrees of unipotent representations constructed in the previous section (5.2.1) into the vector formed by the fake degrees (which are defined in the non-rational case) completed by zeros. And has many other properties of Fourier matrices corresponding to Weyl groups.*

Remark 5.2.9 *Fourier matrices with the necessary properties were also constructed for the groups H_3 and H_4 by Lusztig and Malle [Lus94, Appendix]. Since the Fourier matrix of a direct product of two Coxeter groups can be given as a tensor product of corresponding Fourier matrices we have a construction of Fourier matrix for any finite Coxeter group.*

5.3 G(e,1,n) Case

In this section, following Malle [Mal95], we present a combinatorial contruction in the case $W_n = G(e,1,n)$ of the set parametrizing the "unipotent representations" and give formulae for their generic degrees . We also give Malle's contruction of the Fourier matrix. When $e \leq 2$ the group W_n is of type A_{n-1} (the symmetric group S_n) or of type B_n (hyperoctahedral group) and Malle's constructions and formulae agree with already defined $\mathbf{X}(W_n)$, $(g_{\mathrm{x}}(q))_{\mathrm{x}\in\mathbf{X}(W_n)}$ and Fourier matrices S for Weyl groups.

5.3.1 e-Symbols

We define the main combinatorial object which is used in Malle's constructions and definitions.

Definition 5.3.1 *An ordered family $S = (S_0, \ldots, S_{e-1})$ of e finite sequences $S_i = (0 \leq \lambda_{i,1} < \lambda_{i,2} < \ldots < \lambda_{i,m_i})$ of strictly increasing non-negative integers is called an e-*Symbol *and presented as follows:*

$$S = \begin{pmatrix} \lambda_{0,1} & \ldots & \lambda_{0,m_0} \\ \lambda_{1,1} & \ldots & \lambda_{1,m_1} \\ \vdots & & \vdots \\ \lambda_{e-1,1} & \ldots & \lambda_{e-1,m_{e-1}} \end{pmatrix}. \tag{5.3.2}$$

Now we define different notions related to e-Symbols (or simply, *Symbols*, when it is clear what e we are talking about).

Definition 5.3.3 • *The* content *of the Symbol S in (5.3.2) is $I(S) := m_0 + m_1 + \ldots + m_{e-1}$.*

- *The* rank *of S is*

$$\mathrm{rg}(S) := \sum_{i,j} \lambda_{i,j} - \left\lfloor \frac{(I(S)-1)(I(S)-e+1)}{2e} \right\rfloor.$$

- *For a symbol S with $I(S) \equiv 1 (\mathrm{mod}\ e)$ define*

$$\mathrm{def}(S) = \frac{(e-1)(I(S)-1)}{2} - \sum_{i=0}^{e-1} i|S_i| \pmod{e}.$$

Such a Symbol is called reduced *if* $\mathrm{def}(S) = 0$ *and not all $\lambda_{i,1}$ are zero.*

5.3.2 Unipotent Representations and Generic Degrees.

Now we can introduce Malle's definition of the set of "unipotent representations" and their degrees.

Definition 5.3.4 *The set*

$$\mathcal{U}(W_n) := \{ \text{ reduced } e\text{-Symbols } S \quad | \quad \mathrm{rg}(S) = n, I(S) \equiv 1 \pmod{e}\}$$

is called the set of unipotent representations *of the group* W_n*. Their* generic degrees *are defined by the following formula:*

$$\mathrm{Deg}_S(q) = \frac{(-1)^{\binom{e}{2}\binom{m}{2}} \prod_{i=1}^{n}(q^{ei}-1) \cdot \prod_{i=0}^{e-1}\prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu)\in S_i\times S_j \\ \mu<\lambda \text{ if } i=j}} (q^\lambda\zeta^i - q^\mu\zeta^j)}{\tau(e)^m q^{\binom{e(m-1)+1}{2}+\binom{e(m-2)+1}{2}+\cdots} \cdot \prod_{i=0}^{e-1}\Theta(S_i, q^e)}$$

where

$$m = \frac{I(S)-1}{e}; \quad \tau(e) = \prod_{i=0}^{e-1}\prod_{j=i+1}^{e-1}(\zeta^i - \zeta^j) = \sqrt{-1}^{\binom{e-1}{2}}\sqrt{e}^e$$

and for any finite $A \subset \mathbb{Z}$

$$\Theta(A, q) = \prod_{\substack{\lambda\in A \\ \lambda\geq 1}}\prod_{h=1}^{\lambda}(q^h - 1).$$

Remark 5.3.5 *In his article [Mal95] Gunter Malle gives many evidences supporting the rightness of his construction: he proves that his set* $\mathcal{U}(W_n)$ *and degree* Deg *have properties analogous to those of unipotent representations and their generic degrees in rational case and there are analogues of Harish-Chandra-Theory and of the Fourier matrix. We will talk about his construction of the Fourier matrix later.*

5.3.3 Irreducible Representations and their Degrees

By analogy with the real case one can define an inclusion $\mathrm{Irr}(W_n) \subset \mathcal{U}(W_n)$. Recall that $\mathrm{Irr}(W_n)$ is parametrized by e-partitions of n (cf. (3.3.4)). Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$ be any e-partition of n. By adding zeros if necessary we can think that $\alpha_0 = (0 \leq \alpha_{0,1} \leq \ldots \leq \alpha_{0,m+1})$ and $\alpha_i = (0 \leq \alpha_{i,1} \leq \ldots \leq \alpha_{i,m})$ for $i > 0$ and choose m such that not all $\alpha_{i,1}$ are zero. Define a symbol $S_{\boldsymbol{\alpha}} = (S_0, \ldots, S_{e-1})$ by $S_{i,j} = \alpha_{i,j} + j - 1$. Then $S_{\boldsymbol{\alpha}}$ is a reduced Symbol with $I(S_{\boldsymbol{\alpha}}) = em + 1$ and we obtain an inclusion

$$\mathrm{Irr}(W_n) \hookrightarrow \mathcal{U}(W_n) \quad : \quad \chi_{\boldsymbol{\alpha}} \mapsto S_{\boldsymbol{\alpha}}. \tag{5.3.6}$$

Definition 5.3.7 *For an e-partition* $\boldsymbol{\alpha}$ *we call* $\gamma_{\chi_{\boldsymbol{\alpha}}}(q) := \mathrm{Deg}_{S_{\boldsymbol{\alpha}}}(q)$ *the* generic degree *of the character* $\chi_{\boldsymbol{\alpha}}$.

By analogy with the rational case, the generic degree of a character can be obtained via the symmetrizing trace on a Hecke algebra. Consider the spetsial Hecke algebra $\mathcal{H}$ obtained from the generic Hecke algebra $\mathcal{H}_n^{(e)}(q,-1,u_0,u_1,\dots,u_{e-1})$ (cf. (3.3)) via the specialization

$$u_0 \mapsto q, \quad u_i \mapsto \zeta^i \quad (1 \le i \le e-1).$$

Then $\mathcal{H}$ is a symmetric algebra over $\mathbb{C}[q^{\pm 1}]$, split semisimple over $\mathbb{C}(q)$ and free as a $\mathbb{C}[q^{\pm 1}]$-module. The corresponding symmetrizing form $t_0 : \mathcal{H} \to \mathbb{C}[q^{\pm 1}]$ can be given by $t_0(\mathbf{T_w}) = \delta_{w,1}$ where $\mathbf{w}$ is any reduced expression of $w \in W_n^{(e)}$ (with respect to $S_n^{(e)}$) and $\mathbf{T_w}$ denotes the corresponding product of the standard generators of $\mathcal{H}$ (cf. [GIM00, (4.4)]). By [GIM00, lemma(4.3)]) this form can also be defined as the Markov trace in the sense of (3.3.1) with all parameters equal to zero. The theorem (3.3.5) implies the following:

Proposition 5.3.8 *Let*

$$t_0 = \sum_{\chi \in \mathrm{Irr} W_n} \frac{\chi^{(q)}}{S_\chi}$$

be the decomposition of the symmetrizing form t_0 *into the sum of irreducible characters of* $\mathcal{H}$ *(*$\chi \mapsto \chi^{(q)}$ *means the canonical bijection* $\mathrm{Irr} W_n \to \mathrm{Irr}\mathcal{H}$*). Then we have* $\gamma_\chi(q) = S_{\mathbb{1}}/S_\chi$ *(cf. 5.1.26).*

Let V be the reflection representation of W_n.

Definition 5.3.9 *Let* $S(V^*)_{W_n} = S(V^*)/I$ *be the coinvariant algebra of* W_n *(here* V^* *is the dual space of* V *and* I *is the ideal of* $S(V^*)$ *generated by the homogeneous elements of* $S(V^*)^{W_n}$ *of positive degree). For an e-partition* $\boldsymbol{\alpha}$ *the graded multiplicity* $R_{\chi_{\boldsymbol{\alpha}}}$ *of the character* $\chi_{\boldsymbol{\alpha}}$ *in* $S(V^*)_{W_n}$ *is called the* fake degree *of* $\chi_{\boldsymbol{\alpha}}$*(cf. (5.1.23)).*

Remark 5.3.10 *In the above definition we use* V^* *instead of* V *following the usual notation for complex reflection groups.*

The following proposition (cf. [Mal95, (2.10)]) gives the formula for fake degrees.

Proposition 5.3.11 *For an e-Symbol* S *such that* $|S_0| = m+1, |S_1| = \ldots = |S_{e-1}| = m$ *define an element* $\delta_S(q) \in \mathbb{C}(q)$ *by*

$$\delta_S(q) = \prod_{i=0}^{e-1} \frac{\Delta(S_i, q^e)}{\Theta(S_i, q^e)} \cdot \frac{\prod_{i=1}^{e-1} \prod_{\lambda \in S_i} q^{(e-i)\lambda}}{q^{\binom{e(m-1)+1}{2} + \binom{e(m-2)+1}{2} + \ldots}},$$

where for any finite $A \subset \mathbb{Z}$

$$\Delta(A, q) = \prod_{\substack{\lambda,\lambda' \in A \\ \lambda' < \lambda}} (q^{\lambda} - q^{\lambda'}).$$

Then the fake degree $R_{\chi_{\boldsymbol{\alpha}}}$ of the character $\chi_{\boldsymbol{\alpha}}$ corresponding to the e-partition $\boldsymbol{\alpha}$ of n is equal to

$$\prod_{i=1}^{n} (q^{ei} - 1) \delta_{S_{\boldsymbol{\alpha}}}(q).$$

Remark 5.3.12 *Gunter Malle uses the symmetric algebra $S(V)$ in his definition of fake degrees thus we had to slightly change his formula.*

5.3.4 Lusztig Families and Fourier Transform Matrix

Now we pass to Malle's combinatorial definition of the Lusztig families of the group W_n.

Definition 5.3.13 *Two Symbols S and S' lie in the same* Family *if and only if as multisets S and S' contain the same numbers with the same multiplicities.*

Definition 5.3.14 *Families of Symbols induce a decomposition of the set $\mathcal{U}(W_n)$ into the union of disjoint subsets which are called* Lusztig Families.

Remark 5.3.15 *One can check that whenever W_n is a Coxeter group, the above definition coincides with the definition of the Lusztig families in the real case.*

Definition 5.3.16 *By analogy we say that two irreducible characters $\chi_{\boldsymbol{\alpha}}, \chi_{\boldsymbol{\beta}}$ are in the same* family *(in the real case one says that they* belong to the same two-sided cell*) if the corresponding reduced Symbols $S_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\beta}}$ belong to the same Family.*

We continue with Malle's definition of the Fourier transform matrix.

Let Y be a totally ordered set with $em + 1$ elements, $m > 0$; $\Psi = \Psi(Y)$ is the set of maps

$$\psi : Y \to \{0, \dots, e-1\}, \text{ such that } \sum_{y \in Y} \psi(y) \equiv m \binom{e}{2} \pmod{e},$$

with the subset Ψ_0 of Ψ such that $|\psi^{-1}(i)| = m + \delta_{0,i}$.

There is a natural involution on Ψ

$$^{-} : \Psi \to \Psi, \quad \psi \mapsto \overline{\psi} \text{ with } \overline{\psi}(y) := e - \psi(y) \pmod{e}$$

and a symmetric pairing $\langle\ ,\ \rangle$ on Ψ given by

$$\langle \phi, \psi \rangle = \epsilon(\phi)\epsilon(\psi) \prod_{y \in Y} \zeta^{-\phi(y)\psi(y)}$$

where $\zeta = e^{2\pi i/e}$ and

$$\epsilon(\psi) = (-1)^{c(\psi)}, \quad c(\psi) = |\{(y, y') \in Y \times Y | y < y', \psi(y) < \psi(y')\}|.$$

Definition 5.3.17 *Consider the space $H := R^{\Psi}$ of functions from the set Ψ to some $\mathbb{C}$-algebra R. The operator*

$$T : H \to H, \quad T(f)(\phi) := \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\psi \in \Psi} \langle \phi, \psi \rangle\, f(\psi)$$

is called the Fourier transform *on H ($\tau(e)$ is defined in (5.3.4)).*

In [Mal95, 4A] Malle proves the following important property of the Fourier transform T:

Proposition 5.3.18 *For any $f \in H$ and $\psi \in \Psi$ we have*

$$T^2(f)(\psi) = (-1)^{m\binom{e-1}{2}} \epsilon(\psi)\epsilon(\overline{\psi}) f(\overline{\psi}).$$

where $^-$ is the involution defined above.

Any map $\pi : Y \to \mathbb{Z}_{\geq 0}$ gives rise to an equivalence relation $\sim_\pi$ on Ψ defined as follows

$$\phi \sim_\pi \psi \text{ if } \pi \circ \phi^{-1}(i) = \pi \circ \psi^{-1}(i) \text{ for } 0 \leq i \leq e-1.$$

Definition 5.3.19 *An element $\psi \in \Psi$ is called* π-admissible *if the equalities $\pi(y) = \pi(y')$ and $\psi(y) = \psi(y')$ imply $y = y'$.*

Denote by $[\psi]$ the equivalence class of ψ under the relation $\sim_\pi$.

Proposition 5.3.20 *Let Y be any totally ordered set as above. For a reduced Symbol S with $I(S) = em + 1$ consider any map $\pi : Y \to \mathbb{Z}_{\geq 0}$ such that $|\pi^{-1}(k)| = |\{i | k \in S_i\}|$. Let $\mathcal{F}$ be the Lusztig Family of S. Then the map κ from equivalence classes of π-admissible elements of Ψ to $\mathcal{F}$ given by*

$$\kappa : [\psi] \mapsto S^{[\psi]}, \quad S_i^{[\psi]} := \pi(\psi^{-1}(i))$$

is a well-defined bijection independent of the choice of $S \in \mathcal{F}$.

Proof. It is easy to see that κ is well-defined, injective and independent of the choice of $S \in \mathcal{F}$. Let us prove its surjectivity. Let S' be any Symbol from $\mathcal{F}$. Then $|\pi^{-1}(k)| = |\{i | k \in S_i\}| = |\{i | k \in S'_i\}|$.

For any $k \geq 0$ fix any bijection

$$\beta_k : \pi^{-1}(k) \xrightarrow{\sim} \{i | k \in S_i'\}.$$

Define ψ on $Y = \coprod_{k\geq 0} \pi^{-1}(k)$ by $\psi|_{\pi^{-1}(k)} := \beta_k$. Then it is easy to see that $\psi \in \Psi$ is π-admissible and its image under the map κ is S'. □

Remark 5.3.21 *It follows from the construction in the proof of the above proposition that the equivalence class $[\psi]$ of any π-admissible element ψ consists of π-admissible elements which are parametrized by the choice of the bijections β_k.*

Now we explain how the Fourier transform defined in (5.3.17) provides the Lusztig Fourier transform matrix of the group W_n.

Fix any Lusztig family $\mathcal{F}$. Let Y, π, κ be as in proposition (5.3.20). Consider the R-submodule H_π of H (cf. (5.3.17)) of functions $f \in H$ such that

$$f(\phi) = f(\psi) \text{ if } [\phi] = [\psi], \quad f(\psi) = 0 \text{ for } \psi \text{ not } \pi\text{-admissible}.$$

Then H_π is a free R-module with the basis

$$\left\{ f_{[\phi]} \quad | \quad \phi \text{ is } \pi\text{-admissible }, f_{[\phi]}(\psi) := \delta_{[\phi],[\psi]} \right\}. \tag{5.3.22}$$

Theorem 5.3.23 *The R-submodule H_π is stable under the Fourier transform T defined in (5.3.17). The matrix of $T|_{H_\pi}$ in the basis (5.3.22) is as follows:*

$$T(Y,\pi) := \left(\frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\nu \in [\phi]} \epsilon(\nu)\epsilon(\psi) \prod_{y \in Y} \zeta^{-\nu(y)\psi(y)} \right)_{f_{[\phi]}, f_{[\psi]}}$$

Proof. Cf. [Mal95, (4.8), (4.10)] and remark (5.3.21) □
Define a parametrization of the basis (5.3.22) by $\mathcal{F}$ (cf. Prop. (5.3.20))

$$\iota_{\mathcal{F}} : \mathcal{F} \to H_\pi \quad : \quad S \mapsto f_{[\kappa^{-1}(S)]}. \tag{5.3.24}$$

Definition 5.3.25 *Let $\mathbf{S}$ be a matrix whose lines and columns are indexed by the set $\mathcal{U}(W_n)$ (cf. 5.3.4) and whose entries are defined as follows*

$$\mathbf{S}_{S,S'} := \begin{cases} T(Y,\pi)_{\iota_{\mathcal{F}}(S),\iota_{\mathcal{F}}(S')} & \textit{if } S \textit{ and } S' \textit{ lie in the same Family } \mathcal{F}, \\ 0 & \textit{if } S \textit{ and } S' \textit{ lie in different Families} \end{cases}$$

where Y, π are as in (5.3.20), $T(Y,\pi)$ as in (5.3.23) and $\iota_{\mathcal{F}}$ as in (5.3.24). Then $\mathbf{S}$ is called the Lusztig Fourier transform matrix *(or simply* Fourier matrix*) of the group W_n.*

Remark 5.3.26 *The Fourier matrix* $\mathbf{S}$ *is a symmetric block-diagonal matrix, whose blocks are indexed by the set of Lusztig families and for a given Family* $\mathcal{F}$ *the corresponding block is indexed by the reduced Symbols* $S \in \mathcal{F}$ *with the entries of the block independent of the choice of* Y, π *as in (5.3.20) and* R *as in (5.3.17).*

Remark 5.3.27 *Proposition (5.3.18) implies* $\mathbf{S}^{-1} = \overline{\mathbf{S}}$*.*

Remark 5.3.28 *In the case* $e = 1(W_n = S_n)$ *the Lusztig families consist of one element and the matrix* $\mathbf{S}$ *is the identity matrix. In the case* $e = 2$ *(*W_n *is of type* B_n*) the Fourier matrix just constructed coincides with the Fourier matrix constructed by Lusztig for* B_n*.*

Definition 5.3.29 *Recall that we have an inclusion* $\mathrm{Irr}W_n \hookrightarrow \mathcal{U}(W_n)$ *(cf. 5.3.6). For a reduced Symbol* $S \in \mathcal{U}(W_n)$ *define the* fake degree R_S *to be zero for* $S \notin \mathrm{Irr}W_n$ *and to be* $R_{\chi_{\boldsymbol{\alpha}}}$ *for* $S = S_{\boldsymbol{\alpha}}$ *(cf. (5.3.9)).*

Definition 5.3.30

- *For an e-Symbol* $S = (S_0, S_1, S_2, \ldots, S_{e-1})$ *define the* conjugate *Symbol* $\overline{S} := (S_0, S_{e-1}, S_{e-2}, \ldots, S_1)$*.*
- *By analogy for an e-partition* $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{e-1})$ *define its* conjugate $\overline{\boldsymbol{\alpha}} := (\alpha_0, \alpha_{e-1}, \ldots, \alpha_1)$*.*

Remark 5.3.31 *It can be easily seen that the complex conjugate* $\overline{\chi_{\boldsymbol{\alpha}}}$ *of* $\chi_{\boldsymbol{\alpha}}$ *is* $\chi_{\overline{\boldsymbol{\alpha}}}$*; The set* $\mathcal{U}(W_n)$ *as well as each Lusztig family is stable under conjugation; Conjugation commutes with the inclusion* $\mathrm{Irr}W_n \subset \mathcal{U}(W_n)$*:* $\overline{\chi_{\boldsymbol{\alpha}}} \mapsto \overline{S_{\boldsymbol{\alpha}}}$ *(cf. (5.3.3)).*

In [Mal95, (4.17)] Malle proves the analogue of the proposition (5.2.8) for the group W_n:

Proposition 5.3.32 *The Fourier matrix* $\mathbf{S}$ *of the group* W_n *transforms the vector* $(\mathrm{Deg}S)_{S\in\mathcal{U}(W_n)}$ *formed by generic degrees to the vector* $(R_{\overline{S}})_{S\in\mathcal{U}(W_n)}$ *formed by the fake degrees completed by zeros (cf. (5.3.29)).*

The above proposition combined with the proposition ((5.3.18)) gives us the following

Corollary 5.3.33 *The Fourier matrix* $\mathbf{S}$ *of the group* W_n *transforms the vector* $(R_S)_{S\in\mathcal{U}(W_n)}$ *formed by fake degrees to the vector* $(\mathrm{Deg}S)_{S\in\mathcal{U}(W_n)}$ *formed by generic degrees.*

Proof. It is enough to prove this corollary for any Lusztig family $\mathcal{F}$ and the corresponding block $\mathbf{S}_{\mathcal{F}}$ of the Fourier matrix $\mathbf{S}$ (cf. (5.3.26)). Let Y, π be as in (5.3.20) and $R = \mathbb{C}(q)$. Then from the definition of $\mathbf{S}$ (cf. (5.3.25)), proposition (5.3.32) and proposition (5.3.18) we get

$$(\mathrm{Deg}S)_{S\in\mathcal{F}} = \mathbf{S}_{\mathcal{F}}^3 \left(R_{\overline{S}}\right)_{S\in\mathcal{F}} = \mathbf{S}_{\mathcal{F}} \left(\epsilon(S) R_S\right)_{S\in\mathcal{F}}$$

where for a reduced Symbol $S = (S_0, \ldots, S_{e-1})$ with $|S_i| = m_i$ and $I(S) = em + 1$ we pose $\epsilon(S) := (-1)^c$ with $c = m\binom{e-1}{2} + \sum_{1\leq i<j\leq e-1} m_i m_j$. The vector $(\epsilon(S)R_S)_{S\in\mathcal{F}}$ is equal to the vector $(R_S)_{S\in\mathcal{F}}$ since $R_S = 0$ for $S \notin \mathrm{Irr}W_n$ and $\epsilon(S) = 1$ for $S \in \mathrm{Irr}W_n$ (cf. (5.3.6)). □

5.4 G(e,e,n) Case

In this section, following Malle [Mal95] again, we present a combinatorial contruction in the case $\widetilde{W}_n^{(e)} = G(e,e,n)$ of the set parametrizing the "unipotent representations" and give formulae for their generic degrees. We also give Malle's contruction of the part of the Fourier matrix whose existence is still conjectural for arbitrary e and n. If $e = 2$ then the group $\widetilde{W}_n^{(e)}$ is of type D_n, if $n = 2$ then the group $\widetilde{W}_n^{(e)}$ is a dihedral group and Malle's constructions and formulae agree with the already defined $\mathcal{U}(\widetilde{W}_n^{(e)})$, $(\mathrm{Deg}\rho)_{\rho\in\mathcal{U}(\widetilde{W}_n^{(e)})}$ and Fourier matrices S for finite Coxeter groups.

5.4.1 Equivalence Classes of e-Symbols.

The main combinatorial objects in Malle's construction of "unipotent representations" are not Symbols but their equivalence classes defined as follows:

Definition 5.4.1 *We define* equivalence *of Symbols as the symmetric , transitive closure of the two following operations:* cyclic permutation *of S_i in S and* simultaneous shift *to the right of all S_i given by* $(\lambda_{i,1}, \ldots \lambda_{i,m_i}) \mapsto (0, \lambda_{i,1} + 1, \ldots \lambda_{i,m_i} + 1)$.

Definition 5.4.2 *Let S be an e-Symbol with $I(S) \equiv 0 (\mathrm{mod}\ e)$ (cf. (5.3.1)).*

- *Define its* defect *by*

$$\mathrm{def}(S) = \frac{e-1}{2}I(S) - \sum_{i=0}^{e-1} i|S_i|.$$

- *By analogy with the representations of the group $\widetilde{W}_n^{(e)}$ (cf. (3.4.2)) define $s(S)$ as the cardinality of the set*

$$\{0 \leq i \leq e-1 | \pi^i(S) = S\}$$

where π is the cyclic permutation of S_i in S.

Recall that in (5.3.3) we defined the rank of S. In the case when $I(S) = em$ its formula looks easier:

$$\mathrm{rg}(S) = \sum_{i=0}^{e-1} \sum_{\lambda\in S_i} \lambda - e\binom{m}{2}.$$

Notation 5.4.3 *Denote by* $[S]$ *the equivalence class in the sense of (5.4.1) of a Symbol* S*. Then* $I(S) \bmod e$, $\operatorname{def}(S) \bmod e$, $s(S)$ *and* $\operatorname{rg}(S)$ *are constant on the equivalence class* $[S]$*. We will denote them by* $I[S], \operatorname{def}[S], s[S]$ *and* $\operatorname{rg}[S]$*.*

5.4.2 Unipotent Representations and Generic Degrees.

In this subsection we introduce the set of "unipotent representations" and their degrees for the group $\widetilde{W}_n^{(e)}$ (cf. [Mal95, (6.3)]).

Definition 5.4.4 *Define the multiset*

$$\mathcal{U}(\widetilde{W}_n^{(e)}) := \{ \text{ equivalence classes } [S] \mid \operatorname{rg}[S] = n, I[S] \equiv \operatorname{def}[S] \equiv 0 \pmod{e}\}$$

where the multiplicity of $[S]$ *in* $\mathcal{U}(\widetilde{W}_n^{(e)})$ *is* $s[S]$*. It is called the* set of unipotent representations *of the group* $\widetilde{W}_n^{(e)}$*. Their* generic degrees *are defined as follows.*

Let S *be any representative of the equivalence class* $[S]$*. Then*

$$\operatorname{Deg}_{[S]}(q) := \frac{e}{s(S)} \frac{(-1)^{\binom{e}{2}\binom{m}{2}+\gamma(S)}(q^n-1)\prod_{i=1}^{n-1}(q^{ei}-1)\cdot\prod_{i=0}^{e-1}\prod_{j=i}^{e-1}\prod_{\substack{(\lambda,\mu)\in S_i\times S_j\\ \mu<\lambda \text{ if } i=j}}(q^\lambda\zeta^i - q^\mu\zeta^j)}{\tau(e)^m q^{\binom{e(m-1)}{2}+\binom{e(m-2)}{2}+\cdots}\cdot\prod_{i=0}^{e-1}\Theta(S_i,q^e)}$$

where

$$m = \frac{I(S)}{e}, \gamma(S) = \frac{\operatorname{def}(S)}{e}(em-1), \tau(e) \text{ and } \Theta(S_i, q^e) \text{ are as in (5.3.4).}$$

Remark 5.4.5 *We slightly modified the original definition of* Deg *of Gunter Malle multiplying by* $(-1)^{\gamma(S)}$*. This makes* $\operatorname{Deg}_{[S]}(q)$ *well-defined, i.e. independent of the choice of* $S \in [S]$*.*

Remark 5.4.6 *In order to distinguish different copies of* $[S]$ *in* $\mathcal{U}(\widetilde{W}_n^{(e)})$ *we will denote them by* $[S]_0, [S]_1 \dots, [S]_{s[S]-1}$*.*

Remark 5.4.7 *As it is pointed out in [Mal95, (6.5)] the above definition of the generic degrees of unipotent representations of* $\widetilde{W}_n^{(e)}$ *agrees with the definition of the generic degrees of unipotent representations of* D_n *for* $e = 2$ *and of* $I_2(e)$ *for* $n = 2$ *(cf. (5.1.27) and (5.2.1)).*

5.4.3 Irreducible Representations and their Degrees

By analogy with the real case and the case $G(e,1,n)$ we will define an inclusion $\mathrm{Irr}(\widetilde{W}_n^{(e)}) \subset \mathcal{U}(\widetilde{W}_n^{(e)})$. Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{e-1})$ be any e-partition of n. By adding zeros if necessary we can think that $\alpha_i = (0 \leq \alpha_{i,1} \leq \ldots \leq \alpha_{i,m})$ for $0 \leq i \leq e-1$ and some m. Define a Symbol $\underline{S}_{\boldsymbol{\alpha}} = (S_0, \ldots, S_{e-1})$ by $S_{i,j} = \alpha_{i,j} + j - 1$. Then $\underline{S}_{\boldsymbol{\alpha}}$ is a Symbol of rank n with $I(\underline{S}_{\boldsymbol{\alpha}}) = em$ and $\mathrm{def}(\underline{S}_{\boldsymbol{\alpha}}) = 0$. Denote by $\pi\backslash\Lambda_n^{(e)}$ the set of e-partitions of n modulo the cyclic permutation π. In (3.4.2) we have seen that $\mathrm{Irr}(\widetilde{W}_n^{(e)})$ is the disjoint union:

$$\mathrm{Irr}(\widetilde{W}_n^{(e)}) = \coprod_{\boldsymbol{\alpha}\in\pi\backslash\Lambda_n^{(e)}} \left\{\chi_{(\boldsymbol{\alpha},1)}, \chi_{(\boldsymbol{\alpha},\zeta^{e/s(\boldsymbol{\alpha})})}, \ldots, \chi_{(\boldsymbol{\alpha},(\zeta^{e/s(\boldsymbol{\alpha})})^{(s(\boldsymbol{\alpha})-1)})}\right\}.$$

We define an inclusion of the set $\mathrm{Irr}(\widetilde{W}_n^{(e)})$ into the multiset $\mathcal{U}(\widetilde{W}_n^{(e)})$ as follows:

$$\text{For } \boldsymbol{\alpha} \in \pi\backslash\Lambda_n^{(e)} \text{ the set } \left\{\chi_{(\boldsymbol{\alpha},1)}, \chi_{(\boldsymbol{\alpha},\zeta^{e/s(\boldsymbol{\alpha})})}, \ldots, \chi_{(\boldsymbol{\alpha},(\zeta^{e/s(\boldsymbol{\alpha})})^{(s(\boldsymbol{\alpha})-1)})}\right\}$$
$$\text{goes to the } s(\boldsymbol{\alpha}) = s[\underline{S}_{\boldsymbol{\alpha}}] \text{ copies of } [\underline{S}_{\boldsymbol{\alpha}}] \text{ in } \mathcal{U}(\widetilde{W}_n^{(e)}). \quad (5.4.8)$$

It is easy to see that this is a well-defined inclusion.

Definition 5.4.9 *For an irreducible character $\mu \in \mathrm{Irr}(\widetilde{W}_n^{(e)})$ we define its* generic degree *as $\gamma_\mu(q) := \mathrm{Deg}_{[\underline{S}_{\boldsymbol{\alpha}(\mu)}]}$.*

By analogy with the real and $G(e,1,n)$ cases, the generic degree of a character can be obtained via the symmetrizing trace on a Hecke algebra. Consider the Hecke algebra $\widetilde{\mathcal{H}}_n^{(e)}$ (cf. (3.4.1)) which is symmetric and split semisimple over $\mathbb{C}(q)$. The corresponding symmetrizing form $t_0' : \widetilde{\mathcal{H}}_n^{(e)} \to \mathbb{C}(q)$ can be given as the restriction of the symmetrizing form t_0 on the Hecke algebra $\mathcal{H}_n^{(e)}(q,\zeta)$ (cf. [Mal95, lemma(5.11)]). By [GIM00, lemma(4.3)]) t_0 is a Markov trace in the sense of (3.3.1) with all parameters equal to zero, hence it coincides with the trace $\overline{\tau}$ for $z = 0$ (cf. (3.4.1)). This and definition (3.4.4) imply that t_0' coincides with the special Markov trace $\widetilde{\tau}_e$ for $z = 0$.

Lemma (3.4.11) and theorem (3.3.5) applied to the trace $\overline{\tau}$ give us the following formula for the weights ω_μ' of $\widetilde{\tau}_e$:

$$\omega_\mu' = \frac{(-1)^n \mathrm{Deg}_{[\underline{S}_{\boldsymbol{\alpha}(\mu)}]}(q)}{e(q^n-1)\prod_{i=1}^{n-1}(q^{ei}-1)} \sum_{i=0}^{e-1} R_{\pi^i(\boldsymbol{\alpha}(\mu))}(z, \eta_1, \ldots, \eta_{e-1}), \tag{5.4.10}$$

which implies the following:

Proposition 5.4.11 *Let*

$$t_0' = \sum_{\mu\in\mathrm{Irr}(\widetilde{W}_n^{(e)})} \frac{\mu^{(q)}}{S_\mu}$$

be the decomposition of the symmetrizing form t'_0 *into the sum of irreducible characters of* $\widetilde{\mathcal{H}}_n^{(e)}$. *Then we have* $\gamma_\mu(q) = S_{\mathbb{1}}/S_\mu$ *(cf. (2.2.29)) where* $S_{\mathbb{1}} = \frac{(q^n-1)\prod_{i=1}^{n-1}(q^{ei}-1)}{(q-1)^n}$ *is the Poincare polynomial of* $\widetilde{W}_n^{(e)}$.

Let V be the reflection representation of $\widetilde{W}_n^{(e)}$. As in the case $G(e,1,n)$ we give the following:

Definition 5.4.12 *Let* $S(V^*)_{\widetilde{W}_n^{(e)}} = S(V^*)/I$ *be the coinvariant algebra of* $\widetilde{W}_n^{(e)}$ *where* I *is the ideal of* $S(V^*)$ *generated by the homogeneous elements of* $S(V^*)^{\widetilde{W}_n^{(e)}}$ *of positive degree. For* $\mu \in \mathrm{Irr}(\widetilde{W}_n^{(e)})$ *the graded multiplicity* R_μ *of the character* μ *in* $S(V^*)_{\widetilde{W}_n^{(e)}}$ *is called the* fake degree *of* μ*(cf. (5.1.23) and (5.3.9)).*

The following proposition (cf. [Mal95, (5.6)]) gives the formula for fake degrees.

Proposition 5.4.13 *For an e-Symbol* $\underline{S}$ *such that* $|S_0| = |S_1| = \ldots = |S_{e-1}| = m$ *define an element* $\delta_{\underline{S}}(q) \in \mathbb{C}(q)$ *by*

$$\delta_{\underline{S}}(q) = \frac{1}{s(\underline{S})} \prod_{i=0}^{e-1} \frac{\Delta(S_i, q^e)}{\Theta(S_i, q^e)} \cdot \frac{\sum_{j=0}^{e-1} \prod_{i=1}^{e-1} \prod_{\lambda \in S_{i+j}} q^{(e-i)\lambda}}{q^{\binom{e(m-1)}{2}+\binom{e(m-2)}{2}+\ldots}},$$

where $\Delta(S_i, q^e)$ *are as in (5.3.11). Then the fake degree* R_μ *of a character* $\mu \in \mathrm{Irr}(\widetilde{W}_n^{(e)})$ *is equal to*

$$(q^n - 1) \prod_{i=1}^{n-1} (q^{ei} - 1) \delta_{\underline{S}_{\boldsymbol{\alpha}(\mu)}}(q).$$

Remark 5.4.14 *Gunter Malle uses the symmetric algebra* $S(V)$ *in his definition of fake degrees thus we had to slightly change his formula.*

5.4.4 Lusztig Families and Fourier Transform Matrix

Malle's combinatorial definition of the Lusztig families of the group $\widetilde{W}_n^{(e)}$ is similar to the one in the case of $G(e,1,n)$ (cf. (5.3.4)).

Definition 5.4.15

- *As in (5.3.13) we say that two Symbols* S *and* S' *lie in the same* Family *if and only if as multisets* S *and* S' *contain the same numbers with the same multiplicities.*
- *Two equivalence classes of Symbols* $[S]$ *and* $[S']$ *lie in the same* Family *if and only if there exist Symbols* $S_1 \in [S]$ *and* $S'_1 \in [S']$ *which lie in the same Family.*

Definition 5.4.16 *Families of equivalence classes of Symbols induce a decomposition of the multiset $\mathcal{U}(\widetilde{W}_n^{(e)})$ into the union of disjoint multisubsets which are called* Lusztig families*:*

- *If $s[S] = e$ then all e copies of $[S]$ in $\mathcal{U}(\widetilde{W}_n^{(e)})$ are Lusztig families with one element.*
- *If $[S]$ and $[S']$ lie in the same Family and $s[S] < e$ (which implies that $s[S'] < e$) then all copies of $[S]$ and $[S']$ in $\mathcal{U}(\widetilde{W}_n^{(e)})$ lie in the same Lusztig family.*

Remark 5.4.17 *One can check that whenever $\widetilde{W}_n^{(e)}$ is a Coxeter group, the above definition coincides with the definition of the Lusztig families in the real case.*

Definition 5.4.18 *We say that two irreducible characters $\mu, \mu' \in \operatorname{Irr}(\widetilde{W}_n^{(e)})$ are in the same* family *(in the real case one says that they* belong to the same two-sided cell*) if the corresponding copies of $[\underline{S}_{\boldsymbol{\alpha}(\mu)}]$ and $[\underline{S}_{\boldsymbol{\alpha}(\mu')}]$ belong to the same Lusztig family.*

Remark 5.4.19 *Defining families of characters we follow the result of Maria Chlouveraki (cf. [Chl10, Th. (3.10)]) describing the so-called Rouquier blocks which are a substitute for the families of characters in the case of complex reflection groups (cf. [Rou99]).*

We continue with Malle's definition of the Fourier transform matrix.

Let Y be a totally ordered set with em elements, $m > 0$; $\Psi = \Psi(Y)$ is the set of maps

$$\psi : Y \to \{0, \dots, e-1\}, \text{ such that } \sum_{y \in Y} \psi(y) \equiv m\binom{e}{2} \pmod e,$$

with the subset Ψ_0 of Ψ such that $|\psi^{-1}(i)| = m$.

For $\psi \in \Psi$ let $\gamma(\psi) := \frac{em-1}{e}(m\binom{e}{2} - \sum_{y\in Y} \psi(y))$. There is a natural involution on Ψ

$$^{-} : \Psi \to \Psi, \quad \psi \mapsto \overline{\psi} \text{ with } \overline{\psi}(y) := e - \psi(y) \pmod e$$

and a symmetric pairing $\langle\ ,\ \rangle$ on Ψ given by

$$\langle \phi, \psi \rangle = \epsilon'(\phi)\epsilon'(\psi) \prod_{y \in Y} \zeta^{-\phi(y)\psi(y)}$$

where $\zeta = e^{2\pi i/e}$ and

$$\epsilon'(\psi) = (-1)^{c(\psi)+\gamma(\psi)}, \quad c(\psi) = |\{(y, y') \in Y \times Y | y < y', \psi(y) < \psi(y')\}|.$$

Remark 5.4.20 *As in definition (5.4.4) we have changed Malle's original formula (cf. [Mal95, 6C]) by the sign* $(-1)^{\gamma(\psi)}$*. It won't be hard to see that all the good properties of the Fourier transform remain.*

Remark 5.4.21 *Another small difference with Malle's original construction in [Mal95, 6C] is that we work only with the set* ${}^1\Psi$ *in his notations.*

Definition 5.4.22 *For* $\psi \in \Psi$ *define* $\psi+1$ *by* $(\psi+1)(y) := \psi(y)+1 (\text{mod } e)$*. It is obvious that* $\psi+1$ *is an element of* Ψ*.*

Definition 5.4.23 *Consider the space* $\widetilde{H}$ *of functions from the set* Ψ *to some* $\mathbb{C}$*-algebra* R *with the following property:*

$$f(\psi) = f(\psi+1), \text{ for any } f \in \widetilde{H}, \psi \in \Psi.$$

The operator

$$T : \widetilde{H} \to \widetilde{H}, \quad T(f)(\phi) := \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\psi\in\Psi} \langle \phi, \psi \rangle \, f(\psi)$$

is called the Fourier transform *on* $\widetilde{H}$ *(*$\tau(e)$ *is defined in (5.3.4)).*

In order to show that the image of the Fourier transform is indeed in $\widetilde{H}$ we prove the following:

Lemma 5.4.24 *We have* $\langle \phi+1, \psi \rangle = \langle \phi, \psi \rangle$ *for any* $\phi, \psi \in \Psi$*.*

Proof. By definition we have

$$\langle \phi+1, \psi \rangle = \epsilon'(\phi+1)\epsilon'(\psi) \prod_{y\in Y} \zeta^{-(\phi(y)+1)\psi(y)} =$$
$$(-1)^{\phi^{-1}(e-1)(em-\phi^{-1}(e-1))+(em-1)(\phi^{-1}(e-1)-m)} \epsilon'(\phi)\epsilon'(\psi) \cdot$$
$$\cdot \, \zeta^{-m\binom{e}{2}} \prod_{y\in Y} \zeta^{-\phi(y)\psi(y)} = \langle \phi, \psi \rangle \quad \square$$

Lemma [Mal95, (6.19)] implies the following important property of the Fourier transform T:

Proposition 5.4.25 *For any* $f \in \widetilde{H}$ *and* $\psi \in \Psi$ *we have*

$$T^2(f)(\psi) = (-1)^{m\binom{e-1}{2}} \epsilon'(\psi)\epsilon'(\overline{\psi}) f(\overline{\psi}),$$

where $^-$ *is the involution defined above.*

Definition 5.4.26

- *Any map* $\pi : Y \to \mathbb{Z}_{\geq 0}$ *gives rise to an equivalence relation (* π*-equivalence)* $\sim_\pi$ *on* Ψ *defined as follows*

$$\phi \sim_\pi \psi \text{ if } \pi \circ \phi^{-1}(i) = \pi \circ \psi^{-1}(i) \text{ for } 0 \leq i \leq e-1.$$

Denote by $[\psi]$ *the equivalence class of* ψ *under the relation* $\sim_\pi$*.*

- *An element* $\psi \in \Psi$ *is called* π-admissible *if the equalities* $\pi(y) = \pi(y')$ *and* $\psi(y) = \psi(y')$ *imply* $y = y'$.
- *We also define the equivalence relation (* $\pi+$*-equivalence)* $\sim_{\pi+}$ *on* Ψ *as the transitive closure of* $\sim_\pi$ *and the relation*

$$\phi \sim_+ \psi \text{ if } \phi = \psi + j \text{ for some } j.$$

Denote by $[\psi]_+$ *the equivalence class of* ψ *via the relation* $\sim_{\pi+}$.

Proposition 5.4.27 *Let* Y *be any totally ordered set as above. For a Symbol* S *with* $I(S) = em$ *and* $\operatorname{def}(S) \equiv 0 (\operatorname{mod} e)$ *consider any map* $\pi : Y \to \mathbb{Z}_{\geq 0}$ *such that* $|\pi^{-1}(k)| = |\{i | k \in S_i\}|$.

(a) Let $\mathcal{F}'$ *be the Family of* S*. Then the map* κ *from* π*-equivalence classes of* π*-admissible elements of* Ψ *to* $\mathcal{F}'$ *given by*

$$\kappa : [\psi] \mapsto S^{[\psi]}, \quad S_i^{[\psi]} := \pi(\psi^{-1}(i))$$

is a well-defined bijection independent of the choice of $S \in \mathcal{F}'$.

(b) Let $\mathcal{F}$ *be the Family of the equivalence class* $[S]$*. Then the map* κ' *from* $\pi+$*-equivalence classes of* π*-admissible elements of* Ψ *to* $\mathcal{F}$ *given by*

$$\kappa' : [\psi]_+ \mapsto [S^{[\psi]}], \quad S_i^{[\psi]} = \pi(\psi^{-1}(i))$$

is a well-defined bijection independent of the choice of $[S] \in \mathcal{F}$.

Proof. Same as in (5.3.20) □

Now we explain how the Fourier transform defined in (5.4.23) provides a part of conjectural Lusztig Fourier transform matrix of the group $\widetilde{W}_n^{(e)}$.

Fix any Family $\mathcal{F}$ of equivalence classes of Symbols. Let Y, π, κ' be as in proposition (5.4.27). Consider the R-submodule $\widetilde{H}_{\pi+}$ of $\widetilde{H}$ (cf. (5.4.23)) of functions $f \in \widetilde{H}$ such that

$$f(\phi) = f(\psi) \text{ if } [\phi]_+ = [\psi]_+, \quad f(\psi) = 0 \text{ for } \psi \text{ not } \pi\text{-admissible}.$$

Then $\widetilde{H}_{\pi+}$ is a free R-module with the basis

$$\left\{ f_{[\phi]_+} \quad | \quad \phi \text{ is } \pi\text{-admissible} \ , f_{[\phi]_+}(\psi) := \delta_{[\phi]_+,[\psi]_+} \right\}. \tag{5.4.28}$$

Theorem 5.4.29 *The* R*-submodule* $\widetilde{H}_{\pi+}$ *of* $\widetilde{H}$ *is stable under the Fourier transform* T *defined in (5.4.23). The matrix of* $T|_{H_{\pi+}}$ *in the basis (5.4.28) is as follows:*

$$T(Y, \pi+) := \left(\frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\nu \in [\phi]_+} \epsilon'(\nu)\epsilon'(\psi) \prod_{y \in Y} \zeta^{-\nu(y)\psi(y)} \right)_{f_{[\phi]_+}, f_{[\psi]_+}} \tag{5.4.30}$$

Proof. We follow [Mal95, (4.8)]. By definition

$$T(f_{[\phi]_+})(\psi) = \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\nu\in\Psi} \langle \psi, \nu\rangle\, f_{[\phi]_+}(\nu) = \frac{(-1)^{m(e-1)}\epsilon'(\psi)}{\tau(e)^m} \sum_{\nu\in[\phi]_+} \epsilon'(\nu) \prod_{y\in Y} \zeta^{-\psi(y)\nu(y)}.$$

If ψ is not π-admissible then there exist $y, y' \in Y$ such that $\psi(y) = \psi(y')$ and $\pi(y) = \pi(y')$. Dividing the set $[\phi]_+$ into pairs ν, ν' where $\nu'(y) = \nu(y'), \nu'(y') = \nu(y)$ and is the same as ν on the rest of Y we see that the above sum is equal to zero.

Let ψ be π-admissible. If $\psi' \sim_\pi \psi$ then $T(f_{[\phi]_+})(\psi') = T(f_{[\phi]_+})(\psi)$ by remark (5.3.21). Thus in order to prove the theorem we need to show that if $\psi' = \psi + 1$ then $T(f_{[\phi]_+})(\psi') = T(f_{[\phi]_+})(\psi)$. And this follows from lemma (5.4.24) □

Define a parametrization of the basis (5.4.28) by $\mathcal{F}$ (cf. (b) in (5.4.27))

$$\iota_{\mathcal{F}} : \mathcal{F} \to H_{\pi+} \quad : \quad [S] \mapsto f_{[\kappa'^{-1}([S])]_+}. \tag{5.4.31}$$

Definition 5.4.32 *Denote by $\mathcal{U}'(\widetilde{W}_n^{(e)})$ the set of equivalence classes of Symbols $[S]$ such that $\mathrm{rg}[S] = n, I[S] \equiv \mathrm{def}[S] \equiv 0 \pmod e$. Let $\mathbf{S}'$ be the matrix whose lines and columns are indexed by the set $\mathcal{U}'(\widetilde{W}_n^{(e)})$ and whose entries are defined as follows*

$$\mathbf{S}'_{[S],[S']} := \begin{cases} T(Y, \pi+)_{\iota_{\mathcal{F}}([S]),\iota_{\mathcal{F}}([S'])} & \text{if } [S] \text{ and } [S'] \text{ lie in the same Family } \mathcal{F}, \\ 0 & \text{if } [S] \text{ and } [S'] \text{ lie in different Families} \end{cases}$$

where Y, π are as in (5.4.27), $T(Y, \pi+)$ as in (5.4.29) and $\iota_{\mathcal{F}}$ as in (5.4.31). Then $\mathbf{S}'$ is called pre-Fourier matrix *of the group $\widetilde{W}_n^{(e)}$.*

Define the vector $(R_{[S]})_{[S]\in\mathcal{U}'(\widetilde{W}_n^{(e)})}$ of fake degrees by

$$R_{[S]} := \begin{cases} R_\mu & \text{if } [S] = [\underline{S}_{\boldsymbol{\alpha}(\mu)}] \text{ for } \mu \in \mathrm{Irr}(\widetilde{W}_n^{(e)}) \text{ (cf. (5.4.13))}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.4.33 *The pre-Fourier matrix $\mathbf{S}'$ of the group $\widetilde{W}_n^{(e)}$ is a symmetric matrix which transforms the vector $\left(s[S]\mathrm{Deg}_{[S]}\right)_{[S]\in\mathcal{U}'(\widetilde{W}_n^{(e)})}$ formed by generic degrees to the vector $\left(s[\overline{S}]R_{[\overline{S}]}\right)_{[S]\in\mathcal{U}'(\widetilde{W}_n^{(e)})}$ where $[\overline{S}]$ is the equivalence class of Symbols conjugated to the Symbols in $[S]$ (cf. (5.3.30)). .*

Proof. Follows from [Mal95, (6.26)] and the fact that $\langle\phi, \psi + 1\rangle = \langle\phi, \psi\rangle$ for any $\phi, \psi \in \Psi$ (cf. (5.4.24)). □

The Lusztig Fourier transform matrix of the group $\widetilde{W}_n^{(e)}$ has not been found yet. But the above proposition makes it natural to make, following Malle, the following conjecture:

Conjecture 5.4.34 *The conjectural Lusztig Fourier transform matrix* $\mathbf{S}$ *of the group* $\widetilde{W}_n^{(e)}$ *indexed by the set* $\mathcal{U}(\widetilde{W}_n^{(e)})$ *has the following properties:*

- $\mathbf{S}$ *is a symmetric, unitary matrix whose fourth power is the identity matrix.*
- *If* $[S]$ *and* $[S']$ *are in the same Family then*

$$\sum_{j=0}^{s[S']-1} \mathbf{S}_{[S]_i,[S']_j} = \frac{1}{s[S]}\mathbf{S}'_{[S],[S']} \text{ for any } 0 \leq i \leq s[S]-1,$$

  where $\mathbf{S}'$ *is the pre-Fourier matrix defined in (5.4.32).*
- *If* $[S]_i$ *and* $[S']_j$ *are in different Lusztig Families then* $\mathbf{S}_{[S]_i,[S']_j} = 0$.

Remark 5.4.35 *Note that the conditions in the above conjecture partially define the Lusztig Fourier transform matrix:*

- *If* $s[S'] = 1$ *then* $\mathbf{S}_{[S]_i,[S']_0} = \frac{1}{s[S]}\mathbf{S}'_{[S],[S']}$ *for any* $0 \leq i \leq s[S]-1$.
- *If* $s[S] = e$ *then* $\mathbf{S}_{[S]_i,[S]_j} = 0$ *if* $i \neq j$ *and*

$$\mathbf{S}_{[S]_i,[S]_i} = \frac{(-1)^{m(e-1)}}{\tau(e)^m} \det(\zeta^{-ij})^m_{0\leq i,j\leq e-1} = 1.$$

  The last equality agrees with the fact that $[S]_i$ *are one-element Lusztig families for* $[S]$ *with* $s[S] = e$ *(cf. (5.4.16)).*

Define the vector $(R_{[S]})_{[S]\in\mathcal{U}(\widetilde{W}_n^{(e)})}$ of fake degrees by

$$R_{[S]_i} := \begin{cases} R_\mu & \text{if } [S]_i \text{ is the image of } \mu \in \mathrm{Irr}(\widetilde{W}_n^{(e)}) \text{ via } \mathrm{Irr}(\widetilde{W}_n^{(e)}) \hookrightarrow \mathcal{U}(\widetilde{W}_n^{(e)}), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.4.36 *Any matrix with the properties from the conjecture (5.4.34) transforms the vector* $\left(\mathrm{Deg}_{[S]_i}\right)_{[S]_i\in\mathcal{U}(\widetilde{W}_n^{(e)})}$ *formed by generic degrees to the vector* $\left(R_{[\overline{S}]_i}\right)_{[S]_i\in\mathcal{U}(\widetilde{W}_n^{(e)})}$.

Proof. Since both generic and fake degrees are constant on all copies of $[S]$ in $\mathcal{U}(\widetilde{W}_n^{(e)})$ this proposition is a direct consequence of proposition (5.4.33). □

Remark 5.4.37 *In the case* $e = 2$ *or* $n = 2$ *(*$\widetilde{W}_n^{(e)}$ *is of type* D_n *or* $I_2(e)$*) the Fourier matrix just constructed agrees with the classical constructions by Lusztig.*

The above proposition combined with the proposition (5.4.25) gives us the following:

Corollary 5.4.38 *Any matrix* **S** *with the properties from the conjecture (5.4.34) transforms the vector* $\left(R_{[S]_i}\right)_{[S]_i \in \mathcal{U}(\widetilde{W}_n^{(e)})}$ *to the vector* $\left(\mathrm{Deg}_{[S]_i}\right)_{[S]_i \in \mathcal{U}(\widetilde{W}_n^{(e)})}$ *formed by generic degrees.*

Proof. Similar to the proof of corollary (5.3.33) $\square$

Chapter 6

Gomi's Formula

Let (W, S) be a finite Coxeter system. Denote by $\mathcal{H}_W$ the Iwahori-Hecke algebra of W with the parameters $q_s = q, q'_s = -1$. In chapter 3 we have seen a brief history of appearance and construction of special Markov traces on $\mathcal{H}_W$. The classical example of a special Markov trace is the symmetrizing form t_0 on $\mathcal{H}_W$ (when the parameter $z \mapsto 0$) which can be decomposed into the sum of irreducible characters of $\mathcal{H}_W$ with weights equal to the corresponding generic degrees divided by the Poincare polynomial of W (the Schur element attached to $\mathbb{1}$ (cf. (2.2.29))). In turn the generic degrees can be obtained from the fake degrees using the Fourier matrix of W (cf. (5.1.28) and (5.2.8)) giving the formula for the weights of the symmetrizing form t_0.

This led Yasushi Gomi to the following idea:

- Consider any special Markov trace (for W of type A_n, B_n, D_n or $I_2(m)$) with any parameter z.
- Take as an analogue of the fake degrees (which appear in the Molien series of $S(V)$, where V affords the reflection representation of W) the Molien series of $S(V) \otimes \Lambda(V)$ where $\Lambda(V)$ is the exterior algebra of V.
- Try to obtain a formula for the weights of special Markov traces using the Molien series and the Fourier matrix.
- Using these formulae obtain certain Markov traces for the finite Coxeter groups of exceptional type.

Remark 6.0.1 *As we will see the Markov traces he considers are actually Khovanov-Rozansky traces discussed in chapter 4. (This is still conjectural for H_3 and H_4).*

Following this idea Gomi [Gom06, (4.1)] constructs traces on $\mathcal{H}_W$ giving the formulae for their weights and proves that these traces satisfy the Markov property (cf. 3.2.1) and are equal to special Markov traces for W of non-exceptional type. We recall his construction in the next section.

6.1 Gomi's Theorem

Let (W, S_W) be a finite Coxeter system. Recall that $S = S(V)$ and $L = \Lambda(V)$ are symmetric and exterior algebras of the n-dimensional $\mathbb{C}$-vector space V affording the reflection representation of W. We introduce the usual graduation on S and L posing $\deg(v) = 1$ for $v \in V\backslash\{0\}$ and denote by S_i, L_i their homogeneous components of degree i.

Define a bigraded character of W with values in $\mathbb{C}[\![q]\!][r]$ by

$$\mathrm{Tr}_{S\otimes L}(w) = \sum_{i\geq 0}\sum_{j=0}^{n} \mathrm{Tr}_{S_i\otimes L_j}(w)q^i r^j.$$

Definition 6.1.1 *Consider the decomposition of* $\mathrm{Tr}_{S\otimes L}$ *into the sum of irreducible characters of* W*:*

$$\mathrm{Tr}_{S\otimes L} = \sum_{\chi\in \mathrm{Irr}W} P_\chi(q,r)\chi \quad \text{with } P_\chi(q,r) \in \mathbb{C}[\![q]\!][r].$$

The weight $P_\chi(q,r)$ *is called the* Molien series *of* $S\otimes L$ *for* χ*.*

There is an inclusion $\mathrm{Irr}W \subset \mathcal{U}(W)$ of the set of irreducible characters into the set of unipotent representations (cf (5.1.3) and (5.2.2)). For a pair (χ,μ) of irreducible characters of W we will denote by $S_{\chi,\mu}$ the corresponding entry of the Fourier matrix of W (cf. (5.1.20), (5.2.6) and (5.2.9)).

For $\chi \in \mathrm{Irr}W$ define the weight $\omega^\chi \in \mathbb{C}(q,r)$ by

$$\omega^\chi = \left(\frac{1-q}{1+r}\right)^n \sum_{\mu\in\mathrm{Irr}W} S_{\chi,\mu}P_\mu(q,r). \tag{6.1.2}$$

Let $\mathcal{H}_W$ be the Iwahori-Hecke algebra of W with the parameters $q_s = q, q'_s = -1$. As usual we identify the irreducible characters of W with the irreducible characters of $\mathcal{H}_W$ via $\chi \mapsto \chi^{(q)}$.

In [Gom06, Th. 4.1] Gomi proves the following theorem:

Theorem 6.1.3 (Gomi) *Define a trace* τ_g *on* $\mathcal{H}_W$ *as follows:*

$$\tau_g = \sum_{\chi\in\mathrm{Irr}W} \omega^\chi \chi^{(q)}$$

Then

- τ_g *is a Markov trace with the parameter* $z = \frac{r(q-1)}{1+r}$*.*
- τ_g *coincides with the special Markov traces for* W *of non-exceptional type.*

Definition 6.1.4 *The trace* τ_g *is called the* special *Markov trace or* Gomi trace *for the group* W*.*

In the next section we will consider the similar situation in the case of complex reflection groups $G(e,1,n)$.

6.2 An extension of Gomi's Formula to the Case of G(e,1,n)

In this section we return to the group $W_n = G(e,1,n)$ (cf. (5.3)). Recall that in section (3.3) we gave definition (3.3.10) of the special Markov trace τ_e on the spetsial Hecke algebra $\mathcal{H}$. It seems to be the most natural generalization of the special Markov traces for finite Coxeter groups of type A_{n-1} and B_n. This will be supported by the theorem below, which is the analogue of Gomi's theorem (6.1.3) for finite Coxeter groups.

Definition 6.2.1 *Let V be an n-dimensional $\mathbb{C}$-vector space affording the reflection representaion of W_n. Let $S = S(V^*)$ and $L = \Lambda(V^*)$ be the symmetric and exterior algebras of the dual space of V. For an e-partition $\boldsymbol{\alpha}$ of n let $\chi_{\boldsymbol{\alpha}}$ be the corresponding irreducible character of W_n(cf (3.3.4)). By analogy with the real case we define the* Molien series $P_{\chi_{\boldsymbol{\alpha}}}(q,r) \in \mathbb{C}[\![q]\!][r]$ *for $\chi_{\boldsymbol{\alpha}}$ as the bigraded multiplicity of $\chi_{\boldsymbol{\alpha}}$ in the bigraded algebra $S \otimes L$ (cf (6.1.1)).*

We've seen in (5.3.8) that the generic degrees of the irreducible characters of W_n appear in the weights of the symmetrizing form of the spetsial Hecke algebra $\mathcal{H}$. This and the fact that the algebra $\mathcal{H}$ is the generalization of the Iwahori-Hecke algebra with the parameters $q, -1$ make it natural to ask whether we have the analogue of the formula (6.1.2) for the weights of the special Markov trace τ_e?

The answer is "Yes" and is given by the following

Theorem 6.2.2 *For $\chi_{\boldsymbol{\alpha}} \in \mathrm{Irr}W_n$ define the weight $\omega_{\chi_{\boldsymbol{\alpha}}} \in \mathbb{C}[\![q]\!][r]$ by*

$$\omega^{\chi_{\boldsymbol{\alpha}}} = \left(\frac{1-q}{1+r}\right)^n \sum_{\chi_{\boldsymbol{\alpha}'} \in \mathrm{Irr}W_n} \mathbf{S}_{S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}'}} P_{\chi_{\boldsymbol{\alpha}'}}(q,r), \tag{6.2.3}$$

where $\mathbf{S}_{S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}'}}$ is the entry of the Fourier matrix $\mathbf{S}$ of W_n (cf (5.3.25)) corresponding to the pair $(S_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}'})$ of reduced Symbols associated to the e-partitions $\boldsymbol{\alpha}, \boldsymbol{\alpha}'$ via (5.3.6).

Let τ_g be the trace on $\mathcal{H}$ defined as follows

$$\tau_g = \sum_{\chi_{\boldsymbol{\alpha}} \in \mathrm{Irr}W_n} \omega^{\chi_{\boldsymbol{\alpha}}} {\chi_{\boldsymbol{\alpha}}}^{(q)},$$

where ${\chi_{\boldsymbol{\alpha}}}^{(q)}$ is the irreducible character of the spetsial Hecke algebra $\mathcal{H}$ corresponding to the character $\chi_{\boldsymbol{\alpha}}$. Then τ_g coincides with the special Markov trace τ_e.

Proof. The idea of the proof is totally analogous to Gomi's proof that his trace τ_g is the special trace of type B_n when W is of type B_n (cf. [Gom06, Th. 4.1 Type B]).

We do it in three steps. Each step will be presented as lemma. These three lemmae combined with corollary (5.3.33) will give the proof of the theorem.

Lemma 6.2.4 *For an e-partition* $\boldsymbol{\alpha} = (\alpha_0, \alpha_1 \ldots, \alpha_{e-1})$ *the Molien series for the character* $\chi_{\boldsymbol{\alpha}}$ *is* $P_{\chi_{\boldsymbol{\alpha}}}(q,r) = \frac{R_{\chi_{\boldsymbol{\alpha}}}}{(1-q)^n S_{\mathbb{1}}} \Xi_{\boldsymbol{\alpha}}$ *where*

$$\Xi_{\boldsymbol{\alpha}} = \prod_{k=0}^{e-1} \prod_{x \in \alpha_k} (1 + rq^{ec(x)-1+e\delta_{k,0}}), \tag{6.2.5}$$

$R_{\chi_{\boldsymbol{\alpha}}}$ *is the fake degree of the character* $\chi_{\boldsymbol{\alpha}}$ *(cf. (5.3.9)),* $S_{\mathbb{1}} = \frac{\prod_{i=1}^n (q^{ei}-1)}{(q-1)^n}$ *is the Poincare polynomial of* W_n *(or the Schur element attached to* $\mathbb{1}$ *(cf. (5.3.8))),* $c(x)$ *is the content of* $x \in \alpha_k$ *(cf. (3.2.4)).*

Proof. By definition of $\chi_{\boldsymbol{\alpha}}$ (cf (3.3.4)) and the Frobenius reciprocity we have

$$\begin{aligned} &P_{\chi_{\boldsymbol{\alpha}}}(q,r) \\ &= \langle \mathrm{Tr}_{S\otimes L}, \chi_{\boldsymbol{\alpha}} \rangle_{W_n} \\ &= \langle \mathrm{Tr}_{S\otimes L}|_{W_{\boldsymbol{\alpha}}}, (\{\alpha_0\} \otimes \zeta_0) \boxtimes (\{\alpha_1\} \otimes \zeta_1) \boxtimes \ldots \boxtimes (\{\alpha_{e-1}\} \otimes \zeta_{e-1}) \rangle_{W_{\boldsymbol{\alpha}}} \\ &= \left\langle \mathrm{Tr}_{S(V_0^*)\otimes L(V_0^*)}, \{\alpha_0\} \otimes \zeta_0 \right\rangle_{W_{|\alpha_0|}} \cdot \ldots \\ &\qquad\qquad \cdot \left\langle \mathrm{Tr}_{S(V_{e-1}^*)\otimes L(V_{e-1}^*)}, \{\alpha_{e-1}\} \otimes \zeta_{e-1} \right\rangle_{W_{|\alpha_{e-1}|}} \end{aligned}$$

where $V^* = \oplus_{k=0}^{e-1} V_k^*$ and V_k^* are the dual spaces to the $\mathbb{C}$-vector spaces affording the reflection representations of $W_{|\alpha_k|}$.

Thus in order to calculate $P_{\chi_{\boldsymbol{\alpha}}}(q,r)$ we just need to calculate the expressions of type $\left\langle \mathrm{Tr}_{S\otimes L}, \{\alpha\} \otimes \zeta_k \right\rangle_{W_n}$ where α is a partition of n and $0 \le k \le e-1$.

We have

$$\begin{aligned} \left\langle \mathrm{Tr}_{S\otimes L}, \{\alpha\} \otimes \zeta_k \right\rangle_{W_n} &= |W_n|^{-1} \sum_{g\in(C_e)^n} \sum_{w\in S_n} \{\alpha\}(w)\zeta_k(g) \frac{\det_V(1+rwg)}{\det_V(1-qwg)} \\ &= |S_n|^{-1} \sum_{w\in S_n} \{\alpha\}(w) e^{-n} \sum_{g\in(C_e)^n} \frac{\zeta_k(g)\det_V(1+rwg)}{\det_V(1-qwg)}. \end{aligned}$$

Let us prove the following formula

$$\Delta := e^{-n} \sum_{g\in(C_e)^n} \frac{\zeta_k(g)^{-1}\det_V(1+rwg)}{\det_V(1-qwg)} = q^{kn} \frac{\det_V(1+rq^{-1+e\delta_{k,0}}w)}{\det_V(1-q^e w)} \tag{6.2.6}$$

Since each side is the product of terms attached to the cycles of w we can think without loss of generality that w is the cyclic permutation of the base of V. Thus for any variable x we have $\det_V(1+xwg) = 1-(-x)^n\zeta_1(g)$ for any $g \in (C_e)^n$.

And

$$\begin{aligned}\Delta &= e^{-n}\sum_{g\in(C_e)^n}\frac{\zeta_1(g)^{-k}(1-(-r)^n\zeta_1(g))}{1-q^n\zeta_1(g)} \\ &= e^{-1}\frac{\sum_{i=0}^{e-1}\zeta^{-ik}(1-(-r)^n\zeta^i)\prod_{j\neq i}(1-q^n\zeta^j)}{1-q^{en}}\end{aligned}$$

where the last equality is obtained by taking the sum over $g \in (C_e)^n$ with a fixed $\zeta_1(g)$.

Let $A_k(q,r)$ be the numerator of the above fraction. It is a linear polynomial in r^n and considering indices modulo e we can write $A_k(q,r) = B_k(q) - (-r)^n B_{k-1}(q)$ where $B_k(q) := \sum_{i=0}^{e-1}\zeta^{-ik}\prod_{j\neq i}(1-q^n\zeta^j)$. Note that $B_0(q) = q^{n(e-1)}\sum_{i=0}^{e-1}\prod_{j\neq i}(q^{-n}-\zeta^j) = e$ since the last sum is nothing else but the derivative of $\prod_{j=0}^{e-1}(q^{-n}-\zeta^j) = q^{-en}-1$ with respect to q^{-n}.

From

$$A_k(q,-q) = B_k(q) - q^n B_{k-1}(q) = \Big(\sum_{i=0}^{e-1}\zeta^{ik}\Big)(1-q^{en}) = e\delta_{k,0}(1-q^{en})$$

we obtain $B_k = eq^{kn}$ for $0 \le k \le e-1$.

Finally

$$\Delta = \frac{e^{-1}A_k(q,r)}{1-q^{en}} = q^{kn}\frac{1-(-rq^{-1+e\delta_{k,0}})^n}{1-q^{en}}$$

and using the fact $(1-(-x)^n) = \det_V(1+xw)$ we get the formula (6.2.6).

Now by (6.2.6) we have

$$\begin{aligned}&\Big\langle \mathrm{Tr}_{S\otimes L}, \{\alpha\}\otimes\zeta_k\Big\rangle_{W_n} \\ &\qquad = |S_n|^{-1}\sum_{w\in S_n}\{\alpha\}(w)q^{(e-k-e\delta_{k,0})n}\frac{\det_V(1+rq^{-1+e\delta_{k,0}}w)}{\det_V(1-q^e w)} \\ &\qquad = q^{(e-k-e\delta_{k,0})n}P_{\overline{\{\alpha\}}}(q^e, rq^{-1+e\delta_{k,0}}) = \\ &\qquad = q^{(e-k-e\delta_{k,0})n}P_{\{\alpha\}}(q^e, rq^{-1+e\delta_{k,0}}) = \\ &\qquad = F_{\alpha,k}(q)\prod_{x\in\alpha}(1+rq^{ec(x)-1+e\delta_{k,0}})\end{aligned}$$

where $P_{\{\alpha\}}(q,r)$ is the Molien series for the irreducible character $\{\alpha\}$ of S_n (which is rational and hence the complex conjugate $\overline{\{\alpha\}}$ is $\{\alpha\}$ itself) and $F_{\alpha,k}(q)$ is some element of $\mathbb{C}(q)$ (cf. [Gom06, Th. 2.6]).

Using this we get the formula for the Molien series for the irreducible character $\chi_{\boldsymbol{\alpha}}$ of W_n:

$$P_{\chi_{\boldsymbol{\alpha}}}(q,r) = F(q)\prod_{k=0}^{e-1}\prod_{x\in\alpha_k}(1+rq^{ec(x)-1+e\delta_{k,0}})$$

where $F(q) = \prod_{k=0}^{e-1} F_{\alpha_k,k}(q)$. And from the definition of the Molien series (cf. (6.2.1)) we obtain

$$F(q) = P_{\chi_{\boldsymbol{\alpha}}}(q,0) = \frac{R_{\chi_{\boldsymbol{\alpha}}}}{(1-q)^n S_{\mathbb{1}}}.$$

This finishes the proof of the lemma. □

Lemma 6.2.7 *Let $R_{\boldsymbol{\alpha}}$ be the specialization of $R_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1})$ (cf (3.3.5)) via*

$$u_0 \mapsto q, \quad \xi \mapsto z, \quad u_i \mapsto \zeta^i, \quad \eta_i \mapsto zq^{i-1} \quad (1 \le i \le e-1), \quad z = \frac{r(q-1)}{r+1}. \tag{6.2.8}$$

Then

$$R_{\boldsymbol{\alpha}} = \frac{(1-q)^n}{(r+1)^n} \Xi_{\boldsymbol{\alpha}} \quad \textit{(cf. (6.2.5))} \tag{6.2.9}$$

Proof. Denote by $J_{k,x}$ the polynomials in the square brackets in the defining expression for $R_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1})$ (cf (3.3.5)). Thus we have

$$R_{\boldsymbol{\alpha}}(\xi, \eta_1, \ldots, \eta_{e-1}) = \prod_{k=0}^{e-1} \prod_{x \in \alpha_k} \Big[(-1)^{e-1} \prod_{\substack{l=0 \\ l \ne k}}^{e-1} u_l^{-1} J_{k,x} \Big].$$

In order to calculate $J_{k,x}(q)$, the specializations of $J_{k,x}$ via (6.2.8) we need the following simple fact about the symmetric functions of $q, \zeta, \ldots, \zeta^{e-1}$.

Fact 6.2.10 *If*

$$(X-q)(X-\zeta)\ldots(X-\zeta^{e-1}) = X^e - \sigma_1 X^{e-1} + \sigma_2 X^{e-2} - \ldots + (-1)^e \sigma_e$$

then $\sigma_e = (-1)^{e-1} q$ and $\sigma_i = (-1)^{i-1}(q-1)$ for $0 < i < e$.

We consider two cases:

Case $k = 0$: Here we have

$$\begin{aligned} J_{0,x}(q) &= z(1-q^{c(x)}) \prod_{l=1}^{e-1} (q^{c(x)+1} - \zeta^l) \\ &\quad + (1-q)\Big(\sum_{i=1}^{e-1} (q^{(e-i)c(x)+e-i-1} \sum_{j=1}^{i} (-1)^{i-j} z q^{j-1} \sigma_{i-j}) + 1 \Big) \\ &\overset{(6.2.10)}{=} z(1-q^{c(x)}) \prod_{l=1}^{e-1} (q^{c(x)+1} - \zeta^l) \\ &\quad + (1-q)\Big(z \sum_{i=1}^{e-1} (q^{(c(x)+1)(e-i)-1} + 1 \Big) \end{aligned}$$

For simplicity introduce the notation $Y := q^{c(x)+1}$. Then the last expression is equal to

$$z\left(1-\frac{Y}{q}\right)(Y^{e-1}+\ldots+1) + z(1-q)\left(\frac{Y^{e-1}}{q}+\ldots+\frac{Y}{q}+\left(\frac{1}{q}-\frac{1}{q}\right)\right)+1-q$$

$$\begin{aligned}
&= zq^{-1}(1-Y)(Y^{e-1}+\ldots+1)+z(1-q^{-1})+1-q\\
&= zq^{-1}(1-Y^e)+z(1-q^{-1})+1-q\\
&= z(1-Y^e q^{-1})+1-q\\
&= \frac{1-q}{r+1}(1+rq^{ec(x)-1+e}).
\end{aligned}$$

Case $k \geq 1$: Using (6.2.10) again we have

$$J_{k,x}(q) = z(1-q^{c(x)})(\zeta^k q^{c(x)}-q)\prod_{\substack{l=1\\ l\neq k}}^{e-1}(\zeta^k q^{c(x)}-\zeta^l) + (1-q)\Big(z\sum_{i=1}^{e-1}(\zeta^{k(e-i-1)}q^{(e-i)c(x)})+\zeta^{e-k}q\Big)$$

Making the notation $Y := \zeta^k q^{c(x)}$ we rewrite the last expression as

$$\begin{aligned}
&z(1-\zeta^{-k}Y)(Y-q)\prod_{\substack{l=1\\ l\neq k}}^{e-1}(Y-\zeta^l)+(1-q)\left(z\sum_{i=1}^{e-1}(\zeta^{-k}Y^{e-i})+\zeta^{-k}q\right)\\
&= -\zeta^{-k}\Big[z(Y-q)(Y^{e-1}+\ldots+1)\\
&\qquad + z(q-1)(Y^{e-1}+\ldots+Y+(1-1))+q(q-1)\Big]\\
&= -\zeta^{-k}(z(Y^e-1)-z(q-1)+q(q-1)) = -\zeta^{-k}(z(Y^e-q)+q(q-1))\\
&= \zeta^{-k}q\frac{1-q}{r+1}(1+rq^{ec(x)-1}).
\end{aligned}$$

This combined with (6.2.10) finishes the proof □

Remark 6.2.11 *The above lemma implies that the weight of the special Markov trace τ_e corresponding to the irreducible character $\chi_{\boldsymbol{\alpha}}$ (cf. (3.3.5)) is equal to*

$$\frac{(1-q)^n}{(r+1)^n}D_{\boldsymbol{\alpha}}(q)\Xi_{\boldsymbol{\alpha}}$$

where $D_{\boldsymbol{\alpha}}(q) := D_{S_{\boldsymbol{\alpha}}}(q;1,\zeta,\ldots,\zeta^{e-1})$.

Lemma 6.2.12 *For an e-partition* $\boldsymbol{\alpha}$ *of* n *let* $S_{\boldsymbol{\alpha}} = (S_0, \ldots, S_{e-1})$ *be the corresponding reduced Symbol with* $I(S) = em + 1$ *(cf. (5.3.6)). Then* $\Xi_{\boldsymbol{\alpha}}$ *(cf. (6.2.5)) depends only on the multiset* $S_{\boldsymbol{\alpha}}$ *and thus is constant on the family of* $\chi_{\boldsymbol{\alpha}}$ *(cf. (5.3.16)).*

Proof. In order to prove the lemma we will show that the multiset

$$C_{\boldsymbol{\alpha}} := \{c(x_0) + 1 + m, c(x_1) + m, \ldots, c(x_{e-1}) + m | x_i \in \alpha_i, 0 \leq i \leq e-1\}$$

depends only on the multiset $S_{\boldsymbol{\alpha}}$. Indeed, it is not hard to see that the multiset

$$C_{\alpha_0} := \{c(x) + m + 1 | x \in \alpha_0\}$$

is obtained from the multiset

$$\{S_{0,m+1}, S_{0,m+1} - 1, \ldots, 1, 0, S_{0,m}, S_{0,m} - 1, \ldots, 0, \ldots, S_{0,1}, \ldots, 0\}$$

by removing its multisubset

$$\{m, m-1, \ldots, 1, 0, m-1, m-2, \ldots, 0, \ldots, 1, 0, 0\}.$$

And the multisets

$$C_{\alpha_i} := \{c(x) + m | x \in \alpha_i\} \quad (1 \leq i \leq e-1)$$

are obtained from the multisets

$$\{S_{i,m}, S_{i,m} - 1, \ldots, 1, 0, S_{i,m-1}, S_{i,m-1} - 1, \ldots, 0, \ldots, S_{i,1}, \ldots, 0\}$$

by removing their multisubsets

$$\{m-1, m-2, \ldots, 1, 0, m-2, \ldots, 0, \ldots, 1, 0, 0\} \quad \square$$

Now we are ready to finish the proof of the theorem. If we pose $r = 0$ then $z = 0$ and the trace τ_e is just the symmetrizing form t_0 on the spetsial Hecke algebra $\mathcal{H}$ (cf. [GIM00, Lemma 4.3]).

From (6.2.11) and (5.3.8) we obtain two decompositions of t_0:

$$t_0 = \sum_{\chi_{\boldsymbol{\alpha}} \in \mathrm{Irr} W_n} (1-q)^n D_{\boldsymbol{\alpha}}(q) {\chi_{\boldsymbol{\alpha}}}^{(q)} = \sum_{\chi_{\boldsymbol{\alpha}} \in \mathrm{Irr} W_n} \frac{\gamma_{\chi_{\boldsymbol{\alpha}}}(q)}{S_{\mathbb{1}}} {\chi_{\boldsymbol{\alpha}}}^{(q)}.$$

Thus

$$D_{\boldsymbol{\alpha}}(q) = \frac{\gamma_{\chi_{\boldsymbol{\alpha}}}(q)}{(1-q)^n S_{\mathbb{1}}} \overset{(5.3.33)}{=} \sum_{\chi_{\boldsymbol{\alpha}'} \in \mathrm{Irr} W_n} \mathbf{S}_{S_{\boldsymbol{\alpha}}, S_{\boldsymbol{\alpha}'}} \frac{R_{\chi_{\boldsymbol{\alpha}'}}}{(1-q)^n S_{\mathbb{1}}}.$$

If $\chi_{\boldsymbol{\alpha}}$ and $\chi_{\boldsymbol{\alpha}'}$ are in the same family then by lemmae (6.2.4), (6.2.7) and (6.2.12) we have

$$\Xi_{\boldsymbol{\alpha}} \frac{R_{\chi_{\boldsymbol{\alpha}'}}}{(1-q)^n S_{\mathbb{1}}} \overset{(6.2.12)}{=} \Xi_{\boldsymbol{\alpha}'} \frac{R_{\chi_{\boldsymbol{\alpha}'}}}{(1-q)^n S_{\mathbb{1}}} = P_{\chi_{\boldsymbol{\alpha}'}}(q, r).$$

Let us apply the above equalities to the weight $\left(\frac{1-q}{1+r}\right)^n D_{\boldsymbol{\alpha}}(q)\Xi_{\boldsymbol{\alpha}}$ of τ_e corresponding to the character $\chi_{\boldsymbol{\alpha}}$ (cf. (6.2.11)):

$$\begin{aligned}\left(\frac{1-q}{1+r}\right)^n D_{\boldsymbol{\alpha}}(q)\Xi_{\boldsymbol{\alpha}} &= \left(\frac{1-q}{1+r}\right)^n \sum_{\chi_{\boldsymbol{\alpha}'}\in\mathrm{Irr}W_n} \mathbf{S}_{S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}'}}\Xi_{\boldsymbol{\alpha}}\frac{R_{\chi_{\boldsymbol{\alpha}'}}}{(1-q)^n S_{\mathbb{1}}} \\ &= \left(\frac{1-q}{1+r}\right)^n \sum_{\chi_{\boldsymbol{\alpha}'}\in\mathrm{Irr}W_n} \mathbf{S}_{S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}'}} P_{\chi_{\boldsymbol{\alpha}'}}(q,r) \\ &= \omega^{\chi_{\boldsymbol{\alpha}}}\end{aligned}$$

since $\mathbf{S}_{S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}'}} = 0$ when $\chi_{\boldsymbol{\alpha}}$ and $\chi_{\boldsymbol{\alpha}'}$ are in different families (cf. (5.3.25)). This finishes the proof of the theorem. □

6.3 An extension of Gomi's Formula to the Case of G(e,e,n)

In this section we consider the case of the group $\widetilde{W}_n^{(e)} = G(e,e,n)$. At the moment the Lusztig Fourier matrix for the group $\widetilde{W}_n^{(e)}$ is not completely defined, but we will prove that the conditions on the matrix $\mathbf{S}$ in the conjecture (5.4.34) are sufficient to formulate and prove the analogue of Gomi's theorem (6.1.3) for the group $\widetilde{W}_n^{(e)}$.

Theorem 6.3.1 *For $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}$ define the weight $\omega_\mu \in \mathbb{C}[\![q]\!][r]$ by*

$$\omega_\mu := \left(\frac{1-q}{1+r}\right)^n \cdot \sum_{\mu'\in\mathrm{Irr}\widetilde{W}_n^{(e)}} \mathbf{S}_{\mu,\mu'}P_{\mu'}(q,r), \tag{6.3.2}$$

where $\mathbf{S}_{\mu,\mu'}$ is the entry of the matrix $\mathbf{S}$ from the conjecture (5.4.34) and $P_{\mu'}(q,r)$ is the Molien series for μ' defined by analogy with (6.2.1).

Let τ_g be the trace on the Hecke algebra $\widetilde{\mathcal{H}}_n^{(e)}$ (cf. (3.4.1)) defined as follows

$$\tau_g := \sum_{\mu\in\mathrm{Irr}\widetilde{W}_n^{(e)}} \omega_\mu \mu^{(q)},$$

where $\mu^{(q)}$ is the irreducible character of the Hecke algebra $\widetilde{\mathcal{H}}_n^{(e)}$ corresponding to the character μ of $\widetilde{W}_n^{(e)}$. Then τ_g coincides with the special Markov trace $\widetilde{\tau}_e$ for the group $\widetilde{W}_n^{(e)}$ with the parameter $z = \frac{r(q-1)}{r+1}$ (cf. (3.4.4)).

Proof. In order to prove the theorem we need to show that for any $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}$ the weight ω_μ of τ_g is equal to the weight ω'_μ of $\widetilde{\tau}_e$ for $z = \frac{r(q-1)}{r+1}$ (cf. (3.4.10)). We use some ideas from Gomi's proof of theorem (6.1.3) in the case when W is of type D_n (cf. [Gom06, (4.5)]). We start with the following simple lemma:

Lemma 6.3.3 *For any $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}$ the weight $\omega_\mu \in \mathbb{C}[\![q]\!][r]$ is a rational function from $\mathbb{C}(q,r)$.*

Proof. To prove that $\omega_\mu \in \mathbb{C}(q,r)$ it is enough to prove that $P_{\mu'}(q,r) \in \mathbb{C}(q,r)$ for any $\mu' \in \mathrm{Irr}\widetilde{W}_n^{(e)}$. By definition of the Molien series we have:

$$P_{\mu'}(q,r) = \left\langle \mathrm{Tr}_{S\otimes L}|_{\widetilde{W}_n^{(e)}}, \mu' \right\rangle_{\widetilde{W}_n^{(e)}} = \sum_{w\in\widetilde{W}_n^{(e)}} \mu'(w)\frac{\det_V(1+rw)}{\det_V(1-qw)} \in \mathbb{C}(q,r),$$

where $S \otimes_{\mathbb{C}} L$ is the $\mathbb{C}$-algebra from (6.2.1). □

The above lemma and the fact that $\omega'_\mu \in \mathbb{C}(q,z) = \mathbb{C}(q,r)$ (cf. (3.4.3)) imply that if we prove that $\omega_\mu = \omega'_\mu$ for $r = -q^{em+1}$ for all large enough integers m then we will prove the theorem. We will need the following notation:

Notation 6.3.4 *For an e-Symbol $S = (S_0, \dots, S_{e-1})$ with $S_i = (0 \le \lambda_{i,1} < \dots \lambda_{i,m})$ and $0 \le k \le e-1$ define the Symbols*

$$^kS^{(0)} := \begin{pmatrix} 0 & \lambda_{-k,1}+1 & \dots & \lambda_{-k,m}+1 \\ & \lambda_{1-k,1}+1 & \dots & \lambda_{1-k,m}+1 \\ & \vdots & & \vdots \\ & \lambda_{e-1-k,1}+1 & \dots & \lambda_{e-1-k,m}+1 \end{pmatrix}. \tag{6.3.5}$$

These Symbols are reduced in the sense of (5.3.3).

We proceed with two lemmae giving us expressions for $P_\mu(q, -q^{em+1})$ and ω'_μ for $r = -q^{em+1}$.

Lemma 6.3.6 *Let $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)}$ and $S = \underline{S}_{\boldsymbol{\alpha}(\mu)}$ is the Symbol with em entries attached to an e-partition $\boldsymbol{\alpha}(\mu)$ for μ (cf (5.4.3)). Then we have:*

$$\omega'_\mu = \left(\frac{1-q}{1-q^{em+1}}\right)^n \cdot \frac{\prod_{i=0}^{e-1}\Theta(^0S_i^{(0)}, q^e)}{q^{m^2\binom{e}{2}}U_m\prod_{i=1}^n(q^{ei}-1)} \cdot \frac{1}{s(\boldsymbol{\alpha}(\mu))} \cdot \sum_{k=0}^{e-1}\mathrm{Deg}_{^kS^{(0)}}(q),$$

where $U_m = \left(\prod_{i=1}^{m-1}\prod_{j=1}^{m-i}(q^{ej}-1)\right)^e \cdot \prod_{j=1}^m(q^{ej}-1)$; generic degrees $\mathrm{Deg}_{^kS^{(0)}}(q)$ and $\Theta(^0S_i^{(0)}, q^e)$ are as in (5.3.4).

Proof. By (5.4.10) we have

$$\omega'_\mu = \frac{(-1)^n\mathrm{Deg}_{[\underline{S}_{\boldsymbol{\alpha}(\mu)}]}(q)}{e(q^n-1)\prod_{i=1}^{n-1}(q^{ei}-1)}\sum_{k=0}^{e-1}R_{\pi^k(\boldsymbol{\alpha}(\mu))}(z,\eta_1,\dots,\eta_{e-1}).$$

If $r = -q^{em+1}$ then $z = \frac{q^{em+1}(q-1)}{q^{em+1}-1}$ and by (3.4.1) we get $\eta_i = \frac{q^{m(e-i)}(q-1)}{q^{em+1}-1}$. Denote the entries of the Symbol S by $S_i = (\lambda_{i,1}, \ldots, \lambda_{i,m})$. By (3.3.6) we get

$$R_{\pi^k(\boldsymbol{\alpha}(\mu))}(z, \eta_1, \ldots, \eta_{e-1}) = \left(\frac{1-q}{q^{em+1}-1}\right)^n \cdot$$

$$\cdot \prod_{i=0}^{e-1} \prod_{x \in \alpha(\mu)_{i-k}} (\zeta^i q^{c(x)+m+1} - 1)(\zeta^i q^{c(x)+m} - \zeta) \ldots (\zeta^i q^{c(x)+m} - \zeta^{e-1}) =$$

$$\left(\frac{1-q}{q^{em+1}-1}\right)^n \cdot \prod_{i=0}^{e-1} \left(\prod_{j=1}^{m} \frac{q^{\lambda_{i-k,j}+1}\zeta^i - 1}{q^j \zeta^i - 1} \cdot \prod_{x \in \alpha(\mu)_{i-k}} (q^{e(c(x)+m)} - 1) \right) =$$

$$= (-1)^{m(e-1)} \cdot \left(\frac{1-q}{q^{em+1}-1}\right)^n \cdot \frac{\prod_{i=0}^{e-1} \prod_{j=1}^{m} (q^{\lambda_{i-k,j}+1}\zeta^i - 1) \cdot \prod_{i=0}^{e-1} \Theta(S_i, q^e)}{U_m}.$$

In the last two equalities we used the fact that the multisets

$$\{c(x) + m | x \in \alpha(\mu)_{i-k}\}$$

are obtained from the multisets

$$\{\lambda_{i-k,m}, \lambda_{i-k,m} - 1, \ldots, 1, 0, \lambda_{i-k,m-1}, \lambda_{i-k,m-1} - 1, \ldots, 0, \ldots, \lambda_{i-k,1}, \ldots, 0\}$$

by removing their multisubsets

$$\{m-1, m-2, \ldots, 1, 0, m-2, \ldots, 0, \ldots, 1, 0, 0\}.$$

Combining the expression for $R_{\pi^k(\boldsymbol{\alpha}(\mu))}(z, \eta_1, \ldots, \eta_{e-1})$ and formulae for $\mathrm{Deg}_{[\underline{S}_{\boldsymbol{\alpha}(\mu)}]}(q)$ (cf. (5.4.4)) we obtain:

$$\omega'_\mu = \frac{1}{s(S)} \frac{(-1)^{\binom{e}{2}\binom{m}{2}+m(e-1)} \cdot \prod_{i=0}^{e-1} \prod_{j=i}^{e-1} \prod_{\substack{(\lambda,\mu) \in S_i \times S_j \\ \mu < \lambda \text{ if } i=j}} (q^\lambda \zeta^i - q^\mu \zeta^j)}{\tau(e)^m q^{\binom{e(m-1)}{2} + \binom{e(m-2)}{2} + \cdots} U_m}.$$

$$\cdot \left(\frac{1-q}{1-q^{em+1}}\right)^n \cdot \sum_{k=0}^{e-1} \prod_{i=0}^{e-1} \prod_{j=1}^{m} (q^{\lambda_{i-k,j}+1}\zeta^i - 1).$$

Multiplying the numerator and the denominator of the above fraction by $q^{e\binom{m}{2}+m^2\binom{e}{2}}$ and comparing the summands in the above sum with the formulae for $\mathrm{Deg}_{^kS^{(0)}}(q)$ (cf. (5.3.4)) we finish the proof of the lemma. □

Lemma 6.3.7 *Keeping the notations from the previous lemma we have the following formula:*

$$P_\mu(q, -q^{em+1}) = \frac{\prod_{i=0}^{e-1} \Theta({}^0S_i^{(0)}, q^e)}{q^{m^2\binom{e}{2}} U_m \prod_{i=1}^{n} (q^{ei} - 1)} \cdot \frac{1}{s(\boldsymbol{\alpha}(\mu))} \cdot \sum_{k=0}^{e-1} R_{^kS^{(0)}},$$

where $R_{{}^kS^{(0)}}$ are fake degrees corresponding to ${}^kS^{(0)}$ (cf. (5.3.29)).

Proof. We consider the group $\widetilde{W}_n^{(e)}$ as a subgroup of the group $W_n^{(e)} := G(e,1,n)$. Both $\widetilde{W}_n^{(e)}$ and $W_n^{(e)}$ act on the $\mathbb{C}$-algebra $S \otimes_{\mathbb{C}} L$ (cf. (6.2.1)).

By Frobenius reciprocity we have:

$$P_\mu(q,r) = \left\langle \mathrm{Tr}_{S\otimes L}|_{\widetilde{W}_n^{(e)}}, \mu \right\rangle_{\widetilde{W}_n^{(e)}} = \left\langle \mathrm{Tr}_{S\otimes L}, \mathrm{Ind}_{\widetilde{W}_n^{(e)}}^{W_n^{(e)}} \mu \right\rangle_{W_n^{(e)}} =$$
$$= \left\langle \mathrm{Tr}_{S\otimes L}, \frac{1}{s(\boldsymbol{\alpha}(\mu))} \sum_{k=0}^{e-1} \chi_{\pi^k(\boldsymbol{\alpha}(\mu))} \right\rangle_{W_n^{(e)}} = \frac{1}{s(\boldsymbol{\alpha}(\mu))} \sum_{k=0}^{e-1} P_{\chi_{\pi^k(\boldsymbol{\alpha}(\mu))}}(q,r).$$

For $0 \le k \le e-1$ denote the Symbols $S_{\pi^k(\boldsymbol{\alpha}(\mu))}$ (cf. (5.3.6)) by $\pi^k S$. Combining the formula for Molien series from lemma (6.2.4) and the formula for fake degrees from proposition (5.3.11) we have:

$$P_\mu(q,-q^{em+1}) = \frac{(-1)^n}{s(\boldsymbol{\alpha}(\mu))} \sum_{k=0}^{e-1} \prod_{i=0}^{e-1} \frac{\Delta(\pi^k S_i, q^e)}{\Theta(\pi^k S_i, q^e)} \cdot \frac{\prod_{i=1}^{e-1} \prod_{\lambda \in \pi^k S_i} q^{(e-i)\lambda}}{q^{\binom{e(m-1)+1}{2} + \binom{e(m-2)+1}{2} + \dots}} \cdot$$
$$\prod_{i=0}^{e-1} \prod_{x \in \alpha(\mu)_{i-k}} (1 - q^{e(c(x)+m+e\delta_{i,0})}) =$$
$$= \frac{1}{s(\boldsymbol{\alpha}(\mu))} \sum_{k=0}^{e-1} \frac{\prod_{i=0}^{e-1} \Delta(\pi^k S_i, q^e) \cdot \prod_{i=1}^{e-1} \prod_{\lambda \in \pi^k S_i} q^{(e-i)\lambda}}{q^{\binom{e(m-1)+1}{2} + \binom{e(m-2)+1}{2} + \dots} U_m}.$$

The definitions of ${}^kS^{(0)}$ and $\pi^k S$ imply that the Symbol ${}^kS^{(0)}$ can be obtained from $\pi^k S$ by adding 1 to all entries in the rows $1, \dots, e-1$. Hence multiplying the numerators and the denominators of the fractions in the above sum by $q^{e(e-1)\binom{m}{2} + m\binom{e}{2}}$ and comparing the summands with the formulae for $R_{{}^kS^{(0)}}$ (cf. (5.3.11)) we finish the proof of the lemma. □

Let $\mathcal{F}^\mu$ be the Family of the equivalence class $[\underline{S}_{\boldsymbol{\alpha}(\mu)}]$ (cf. (5.4.15)). Denote by $\mathcal{F}_0^\mu$ the subset of $\mathcal{F}^\mu$ of equivalence classes whose Symbols have the same number of elements in each row. It is easy to see that $\mathcal{F}_0^\mu$ is the intersection of $\mathcal{F}^\mu$ and $\{[\underline{S}_{\boldsymbol{\alpha}(\mu')}] \mid \mu' \in \mathrm{Irr}\widetilde{W}_n^{(e)}\}$.

Let as above $S = \underline{S}_{\boldsymbol{\alpha}(\mu)}$ be a Symbol with em entries. We will work with the group $W_{n+m(e-1)} = G(e,1,n+m(e-1))$. Denote by $\mathcal{F}$ the Lusztig Family in the sense of (5.3.14) of the reduced symbol ${}^0S^{(0)} \in \mathcal{U}(W_{n+m(e-1)})$ and by $\mathcal{F}_0$ the intersection $\mathcal{F} \cap \mathrm{Irr}W_{n+m(e-1)}$ where $\mathrm{Irr}W_{n+m(e-1)}$ is considered as a subset of $\mathcal{U}(W_{n+m(e-1)})$. For an equivalence class $[S'] \in \mathcal{F}_0^\mu$ denote by S' any of its representatives with em elements. Then one can see that the disjoint union

$$\mathcal{F}_0^{(0)} := \coprod_{[S'] \in \mathcal{F}_0^\mu} \{{}^0S'^{(0)}, {}^1S'^{(0)}, \dots, {}^{e/s[S']-1}S'^{(0)}\}$$

is equal to the subset of $\mathcal{F}_0$ made up by Symbols with zero in the upper row.

Let $\mathbf{S}^*$ be the Lusztig Fourier transform matrix of the group $W_{n+m(e-1)}$ (cf. (5.3.25)). By corollary (5.3.33) the matrix $\mathbf{S}^*$ indexed by the set $\mathcal{U}(W_{n+m(e-1)})$ sends the vector $(R_{S'})_{S'\in\mathcal{U}(W_{n+m(e-1)})}$ to the vector $(\mathrm{Deg}_{S'})_{S'\in\mathcal{U}(W_{n+m(e-1)})}$. Restricting ourselves to the block of $\mathbf{S}^*$ corresponding to the Family $\mathcal{F}$ we see that the part of $\mathbf{S}^*$ indexed by the set $\mathcal{F}_0$ sends the vector $(R_{S'})_{S'\in\mathcal{F}_0}$ to the vector $(\mathrm{Deg}_{S'})_{S'\in\mathcal{F}_0}$ since the vector $(R_{S'})_{S'\in\mathcal{F}}$ is zero outside of $\mathcal{F}_0$. In particular

$$\mathrm{Deg}_{{}^kS^{(0)}} = \sum_{S'\in\mathcal{F}_0} \mathbf{S}^*_{{}^kS^{(0)},S'} R_{S'}, \quad 0 \le k \le e-1.$$

Taking the sum of the above equalities for $0 \le k \le e-1$ we get

$$\sum_{k=0}^{e-1} \mathrm{Deg}_{{}^kS^{(0)}} = \sum_{S'\in\mathcal{F}_0} \Big(\sum_{k=0}^{e-1} \mathbf{S}^*_{{}^kS^{(0)},S'}\Big) R_{S'}. \tag{6.3.8}$$

Lemma 6.3.9 *Let $[S]$ be in $\mathcal{F}_0^\mu$ and S be a representative of $[S]$ with em elements. Let $S' \in \mathcal{F}_0$.*

(a) If $S' \notin \mathcal{F}_0^{(0)}$ then $\sum_{k=0}^{e-1} \mathbf{S}^*_{{}^kS^{(0)},S'} = 0$.

(b) If $S' \in \mathcal{F}_0^{(0)}$ define S'' by $S' = {}^iS''^{(0)}$ for some i. Then

$$\sum_{k=0}^{e-1} \mathbf{S}^*_{{}^kS^{(0)},S'} = \mathbf{S}'_{[S],[S'']}$$

where $\mathbf{S}'$ is the pre-Fourier matrix of the group $\widetilde{W}_n^{(e)}$ defined in (5.4.32).

Proof. Let Y be a totally ordered set with em elements and $\pi : Y \to \mathbb{Z}_{\ge 0}$ be a map corresponding to the Lusztig Family $\mathcal{F}^\mu$ (cf. (5.4.27)). Denote by Y^* the totally ordered set $Y \sqcup \{\overline{y}\}$ where $\overline{y}$ is the maximal element. Let $\pi^* : Y^* \to \mathbb{Z}_{\ge 0}$ be the map defined by

$$\pi^*(y) = \begin{cases} \pi(y)+1 & \text{if } y \in Y, \\ 0 & \text{if } y = \overline{y}. \end{cases}$$

Then the Fourier transform $T(Y^*, \pi^*)$ can be used to construct the block corresponding to the Family $\mathcal{F}$ of the Lusztig Fourier transform matrix $\mathbf{S}^*$ (cf. (5.3.25)). Denote by Ψ^* the set of maps

$$\psi : Y^* \to \{0,\ldots,e-1\}, \text{ such that } \sum_{y\in Y^*} \psi(y) \equiv m\binom{e}{2} \pmod e,$$

with the subset Ψ_0^* of Ψ^* such that $|\psi^{-1}(i)| = m + \delta_{0,i}$ (cf. (5.3.4)). For $\psi \in \Psi^*$ we will denote by $\psi \oplus j$ the element of Ψ^* such that $(\psi \oplus k)(y) \equiv \psi(y) + k - k\delta_{y,\overline{y}} (\mathrm{mod}\ e)$. We keep unchanged other notations from (5.3.4).

Let $[\phi] = \kappa^{-1}({}^0S^{(0)})$ where κ is from (5.3.20)). Then $[\phi \oplus k] = \kappa^{-1}({}^kS^{(0)})$ for any $0 \leq k \leq e-1$. By theorem (5.3.23) and definition (5.3.25) of the Lusztig Fourier transform matrix $\mathbf{S}^*$ we have:

$$\sum_{k=0}^{e-1} \mathbf{S}^*_{{}^kS^{(0)},S'} = \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{k=0}^{e-1} \sum_{\nu \in [\phi \oplus k]} \epsilon(\nu)\epsilon(\psi) \prod_{y \in Y^*} \zeta^{-\nu(y)\psi(y)} \tag{6.3.10}$$

where ψ is any representative of the equivalence class $\kappa^{-1}(S')$.

It is not hard to see that if $\nu \in [\phi] \subset \Psi^*$ then $\epsilon(\nu \oplus k) = (-1)^{km(e-1)}\epsilon(\nu)$ and $\nu \oplus k \in [\phi \oplus k]$. Hence the above sum is equal to

$$\frac{(-1)^{m(e-1)}}{\tau(e)^m}\epsilon(\psi) \sum_{\nu \in [\phi]} \epsilon(\nu) \sum_{k=0}^{e-1} (-1)^{km(e-1)} \prod_{y \in Y^*} \zeta^{-(\nu(y) \oplus k)\psi(y)}. \tag{6.3.11}$$

We have

$$\sum_{k=0}^{e-1} (-1)^{km(e-1)} \prod_{y \in Y^*} \zeta^{-(\nu(y) \oplus k)\psi(y)} =$$
$$\prod_{y \in Y^*} \zeta^{-\nu(y)\psi(y)} \cdot \sum_{k=0}^{e-1} (-1)^{km(e-1)} \prod_{y \in Y} \zeta^{-k\psi(y)} =$$
$$\prod_{y \in Y^*} \zeta^{-\nu(y)\psi(y)} \cdot \sum_{k=0}^{e-1} \zeta^{k\psi(\overline{y})},$$

where the second equality comes from the fact that $\psi(\overline{y}) + \sum_{y \in Y} \psi(y) \equiv m\binom{e}{2} (\mathrm{mod}\ e)$. The fact $S' \notin \mathcal{F}_0^{(0)}$ implies that $\psi(\overline{y}) \neq 0$ hence $\sum_{k=0}^{e-1} \zeta^{k\psi(\overline{y})} = 0$. This proves the first part of the lemma.

In order to prove the second part we will use the notations from (5.4.4). Let Ψ be the set of maps

$$\psi : Y \to \{0, \ldots, e-1\}, \text{ such that } \sum_{y \in Y} \psi(y) \equiv m\binom{e}{2} \quad (\mathrm{mod}\ e),$$

with the subset Ψ_0 of Ψ such that $|\psi^{-1}(i)| = m$. Then there is the obvious bijection

$$\Psi_0^{*(0)} := \{\psi \in \Psi_0^* \mid \psi(\overline{y}) = 0\} \longrightarrow \Psi_0 \text{ given by } \psi \mapsto \widetilde{\psi} : \widetilde{\psi}(y) = \psi(y), y \in Y.$$

It is obvious that if $\psi = \phi \oplus k$ then $\widetilde{\psi} = \widetilde{\phi} + k$. The fact that $\overline{y}$ is maximal in Y^* implies that $\epsilon(\psi) = \epsilon'(\widetilde{\psi})$ for $\psi \in \Psi_0^{*(0)}$. Let ψ be any representative of the equivalence class $\kappa^{-1}({}^iS''^{(0)})$. Then formula (6.3.10) implies

$$\sum_{k=0}^{e-1} \mathbf{S}^*_{{}^kS^{(0)},S'} = \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{k=0}^{e-1} \sum_{\widetilde{\nu}\in[\widetilde{\phi}]_+} \epsilon'(\widetilde{\nu})\epsilon'(\widetilde{\psi}) \prod_{y\in Y} \zeta^{-\widetilde{\nu}(y)\widetilde{\psi}(y)}$$

Comparing the above expression with the formula (5.4.30) we see that it coincides with the entry indexed by $(f_{[\widetilde{\phi}]_+}, f_{[\widetilde{\psi}]_+})$ of the matrix $T(Y,\pi+)$. The facts that $[\phi] = \kappa^{-1}({}^0S^{(0)})$ and $[\psi] = \kappa^{-1}({}^iS''^{(0)})$ imply that $\iota_{\mathcal{F}}([S]) = f_{[\widetilde{\phi}]_+}$ and $\iota_{\mathcal{F}}([S'']) = f_{[\widetilde{\psi}]_+}$ where $\iota_{\mathcal{F}}$ is defined in (5.4.31). Hence by (5.4.32) we have $T(Y,\pi+)_{f_{[\widetilde{\phi}]_+}, f_{[\widetilde{\psi}]_+}} = \mathbf{S}'_{[S],[S'']}$. This finishes the proof of the lemma. □

Formula (6.3.8) and the above lemma imply

$$\sum_{k=0}^{e-1} \mathrm{Deg}_{{}^kS^{(0)}} = \sum_{S'\in\mathcal{F}_0^{(0)}} \Big(\sum_{k=0}^{e-1} \mathbf{S}^*_{{}^kS^{(0)},S'}\Big) R_{S'} =$$

$$\sum_{[S'']\in\mathcal{F}_0^{\mu}} \mathbf{S}'_{[S],[S'']} \sum_{i=0}^{e/s[S'']-1} R_{{}^iS''^{(0)}} = \sum_{[S'']\in\mathcal{F}_0^{\mu}} \frac{\mathbf{S}'_{[S],[S'']}}{s[S'']} \sum_{k=0}^{e-1} R_{{}^kS''^{(0)}}.$$

Now we are ready to prove that $\omega_\mu = \omega'_\mu$ for $r = -q^{em+1}$.
We start with the case when $s(\boldsymbol{\alpha}(\mu)) < e$. Let $\mathcal{F}'_\mu$ be the family of $\mu \in \mathrm{Irr}\widetilde{W}_n^{(e)} \subset \mathcal{U}(\widetilde{W}_n^{(e)})$ (cf. (5.4.18)). Then

$$\mathcal{F}'_\mu = \coprod_{[S']\in\mathcal{F}_0^{\mu}} \{[S']_0, \ldots, [S']_{s[S']-1}\}.$$

We denote by $\nu(\mathcal{F}'_\mu)$ the following expression figuring in lemmae (6.3.6) and (6.3.7):

$$\frac{\prod_{i=0}^{e-1} \Theta({}^0S_i^{(0)}, q^e)}{q^{m^2\binom{e}{2}} U_m \prod_{i=1}^{n}(q^{ei}-1)}.$$

It is not hard to see that it is independent of the choice of μ in the family $\mathcal{F}'_\mu$.

Properties of the matrtix $\mathbf{S}$ from the conjecture (5.4.34) and the formula for $P_{\mu'}(q, -q^{em+1})$ in lemma (6.3.7) imply:

$$\omega_\mu = \left(\frac{1-q}{1-q^{em+1}}\right)^n \cdot \sum_{\mu'\in\mathrm{Irr}\widetilde{W}_n^{(e)}} \mathbf{S}_{\mu,\mu'} P_{\mu'}(q, -q^{em+1}) =$$

$$\left(\frac{1-q}{1-q^{em+1}}\right)^n \cdot \sum_{\mu'\in\mathcal{F}'_\mu} \mathbf{S}_{\mu,\mu'} P_{\mu'}(q, -q^{em+1}) =$$

$$\left(\frac{1-q}{1-q^{em+1}}\right)^n \nu(\mathcal{F}'_\mu) \sum_{[S']\in\mathcal{F}_0^\mu} \frac{1}{s[S']} \sum_{j=0}^{s[S']-1} \mathbf{S}_{\mu,[S']_j} \sum_{k=0}^{e-1} R_{{}^k S'^{(0)}} =$$

$$\left(\frac{1-q}{1-q^{em+1}}\right)^n \nu(\mathcal{F}'_\mu) \cdot \frac{1}{s(\boldsymbol{\alpha}(\mu))} \cdot \sum_{[S']\in\mathcal{F}_0^\mu} \frac{\mathbf{S}'_{[S],[S']}}{s[S']} \sum_{k=0}^{e-1} R_{{}^k S'^{(0)}} =$$

$$\left(\frac{1-q}{1-q^{em+1}}\right)^n \nu(\mathcal{F}'_\mu) \cdot \frac{1}{s(\boldsymbol{\alpha}(\mu))} \cdot \sum_{k=0}^{e-1} \mathrm{Deg}_{{}^k S^{(0)}} \overset{(6.3.6)}{=} \omega'_\mu.$$

We pass to the case when $s(\boldsymbol{\alpha}(\mu)) = e$. In this case we have $\mathcal{F}'_\mu = \{\mu\}, \mathcal{F}_0^\mu = \{[S]\}$ and $\mathcal{F} = \{{}^0 S^{(0)}\}$. Hence

$$\omega_\mu = \left(\frac{1-q}{1-q^{em+1}}\right)^n P_\mu(q, -q^{em+1}) \overset{(6.3.7)}{=} \left(\frac{1-q}{1-q^{em+1}}\right)^n \nu(\mathcal{F}'_\mu) R_{{}^0 S^{(0)}}.$$

Corollary (5.3.33) combined with the fact that the Lusztig family of ${}^0 S^{(0)}$ has only one element implies $\mathrm{Deg}_{{}^0 S^{(0)}} = R_{{}^0 S^{(0)}}$. Thus $\omega_\mu = \omega'_\mu$ and the theorem is proved. □

Chapter 7

Gomi and Khovanov-Rozansky Traces

This chapter is devoted to the proof of the following conjecture:

Conjecture 7.0.1 (Jean Michel) *Let (W,S) be a finite Coxeter system. Then the Khovanov-Rozansky trace τ_{kr} (cf. (4.4.8)) coincides with the special Markov trace τ_g for the group W introduced by Gomi (cf. (6.1.3)).*

This conjecture tells us that the special Markov trace, the main subject of the study in this thesis, can be constructed using the Hochschild homology of Soergel bimodules from chapter 4.

7.1 Case of Weyl Groups

In this section we present a sketch of Ben Webster and Geordie Williamson's proof of Jean Michel's conjecture in the case when W is a Weyl group. Their proof uses some difficult geometric notions which we will not define in this thesis. Interested readers can look at their article [WW09].

Let $\mathbf{G}$ be a connected affine algebraic group over $\overline{\mathbb{F}}_q$, split over $\mathbb{F}_q$, containing a rational Borel subgroup $\mathbf{B}$ and a split maximal torus $\mathbf{T}_1 \subset \mathbf{B}$ such that $W = N_{\mathbf{G}}(\mathbf{T}_1)/\mathbf{T}_1$. For $w \in W$ denote by $\dot{w}$ a representative in $N_{\mathbf{G}}(\mathbf{T}_1)$ of w.

Definition 7.1.1 *Let $\mathbf{IC}_w$ denote the intersection cohomology complex corresponding to the $\mathbf{B} \times \mathbf{B}$ orbit $\mathbf{B}\dot{w}\mathbf{B}$, normalized so that the restriction of $\mathbf{IC}_w$ to $\mathbf{B}\dot{w}\mathbf{B}$ is the constant sheaf $\underline{\overline{\mathbb{Q}}_l}_{\mathbf{B}\dot{w}\mathbf{B}}$ shifted by $l(w)$ and Tate-twisted by $l(w)/2$.*

Let $\mathbf{B}_\Delta \subset \mathbf{B} \times \mathbf{B}$ be the diagonal subgroup. Denote by $\mathbb{H}^*_{\mathbf{B}_\Delta}(\mathbf{IC}_w)$ the $\mathbf{B}_\Delta$-equivariant cohomology of $\mathbf{IC}_w$. Webster and Williamson use that it is a bigraded vector space by the weight of the Frobenius map and the degree of the cohomology. This allows them to construct a trace Tr on the Hecke algebra $\mathcal{H}_W$ ($\mathbf{H}_\Gamma$ in their notation) as follows:

Theorem 7.1.2 (Webster-Williamson) *For $w \in W$ denote by C_w^* the corresponding Kazhdan-Lusztig element in $\mathcal{H}_w$. Let $\mathbb{H}^{i,j}_{\mathbf{B}_\Delta}(\mathbf{IC}_w)$ be the subspace of $\mathbb{H}^i_{\mathbf{B}_\Delta}(\mathbf{IC}_w)$ of weight j. Then the linear extension of the function $\mathrm{Tr}(C_w^*) = \sum_{i,j} \dim \mathbb{H}^{i,j}_{\mathbf{B}_\Delta}(\mathbf{IC}_w) v^i t^{j-i}$ is a trace on $\mathcal{H}_W$.*

Proof. Part of theorem [WW09, Th. 6] □
Theorem [WW08, Th 1.4] and lemma (7.2.6) (which will be proven in the next section) show us that the above trace Tr multiplied by $\left(\frac{1-v^2}{1+vt}\right)^{|S|}$ coincides with the Khovanov-Rozansky trace τ_{kr} (cf. (4.4.8)).

Remark 7.1.3 *To evade possible confusion in the notations we should point out that the variable t in theorem (7.1.2) is different from the variable t used in chapter 4 and in the next section.*

Webster and Williamson then show that $\mathbb{H}^*_{\mathbf{B}_\Delta}(\mathbf{IC}_w) \cong \mathbb{H}^*_{\mathbf{G}_\Delta}(\mathbf{K}_w)$, where $\mathbf{K}_w = \mathrm{ind}_{\mathbf{B}_\Delta}^{\mathbf{G}_\Delta} \mathbf{IC}_w$ is a character sheaf studied by Lusztig whose decomposition into Springer sheaves is described by the Lusztig-Fourier transform. They finally prove that the $\mathbf{G}_\Delta$-equivariant cohomology of a Springer sheaf attached to $\chi \in \mathrm{Irr}W$ is described by the Molien series $P_\chi(v^2, vt)$. This implies the following theorem:

Theorem 7.1.4 (Webster-Williamson) *The weight ω_χ of the character χ in* Tr *is*

$$\omega_\chi = \sum_{\mu \in \mathrm{Irr}W} S_{\chi,\mu} P_\mu(v^2, vt)$$

where $S_{\chi,\mu}$ are the entries of Lusztig's Fourier transform matrix S of W (cf. (5.1.20)).

Comparing the weights ω_χ of Tr with the weights ω^χ of Gomi's trace τ_g (cf. (6.1.2)) we see that τ_g coincides with the trace Tr multiplied by $\left(\frac{1-v^2}{1+vt}\right)^{|S|}$. Thus it coincides with the Khovanov-Rozansky trace τ_{kr} and the conjecture (7.0.1) is proven in the Weyl group case.

7.2 Case of Dihedral Groups

In this section we will work with the *dihedral groups* $I_2(m)$, an important example of Coxeter groups. Recall its definition.

Definition 7.2.1 *Let m be a positive integer. The dihedral group $I_2(m)$ is the Coxeter group with the following presentation*

$$I_2(m) = \left\langle s_0, s_1 \mid s_0^2 = s_1^2 = (s_0 s_1)^m = 1 \right\rangle.$$

Our goal is to prove Jean Michel's conjecture (7.0.1) for dihedral groups.

Remark 7.2.2 *In our case $I_2(m)$ the trace τ_g is the special Markov trace τ_m of type $I_2(m)$ (cf. (3.2.19) and (6.1.3)).*

Let v, r be variables and $z = \frac{r(v^2-1)}{r+1}, t = rv^{-2}$. Let $K = \mathbb{Q}(v,r) = \mathbb{Q}(v,z) = \mathbb{Q}(v,t)$. Denote by $\mathcal{H}_m$ the Iwahori-Hecke algebra of type $I_2(m)$ over the field K with the parameters $q_{s_0} = q_{s_1} = v^2, q'_{s_0} = q'_{s_1} = -1$. Recall that it has the following presentation:

$$\mathcal{H}_m = \Big\langle T_0, T_1 \,|\, \underbrace{T_0T_1T_0\cdots}_{m\ terms} = \underbrace{T_1T_0T_1\cdots}_{m\ terms},\ (T_i - v^2)(T_i+1) = 0 \quad i \in \{0,1\} \Big\rangle$$

We denote by $\mathcal{H}'_m$ the Iwahori-Hecke algebra of type $I_2(m)$ with the parameters $q_{s_0} = q_{s_1} = v^{-2}, q'_{s_0} = q'_{s_1} = -1$.

Remark 7.2.3 *Some results in this section are true for any Coxeter system (W,S). We will denote by $\mathcal{H}_W$ (by $\mathcal{H}'_W$) the corresponding Iwahori-Hecke algebra over K with the parameters $q_s = v^2, q'_s = -1$ (with the parameters $q_s = v^{-2}, q'_s = -1$) and by $\{\mathbf{T}_w\}_{w\in W}$ their standard K-bases.*

Recall that in (2.2.15) we have seen the definition of the Kazhdan-Lusztig basis, an alternative K-basis of $\mathcal{H}_W$. In the case of dihedral groups it can be given explicitly. For any $x \in I_2(m)$ the corresponding element of the Kazhdan-Lusztig basis of $\mathcal{H}_m$ is $C'_x = v^{-l(x)} \sum_{y\le x} T_y$ where $l(x)$ is the length of x and $\le$ is the Bruhat order on $I_2(m)$. By analogy the Kazhdan-Lusztig basis of $\mathcal{H}'_W$ is given by $C'_x = v^{l(x)} \sum_{y\le x} T_y$ for $x \in W$.

Remark 7.2.4 *In the next two subsections we will calculate the values of the Khovanov-Rozansky trace τ_{kr} and the special Markov trace τ_m of dihedral type on the Kazhdan-Lusztig basis elements C'_x. We will see that the values of these traces coincide. This proves conjecture (7.0.1) in the dihedral case.*

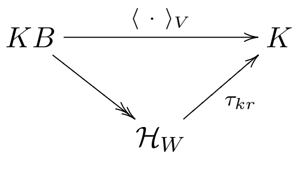
7.2.1 Values of the Khovanov-Rozansky Trace.

Our aim in this subsection is to prove the following

Proposition 7.2.5 *Let $x \in I_2(m)$ and C'_x the Kazhdan-Lusztig basis element of $\mathcal{H}_m$. Then the value of the Khovanov-Rozansky trace τ_{kr} on C'_x is as follows:*

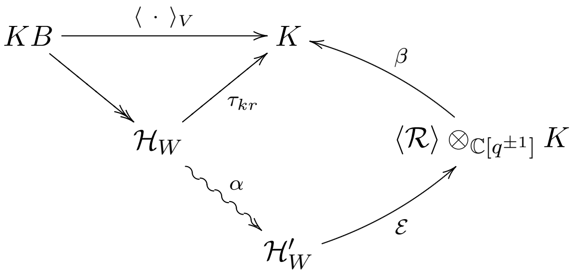
$$\tau_{kr}(C'_x) = v^{-l(x)}(z+1)\left(\frac{v^{2l(x)-2}-1}{v^2-1}z + 1\right).$$

In (4.4.8) we defined the Khovanov-Rozansky trace τ_{kr} using the following diagram:



where KB is the group algebra of the braid group B corresponding to W. In fact the above diagram can be enlarged:

Lemma 7.2.6 *The following diagram is commutative:*



- *Here* α *is a ring isomorphism defined by* $\alpha(T_i) = T_i,\ i \in \{0, 1\}$ *and* $\alpha(v) = v^{-1}, \alpha(t) = t$*;*
- $\mathcal{E}$ *is Soergel's "categorification" homomorphism (cf. (4.2.2));*
- *Morphism* β *is the homomorphism of* K*-vector spaces defined by*

$$\beta\left(\langle M\rangle\right) = \left(\frac{1-v^2}{1+r}\right)^n \spadesuit_{v,t}(M) \qquad \textit{for any } M \in \mathcal{R}.$$

  where $\spadesuit_{v,t}(M)$ *is the bigraded dimension of the Hochschild homology of* M *(cf. (4.4.7)),* $n = |S|$*.*

Proof. Since both τ_{kr} and $\beta \circ \mathcal{E} \circ \alpha$ are K-linear it is enough to check the commutativity on the basis elements $\{\mathbf{T}_w\}_{w\in W}$. Consider any reduced decomposition of w into a product of simple generators $s_i \in S$ of the group W: $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, where $k = l(w)$. Let $\sigma = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ be the braid group element whose image under the canonical surjection $B \twoheadrightarrow W$ is w. Then $\mathbf{T}_w = \mathbf{T}_{s_{i_1}}\mathbf{T}_{s_{i_2}}\ldots\mathbf{T}_{s_{i_k}}$ and the corresponding Rouquier complex $F(\sigma)$ has the following components (cf. (4.3.1)):

$$F^j(\sigma) = \begin{cases} \bigoplus\limits_{1\le i_1<i_2<\ldots<i_{k-j}\le k} B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \cdots \otimes_R B_{s_{i_{k-j}}} & \text{for } 0 \le j < k, \\ R & \text{for } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

From the other side

$$\beta \circ \mathcal{E} \circ \alpha(\mathbf{T}_w) = \beta \circ \mathcal{E}(\mathbf{T}_w) = \beta\left(\prod_{i=1}^{k}\left(\langle B_{s_i}\rangle - \langle R\rangle\right)\right)$$

$$= \beta\left(\bigoplus_{j=0}^{k}\bigoplus_{1\le i_1<i_2<\ldots<i_{k-j}\le k}(-1)^j\left\langle B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \cdots \otimes_R B_{s_{i_{k-j}}}\right\rangle\right).$$

Comparing the definitions of β and $\langle \cdot \rangle_V$ (cf.(4.4.3)) we have $\tau_{kr}(\mathbf{T}_w) = \beta \circ \mathcal{E} \circ \alpha(\mathbf{T}_w)$. □

Notation 7.2.7 *For any graded finitely generated R-module M its* graded dimension $\sum_i \dim_{\mathbb{C}} M_i v^i$ *is denoted by* $\dim_v M$*.*

We also define $\dim_v$ on the Grothendieck group $\langle \mathcal{R} \rangle$ by $\dim_v \langle M \rangle := \dim_v M$ and extend it on all $\langle \mathcal{R} \rangle$ by linearity.

By lemma (7.2.6) and Soergel's theorem (4.2.8) we get that in order to calculate the value of the Khovanov-Rozansky trace τ_{kr} on the Kazhdan-Lusztig basis element C'_x we must find the value of $\beta(\langle R_{\leq x}(l(x))\rangle)$ or simply the value of $\spadesuit_{v,t}(R_{\leq x}(l(x))) = \sum_{j=0}^{n} \dim_v \mathrm{HH}_j(R,\ R_{\leq x}(l(x))\ )t^j$.

In the case of dihedral groups $n = 2$ and $R = \mathbb{C}[x_1, x_2]$. Thus the Koszul resolution of R consists of three non-trivial terms (cf. (4.1)):

$$0 \longrightarrow R^e(-4) \longrightarrow R^e(-2) \oplus R^e(-2) \longrightarrow R^e \longrightarrow 0.$$

Tensoring it with $R_{\leq x}(l(x))$ we get:

$$0 \to R_{\leq x}(l(x)-4) \to \\ R_{\leq x}(l(x)-2) \oplus R_{\leq x}(l(x)-2) \to R_{\leq x}(l(x)) \to 0 \quad (7.2.8)$$

and we only need to calculate $\dim_v \mathrm{HH}_j(R,\ R_{\leq x}(l(x))\ )$ for $j \in \{0,1,2\}$ since all higher Hochschild homology groups are trivial.

Calculation of $\dim_v \mathrm{HH}_0(R,\ R_{\leq x}(l(x))\ )$.

Since the above resolution corresponds to the R-bimodule R we have

$$\mathrm{HH}_0(R, R_{\leq x}(l(x))) = R \otimes_{R^e} R_{\leq x}(l(x)) = R(l(x)).$$

The second equality can be obtained by applying the functor $\otimes_{R^e} R$ to the short sequence of surjective ring homomorphisms:

$$R^e = R \otimes R \twoheadrightarrow R_{\leq x} \twoheadrightarrow R.$$

Thus $\dim_v \mathrm{HH}_0(R,\ R_{\leq x}(l(x))\ ) = \frac{v^{-l(x)}}{(1-v^2)^2}$.

Calculation of $\dim_v \mathrm{HH}_2(R,\ R_{\leq x}(l(x))\ )$.

Consider the functor $D = \mathrm{Hom}_{-R}(\ , R)$ from the category $\mathcal{R}$ to the category of R-bimodules. Here we provide our space of homomorphisms of right R-modules with the obvious $\mathbb{Z}$-grading. For $M \in \mathcal{R}$ define the right and the left action of R on DM via the right and left action on the bimodule M by $(r_1 f r_2)(m) = f(r_1 m r_2)$ for $m \in M$.

Proposition 7.2.9 (Soergel) *The functor D has the following properties:*

- $D(M(k)) = D(M)(-k)$

- $D(R_{\leq x}(l(x))) \cong R_{\leq x}(l(x))$ *for any* $x \in I_2(w)$.

We apply the functor D to the short sequence (7.2.8). By the proposition (7.2.9) we obtain the sequence

$$0 \longleftarrow R_{\leq x}(l(x)+4) \longleftarrow R_{\leq x}(l(x)+2) \oplus R_{\leq x}(l(x)+2) \longleftarrow R_{\leq x}(l(x)) \longleftarrow 0,$$

which is up to the shift by -4 isomorphic to the initial short sequence. Using this and the fact that kernels, images and homologies in the short sequence (7.2.8) are free as right R-modules we get

$$D(\mathrm{HH}_2(R, R_{\leq x}(l(x)))) \cong \mathrm{HH}_0(R, R_{\leq x}(l(x)+4)).$$

And since the functor D alternates the shifts we have

$$\dim_v \mathrm{HH}_0(R, R_{\leq x}(l(x)))v^{-4} = \dim_{v^{-1}} \mathrm{HH}_2(R, R_{\leq x}(l(x))) \cdot \frac{(1-v^{-2})^2}{(1-v^2)^2}$$

Hence

$$\dim_v \mathrm{HH}_2(R, R_{\leq x}(l(x))) = \frac{v^{l(x)}}{(1-v^{-2})^2}.$$

Calculation of $\dim_v \mathrm{HH}_1(R,\ R_{\leq x}(l(x))\ )$.

First of all we note that taking $t=-1$ we have that $\spadesuit_{v,-1}(R_{\leq x}(l(x)))$ is the Euler characteristic of the complex

$$0 \to R_{\leq x}(l(x)-4) \to R_{\leq x}(l(x)-2) \oplus R_{\leq x}(l(x)-2) \to R_{\leq x}(l(x)) \to 0$$

and thus is equal to $(1-v^2)^2 \dim_v R_{\leq x}(l(x))$.

Lemma 7.2.10

$$\dim_v R_{\leq x}(l(x)) = \frac{(1-v^4)(v^{-l(x)} - v^{l(x)})}{(1-v^2)^4}.$$

Proof. We will consider the indices modulo 2: for index $i \in \{0,1\}$ the index $i+1 \in \{0,1\} \setminus \{i\}$. Let $\alpha : \mathcal{H}_W \to \mathcal{H}'_W$ be as in lemma (7.2.6). By Soergel's theorem (4.2.8) we have

$$\langle R_{\leq x}(l(x)) \rangle = \mathcal{E} \circ \alpha(C'_x) = \mathcal{E}\Big(q^{l(x)} \sum_{y \leq x} \mathbf{T}_y\Big) = \sum_{y \leq x} \mathcal{E}\left(q^{l(x)}\mathbf{T}_y\right).$$

Taking a presentation of any $\mathbf{T}_y$ as a product of the generators $\mathbf{T}_0, \mathbf{T}_1$: $\mathbf{T}_y = \underbrace{\mathbf{T}_{i_y}\mathbf{T}_{i_y+1}\mathbf{T}_{i_y}\ldots}_{l(y)\ terms}$, we can write

$$\mathcal{E}\left(q^{l(x)}\mathbf{T}_y\right) = \mathcal{E}\Big(q^{l(x)}\underbrace{\mathbf{T}_{i_y}\mathbf{T}_{i_y+1}\mathbf{T}_{i_y}\ldots}_{l(y)\ terms}\Big) = \mathcal{E}(q^{l(x)})\underbrace{\mathcal{E}(\mathbf{T}_{i_y})\mathcal{E}(\mathbf{T}_{i_y+1})\mathcal{E}(\mathbf{T}_{i_y})\ldots}_{l(y)\ terms}$$
$$= \langle R(l(x))\rangle \underbrace{\left(\langle B_{i_y}\rangle - \langle R\rangle\right)\left(\langle B_{i_y+1}\rangle - \langle R\rangle\right)\left(\langle B_{i_y}\rangle - \langle R\rangle\right)\cdots}_{l(y)\ terms},$$

where $B_i = R \otimes_{R^{s_i}} R$ (cf. (4.2)).

Since $\dim_v R = (1-v^2)^{-2}$, it is not hard to see that $\dim_v$ has the following property:

$$\dim_v \langle M \otimes_R N \rangle = (1-v^2)^2 \dim_v \langle M \rangle \dim_v \langle N \rangle$$

for any R-bimodules $M, N \in \mathcal{R}$ such that M is free as a right R-module, N is free as a left R-module. Hence, taking into account the easy formulae

$$\dim_v R = \frac{1}{(1-v^2)^2} \qquad \dim_v B_i = \frac{1+v^2}{(1-v^2)^2},$$

we get

$$\dim_v R_{\leq x}(l(x)) = v^{-l(x)} \sum_{y \leq x} \frac{v^{2l(y)}}{(1-v^2)^2} = \frac{(1-v^4)(v^{-l(x)} - v^{l(x)})}{(1-v^2)^4}. \quad \square$$

From this lemma we obtain

$$\begin{aligned} &\dim_v \mathrm{HH}_1(R, R_{\leq x}(l(x))) = \\ &= \dim_v \mathrm{HH}_0(R, R_{\leq x}(l(x))) + \dim_v \mathrm{HH}_2(R, R_{\leq x}(l(x))) - \spadesuit_{v,-1}(R_{\leq x}(l(x))) \\ &= \frac{v^{-l(x)} + v^{l(x)+4} - (1-v^4)(v^{-l(x)} - v^{l(x)})}{(1-v^2)^2}. \end{aligned}$$

We can finally calculate the value of the Khovanov-Rozansky trace $\tau_{kr}(C'_x)$:

$$\begin{aligned} &\tau_{kr}(C'_x) \\ &= \left(\frac{1-v^2}{1+tv^2}\right)^2 \cdot \frac{v^{-l(x)} + tv^{-l(x)} + tv^{l(x)+4} - t(1-v^4)(v^{-l(x)} - v^{l(x)}) + t^2 v^{l(x)+4}}{(1-v^2)^2} \\ &= \frac{v^{-l(x)}(1+tv^4)(1+tv^{2l(x)})}{(1+tv^2)^2} = v^{-l(x)}(z+1)\left(\frac{v^{2l(x)-2}-1}{v^2-1} z + 1\right), \end{aligned}$$

where $z = \frac{tv^2(v^2-1)}{tv^2+1}$ is a Markov trace parameter.

7.2.2 Values of the Special Markov Trace of Dihedral Type.

In this subsection we prove that the special Markov trace τ_m takes the same values on the Kazhdan-Lusztig basis of $\mathcal{H}_m$:

Proposition 7.2.11 *Let $x \in I_2(m)$ and C'_x the Kazhdan-Lusztig basis element of $\mathcal{H}_m$. Then the value of the special Markov trace τ_m on C'_x is as follows*

$$\tau_m(C'_x) = v^{-l(x)}(z+1)\left(\frac{v^{2l(x)-2}-1}{v^2-1} z + 1\right).$$

Let $q = v^2$. Recall that by Gomi's theorem (6.1.3) we have

$$\tau_m = \tau_g = \sum_{\phi \in \mathrm{Irr} I_2(m)} \omega^\phi \phi^{(q)} \tag{7.2.12}$$

where

$$\omega^\phi = \left(\frac{1 - v^2}{1 + r} \right)^2 \sum_{\mu \in \mathrm{Irr} I_2(m)} S_{\phi,\mu} P_\mu(v^2, r),$$

$P_\mu(v^2, r)$ is the Molien series for μ (cf. (6.1.1)), $S_{\phi,\mu}$ is the entry of the Fourier matrix S of $I_2(m)$ corresponding to the pair (ϕ, μ) of irreducible characters of $I_2(m)$ (cf. (5.2.2) and (5.2.6)).

We prove proposition (7.2.11) by direct calculation. We use the notation of the CHEVIE system for the irreducible characters of $I_2(m)$:

$$\mathrm{Irr} I_2(m) = \begin{cases} \{\phi_{1,0}, \phi_{1,m}, \phi_{2,1}, \phi_{2,2}, \ldots, \phi_{2,\frac{m-1}{2}}\}, & \text{if } m \text{ is odd,} \\ \{\phi_{1,0}, \phi_{1,m}, \phi'_{1,\frac{m}{2}}, \phi''_{1,\frac{m}{2}}, \phi_{2,1}, \phi_{2,2}, \ldots, \phi_{2,\frac{m-2}{2}}\}, & \text{if } m \text{ is even.} \end{cases}$$

Let ξ be an m-th primitive root of 1. The Iwahori-Hecke algebra $\mathcal{H}_m$ is split semisimple over $K(\xi)$ and its irreducible characters are in bijection with the irreducible characters of $I_2(m)$. As usual we denote by $\phi^{(q)}$ the irreducible character of $\mathcal{H}_m$ corresponding to $\phi \in \mathrm{Irr} I_2(m)$.

The following theorem can be found in [GP00, Th. (8.3.1)]:

Theorem 7.2.13 (Kilmoyer-Solomon) *For an integer j let $c_j, c'_j \in \mathbb{C}(v)$ be such that*

$$c_j c'_j = v^2(2 + \xi^j + \xi^{-j}).$$

Then up to equivalence, the irreducible representations of $\mathcal{H}_m$ are as follows:

- *Let m be odd. Then there are two one-dimensional representations given by*

$$\phi_{1,0}^{(q)} : \mathbf{T}_0 \to v^2, \mathbf{T}_1 \to v^2, \qquad \phi_{1,m}^{(q)} : \mathbf{T}_0 \to -1, \mathbf{T}_1 \to -1,$$

  and $(m-1)/2$ irreducible representations of dimension two given by

$$\phi_{2,j}^{(q)} : \mathbf{T}_0 \to \begin{bmatrix} -1 & 0 \\ c_j & v^2 \end{bmatrix}; \qquad \mathbf{T}_1 \to \begin{bmatrix} v^2 & c'_j \\ 0 & -1 \end{bmatrix} \textit{ for } 1 \le j \le (m-1)/2.$$

- *Let m be even. Then there are four one-dimensional representations given by*

$$\begin{array}{llll} \phi_{1,0}^{(q)} : & \mathbf{T}_0 \to v^2, \quad \mathbf{T}_1 \to v^2 & \phi_{1,\frac{m}{2}}^{'(q)} : & \mathbf{T}_0 \to -1, \quad \mathbf{T}_1 \to v^2 \\ \phi_{1,m}^{(q)} : & \mathbf{T}_0 \to -1, \quad \mathbf{T}_1 \to -1 & \phi_{1,\frac{m}{2}}^{''(q)} : & \mathbf{T}_0 \to v^2, \quad \mathbf{T}_1 \to -1 \end{array}.$$

  and $(m-2)/2$ irreducible representations of dimension two given by

$$\phi_{2,j}^{(q)} : \mathbf{T}_0 \to \begin{bmatrix} -1 & 0 \\ c_j & v^2 \end{bmatrix}; \qquad \mathbf{T}_1 \to \begin{bmatrix} v^2 & c'_j \\ 0 & -1 \end{bmatrix} \textit{ for } 1 \le j \le (m-2)/2$$

In order to give the entries of the Fourier matrix S of $I_2(m)$ (cf. (5.2.6)) we need to synchronize our notations with Lusztig's notations of unipotent representations $\mathcal{U}(I_2(m))$ (cf. (5.2.1)) and the inclusion $\mathrm{Irr} I_2(m) \subset \mathcal{U}(I_2(m))$ (cf. (5.2.2)):

- Case $m = 2k+1$: $\begin{cases} \phi_{1,0} \leftrightarrow 1, \\ \phi_{1,m} \leftrightarrow \sigma, \\ \phi_{2,i} \leftrightarrow \rho_i \leftrightarrow (0,i) \text{ for } 1 \le i \le k, \end{cases}$

- Case $m = 2k+2$: $\begin{cases} \phi_{1,0} \leftrightarrow 1, \\ \phi_{1,m} \leftrightarrow \sigma, \\ \phi_{2,i} \leftrightarrow \rho_i \leftrightarrow (0,i) \text{ for } 1 \le i \le k, \\ \phi'_{1,\frac{m}{2}} \leftrightarrow \epsilon' \leftrightarrow (0, m/2)', \\ \phi''_{1,\frac{m}{2}} \leftrightarrow \epsilon'' \leftrightarrow (0, m/2)''. \end{cases}$

The set of irreducible characters $\mathrm{Irr} I_2(m)$ is decomposed into two singletons $\{\phi_{1,0}\}$, $\{\phi_{1,m}\}$ and a family $\mathcal{F}$ which consists of the remaining characters. By the above correspondence the entries of the Fourier matrix S corresponding to the family $\mathcal{F}$ are as follows (cf. (5.2.2)):

$$S_{\phi_{2,j},\phi_{2,k}} = S_{\phi_{2,k},\phi_{2,j}} = \frac{2 - \xi^{jk} - \xi^{-jk}}{m}, \text{ for } 1 \le j, k \le \left[\frac{m-1}{2}\right],$$

$$\begin{aligned} S_{\phi_{2,j},\phi'_{1,\frac{m}{2}}} = S_{\phi'_{1,\frac{m}{2}},\phi_{2,j}} = S_{\phi_{2,j},\phi''_{1,\frac{m}{2}}} = S_{\phi''_{1,\frac{m}{2}},\phi_{2,j}} = \\ = \frac{1-(-1)^j}{m}, \text{ for } 1 \le j \le \frac{m-2}{2}, \end{aligned}$$

$$\begin{aligned} S_{\phi'_{1,\frac{m}{2}},\phi'_{1,\frac{m}{2}}} &= S_{\phi''_{1,\frac{m}{2}},\phi''_{1,\frac{m}{2}}} = \frac{1-(-1)^{\frac{m}{2}}+m}{2m}, \\ S_{\phi'_{1,\frac{m}{2}},\phi''_{1,\frac{m}{2}}} &= S_{\phi''_{1,\frac{m}{2}},\phi'_{1,\frac{m}{2}}} = \frac{1-(-1)^{\frac{m}{2}}-m}{2m}. \end{aligned}$$

Let $\omega_{i,j} = \omega^{\phi}$ (respectively $\omega'_{1,\frac{m}{2}}, \omega''_{1,\frac{m}{2}}$) be the weight corresponding to $\phi = \phi_{i,j}$ (respectively $\phi'_{1,\frac{m}{2}}, \phi''_{1,\frac{m}{2}}$). Direct calculation shows that

$$\omega_{1,0} = \frac{(z+1)((v^{2m-2}-1)z + v^2 - 1)}{(v^2+1)(v^{2m}-1)},$$

$$\omega_{1,m} = \frac{(z-v^2)((v^{2m-2}-1)z - v^{2m-2}(v^2-1))}{(v^2+1)(v^{2m}-1)},$$

$$\omega_{2,j} = \frac{(z+1)(v^2-z)(2-\xi^j-\xi^{-j})}{m(v^2-\xi^j)(v^2-\xi^{-j})}, \text{ for } 1 \le j \le \left[\frac{m-1}{2}\right],$$

$$\omega'_{1,\frac{m}{2}} = \omega''_{1,\frac{m}{2}} = \frac{2(z+1)(v^2-z)}{m(v^2+1)^2}.$$

Let $x \in I_2(m)$. We are ready to calculate $\tau_m(C'_x)$ using the formula (7.2.12). We may assume that x has even length $2p$ (once we obtain formulae for such x's we will easily get the formulae for all $x \in I_2(m)$ by induction). Suppose (without loss of generality) $x = \underbrace{\mathbf{T}_0\mathbf{T}_1\mathbf{T}_0\cdots\mathbf{T}_1}_{2p\ terms}$.

2.4.1) Case $\mathbf{m = 2k+1}$ is odd.

By Gomi's formula (7.2.12) we have

$$\rho := v^{2p}\tau(C'_x) = \omega_{1,0}\sum_{y\le x}\phi_{1,0}(\mathbf{T}_y)+\omega_{1,m}\sum_{y\le x}\phi_{1,m}(\mathbf{T}_y)+\sum_{j=1}^{\frac{m-1}{2}}\omega_{2,j}\sum_{y\le x}\phi_{2,j}(\mathbf{T}_y)$$
$$= \frac{(z+1)((v^{2m-2}-1)z+v^2-1)}{(v^2+1)(v^{2m}-1)}\cdot(1+v^2)\cdot\frac{v^{4p}-1}{v^2-1}+\omega_{1,m}\cdot 0$$
$$+\frac{(z+1)(v^2-z)}{m}\sum_{j=1}^{\frac{m-1}{2}}\frac{(1-\xi^j)(1-\xi^{-j})}{(v^2-\xi^j)(v^2-\xi^{-j})}\times$$
$$\times\Bigg[2+2v^2(\xi^j+\xi^{-j})+2v^4(\xi^{2j}+\xi^{-2j})+\dots$$
$$+2v^{2(p-1)}(\xi^{j(p-1)}+\xi^{-j(p-1)})+v^{2p}(\xi^{jp}+\xi^{-jp})$$
$$+2\frac{v^2-1}{\xi^j-1}(\,\xi^j-1+v^2(\xi^{2j}-\xi^{-j})+v^4(\xi^{3j}-\xi^{-2j})+\dots$$
$$+v^{2(p-1)}(\xi^{jp}-\xi^{-j(p-1)}))\Bigg]$$

The expression in the square brackets can be reduced to

$$(1+v^2\xi^j)\frac{v^{2p}\xi^{jp}-1}{v^2\xi^j-1}+(1+v^2\xi^{-j})\frac{v^{2p}\xi^{-jp}-1}{v^2\xi^{-j}-1}+$$
$$2\frac{v^2-1}{\xi^j-1}\left(\xi^j\frac{v^{2p}\xi^{jp}-1}{v^2\xi^j-1}-\frac{v^{2p}\xi^{-jp}-1}{v^2\xi^{-j}-1}\right)$$
$$=\frac{\xi^j+1}{\xi^j-1}(v^{2p}\xi^{jp}-1-(v^{2p}\xi^{-jp}-1))=v^{2p}\frac{\xi^j+1}{\xi^j-1}(\xi^{jp}-\xi^{-jp}).$$

Thus

$$\rho=\frac{(z+1)((v^{2m-2}-1)z+v^2-1)(v^{4p}-1)}{(v^2-1)(v^{2m}-1)}$$
$$+\frac{v^{2p}(z+1)(v^2-z)}{m}\sum_{j=1}^{m-1}\frac{\xi^{jp}-\xi^{-jp}}{v^2-\xi^j}.$$

To continue the calculation we need the following lemma

Lemma 7.2.14 *Let x be a variable, ξ an m-th primitive root of unity and $1 \leq p \leq m-1$. Then*

$$\sum_{j=1}^{m-1} \frac{\xi^{-jp} - \xi^{jp}}{x - \xi^j} = \frac{m(x^{m-p-1} - x^{p-1})}{x^m - 1}.$$

Proof. After multiplying both sides by $x^m - 1$ we need to compare two polynomials of degree less than m. Since these polynomials take the same values for $x = \xi^j, 0 \leq j \leq m-1$, they are the same. □
Applying this lemma for $x = v^2$ we get

$$\rho = \frac{z+1}{v^{2m}-1} \cdot \left(\frac{((v^{2m-2}-1)z + v^2 - 1)(v^{4p}-1)}{q^2-1} + v^{2p}(v^2 - z)(v^{2m-2p-2} - v^{2p-2}) \right)$$

$$= (z+1)\left(\frac{v^{4p-2}}{v^2-1} z + 1 \right).$$

2.4.2) Case $\mathbf{m = 2k}$ is even.

Again by Gomi's formula (7.2.12) we have

$$\rho = v^{2p}\tau(C'_x) = \omega_{1,0} \sum_{y \leq x} \phi_{1,0}(\mathbf{T}_y) + \omega_{1,m} \sum_{y \leq x} \phi_{1,m}(\mathbf{T}_y) + \omega'_{1,\frac{m}{2}} \sum_{y \leq x} \phi'_{1,\frac{m}{2}}(\mathbf{T}_y) + \omega''_{1,\frac{m}{2}} \sum_{y \leq x} \phi''_{1,\frac{m}{2}}(\mathbf{T}_y) + \sum_{j=1}^{\frac{m-2}{2}} \omega_{2,j} \sum_{y \leq x} \phi_{2,j}(\mathbf{T}_y)$$

$$= \frac{(z+1)((v^{2m-2}-1)z + v^2 - 1)}{(v^2+1)(v^{2m}-1)} \cdot (1+v^2) \cdot \frac{v^{4p}-1}{v^2-1} + \omega_{1,m} \cdot 0 + \omega'_{1,\frac{m}{2}} \cdot 0 + \omega''_{1,\frac{m}{2}} \cdot 0 + \sum_{j=1}^{\frac{m-2}{2}} \omega_{2,j} \sum_{y \leq x} \phi_{2,j}(\mathbf{T}_y).$$

Using the same formulae, lemma (7.2.14) and the obvious fact $\xi^{-\frac{mp}{2}} - \xi^{\frac{mp}{2}} = 0$ we obtain

$$\rho = \frac{(z+1)((v^{2m-2}-1)z + v^2 - 1)(v^{4p}-1)}{(v^2-1)(v^{2m}-1)} + \frac{v^{2p}(z+1)(v^2 - z)}{m} \sum_{j=1}^{m-1} \frac{\xi^{jp} - \xi^{-jp}}{v^2 - \xi^j}$$

$$= \frac{z+1}{v^{2m}-1} \cdot \left(\frac{((v^{2m-2}-1)z + v^2 - 1)(v^{4p}-1)}{v^2-1} \right.$$

$$\left. + v^{2p}(v^2 - z)(v^{2m-2p-2} - v^{2p-2}) \right)$$

$$= (z+1)\left(\frac{v^{4p-2}}{v^2-1} z + 1 \right).$$

The proposition (7.2.11) is proven.

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