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Autour des algèbres de battages quantiques : idéaux de définition, spécialisation et cohomologie

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dirigée par Marc ROSSO

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Résumé

La partie principale de cette thèse est consacrée à l'étude de certaines constructions et de structures liées aux algèbres de battages quantiques : algèbres différentielles et les opérateurs de Kashiwara ; idéaux de définition et le problème de spécialisation ; homologie de coHochschild et théorème de type Borel-Weil-Bott. Dans le dernier chapitre, on obtient une famille d'identités entre les puissances de la fonction η de Dedekind et la trace de l'élément de Coxeter du groupe de tresses d'Artin agissant sur les algèbres de coordonnées quantiques.

Mots-clefs

Algèbres q-Bosons, algèbres de Nichols, algèbres de battages quantiques, algèbres de Weyl quantiques, fonction η de Dedekind, homologie de coHochschild, groupes quantiques.

Around quantum shuffle algebras: defining ideals, specializations and cohomology

Abstract

The main part of this thesis is devoted to study some constructions and structures around quantum shuffle algebras: differential algebras and Kashiwara operators; defining ideals and specialization problem; coHochschild homology and an analogue of Borel-Weil-Bott theorem. In the last chapter we prove a family of identities relating powers of Dedekind η-function and the trace of the Coxeter element in the Artin braid groups acting on quantum coordinate algebras.

Keywords

q-Boson algebras, coHochschild homology, Dedekind η-function, Nichols algebras, quantum groups, quantum shuffle algebras, quantum Weyl algebras.

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Chapitre 1

Introduction

Cette thèse est dédiée à l'étude de quelques constructions autour des objets provenant des algèbres enveloppantes quantiques : les algèbres de battages quantiques (ou dualement, les algèbres de Nichols) et les liens avec les q-séries. Elle contient quatre parties dont les trois premières sont reliées. Les résultats principaux sont expliqués brièvement ci-dessous.

1.1 Les algèbres q-Bosons

Cette partie reprend les chapitres 2 et 3.

1.1.1 Motivation

Les algèbres q-Bosons $B_q(\mathfrak{g})$, comme les extensions des algèbres de Weyl quantiques $W_q(\mathfrak{g})$, sont construites initialement dans les travaux de M. Kashiwara [44] sur les bases cristallines, ayant pour but de définir les «opérateurs de Kashiwara» agissant sur la partie négative $U_q^{<0}(\mathfrak{g})$ d'un groupe quantique associé à une algèbre de Kac-Moody symétrizable $\mathfrak{g}$. C'est dans ce même article que la simplicité de $U_q^{<0}(\mathfrak{g})$ comme un $W_q(\mathfrak{g})$-module est démontrée en utilisant les calculs concernant les relations de commutation entre les générateurs et l'élément de Casimir ; la semi-simplicité de la catégorie $\mathcal{O}(W_q(\mathfrak{g}))$ y est conjecturée être un exercice simple.

Or, ce problème n'est pas aussi simple que l'indication donnée par Kashiwara peut le laisser croire : l'article [65] donne une «preuve» insuffisante.

La première démonstration complète est publiée dans [66] plus tard environ dix ans, en appliquant un outil de «projecteurs extrêmaux», avec de gros calculs pour vérifier ses propriétés.

1.1.2 Une esquisse

L'essentiel de la première partie de cette thèse est dédiée à donner une démonstration conceptuelle du théorème structurel de la catégorie $\mathcal{O}(W_q(\mathfrak{g}))$: on expliquera pourquoi la semi-simplicité de $\mathcal{O}(W_q(\mathfrak{g}))$ provient de la dualité intrinsèque de l'algèbre $W_q(\mathfrak{g})$.

Dans notre approche, la trivialité de la catégorie $\mathcal{O}(W_q(\mathfrak{g}))$ dépend fortement d'une construction plus fonctorielle de l'algèbre $W_q(\mathfrak{g})$ comme un double. En voici une explication rapide.

Soient A et B deux algèbres de Hopf et $\varphi : A \times B \to k$ un accouplement de Hopf généralisé. Le double quantique et le double de Heisenberg associés sont notés par $D_\varphi(A, B)$ et $H_\varphi(A, B)$ respectivement. Munis des structures de $D_\varphi(A, B)$-module et comodule sur $H_\varphi(A, B)$ définies dans la Section 2.2.8, on a

Proposition 1 (Proposition 2.3)**.** $H_\varphi(A, B)$ est un $D_\varphi(A, B)$-Yetter-Drinfel'd module algèbre.

L'avantage d'avoir la structure de Yetter-Drinfel'd provient de l'existence d'un tressage

$$\sigma : H_\varphi(A, B) \otimes H_\varphi(A, B) \to H_\varphi(A, B) \otimes H_\varphi(A, B)$$

qui munit l'espace vectoriel $H_\varphi(A, B) \otimes H_\varphi(A, B)$ d'une structure d'algèbre en remplaçant le flip usuel par le tressage ci-dessus. On le notera $H_\varphi(A, B) \underline{\otimes} H_\varphi(A, B)$ pour souligner cette structure d'algèbre.

Lorsque $A = U_q^{\geq 0}(\mathfrak{g})$ et $B = U_q^{\leq 0}(\mathfrak{g})$ sont les parties positive et négative d'un groupe quantique, en identifiant les deux parties tores et faisant un changement de variables, le double de Heisenberg n'est rien d'autre que $B_q(\mathfrak{g})$. À ce moment, la proposition ci-dessus s'écrit comme

Proposition 2 (Proposition 2.7)**.** $W_q(\mathfrak{g})$ est un $U_q(\mathfrak{g})$-Yetter-Drinfel'd module algèbre.

Comme conséquence immédiate, le tressage

$$\sigma : W_q(\mathfrak{g}) \otimes W_q(\mathfrak{g}) \to W_q(\mathfrak{g}) \otimes W_q(\mathfrak{g})$$

qui munit $W_q(\mathfrak{g}) \underline{\otimes} W_q(\mathfrak{g})$ d'une structure d'algèbre, est bien défini. De plus, l'algèbre $W_q(\mathfrak{g})$ se retrouve ainsi : soient $B_q^{<0}(\mathfrak{g})$ et $B_q^{>0}(\mathfrak{g})$ les images de $U_q^{<0}(\mathfrak{g})$ et $U_q^{>0}(\mathfrak{g})$ dans $W_q(\mathfrak{g})$, alors

Proposition 3 (Proposition 2.8)**.** Il existe un isomorphisme d'algèbre

$$B_q^{<0}(\mathfrak{g}) \underline{\otimes} B_q^{>0}(\mathfrak{g}) \cong W_q(\mathfrak{g}).$$

En utilisant cette construction, une version tressée du théorème structurel des modules de Hopf peut être appliqué aux $W_q(\mathfrak{g})$-modules dans $\mathcal{O}(W_q(\mathfrak{g}))$ et finalement on obtient

Théorème 1 (Theorem 2.1)**.** Il existe une équivalence de catégorie

$$\mathcal{O}(W_q(\mathfrak{g})) \sim \mathbf{Vect}$$

où **Vect** est la catégorie des espaces vectoriels. Plus précisément, cette équivalence est donnée par :

$$M \mapsto M^{cop}, \quad V \mapsto B_q^{<0} \otimes V$$

pour $M \in \mathcal{O}(W_q(\mathfrak{g}))$ et $V \in \mathbf{Vect}$, où $M^{cop} = \{m \in M \mid \rho(m) = m \otimes 1\}$ est l'ensemble des coinvariants à droite dans M.

Le théorème structurel de la catégorie $\mathcal{O}(B_q(\mathfrak{g}))$ provient du même principe, voir Theorem 2.2 pour un énoncé complet. De plus, la semi-simplicité de $\mathcal{O}(B_q(\mathfrak{g}))$ et la classification des objets simples en sont des corollaires immédiats.

1.1.3 Les avantages

La construction ci-dessus nous permet d'interpréter plusieurs notions importantes d'une manière plus compacte et jolie.

1. L'ensemble des vecteurs extrêmaux dans M s'identifie à M^{cop}.
2. Dans la démonstration du théorème structurel des modules de Hopf, un projecteur $P : M \to M^{cop}$ apparaît. Dans la situation ci-dessus, ce projecteur coïncide avec celui défini par Kashiwara dans le cas sl_2 ; de plus, à une bar involution près, il est rien d'autre que le «projecteur extrêmal» au sens de Nakashima.
3. Les opérateurs de Kashiwara sur $U_q^{<0}(\mathfrak{g})$ peuvent être écrits comme une convolution qui est plus fonctorielle que la définition originale et ceci s'étend à un cadre plus général.

1.1.4 Sous-algèbres unipotentes

Une grande partie de la construction ci-dessus peut se généraliser au cas «cellules de Bruhat», ce qui forme le contenu principal du chapitre 3.

Supposons que $\mathfrak{g}$ est une algèbre de Lie semi-simple de dimension finie. Soient $w \in W$ un élément appartenant au groupe de Weyl de $\mathfrak{g}$ et $U_q^{<0}[w]$ la sous-algèbre unipotente associée contenue dans $U_q^{<0}$. En restreignant l'accouplement de Hopf φ entre $U_q^{>0}$ et $U_q^{<0}$ au sous-espace $U_q^{>0} \times U_q^{<0}[w]$, le double de Heisenberg $H_\varphi[w]$ est bien défini.

On peut similairement définir la catégorie $\mathcal{O}(H_\varphi[w])$ et prouver le théorème suivant.

Théorème 2 (Corollary 3.1)**.** Il existe une équivalence de catégorie

$$\mathcal{O}(H_\varphi[w]) \sim {}^{(U_q^{<0}[w])^\perp}\mathcal{M}$$

où $(U_q^{<0}[w])^\perp$ est une cogèbre «complémentaire» définie dans la Section 3.2.2 et ${}^{(U_q^{<0}[w])^\perp}\mathcal{M}$ est la catégorie des $(U_q^{<0}[w])^\perp$-comodules à gauche.

En particulier, si $w = w_0$ est le plus long élément dans W, l'algèbre $U_q^{<0}[w]$ est l'algèbre $U_q^{<0}(\mathfrak{g})$ toute entière et la catégorie ${}^{(U_q^{<0}[w])^\perp}\mathcal{M}$ se réduit à la catégorie **Vect** : ceci n'est rien d'autre que le cas du chapitre 2.

1.2 L'idéal de définition d'une algèbre de Nichols

Cette partie reprend les chapitres 4 et 5.

1.2.1 Motivation

En gros, une algèbre de Nichols est un objet dans la catégorie des algèbres de Hopf tressées naturellement associé à un module de Yetter-Drinfel'd V sur une algèbre de Hopf H (ou plus général, à un espace vectoriel tressé). Plusieurs algèbres importantes se trouvent dans ce cadre en choisissant H et V proprement : par exemple, les algèbres extérieures, les algèbres symétriques et les parties négatives (ou positives) des groupes quantiques.

Plus précisément, une algèbre de Nichols peut être construite à partir d'une algèbre tensorielle tressée $T(V)$, qui est une algèbre de Hopf tressée en remplaçant le flip par le tressage provenant de la structure de Yetter-Drinfel'd pour munir $T(V) \otimes T(V)$ d'une structure d'algèbre associative. L'algèbre de Nichols $\mathfrak{N}(V)$ se définit comme le quotient de $T(V)$ par un idéal de définition $\mathfrak{I}(V)$ défini comme le plus grand coidéal contenu dans le sous-espace de $T(V)$ engendré par les tenseurs de degrés supérieurs ou égaux à deux. Comme conséquence, nous pourrons dire que l'algèbre de Hopf tressée $\mathfrak{N}(V)$ est engendrée par les générateurs dans V et les relations dans $\mathfrak{I}(V)$.

Les deux problèmes suivants sont initialement posés par N. Andruskiewitsch dans [2], les versions ici sont légèrement modifiées :

1. Trouver un ensemble de générateurs «agréables» dans $\mathfrak{I}(V)$.
2. Sous quelles conditions l'idéal $\mathfrak{I}(V)$ est-il finiment engendré ?

Ces deux chapitres sont dédiés à les étudier.

1.2.2 Éléments de niveau n

Les études des idéaux $\mathfrak{I}(V)$ commençent dans les travaux de Andruskiewitsch-Graña [3], M. Rosso [73] et P. Schauenburg [79].

Théorème 3. Soit $S_n : T(V) \to T(V)$ l'opérateur de symétrisation totale. Alors

$$\mathfrak{N}(V) = \bigoplus_{n\geq 0} \left(V^{\otimes n}/\ker(S_n)\right).$$

Ce théorème sert de point de départ aux travaux dans les chapitres 3 et 4 pour étudier le noyau de chaque S_n.

Dans le chapitre 3, nous introduissons la notion «d'éléments de niveau n» en considérant une décomposition de S_n et étudions ses propriétés.

Pour $n \geq 2$ un entier, notons $\mathfrak{B}_n$ le groupe de tresses en n brins engendré par les générateurs $\sigma_1, \cdots, \sigma_{n-1}$ et les relations bien connues. Soient $1 < s < n$ un entier et $i : \mathfrak{B}_s \to \mathfrak{B}_n$ un morphisme injectif de groupes. On l'appelle un «plongement positionel» s'il existe un entier $0 \leq r \leq n-s$ tel que $i(\sigma_t) = \sigma_{t+r}$ pour tout $1 \leq t \leq s-1$.

Fixons un entier $n \geq 2$ et soit $v \in V^{\otimes n}$ un élément non-nul. Puisque V est un espace tressé, l'espace vectoriel $V^{\otimes n}$ admet une structure de $k[\mathfrak{B}_n]$-module. On notera $k[X_v]$ le $k[\mathfrak{B}_n]$ sous-module de $V^{\otimes n}$ engendré par v, $S_n : k[X_v] \to k[X_v]$ est bien défini.

Préservons les hypothèses concernant v comme ci-dessus ; on l'appelle «de niveau n» si

1. v est annulé par S_n ;
2. Pour tout plongement positionel $\iota : \mathfrak{B}_s \to \mathfrak{B}_n$, l'équation $\iota(\theta_s)x = x$ n'admet aucune solution dans $k[X_v]$, où θ_s est l'élément engendrant le centre de $\mathfrak{B}_s$.

L'avantage de considérer les éléments «de niveau n» s'explique par le théorème suivant :

Théorème 4 (Theorem 4.2)**.** Les éléments «de niveau n» sont primitifs.

Soit Δ_n l'élément de Garside dans $\mathfrak{B}_n$. Alors d'après [48], Theorem 1.24, $\theta_n = \Delta_n^2$ engendre le centre $Z(\mathfrak{B}_n)$ de $\mathfrak{B}_n$. Les éléments «aux niveaux n» sont fortement reliés aux points fixes de l'action de θ_n sur $V^{\otimes n}$. Les algorithmes pour calculer ces éléments à partir des points fixes de θ_n sont proposés dans la Section 4.5.3.

Il faudrait remarquer que si la structure du module de Yetter-Drinfel'd est du type diagonal provenant d'une matrice de Cartan symétrisable, alors toutes les relations de Serre quantiques dans $T(V)$ sont clairement de niveau n, donc primitives.

1.2.3 Cas diagonal

Les résultats principaux du chapitre 4 sont valables pour les algèbres de Nichols de type quelconque. Lorsqu'on se restreint à des cas particuliers (par exemple, ceux de type diagonal), il y doit avoir des renforcements provenant des propriétés supplémentaires imposées au tressage.

Dans le chapitre 5, sous l'hypothèse que les tressages sont de type diagonal, on introduit un espace vectoriel : «l'espace des pré-relations», qui engendre l'idéal de définition $\mathfrak{I}(V)$ et qui est de taille «assez petite». Les conditions proposées dans la définition de ces relations nous permettent d'étudier le problème de spécialisation de ces algèbres quand le tressage est associé à une matrice de Cartan généralisée qui n'est pas forcément symétrisable.

De plus, on expliquera la raison pour laquelle l'espace des pré-relations est «assez petit» : on établit un lien entre la taille de cet espace et les solutions entières d'une forme quadratique entière.

1.2.4 Application au problème de spécialisation

Cette application est une étape vers la compréhension des groupes quantiques associés aux matrices de Cartan généralisées non-symétrisables, la motivation provenant du problème suivant posé par M. Kashiwara [45], Section 13, Problem 3 :

Est-ce qu'un graphe cristallin pour $\mathfrak{g}$ non-symétrisable a un sens ?

À cause du fait que les groupes quantiques ont initialement définis par Drinfel'd et Jimbo en quantifiant de la présentation de Chevalley-Serre des bigèbres de Lie et cette construction n'est valable que dans le cas symétrisable, on ne connaissait pas de définition possible jusqu'à l'apparition de la recherche sur les algèbres de Nichols [67] ou dualement des algèbres de battages quantiques [73]. Cet possibilité d'avoir une définition n'implique jamais de connaissance sur sa structure et ses représentations. En effet, même dans le cas non-quantifié (c'est-à-dire, les algèbres de Kac-Moody), ce problème (donner une description explicite de $\mathfrak{g}$ par générateurs et relations) reste ouvert (il faut rappeler que dans le cas symétrisable, c'est le théorème de Gabber-Kac).

Dans le chapitre 5, on cherche à comprendre le problème de spécialisation de ces groupes quantiques en étudiant l'idéal de définition.

Plus précisément, on propose un sous-espace vectoriel dans l'idéal de définition qui s'appelle «pré-relations». Les restrictions posées sont triples : une pré-relation est un élément non-nul dans $T(V)$ vérifiant :

1. il est annulé par tous les opérateurs différentiels ;
2. il est obtenu comme crochets itérés ;
3. il est un point fixe sous l'action du centre du groupe de tresse.

Théorème 5 (Theorem 5.2)**.** L'idéal de Hopf engendré par les pré-relations à droite (ou à gauche) est l'idéal de définition $\mathfrak{I}(V)$.

Ensuite, ce théorème est appliqué à l'étude du morphisme de spécialisation. Les contre-exemples sont construits pour expliquer que ce morphisme n'est pas bien défini pour toutes les matrices de Cartan généralisées. Passer à la matrice moyenne donne une solution de ce problème : en effet, le théorème suivant est démontré :

Théorème 6 (Theorem 5.3)**.** Le morphisme de spécialisation est bien défini une fois qu'on passe à la matrice de Cartan généralisée moyenne. De plus, il est surjectif.

1.3 Un théorème du type Borel-Weil-Bott pour les algèbres de battages quantiques

Cette partie reprend le chapitre 6.

1.3.1 Motivations

L'un des problèmes centraux dans la théorie des représentations des groupes et algèbres est de construire les représentations irréductibles et indécomposables. Dans le monde analytique, (par exemple, groupes de Lie), il existe deux outils principaux pour étudier ce problème : les «théorème de Peter-Weyl» et «théorème de Borel-Weil-Bott» qui permettent de réaliser les représentations irréductibles à partir des fonctions sur le groupe.

À la fin du dernier siècle, les algèbres enveloppantes associées aux algèbres de Kac-Moody symétrisables sont déformées parfaitement comme les algèbres de Hopf dans les travaux de Drinfel'd et Jimbo et les résultats sont nommés «algèbres enveloppantes quantiques» ou «groupes quantiques».

De plus, cette procédure déforme simultanément les représentations irréductibles des algèbres enveloppantes, ce qui nous motive à nous poser la question suivante : y a-t-il des analogues des théorèmes de Peter-Weyl et Borel-Weil-Bott dans le cadre des groupes quantiques ?

Un analogue du théorème de Borel-Weil-Bott a été formalisé rapidement par Anderson, Polo et Wen dans [1] : ils ont étudié un analogue de la variété de drapeaux G/B et considéré un caractère dessus comme un fibré en droite. Finalement, le théorème a été généralisé en utilisant des techniques provenant de la théorie des groupes algébriques. Par ailleurs, en 1994, un théorème de Peter-Weyl «postiche» est démontré par A. Joseph et G. Letzter dans [41].

Dans ce chapitre, nous considérons une autre construction dans le cadre «algèbres de battages quantiques» pour donner une généralisation différente du théorème de Borel-Weil-Bott aux groupes quantiques.

1.3.2 Une esquisse

Etant donnée une algèbre de Hopf H et un module de Hopf M sur celle-ci, l'ensemble des coinvariants à droite $V = M^{coR}$ dans M admet une structure de H-module au sens de Yetter-Drinfel'd qui le munit d'une structure d'espace tressé. Ensuite, la machine d'algèbres de battages quantiques construite par M. Rosso dans [73] peut s'appliquer pour fabriquer fonctoriellement une algèbre de Hopf tressée $S_\sigma(V)$. Lorsque les choix de H et M sont faits proprement, $S_\sigma(V)$ est isomorphe à la partie strictement négative (ou positive) d'un groupe quantique. Cette algèbre $S_\sigma(V)$ sert comme analogue de la «variété de drapeau» dans notre généralisation.

Le bicomodule sur lequel l'homologie prend sa valeur est construit en élargissant l'algèbre de Hopf H par un élément group-like K_λ paramétré par un poids dominant $\lambda \in \mathcal{P}_{++}$ et le module de Hopf M par un vecteur v_λ avec les actions et coactions bien choisies (voir Section 6.4.1 pour les détails), puis l'algèbre de battages quantiques $S_{\widetilde{\sigma}}(W)$ sort de la machine avec $W = \mathrm{span}(V, v_\lambda)$ et un tressage $\widetilde{\sigma}$ qui contient $S_\sigma(V)$ comme une sous-algèbre de Hopf tressée. Une graduation prenant en compte l'apparition de v_λ est considérée en mettant degré 0 sur les éléments dans $S_\sigma(V)$ et mettant en degré 1 v_λ : notons $S_{\widetilde{\sigma}}(W)_{(n)}$ l'ensemble des éléments de degré n dans $S_{\widetilde{\sigma}}(W)$. Une conséquence immédiate de cette construction affirme que les $S_{\widetilde{\sigma}}(W)_{(n)}$ admettent les structures de $S_\sigma(V)$-bimodules de Hopf, où $S_{\widetilde{\sigma}}(W)_{(1)}$ sert comme un «fibré en droite» non-commutatif sur $S_\sigma(V)$. L'avantage de cette construction se trouve dans le fait qu'elle est plus proche que la géométrie dans le cas commutatif et qu'elle est plus fonctorielle.

Notons que la théorie d'homologie qu'on utilisera est celle du cadre dual : l'homologie de coHochschild des cogèbres prenant ses valeurs dans les bicomodules au-dessus.

1.3.3 Résultats principaux

Supposons que $\mathfrak{g}$ est une algèbre de Lie simple de dimension finie.

Théorème 7 (Theorem 6.7)**.** Le groupe d'homologie de coHochschild de $S_\sigma(V)$ prenant valeurs dans $S_{\widetilde{\sigma}}(W)_{(1)}$ est donné par :

1. Si q n'est pas une racine de l'unité et $\lambda \in \mathcal{P}_{++}$ est un poids dominant,

$$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ 0, & n \neq 0, \end{cases}$$

comme $U_q(\mathfrak{g})$-modules.

2. Si $q^l = 1$ est une racine primitive de l'unité et $\lambda \in \mathcal{P}^l_{++}$ est un poids dominant ayant les coefficients par rapport aux racines simples moins que l, on a :

$$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ \wedge^n(\mathfrak{n}_-), & n \geq 1, \end{cases}$$

comme $U_q(\mathfrak{g})$-modules, où $\mathfrak{n}_-$ s'identifie avec la partie négative de $\mathfrak{g}$.

La démonstration de ce théorème comporte deux parties : le calcul de l'homologie en degré 0 provient d'un théorème dû à M. Rosso qui décrit l'espace des coinvariants à droite de la structure de $S_\sigma(V)$-module de Hopf sur $S_{\widetilde{\sigma}}(W)_{(1)}$. L'annulation de l'homologie en degré plus grand s'obtient en utilisant les outils suivants :

1. L'autodualité de l'algèbre de battages quantiques $S_\sigma(V)$ pour relier l'homologie d'algèbre et de cogèbre.
2. La filtration de PBW de $S_\sigma(V)$, (resp. $S_{\widetilde{\sigma}}(W)_{(1)}$), et l'algèbre (le module) gradué(e) associé(e).
3. La dualité de Koszul pour obtenir une résolution de $\mathrm{gr}(S_{\widetilde{\sigma}}(W)_{(1)})$.
4. Une homotopie explicite pour prouver l'acyclicité du complexe ci-dessus.
5. Revenir au cas filtré par un argument de la suite spectrale.
6. Au cas racine de l'unité, on extrait un sous-complexe dans le complexe de Koszul ayant une différentielle nulle et tel que le reste est acyclique.

Cette construction admet divers avantages :

1. Elle nous permet d'étudier les «fibrés» de degrés plus hauts : $S_{\widetilde{\sigma}}(W)_{(2)}$, $S_{\widetilde{\sigma}}(W)_{(3)}$, $\cdots$:

   Théorème 8 (Theorem 6.10)**.** Sous l'hypothèse du point (1) dans le théorème précédent, on a :

   (a) Si pour tout $i \in I$, $(\lambda, \alpha_i^\vee) = 1$, alors comme $U_q(\mathfrak{g})$-modules,

   $$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)}) = \begin{cases} L(\lambda)\otimes L(\lambda) & n = 0; \\ 0 & n \neq 0. \end{cases}$$

   (b) Si $J \subset I$ est le sous-ensemble contenant les $j \in I$ tel que $(\lambda, \alpha_j^\vee) = 1$, alors comme $U_q(\mathfrak{g})$-modules,

   $$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)}) = \begin{cases} (L(\lambda)\otimes L(\lambda)) \Big/ \bigoplus_{j\in J} L(2\lambda - \alpha_j) & n = 0; \\ 0 & n \neq 0. \end{cases}$$

2. Elle donne une construction inductive des parties négatives (positives) des groupes quantiques et simultanément les bases PBW : voir Sections 6.8 et 6.9.
3. Elle peut se généraliser à un cadre plus large : les algèbres affines quantiques, les algèbres de Hall sphériques, etc. (On espère y revenir dans le futur.)

1.4 Fonction η de Dedekind et groupes quantiques

Cette partie contient le chapitre 7.

1.4.1 Motivations historiques

La fonction de partition $p(n)$ d'un nombre naturel n est un object mathématique important. L'inverse de sa série génératrice, notée par $\varphi(x)$, admet une expression simple et compacte $\prod_{n\geq 1}(1-x^n)$ qui est reliée avec la fonction η de Dedekind par la relation $\eta(x) = x^{\frac{1}{24}}\varphi(x)$. La 24-ième puissance de $\eta(x)$ est une forme modulaire de poids 12 qui contient les fonctions τ de Ramanujan comme les coefficients dans sa série de Taylor.

Quelques puissances de $\varphi(x)$ sont étudiées par Euler puis par Jacobi dans ses travaux sur les fonctions θ et les fonctions elliptiques. Par exemple, il est démontré par Jacobi que

$$\varphi(x)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{n(n+1)}{2}}.$$

Ces formules concernant les puissances de $\varphi(x)$ et $\eta(x)$ sont largement élargies dans les travaux de I. MacDonald en les expliquant comme cas particuliers de la formule de dénominateur de Weyl associée aux systèmes de racines affines. Par exemple, la formule de Jacobi s'obtient à partir des informations combinatoires du système de racine du type A_1.

En 1976, les formules de MacDonald sont réintérprétées par B. Kostant en utilisant la théorie des représentations des groupes de Lie compacts : par exemple, si G est simplement lacé, il les réécrit comme une somme sur les poids dominants :

$$\eta(x)^{\dim G} = \sum_{\lambda\in\mathcal{P}_+}\mathrm{Tr}(c, V_1(\lambda)_0)\dim V_1(\lambda)x^{(\lambda+\rho,\lambda+\rho)},$$

où c est un élément de Coxeter dans le groupe de Weyl, $V_1(\lambda)$ est la représentation de $\mathfrak{g}$ de plus haut poids λ et $V_1(\lambda)_0$ est son sous-espace de poids 0.

1.4.2 Groupes quantiques

Avec comme but de construire les solutions de l'équation de Yang-Baxter, les groupes quantiques sont construits par Drinfel'd et Jimbo comme déformations formelles des algèbres enveloppantes au milieu des années 1980. Cette procédure déforme non seulement les algèbres mais aussi les représentations intégrables et les groupes de Weyl.

De nouvelles structures et de nouveaux outils paraissent après cette procédure : la R-matrice universelle, l'action du groupe de tresse, les bases canoniques (cristallines), etc. De plus, l'apparition du paramètre q enrichit la structure interne de l'algèbre enveloppante : cette liberté nous permet de marquer les croisements différents dans les diagrammes planaires des nœuds pour obtenir les invariants quantiques.

1.4.3 Groupe de Weyl quantique

Le groupe de Weyl associé à une algèbre de Lie semi-simple contrôle la symétrie interne de cette algèbre et de ses représentations. Une déformation de ces symétries est construite parfaitement dans les travaux de Kirillov-Reshetikhin et Levendorski-Soibelman pour obtenir une forme explicite de la R-matrice universelle. Ceci munit toutes les représentations intégrables d'une symétrie donnée par le groupe de tresse d'Artin.

1.4.4 La fonction η de Dedekind et groupes quantiques

Le chapitre 7 est dédié à bien comprendre et généraliser les formules de Jacobi, MacDonald et Kostant en utilisant les nouveaux outils offerts par la théorie des groupes quantiques. Plus précisément, on écrira les puissances de la fonction η de Dedekind comme la trace d'un opérateur agissant sur l'algèbre quantique des coordonnées associée à un groupe quantique.

En effet, soient $\mathfrak{g}$ une algèbre de Lie simple complexe de rang l, W son groupe de Weyl, q un paramètre formel, $U_q(\mathfrak{g})$ le groupe quantique associé à $\mathfrak{g}$, $\mathfrak{B}_\mathfrak{g}$ le groupe de tresse d'Artin associé à W avec générateurs $\sigma_1, \cdots, \sigma_l$, $\mathbb{C}_q[G]$ l'anneau des coordonées quantiques, $\lambda \in \mathcal{P}_+$ un poids dominant, $V(\lambda)$ la représentation irréductible de $U_q(\mathfrak{g})$ de plus haut poids λ du type 1.

Le groupe $\mathfrak{B}_\mathfrak{g}$ agit sur $V(\lambda)$ donc sur

$$\mathbb{C}_q[G] \cong \bigoplus_{\lambda \in \mathcal{P}_+} V(\lambda) \otimes V(\lambda)^*$$

via le plongement

$$\mathfrak{B}_\mathfrak{g} \to \mathrm{Aut}(\mathbb{C}_q[G]), \quad \sigma_i \mapsto \sigma_i \otimes \mathrm{id}.$$

Notons $\Pi = \sigma_1 \cdots \sigma_l \in \mathfrak{B}_\mathfrak{g}$ un élément de Coxeter dans $\mathfrak{B}_\mathfrak{g}$ et h le nombre de Coxeter de W.

Théorème 9 (Theorem 7.3)**.** On a l'identité suivante :

$$\mathrm{Tr}(\Pi \otimes \mathrm{id}, \mathbb{C}_q[G]) = \left(\prod_{i=1}^{l} \varphi(q^{(\alpha_i, \alpha_i)}) \right)^{h+1}.$$

Chapitre 2

q-Boson algebras

Contents of this chapter is published in [24].

2.1 Introduction

In his article [44], M.Kashiwara defined crystal bases for quantized enveloping algebras. To show the existence of such bases for the strictly negative parts $U_q^{<0}(\mathfrak{g})$ of quantized enveloping algebras, he constructed an associative algebra generated by operators on $U_q^{<0}(\mathfrak{g})$, which is a q-analogue of boson. In fact, this algebra is a quantized version of the usual Weyl algebra and with the help of such algebra, he proved that $U_q^{<0}(\mathfrak{g})$, viewed as a module over this "quantized Weyl algebra", is simple. Moreover, he affirmed without proof that imposing a finiteness condition on modules over "quantized Weyl algebra" will lead to semi-simplicity results.

Later, in his article [65], T.Nakashima defined the so called "q-Boson algebra" $B_q(\mathfrak{g})$, an extension of the quantized Weyl algebra $W_q(\mathfrak{g})$ by a torus, and studied these algebras. Finally, in [66], he archived in proving the semi-simplicity of $\mathcal{O}(B_q)$, the category of modules over $B_q(\mathfrak{g})$ with some finiteness conditions, where the main tool is an "extremal projector" defined therein. But we should point out that the proof in [66] depends on the "Casimir-like" element of a pairing ; to get the desired properties, the author has to use a large quantity of computation, see for example [83], [65] and [66].

In this article, we will construct quantized enveloping algebras(quantum groups), q-Boson algebras and quantized Weyl algebras in a unified method and give an action of quantum groups on quantized Weyl algebras by the Schrödinger representation. This enables us to give another construction of the quantized Weyl algebra with the help of the braiding in some Yetter-Drinfel'd module category. With this construction, we can obtain a structural result for all $W_q(\mathfrak{g})$-modules with a natural finiteness condition, which will lead directly to the semi-simplicity of $\mathcal{O}(B_q)$ and the classification of all simple objects in it. Moreover, the proof we give here is more conceptual : it means that the structure of category $\mathcal{O}(B_q)$ depends heavily on the intrinsic duality of $B_q(\mathfrak{g})$. As a byproduct, we prove the semi-simplicity of $W_q(\mathfrak{g})$-modules with a finiteness condition and classify all simple modules of this type.

This work is inspired by an observation in the finite dimensional case : once we have a nondegenerate pairing between two Hopf algebras, we may form the smash product of them, where the "module algebra type" action is given by this pairing. If we have a finite dimensional module over this smash product, from the duality, we will obtain simultaneously a module and a comodule structure, and the construction of smash product is exactly the compatibility condition of the module and comodule structures to yield a Hopf module. As showed in [81], all Hopf modules are trivial, that is to say, a free module over the original Hopf algebra, and blocks are parameterized by a vector space called "coinvariants".

We would like to generalize this observation to a more general case, for example, quantized Weyl algebras or q-Boson algebras. But unfortunately, it does not work well ase the action of torus part is not locally nilpotent. Our main idea for overcoming this difficulty is to hide the "torus part" behind the construction with the help of a braiding originated in a quantum group action. This is the main reason for our use of the technical language of Yetter-Drinfel'd modules and braided Hopf algebras.

We want to be more precise : for any module M in $\mathcal{O}(B_q)$, it is possible to restrict it to the quantized Weyl algebra to obtain a $W_q(\mathfrak{g})$-module with a finiteness condition. In Section 2.4.1, we will realize $W_q(\mathfrak{g})$ as an algebra obtained from its negative and positive parts with a braiding, this enables us to get a module and comodule structure on M. Unfortunately again, these structures are not compatible, but it is not too far away : they are compatible in the sense of braiding in this case ; we may still prove a trivialization result, which gives out the structural theorem of all $W_q(\mathfrak{g})$-modules with finiteness condition and will lead easily to the structure theory of category $\mathcal{O}(B_q)$.

In the proof of the structural theorem of Hopf modules, there exists a projection from the Hopf module to the set of its coinvariants, which will be shown to be exactly the "extremal projector" in [66] and the projection given in [44], (3.2.2) in the $\mathfrak{sl}_2$ case. This explains the "extremal projector" in a more natural way.

The constitution of this chapter is as follows. In Section 2.2, we recall some notions in Hopf algebras and give out an action of quantum doubles on Heisenberg doubles with the help of Schrödinger representations. In Section 2.3, we construct quantum groups and q-Boson algebras concretely and calculate the action between them in the case of $\mathfrak{sl}_2$. In Section 2.4, we give constructions of quantized Weyl algebras from the braiding in Yetter-Drinfel'd category and prove the main theorem on the structure of $\mathcal{O}(B_q)$, at last, we compare our projection with those defined in [44] and [66].

At last, we should remark that in the preparation of this chapter, the preprint of A.M. Semikhatov [77] came into our sight, he got essentially same results as in the Section 2.2 of this chapter, though with a different objective and point of view.

2.2 Hopf pairings and double constructions

From now on, suppose that we are working on the complex field $\mathbb{C}$. Results in this section hold for any field with characteristic 0. All tensor products are over $\mathbb{C}$ if not specified otherwise.

2.2.1 Yetter-Drinfel'd modules

Let H be a Hopf algebra. A vector space V is called a (left) H-Yetter-Drinfel'd module if it is simultaneously an H-module and an H-comodule satisfying the Yetter-Drinfel'd compatibility condition : for any $h \in H$ and $v \in V$,

$$\sum h_{(1)} v_{(-1)} \otimes h_{(2)}.v_{(0)} = \sum (h_{(1)}.v)_{(-1)} h_{(2)} \otimes (h_{(1)}.v)_{(0)},$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and $\rho(v) = \sum v_{(-1)} \otimes v_{(0)}$ are Sweedler notations for coproduct and comodule structure maps.

Morphisms between two H-Yetter-Drinfel'd modules are linear maps preserving H-module and H-comodule structures.

We let ${}^H_H\mathcal{YD}$ denote the category of H-Yetter-Drinfel'd modules; it is a tensor category.

The advantage of Yetter-Drinfel'd module is : for $V, W \in {}^H_H\mathcal{YD}$, there exists a braiding $\sigma : V \otimes W \to W \otimes V$, given by $\sigma(v \otimes w) = \sum v_{(-1)}.w \otimes v_{(0)}$. If both V and W are H-module algebras, $V \otimes W$ will have an algebra structure if we use σ instead of the usual flip. We let $V \underline{\otimes} W$ denote this algebra.

2.2.2 Braided Hopf algebras in ${}^H_H\mathcal{YD}$

In [69], D.Radford constructed the biproduct of two Hopf algebras when there exists an action and coaction between them and obtained the necessary and sufficient conditions for the existence of a Hopf algebra structure on this biproduct. See Theorem 1 and Proposition 2 in [69].

Once the language of Yetter-Drinfel'd module has been adopted, conditions in [69] can be easily rewritten.

Definition 2.1 ([6], Section 1.3)**.** A braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ is a collection $(A, m, \eta, \Delta, \varepsilon, S)$ such that :

1. (A, m, η) is an algebra in ${}^H_H\mathcal{YD}$; (A, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$. That is to say, $m, \eta, \Delta, \varepsilon$ are morphisms in ${}^H_H\mathcal{YD}$;
2. $\Delta : A \to A \underline{\otimes} A$ is a morphism of algebras;
3. $\varepsilon : A \to \mathbb{C}$, $\eta : \mathbb{C} \to A$ are morphisms of algebras;
4. S is the convolution inverse of $\mathrm{Id}_A \in End(A)$.

Remark 2.1.

1. Once a braided Hopf algebra A has been given, we can form the tensor product $A \otimes H$, it yields a Hopf algebra structure, as shown in [69].
2. An important example here is the construction of the "positive part" of a quantized enveloping algebra as a twist of a braided Hopf algebra with primitive coproduct by a commutative group algebra.
3. For a general construction in the framework of Hopf algebras with a projection, see [6], Section 1.5.

2.2.3 Braided Hopf modules

Let B be a braided Hopf algebra in some Yetter-Drinfel'd module category. For a left braided B-Hopf module M, we mean a left B-module and a left B-comodule satisfying compatibility condition as follows :

$$\rho \circ l = (m \otimes l) \circ (id \otimes \sigma \otimes id) \circ (\Delta \otimes \rho) : B \otimes M \to B \otimes M,$$

where m is the multiplication in B, $l : B \otimes M \to M$ is the module structure map, $\rho : M \to B \otimes M$ is the comodule structure map and σ is the braiding in the fixed Yetter-Drinfel'd module category.

Example 2.1. Let V be a vector space over $\mathbb{C}$. Then $B \otimes V$ admits a trivial B-braided Hopf module structure given by : for $b, b' \in B$ and $v \in V$,

$$b'.(b \otimes v) = b'b \otimes v, \quad \rho(b \otimes v) = \sum b_{(1)} \otimes b_{(2)} \otimes v \in B \otimes (B \otimes V).$$

We let ${}^B_B\mathcal{M}$ denote the category of left B-braided Hopf modules. The following proposition gives the triviality of such kind of modules.

Proposition 2.1. Let $M \in {}^B_B\mathcal{M}$ be a braided Hopf module, $\rho : M \to B \otimes M$ be the structural map, $M^{cop} = \{m \in M |\ \rho(m) = 1 \otimes m\}$ be the set of coinvariants. Then there exists an isomorphism of B-braided Hopf modules :

$$M \cong B \otimes M^{cop},$$

where the right hand side admits the trivial Hopf module structure as in the example above. Moreover, maps in two directions are given by :

$$M \to B \otimes M^{cop}, \quad m \mapsto \sum m_{(-1)} \otimes P(m_{(0)}),$$
$$B \otimes M^{cop} \to M, \quad b \otimes m \mapsto bm,$$

where $m \in M$, $b \in B$ and $P : M \to M^{cop}$ is defined by : $P(m) = \sum S(m_{(-1)})m_{(0)}$.

The proof for the triviality of Hopf modules given in [81] can be adopted to the braided case.

Remark 2.2. Proposition 2.1 can be translated into the categorical language, which says that there exists an equivalence of category ${}^B_B\mathcal{M} \sim \mathbf{Vect}$, where $\mathbf{Vect}$ is the category of vector spaces, given by $M \mapsto M^{cop}$ and $V \mapsto B \otimes V$ for $M \in {}^B_B\mathcal{M}$ and $V \in \mathbf{Vect}$.

2.2.4 Generalized Hopf pairings

Generalized Hopf pairings give dualities between Hopf algebras.

Let A and B be two Hopf algebras with invertible antipodes. A generalized Hopf pairing between A and B is a bilinear form $\varphi : A \times B \to \mathbb{C}$ satisfying :

1. for any $a \in A$, $b, b' \in B$, $\varphi(a, bb') = \sum \varphi(a_{(1)}, b)\varphi(a_{(2)}, b')$;
2. for any $a, a' \in A$, $b \in B$, $\varphi(aa', b) = \sum \varphi(a, b_{(2)})\varphi(a', b_{(1)})$;
3. for any $a \in A$, $b \in B$, $\varphi(a, 1) = \varepsilon(a)$, $\varphi(1, b) = \varepsilon(b)$.

Remark 2.3. From the uniqueness of the antipode and conditions (1)-(3) above, we have : for any $a \in A$, $b \in B$, $\varphi(S(a), b) = \varphi(a, S^{-1}(b))$.

2.2.5 Quantum doubles

Let A and B be two Hopf algebras with invertible antipodes and φ be a generalized Hopf pairing between them. The quantum double $D_\varphi(A,B)$ is defined by :

1. as a vector space, it is $A \otimes B$;
2. as a coalgebra, it is the tensor product of coalgebras A and B ;
3. as an algebra, the multiplication is given by :

$$(a \otimes b)(a' \otimes b') = \sum \varphi(S^{-1}(a'_{(1)}), b_{(1)})\varphi(a'_{(3)}, b_{(3)})aa'_{(2)} \otimes b_{(2)}b'.$$

2.2.6 Schrödinger Representations

The prototype of Schrödinger representation in physics is the momentum group G action on a position space M ; this will give out an action of $\mathbb{C}(M) \rtimes \mathbb{C}(G)$ on $\mathbb{C}(M)$. Details of this view point can be found in the Chapter 6 of [60].

The definitions and propositions in this subsection are essentially in [60], Example 7.1.8.

The Schrödinger representation of $D_\varphi(A,B)$ on A is given by : for $a, x \in A$, $b \in B$,

$$(a \otimes 1).x = \sum a_{(1)}xS(a_{(2)}),$$

$$(1 \otimes b).x = \sum \varphi(x_{(1)}, S(b))x_{(2)}.$$

The Schrödinger representation of $D_\varphi(A,B)$ on B is given by : for $a \in A$, $b, y \in B$,

$$(a \otimes 1).y = \sum \varphi(a, y_{(1)})y_{(2)},$$

$$(1 \otimes b).y = \sum b_{(1)}yS(b_{(2)}).$$

So

$$(a \otimes b).x = \sum \varphi(x_{(1)}, S(b))a_{(1)}x_{(2)}S(a_{(2)}),$$

$$(a \otimes b).y = \sum \varphi(a, b_{(1)}y_{(1)}S(b_{(4)}))b_{(2)}y_{(2)}S(b_{(3)}).$$

Proposition 2.2 ([60], Example 7.1.8)**.** With the definition above, both A and B are $D_\varphi(A,B)$-module algebras.

2.2.7 Heisenberg doubles

Keep assumptions in previous sections. Now we construct the Heisenberg double between A and B ; it is the smash product of them where the module algebra type action of A on B is given by the Hopf pairing. For the background of this double, see [54].

The Heisenberg double $H_\varphi(A,B)$ is an algebra defined as follows :

1. as a vector space, it is $B \otimes A$ and we denote the pure tensor by $b\sharp a$;
2. the product is given by : for $a, a' \in A$, $b, b' \in B$,

$$(b\sharp a)(b'\sharp a') = \sum \varphi(a_{(1)}, b'_{(1)})bb'_{(2)}\sharp a_{(2)}a'.$$

Remark 2.4. In general, $H_\varphi(A,B)$ has no Hopf algebra structure.

2.2.8 Quantum double action on Heisenberg double

We define an action of $D_\varphi(A,B)$ on $H_\varphi(A,B)$ as follows : for $a,a' \in A$, $b,b' \in B$,

$$(a \otimes b).(b' \sharp a') = \sum (a_{(1)} \otimes b_{(1)}).b' \sharp (a_{(2)} \otimes b_{(2)}).a',$$

this is a diagonal type action. Moreover, we have the following result :

Proposition 2.3. With this action, $H_\varphi(A,B)$ is a $D_\varphi(A,B)$-module algebra.

To be more precise, the above action can be written as :

$$(a \otimes b).(b' \sharp a') = \sum \varphi(a_{(1)}, b_{(1)} b'_{(1)} S(b_{(4)})) \varphi(a'_{(1)}, S(b_{(5)})) b_{(2)} b'_{(2)} S(b_{(3)}) \sharp a_{(2)} a'_{(2)} S(a_{(3)}).$$

Remark 2.5. This proposition gives a family of examples for Yang-Baxter algebras ; for the definition and fundamental properties, see [37]. Properties of such kind of algebra make it possible to define a braiding on the tensor product of $H_\varphi(A,B)$, which gives an algebra structure on $H_\varphi(A,B)^{\otimes n}$. Equivalently, we can translate this braiding in the framework of Yetter-Drinfel'd modules, which is much more useful for future applications.

We define a $D_\varphi(A,B)$-comodule structure on both A and B as follows :

$$A \to D_\varphi(A,B) \otimes A, \quad a \mapsto \sum a_{(1)} \otimes 1 \otimes a_{(2)},$$

$$B \to D_\varphi(A,B) \otimes B, \quad b \mapsto \sum 1 \otimes b_{(1)} \otimes b_{(2)}.$$

Proposition 2.4. With Schrödinger representations and comodule structure maps defined above, both A and B are in the category ${}^{D_\varphi}_{D_\varphi}\mathcal{YD}$.

More generally, we have the following result.

Proposition 2.5. With the comodule structure map defined by :

$$\delta : H_\varphi(A,B) \to D_\varphi(A,B) \otimes H_\varphi(A,B), \quad b \sharp a \mapsto \sum ((1 \otimes b_{(1)})(a_{(1)} \otimes 1)) \otimes b_{(2)} \sharp a_{(2)},$$

for $a \in A$, $b \in B$, $H_\varphi(A,B)$ is in the category ${}^{D_\varphi}_{D_\varphi}\mathcal{YD}$.

The rest part of this section will be devoted to giving proofs of these propositions using the Miyashita-Ulbrich action. This is recommended by the referee.

2.2.9 Twisted product

Let H be a Hopf algebra, $\sigma : H \otimes H \to \mathbb{C}$ be a 2-cocycle which is invertible in $(H \otimes H)^*$ (for a definition, see [20] or [54]). Then we can form the following two twisted products on H.

Definition 2.2. The twisted algebra H^σ is defined as follows :

1. as a vector space, it is H itself ;

2. for any $x, y \in H$, the product on H^σ is given by :

$$x \bullet y = \sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \sigma^{-1}(x_{(3)}, y_{(3)}),$$

where σ^{-1} is the inverse of σ in $(H \otimes H)^*$.

With the original coproduct on H, H^σ is a Hopf algebra.

Definition 2.3. The twisted algebra ${}_\sigma H$ is defined as follows :

1. as a vector space, it is H itself ;
2. for any $x, y \in H$, the product in ${}_\sigma H$ is given by :

$$x \circ y = \sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)}.$$

The coproduct on H gives ${}_\sigma H$ a left H^σ-comodule algebra structure. Moreover, ${}_\sigma H$ is cleft H^σ-Hopf-Galois extension over $\mathbb{C}$ on the left such that the identity map $\gamma : H^\sigma \to {}_\sigma H$ is a convolution-invertible H^σ-comodule morphism (see, for example, Theorem 4.3 in [64]). We let γ^{-1} denote the convolution-inverse of γ.

For $x \in H^\sigma$, $y \in {}_\sigma H$,

$$x \rightharpoonup y = \sum \gamma(x_{(1)}) \circ y \circ \gamma^{-1}(x_{(2)})$$

gives the Miyashita-Ulbrich action of H^σ on ${}_\sigma H$ (see [78] for the definition).

From Corollary 3.1 in [78], the Miyashita-Ulbrich action of H^σ on ${}_\sigma H$ and the original coaction make ${}_\sigma H$ into an algebra object in the category ${}^{H^\sigma}_{H^\sigma}\mathcal{YD}$.

2.2.10 Application to the double construction

We preserve notations in previous sections.

Let A, B be two Hopf algebras and $H = B \otimes A$ be their tensor product. Suppose that there exists a Hopf pairing φ between A and B. Then we can define a 2-cocycle using this pairing : for $a, a' \in A$ and $b, b' \in B$,

$$\sigma : H \otimes H \to \mathbb{C}, \quad \sigma(b \otimes a, b' \otimes a') = \varepsilon(b)\varphi(a, b')\varepsilon(a').$$

Moreover, the inverse of σ is given by :

$$\sigma^{-1} : H \otimes H \to \mathbb{C}, \quad \sigma^{-1}(b \otimes a, b' \otimes a') = \varepsilon(b)\varphi(a, S(b'))\varepsilon(a').$$

Proposition 2.6 ([20],[54])**.** (1). There exists an isomorphism of Hopf algebras :

$$D_\varphi(A, B) \to H^\sigma, \quad a \otimes b \mapsto (1 \otimes a) \bullet (b \otimes 1).$$

(2). As algebras, $H_\varphi(A, B) = {}_\sigma H$.

Thus H_φ is a cleft D_φ-Hopf-Galois extension over $\mathbb{C}$ on the left.

In this case, we compute the Miyashita-Ulbrich action explicitly. Note that D_φ includes A, B as Hopf subalgebras, and that H_φ includes A (resp., B) as a left A- (resp.,

B-)comodule subalgebra. It results that the identity map $D_\varphi \xrightarrow{\sim} H^\sigma \to {}_\sigma H = H_\varphi$, when restricted to A, B, has antipodes of A, B as convolution inverses.

Let $a, a' \in A$ and $b, b' \in B$. Then

$$\begin{aligned}(a \otimes 1) \rightharpoonup (b \otimes 1) &= \sum (1 \otimes a_{(1)})(b \otimes 1)(1 \otimes S(a_{(2)})) \\ &= \sum \varphi(a, b_{(1)}) b_{(2)} \otimes 1;\end{aligned}$$

$$\begin{aligned}(a \otimes 1) \rightharpoonup (1 \otimes a') &= \sum (1 \otimes a_{(1)})(1 \otimes a')(1 \otimes S(a_{(2)})) \\ &= 1 \otimes \sum a_{(1)} a' S(a_{(2)});\end{aligned}$$

$$\begin{aligned}(1 \otimes b) \rightharpoonup (1 \otimes a) &= \sum (b_{(1)} \otimes 1)(1 \otimes a)(S(b_{(2)} \otimes 1) \\ &= 1 \otimes \sum \varphi(a_{(1)}, S(b)) a_{(2)};\end{aligned}$$

$$\begin{aligned}(1 \otimes b) \rightharpoonup (b' \otimes 1) &= \sum (b_{(1)} \otimes 1)(b' \otimes 1)(S(b_{(2)} \otimes 1) \\ &= \sum b_{(1)} b' S(b_{(2)}) \otimes 1.\end{aligned}$$

These recover Schrödinger representations of $D_\varphi(A, B)$ on A and B.

Now Proposition 2.2, 2.3 and 2.4 are direct corollaries of Corollary 3.1 in [78]. Proposition 2.5 comes from the same corollary in [78] and Proposition 2.6.

2.3 Construction of quantum algebras

This section is devoted to the construction of three important quantum algebras : quantum groups, quantized Weyl algebras and q-Boson algebras from the machinery built in the last section.

2.3.1 Definitions and notations

Assume that $q \in \mathbb{C}^*$ is not a root of unity. The q-numbers are defined by :

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = \prod_{i=1}^{n} [i], \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, $\mathfrak{h}$ be its Cartan subalgebra, $n = dim\mathfrak{h} = rank(\mathfrak{g})$, Q be its root lattice, Q_+ be the set of its positive roots, $\mathcal{P}$ be its weight lattice, $\Delta = \{\alpha_1 \cdots, \alpha_n\}$ be the set of simple roots, $\mathcal{P}_+ = \mathcal{P} \cap (\mathbb{Q} \otimes_{\mathbb{Z}} Q)$. We denote $(\cdot, \cdot)$ the standard inner product on $\mathfrak{h}^*$ and define $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$.

Bricks of our construction are Hopf algebras $\widetilde{U_q^{\geq 0}}$ and $\widetilde{U_q^{\leq 0}}$, which are defined by generators and relations :

1. $\widetilde{U_q^{\geq 0}}$ is generated by E_i, $(i = 1, \cdots, n)$, $K_\lambda^{\pm 1}$ $(\lambda \in \mathcal{P}_+)$ with relations :

$$K_\lambda E_i K_\lambda^{-1} = q^{(\alpha_i,\lambda)} E_i, \quad K_\lambda K_\lambda^{-1} = K_\lambda^{-1} K_\lambda = 1.$$

   It has a Hopf algebra structure given by :

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \quad \Delta(E_i) = E_i \otimes K_{\alpha_i}^{-1} + 1 \otimes E_i,$$

$$\varepsilon(E_i) = 0, \quad \varepsilon(K_\lambda) = 1, \quad S(E_i) = -E_i K_{\alpha_i}, \quad S(K_\lambda) = K_\lambda^{-1}.$$

2. $\widetilde{U_q^{\leq 0}}$ is generated by F_i, $(i = 1, \cdots, n)$, ${K'_\lambda}^{\pm 1}$ $(\lambda \in \mathcal{P}_+)$ with relations :

$$K'_\lambda F_i {K'_\lambda}^{-1} = q^{-(\alpha_i,\lambda)} F_i, \quad K'_\lambda {K'_\lambda}^{-1} = {K'_\lambda}^{-1} K'_\lambda = 1.$$

   It has a Hopf algebra structure given by :

$$\Delta(K'_\lambda) = K'_\lambda \otimes K'_\lambda, \quad \Delta(F_i) = F_i \otimes 1 + K'_{\alpha_i} \otimes F_{\alpha_i},$$

$$\varepsilon(F_i) = 0, \quad \varepsilon(K'_\lambda) = 1, \quad S(F_i) = -F_i {K'_{\alpha_i}}^{-1}, \quad S(K'_\lambda) = {K'_\lambda}^{-1}.$$

   If $\lambda = \alpha_i$ for some i, we denote $K_i := K_{\alpha_i}$.

2.3.2 Construction of quantum groups

The construction in this section can be found in [47].

We let $D_\varphi(\widetilde{U_q^{\geq 0}}, \widetilde{U_q^{\leq 0}})$ denote the quantum double of $\widetilde{U_q^{\geq 0}}$ and $\widetilde{U_q^{\leq 0}}$, where the generalized Hopf pairing $\varphi : \widetilde{U_q^{\geq 0}} \times \widetilde{U_q^{\leq 0}} \to \mathbb{C}$ is defined by :

$$\varphi(E_i, F_j) = \frac{\delta_{ij}}{q_i^{-1} - q_i}, \quad \varphi(K_\lambda, K'_\mu) = q^{-(\lambda,\mu)}, \quad \varphi(E_i, K'_\lambda) = \varphi(K_\lambda, F_i) = 0,$$

$$\varphi(E', 1) = \varepsilon(E'), \quad \varphi(1, F') = \varepsilon(F'), \quad \forall E' \in \widetilde{U_q^+}, \ F' \in \widetilde{U_q^-}.$$

Now, from the definition of the multiplication in quantum double,

$$(1 \otimes F_j)(E_i \otimes 1) = \varphi(E_i, F_j) 1 \otimes K'_j + E_i \otimes F_i + \varphi(S^{-1}(E_i), F_j) K_i^{-1} \otimes 1,$$

that is to say,

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K'_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

With the same kind of computation, we also have :

$$K'_\lambda E_i = q^{(\lambda,\alpha_i)} E_i K'_\lambda, \quad F_i K_\lambda = q^{(\lambda,\alpha_i)} K_\lambda F_i.$$

The quantum group $U_q(\mathfrak{g})$ can be obtained as follows : at first, to get a non-degenerate pairing, we need to do the quotient by its left and right radical, denoted I_l and I_r respectively. Denote $U_q^{\geq 0} = \widetilde{U_q^{\geq 0}}/I_l$ and $U_q^{\leq 0} = \widetilde{U_q^{\leq 0}}/I_r$. Then φ induces a nondegenerate pairing on $U_q^{\geq 0} \otimes U_q^{\leq 0}$, which is also denoted by φ. The quantum group associated to $\mathfrak{g}$ is the quotient :

$$U_q(\mathfrak{g}) = D_\varphi(U_q^{\geq 0}, U_q^{\leq 0})/(K_\lambda - K'_\lambda | \ \lambda \in \mathcal{P}).$$

Remark 2.6. I_l (resp. I_r) is generated by quantized Serre relations in $\widetilde{U_q^{\geq 0}}$ (resp. $\widetilde{U_q^{\leq 0}}$). Indeed, it is shown in [3] that the radical of this Hopf pairing coincides with the defining ideal of the associated Nichols algebra, which is well-known to be generated by the quantized Serre relations.

2.3.3 Heisenberg double and q-Boson algebras

The procedure above, once applied to the Heisenberg double, will give q-Boson algebra.

In this section, we directly adopt notations $U_q^{\geq 0}$ and $U_q^{\leq 0}$ (that is to say, we add quantized Serre relations) and generators $U_q^{\geq 0} =< e_i, t_\lambda^{\pm 1} >$, $U_q^{\leq 0} =< f_i, t'^{\pm 1}_\lambda >$ for making it distinct from the quantum double case. Moreover, $t_i := t_{\alpha_i}$.

Now we compute the multiplication structure between $U_q^{\geq 0}$ and $U_q^{\leq 0}$:

$$(1\sharp t_\lambda)(f_i\sharp 1) = \varphi(t_\lambda, t'_i) f_i \sharp t_\lambda = q^{-(\alpha_i,\lambda)} f_i \sharp t_\lambda,$$

$$(1\sharp e_i)(t'_\lambda \sharp 1) = t'_\lambda \sharp e_i.$$

For this reason, it is better to adopt generators $e'_i = (q_i^{-1} - q_i) t_i e_i$, which leads to :

$$\Delta(e'_i) = e'_i \otimes 1 + t_i \otimes e'_i, \quad t_\lambda e'_i t_\lambda^{-1} = q^{(\alpha_i,\lambda)} e'_i, \quad \varphi(e'_i, f_j) = \delta_{ij}.$$

With these modified notations,

$$(1\sharp e'_i)(t'_\lambda \sharp 1) = \varphi(t_i, t'_\lambda) t'_\lambda \sharp e'_i = q^{-(\lambda,\alpha_i)} t'_\lambda \sharp e'_i,$$

this is what we desired.

We calculate the relation between e'_i and f_j :

$$(1\sharp e'_i)(f_j \sharp 1) = \varphi(e'_i, f_j) + \varphi(t_i, t'_j) f_j \sharp e'_i = q^{-(\alpha_i,\alpha_j)} f_j \sharp e'_i + \delta_{ij}.$$

A simplification of the notation gives :

$$e'_i f_j = q^{-(\alpha_i,\alpha_j)} f_j e'_i + \delta_{ij}.$$

Then all relations in q-Boson algebra have beed recovered and

$$B_q(\mathfrak{g}) \cong H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})/(t_\lambda - t'_\lambda |\ \lambda \in \mathcal{P}),$$

where $B_q(\mathfrak{g})$ is the q-Boson algebra defined in [65] and [66].

2.3.4 Action of quantum doubles on Heisenberg doubles

Proposition 2.5 gives an action of $D_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ on $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ such that $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ is a $D_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$-Yetter-Drinfel'd module. In this section, we will show that, this action gives a $U_q(\mathfrak{g})$-module algebra structure on $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$, but it can not pass to the quotient to get an action on $B_q(\mathfrak{g})$. So more naturally, we need to introduce the quantized Weyl algebra $W_q(\mathfrak{g})$: this is a subalgebra of $B_q(\mathfrak{g})$, a $U_q(\mathfrak{g})$-Yetter-Drinfel'd module and a $U_q(\mathfrak{g})$-module algebra.

At first, we calculate the action of K_λ and K'_λ :

$$K_\lambda.e'_i = adK_\lambda(e'_i) = q^{(\lambda,\alpha_i)}e'_i, \quad K'_\lambda.e'_i = \varphi(t_i, {K'_\lambda}^{-1})e'_i = q^{(\lambda,\alpha_i)}e'_i,$$

$$K_\lambda.f_i = \varphi(K_\lambda, t'_i)f_i = q^{-(\lambda,\alpha_i)}f_i, \quad K'_\lambda.f_i = adK'_\lambda(f_i) = q^{-(\lambda,\alpha_i)}f_i.$$

So the action of $D_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ on $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ may pass to the quotient to give a $U_q(\mathfrak{g})$-module structure on $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$.

But this in general can not give an action of $U_q(\mathfrak{g})$ on $B_q(\mathfrak{g})$ as we will show in an example later. Let $W_q(\mathfrak{g})$ denote the subalgebra of $B_q(\mathfrak{g})$ generated by e'_i and f_j. It is a quantized version of classical Weyl algebra : taking the Cartan matrix $C = 0$ and $q = 1$ will recover the usual Weyl algebra. (The condition $C = 0$ has to do with quantized Serre relations.) The name "quantized Weyl algebra" is proposed by A.Joseph in [42]. In [44], M.Kashiwara calls it "q-analogue of Boson".

From the definition of Schrödinger representation and Proposition 2.5, we have :

Proposition 2.7. $W_q(\mathfrak{g})$ is a $U_q(\mathfrak{g})$-module algebra a $U_q(\mathfrak{g})$-Yetter-Drinfel'd module.

2.3.5 Example

In this section, we compute the action of the quantum double on the Heisenberg double in the case of $\mathfrak{g} = \mathfrak{sl}_2$. Generators of $D_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ are $E, F, K^{\pm 1}, K'^{\pm 1}$; for $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$, they are $e, f, t^{\pm 1}, t'^{\pm 1}$.

At first, we calculate the action of K and K' :

$$K.e' = adK(e') = q^2 e', \quad K'.e' = \varphi(t, K'^{-1})e' = q^2 e',$$

$$K.f = \varphi(K, t')f = q^{-2}f, \quad K'.f = adK'(f) = q^{-2}f.$$

This verifies that the action of $D_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ on $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$ may pass to the quotient to give a $U_q(\mathfrak{sl}_2)$-module structure on $H_\varphi(U_q^{\geq 0}, U_q^{\leq 0})$.

But in general, it is not possible to obtain an action of $U_q(\mathfrak{sl}_2)$ on $B_q(\mathfrak{sl}_2)$ after the following computation :

$$E.t = adE(t) = (1-q^2)et^2, \quad E.t' = 0.$$

So it is natural to consider the action of $U_q(\mathfrak{g})$ on $W_q(\mathfrak{g})$.

In the end, it is better to write down formulas for all other actions : first recall that $e = \frac{t^{-1}e'}{q^{-1}-q}$ and suppose that $m \leq n$:

$$E^m.e'^n = \frac{[n+m-1]!}{[n-1]!}q^{-\frac{(2n+3+m)m}{2}}e'^{n+m},$$

$$E^m.f^n = \frac{1}{(q^{-1}-q)^m}\frac{[n]!}{[n-m]!}q^{\frac{(2n-m-1)m}{2}}f^{n-m},$$

$$F^m.e'^n = (-1)^m\frac{[n]!}{[n-m]!}q^{\frac{(2n+3-m)m}{2}}e'^{n-m}, \quad F^m.f^n = \prod_{i=0}^{m-1}(1-q^{-2(n+i)})f^{n+m}.$$

These formulas will be useful for the calculation in Section 2.4.1.

2.4 Modules over q-Boson algebras

Sometimes, we use notations U_q, B_q and W_q instead of $U_q(\mathfrak{g})$, $B_q(\mathfrak{g})$ and $W_q(\mathfrak{g})$ for short. Capital letters will be used for elements in $U_q(\mathfrak{g})$, lowercases for $B_q(\mathfrak{g})$ and $W_q(\mathfrak{g})$.

Let $B_q^{>0}$ and $B_q^{<0}$ denote subalgebras of $B_q(\mathfrak{g})$ generated by e_i', f_j $(1 \leq i,j \leq n)$ respectively and B_q^0 the subalgebra generated by $t_\lambda^{\pm 1}$ $(\lambda \in \mathcal{P}_+)$. Let U_q^0 denote the sub-Hopf algebra of U_q generated by $K_\lambda^{\pm 1}$ $(\lambda \in \mathcal{P}_+)$.

2.4.1 Construction of $W_q(\mathfrak{g})$ from braiding

We have seen in the previous section that $W_q(\mathfrak{g})$ is in ${}^{U_q}_{U_q}\mathcal{YD}$.

There exists a U_q-Yetter-Drinfel'd module algebra structure on $B_q^{>0}$: the U_q-module structure is given by the Schrödinger representation and the U_q-comodule structure is given by $\delta(e_i') = (q_i^{-1} - q_i)K_iE_i \otimes 1 + K_i \otimes e_i'$. It is easy to see that $B_q^{>0}$ is indeed a U_q-module as the adjoint action preserves it. These structures are compatible as δ is just Δ in U_q.

In the category ${}^{U_q}_{U_q}\mathcal{YD}$, we can use the braiding arising from the Yetter-Drinfel'd structure to give the tensor product of two module algebras an algebra structure. We will consider $W_q \otimes W_q$ and denote the braiding by σ ; then $(m \otimes m) \circ (id \otimes \sigma \otimes id)$:

$$W_q \otimes W_q \otimes W_q \otimes W_q \to W_q \otimes W_q \otimes W_q \otimes W_q \to W_q \otimes W_q,$$

gives $W_q \otimes W_q$ an algebra structure. We denote this algebra by $W_q \underline{\otimes} W_q$.

We want to restrict this braiding to the subspace $B_q^{<0} \otimes B_q^{>0} \subset W_q \otimes W_q$. After the definition of the braiding, this requires to restrict the U_q-comodule structure on W_q to $B_q^{>0}$ and the U_q-module structure on W_q to $B_q^{<0}$. The comodule structure could be directly restricted as we did in the beginning of this section ; the possibility for the restriction of the module structure comes from the fact that the Schrödinger representation makes $B_q^{<0}$ stable. As a consequence, we obtain an algebra $B_q^{<0} \underline{\otimes} B_q^{>0}$.

We precise the algebra structure on $B_q^{<0} \underline{\otimes} B_q^{>0}$:

$$(f_i \otimes 1)(1 \otimes e_j') = f_i \otimes e_j',$$

$$\begin{aligned}
(1 \otimes e_i')(f_j \otimes 1) &= \sum (e_i')_{(-1)} \cdot f_j \otimes (e_i')_{(0)} \\
&= ((q^{-1} - q)K_iE_i) \cdot f_j \otimes 1 + K_i \cdot f_j \otimes e_i' \\
&= \delta_{ij} + q^{-(\alpha_i,\alpha_j)} f_j \otimes e_i'.
\end{aligned}$$

These are nothing but relations in the quantized Weyl algebra $W_q(\mathfrak{g})$. Moreover, as a vector space, $W_q(\mathfrak{g})$ has a decomposition $W_q(\mathfrak{g}) \cong B_q^{<0} \otimes B_q^{>0}$, where the inverse map is given by the multiplication. This proves the following proposition :

Proposition 2.8. There exists an algebra isomorphism :

$$B_q^{<0} \underline{\otimes} B_q^{>0} \cong W_q(\mathfrak{g}), \quad f \otimes e \mapsto fe,$$

where $f \in B_q^{<0}$, $e \in B_q^{>0}$.

2.4.2 Modules over $W_q(\mathfrak{g})$

This subsection is devoted to studying modules over $W_q(\mathfrak{g})$ with finiteness conditions.

We define the category $\mathcal{O}(W_q)$ as a full subcategory of $W_q(\mathfrak{g})$-modules containing those $W_q(\mathfrak{g})$-modules satisfying the following locally nilpotent condition : for any M in $\mathcal{O}(W_q)$ and $m \in M$, there exists an integer $l > 0$ such that for any $1 \leq i_1, \cdots, i_l \leq n$, $e'_{i_1}e'_{i_2}\cdots e'_{i_l}.m = 0$.

Let M be a $W_q(\mathfrak{g})$-module in $\mathcal{O}(W_q)$. The braided Hopf algebras $B_q^{>0}$ and $B_q^{<0}$ are both $\mathbb{N}$-graded by defining $\deg(e'_i) = \deg(f_i) = 1$. We let $B_q^{>0}(n)$ denote the finite dimensional subspace of $B_q^{>0}$ containing elements of degree n and $(B_q^{>0})^g = \bigoplus_{n\geq 0} B_q^{>0}(n)^*$ the graded dual coalgebra of $B_q^{>0}$.

Recall that there exists a pairing between $B_q^{>0}$ and $B_q^{<0}$ given by $\varphi(e'_i, f_j) = \delta_{ij}$. As $\varphi(B_q^{>0}(n), B_q^{<0}(m)) = 0$ for $m \neq n$ and the restriction of φ to $B_q^{>0}(n) \times B_q^{<0}(n)$, $n \geq 0$, is non-degenerate, the graded dual of $B_q^{>0}$ is anti-isomorphic to $B_q^{<0}$ as graded coalgebras. The prefix "anti" comes from Lemma 2.5 in [61] for the restricted pairing $B_q^{>0} \times B_q^{<0} \to \mathbb{C}$ on braided Hopf algebras. Thus we obtain an isomorphism of graded braided Hopf algebras

$$(B_q^{>0})^g \cong B_q^{<0}.$$

From the definition, the action of $B_q^{>0}$ on M is locally nilpotent, then from duality, we obtain a left $(B_q^{>0})^g$-comodule structure on M. With the help of the isomorphism above, there is a left $B_q^{<0}$-comodule structure on M given in the following way : if we adopt the Sweedler notation for $\rho : M \to B_q^{<0} \otimes M$ as $\rho(m) = \sum m_{(-1)} \otimes m_{(0)}$, then for $e \in B_q^{>0}$,

$$e.m = \sum \varphi(e, m_{(-1)})m_{(0)}.$$

Thus from a W_q-module, we obtain a $B_q^{<0}$-module which is simultaneously a $B_q^{<0}$-comodule, and is, moreover, a braided Hopf module.

Remark 2.7. For the left $B_q^{<0}$-comodule structure on M, it is needed to consider the braided Hopf algebra structure on $B_q^{<0}$, that is to say, we use the primitive coproduct and twist the algebra structure by the braiding, i.e., $\Delta_0 : B_q^{<0} \to B_q^{<0}\underline{\otimes} B_q^{<0}$, $\Delta_0(f_i) = f_i \otimes 1 + 1 \otimes f_i$. This gives us a good duality between left $B_q^{>0}$-modules and right $B_q^{<0}$-comodules. For the left $B_q^{<0}$-module structure on M, we keep the ordinary coproduct $\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i \in B_q^- \otimes B_q^{<0}$.

We define a linear projection $\pi : B_q^- \to B_q^{<0}$ by $ft \mapsto f\varepsilon(t)$, where $f \in B_q^{<0}$ and $t \in B_q^0$.

Proposition 2.9. The following compatibility relation between the module and comodule structures defined above holds : for $f \in B_q^{<0}$ and $m \in M$,

$$\rho(f.m) = \sum \pi(f_{(1)}m_{(-1)}) \otimes f_{(2)}.m_{(0)} = \Delta_0(f)\rho(m). \tag{2.1}$$

Proof. At first we compute $\rho(f.m)$: for any $e \in B_q^{>0}$,

$$
\begin{aligned}
e(f.m) = (ef).m &= \sum (e_{(-1)} \cdot f) e_{(0)} m \\
&= \sum \varphi(e_{(0)}, m_{(-1)}) (e_{(-1)} \cdot f) m_{(0)} \\
&= \sum \varphi(e_{(-1)}, f_{(1)}) \varphi(e_{(0)}, m_{(-1)}) f_{(2)}.m_{(0)} \\
&= \sum \varphi(e, f_{(1)} m_{(-1)}) f_{(2)} m_{(0)}.
\end{aligned}
$$

In this computation, although $f_{(1)}m_{(-1)}$ is not necessary in $B_q^{<0}$, we always have

$$\varphi(e, f_{(1)}m_{(-1)}) = \varphi(e, \pi(f_{(1)}m_{(-1)})),$$

which gives the first equality.

We prove the second one : from the definition of the braiding,

$$\Delta_0(f)\rho(m) = \sum f^{(1)}((f^{(2)})_{(-1)}.m_{(-1)}) \otimes (f^{(2)})_{(0)} m_{(0)}, \tag{2.2}$$

where $\Delta_0(f) = \sum f^{(1)} \otimes f^{(2)}$.

As explained in Remark 2.7, we look $B_q^{<0}$ as a braided Hopf algebra when considering the comodule structure, so $(f^{(2)})_{(0)} = f^{(2)}$ and from the definition of Δ in $B_q^{<0}$,

$$\pi(f_{(1)})((f_{(2)})_{(-1)}.m_{(-1)}) = \pi(f_{(1)}m_{(-1)}).$$

Notice that $\Delta_0(f) = (\pi \otimes id)(\Delta(f))$. So

$$\sum f^{(1)} \otimes f^{(2)} = \sum \pi(f_{(1)}) \otimes f_{(2)}$$

and the formula above gives the second equality. □

In the categorical language, the proposition above is interpreted as :

Corollary 2.1. There exists an equivalence of category $\mathcal{O}(W_q) \sim {}^{B_q^{<0}}_{B_q^{<0}}\mathcal{M}$.

The following theorem gives the structural result for W_q-modules with finiteness conditions.

Theorem 2.1. There exists an equivalence of category $\mathcal{O}(W_q) \sim \mathbf{Vect}$, where $\mathbf{Vect}$ is the category of vector spaces. The equivalence is given by :

$$M \mapsto M^{cop}, \quad V \mapsto B_q^{<0} \otimes V,$$

where $M \in \mathcal{O}(W_q)$, $V \in \mathbf{Vect}$, $M^{cop} = \{m \in M | \ \rho(m) = 1 \otimes m\}$ is the set of right coinvariants.

Proof. We have seen in Corollary 2.1 that categories $\mathcal{O}(W_q)$ and ${}^{B_q^{<0}}_{B_q^{<0}}\mathcal{M}$ are equivalent, so the theorem comes from the triviality of the braided Hopf modules as showed in Proposition 2.1. □

It is better to write down an explicit formula for ρ.

For $\beta \in Q_+ \cup \{0\}$, we denote

$$(B_q^{>0})_\beta = \{x \in B_q^{>0} |\ t_\lambda x t_\lambda^{-1} = q^{(\beta,\lambda)}x, \forall t_\lambda \in U_q^0\}.$$

Moreover, $(B_q^{<0})_{-\beta}$ can be similarly defined. Elements in $(B_q^{>0})_\beta$ (resp, $(B_q^{<0})_{-\beta}$) are called of degree β (resp, $-\beta$). Let $e_{\alpha,i} \in (B_q^{>0})_\alpha$, $1 \leq i \leq dim((B_q^{>0})_\alpha)$ be a basis of $B_q^{>0}$, $f_{\beta,j}$ be the dual basis respected to φ, such that

$$\varphi(e_{\alpha,i}, f_{\beta,j}) = \delta_{ij}\delta_{\alpha\beta}.$$

We define formally the Casimir element $\mathcal{R} = \sum_{i,\alpha} f_{\alpha,i} \otimes e_{\alpha,i}$. As $M \in \mathcal{O}(B_q)$, $\mathcal{R}(1 \otimes m)$ is well-defined for any $m \in M$. This element $\mathcal{R}$, up to a normalization, coincides with the Casimir element in the Chapter 4 of [58].

Proposition 2.10. For any $m \in M$, $\rho(m) = \mathcal{R}(1 \otimes m)$.

Proof. For $m \in M$, we suppose that $\rho(m) = \sum_{\alpha,j} f_{\alpha,j} \otimes m_{\alpha,j}$, then from the definition of the left comodule structure,

$$e_{\beta,i}.m = \sum_{\alpha,j} \varphi(e_{\beta,i}, f_{\alpha,j})m_{\alpha,j} = m_{\beta,i},$$

which gives $\rho(m) = \sum_{\alpha,i} f_{\alpha,i} \otimes e_{\alpha,i}.m = \mathcal{R}(1 \otimes m)$. □

We verify formula (2.1) in an example.

Example 2.2. Consider the $\mathfrak{sl}_2$ case, generators of $B_q(\mathfrak{sl}_2)$ will be denoted by $e, f, t^{\pm 1}$. We choose $m \in M$ such that $e.m \neq 0$ and $e^2.m = 0$. Then

$$\rho(f.m) = 1 \otimes fm + f \otimes efm + f^2 \otimes \frac{1}{q^{-2}+1} e^2 fm = 1 \otimes fm + f \otimes q^{-2} fem + f \otimes m + f^2 \otimes em,$$

as $e^2 f = q^{-4} fe^2 + (q^{-2}+1)e$. This gives

$$\begin{aligned}(\pi \otimes id)(\Delta(f)\rho(m)) &= (\pi \otimes id)(f \otimes m + f^2 \otimes em + t \otimes fm + tf \otimes fem) \\ &= f \otimes m + f^2 \otimes em + 1 \otimes fm + f \otimes q^{-2} fem.\end{aligned}$$

On the other side, for the primitive coproduct,

$$\Delta_0(f)\rho(m) = f \otimes m + f^2 \otimes em + 1 \otimes fm + q^{-2} f \otimes fem.$$

For a W_q-module M, $0 \neq m \in M$ is called a maximal vector if it is annihilated by all e_i'. The set of all maximal vectors in M is denoted by $K(M)$.

The following lemma is a direct consequence of the definition.

Lemma 2.1. Suppose that $m \in M^{cop}$. Then for any non-constant element $e \in B_q^{>0}$, $e.m = 0$.

The following lemma is well-known.

Lemma 2.2. Let $f \in B_q^{<0}$, $f \notin \mathbb{C}^*$, such that for any i, $e_i'.f = 0$. Then $f = 0$.

Proof. If $e_i'.f = 0$ for any i, f is annihilated by all non-constant elements in $B_q^{>0}$. For any $e \in B_q^{>0}$, $e.f = \sum \varphi(e, f_{(1)})f_{(2)}$, where we can suppose that these $f_{(2)}$ are linearly independent, so $\varphi(e, f_{(1)}) = 0$ for any $f_{(1)}$ and any non-constant $e \in B_q^{>0}$, the non-degeneracy of the Hopf pairing forces $f_{(1)}$ to be constants.

So $f = (id \otimes \varepsilon)\Delta(f) = \sum f_{(1)}\varepsilon(f_{(2)}) \in \mathbb{C}$ and it must be 0 after the hypothesis. □

Combined with Theorem 2.1 above, Lemma 2.1 and 2.2 give :

Corollary 2.2. Let $M \in \mathcal{O}(W_q)$ be a W_q-module. Then $M^{cop} = K(M)$.

Remark 2.8. The corollary above gives another interpretation of the "extremal vectors" defined in [66] from a dual point of view.

2.4.3 Modules over $B_q(\mathfrak{g})$

We recall the definition of the category $\mathcal{O}(B_q)$ in [66] : it is a full subcategory of left module category over $B_q(\mathfrak{g})$ containing objects satisfying the following conditions :
(i). Any object M in $\mathcal{O}(B_q)$ has a weight space decomposition :

$$M = \bigoplus_{\lambda \in \mathcal{P}} M_\lambda, \quad M_\lambda = \{m \in M |\ t_\mu.m = q^{(\mu,\lambda)}m\}.$$

(ii). For any M in $\mathcal{O}(B_q)$ and any $m \in M$, there exists an integer $l > 0$ such that for any $1 \leq i_1, \cdots, i_l \leq n$, $e_{i_1}'e_{i_2}' \cdots e_{i_l}'.m = 0$.

Moreover, we let $\mathcal{O}'(B_q)$ denote the full sub-category of the category of B_q-modules containing objects satisfying only (ii) above. The category $\mathcal{O}(B_q)$ is a sub-category of $\mathcal{O}'(B_q)$.

The main theorem of this chapter is the following structural result.

Theorem 2.2. There exists an equivalence of category $\mathcal{O}'(B_q) \sim {}_{U_q^0}Mod$. The equivalence is given by :

$$M \mapsto K(M), \quad V \mapsto B_q^{<0} \otimes V,$$

where $M \in \mathcal{O}'(B_q)$, V is a U_q^0-module and $K(M)$ is the set of maximal vectors in M, when it is looked as a W_q-module.

Moreover, when restricted to the subcategory $\mathcal{O}(B_q)$, the equivalence above gives $\mathcal{O}(B_q) \sim {}_{\mathcal{P}}Gr$, where the latter is the category of $\mathcal{P}$-graded vector spaces.

Remark 2.9. Results in this chapter will be generalized in Chapter 4 to give a similar result for Nichols algebras associated to Yetter-Drinfel'd modules.

The next subsection is devoted to giving a the proof of this result.

2.4.4 Proof of Theorem 2.2

We proceed to the proof of Theorem 2.2.

At first, for any U_q^0-module V, we may look it as a vector space through the forgetful functor. From Theorem 2.1, $N = B_q^{<0} \otimes V$ admits a locally nilpotent W_q-module structure such that $N^{cop} = V$. Moreover, if the U_q^0-module structure on V is under consideration, there exists a B_q-module structure on $B_q^{<0} \otimes V$ giving by : for $v \in V$, $x, f \in B_q^{<0}$, $e \in B_q^{>0}$, $t \in B_q^0$,

$$e.(x \otimes v) = \sum \varphi(e, x_{(1)})x_{(2)} \otimes v, \quad f.(x \otimes v) = fx \otimes v, \quad t.(x \otimes v) = txt^{-1} \otimes tv.$$

As a summary, the discussion above gives a functor ${}_{U_q^0}\mathbf{Mod} \to \mathcal{O}'(B_q)$.

From now on, let $M \in \mathcal{O}'(B_q)$ be a B_q-module with finiteness condition.

The restriction from B_q-modules to W_q-modules gives a functor $\mathcal{O}'(B_q) \to \mathcal{O}(W_q)$, which gives functor $\mathcal{O}'(B_q) \to {}_{U_q^0}\mathbf{Mod}$ by composing with the equivalence functor $\mathcal{O}(W_q) \to \mathbf{Vect}$.

From Theorem 2.1 and the module structures defined above, these two functors give an equivalence of category $\mathcal{O}'(B_q) \sim {}_{U_q^0}\mathbf{Mod}$, and the first point of Theorem 2.2 comes from Corollary 2.2.

The second assertion in the theorem comes from the equivalence of the U_q^0-modules satisfying condition (i) in $\mathcal{O}(B_q)$ and $\mathcal{P}$-graded vector spaces.

2.4.5 Semi-simplicity

Now it is easy to deduce the structural results for $\mathcal{O}(B_q)$ as showed in [66].

The following result is a direct corollary of Theorem 2.2 and Lemma 2.1.

Corollary 2.3. Let $M \in \mathcal{O}(B_q)$ be a non-trivial B_q-module. There exist non-zero maximal vectors in M.

For $\lambda \in \mathcal{P}$, we define a left ideal of B_q by :

$$I_\lambda = \sum_i B_q e_i' + \sum B_q(t_\alpha - q^{(\lambda,\alpha)}),$$

and denote $H(\lambda) = B_q/I_\lambda$, then $H(\lambda)$ is a free $B_q^{<0}$-module of rank 1, generated by 1. The following corollaries follow from Theorem 2.2.

Corollary 2.4. Let $M \in \mathcal{O}(B_q)$, $v \in M$ be a maximal vector of weight λ. Then $B_q^{<0} \otimes \mathbb{C}v \to H(\lambda)$, $F \otimes v \mapsto F.v$ is an isomorphism of B_q-modules. In particular, $H(\lambda)$ are all simple objects in $\mathcal{O}(B_q)$.

Corollary 2.5. (1). Let $M \in \mathcal{O}(B_q)$ be a simple B_q-module. Then there exists some λ such that $M \cong H(\lambda)$.
(2). Suppose that $M \in \mathcal{O}(B_q)$, then M is semi-simple.

2.4.6 Extremal projector

This section is devoted to the computation of the projection P in the $\mathfrak{g} = \mathfrak{sl}_2$ case, and show that it is exactly the operator given by the formula (3.2.2) of [44]. Moreover, changing q by q^{-1} leads us to the formula in Example 5.1 of [66].

At first, we calculate $\Delta_0 : B_q^{<0} \to B_q^{<0} \underline{\otimes} B_q^{<0}$ and the antipode.

Lemma 2.3. (1). $\Delta_0(f^n) = \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix} q^{p^2-np} f^p \otimes f^{n-p}$;
(2). $S(f^n) = (-1)^n q^{-n(n-1)} f^n$.

Proof. (1). By induction on n and use the identity

$$\begin{bmatrix} n+1 \\ p+1 \end{bmatrix} = q^{p+1} \begin{bmatrix} n \\ p+1 \end{bmatrix} + q^{p-n} \begin{bmatrix} n \\ p \end{bmatrix}.$$

(2). Applying $S \otimes id$ on the formula of $\Delta_0(f^n)$ then use induction and the following identity

$$\sum_{i=0}^{r} (-1)^i q^{-i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix} = 0.$$

□

After Proposition 2.10, we obtain a well-defined comodule structure map in the $\mathfrak{sl}_2$ case :

$$\rho(m) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{f^n}{[n]!} \otimes e^n.m, \tag{2.3}$$

then the projection is given by

$$P(m) = \sum_{n=0}^{\infty} (-1)^n q^{-\frac{n(n-1)}{2}} \frac{f^n}{[n]!} e^n.m. \tag{2.4}$$

It is exactly the operator defined in [44], (3.2.2) and almost the extremal projector Γ in [66].

Chapitre 3

q-Boson algebras of Schubert cells and Kashiwara operators

In this chapter, using the notion of relative Hopf modules, we generalize the main result in the last chapter to the case of unipotent subalgebras associated to elements in the Weyl group. As a complement of the last chapter, we define the Kashiwara operator as a "convolution type" map which coincide with the original definition of Kashiwara but can be generalized to the case of Nichols algebras. Moreover, some formulas in [44] based on detailed calculations can be derived easily from our approach.

We fix $\mathbb{C}$ as the base field.

3.1 General results for relative Hopf modules

3.1.1 Cotensor product

The cotensor product over a coalgebra C is a dual version of the tensor product over some fixed algebra A. We recall the definition of cotensor product in this subsection.

Let C be a coalgebra, M be a right C-comodule and N be a left C-comodule. The cotensor product of M and N is a $\mathbb{C}$-vector space defined as follows : we consider two linear maps $\delta_R \otimes id_N$, $id_M \otimes \delta_L :\ M \otimes N \to M \otimes C \otimes N$; the cotensor product of M and N, which is denoted by $M \square_C N$, is the equaliser of $\delta_R \otimes id_N$ and $id_M \otimes \delta_L$.

3.1.2 Relative Hopf modules

We explain the notion of relative Hopf modules after [82].

For our purpose, we consider only relative Hopf modules associated to a Hopf algebra A and its coideal subalgebra B in the following discussion.

Let A be a Hopf algebra and $B \subset A$ be a left coideal subalgebra (i.e. B is a subalgebra of A satisfying $\Delta(B) \subset A \otimes B$).

We define an abelian category ${}^A_B\mathcal{M}$ as follows :

1. An object in ${}^A_B\mathcal{M}$ is a $\mathbb{C}$-vector space M, equipped with a left A-comodule and a left B-module structure (where $a : B \otimes M \to M$ and $\rho : M \to A \otimes M$ are

structural morphisms) such that the following diagram is commutative :

$$\begin{array}{ccccc} B\otimes M & \xrightarrow{a} & M & \xrightarrow{\rho} & A\otimes M \\ \downarrow{\scriptstyle \Delta\otimes\rho} & & & & \uparrow{\scriptstyle m\otimes a} \\ A\otimes B\otimes A\otimes M & & \xrightarrow{id\otimes\sigma\otimes id} & & A\otimes A\otimes B\otimes M. \end{array}$$

2. Morphisms are linear maps which are simultaneously an A-comodule and a B-module map.

Objects in ${}^{A}_{B}\mathcal{M}$ are called (A,B)-Hopf modules.

We let $B^{+}=B\cap\ker\varepsilon_A$ denote the augmentation part in B and define a projection $\pi : A\to A/B^{+}A$, where $B^{+}A$ is the right ideal in A generated by B^{+}, thus $B^{+}A$ is a right regular submodule (i.e., a sub-representation of the right regular representation of A on itself) of A and a bilateral coideal, which implies that π is a right A-module coalgebra morphism.

3.1.3 Equivalence of categories

In this subsection, we briefly recall a theorem describing (A,B)-Hopf modules due to Takeuchi [82].

We let ${}^{\pi}\mathcal{M}$ denote the category of left $\pi(A)$-comodules. In [82], Takeuchi defined two functors :

$$\Phi : {}^{A}_{B}\mathcal{M}\to {}^{\pi}\mathcal{M},\quad \Psi : {}^{\pi}\mathcal{M}\to {}^{A}_{B}\mathcal{M}.$$

1. Let $M\in {}^{A}_{B}\mathcal{M}$ be an (A,B)-Hopf module. Then $(\pi\otimes id)\circ\rho$ gives M a $\pi(A)$-comodule structure. Moreover, $B^{+}M$ is a $\pi(A)$-subcomodule because of the compatibility condition in ${}^{A}_{B}\mathcal{M}$ and so is $\overline{M}=M/B^{+}M$ as a quotient by a $\pi(A)$-subcomodule. The functor $\Phi : {}^{A}_{B}\mathcal{M}\to {}^{\pi}\mathcal{M}$ is defined by $M\mapsto\overline{M}$.
2. Let $V\in {}^{\pi}\mathcal{M}$ be a $\pi(A)$-comodule. We remark that A itself is a right $\pi(A)$-comodule given by the projection π. Thus we obtain an (A,B)-Hopf module $A\square_\pi V$ as the cotensor product on $\pi(A)$, where we use $A\square_\pi V$ to simplify the notation $A\square_{\pi(A)}V$. The functor $\Psi : {}^{\pi}\mathcal{M}\to {}^{A}_{B}\mathcal{M}$ is defined by $V\mapsto A\square_\pi V$.

Theorem 3.1 ([82])**.** Let $B\subset A$ be a left coideal subalgebra and $\pi : A\to A/B^{+}A$ the projection as defined above. Suppose that there is a right A-module N which is a faithfully flat right B-module. Then two functors Ψ and Φ above establish an equivalence of category ${}^{A}_{B}\mathcal{M}\sim {}^{\pi}\mathcal{M}$.

It is better to make these two isomorphisms $\Phi\circ\Psi\sim\mathrm{Id}$ and $\Psi\circ\Phi\sim\mathrm{Id}$ explicit. For $M\in {}^{A}_{B}\mathcal{M}$, the isomorphism $M\to A\square_\pi\overline{M}$ is given by : $m\mapsto\sum m_{(-1)}\otimes\overline{m_{(0)}}$. For $V\in {}^{\pi}\mathcal{M}$, $\overline{A\square_\pi V}\to V$ is given by $\sum a_i\otimes v_i\mapsto\sum\varepsilon(a_i)v_i$.

After Takeuchi, we let B_π denote the set

$$B_\pi=\{a\in A|\ \sum a_{(1)}\otimes\pi(a_{(2)})=a\otimes\pi(1)\},$$

this is a left coideal subalgebra of A.

Proposition 3.1 ([82])**.** With the same hypothesis as in the theorem above, $B=B_\pi$.

3.2 Application to $U_q^{<0}[w]$

3.2.1 Construction of $H_\varphi[w]$

In this subsection, we will recall the construction of a subalgebra of the q-Boson algebra associated to an element w in the Weyl group W.

We begin with a general remark : since the construction of the quantum double will not be used in this chapter, we will suppose that the generator E_i (resp. F_i) here is e_i' (resp. f_i) used in the last chapter for a q-Boson algebra.

A good reference of the following statements is Lusztig's book [58].

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, W be the corresponding Weyl group. Let $w \in W$ be an element in the Weyl group and $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression of w. By a simple argument (the Weyl group acts simply transitively on the set of Weyl chambers), there exists a reduced decomposition of $w_0 = s_{j_1} \cdots s_{j_N}$ such that $w = s_{j_{N-k+1}} \cdots s_{j_N}$ is its tail, where w_0 is the longest element in the Weyl group W.

For this $w \in W$, we can associate with it a subalgebra $U_q^{<0}[w]$ of the algebra $U_q^{<0}(\mathfrak{g})$, where $U_q^{<0}(\mathfrak{g})$ is the braided Hopf algebra generated by F_i, $i \in I$, satisfying Serre relations.

We fix some notations here. For the reduced decomposition of w above, we define $\beta_1 = \alpha_{i_1}, \cdots, \beta_t = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}), \cdots$ and F_{β_t} the corresponding PBW root vector in $U_q^{<0}$ associated to β_t. Then $U_q^{<0}[w]$ is generated by these F_{β_i}, $1 \leq i \leq k$, as an algebra and $\{F_{\beta_k}^{m_k} \cdots F_{\beta_1}^{m_1} |\ m_i \geq 0,\ i = 1, \cdots, k\}$ is a linear basis of $U_q^{<0}[w]$. Moreover, we can continue this procedure to obtain all PBW root vectors associated to the reduced decomposition of w_0 as above. They are denoted by $\{\beta_1, \cdots, \beta_k, \beta_{k+1}, \cdots, \beta_N\}$. Thus $U_q^{<0}$ has a linear basis $\{F_{\beta_N}^{m_N} \cdots F_{\beta_1}^{m_1} |\ m_i \geq 0,\ i = 1, \cdots, N\}$.

Proposition 3.2. With the comodule structure given by the coproduct in $U_q^{<0}$, $U_q^{<0}[w]$ is a left $U_q^{<0}(\mathfrak{g})$-comodule algebra (that is to say, $U_q^{<0}[w]$ is a left coideal subalgebra of $U_q^{<0}$).

Let $\varphi : U_q^{\geq 0} \times U_q^{\leq 0} \to \mathbb{C}$ be the generalized Hopf pairing defined in the last chapter. Being restricted to $U_q^{>0} \times U_q^{<0}[w]$, we obtain a bilinear form over it ; as $U_q^{<0}[w]$ is a coideal subalgebra of $U_q^{<0}$, this restriction is also a generalized Hopf pairing (we remark that there is no adjoint relation for the antipode S since the latter is not a Hopf algebra).

From the definition of the Heisenberg double in 2.2.7, the multiplication is given by :

$$(1\sharp a)(b\sharp 1) = \sum \varphi(a_{(1)}, b_{(1)}) b_{(2)} \sharp a_{(2)}.$$

From the argument above, the Heisenberg double of $U_q^{>0}$ and $U_q^{<0}[w]$ is well defined and thus we obtain an associative subalgebra $H_\varphi[w]$ of the whole Heisenberg double H_φ of U_q^+ and U_q^-.

3.2.2 Category $\mathcal{O}(H_\varphi[w])$

We study the category $\mathcal{O}(H_\varphi[w])$ in this subsection.

The category $\mathcal{O}(H_\varphi[w])$ is defined as a full subcategory of $H_\varphi[w]$-module satisfying the following nilpotency condition : for any $M \in \mathcal{O}(H_\varphi[w])$ and $m \in M$, there exists an integer $l > 0$ such that for any $1 \leq i_1, \cdots, i_l \leq n$, $E_{i_1} \cdots E_{i_l}.m = 0$.

Let M be an $H_\varphi[w]$-module which is contained in $\mathcal{O}(H_\varphi[w])$. Then M is both a $U_q^{>0}$-module and a $U_q^{<0}[w]$-module. For a $U_q^{>0}$-module with the finiteness condition as in category $\mathcal{O}$, we can give it a $U_q^{<0}$-comodule structure (we need to define a linear map

$$\rho : M \to U_q^{<0} \otimes M, \quad m \mapsto \sum m_{(-1)} \otimes m_{(0)},$$

which is determined by : $e.m = \sum \varphi(e, m_{(-1)})m_{(0)}$, for any $e \in U_q^{>0}$; this is a well-defined comodule structural map as the generalized Hopf pairing between $U_q^{>0}$ and $U_q^{<0}$ is graded non-degenerate). As a summary, we obtain a $U_q^{<0}[w]$-module and a $U_q^{<0}$-comodule structure on M.

Now we want to show that the comodule structure and the module structure defined above satisfy the compatibility condition. In the last chapter, to construct the quantized Weyl algebra $W_q(\mathfrak{g})$, we considered the algebra $B_q^{--}\underline{\otimes}B_q^{++}$ by restricting the braiding on $W_q \otimes W_q$ to these two components. Now we try to restrict it on $U_q^{>0} \otimes U_q^{<0}[w]$. Recall the definition of the braiding in a Yetter-Drinfel'd module category : $\sigma(v \otimes w) = \sum v_{(-1)}.w \otimes v_{(0)}$. It suffices to show that the action of $U_q^{>0}$ on $U_q^{<0}[w]$ preserves the latter, this comes from the definition of Schrödinger representation in the last chapter : $(a \otimes 1).y = \sum \varphi(a, y_{(1)})y_{(2)}$ and the fact that $U_q^{<0}[w]$ is a left $U_q^{<0}$-comodule.

From the argument above, we obtain a subalgebra

$$U_q^{<0}[w]\underline{\otimes}U_q^{>0} \subset U_q^{<0}\underline{\otimes}U_q^{>0} \cong W_q(\mathfrak{g}).$$

Thus $U_q^{<0}[w]\underline{\otimes}U_q^{>0}$ is isomorphic to $H_\varphi[w]$ as an algebra, so the compatibility condition as Proposition 2.9 in the last chapter holds in this restricted framework. As $U_q^{<0}[w] \subset U_q^{<0}$ is a left coideal subalgebra, we obtain the following proposition :

Proposition 3.3. Let $M \in \mathcal{O}(H_\varphi[w])$. Then M is a $(U_q^{<0}, U_q^{<0}[w])$-Hopf module.

To apply Theorem 3.1, we need to find a right $U_q^{<0}$-module N which is faithfully flat when restricted to the subalgebra $U_q^{<0}[w]$. As a candidate, we consider the right regular $U_q^{<0}$-module $U_q^{<0}$, the restriction to $U_q^{<0}[w]$ gives a right $U_q^{<0}[w]$-module.

Proposition 3.4. As a right $U_q^{<0}[w]$-module, $U_q^{<0}$ is faithfully flat.

Proof. We remark that if A is an algebra, then all free A-modules are faithfully flat over A.

Now we proceed to prove that as a right $U_q^{<0}[w]$-module, $U_q^{<0}$ is free. It suffices to produce a basis : as we remarked at the beginning of this section, $U_q^{<0}$ has a linear basis $\{F_{\beta_N}^{m_N} \cdots F_{\beta_1}^{m_1}\}$ and $U_q^{<0}[w]$ has a linear basis $\{F_{\beta_k}^{m_k} \cdots F_{\beta_1}^{m_1}\}$. As $U_q^{<0}[w]$ itself is an algebra, the vector space decomposition

$$U_q^{<0} = \bigoplus_{m_{k+1}, \cdots, m_N \geq 0} F_{\beta_N}^{m_N} \cdots F_{\beta_{k+1}}^{m_{k+1}} \otimes U_q^{<0}[w]$$

is a decomposition of right $U_q^{<0}[w]$-modules. Moreover, for each $(m_N, \cdots, m_{k+1}) \in \mathbb{N}_{\geq 0}^{N-k}$, $F_{\beta_N}^{m_N} \cdots F_{\beta_{k+1}}^{m_{k+1}} \otimes U_q^{<0}[w]$ is a free right $U_q^{<0}[w]$-module of rank 1. Thus $U_q^{<0}$ is a right free $U_q^{<0}[w]$-module, so faithfully flat over it. □

Before giving the main theorem, we want to give a glimpse to the quotient A/B^+A : in this case, it is $U_q^{<0}/(U_q^{<0}[w])^+U_q^{<0}$.

As a vector space, $U_q^{<0}$ has another linear basis $\{F_{\beta_1}^{m_1}\cdots F_{\beta_N}^{m_N} \mid m_1,\cdots,m_N \geq 0\}$ and $U_q^{<0}[w]$ has another linear basis $\{F_{\beta_1}^{m_1}\cdots F_{\beta_k}^{m_k} \mid m_1,\cdots,m_k \geq 0\}$, they differ from reversing the order on the root system coming from the fixed decomposition of w_0.

Thus as a vector space, $(U_q^{<0}[w])^+U_q^{<0}$ contains elements $F_{\beta_1}^{m_1}\cdots F_{\beta_k}^{m_k}F_{\beta_{k+1}}^{m_{k+1}}\cdots F_{\beta_N}^{m_N}$ with $m_1,\cdots,m_k$ not simultaneous zero. So the quotient, as a vector space, has a linear basis $\{F_{\beta_{k+1}}^{m_{k+1}}\cdots F_{\beta_N}^{m_N} \mid m_{k+1},\cdots,m_N \geq 0\}$. This vector space admits a $U_q^{<0}$-module and a coalgebra structure given by the quotient.

We let $(U_q^{<0}[w])^\perp$ denote this right $U_q^{<0}$-module coalgebra. Thus the part $\pi(A)$ in Theorem 3.1 is isomorphic to $(U_q^{<0}[w])^\perp$.

Thus the following theorem comes from Theorem 3.1 and the discussion above.

Theorem 3.2. There exists an equivalence of category

$$ {}^{U_q^{<0}}_{U_q^{<0}[w]}\mathcal{M} \sim {}^{(U_q^{<0}[w])^\perp}\mathcal{M} $$

given by $M \mapsto M/(U_q^{<0}[w])^+M$ and $V \mapsto U_q^{<0}\square_{(U_q^{<0}[w])^\perp}V$ for $M \in {}^{U_q^{<0}}_{U_q^{<0}[w]}\mathcal{M}$ and $V \in {}^{(U_q^{<0}[w])^\perp}\mathcal{M}$.

This theorem gives us an isomorphism of $(U_q^{<0}, U_q^{<0}[w])$-Hopf modules :

$$ M \xrightarrow{\sim} U_q^{<0}\square_{(U_q^{<0}[w])^\perp}\left(M/(U_q^{<0}[w])^+M\right), $$

which is defined by $m \mapsto \sum m_{(-1)} \otimes \overline{m_{(0)}}$. Combining results above, we obtain that

Corollary 3.1. We have an equivalence of category

$$ \mathcal{O}(H_\varphi[w]) \sim {}^{(U_q^{<0}[w])^\perp}\mathcal{M}. $$

3.2.3 Particular cases

In this subsection, we consider some particular cases of Theorem 3.2 and Corollary 3.1 above.

We start from considering $U_q^{<0}[w] \in \mathcal{O}(H_\varphi[w])$. Recall that the $U_q^{>0}$-module structure on $U_q^{<0}[w]$ is given by the Schrödinger representation : for $a \in U_q^{>0}$ and $y \in U_q^{<0}[w]$,

$$ a.y = \sum \varphi(a, y_{(1)})y_{(2)}. $$

At first, it is clear that $U_q^{<0}[w]/(U_q^{<0}[w])^+U_q^{<0}[w] \cong \mathbb{C}$. Thus after the definition of the cotensor product,

$$ U_q^{<0}\square_{(U_q^{<0}[w])^\perp}\mathbb{C} \cong \{x \in U_q^{<0} \mid \sum x_{(1)} \otimes \pi(x_{(2)}) = x \otimes 1\}. $$

Once Proposition 3.1 is applied, we obtain that the set in the right hand side is exactly $U_q^{<0}[w]$.

More generally, for $M \in \mathcal{O}(H_\varphi[w])$, if the left $(U_q^{<0}[w])^\perp$-comodule structure on $V = M/(U_q^{<0}[w])^+M$ is trivial (that is to say, for any $v \in V$, $\rho(v) = 1 \otimes v$), then as $(U_q^{<0}, U_q^{<0}[w])$-Hopf modules, we have :

$$M \cong U_q^{<0}[w] \otimes V,$$

where the right hand side admits a trivial Hopf module structure.

At last, we discuss when this comodule structure is trivial.

We consider the PBW basis elememt $F(\underline{m}) = F_{\beta_N}^{m_N} \cdots F_{\beta_1}^{m_1}$ for some $\underline{m} = (m_1, \cdots, m_N) \in \mathbb{N}_{\geq 0}^N$ and let $E(\underline{m})$ denote its dual basis in $U_q^{>0}$ with respect to the generalized Hopf pairing. Then for $M \in \mathcal{O}(H_\varphi[w])$, the $U_q^{<0}$-comodule structure on M is given by : $\rho : M \to U_q^{<0} \otimes M$, for $x \in M$,

$$\rho(x) = \sum_{\underline{m} \in \mathbb{N}_{\geq 0}^N} F(\underline{m}) \otimes E(\underline{m}).x \ .$$

The $(U_q^{<0}[w])^\perp$-comodule structure on $V = M/(U_q^{<0}[w])^+M$ is trivial means that the first component in the formula of ρ above could not be contained in $(U_q^{<0}[w])^\perp \cap \ker \varepsilon$ when the second component is zero in the quotient $M/(U_q^{<0}[w])^+M$, which forces for any $\underline{m} = (m_1, \cdots, m_N) \in \mathbb{N}_{\geq 0}^N$ such that $m_1 = \cdots = m_k = 0$ and any $x \in M$, we have $E(\underline{m}).x \in (U_q^{<0}[w])^+M$.

As a summary, we obtain the following result.

Proposition 3.5. Let $M \in \mathcal{O}(H_\varphi[w])$. If for any $\underline{m} = (m_1, \cdots, m_N) \in \mathbb{N}_{\geq 0}^N$ such that $m_1 = \cdots = m_k = 0$, $E(\underline{m})M \subset (U_q^{<0}[w])^+M$, then there exists a vector space $V = M/(U_q^{<0}[w])^+M$ such that as $(U_q^{<0}, U_q^{<0}[w])$-Hopf module, we have $M \cong U_q^{<0}[w] \otimes V$, where the right hand side admits a trivial Hopf module structure.

3.3 Kashiwara operators

In this section, we keep notations in the last chapter.

3.3.1 Application to sl_2-copies

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and $U_q(\mathfrak{g})$ the corresponding quantized enveloping algebra with a formal parameter q. Let $B_q(\mathfrak{g})$ and $W_q(\mathfrak{g})$ denote respectively the corresponding q-Boson algebra and quantized Weyl algebra. Let $U_q^{<0}$ denote the strictly negative part of $U_q(\mathfrak{g})$.

In the previous chapter, we have proved the semi-simplicity of the category $\mathcal{O}(B_q(\mathfrak{g}))$ and $\mathcal{O}(W_q(\mathfrak{g}))$ and showed that there exists an extremal projection operator given by the projection in the structure theorem of Hopf modules.

We choose a $U_q(sl_2)$-copy in $U_q(\mathfrak{g})$ corresponding to the simple root α_i and let $U_{q,i}$ denote it. Then we have the corresponding subalgebras $B_{q,i}$ and $W_{q,i}$ in $B_q(\mathfrak{g})$ and $W_q(\mathfrak{g})$ respectively.

It is clear that the strictly negative part $U_q^{<0}$, once looked as a module over $B_{q,i}$, is in the category $\mathcal{O}(B_{q,i})$. So after the structural theorem of Hopf modules, we obtain an isomorphism

$$\varphi_i : U_q^{<0} \xrightarrow{\sim} B_{q,i}^{<0} \otimes V,$$

$$x \mapsto \sum x_{(-1)} \otimes P(x_{(0)}),$$

where

1. V is the set of extremal vectors corresponding to the i-th direction in $U_q^{<0}$, which are exactly those elements in $U_q^{<0}$ annihilated by the action of e_i' ;
2. the projection $P : U_q^{<0} \to V$ is given by $P(x) = \sum S(x_{(-1)})x_{(0)}$;
3. the inverse of φ_i is given by the action of $B_{q,i}^{<0}$ on V :

$$a : B_{q,i}^{<0} \otimes V \to U_q^{<0}, \quad x \otimes v \mapsto x.v;$$

4. as shown in the formula (2.4), the operator P has the following explicit form :

$$P(x) = \sum_{n=0}^{\infty} q_i^{-\frac{n(n-1)}{2}} f_i^{(n)} e_i'^n.x.$$

3.3.2 Convolution product

Let $A \in \mathrm{End}(B_{q,i}^{<0})$ be an endomorphism. We can lift it to an endomorphism of $U_q^{<0}$ as follows :

$$U_q^{<0} \xrightarrow{\varphi_i} B_{q,i}^{<0} \otimes V \xrightarrow{A\otimes id} B_{q,i}^{<0} \otimes V \xrightarrow{a} U_q^{<0}.$$

It results a linear map

$$\Psi_i : \mathrm{End}(B_{q,i}^{<0}) \to \mathrm{End}(U_q^{<0}),$$

where both sides are associative algebras under the composition of endomorphisms. We have :

Lemma 3.1. For any $i \in I$, Ψ_i is an algebra morphism.

Proof. Since a is the inverse of φ_i, Ψ_i maps identity to identity. Taking $f, g \in \mathrm{End}(B_{q,i}^{<0})$, then

$$\Psi_i(f \circ g) = \varphi_i \circ f \circ a \circ \varphi_i \circ g \circ a = \Psi_i(f) \circ \Psi_i(g).$$

□

We denote the normalized power $f_i^{(n)} = \frac{f_i^n}{[n]_q!}$.

Let D_i be the endomorphism in $\mathrm{End}(B_{q,i}^{<0})$ sending $f_i^{(n)}$ to $f_i^{(n-1)}$ (differentiation). Let I_i be the endomorphism in $\mathrm{End}(B_{q,i}^{<0})$ sending $f_i^{(n)}$ to $f_i^{(n+1)}$ (integration).

Definition 3.1. For any $i \in I$, the images of endomorphisms D_i and I_i under Ψ_i in $\mathrm{End}(U_q^{<0})$ are called Kashiwara operators of rank 1. We let $\widetilde{e}_i$ and $\widetilde{f}_i$ denote them, respectively.

After the definition and the lemma above, the relation $\widetilde{e}_i\widetilde{f}_i = 1$ is clear.

It is also clear that these operators $\widetilde{e}_i$, $\widetilde{f}_i$ coincide with the Kashiwara operators defined in [44]. The following lemma gives the explicit form of the decomposition in the definition of Kashiwara operators :

Lemma 3.2. Let $u \in U_q^{<0}$. If $u = \sum_{n\geq 0} f_i^{(n)} u_n$ and $e_i' u_n = 0$, then

$$u_n = \sum_{m=0}^{\infty} (-1)^m q_i^{\frac{n^2-n-m^2+m}{2}} f_i^{(m)} e_i'^{m+n} u.$$

Proof. After the lemma above, $\Psi_i(\mathrm{id})(u) = u$. The left hand side can be computed in another way as $a \circ (\mathrm{id} \otimes P) \circ \rho$. Applying the formulas (2.3) and (2.4) gives

$$u = \sum_{n=0}^{\infty} f_i^{(n)} \left(\sum_{m=0}^{\infty} (-1)^m q_i^{\frac{n^2-n-m^2+m}{2}} f_i^{(m)} e_i'^{m+n} u \right),$$

from which the lemma. □

We let $P_{i,n}$ denote the endomorphism in $\mathrm{End}(U_q^{<0})$ defined by

$$P_{i,n} = \sum_{m=0}^{\infty} (-1)^m q_i^{\frac{n^2-n-m^2+m}{2}} f_i^{(m)} e_i'^{m+n},$$

then the decomposition above can be written as

$$u = \sum_{n=0}^{\infty} f_i^{(n)} P_{i,n}(u).$$

This recovers the formula in Proposition 3.2.1 of [44].

Chapitre 4

On defining ideals and differential algebras of Nichols algebras

Contents of this chapter is published in [25].

4.1 Introduction

Nichols algebras, as their name says, are constructed by W.D. Nichols in [67] with the name "bialgebra of type one" for classifying finite dimensional graded Hopf algebras generated by elements in degree 0 and 1. No much attention was paid to his work at that time until quantized enveloping algebras are constructed by Drinfel'd and Jimbo in the middle of eighties.

The construction of Nichols, after having slept for about 15 years, is highlighted by M. Rosso in his article [73] to give a functorial and coordinate free construction of quantized enveloping algebras, which meanwhile gives the motivation and another point of view to researches on pointed Hopf algebras.

The construction in [73] starts with a Hopf algebra H and an H-Hopf bimodule M. Then the set of right coinvariants M^R admits a braiding $\sigma_M : M^R \otimes M^R \to M^R \otimes M^R$. Once the usual flip is replaced by this braiding, the classical construction of shuffle algebra will give a new Hopf algebra whose structure is controlled by this braiding, and is called quantum shuffle algebra. This construction gives many interesting examples : the positive part of a quantized enveloping algebra associated to a symmetrizable Cartan matrix can be found as a Hopf sub-algebra generated by H and M^R in the quantum shuffle algebra with some particular Hopf algebra H and H-Hopf bimodule M.

A well-known result affirms that there is an equivalence of braided category between the category of Hopf bimodules over H and that of left H-Yetter-Drinfel'd modules, given by sending a Hopf bimodule M to the set of its right coinvariants M^R; so it is possible to work in the context of Yetter-Drinfel'd modules at the very beginning. This gives a translation of language between Nichols-Rosso and Andruskiewitsch-Schneider.

The dual construction of quantum shuffle algebra is easier to understand : it is the braided tensor algebra $T(V)$ for $V = M^R$ and the Nichols algebra $\mathfrak{N}(V)$ is the quotient of $T(V)$ by some Hopf ideal $\mathfrak{I}(V)$. As an example, it gives the strict positive part of quantized enveloping algebras when H and V are properly chosen.

In the latter case, as a quotient of the braided Hopf algebra $T(V)$, $\mathfrak{N}(V)$ can be viewed as imposing some relations in $T(V)$ which have good behavior under the coproduct. It is natural to ask for the structure of such ideals, but unfortunately, this Hopf ideal is defined as the maximal coideal contained in the set of elements in $T(V)$ of degree no less than 2 and it is very difficult to read out these relations directly from such an abstract definition.

Similar problems arise in some other places in mathematics. For example, the Gabber-Kac theorem in the theory of Kac-Moody Lie algebras is of the same philosophy : starting with a symmetrizable Cartan matrix and some Chevalley generators, it is possible to construct a Lie algebra with Chevalley relations ; but to get a Kac-Moody Lie algebra, one is forced to do the quotient by an ideal with some maximal properties and of course with mysteries. It is in Gabber-Kac [30] that they proved that this ideal is generated by Serre relations, which completes the whole story. It should be remarked that this result is not simple at all : the proof clarifies some structures and uses several tools in Kac-Moody Lie algebras, such as generalized Casimir elements and Verma modules.

As we know, for Nichols algebras, the problem of deciding generating relations in the ideal $\mathfrak{I}(V)$ is still open and the best general result can be found in the literature is due to M. Rosso [73], P. Schauenburg [79] and others, which affirms that elements of degree n in $\mathfrak{I}(V)$ are those annihilated by the symmetrization operator S_n. However, it is still difficult to make these relations explicit because of the appearance of large amount of equations when a basis is chosen.

The main objective of this chapter is to give some restrictions on elements in $\mathfrak{I}(V)$ to obtain a number of important relations imposed, and we conjecture that these relations generate $\mathfrak{I}(V)$ as an ideal.

To be more precise, the restriction is at first given by some operators P_n, called Dynkin operators, on $T(V)$; these operators can be viewed as analogues of the Dynkin projection operators in the characterization of free Lie algebras. Our first restriction is passing from $\ker(S_n)$ to $\ker(S_n) \cap Im(P_n)$: though the latter is somehow subtle at a first glimpse, in some important cases, they will generate the Hopf ideal $\mathfrak{I}(V)$; in general, if a conjecture of Andruskiewitsch-Schneider is true, the statement above holds for any Nichols algebras. This is proved by showing that all primitive elements of degree n are eigenvectors of P_n with eigenvalue n.

Another restriction is given by concentrating on some levels in $T(V)$ having their origin in the decomposition of the element S_n in the group algebra $\mathbb{C}[\mathfrak{B}_n]$, where $\mathfrak{B}_n$ is the braid group on n strands. The main idea here is building a bridge to connect some solutions of equation $S_n x = 0$ in $V^{\otimes n}$ with the invariant space for the central element θ_n of $\mathfrak{B}_n$, which are much easier to understand and compute. Moreover, it throws some light on understanding the structure of Nichols algebras from the representation theory of braid groups, though the latter is difficult indeed.

When constructing this bridge, we captured the appearance of the Dynkin operator P_n as an important ingredient. Moreover, it is a central tool when constructing solutions of $S_n x = 0$ from solutions of the equation $\theta_n x = x$. As an example, we obtain quantized Serre relations from a pedestrian point of view : that is to say, the machinery will tell us what these relations are by assuming almost no knowledge on Lie theory and quantized enveloping algebra.

A natural question is posed when observing the calculation in the exterior algebra and quantized enveloping algebras : whether elements with levels in $T(V)$ are primitive ?

The second part of this chapter is devoted to give a positive answer in the general case : that is to say, for any braiding coming from a Yetter-Drinfel'd structure.

The proof is based on the construction of the differential algebra of a Nichols algebra, which can be viewed as a generalization of the construction of the quantized Weyl algebra over a quantized enveloping algebra given in [24]. The advantage of our approach is : at the very beginning, we never make hypotheses on the type of the braiding, so this is a general construction shared by all kinds of Nichols algebras coming from a Yetter-Drinfel'd module.

Once restricting ourselves to the diagonal case, with the help of these differential operators, we proved the classical Taylor lemma, which generalizes a result in Heckenberger [34]. Moreover, when the derivation is given by a primitive element of degree 1, a decomposition theorem of $T(V)$ as a braided algebra is obtained, which can be viewed as a generalization of a result in classical Weyl algebra given by A. Joseph [39] in solving the Gelfan'd-Kirillov conjecture.

At last, we show that elements with levels are all primitive with the help of these differential operators.

The organization of this chapter is as follows :

In Section 2, some notions in Hopf algebras are recalled and notations are introduced. Section 3 is devoted to defining Dynkin operators and proving the "convolution invariance" of these operators. In Section 4, we are concerned with the decomposition of specific elements in the group algebra of braid groups, which is an algebraic preparation for solving equation $S_n x = 0$. The construction of the bridge mentioned above is given in Section 5 and some properties of $\ker(S_n) \cap Im(P_n)$ are obtained. Section 6 contains examples in the diagonal case ; a concrete calculation of quantized Serre relations in the case $\mathcal{U}_q(\mathfrak{sl}_3)$ is among examples. In Section 7, the differential algebra of a Nichols algebra is constructed and then the Taylor Lemma and a decomposition theorem are proved as an application in Section 8. Finally, the main theorem on primitive elements is demonstrated with the help of the differential algebra in Section 9.

4.2 Recollections on Hopf algebras

From now on, suppose that we are working in the complex field $\mathbb{C}$. All algebras and modules concerned are over $\mathbb{C}$. Results in this section will hold for any field of characteristic 0. All tensor products are over $\mathbb{C}$ if not specified otherwise.

This section is devoted to giving a recollection on some constructions in Hopf algebras and fixing notations.

4.2.1 Yetter-Drinfel'd modules

Let H be a Hopf algebra. A vector space V is called a (left) H-Yetter-Drinfel'd module if :

1. It is simultaneously an H-module and an H-comodule ;
2. These two structures satisfy the Yetter-Drinfel'd compatibility condition : for any $h \in H$ and $v \in V$,
$$\sum h_{(1)}v_{(-1)} \otimes h_{(2)}.v_{(0)} = \sum (h_{(1)}.v)_{(-1)}h_{(2)} \otimes (h_{(1)}.v)_{(0)},$$
where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and $\rho(v) = \sum v_{(-1)} \otimes v_{(0)}$ are Sweedler notations for coproduct and comodule structure map.

Morphisms between two H-Yetter-Drinfel'd modules are linear maps preserving H-module and H-comodule structures.

We denote the category of H-Yetter-Drinfel'd modules by ${}^H_H\mathcal{YD}$; it is a tensor category.

The advantage of working in the category of Yetter-Drinfel'd module is : for $V, W \in {}^H_H\mathcal{YD}$, there exists a braiding $\sigma_{V,W} : V \otimes W \to W \otimes V$, given by $\sigma_{V,W}(v \otimes w) = \sum v_{(-1)}.w \otimes v_{(0)}$. This braiding gives ${}^H_H\mathcal{YD}$ a braided tensor category structure.

Let A and B be two algebras in ${}^H_H\mathcal{YD}$. Then $A \otimes B$ admits an algebra structure with the following multiplication map :

$$A \otimes B \otimes A \otimes B \xrightarrow{id \otimes \sigma_{B,A} \otimes id} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B,$$

where m_A and m_B are multiplications in A and B, respectively.

4.2.2 Braided Hopf algebras in ${}^H_H\mathcal{YD}$

Definition 4.1 ([6], Section 1.3)**.** A braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ is a collection $(A, m, \eta, \Delta, \varepsilon, S)$ such that :

1. (A, m, η) is an algebra in ${}^H_H\mathcal{YD}$; (A, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$. It is to say, $m, \eta, \Delta, \varepsilon$ are morphisms in ${}^H_H\mathcal{YD}$;
2. $\Delta : A \to A\underline{\otimes}A$ is a morphism of algebra. The notation $\underline{\otimes}$ stands for the tensor product of two Yetter-Drinfel'd module algebras, where we use the braiding in ${}^H_H\mathcal{YD}$ instead of the usual flip ;
3. $\varepsilon : A \to \mathbb{C}$ is a morphism of algebra ;
4. S is the convolution inverse of $id_A \in End(A)$.

The most important example of a braided Hopf algebra is the braided tensor Hopf algebra defined as follows.

Example 4.1 ([6])**.** Let $V \in {}^H_H\mathcal{YD}$ be an H-Yetter-Drinfel'd module. There exists a braided Hopf algebra structure on the tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

1. The multiplication is the original one on $T(V)$ given by the concatenation.
2. The coalgebra structure is defined on V by : for any $v \in V$, $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon(v) = 0$. Then it can be extended to the whole $T(V)$ by the universal property of $T(V)$ as an algebra.

4.2.3 Nichols algebras

Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional $\mathbb{C}$-vector space and $T(V)$ be the braided tensor Hopf algebra over V as defined in Example 5.1 ; it is $\mathbb{N}$-graded.

We will recall briefly the definition and the explicit construction of Nichols algebras, which dates back to [67] and is given in [6]. Another definition in a dual point of view is given in [73] under the name quantum shuffle algebra and is denoted by $S_\sigma(V)$. The difference between them is : the construction in [73] is in the graded dual of $T(V)$, so instead of being a quotient object, it will be a sub-object in the graded dual. But they are isomorphic as braided Hopf algebra up to a symmetrization morphism.

Definition 4.2 ([6])**.** A graded braided Hopf algebra $R = \bigoplus_{n=0}^{\infty} R(n)$ is called a Nichols algebra of V if

1. $R(0) \cong \mathbb{C}$, $R(1) \cong V \in {}^H_H\mathcal{YD}$;
2. R is generated as an algebra by $R(1)$;
3. $R(1)$ is the set of all primitive elements in R.

We let $\mathfrak{N}(V)$ denote this braided Hopf algebra.

Remark 4.1. It is conjectured by Andruskiewitsch and Schneider in [6] that (3) implies (2) when R is finite dimensional.

There is a construction of $\mathfrak{N}(V)$ from $T(V)$ as shown in [6] : let

$$T^{\geq 2}(V) = \bigoplus_{n \geq 2} V^{\otimes n}$$

and $\mathfrak{I}(V)$ be the maximal coideal of $T(V)$ contained in $T^{\geq 2}(V)$. Then $\mathfrak{I}(V)$ is also a two-sided ideal ; the Nichols algebra $\mathfrak{N}(V)$ of V can be constructed as $T(V)/\mathfrak{I}(V)$. We denote S the convolution inverse of $id : \mathfrak{N}(V) \to \mathfrak{N}(V)$.

For $k \in \mathbb{N}$, let $\mathfrak{N}(V)_k$ denote the subspace of degree k elements in $\mathfrak{N}(V)$; so from the definition, $\mathfrak{N}(V)_0 = \mathbb{C}$ and $\mathfrak{N}(V)_1 = V$ is the set of all primitive elements in $\mathfrak{N}(V)$.

4.2.4 Nichols algebras of diagonal type

In this subsection, we recall a particular type of Nichols algebra, which will be a good source of examples in our later discussions. A concrete approach can be found in [6].

Let G be an abelian group and $H = \mathbb{C}[G]$ be its group algebra : it is a commutative and cocommutative Hopf algebra. We let $\widehat{G}$ denote the character group of G. Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional H-Yetter-Drinfel'd module and $dimV = n$. Let $T(V)$ denote the braided tensor Hopf algebra and $\mathfrak{N}(V)$ denote the associated Nichols algebra.

As shown in [67] or Remark 1.5 in [6], the category ${}^H_H\mathcal{YD}$ is made of a G-graded vector space $V = \bigoplus_{g\in G} V_g$ such that for any $h \in G$ and $v \in V_g$, $h.v \in V_g$. The comodule structure is given by : for V in ${}^H_H\mathcal{YD}$ and $v \in V_g$ in the decomposition above, the comodule structure map $\delta : V \to H \otimes V$ is $\delta(v) = g \otimes v$.

Definition 4.3. Let $V \in {}^H_H\mathcal{YD}$ be a finite dimensional H-Yetter-Drinfel'd module. V is called of diagonal type if there exists a basis $\{v_1, \cdots, v_n\}$ of V, $g_1, \cdots, g_n \in G$ and $\chi_1, \cdots, \chi_n \in \widehat{G}$ such that for any $g \in G$ and $v_i \in V_{g_i}$,

$$g.v_i = \chi_i(g)v_i.$$

Sometimes, we call $T(V)$ of diagonal type if V is of diagonal type.

In this case, the braiding in ${}^H_H\mathcal{YD}$ is given by : for $V, W \in {}^H_H\mathcal{YD}$,

$$\sigma_{V,W} : V \otimes W \to W \otimes V, \quad \sigma_{V,W}(v \otimes w) = (g.w) \otimes v$$

for any $g \in G$, $v \in V_g$ and $w \in W$.

In particular, if we choose $V = W$ and $v_1, \cdots, v_n$ be a basis of V as in the definition above, the braiding, when acting on basis elements, is given by : for $1 \leq i, j \leq n$,

$$\sigma_{V,V}(v_i \otimes v_j) = \chi_j(g_i)v_j \otimes v_i.$$

So $\sigma_{V,V}$ is completely determined by the matrix $(\chi_j(g_i))_{1\leq i,j\leq n}$. We denote $q_{ij} = \chi_j(g_i)$ and call $(q_{ij})_{1\leq i,j\leq n}$ a braiding matrix.

It is convenient to define a bicharacter χ over G when $G = \mathbb{Z}^n$ to rewrite the braiding above.

Definition 4.4. A bicharacter over an abelian group A is a map $\chi : A \times A \to \mathbb{C}^*$ such that :

$$\chi(a + b, c) = \chi(a, c)\chi(b, c), \quad \chi(a, b + c) = \chi(a, b)\chi(a, c),$$

for any $a, b, c \in A$.

Suppose that $G = \mathbb{Z}^n$ and $V \in {}^H_H\mathcal{YD}$. Then V, $T(V)$ and $\mathfrak{N}(V)$ are all $\mathbb{Z}^n$-graded. Let $v_1, \cdots, v_n$ be a basis of V as in Definition 5.2, $\alpha_1, \cdots, \alpha_n$ be a free basis of $\mathbb{Z}^n$ and $deg(v_i) = \alpha_i$ be their grade degrees in $\mathbb{Z}^n$.

If this is the case, a bicharacter over $\mathbb{Z}^n$ can be defined using the braiding matrix : for any $1 \leq i, j \leq n$, $\chi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}^*$ is determined by $\chi(\alpha_i, \alpha_j) = q_{ij}$.

4.2.5 Radford's biproduct

Let $A \in {}^H_H\mathcal{YD}$ be a braided Hopf algebra. Then $A \otimes H$ admits a Hopf algebra structure from a construction in Radford [69], which is called the biproduct of A and H.

These structures are defined by :

1. The multiplication is given by the crossed product : for $a, a' \in A$, $h, h' \in H$,

$$(a \otimes h)(a' \otimes h') = \sum a(h_{(1)}.a') \otimes h_{(2)}h';$$

2. The comultiplication is given by : for $a \in A$ and $h \in H$,

$$\Delta(a \otimes h) = \sum (a_{(1)} \otimes (a_{(2)})_{(-1)} h_{(1)}) \otimes ((a_{(2)})_{(0)} \otimes h_{(2)});$$

3. The antipode is completely determined by : for $a \in A$ and $h \in H$,

$$S(a \otimes h) = \sum (1 \otimes S_H(h) S_H(a_{(-1)}))(S_A(a_{(0)}) \otimes 1).$$

Proposition 4.1 (Radford)**.** With structures defined above, $A \otimes H$ is a Hopf algebra. We let $A \sharp H$ denote it.

4.2.6 Quantum doubles and Heisenberg doubles

We recall first the definition of a generalized Hopf pairing, which gives duality between Hopf algebras.

Definition 4.5 ([47])**.** Let A and B be two Hopf algebras with invertible antipodes. A generalized Hopf pairing between A and B is a bilinear form $\varphi : A \times B \to \mathbb{C}$ such that :

1. For any $a \in A$, $b, b' \in B$, $\varphi(a, bb') = \sum \varphi(a_{(1)}, b) \varphi(a_{(2)}, b')$;
2. For any $a, a' \in A$, $b \in B$, $\varphi(aa', b) = \sum \varphi(a, b_{(2)}) \varphi(a', b_{(1)})$;
3. For any $a \in A$, $b \in B$, $\varphi(a, 1) = \varepsilon(a)$, $\varphi(1, b) = \varepsilon(b)$.

Remark 4.2. From the uniqueness of the antipode and conditions (1)-(3) above, we have : for any $a \in A$, $b \in B$, $\varphi(S(a), b) = \varphi(a, S^{-1}(b))$.

Starting with a generalized Hopf pairing between two Hopf algebras, we can define the quantum double and the Heisenberg double of them, which will be essential in our later construction of differential algebras of Nichols algebras.

Definition 4.6 ([47])**.** Let A, B be two Hopf algebras with invertible antipodes and φ be a generalized Hopf pairing between them. The quantum double $D_\varphi(A, B)$ is defined by :

1. As a vector space, it is $A \otimes B$;
2. As a coalgebra, it is the tensor product of coalgebras A and B ;
3. As an algebra, the multiplication is given by :

$$(a \otimes b)(a' \otimes b') = \sum \varphi(S^{-1}(a'_{(1)}), b_{(1)}) \varphi(a'_{(3)}, b_{(3)}) a a'_{(2)} \otimes b_{(2)} b'.$$

Then we construct the Heisenberg double of A and B : it is a crossed product of them where the module algebra type action of A on B is given by the Hopf pairing.

Definition 4.7 ([54])**.** The Heisenberg double $H_\varphi(A, B)$ of A and B is an algebra defined as follows :

1. As a vector space, it is $B \otimes A$ and we let $b \sharp a$ denote a pure tensor ;
2. The product is given by : for $a, a' \in A$, $b, b' \in B$,

$$(b \sharp a)(b' \sharp a') = \sum \varphi(a_{(1)}, b'_{(1)}) b b'_{(2)} \sharp a_{(2)} a'.$$

4.3 Dynkin operators and their properties

In this section, we will define Dynkin operators in the group algebras of braid groups. The definition of these operators is motivated by the iterated brackets in Lie algebras which are used by Dynkin in the proof of the Dynkin-Wever-Spechet theorem for characterizing elements in free Lie algebras (for example, see [70]).

As will be shown in this section, Dynkin operators have good properties under the convolution product, which generalizes the corresponding result in free Lie algebras. This will be used to detect primitive elements later.

4.3.1 Definition of Dynkin operators

We suppose that $n \geq 2$ is an integer.

Let $\mathfrak{S}_n$ denote the symmetric group : it acts on an alphabet with n letters by permuting their positions. It can be generated by the set of transpositions $\{s_i = (i, i+1) |\ 1 \leq i \leq n-1\}$.

Let $\mathfrak{B}_n$ be the braid group of n strands, it is defined by generators σ_i for $1 \leq i \leq n-1$ and relations :

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for } |i-j| \geq 2; \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for } 1 \leq i \leq n-2.$$

Let $\pi : \mathfrak{B}_n \to \mathfrak{S}_n$ be the canonical surjection which maps $\sigma_i \in \mathfrak{B}_n$ to $s_i = (i, i+1) \in \mathfrak{S}_n$.

We consider the group $\mathfrak{S}_n^{\pm} = \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_n$, where $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ is the signature. We are going to define a subset $\mathcal{P}_{i,j} \subset \mathfrak{S}_n^{\pm}$ by induction on $|i-j|$ for $1 \leq i \leq j \leq n$.

We omit the signature 1. Define $\mathcal{P}_{i,i} = \{(1)\}$, $\mathcal{P}_{i,i+1} = \{(1), -(i, i+1)\}$, and

$$\mathcal{P}_{i,j} = \mathcal{P}_{i+1,j} \cup (\mathcal{P}_{i,j-1} \circ -(i, j, j-1, \cdots, i+1)),$$

where $\circ$ is the product in $\mathbb{C}[\mathfrak{S}_n]$.

Moreover, we define

$$P_{i,j} = \sum_{(\varepsilon,\omega)\in\mathcal{P}_{i,j}} \varepsilon\omega \in \mathbb{C}[\mathfrak{S}_n].$$

Let $\sigma \in \mathfrak{S}_n$ and $\sigma = s_{i_1} \cdots s_{i_r}$ be a reduced expression of σ. It is possible to define a corresponding lifted element $T_\sigma = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathfrak{B}_n$. This gives a linear map $T : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}[\mathfrak{B}_n]$ called Matsumoto section. For $0 \leq k \leq n$, let $\mathfrak{S}_{k,n-k} \subset \mathfrak{S}_n$ denote the set of $(k, n-k)$-shuffles in $\mathfrak{S}_n$ defined by :

$$\mathfrak{S}_{k,n-k} = \{\sigma \in \mathfrak{S}_n |\ \sigma^{-1}(1) < \cdots < \sigma^{-1}(k),\ \sigma^{-1}(k+1) < \cdots < \sigma^{-1}(n)\}.$$

Example 4.2. We explain the definition of these elements $\mathcal{P}_{i,j}$ in an example for $\mathfrak{S}_4$:

$$\mathcal{P}_{1,2} = \{(1), -(12)\}, \quad \mathcal{P}_{1,3} = \mathcal{P}_{2,3} \cup (\mathcal{P}_{1,2} \circ -(132)) = \{(1), -(23), -(132), (13)\},$$
$$\mathcal{P}_{2,4} = \{(1), -(34), -(243), (24)\},$$

$$\begin{aligned} \mathcal{P}_{1,4} &= \mathcal{P}_{2,4} \cup (\mathcal{P}_{1,3} \circ -(1432)) \\ &= \{(1), -(34), -(243), (24), -(1432), (142), (1423), -(14)(23)\}. \end{aligned}$$

These elements $\mathcal{P}_{i,j}$ and $P_{i,j}$ come from iterated brackets :

$$[a,[b,[c,d]]] = abcd - abdc - acdb + adcb - bcda + bdca + cdba - dcba.$$

When $\mathfrak{S}_4^{\pm}$ acts on letters $abcd$ by permuting their position and then multiplying by the signature, an easy computation gives :

$$[a,[b,[c,d]]] = T_{P_{1,4}}(abcd).$$

Definition 4.8. We call these $P_{i,j}$ Dynkin operators in $\mathbb{C}[\mathfrak{S}_n]$ and corresponding elements $T_{P_{i,j}}$ Dynkin operators in $\mathbb{C}[\mathfrak{B}_n]$.

For $\sigma \in \mathfrak{S}_n$, let $l(\sigma)$ denote the length of σ. It is exactly the number of generators appearing in any reduced expression of σ.

Remark 4.3. In general, the Matsumoto section $T : \mathfrak{S}_n \to \mathfrak{B}_n$ is not a group homomorphism, but we have the following property : for $w, w' \in \mathfrak{S}_n$, if $l(ww') = l(w)+l(w')$, then $T_w T_{w'} = T_{ww'}$.

Lemma 4.1. Let $w \in \mathcal{P}_{1,k}$ and $\sigma \in \mathfrak{S}_{k,n-k}$. Then $T_{w\sigma} = T_w T_\sigma$.

Proof. Recall that the length of an element in $\mathfrak{S}_n$ equals to the number of inversions of its action on $\{1, \cdots, n\}$. As w permutes only the first k positions, and the $(k, n-k)$-shuffle σ preserves the order of first k positions, the number of inversions of $w\sigma$ is the sum of those for w and σ, which means that $l(w\sigma) = l(w) + l(\sigma)$ and then the lemma comes from the remark above. □

The following lemma is helpful for the understanding of the operator $P_{i,j}$ and for our further applications.

Lemma 4.2. For $n \geq 2$ and $1 \leq i < j \leq n$, the following identity holds in $\mathbb{C}[\mathfrak{B}_n]$:

$$P_{i,j} = (1 - \sigma_{j-1}\sigma_{j-2}\cdots\sigma_i)(1 - \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\cdots(1 - \sigma_{j-1}).$$

Proof. It suffices to show it for $i = 1$ and $j = n$. We prove it by induction on n. The case $n = 2$ is clear.

Suppose that the lemma holds for $n-1$. From the definition of $P_{1,n}$, $P_{1,n} = P_{2,n} - P_{1,n-1} \circ (1, n, \cdots, 2)$, so $P_{1,n} = P_{2,n} - (1, n, \cdots, 2) \circ P_{2,n}$ and then

$$T_{P_{1,n}} = (1 - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1) T_{P_{2,n}}.$$

From the induction hypothesis,

$$T_{P_{2,n}} = (1 - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(1 - \sigma_{n-1}\sigma_{n-2})(1 - \sigma_{n-1}),$$

which finishes the proof. □

4.3.2 Properties of Dynkin operators

We treat $T(V)$ as a braided Hopf algebra as in Section 4.2.

At first, we define Dynkin operators on $T(V)$.

Definition 4.9. We define a graded endomorphism $\Phi \in \bigoplus_{n=0}^{\infty} End(V^{\otimes n})$ by : $\Phi(1) = 0$ and for $x \in V^{\otimes n}$ with $n \geq 1$,

$$\Phi(x) = T_{P_{1,n}}(x) \in V^{\otimes n}.$$

It can be viewed as a linear map $\Phi : T(V) \to T(V)$ and is called a Dynkin operator on $T(V)$.

Using this notation, we can deduce from Lemma 4.2 the following inductive characterization of Φ : for $v \in V$ and $w \in V^{\otimes n}$, we have :

$$\Phi(vw) = \begin{cases} vw, & \text{if } w \in \mathbb{C}, \\ (1 - T_{(1,n+1,\cdots,2)})(v\Phi(w)), & \text{if } n \geq 1. \end{cases} \tag{4.1}$$

Moreover, the following identity is clear :

$$T_{(1,n+1,\cdots,2)}(v\Phi(w)) = (\Phi|_{V^{\otimes n}} \otimes id)(T_{(1,n+1,\cdots,2)}(vw)). \tag{4.2}$$

The following proposition can be viewed as a generalization of a classical result for free Lie algebras in [70]. As $T(V)$ is a braided Hopf algebra, we let $*$ denote the convolution product in $End(T(V))$.

Theorem 4.1. Let $x \in V^{\otimes n}$. Then

$$(\Phi * id)(x) = nx.$$

Proof. The proof is given by induction on the degree n. The case $n = 1$ is trivial.

Let $n \geq 2$. Suppose that the theorem holds for all elements of degree $n-1$. It suffices to show that for any $v \in V$ and $v \in V^{\otimes n-1}$,

$$(\Phi * id)(vw) = nvw.$$

We write $\Delta(w) = 1 \otimes w + \sum w' \otimes w''$ where $w' \in \ker \varepsilon = \bigoplus_{k=1}^{\infty} V^{\otimes k}$. For a homogeneous element $x \in T(V)$, we let $l(x)$ denote its degree. As Δ is an algebra morphism, with these notations,

$$\Delta(vw) = v \otimes w + 1 \otimes vw + \sum vw' \otimes w'' + (1 \otimes v)(\sum w' \otimes w''),$$

and then

$$(\Phi * id)(vw) = \Phi(v)w + \sum \Phi(vw')w'' + (\Phi|_{V^{\otimes l(w')}} \otimes id)(T_{(1,l(w')+1,\cdots,2)}(vw'))w''.$$

By the induction hypothesis,

$$(n-1)w = (\Phi * id)(w) = \sum \Phi(w')w'',$$

then after (4.1),

$$\sum \Phi(vw')w'' = \sum v\Phi(w')w'' - \sum T_{(1,l(w')+1,\cdots,2)}(v\Phi(w'))w''.$$

In this formula, the first term is $(n-1)vw$ and the second one, after (4.2), equals to

$$(\Phi|_{V^{\otimes l(w')}} \otimes id)(T_{(1,l(w')+1,\cdots,2)}(vw'))w''.$$

Combining these formulas terminates the proof of the theorem. □

To write down the formula in a more compact form, we define the number operator :

Definition 4.10. The number operator $\mathcal{N} : T(V) \to T(V)$ is the linear map defined by : $\mathcal{N}(1) = 0$ and for any $x \in V^{\otimes n}$ with $n \geq 1$,

$$\mathcal{N}(x) = nx.$$

So the formula in Theorem 4.1 can be written as

$$(\Phi * id)(x) = \mathcal{N}(x).$$

As S is the convolution inverse of the identity map, we have :

Corollary 4.1. Let $x \in T(V)$. Then :

$$(\mathcal{N} * S)(x) = \Phi(x).$$

As an application of Corollary 4.1, we may descend Φ from braided tensor Hopf algebra $T(V)$ to Nichols algebra $\mathfrak{N}(V)$.

Proposition 4.2. $\Phi(\mathfrak{I}(V)) \subset \mathfrak{I}(V)$, so Φ induces a linear map $\Phi : \mathfrak{N}(V) \to \mathfrak{N}(V)$.

Proof. From the definition of $\mathfrak{I}(V)$, it is both a coideal and a two-sided ideal of $T(V)$. So the coproduct on it satisfies :

$$\Delta(\mathfrak{I}(V)) \subset \mathfrak{I}(V) \otimes T(V) + T(V) \otimes \mathfrak{I}(V).$$

From Corollary 4.1, $\Phi(\mathfrak{I}(V)) = (\mathcal{N} * S)(\mathfrak{I}(V)) \subset \mathfrak{I}(V)$ because $S(\mathfrak{I}(V)) \subset \mathfrak{I}(V)$ and $\mathcal{N}$ respects $\mathfrak{I}(V)$ (note that $\mathfrak{I}(V)$ is a homogeneous ideal). □

4.4 Decompositions in braid groups

The objective of this section is to give a preparation for results that will be used in our investigations on the Dynkin operators and their relations with the structure of Nichols algebras. As such operators live in the group algebra $\mathbb{C}[\mathfrak{B}_n]$, we would like to give first some decomposition results for some specific elements in $\mathbb{C}[\mathfrak{B}_n]$.

4.4.1 Central element

Let $n \geq 2$ be an integer and $Z(\mathfrak{B}_n)$ denote the center of $\mathfrak{B}_n$.
In the braid group $\mathfrak{B}_n$, there is a Garside element

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1.$$

The following characterization of $Z(\mathfrak{B}_n)$ is well known.

Proposition 4.3 ([48], Theorem 1.24)**.** For $n \geq 3$, let $\theta_n = \Delta_n^2$. Then $Z(\mathfrak{B}_n)$ is generated by θ_n.

For the particular case $n = 2$, we have $\theta_2 = \Delta_2^2 = \sigma_1^2$.
Between lines of the proof of the proposition above in [48], the following lemma is obtained.

Lemma 4.3 ([48])**.** For any $1 \leq i \leq n-1$, $\sigma_i \Delta_n = \Delta_n \sigma_{n-i}$.

Lemma 4.4. The following identities hold :

1. For any $1 \leq s \leq n-2$,

$$\sigma_s(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1) = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)\sigma_{s+1}.$$

2. The element Δ_n has another presentation :

$$\Delta_n = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1})\cdots(\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}).$$

3. The element θ_n has another presentation :

$$\theta_n = \Delta_n^2 = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^n.$$

Proof. 1. This can be proved by a direct verification.

2. Δ_n is the image under the Matsumoto section of the element $\sigma \in \mathfrak{S}_n$ such that for any $1 \leq i \leq n$, $\sigma(i) = n - i + 1$. It is easy to check that once projected to $\mathfrak{S}_n$, the element on the right hand side is exactly σ. Moreover, this decomposition is reduced because both sides have the same length, which finishes the proof.
3. This identity comes from a direct computation using Lemma 4.3 :

$$\begin{aligned}
\Delta_n^2 &= \Delta_n\Delta_n \\
&= \Delta_n(\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1 \\
&= (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})\sigma_{n-1}\Delta_n \\
&= (\sigma_{n-1}\cdots\sigma_1)\cdots(\sigma_{n-1}\sigma_{n-2})\sigma_{n-1}(\sigma_1\cdots\sigma_{n-1})\cdots(\sigma_1\sigma_2)\sigma_1 \\
&= (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^n.
\end{aligned}$$

□

The following proposition is the main result of this subsection.

Proposition 4.4. The element θ_n has another presentation :

$$\theta_n = \Delta_n^2 = (\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)^{n-1}.$$

Proof. From Lemma 4.4,

$$\Delta_n^2 = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)\cdots(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1).$$

At first, using Lemma 4.4, it is possible to move the first σ_1 towards right until it can not move anymore. We exchange it with $(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$ for $n-2$ times, which gives σ_{n-1} finally and so :

$$\Delta_n^2 = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)\cdots(\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1).$$

Repeating this procedure with the help of Lemma 4.4 for the first $\sigma_2,\cdots,\sigma_{n-2}$, we will obtain the presentation as announced in the proposition. □

4.4.2 Decompositions in the group algebra

In this subsection, we will work in the group algebra $\mathbb{C}[\mathfrak{B}_n]$ for some $n \geq 2$.

The symmetrization operator in $\mathbb{C}[\mathfrak{B}_n]$ is defined by :

$$S_n = \sum_{\sigma\in\mathfrak{S}_n} T_\sigma \in \mathbb{C}[\mathfrak{B}_n].$$

Because V is a braided vector space and $\mathfrak{B}_n$ acts naturally on $V^{\otimes n}$, we may treat S_n as a linear operator in $End(V^{\otimes n})$.

For $1 \leq i \leq n-1$, let $(i, i+1) \in \mathfrak{S}_n$ be a transposition. We have seen that $T_{(i,i+1)} = \sigma_i$.

The element $S_n \in \mathbb{C}[\mathfrak{B}_n]$ has a remarkable decomposition as shown in [23]. For any $2 \leq m \leq n$, we define

$$T_m = 1 + \sigma_{m-1} + \sigma_{m-1}\sigma_{m-2} + \cdots + \sigma_{m-1}\sigma_{m-2}\cdots\sigma_1 \in \mathbb{C}[\mathfrak{B}_n].$$

Proposition 4.5 ([23])**.** For any $n \geq 2$,

$$S_n = T_2T_3\cdots T_n \in \mathbb{C}[\mathfrak{B}_n].$$

In fact, this proposition is true when being projected to $\mathbb{C}[\mathfrak{S}_n]$; then notice that the expansion of the product on the right hand side contains only reduced terms.

Recall the definition of $P_{1,n}$ in Section 4.3.1. To simplify the notation, we denote

$$P_n = T_{P_{1,n}} \in \mathbb{C}[\mathfrak{B}_n].$$

This element will be an important ingredient in our further discussion.

For $n \geq 2$, recall the decomposition of P_n given in Lemma 4.2 :

$$P_n = (1-\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(1-\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(1-\sigma_{n-1}).$$

This element P_n permits us to give a much more refined structure of T_n. We introduce another member

$$T_n' = (1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2)\cdots(1-\sigma_{n-1}^2) \in \mathbb{C}[\mathfrak{B}_n].$$

Proposition 4.6. For $n \geq 2$, the decomposition $T_n P_n = T_n'$ holds in $\mathbb{C}[\mathfrak{B}_n]$.

Proof. The Proposition 6.11 in [23] affirms that if all inverses appearing are well defined, then

$$T_n = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) \cdots (1 - \sigma_{n-1}^2)(1 - \sigma_{n-1})^{-1} \cdots (1 - \sigma_{n-1} \cdots \sigma_1)^{-1}.$$

So the proposition follows from Lemma 4.2. □

Corollary 4.2. The following identity holds in $\mathbb{C}[\mathfrak{B}_n]$:

$$\left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1) = 1 - \Delta_n^2 = 1 - \theta_n.$$

Moreover, for $3 \leq s \leq n-1$, let $\iota_s : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$ be the canonical embedding of braid groups on the last s strands. If θ_s is the central element in $\mathfrak{B}_s$, we denote $\theta_s^{\iota_s} = \iota_s(\theta_s)$ and $\theta_2^{\iota_2} = \sigma_{n-1}^2$, then there exists an element

$$L_n = \left(\sum_{k=0}^{1} (\sigma_{n-1}^2 \sigma_{n-2})^k \right) \cdots \left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) \in \mathbb{C}[\mathfrak{B}_n],$$

such that in $\mathbb{C}[\mathfrak{B}_n]$,

$$L_n T_n' = (1 - \theta_n)(1 - \theta_{n-1}^{\iota_{n-1}}) \cdots (1 - \theta_2^{\iota_2}).$$

4.5 The study of the ideal $\mathfrak{I}(V)$

We keep assumptions and notations in previous sections.

4.5.1 A result on the quotient ideal

As we have seen, the Nichols algebra associated to an H-Yetter-Drinfel'd module V is a quotient of the braided tensor Hopf algebra $T(V)$ by a maximal coideal $\mathfrak{I}(V)$ contained in $T^{\geq 2}(V)$. This definition tells us almost nothing about the concrete structure of $\mathfrak{I}(V)$: as $\mathfrak{I}(V)$ is an ideal, the Nichols algebra $\mathfrak{N}(V)$ can be viewed as imposing some relations in $T(V)$, but such relations can never be read directly from the definition.

As we know, the best result for the structure of $\mathfrak{I}(V)$ is obtained by M. Rosso in [73] in a dual point of view and by P. Schaurenburg in [79]. We recall this result in this subsection.

Let $S_n : V^{\otimes n} \to V^{\otimes n}$ be the element in $\mathbb{C}[\mathfrak{B}_n]$ defined in Section 4.4.2.

Proposition 4.7 ([73], [79], [85])**.** Let V be an H-Yetter-Drinfel'd module. Then

$$\mathfrak{N}(V) = \bigoplus_{n \geq 0} \left(V^{\otimes n} / \ker(S_n) \right).$$

So to make more precise the structure of $\mathfrak{I}(V)$, it suffices to study each subspace $\ker(S_n)$. In the following part of this section, we want to characterize a part of elements in $\ker(S_n)$ and show that in cases of great interest, this part is the essential one for understanding the structure of $\ker(S_n)$.

4.5.2 General assumption

From now on, assume that $n \geq 2$ is an integer. To study the structure of $\ker(S_n)$, we want first to concentrate on some essential "levels" in it.

Definition 4.11. Let $1 < s < n$ be an integer and $i : \mathfrak{B}_s \to \mathfrak{B}_n$ be an injection of groups. We call i a positional embedding if there exists some integer $0 \leq r \leq n - s$ such that for any $1 \leq t \leq s-1$, $i(\sigma_t) = \sigma_{t+r}$.

For an element $v \in V^{\otimes n}$, if $v \in \ker(S_n)$, there are two possibilities :

1. There exists some $2 \leq s < n$ and some positional embedding of groups $\iota : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$ such that v is annihilated by $\iota(S_s)$;
2. For any s and positional embedding ι as above, v is not in $\ker(\iota(S_s))$.

Elements falling in the case (2) are much more interesting in our framework. So we would like to give a more concrete assumption for the purpose of concentrating on such elements ; here, we want to impose a somehow stronger restriction.

Let $v \in V^{\otimes n}$ be a non-zero element and $\mathbb{C}[X_v]$ denote the $\mathbb{C}[\mathfrak{B}_n]$-submodule of $V^{\otimes n}$ generated by v, that is to say, $\mathbb{C}[X_v] = \mathbb{C}[\mathfrak{B}_n].v$. Because $\mathbb{C}[X_v]$ is a $\mathbb{C}[\mathfrak{B}_n]$-module, $S_n : \mathbb{C}[X_v] \to \mathbb{C}[X_v]$ is well defined.

We fix this $v \in V^{\otimes n}$ as above, the restriction on v we want to impose is as follows :

Definition 4.12. An element $v \in V^{\otimes n}$ is called of level n if $S_n v = 0$ and for any $2 \leq s \leq n-1$ and any positional embedding $\iota : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$, the equation $\iota(\theta_s)x = x$ has no solution in $\mathbb{C}[X_v]$.

Proposition 4.8. If $v \in V^{\otimes n}$ is a non-zero solution of equation $(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)x = 0$, then $\theta_n.v = v$.

Proof. If $(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)v = 0$, from Corollary 4.2,

$$0 = \left(\sum_{k=0}^{n-2}(\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)^k\right)(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)v = (1-\theta_n)v.$$

□

Remark 4.4. As $V^{\otimes n}$ is a $\mathbb{C}[\mathfrak{B}_n]$-module, we can define $\mathcal{H} \subset V^{\otimes n}$ as the subspace of $V^{\otimes n}$ formed by eigenvectors of θ_n with eigenvalue 1 (that is to say, $\mathcal{H} = \{w \in V^{\otimes n} | \ \theta_n w = w\}$).

As $\theta_n \in Z(\mathfrak{B}_n)$, for any $w \in \mathcal{H}$ and $Y \in \mathbb{C}[\mathfrak{B}_n]$, we have $\theta_n Y w = Y w$, thus $\mathcal{H}$ is a $\mathbb{C}[\mathfrak{B}_n]$-submodule of $V^{\otimes n}$. It means that if $v \in \mathcal{H}$, then $\mathbb{C}[X_v] \subset \mathcal{H}$.

Lemma 4.5. Let $v \in V^{\otimes n}$ be a non-zero element of level n. Then for any $2 \leq i \leq n-1$ and any positional embedding $\iota_i : \mathfrak{B}_i \hookrightarrow \mathfrak{B}_n$, $\iota_i(S_i)x = 0$ has no solution on $\mathbb{C}[X_v]$.

Proof. Let $\iota_i : \mathfrak{B}_i \hookrightarrow \mathfrak{B}_n$ be a positional embedding such that $\iota_i(S_i)x = 0$ has a solution in $\mathbb{C}[X_v]$. Then $\iota_i(S_i) = \iota_i(T_2)\cdots\iota_i(T_i) \in \mathbb{C}[\mathfrak{B}_n]$. The equation $\iota_i(S_i)x = 0$ has a solution in $\mathbb{C}[X_v]$ means that $det(\iota_i(S_i)) = 0$, so there exists some $2 \leq j \leq i$ such that $det(\iota_i(T_j)) = 0$.

Because $\iota_i(T_j)\iota_i(P_j) = \iota_i(T_j')$, we obtain that $det(\iota_i(T_j')) = 0$. From the definition of T_j', there exists some $1 \leq k \leq j-1$ such that

$$det(1 - \iota_i(\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_k)) = 0.$$

So we can choose another positional embedding $\iota : \mathfrak{B}_{j-k+1} \to \mathfrak{B}_n$ such that for the action of $\mathbb{C}[\mathfrak{B}_n]$ on $\mathbb{C}[X_v]$,

$$det(1 - \iota(\sigma_{j-k}^2\sigma_{j-k-1}\cdots\sigma_1)) = 0.$$

So, from Proposition 4.8, $\iota(\theta_{j-k+1})x = x$ has a non-zero solution on $\mathbb{C}[X_v]$, which contradicts to the assumption that v is of level n. □

4.5.3 Solutions

Fix some $n \geq 2$, we want to solve the equation $S_n x = 0$ on $\mathbb{C}[X_v]$ for some non-zero element $v \in V^{\otimes n}$.

We define an element in $\mathbb{C}[\mathfrak{B}_n]$:

$$X = (1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2)\cdots(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_3)\cdots(1 - \sigma_{n-1}^2\sigma_{n-2})(1 - \sigma_{n-1}^2).$$

Proposition 4.9. If $v \in V^{\otimes n}$ is a non-zero element of level n, then X is invertible on $\mathbb{C}[X_v]$.

Proof. We may view X as an element in $End(\mathbb{C}[X_v])$. If X is not invertible, $det(X) = 0$. From the definition, there must exist some term, say $(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_i)$, for some $2 \leq i \leq n-1$, having determinant 0. So there exists some nonzero element $w \in \mathbb{C}[X_v]$ such that $(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_i)w = 0$. But from Proposition 4.8, we may find some positional embedding $\iota : \mathfrak{B}_{n-i+1} \hookrightarrow \mathfrak{B}_n$ such that $\iota(\theta_{n-i+1})w = w$, which contradicts to the assumption that v is of level n. □

The level n assumption we are working with will give more information on solutions of equation $S_n x = 0$.

Proposition 4.10. Let $v \in V^{\otimes n}$ be a non-zero element of level n.

1. There exists a bijection between nonzero solutions of equation $T_n' x = 0$ and of the equation $(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)x = 0$ in $\mathbb{C}[X_v]$.
2. Equations $S_n x = 0$ and $T_n x = 0$ have the same solutions in $\mathbb{C}[X_v]$.

Proof. 1. From Proposition 4.9, $X^{-1} : \mathbb{C}[X_v] \to \mathbb{C}[X_v]$ is well defined. So this proposition comes from the identity : $T_n' = (1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)X$.

2. Let w be a non-zero solution of $T_n x = 0$, then from Proposition 4.5, $S_n w = 0$. Conversely, let u be a non-zero solution of $S_n x = 0$. If $T_n u \neq 0$, again from Proposition 4.5, $T_n u$ will be a non-zero solution of equation $T_2 \cdots T_{n-1} x = 0$ on $\mathbb{C}[X_v]$, so $S_{n-1} x = 0$ has a non-zero solution on $\mathbb{C}[X_v]$, contradicts Lemma 4.5 above.

□

Recall that $P_n = T_{P_{1,n}} \in \mathbb{C}[\mathfrak{B}_n]$ as defined in the last section, $P_n \in End(\mathbb{C}[X_v])$.

Now, let $w \in \ker(S_n) \cap Im(P_n)$ be a non-zero element of level n. Then from Lemma 4.5, w satisfies $T_n w = 0$. Moreover, because it is in $Im(P_n)$, we can choose some w' such that $P_n(w') = w$, then

$$T_n' w' = T_n w = 0.$$

From the identity $T_n' = (1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)X$, Xw' is a solution of the equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)x = 0$, so from Proposition 4.8, $\theta_n X w' = X w'$. This discussion gives the following proposition.

Proposition 4.11. Let $w \in \ker(S_n) \cap Im(P_n)$ be an element of level n. Then $\theta_n w = w$.

Proof. From the definition of w', if θ_n fixes w', then it fixes w. So if $\theta_n w \neq w$, then it does not fix w' and then Xw' (see Remark 4.4), which is a contradiction. □

Remark 4.5. Let $\mathcal{H}$ denote the eigenspace of θ_n corresponding to the eigenvalue 1 as in the Remark 4.4 above. If we let E_n denote the set of elements in $Im(P_n)$ with level n in $V^{\otimes n}$, then the proposition above implies that $E_n \subset P_n(\mathcal{H})$.

We have constructed solutions of equation $\theta_n x = x$ on $\mathbb{C}[X_v]$ from some kinds of elements in $\ker(S_n) \cap Im(P_n)$. Now we proceed to consider the construction in the opposite direction.

Let $w \in \mathbb{C}[X_v]$ be a solution of $\theta_n x = x$. If

$$u = \left(\sum_{k=0}^{n-2} (\sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)^k \right) w \neq 0,$$

it will be a solution of the equation $(1 - \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1)x = 0$, then $X^{-1}u$ is a solution of the equation $T_n' x = 0$ and $P_n X^{-1} u$ will be a non-trivial solution of $S_n x = 0$ if it is not zero ; moreover, it is in $Im(P_n)$, from which we obtain an element in $\ker(S_n) \cap Im(P_n)$.

There are some possibilities for the appearance of zero elements when passing from the solutions of $\theta_n x = x$ to those of $S_n x = 0$. The appearance of zeros mostly comes from the fact that an element satisfying $\theta_n x = x$ may be the solution of $\iota_s(\theta_s)x = x$ for some $2 \leq s \leq n-1$ and some positional embedding $\iota_s : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_n$.

The subspace $\ker(S_n) \cap Im(P_n)$ is sufficiently important, as will be shown in the next subsection.

4.5.4 Properties of $\ker(S_n) \cap Im(P_n)$

In this subsection, suppose that $n \geq 2$ is an integer.

Instead of $End(\mathbb{C}[X_v])$, it is better in this subsection to view S_n, P_n as elements in $End(V^{\otimes n})$. We want to show that $\ker(S_n) \cap Im(P_n)$ contains all primitive elements and in some special cases (for example, the diagonal case), it generates $\ker(S_n)$.

Proposition 4.12. Let $v \in V^{\otimes n}$ be a homogeneous primitive element of degree n. Then $v \in \ker(S_n) \cap Im(P_n)$.

Proof. The fact $v \in \ker(S_n)$ is a corollary of the definition of Nichols algebra and Proposition 5.1. So it suffices to show that $v \in Im(P_n)$.

The element v is primitive means that $\Delta(v) = v \otimes 1 + 1 \otimes v$. From Theorem 4.1, $\Phi * id = \mathcal{N}$, so

$$nv = \Phi * id(v) = m \circ (\Phi \otimes id)\Delta(v) = \Phi(v),$$

and then

$$v = \frac{1}{n}\Phi(v) = \frac{1}{n}P_n(v) \in Im(P_n).$$

□

The second property we want to establish is that in the diagonal case, these subspaces $\ker(S_n) \cap Im(P_n)$ will generate the ideal (also coideal) $\mathfrak{I}(V)$.

Recall that from the definition of Nichols algebra and Proposition 5.1, the subspace

$$\mathfrak{I}(V) = \bigoplus_{n\geq 2} \ker(S_n) \subset T(V)$$

is a maximal coideal contained in $T^{\geq 2}(V)$. Moreover, it is a homogeneous ideal.

Let $J \subset T^{\geq 2}(V)$ be a coideal in ${}^H_H\mathcal{YD}$ containing the subspace

$$\bigoplus_{n\geq 2} \left(\ker(S_n) \cap Im(P_n)\right).$$

Such a coideal does exist as $\mathfrak{I}(V)$ satisfies these conditions.

Proposition 4.13. Let $T(V)$ be of diagonal type. Then the ideal generated by J in $T(V)$ is $\mathfrak{I}(V)$.

Proof. Let K be the two-sided ideal generated by $J \subset T^{\geq 2}(V)$ in $T(V)$. Then K is also an ideal in $T^{\geq 2}(V)$. As a two-sided ideal generated by a coideal, K is also a coideal. From the maximality of $\mathfrak{I}(V)$, $K \subset \mathfrak{I}(V)$.

We proceed to prove that $T(V)/K \cong \mathfrak{N}(V)$. For this purpose, the following lemma is needed.

Lemma 4.6 ([35])**.** Suppose that the Nichols algebra is of diagonal type. Let $K \subset T^{\geq 2}(V)$ be simultaneously an ideal, a coideal and an H-Yetter-Drinfel'd module. If all primitive elements in $T(V)/K$ are concentrated in V, then $T(V)/K \cong \mathfrak{N}(V)$.

From this lemma, it suffices to show that there is no non-zero primitive element of degree greater than 1 in $T(V)/K$.

Suppose that v is such a non-zero element which is moreover homogeneous of degree n, so in $T(V)$,

$$\Delta(v) \in v \otimes 1 + 1 \otimes v + K \otimes T(V) + T(V) \otimes K.$$

As $K \subset \mathfrak{I}(V)$,

$$\Delta(v) \in v \otimes 1 + 1 \otimes v + \mathfrak{I}(V) \otimes T(V) + T(V) \otimes \mathfrak{I}(V).$$

But in $T(V)/\mathfrak{I}(V)$, from the definition of Nichols algebra, there is no such element, which forces $v \in \mathfrak{I}(V)$ and then $S_n v = 0$.

We need to show that in fact $v \in K$. From Corollary 4.1,

$$P_n(v) = \mathcal{N} * S(v) \in nv + K$$

and then $v - k \in Im(P_n)$ for some $k \in K$. As $S_n v = 0$ and $K \subset \ker(S_n)$, $v - k \in \ker(S_n) \cap Im(P_n) \subset K$; this implies $v \in K$. □

This proposition shows the importance of these subspaces $\ker(S_n) \cap Im(P_n)$ in the study of the defining ideal.

Remark 4.6. This proposition holds for the general case if the conjecture in Remark 4.1 is true.

4.5.5 Main theorem

The main result of this chapter is :

Theorem 4.2. Elements of level n are primitive.

From this theorem, level n solutions of $S_n x = 0$ are primitive elements of degree n, so they are in $\ker(S_n) \cap Im(P_n)$. Moreover, this introduces a method to find primitive elements in $T(V)$.

The proof of this theorem will be given in the end of this chapter, after introducing the differential algebra of a Nichols algebra.

4.6 Applications

In this section, we give some applications of the machinery constructed above.

Though the discussion in the last section is somehow elementary, it may give remarkable results and good points of view once being applied to some concrete examples.

4.6.1 A general application for the diagonal type

Let H be the group algebra of an abelian group G, $V \in {}^H_H\mathcal{YD}$ be of diagonal type, $T(V)$ and $\mathfrak{N}(V)$ be the braided tensor Hopf algebra and Nichols algebra, respectively.

Suppose that $dimV = m$, with basis $v_1, \cdots, v_m$ such that

$$\sigma(v_i \otimes v_j) = q_{ij} v_j \otimes v_i.$$

From the definition of the braiding, the action of $\mathbb{C}[\mathfrak{B}_n]$ on $V^{\otimes n}$ has the following decomposition

$$V^{\otimes n} = \bigoplus_{\underline{i} \in I} \mathbb{C}[\mathfrak{B}_n].v_1^{i_1} \cdots v_m^{i_m},$$

where the sum runs over $I = \{\underline{i} = (i_1, \cdots, i_m) |\ i_1 + \cdots + i_m = n\}$.

We fix some $\underline{i} = (i_1, \cdots, i_m) \in \mathbb{N}^m$ such that $\underline{i} \in I$ and a monomial $v_{\underline{i}} = v_1^{i_1} \cdots v_m^{i_m}$, then in

$$\mathbb{C}[X_{\underline{i}}] = \mathbb{C}[\mathfrak{B}_n].v_{\underline{i}},$$

if $\theta_n x = x$ has a solution, we must obtain $\theta_n v_{\underline{i}} = v_{\underline{i}}$.

Indeed, when projected canonically to $\mathfrak{S}_n$, the element $\theta_n \in \mathfrak{B}_n$ corresponds to 1, so if $\theta_n x = x$, θ_n will stablize all components of x. From the decomposition above, for any component x_0 of x, there exists a nonzero constant c and an element $\sigma \in \mathfrak{B}_n$ such that $cx_0 = \sigma(v_{\underline{i}})$, thus $v_{\underline{i}} = \sigma^{-1}(cx_0)$. Thus all level n elements are contained in the sum of some $\mathbb{C}[X_{\underline{i}}]$ for some $v_{\underline{i}}$ satisfying $\theta_n v_{\underline{i}} = v_{\underline{i}}$.

To exclude those elements which have not level n but are stable under the action of θ_n, some notations are needed.

We fix some $\underline{i}$ and $v_{\underline{i}}$. It is more convenient to write $v_{\underline{i}} = e_1 \cdots e_n$, where e_i are some v_j's. Then let $T_{\underline{i}} = (t_{ij})$ denote a matrix in $M_n(\mathbb{C})$ with $t_{ii} = 1$ and for $i \neq j$, t_{ij} are defined by

$$\sigma(e_i \otimes e_j) = t_{ij} e_j \otimes e_i.$$

For some $2 \leq s \leq n$, $1 \leq k_1 < \cdots < k_s \leq n$, $\underline{k} = (k_1, \cdots, k_s)$, we define :

$$\Pi_s^{\underline{k}} = \prod_{i=1}^{s} \prod_{j=1}^{s} t_{k_i, k_j}.$$

The following proposition is an easy consequence of the definition.

Proposition 4.14. With the notations above, we have :

1. $\theta_n v_{\underline{i}} = v_{\underline{i}}$ if and only if $\Pi_n^{\underline{k}} = 1$.
2. If for any $2 \leq s \leq n-1$ and any $\underline{k}$, $\Pi_s^{\underline{k}} \neq 1$, then $v_{\underline{i}}$ satisfies the assumption in Definition 4.12. Moreover, all elements in $\ker(S_n) \cap \mathbb{C}[X_{v_{\underline{i}}}]$ are of level n.

Remark 4.7. Under the assumptions (2) in the proposition above,

1. From Theorem 4.2, all elements in $\ker(S_n) \cap \mathbb{C}[X_{v_{\underline{i}}}]$ are primitive.
2. From Remark 4.5, all elements in $\ker(S_n) \cap \mathbb{C}[X_{v_{\underline{i}}}]$ can be constructed from $\mathbb{C}[X_{v_{\underline{i}}}]$ by the method given in the end of Section 4.5.3. So such a family of primitive elements can be easily and directly computed.

Remark 4.8. If $v \in V \otimes V$ is an element in $\ker(S_2) \cap Im(P_2)$, then it must be of level 2 and so primitive. Moreover, in the diagonal case, level 2 elements in $\ker(S_2) \cap Im(P_2)$ can be obtained from monomials stablized by θ_2 by applying P_2.

4.6.2 Exterior algebras

In this subsection, as a warm up, we apply results of the previous section to the construction of exterior algebras.

The main ingredient is the Hopf algebra $H = \mathbb{C}[G]$, where $G = \mathbb{Z}/2\mathbb{Z} = \{1, \varepsilon\}$. Let V be a finite dimensional vector space with basis $v_1, \cdots, v_m$.

1. The action of H on V is given by : for $v \in V$, $\varepsilon.v = -v$;
2. The coaction is given by : $\delta(v) = \varepsilon \otimes v$, where $\delta : V \to H \otimes V$.

This makes V an H-Yetter-Drinfel'd module.

We form the braided Hopf algebra $T(V)$ and want to calculate relations appearing in the ideal $\mathfrak{I}(V)$.

At first, we consider relations in $V^{\otimes n}$ of level n. In fact, for $n \geq 3$, there are no such relations because if $v = v_{i_1} \cdots v_{i_n} \in V^{\otimes n}$ is a pure tensor such that $\theta_n v = v$, from the definition of the braiding, there must exist some $1 \leq s < t \leq n$ such that $\sigma^2(v_{i_s} \otimes v_{i_t}) = v_{i_s} \otimes v_{i_t}$, which contradicts the definition of level n relations.

So it suffices to consider relations of level 2 in $V^{\otimes 2}$. We start from considering all solutions of $\theta_2 x = x$ in $V^{\otimes 2}$. These solutions are : $v_i v_j$, for $1 \leq i, j \leq n$.

As in the procedure of constructing solutions of $S_n x = 0$ from $\theta_n x = x$ given in the last section, the action of P_2 on these elements gives :

$$P_2(v_i v_j) = v_i v_j + v_j v_i,$$

so $v_i v_j + v_j v_i \in \ker(S_2) \cap Im(P_2)$. Moreover, from Remark 4.8 in the last subsection, we obtain

$$\ker(S_2) \cap Im(P_2) = \mathrm{span}\{v_i v_j + v_j v_i |\ 1 \leq i, j \leq n\}.$$

4.6.3 Quantized enveloping algebras

In this subsection, we will discover the quantized Serre relations in the definition of the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ associated to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ by assuming almost no knowledge about the existence of such relations.

Let q be a nonzero complex number such that for any $N \geq 1$, $q^N \neq 1$. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra of rank n, $C = (C_{ij})_{n \times n}$ be its generalized Cartan matrix and $A = DC$ be the symmetrization of the Cartan matrix by some diagonal matrix $D = (d_1, \cdots, d_n)$ with d_i positive integers which are relatively prime. We denote $A = (a_{ij})_{n \times n}$.

At first, we briefly recall the construction of the strict positive part of $\mathcal{U}_q(\mathfrak{g})$ in the framework of Nichols algebras. This construction is due to M. Rosso and can be found in [73] with a slightly different language.

Let $H = \mathbb{C}[G]$ be the group algebra where G is the abelian group $\mathbb{Z}^n$. Let $K_1, \cdots, K_n$ denote a basis of $\mathbb{Z}^n$. Then H is a commutative and cocommutative Hopf algebra.

Let V be a $\mathbb{C}$-vector space of dimension n with basis $E_1, \cdots, E_n$. We define an H-Yetter-Drinfel'd module structure on V by :

1. The action of K_i on E_j is given by : $K_i.E_j = q^{a_{ij}} E_j$;
2. The coaction of E_i is given by : $\delta(E_i) = K_i \otimes E_i$, where $\delta : V \to H \otimes V$ is the structure map of left H-comodule structure on V.

Starting with this $V \in {}^H_H\mathcal{YD}$, the braided tensor algebra $T(V)$ and the corresponding Nichols algebra $\mathfrak{N}(V)$ can be constructed. The defining ideal is denoted by $\mathfrak{I}(V)$.

Assume that we know nothing about this ideal $\mathfrak{I}(V)$ (because from the general theory of quantized enveloping algebras, we know that $\mathfrak{I}(V)$ is generated by quantized Serre relations). Here, our point of view is much more pedestrian : if we do not know them, how to find ?

Results in the previous section will offer us a method.

At first, we want to concentrate on the case $\mathcal{U}_q(\mathfrak{sl}_3)$, the simplest one which has such quantized Serre relations. In this case, $H = \mathbb{Z}^2$ with basis K_1, K_2, V is of dimension 2 with basis E_1, E_2.

We would like to compute the level 3 relations in $\mathfrak{I}(V)$.

At first, we write down all monomials of degree 3 which are stabilized by the action of θ_3 but not for all θ_2 with possible embeddings. They are :

$$E_1^2E_2,\quad E_1E_2E_1,\quad E_2E_1^2,\quad E_1E_2^2,\quad E_2E_1E_2,\quad E_2^2E_1.$$

After the action of $1+\sigma_2^2\sigma_1$, we obtain :

$$2E_1^2E_2,\quad E_1E_2E_1+q^3E_2E_1^2,\quad E_2E_1^2+q^{-3}E_1E_2E_1,$$

$$E_1E_2^2+q^{-3}E_2E_1E_2,\quad E_2E_1E_2+q^3E_1E_2^2,\quad 2E_2^2E_1.$$

In this case, $X = 1-\sigma_2^2$, so the action of X^{-1} on these elements will give :

$$x_1 = \frac{2}{1-q^{-2}}E_1^2E_2,\quad x_2 = \frac{1}{1-q^{-2}}E_1E_2E_1 + \frac{q^3}{1-q^4}E_2E_1^2,$$

$$x_3 = \frac{1}{1-q^4}E_2E_1^2 + \frac{q^{-3}}{1-q^{-2}}E_1E_2E_1,\quad x_4 = \frac{1}{1-q^4}E_1E_2^2 + \frac{q^{-3}}{1-q^{-2}}E_2E_1E_2,$$

$$x_5 = \frac{1}{1-q^{-2}}E_2E_1E_2 + \frac{q^3}{1-q^4}E_1E_2^2,\quad x_6 = \frac{1}{1-q^{-2}}E_2^2E_1.$$

It is easy to compute the action of P_3 on all possible monomials :

$$P_3(E_1^2E_2) = E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2,$$

$$P_3(E_1E_2E_1) = 2E_1E_2E_1 - q^{-1}E_1^2E_2 - qE_2E_1^2,$$
$$P_3(E_2E_1^2) = (1-q^2)E_2E_1^2 - (q^{-2}-1)E_1^2E_2,$$
$$P_3(E_1E_2^2) = (1-q^2)E_1E_2^2 - (q^{-2}-1)E_2^2E_1,$$
$$P_3(E_2E_1E_2) = 2E_2E_1E_2 - qE_1E_2^2 - q^{-1}E_2^2E_1,$$
$$P_3(E_2^2E_1) = E_2^2E_1 - (q+q^{-1})E_2E_1E_2 + E_1E_2^2.$$

And then

$$P_3(x_1) = \frac{2}{1-q^{-2}}(E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2),$$

$$P_3(x_2) = -\frac{2q^{-1}}{1-q^{-4}}(E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2),$$

$$P_3(x_3) = \frac{2}{1-q^4}(E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2),$$

$$P_3(x_4) = \frac{2}{1-q^4}(E_2^2E_1 - (q+q^{-1})E_2E_1E_2 + E_1E_2^2),$$

$$P_3(x_5) = -\frac{2q^{-1}}{1-q^{-4}}(E_2^2E_1 - (q+q^{-1})E_2E_1E_2 + E_1E_2^2),$$

$$P_3(x_6) = \frac{2}{1-q^{-2}}(E_2^2E_1 - (q+q^{-1})E_2E_1E_2 + E_1E_2^2).$$

So starting with solutions of $\theta_3 x = x$ with level 3, the solutions in $Im(P_3)$ with level 3 we obtained for the equation $S_3x = 0$ are exactly the quantized Serre relations of degree 3.

Moreover, we show that there are no other relations of level 3. If $w \in \ker(S_3)$ is such an element, it will be stable under the action of θ_3, so it is a linear combination of monomials above, then it must be a linear combination of degree 3 Serre relations.

Finally, we turn to the level n elements for an arbitrary integer $n \geq 2$. As explained in Section 4.6.1, it suffices to consider a monomial of form $E_1^sE_2^t$ for some positive integers s and t.

The action of θ_{s+t} on this monomial gives :

$$\theta_{s+t}(E_1^sE_2^t) = q^{s^2-s+t^2-t-st}E_1^sE_2^t.$$

So this monomial is stablized by θ_{s+t} if and only if

$$s^2 - s + t^2 - t - st = \frac{1}{2}\Big((s-t)^2 + (s-1)^2 + (t-1)^2 - 2\Big) = 0.$$

The only possible positive integer solutions (s,t) of the equation $(s-t)^2 + (s-1)^2 + (t-1)^2 = 2$ are $(2,2)$, $(2,1)$ and $(1,2)$. But $(s,t) = (2,2)$ is not of level 4 because we can always find a subword which is fixed by θ_3.

As a conclusion, the only possible level in this case is 3 and all possible relations coming from level 3 elements are quantized Serre relations as shown above.

4.6.4 Primitivity of Serre relations

As an application of the main theorem, we deduce a short proof for the primitivity of Serre relations with little computation. A direct proof can be found in the appendix of [5].

Let $A = (a_{ij})_{n\times n}$ be a symmetrized Cartan matrix and V be a $\mathbb{Z}^n$-Yetter-Drinfeld module of diagonal type with dimension n. Notations in the previous subsection are adopted. Moreover, suppose that the braiding matrix (q_{ij}) satisfies :

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \quad 1 \leq i,j \leq n. \tag{4.3}$$

Proposition 4.15. For any $1 \leq i,j \leq n$, $i \neq j$, we denote $N = 1 - a_{ij}$. Then $P_{N+1}(v_i^N v_j)$ is a primitive element, where P_{N+1} is the Dynkin operator.

Proof. At first, it is easy to show that

$$(1 - \sigma_N^2\sigma_{N-1}\cdots\sigma_1)(v_i^Nv_j) = (1 - q_{ii}^{-a_{ij}}q_{ij}q_{ji})v_i^Nv_j.$$

From the hypothesis (4.3) above, the right hand side is 0, so from Proposition 4.8, $\theta_{N+1}(v_i^Nv_j) = v_i^Nv_j$.

Moreover, it is obvious that for any $1 < s < N+1$ and any positional embedding $\iota : \mathfrak{B}_s \hookrightarrow \mathfrak{B}_{N+1}$, $\iota(\theta_s)(v_i^Nv_j) \neq v_i^Nv_j$. As a consequence, in the algorithm after Remark 4.5, X^{-1} is well defined and from the definition of X, $X^{-1}(v_i^Nv_j) = \lambda v_i^Nv_j$ for some non-zero constant λ. Then $P_{N+1}(v_i^Nv_j)$ is a nonzero solution of $S_{N+1}x = 0$ of level n, so it is primitive by Theorem 4.2. $\square$

4.6.5 Quantized enveloping algebras revisited

We keep notations in the beginning of Section 4.6.3.

Because $A = DC$ is a symmetrized Cartan matrix, for any $1 \leq i, j \leq n$, we have $d_i c_{ij} = d_j c_{ji}$. Then from the definition of q_{ij}, the following lemma is clear.

Lemma 4.7. Let $A = DC$ be a symmetrized Cartan matrix. Then for any $1 \leq i, j \leq n$,

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}.$$

Combined with Proposition 4.15, this lemma gives :

Corollary 4.3. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Then degree n quantized Serre relations in $\mathcal{U}_q(\mathfrak{g})$ are of level n. Moreover, the union of level n elements for $n \geq 2$ generates $\mathfrak{I}(V)$ as an ideal.

Remark 4.9. The corollary above explains the reason for the importance of symmetrizable Kac-Moody algebras : they contain sufficient Serre relations. This gives a strong constraint on the representation theory of such Lie algebras.

4.7 Differential algebras of Nichols alegbras

4.7.1 Pairings between Nichols algebras

In this subsection, we want to recall a result of [13] and [61]. It should be remarked that these two constructions, though in different languages (one is dual to the other), are essentially the same.

Let

$$H = \bigoplus_{n=0}^{\infty} H_n, \quad B = \bigoplus_{n=0}^{\infty} B_n$$

be two graded Hopf algebras with finite dimensional graded components.

Definition 4.13. A generalized Hopf pairing $\phi : H \times B \to \mathbb{C}$ is called graded if for any $i \neq j$, $\phi(H_i, B_j) = 0$.

We fix a graded Hopf pairing $\phi_0 : H \times B \to \mathbb{C}$ between H and B and assume moreover that ϕ_0 is non-degenerate.

Let $V \in {}^H_H\mathcal{YD}$ and $W \in {}^B_B\mathcal{YD}$ be two Yetter-Drinfel'd modules, $\phi_1 : V \times W \to \mathbb{C}$ be a non-degenerate bilinear form such that for any $h \in H$, $b \in B$, $v \in V$ and $w \in W$,

$$\phi_1(h.v, w) = \sum \phi_0(h, w_{(-1)}) \phi_1(v, w_{(0)}), \tag{4.4}$$

$$\phi_1(v, b.w) = \sum \phi_0(v_{(-1)}, b) \phi_1(v_{(0)}, w), \tag{4.5}$$

where $\delta_V(v) = \sum v_{(-1)} \otimes v_{(0)}$ and $\delta_W(w) = \sum w_{(-1)} \otimes w_{(0)}$ are H-comodule and B-comodule structure maps, respectively.

Let $T(V)$, $T(W)$ be the corresponding braided tensor Hopf algebras and $\mathfrak{N}(V)$, $\mathfrak{N}(W)$ be Nichols algebras associated to V and W, respectively. Let $\mathfrak{B}_H(V) = \mathfrak{N}(V) \sharp H$ and $\mathfrak{B}_B(W) = \mathfrak{N}(W) \sharp B$ denote crossed biproducts defined in Section 4.2.5.

Theorem 4.3 ([13],[61])**.** There exists a unique graded Hopf pairing

$$\phi : \mathfrak{B}_H(V) \times \mathfrak{B}_B(W) \to \mathbb{C},$$

extending ϕ_0 and ϕ_1. Moreover, it is non-degenerate.

In the following argument, attention will be paid to a particular case of this theorem. In our framework, we take $H = B$ and $V = W$ in Theorem 4.3, $\phi_0 : H \times H \to \mathbb{C}$ a non-degenerate graded Hopf pairing and $\phi_1 : V \times V \to \mathbb{C}$ a non-degenerate bilinear form satisfying the compatibility conditions above. So the machinery in Theorem 4.3 produces a non-degenerate graded Hopf pairing

$$\phi : \mathfrak{B}_H(V) \times \mathfrak{B}_H(V) \to \mathbb{C}.$$

This will be the main tool in our further construction.

4.7.2 Double construction and Schrödinger representation

In this subsection, as a review, we will apply results from [24], Section 2 to the case of Nichols algebras.

Suppose that H is a graded Hopf algebra, $V \in {}^H_H\mathcal{YD}$ is an H-Yetter-Drinfel'd module and $\phi : \mathfrak{B}_H(V) \times \mathfrak{B}_H(V) \to \mathbb{C}$ is the non-degenerate graded Hopf pairing constructed in the last section.

We recall some results from [24] briefly.

To indicate their positions, we denote $\mathfrak{B}_H^+(V) = \mathfrak{B}_H(V)$, $\mathfrak{B}_H^-(V) = \mathfrak{B}_H(V)$ and

$$D_\phi(\mathfrak{B}_H(V)) = D_\phi(\mathfrak{B}_H^+(V), \mathfrak{B}_H^-(V))$$

their quantum double. The Schrödinger representation defined in [24] gives a module algebra type action of $D_\phi(\mathfrak{B}_H(V))$ on these two components.

1. On $\mathfrak{B}_H^+(V)$, the action is given by : for $a, x \in \mathfrak{B}_H^+(V)$ and $b \in \mathfrak{B}_H^-(V)$,

$$(a \otimes 1).x = \sum a_{(1)} x S(a_{(2)}),$$

$$(1 \otimes b).x = \sum \varphi(x_{(1)}, S(b)) x_{(2)}.$$

2. On $\mathfrak{B}_H^-(V)$, the action is given by : for $a \in \mathfrak{B}_H^+(V)$ and $b, y \in \mathfrak{B}_H^-(V)$,

$$(a \otimes 1).y = \sum \varphi(a, y_{(1)}) y_{(2)},$$

$$(1 \otimes b).y = \sum b_{(1)} y S(b_{(2)}).$$

As has been shown in [24], these actions give both $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ a $D_\phi(\mathfrak{B}_H(V))$-module algebra structure.

Moreover, we can construct the Heisenberg double

$$H_\phi(\mathfrak{B}_H(V)) = H_\phi(\mathfrak{B}_H^+(V), \mathfrak{B}_H^-(V)),$$

which, in general, is not a Hopf algebra.

In [24], we defined an action of $D_\phi(\mathfrak{B}_H(V))$ on $H_\phi(\mathfrak{B}_H(V))$ by : for $a, a' \in \mathfrak{B}_H^+(V)$ and $b, b' \in \mathfrak{B}_H^-(V)$,

$$(a \otimes b).(b'\sharp a') = \sum (a_{(1)} \otimes b_{(1)}).b'\sharp(a_{(2)} \otimes b_{(2)}).a',$$

which makes $H_\phi(\mathfrak{B}_H(V))$ a $D_\phi(\mathfrak{B}_H(V))$-module algebra.

The following two results are also obtained in [24].

Proposition 4.16 ([24])**.** We define a $D_\phi(\mathfrak{B}_H(V))$-comodule structure on $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ by : for $a \in \mathfrak{B}_H^+(V)$ and $b \in \mathfrak{B}_H^-(V)$,

$$\mathfrak{B}_H^+(V) \to D_\phi(\mathfrak{B}_H(V)) \otimes \mathfrak{B}_H^+(V), \quad a \mapsto \sum a_{(1)} \otimes 1 \otimes a_{(2)},$$

$$\mathfrak{B}_H^-(V) \to D_\phi(\mathfrak{B}_H(V)) \otimes \mathfrak{B}_H^-(V), \quad b \mapsto \sum 1 \otimes b_{(1)} \otimes b_{(2)}.$$

Then with the Schrödinger representation and comodule structures defined above, both $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ are in the category ${}^{D_\phi}_{D_\phi}\mathcal{YD}$.

Moreover, the Heisenberg double $H_\phi(\mathfrak{B}_H(V))$ is in the $D_\phi(\mathfrak{B}_H(V))$-Yetter-Drinfel'd module category.

Theorem 4.4 ([24])**.** We define a $D_\phi(\mathfrak{B}_H(V))$-comodule structure on $H_\phi(\mathfrak{B}_H(V))$ by : for $a \in \mathfrak{B}_H^+(V)$ and $b \in \mathfrak{B}_H^-(V)$,

$$H_\phi(\mathfrak{B}_H(V)) \to D_\phi(\mathfrak{B}_H(V)) \otimes H_\phi(\mathfrak{B}_H(V)),$$

$$b\sharp a \mapsto \sum (1 \otimes b_{(1)}).(a_{(1)} \otimes 1) \otimes b_{(2)}\sharp a_{(2)}.$$

Then with the module structure defined above and this comodule structure, $H_\phi(\mathfrak{B}_H(V))$ is in the category ${}^{D_\phi}_{D_\phi}\mathcal{YD}$.

4.7.3 Construction of differential algebras

In this section, we want to construct the differential algebra of a Nichols algebra. It generalizes the construction of quantized Weyl algebra in [24].

But it should be remarked that the construction in [24] concentrates on a specific Hopf algebra, say $\mathbb{C}[\mathbb{Z}^n]$ and a special action on the Nichols algebra. So to generalize it, we need some more work.

Let $\mathfrak{N}^+(V)$ and $\mathfrak{N}^-(V)$ be Nichols algebras contained in $\mathfrak{B}_H^+(V)$ and $\mathfrak{B}_H^-(V)$ respectively. We would like to give both $\mathfrak{N}^+(V)$ and $\mathfrak{N}^-(V)$ a $D_\phi(\mathfrak{B}_H(V))$-Yetter-Drinfel'd module algebra structure.

For $\mathfrak{N}^+(V)$, the $D_\phi(\mathfrak{B}_H(V))$-module structure is given by the Schrödinger representation and the $D_\phi(\mathfrak{B}_H(V))$-comodule structure is given by :

$$\mathfrak{N}^+(V) \to D_\phi(\mathfrak{B}_H(V)) \otimes \mathfrak{N}^+(V),$$

$$b \mapsto \sum (b_{(1)}\sharp(b_{(2)})_{(-1)} \otimes 1) \otimes (b_{(2)})_{(0)}.$$

This is obtained from the formula in Proposition 4.16.

Proposition 4.17. With the structures defined above, $\mathfrak{N}^+(V)$ is a $D_\phi(\mathfrak{B}_H(V))$-Yetter-Drinfel'd module algebra.

Proof. At first, we need to show that the Schrödinger representation preserves $\mathfrak{N}^+(V)$.

Let $a \in \mathfrak{B}_H(V)$. It suffices to prove that if $x \in \mathfrak{N}^+(V)$, then both $(a \otimes 1).x$ and $(1 \otimes a).x$ are contained in $\mathfrak{N}^+(V)$. For this purpose, because the action is linear, we write $a = b\sharp h \in \mathfrak{N}^+(V)\sharp H$. From the formula given in Radford's crossed biproduct,

$$\begin{aligned}
((b\sharp h) \otimes 1).x &= \sum (b\sharp h)_{(1)} x S((b\sharp h)_{(2)}) \\
&= \sum (b_{(1)}\sharp (b_{(2)})_{(-1)} h_{(1)})(x\sharp 1) S((b_{(2)})_{(0)}\sharp h_{(2)}) \\
&= \sum (b_{(1)}((b_{(2)})_{(-3)} h_{(1)}.x)\sharp (b_{(2)})_{(-2)} h_{(2)})(1\sharp S(h_{(3)}) S((b_{(2)})_{(-1)}))(S((b_{(2)})_{(0)})\sharp 1) \\
&= \sum (b_{(1)}((b_{(2)})_{(-3)} h_{(1)}.x)\sharp (b_{(2)})_{(-2)} h_{(2)} S(h_{(3)}) S((b_{(2)})_{(-1)}))(S((b_{(2)})_{(0)})\sharp 1) \\
&= \sum b_{(1)}((b_{(2)})_{(-1)} h_{(1)}.x) S((b_{(2)})_{(0)})\sharp 1,
\end{aligned}$$

which is in $\mathfrak{N}^+(V)$.

For the other action, we have :

$$(1 \otimes (b\sharp h)).x = \sum \phi(x_{(1)}, S(b\sharp h)) x_{(2)}.$$

From the definition of the crossed biproduct, when restricted to $\mathfrak{N}^+(V)$, the coproduct gives $\Delta : \mathfrak{N}^+(V) \to \mathfrak{B}_H(V) \otimes \mathfrak{N}^+(V)$, so the result is in $\mathfrak{N}^+(V)$.

Thus the action and coaction of $D_\phi(\mathfrak{B}_H(V))$ on $\mathfrak{N}^+(V)$ are both well defined and as a consequence, $\mathfrak{N}^+(V)$ is a $D_\phi(\mathfrak{B}_H(V))$-module algebra.

These structures are compatible because we have seen that the coaction defined above is just the restriction of the coproduct in $D_\phi(\mathfrak{B}_H(V))$ on $\mathfrak{N}^+(V)$. □

The same argument, once applied to $\mathfrak{N}^-(V)$, implies that $\mathfrak{N}^-(V)$ is a $D_\phi(\mathfrak{B}_H(V))$-Yetter-Drinfel'd module algebra.

As remarked after the definition of Yetter-Drinfel'd modules, we may use the natural braiding in the category ${}^{D_\phi}_{D_\phi}\mathcal{YD}$ to give $\mathfrak{N}^-(V) \otimes \mathfrak{N}^+(V)$ an associative algebra structure, which is denoted by $W_\phi(V)$ and is called the differential algebra of the Nichols algebra $\mathfrak{N}(V) = \mathfrak{N}^-(V)$.

This gives a natural action of $W_\phi(V)$ on $\mathfrak{N}^-(V)$, where $\mathfrak{N}^+(V) \subset W_\phi(V)$ acts by "differential".

Remark 4.10.

1. It should be pointed out that $\mathfrak{N}^-(V) \otimes \mathfrak{N}^+(V)$ is a subalgebra of $H_\phi(\mathfrak{B}_H(V))$. This can be obtained from the definition of the braiding σ in the category ${}^{D_\phi}_{D_\phi}\mathcal{YD}$ and the formula for the action of $D_\phi(\mathfrak{B}_H(V))$ on $\mathfrak{B}_H^-(V)$.Moreover, it is exactly the subalgebra of $H_\phi(\mathfrak{B}_H(V))$ generated by $\mathfrak{N}^-(V)$ and $\mathfrak{N}^+(V)$.
2. This action of $W_\phi(V)$ on $\mathfrak{N}^-(V)$ can be explained as follows : we consider the trivial $\mathfrak{N}^+(V)$-module $\mathbb{C}$ given by the counit ε, then

$$\mathrm{Ind}_{\mathfrak{N}^+(V)}^{W_\phi(V)}(\mathbb{C}.1) = W_\phi(V) \otimes_{\mathfrak{N}^+(V)} \mathbb{C}.1$$

is isomorphic to $\mathfrak{N}^-(V)$ as a vector space and from this, $\mathfrak{N}^-(V)$ can be regarded as a $W_\phi(V)$-module.
3. A direct construction of the differential algebra of a Nichols algebra without passing to the quantum double can be found in [4].

4.7.4 Non-degeneracy assumption

We should point out that results in previous sections do not depend on the non-degeneracy of the generalized Hopf pairing. Some results concerning with this property will be discussed in this subsection.

Recall the notation $\mathfrak{N}(V) = \mathfrak{N}^-(V)$. So $W_\phi(V)$ acts on $\mathfrak{N}(V)$ by : for $x \in \mathfrak{N}^+(V) \subset W_\phi(V)$ and $y \in \mathfrak{N}(V)$,

$$x.y = \sum \phi(x, y_{(1)})y_{(2)}.$$

Because the generalized Hopf pairing is graded and non-degenerate, results in [24] can be generalized to the present context.

Let $v_1, \cdots, v_n$ be a basis of V.

Lemma 4.8. Let $y \in \mathfrak{N}(V)$, $y \notin \mathbb{C}^*$ such that for any basis element $v_i \in \mathfrak{N}^+(V) \subset W_\phi(V)$, $v_i.y = 0$. Then $y = 0$.

Proposition 4.18. Let $y \in \mathfrak{N}(V)$ such that $y \neq 0$. Then there exists $x \in \mathfrak{N}^+(V) \subset W_\phi(V)$ such that $x.y$ is a non-zero constant.

4.8 Applications to Nichols algebras

At first, we want to show that the differential algebra above generalizes the skew-derivation defined by Nichols [67].

4.8.1 Derivations

We keep notations from previous sections and fix a basis $v_1, \cdots, v_n$ of V.

Definition 4.14. For $a \in \mathfrak{N}^+(V)$, we define the left derivation $\partial_a^L : \mathfrak{N}(V) \to \mathfrak{N}(V)$ by : for $y \in \mathfrak{N}(V)$, $\partial_a^L(y) = a.y$. If $a = v_i$, the notation ∂_i^L is adopted for $\partial_{v_i}^L$.

In the proposition below, we suppose that $\phi_1 : V \times V \to \mathbb{C}$ is given by the natural duality between V^* and V and the Nichols algebra is of diagonal type.

Proposition 4.19. For $i = 1, \cdots, n$, the definition of ∂_i^L above coincides with the one given in [34].

Proof. If the Nichols algebra is of diagonal type,

$$\partial_i^L(v_{i_1} \cdots v_{i_k}) = \sum \phi(v_i, (v_{i_1} \cdots v_{i_k})_{(1)})(v_{i_1} \cdots v_{i_k})_{(2)}.$$

From the definition of the coproduct in Nichols algebras, the fact that σ is of diagonal type and ϕ is graded, we obtain that terms satisfying

$$\phi(v_i, (v_{i_1} \cdots v_{i_k})_{(1)}) \neq 0$$

are exactly those given by the shuffle action $\mathfrak{S}_{1,k-1}$ in the coproduct formula. A simple calculation shows that

$$\mathfrak{S}_{1,k-1} = \{1, \sigma_1, \sigma_1\sigma_2, \cdots, \sigma_1 \cdots \sigma_{k-1}\},$$

which gives exactly the formula in the definition after Heckenberger. □

Remark 4.11. The advantage of our definition for differential operators on Nichols algebras are twofold :

1. This is a global and functorial construction, we never need to work in a specific coordinate system at the beginning;
2. We make no assumption on the type of the braiding, it has less restriction and can be applied to more general cases, for example : Hecke type, quantum group type, and so on.

Remark 4.12. In the same spirit, for $a \in \mathfrak{N}^+(V)$, the right differential operator ∂_a^R can be similarly defined by considering the right action : for $y \in \mathfrak{N}(V)$, $a.y = \sum y_{(1)}\phi(a, y_{(2)})$.

Some results from [34] can be generalized with simple proofs.

Lemma 4.9. Let $x \in \mathfrak{N}(V)$ and $a \in \mathfrak{N}^+(V)$. Then :

$$\Delta(\partial_a^L(x)) = \sum \partial_a^L(x_{(1)}) \otimes x_{(2)}, \quad \Delta(\partial_a^R(x)) = \sum x_{(1)} \otimes \partial_a^R(x_{(2)}).$$

Proof. We prove it for ∂_a^L :

$$\Delta(\partial_a^L(x)) = \Delta\left(\sum \phi(a, x_{(1)})x_{(2)}\right) = \sum \phi(a, x_{(1)})x_{(2)} \otimes x_{(3)} = \sum \partial_a^L(x_{(1)}) \otimes x_{(2)}.$$

□

Moreover, we have following results :

Lemma 4.10. For any $a, b \in \mathfrak{N}^+(V)$, $\partial_a^L\partial_b^R = \partial_b^R\partial_a^L$.

Lemma 4.11. For $x, y \in \mathfrak{N}(V)$ and $1 \leq i \leq n$, we have :

$$\partial_i^L(xy) = \partial_i^L(x)y + \sum x_{(0)}\phi((v_i)_{(-1)}, x_{(-1)})\partial^L_{(v_i)_{(0)}}(y).$$

Proof. At first, from the definition,

$$\Delta(xy) = \sum x_{(1)}((x_{(2)})_{(-1)}.y_{(1)}) \otimes (x_{(2)})_{(0)}y_{(2)}.$$

so the action of ∂_i^L gives :

$$\begin{aligned}
\partial_i^L(xy) &= \sum \phi(v_i, x_{(1)}((x_{(2)})_{(-1)}.y_{(1)}))(x_{(2)})_{(0)}y_{(2)} \\
&= \sum \left(\phi(v_i, x_{(1)})\varepsilon((x_{(2)})_{(-1)}.y_{(1)}) + \varepsilon(x_{(1)})\phi(v_i, (x_{(2)})_{(-1)}.y_{(1)})\right)(x_{(2)})_{(0)}y_{(2)} \\
&= \sum \phi(v_i, x)y + \phi(v_i, x_{(-1)}.y_{(1)})x_{(0)}y_{(2)} \\
&= \partial_i^L(x)y + \sum \phi((v_i)_{(-1)}, x_{(-1)})x_{(0)}\partial^L_{(v_i)_{(0)}}(y).
\end{aligned}$$

□

4.8.2 Taylor Lemma

This subsection is devoted to generalizing the Taylor Lemma in [40] to diagonal type Nichols algebras. We keep notations from the last subsection and suppose that $G = \mathbb{Z}^n$.

Remark 4.13. As remarked in the beginning of Section 4.7.4, derivations constructed in Section 4.8.1 can be similarly defined on $T(V)$.

For any $a \in T(V)$, we have defined $\partial_a^L : T(V) \to T(V)$, which will be denoted by ∂_a in this subsection.

From now on, we fix a homogeneous element $w \in T(V)$ of degree α (the degree given by $\mathbb{Z}^n$), denote $q_{\alpha,\alpha} = \chi(\alpha, \alpha)$ and

$$T(V)^{\partial_w} = \{v \in T(V) | \ \partial_w(v) = 0\}.$$

Remark 4.14. It is easy to see that if $w \in T(V)$ is a non-constant element, ∂_w is a locally nilpotent linear map because it decreases the degree when acts on an element.

Lemma 4.12 (Taylor Lemma)**.** Suppose that q is not a root of unity. If there exists some homogeneous element $a \in T(V)$ such that $\partial_w(a) = 1$, then a is free over $T(V)^{\partial_w}$ and as vector spaces, we have :

$$T(V) = T(V)^{\partial_w} \otimes_{\mathbb{C}} \mathbb{C}[a].$$

Proof. Recall that α is the degree of w.

Because $T(V)$ is $\mathbb{Z}^n$-graded and ∂_w is a linear map of degree $-\alpha$, we may suppose that a is homogeneous of degree α.

It is clear that $T(V)^{\partial_w} \otimes \mathbb{C}[a] \subset T(V)$. Now we prove the other inclusion.

Because we are working under the diagonal hypothesis, a simple computation gives that

$$\partial_w^n(a^n) = (n)_{q_{\alpha,\alpha}}!.$$

Then if for some $x_i \in T(V)^{\partial_w}$, $\sum x_i a^i = 0$, applying ∂_w sufficiently many times will force all x_i to be zero.

Let $x \in T(V)$ be a homogeneous element. Then $\partial_w(x)$ is also homogeneous. We let μ denote the degree of x and $n \in \mathbb{N}$ a positive integer such that $\partial_w^n(x) \neq 0$ but $\partial_w^{n+1}(x) = 0$. If $x \in T(V)^{\partial_w}$, the lemma is proved. Now we suppose that $x \notin T(V)^{\partial_w}$, which implies that $n > 0$.

We let λ denote the degree of $\partial_w^n(x)$ and $q_{\alpha,\lambda} = \chi(\alpha, \lambda)$, then

$$\partial_w^n(\partial_w^n(x)a^n) = q_{\alpha,\lambda}^n \partial_w^n(x)\partial_w^n(a^n).$$

If we define

$$X = x - \frac{1}{(n)_{q_{\alpha,\alpha}}!}\frac{1}{q_{\alpha,\lambda}^n}\partial_w^n(x)a^n,$$

then $X \equiv x \pmod{T(V)^{\partial_w} \otimes \mathbb{C}[a]}$ and

$$\partial_w^n(X) = \partial_w^n(x) - \frac{1}{(n)_{q_{\alpha,\alpha}}!}\frac{1}{q_{\alpha,\lambda}^n}q_{\alpha,\lambda}^n\partial_w^n(x)\partial_w^n(a^n) = 0.$$

So the lemma follows by induction on the nilpotent degree of x. □

For a homogeneous element $b \in T(V)$, let $\partial^\circ b$ denote its degree.

Remark 4.15. Let $w \in T(V)$ be an element such that it has non-zero image in $\mathfrak{N}(V)$. As the pairing we are considering is non-degenerate, the element a always exists if q is not a root of unity.

We denote $q_{ii} = \chi(\alpha_i, \alpha_i)$ and suppose that q is not a root of unity.

Theorem 4.5. Let $w = v_i$ for some $1 \leq i \leq n$ and $a \in T(V)$ be a homogeneous element satisfying $\partial_i(a) = 1$. We dispose $-\partial^\circ v_i$ the degree of a and define the adjoint action of $\mathbb{C}[a]$ on $T(V)^{\partial_i}$ by : for a homogeneous element $b \in T(V)^{\partial_i}$,

$$a \cdot b = ab - \chi(\partial^\circ a, \partial^\circ b)ba.$$

Then we can form the crossed product of $\mathbb{C}[a]$ and $T(V)^{\partial_i}$ with the help of this action, which is denoted by $T(V)^{\partial_i}\sharp\mathbb{C}[a]$. With this construction, the multiplication gives an isomorphism of algebra :

$$T(V) \cong T(V)^{\partial_i}\sharp\mathbb{C}[a].$$

Proof. At first, we should show that the action defined above preserves $T(V)^{\partial_i}$. Let $b \in T(V)^{\partial_i}$ and denote $q_{i,b} = \chi(\partial^\circ v_i, \partial^\circ b)$. Then

$$\begin{aligned} \partial_i(a \cdot b) &= \partial_i(ab) - q_{i,b}^{-1}\partial_i(ba) \\ &= \partial_i(a)b + \chi(\partial^\circ v_i, \partial^\circ a)a\partial_i(b) - q_{i,b}^{-1}\partial_i(b)a - q_{i,b}^{-1}q_{i,b}b\partial_i(a) \\ &= \partial_i(a)b - b\partial_i(a) = 0, \end{aligned}$$

where equations $\partial_i(b) = 0$ and $\partial_i(a) = 1$ are used. So the crossed product is well defined.

Note that a is primitive. We proceed to prove that the multiplication is a morphism of algebra : what needs to be demonstrated is that for any $m, n \in \mathbb{N}$ and homogeneous elements $x, y \in T(V)^{\partial_w}$,

$$(x \otimes a^m)(y \otimes a^n) = xa^m ya^n.$$

From the definition, it suffices to show that

$$(1 \otimes a^m)(y \otimes 1) = a^m y.$$

At first, it should be pointed out that from the definition, the action of $\mathbb{C}[a]$ on $T(V)^{\partial_i}$ is just the commutator coming from a braiding. If we let Φ denote the linear map in Theorem 4.1, then :

$$(1 \otimes a^m)(y \otimes 1) = (\Phi \otimes id)(\Delta(a^m)(y \otimes 1)).$$

So it suffices to prove that

$$m \circ (\Phi \otimes id)(\Delta(a^m)(y \otimes 1)) = a^m y.$$

We proceed to show this by induction.

For $m = 1$,

$$\begin{aligned} m \circ (\Phi \otimes id)(\Delta(a)(y \otimes 1)) &= m \circ (\Phi \otimes id)(ay \otimes 1 + \chi(\partial^\circ a, \partial^\circ y) y \otimes a) \\ &= ay - \chi(\partial^\circ a, \partial^\circ y) ya + \chi(\partial^\circ a, \partial^\circ y) ya \\ &= ay. \end{aligned}$$

For the general case, we denote $\Delta(a^{m-1})(y \otimes 1) = \sum x' \otimes x''$, then

$$\begin{aligned} & m \circ (\Phi \otimes id)(\Delta(a^m)(y \otimes 1)) \\ = \; & m \circ (\Phi \otimes id) \left((a \otimes 1 + 1 \otimes a) \left(\sum x' \otimes x'' \right) \right) \\ = \; & m \circ (\Phi \otimes id) \left(\sum a x' \otimes x'' + \sum \chi(\partial^\circ a, \partial^\circ x') x' \otimes a x'' \right) \\ = \; & \sum a \Phi(x') x'' - \sum \chi(\partial^\circ a, \partial^\circ x') \Phi(x') a x'' + \sum \chi(\partial^\circ a, \partial^\circ x') \Phi(x') a x'' \\ = \; & \sum a \Phi(x') x'' \\ = \; & a^m y, \end{aligned}$$

where the last equality comes from the induction hypothesis. □

Remark 4.16. In the theorem, we need to take the opposite of the degree of a because it acts as a differential operator, which has negative degree. But to get an isomorphism, a positive one is needed.

4.9 Primitive elements

This last section is devoted to giving a proof of Theorem 4.2.

At first, as in the construction of Section 4.7, let $\phi : \mathfrak{N}(V) \otimes \mathfrak{N}(V) \to \mathbb{C}$ be a non-degenerate pairing between Nichols algebras.

For $x \in T(V)$, we let $\Delta_{i,j}(x)$ denote the component of $\Delta(x)$ of bidegree (i, j).

Proposition 4.20. Let $x \in \ker(S_n)$ be a non-zero solution of equation $S_n x = 0$ of level n. Then for any $i = 1, \cdots, n$, $\partial_i^R(x) = 0$.

Proof. At first, we prove that $\partial_i^R(x) = 0$. From the definition of ∂_i^R and the fact that ϕ is graded, the possible non-zero terms in $\partial_i^R(x)$ are those belonging to $\Delta_{n-1,1}(x)$ in $\Delta(x)$.

From the definition of the coproduct, $\Delta_{n-1,1}$ corresponds to the action of the element

$$\sum_{\sigma \in \mathfrak{S}_{n-1,1}} T_\sigma.$$

It is clear that

$$\mathfrak{S}_{n-1,1} = \{1, \sigma_{n-1}, \sigma_{n-1}\sigma_{n-2}, \cdots, \sigma_{n-1} \cdots \sigma_1\},$$

so in fact, $\Delta_{n-1,1}$ corresponds to the part T_n in the decomposition of S_n. The condition of x being of level n means that $T_n x = 0$, thus

$$(id \otimes \phi(v_i, \cdot)) \circ \Delta_{n-1,1}(x) = 0,$$

and so $\partial_i^R(x) = 0$. □

Corollary 4.4. With the assumption in the last proposition, for any non-constant $a \in T(V)$, we have $\partial_a^R(x) = 0$.

Proposition 4.21. Let $x \in T(V)$ be a homogeneous element which is not a constant. If for any non-constant element $a \in T(V)$, $\partial_a^R(x) = 0$, then $\Delta(x) - x \otimes 1 \in T(V) \otimes \mathfrak{I}(V)$, $x \in \mathfrak{I}(V)$.

Proof. If for any $a \in T(V)$, $\partial_a^R(x) = 0$, then $\sum x_{(1)}\phi(a, x_{(2)}) = 0$ for any a. We choose $x_{(1)}$ to be linearly independent. If $x_{(2)}$ is not a constant, it must be in the right radical of ϕ, which is exactly $\mathfrak{I}(V)$.

So $\Delta(x) - x \otimes 1 \in T(V) \otimes \mathfrak{I}(V)$. We obtain that $x \in \mathfrak{I}(V)$ by applying $\varepsilon \otimes id$ on both sides. □

It is clear that if $x \in \ker(S_2)$ be a non-zero solution of $S_2 x = 0$ with level 2, then x is primitive.

Theorem 4.6. Let $n \geq 2$ and $x \in \ker(S_n)$ be a non-zero solution of equation $S_n x = 0$ with level n. Then x is primitive and it is in $Im(P_n)$.

Proof. The case $n = 2$ is clear.

Let $n > 3$ and x be a solution of equation $S_n x = 0$ with level n. It suffices to prove that $\Delta_{i,n-i}(x) = 0$ for any $2 \leq i \leq n-2$.

From the definition, components in $\Delta_{i,n-i}(x)$ can be obtained by acting a shuffle element on x, we want to show that $\Delta_{i,n-i}(x) = \sum x' \otimes x'' = 0$.

From Corollary 4.4, x is of level n implies that for any non-constant $a \in T(V)$, $\partial_a^R(x) = 0$. So from Proposition 4.21, $\Delta(x) - x \otimes 1 \in T(V) \otimes \mathfrak{I}(V)$ and then $x'' \in \mathfrak{I}(V)$, $S_{n-i}x'' = 0$. It is easy to see that there exists a positional embedding $\iota : \mathfrak{B}_{n-i} \hookrightarrow \mathfrak{B}_n$ such that

$$\iota(S_{n-i}) \left(\sum_{\sigma \in \mathfrak{S}_{i,n-i}} T_\sigma \right) (x) = 0,$$

which means that the equation $\iota(S_{n-i})v = 0$ has a non-zero solution in $\mathbb{C}[X_x]$. It contradicts Lemma 4.5. □

Corollary 4.5. Let $n \geq 2$ and E_n be the set of level n solutions of equation $S_n x = 0$ in $V^{\otimes n}$. Then E_n is a subspace of $T(V)$. If we denote $P = \bigoplus_{n \geq 2} E_n$, then P is a coideal and the ideal K generated by P in $T(V)$ is contained in $\mathfrak{I}(V)$.

Chapitre 5

Specialization of quantum groups : non-symmetrizable case

5.1 Introduction

5.1.1 Motivations

Quantized enveloping algebras (quantum groups) $U_q(\mathfrak{g})$ are constructed by V. Drinfel'd and M. Jimbo in the eighties of the last century in the aim of deforming the usual enveloping algebra associated to symmetrizable Kac-Moody Lie algebras $\mathfrak{g}$ to construct solutions of Yang-Baxter equations. They motivate numerous work in the last decade such as pointed Hopf algebras, canonical (crystal) bases, knot invariants, quiver representations and Hall algebras, (quantum) cluster algebras, Hecke algebras, quantum affine and toroidal algebras, and so on.

In the original definition of quantum groups in generators and relations, the symmetrizable condition is essential in writing down explicitly the quantized Serre relations. With this explicit expression, it is not difficult to construct a specialization map sending q to 1 to recover the enveloping algebra $U(\mathfrak{g})$. It is shown to be an isomorphism of Hopf algebra in Lusztig [55]. It should be remarked that the well-definedness of the specialization map depends on the knowledge of the quantized Serre relations and the Gabber-Kac theorem in Kac-Moody Lie algebras.

In a survey article [45], M. Kashiwara posed the following problem : has a crystal graph for non-symmetrizable $\mathfrak{g}$ a meaning ? He also remarked that the quantum group $U_q(\mathfrak{g})$ associated to a general Kac-Moody Lie algebra $\mathfrak{g}$ is not known at that time.

The first functorial (coordinate-free) construction of (the positive or negative part of) a quantum group appears in the work of M. Rosso [73] with the name "quantum shuffle algebras" and then explained in a dual language by Andruskiewitsch-Schneider [6] named "Nichols algebras". These constructions largely generalize the definition of the usual quantum group and can be applied in particular to the non-symmetrizable case to obtain a half of the quantum group. Then the machinery of the quantum double construction can be used to combine the positive and negative parts to yield the whole quantum group.

Compared with the definition in generators and relations, although having many advantages, this construction does not fit for studying the specialization problem as neither defining relations of the quantum group nor explicit relations in the Kac-Moody Lie algebra is known in the non-symmetrizable case.

This chapter is devoted to understanding the specialization problem in the non-symmetrizable case through studying the defining ideal of the corresponding Nichols algebra.

5.1.2 Defining ideals in Nichols algebras

Let (V, σ) be a braided vector space. The tensor algebra $T(V)$ admits a braided Hopf algebra structure by imposing a coproduct making elements in V primitive ; it can be then generalized to the entire $T(V)$ in replacing the usual flip by the braiding.

If the braiding σ comes from an H-Yetter-Drinfel'd module structure on V over a Hopf algebra H, the Nichols algebra can be defined as the quotient of $T(V)$ by some maximal ideal and coideal $\mathfrak{I}(V)$ contained in the set of elements of degree no less than 2. We call $\mathfrak{I}(V)$ the defining ideal of the Nichols algebra $\mathfrak{N}(V)$.

As an example, for a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, the negative part $U_q^-(\mathfrak{g})$ of the corresponding quantum group is a Nichols algebra, in which case the defining ideal $\mathfrak{I}(V)$ is generated as a Hopf ideal by quantized Serre relations. In general, it is very difficult to find out a minimal generating set of $\mathfrak{I}(V)$ as a Hopf ideal in $T(V)$.

In [2], Andruskiewitsch posed some problems which guide the researches of this domain and the following ones concerning defining ideals appear therein :

1. For those $\mathfrak{N}(V)$ having finite Gelfan'd-Kirillov dimension, decide a minimal generating set of $\mathfrak{I}(V)$.
2. When is the ideal $\mathfrak{I}(V)$ finitely generated ?

The first general result on the study of the defining ideal is due to M. Rosso [73] and P. Schauenburg [79] who characterizes it as the kernel of the total symmetrization operators.

In [25], we proposed the notion of "level n" elements with the help of a decomposition of the total symmetrization operator in the braid group and proved the primitivity of them. These elements could be easily computed and the degrees where they appear are strongly restricted. This construction works for Nichols algebra of arbitrary type, but we must pay the price that they may not generate the defining ideal, which is essentially due to the strong condition we have posed.

Once restricted to some particular cases, with a slight modification on the conditions of "level n" elements, we will show that it gives indeed a generating set.

5.1.3 Main ideas and results

The main part of this chapter is devoted to proposing some methods in the study of a slightly modified version of the problems above. Firstly, we will restrict ourselves to the Nichols algebras of diagonal type having infinite Gelfan'd-Kirillov dimensions. Secondly, our principle has a pragmatic feature : we do not always desire a minimal generating set of the defining ideal but are satisfied with finding generating subsets fitting for solving concrete problems.

We propose four subsets of the defining ideal $\mathfrak{I}(V)$: left and right constants, left and right pre-relations. The first two sets are defined as the intersection of kernels of left and right differential operators and the last two are their subsets obtained by selecting elements which are simultaneously contained in images of the Dynkin operators and fixed by the action of the centre of the braid group.

Two main results (Theorem 5.1 and Theorem 5.2) in this chapter affirm that they are all generating set of the defining ideal.

These results are then applied to the study of the specialization problem. In general, if the generalized Cartan matrix C is not symmetric, we show in a counterexample that the natural specialization map may not be well-defined. So in our approach, the first step is passing to a symmetric matrix $Av(C)$ by taking the average of the Cartan matrix. A result due to Andruskiewitsch and Schneider ensures that this procedure does not lose too many information.

Once passed to the averaged matrix, we prove in Theorem 5.3 that the specialization map $U_q(Av(C)) \to U(\mathfrak{g}(Av(C)))$ is well-defined and is surjective.

As the other application, we relate the degrees where pre-relations may appear with integral points of some quadratic forms arising from the action of the centre of the braid group. This allows us to reprove some well-known results in a completely different way which we hope could shed light on the finite generation problem of $\mathfrak{I}(V)$.

5.1.4 Constitution of this chapter

After giving some recollections on Nichols algebras and braid groups in Section 5.2 and 5.3, we define the constants and pre-relations in Section 5.4 and 5.5 and show that they are indeed generating sets. These results are then applied to study the specialization problem in Section 5.6 and 5.7. Another application to the finitely generating property is given in Section 5.8.

5.2 Recollections on Nichols algebras

Let k be an algebraically closed field of characteristic 0. All algebras and vector spaces, if not specified, are over the field k.

5.2.1 Nichols algebras

Let H be a Hopf algebra and ${}^H_H\mathcal{YD}$ be the category of H-Yetter-Drinfel'd modules ([6]).

Definition 5.1 ([6])**.** A graded braided Hopf algebra $R = \bigoplus_{n=0}^{\infty} R(n)$ is called a Nichols algebra of V if

1. $R(0) \cong k$, $R(1) \cong V \in {}^H_H\mathcal{YD}$;
2. R is generated as an algebra by $R(1)$;
3. $R(1)$ is the set of all primitive elements in R.

We let $\mathfrak{N}(V)$ denote this braided Hopf algebra.

A concrete construction of Nichols algebras can be given as a quotient of the braided Hopf algebra $T(V)$.

Example 5.1 ([6])**.** Let $V \in {}_H^H\mathcal{YD}$ be an H-Yetter-Drinfel'd module. There exists a braided tensor Hopf algebra structure on the tensor algebra

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

1. The multiplication on $T(V)$ is given by the concatenation.
2. The coalgebra structure is defined on V by : for any $v \in V$, $\Delta(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon(v) = 0$. Then it can be extended to the whole $T(V)$ by the universal property of $T(V)$ as an algebra.

For $k \geq 2$, let $T^{\geq k}(V) = \bigoplus_{n \geq k} V^{\otimes n}$ and $\mathfrak{I}(V)$ be the maximal coideal of $T(V)$ contained in $T^{\geq 2}(V)$. Then $\mathfrak{I}(V)$ is also a two-sided ideal ; the Nichols algebra $\mathfrak{N}(V)$ associated with V is isomorphic to $T(V)/\mathfrak{I}(V)$ as a braided Hopf algebra. We let S denote the convolution inverse of the identity map on $\mathfrak{N}(V)$.

5.2.2 Nichols algebras of diagonal type

Let $G = \mathbb{Z}^N$ be an abelian group and $H = k[G]$ be its group algebra. We let $\widehat{G}$ denote the character group of G. Let $V \in {}_H^H\mathcal{YD}$ be an H-Yetter-Drinfel'd module of dimensional N. Then V has a decomposition into linear subspaces $V = \bigoplus_{g \in G} V_g$ where $V_g = \{v \in V |\ \delta(v) = g \otimes v\}$.

We let $I = \{1, \cdots, N\}$ denote the index set.

Definition 5.2. Let $V \in {}_H^H\mathcal{YD}$ be an H-Yetter-Drinfel'd module of dimension N. V is called of diagonal type if there exists a basis $\{v_1, \cdots, v_N\}$ of V, elements $g_1, \cdots, g_N \in G$ and characters $\chi_1, \cdots, \chi_N \in \widehat{G}$ such that $v_i \in V_{g_i}$ and for any $g \in G$,

$$g.v_i = \chi_i(g) v_i.$$

We call $T(V)$ and $\mathfrak{N}(V)$ of diagonal type if V is so.

If $V \in {}_H^H\mathcal{YD}$ is of diagonal type, the braiding $\sigma_{V,V}$ has an explicit form : for $1 \leq i, j \leq N$,

$$\sigma_{V,V}(v_i \otimes v_j) = \chi_j(g_i) v_j \otimes v_i.$$

As a consequence, $\sigma_{V,V}$ is completely determined by the matrix $(\chi_j(g_i))_{1 \leq i,j \leq N} \in M_N(k)$. We denote $q_{ij} = \chi_j(g_i)$ and call $(q_{ij})_{1 \leq i,j \leq N}$ the braiding matrix associated to $\sigma_{V,V}$.

Let $k^\times = k \backslash \{0\}$. For an arbitrary matrix $A = (q_{ij}) \in M_N(k)$ with coefficients in $k^\times$, we let $\mathfrak{N}(V_A)$ denote the Nichols algebra associated to the H-Yetter-Drinfel'd module V of diagonal type with the braiding matrix A. If the matrix A under consideration is fixed, we denote it by $\mathfrak{N}(V)$ for short.

5.2.3 Differential operators

Let $v_1, \cdots, v_N$ be a basis of V as fixed in the last subsection.

Definition 5.3 ([47])**.** Let A and B be two Hopf algebras with invertible antipodes. A generalized Hopf pairing between A and B is a bilinear form $\varphi : A \times B \to k$ such that :

1. For any $a \in A$, $b, b' \in B$, $\varphi(a, bb') = \sum \varphi(a_{(1)}, b)\varphi(a_{(2)}, b')$;
2. For any $a, a' \in A$, $b \in B$, $\varphi(aa', b) = \sum \varphi(a, b_{(2)})\varphi(a', b_{(1)})$;
3. For any $a \in A$, $b \in B$, $\varphi(a, 1) = \varepsilon(a)$, $\varphi(1, b) = \varepsilon(b)$.

Let φ be a generalized Hopf pairing on $T(V)$ such that $\varphi(v_i, v_j) = \delta_{ij}$ (Kronecker delta notation). This pairing is in general not non-degenerate and its kernel is given by the defining ideal $\mathfrak{I}(V)$, so it may pass the quotient to give a non-degenerate generalized Hopf pairing on $\mathfrak{N}(V)$ (see, for example, Section 3.2 in [3] for details).

Definition 5.4 ([25])**.** For any $a \in T(V)$, the left and right differential operators associated to the element a are defined by :

1. $\partial_a^L : T(V) \to T(V)$, $\partial_a^L(x) = \sum \varphi(a, x_{(1)})x_{(2)}$;
2. $\partial_a^R : T(V) \to T(V)$, $\partial_a^R(x) = \sum x_{(1)}\varphi(a, x_{(2)})$.

If $a = v_i$ for some $i \in I$, we will denote them by ∂_i^L and ∂_i^R, respectively.

These differential operators descend to $\mathfrak{N}(V)$ to give an endomorphism of $\mathfrak{N}(V)$. We keep the notations ∂_a^L and ∂_a^R for this induced endomorphsm.

The following known lemma, whose proof is trivial, will be useful. It also holds when $T(V)$ is replaced by $\mathfrak{N}(V)$.

Lemma 5.1. 1. For any $a, x \in T(V)$, we have :

$$\Delta(\partial_a^L(x)) = \sum \partial_a^L(x_{(1)}) \otimes x_{(2)}, \quad \Delta(\partial_a^R(x)) = \sum x_{(1)} \otimes \partial_a^R(x_{(2)}).$$

2. For any $a, b \in T(V)$, $\partial_a^L \partial_b^R = \partial_b^R \partial_a^L$ holds.

5.3 Identities in braid groups

5.3.1 Braid groups

We suppose that $n \geq 2$ is an integer. Let $\mathfrak{S}_n$ denote the symmetric group : it acts on an alphabet with n letters by permuting their positions. It can be generated by the set of transpositions $\{s_i = (i, i+1) |\ 1 \leq i \leq n-1\}$.

Let $\mathfrak{B}_n$ denote the braid group of n strands. It is defined by generators σ_i for $1 \leq i \leq n-1$ and relations :

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for } |i-j| \geq 2; \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for } 1 \leq i \leq n-2.$$

Let $\sigma \in \mathfrak{S}_n$ and $\sigma = s_{i_1} \cdots s_{i_r}$ be a reduced expression of σ. We denote the corresponding lifted element $T_\sigma = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathfrak{B}_n$. This gives a linear map $T : k[\mathfrak{S}_n] \to k[\mathfrak{B}_n]$ called Matsumoto section.

5.3.2 Defining ideals

The total symmetrization operator in $k[\mathfrak{B}_n]$ is defined by :

$$S_n = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \in k[\mathfrak{B}_n].$$

As $V \in {}^H_H\mathcal{YD}$, $\mathfrak{B}_n$ acts naturally on $V^{\otimes n}$. We look S_n as a linear operator in $\mathrm{End}(V^{\otimes n})$.

Proposition 5.1 ([73], [79])**.** Let V be an H-Yetter-Drinfel'd module. Then

$$\mathfrak{N}(V) = \bigoplus_{n \geq 0} \left(V^{\otimes n} / \ker(S_n) \right).$$

After this proposition, $\mathfrak{N}(V)$ can be viewed as imposing some relations in $T(V)$, locating defining relations of $\mathfrak{N}(V)$ can be reduced to the study of each subspace $\ker(S_n)$.

5.3.3 Particular elements in braid groups and their relations

We start by introducing some particular elements in the group algebra of braid groups.

Definition 5.5. Let $n \geq 2$ be an integer. We define the following elements in $k[\mathfrak{B}_n]$:

$$\text{Garside element} : \Delta_n = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1;$$

$$\text{Central element} : \theta_n = \Delta_n^2;$$

$$\text{Right differential element} : T_n = 1 + \sigma_{n-1} + \sigma_{n-1}\sigma_{n-2} + \cdots + \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1;$$

$$\text{Right Dynkin element} : P_n = (1 - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(1 - \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(1 - \sigma_{n-1});$$

$$T_n' = (1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2)\cdots(1 - \sigma_{n-1}^2);$$

$$\text{Left differential element} : U_n = 1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2\cdots\sigma_{n-1};$$

$$\text{Left Dynkin element} : Q_n = (1 - \sigma_1\sigma_2\cdots\sigma_{n-1})(1 - \sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(1 - \sigma_1);$$

$$U_n' = (1 - \sigma_1^2\sigma_2\cdots\sigma_{n-1})(1 - \sigma_1^2\sigma_2\cdots\sigma_{n-2})\cdots(1 - \sigma_1^2).$$

We give a summary for some known results on the relations between these elements :

Proposition 5.2 ([25], [48])**.** The following identities hold :

1. for $n \geq 3$, $Z(\mathfrak{B}_n)$, the centre of $\mathfrak{B}_n$, is generated by θ_n ;
2. for any $1 \leq i \leq n-1$, $\sigma_i\Delta_n = \Delta_n\sigma_{n-i}$;
3. $\theta_n = \Delta_n^2 = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^n = (\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)^{n-1}$;
4. $\left(\sum_{k=0}^{n-2}(\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)^k\right)(1 - \sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1) = 1 - \Delta_n^2 = 1 - \theta_n$;
5. $S_n = T_2T_3\cdots T_n = U_2U_3\cdots U_n$;
6. $T_nP_n = T_n',\ U_nQ_n = U_n'$.

We fix the notation for the embeddings of braid groups in a fixed position.

Definition 5.6. For $m \geq 3$ and $2 \leq k \leq m-1$, we let $\iota_k^m : \mathfrak{B}_k \to \mathfrak{B}_m$ denote the group homomorphism satisfying $\iota_k^m(\sigma_i) = \sigma_{m-k+i}$.

5.3.4 Relations with differential operators

The following lemma explains the relation between the operator T_n and the differential operator ∂_i^R defined in the Section 5.2.3.

Lemma 5.2. Let $x \in T^n(V)$. The following statements are equivalent :

1. $T_n x = 0$;
2. For any $i \in I$, $\partial_i^R(x) = 0$.

Proof. It comes from the following identity, which is clear from definition : for any $x \in T^n(V)$,

$$T_n x = \sum_{i \in I} \partial_i^R(x) v_i.$$

□

Remark 5.1. The same result holds for left operators : $U_n x = 0$ if and only if for any $i \in I$, $\partial_i^L(x) = 0$.

5.3.5 Tensor space representation of $\mathfrak{B}_n$

We call $\underline{i} = (i_1, \cdots, i_m) \in \mathbb{N}^m$ a partition of n, denote by $\underline{i} \vdash n$, if $i_1 + \cdots + i_m = n$.

As V is an H-Yetter-Drinfel'd module, the braid group $\mathfrak{B}_n$ acts on $V^{\otimes n}$, making it a $k[\mathfrak{B}_n]$-module.

As the braiding is of diagonal type, we have the following decomposition of $V^{\otimes n}$ into its submodules :

$$V^{\otimes n} = \bigoplus_{\underline{i} \in I} k[\mathfrak{B}_n].v_1^{i_1} \cdots v_m^{i_m}, \tag{5.1}$$

where $I = \{\underline{i} = (i_1, \cdots, i_m) |\ \underline{i} \vdash n\}$.

To simplify the notation, for $\underline{i} = (i_1, \cdots, i_m)$, we denote $v_{\underline{i}} = v_1^{i_1} \cdots v_m^{i_m}$ and the $k[\mathfrak{B}_n]$-module $k[X_{\underline{i}}] = k[\mathfrak{B}_n].v_{\underline{i}}$.

We let $\mathcal{H}$ denote the set of invariants $(V^{\otimes n})^{\theta_n}$ under the action of the central element θ_n. As $\theta_n \in Z(\mathfrak{B}_n)$, $\theta_n v_{\underline{i}} = v_{\underline{i}}$ implies that $k[X_{\underline{i}}] \subset \mathcal{H}$. Moreover, there exists some subset $J \subset I$ such that

$$\mathcal{H} = \bigoplus_{\underline{i} \in J} k[X_{\underline{i}}]$$

(see the argument in the Section 6.1 of [25]).

We end this subsection by the following remark, which will be frequently used in the following discussions.

Remark 5.2. Let $\underline{i} \vdash n$, $x \in k[X_{\underline{i}}]$ and $v \in V$. Then $(\mathrm{id} - \sigma_n \cdots \sigma_1)(vx)$ is in the ideal generated by x.

To show this, notice that the coefficient λ such that $\sigma_n \cdots \sigma_1(vx) = \lambda xv$ only depends on the chosen partition $\underline{i}$, so is a constant for any $x \in k[X_{\underline{i}}]$.

5.3.6 Defining ideals of degree 2

Elements of degree two in the defining ideal can be tackled by hand. They are characterized by the following proposition :

Proposition 5.3. The following statements are equivalent :

1. $q_{ij}q_{ji} = 1$;
2. $v_iv_j - q_{ij}v_jv_i \in \ker S_2$;
3. $\theta_2(v_iv_j) = v_iv_j$.

Proof. It suffices to prove that (2) is equivalent to (3). Notice that $v_iv_j - q_{ij}v_jv_i = P_2(v_iv_j)$ and $S_2 = T_2$. Then $T_2P_2(v_iv_j) = 0$ if and only if $T_2'(v_iv_j) = 0$ if and only if $\theta_2(v_iv_j) = v_iv_j$. □

5.4 Another characterization of $\mathfrak{I}(V)$

In this section, we give a characterization for a generating set of the defining ideal $\mathfrak{I}(V)$ using kernels of operators T_n. This is motivated by the work of Frønsdal and Galindo [29]. In fact, we could use the left or right differential operators to give a complete characterization of elements generating the defining ideal.

The following definition is due to Frønsdal-Galindo [29] :

Definition 5.7. An element $w \in T^n(V)$ is called a right (left) constant of degree n if $T_nx = 0$ ($U_nx = 0$). We let Con_n^R (Con_n^L) denote the vector space generated by all right (left) constants of degree n and for any $m \geq 2$,

$$\mathrm{Con}_{\leq m}^R = \mathrm{span}\left(\bigcup_{2\leq n\leq m} \mathrm{Con}_n^R\right), \quad \mathrm{Con}^R = \mathrm{span}\left(\bigcup_{n\geq 2} \mathrm{Con}_n^R\right),$$

$$\mathrm{Con}_{\leq m}^L = \mathrm{span}\left(\bigcup_{2\leq n\leq m} \mathrm{Con}_n^L\right), \quad \mathrm{Con}^L = \mathrm{span}\left(\bigcup_{n\geq 2} \mathrm{Con}_n^L\right),$$

where the notation $\mathrm{span}(X)$ stands for the vector space generated by the set X.

The main technical tool is the following non-commutative version of the Taylor lemma for the diagonal braiding.

Lemma 5.3 (Taylor Lemma, [29]). 1. (Left version) For any integer $l \geq 1$ and $\underline{i} = (i_1, \cdots, i_l) \in \{1, \cdots, l\}^l$, there exists

$$A^{\underline{i}} = \sum_{\sigma\in\mathfrak{S}_l} A^\sigma v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(l)}} \in T^l(V)$$

with $A^\sigma \in k$ such that for any $x \in T^m(V)$,

$$x = c(x) + \sum_{l\geq 1}\sum_{\underline{i}\in\{1,\cdots,l\}^l} A^{\underline{i}}\partial_{i_1}^L \cdots \partial_{i_l}^L(x),$$

where $c(x) \in T^m(V)$ satisfying $\partial_i^L(c(x)) = 0$ for any $i \in I$.

2. (Right version) For any integer $l \geq 1$ and $\underline{i} = (i_1, \cdots, i_l) \in \{1, \cdots, l\}^l$, there exists
$$B^{\underline{i}} = \sum_{\sigma \in \mathfrak{S}_l} B^\sigma v_{i_{\sigma(1)}} \cdots v_{i_{\sigma(l)}} \in T^l(V)$$
with $B^\sigma \in k$ such that for any $x \in T^m(V)$,
$$x = d(x) + \sum_{l \geq 1} \sum_{\underline{i} \in \{1, \cdots, l\}^l} \partial_{i_1}^R \cdots \partial_{i_l}^R(x) B^{\underline{i}},$$
where $d(x) \in T^m(V)$ satisfying $\partial_i^R(d(x)) = 0$ for any $i \in I$.

Lemma 5.4. For any $m \geq 2$, $\mathrm{Con}_{\leq m}^L$ and $\mathrm{Con}_{\leq m}^R$ are coideals in the coalgebra $T^{\leq m}(V)$.

Proof. We prove it for $\mathrm{Con}_{\leq m}^R$. Let $x \in \ker T_n$ for some n. Then for any $i \in I$, $\partial_i^R(x) = 0$, which implies that
$$0 = \Delta(\partial_i^R(x)) = \sum x_{(1)} \otimes \partial_i^R(x_{(2)})$$
and then for any $i \in I$, $\partial_i^R(x_{(2)}) = 0$. This gives $\Delta(x) - x \otimes 1 \in T^{\leq m}(V) \otimes \mathrm{Con}_{\leq m}^R$ and
$$\Delta(x) \in \mathrm{Con}_{\leq m}^R \otimes T^{\leq m}(V) + T^{\leq m}(V) \otimes \mathrm{Con}_{\leq m}^R.$$
□

For a ring R and a subset $X \subset R$, we let $< X >_{\mathrm{ideal}}$ denote the ideal in R generated by X.

Theorem 5.1. For any $m \geq 2$, let
$$R^m = \left\langle \mathrm{Con}_{\leq m}^R \right\rangle_{\mathrm{ideal}} \cap T^m(V).$$
Then $R^m = \ker S_m$.

Proof. Since $S_r = T_2 T_3 \cdots T_r$ and $\ker(T_r : T^r(V) \to T^r(V)) \subset \ker S_r \subset \mathfrak{I}(V)$, the inclusion $R^m \subset \ker S_m$ comes from the fact that $\mathfrak{I}(V) = \bigoplus_{m \geq 2} \ker S_m$ is an ideal. It suffices to prove the other inclusion. Let $x \in \ker S_m$. We prove that $x \in R^m$ by induction on m. The case $m = 2$ is clear as $T_2 = S_2$.

Suppose that for any $2 \leq k \leq m-1$, $R^k = \ker S_k$. We affirm that it suffices to show that if for any $i \in I$, $\partial_i^R(x) \in \ker S_{m-1}$, then $x \in R^m$. Indeed, for an element $x \in \ker S_m$, there are two cases :

1. $T_m x = 0$; in this case, $x \in R^m$ is clear from definition.
2. $T_m x \neq 0$; from the decomposition of S_n, $T_m x \in \ker S_{m-1}$, which implies that for any $i \in I$, $\partial_i^R(x) \in \ker S_{m-1}$. The proof will be terminated if the affirmation above is proved.

We proceed to show the affirmation above. The following lemma is needed.

Lemma 5.5. For any $k \geq 3$, if $x \in R^k$, then $\partial_i^R(x) \in R^{k-1}$ for any $i \in I$.

We continue the proof of the theorem. Let $x \in T^m(V)$ satisfying that for any $i \in I$, $\partial_i^R(x) \in \ker S_{m-1} = R^{m-1}$. From the right version of the Taylor lemma,

$$x = d(x) + \sum_{l \geq 1} \sum_{\underline{i} \in \{1, \cdots, l\}^l} \partial_{i_1}^R \cdots \partial_{i_l}^R (x) B^{\underline{i}}.$$

The first term $d(x)$ in the right hand side satisfies $T_m(d(x)) = 0$ so it is in R^m. Moreover, the hypothesis on $\partial_i^R(x)$ and the lemma above force $\partial_{i_1}^R \cdots \partial_{i_n}^R(x)$ to be in R^{m-n}, so the second term is in R^m. □

Now it suffices to prove the lemma.

Proof of the lemma. It suffices to deal with the case where $x = urw \in R^k$ such that $r \in \ker T_s \cap k[X_{\underline{i}}]$ for some $\underline{i} \vdash s$, $u \in T^p(V)$ and $w \in T^q(V)$ satisfying $k = s + p + q$.

We have the following decomposition of T_k : let

$$T_k^1 = 1 + \sigma_{k-1} + \sigma_{k-1}\sigma_{k-2} + \cdots + \sigma_{k-1} \cdots \sigma_{p+s+1},$$

$$T_k^2 = \sigma_{k-1} \cdots \sigma_{p+s}(\iota_s^{p+s}(T_s)),$$

$$T_k^3 = \sigma_{k-1} \cdots \sigma_{p-1} + \cdots + \sigma_{k-1} \cdots \sigma_1.$$

Then $T_k = T_k^1 + T_k^2 + T_k^3$.

It is clear that $T_k^2 x = 0$. On the other hand, $T_k^1 x$ and $T_k^3 x$ are in R^k as from Remark 5.2, they are in the ideal generated by r. Moreover, it should be remarked that in $T_k x$, r is always contained in the first $k-1$ tensor terms. This is true from the definition of T_k^1 and T_k^3.

As a conclusion, we have shown that $T_k x \in R^k$, so $\partial_i^R(x) \in R^{k-1}$ for any $i \in I$, after the relation between T_k and ∂_i^R given in the proof of Lemma 5.2. □

In the proof of the previous theorem, we have already shown the following proposition, which can be looked as a sort of "invariance under integration".

Proposition 5.4. Let $x \in T^m(V)$ such that for any $i \in I$, $\partial_i^R(x) \in R^{m-1}$. Then $x \in R^m$.

Combined this proposition with the lemma above, we obtain the following characterization of R^m or $\ker S_m$.

Corollary 5.1. For $x \in T^m(V)$, the following statements are equivalent :

1. $x \in R^m$;
2. for any $i \in I$, $\partial_i^R(x) \in R^{m-1}$.

These results are still correct when the prefix "right" is replaced by "left". The proof above can be adapted by using the left version of the Taylor lemma. We omit these statements but end with the following corollary.

Corollary 5.2. Let

$$L^m = \left\langle \mathrm{Con}_{\leq m}^L \right\rangle_{\mathrm{ideal}} \cap T^m(V).$$

Then $R^m = L^m = \ker S_m$.

As a conclusion, to find the generating relations, it suffices to consider those in the intersection of $\ker \partial_i^R$ for all $i \in I$, or the intersection of $\ker \partial_i^L$ for all $i \in I$. We will see in the next section a sharper result permitting us to get more constraints.

Remark 5.3. Globally, there is no difference between the left and right cases. But it should be remarked that an element annihilated by all right differentials is not necessarily contained in the kernel of all ∂_i^L. We will return to this problem in Section 5.5.2.

The following lemma is an easy consequence of Lemma 5.1.

Lemma 5.6. For any $i \in I$ and $m \geq 3$, ∂_i^R (∂_i^L) sends $\mathrm{Con}_{\leq m}^L$ ($\mathrm{Con}_{\leq m}^R$) to $\mathrm{Con}_{\leq m-1}^L$ ($\mathrm{Con}_{\leq m-1}^R$).

5.5 Defining relations in the diagonal type

5.5.1 More constraints

In this subsection, we propose an another set of generators in $\mathfrak{I}(V)$ by giving more constraints on the left and right constants, these constraints give a restriction on the degrees where generators may appear. It should be mentioned that under the finite dimensional assumption, a generating set of the defining ideal is found recently by I. Angiono ([7], [8]). We start from some motivations of the main definition.

Proposition 5.5 ([25])**.** The Hopf ideal in ${}^H_H\mathcal{YD}$ generated by $\bigoplus_{n\geq 2} (\ker(S_n) \cap \mathrm{Im}(P_n))$ is $\mathfrak{I}(V)$.

The main result in the last section could be used to give more constraints.

Corollary 5.3. The Hopf ideal in ${}^H_H\mathcal{YD}$ generated by $\bigoplus_{n\geq 2} (\ker(T_n) \cap \mathrm{Im}(P_n))$ is $\mathfrak{I}(V)$.

From this corollary, to find relations imposed in $\mathfrak{N}(V)$, it suffices to concentrate on elements in $\mathrm{Im}(P_n)$ which are killed by all right differentials ∂_i^R.

After Proposition 5.2, to find a generating set of $\mathfrak{I}(V)$, it suffices to consider the solution of the equation $T_n' x = T_n P_n x = 0$ in $T^n(V)$. This observation motivates the following definition :

Definition 5.8. We call a non-zero element $v \in T^n(V)$ a right pre-relation of degree n if

1. $T_n v = 0$ and $\iota_{n-1}^n(T_{n-1}')v \neq 0$;
2. $v \in \mathrm{Im}(P_n)$;
3. $\theta_n v = v$.

Let Rel_r^n denote the vector space generated by all right pre-relations of degree n and Rel_r denote the vector space generated by $\bigcup_{n\geq 2} \mathrm{Rel}_r^n$. Elements in Rel_r are called right pre-relations.

We can similarly define left pre-relations of degree n by replacing T_n by U_n, T'_{n-1} by U'_{n-1} and P_n by Q_n in the definition above. Let Rel_l^n denote the vector space generated by all left pre-relations of degree n and Rel_l denote the vector space generated by $\bigcup_{n\geq 2}\mathrm{Rel}_l^n$. Elements in Rel_l are called left pre-relations.

Remark 5.4. We call them pre-relations as they may be redundant.

The main result of this section is the following theorem.

Theorem 5.2. The Hopf ideal generated by Rel_r is $\mathfrak{I}(V)$.

The rest of this subsection is devoted to giving the proof of this theorem. The main idea of the proof is to exclude redundant elements in the union of all $\ker T_n$.

Recall the definition of T'_n :

$$T'_n = (1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2)\cdots(1-\sigma_{n-1}^2\sigma_{n-2})(1-\sigma_{n-1}^2).$$

We define the following elements in $k[\mathfrak{B}_n]$:

$$X_{m,n} = (1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_{n-m})\cdots(1-\sigma_{n-1}^2\sigma_{n-2})(1-\sigma_{n-1}^2) = \iota_{m+1}^n(T'_{m+1}).$$

Then $X_{1,n} = (1-\sigma_{n-1}^2)$ and $X_{n-1,n} = T'_n$.

Lemma 5.7. If $T'_n v = 0$ and $X_{n-2,n}v \neq 0$, then $\theta_n v = v$.

Proof. From definition, $T'_n v = (1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)X_{n-2,n}v$. If $X_{n-2,n}v \neq 0$, it will be a solution of the equation $(1-\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)x = 0$. After Proposition 5.2, multiplying both sides by $\sum_{k=0}^{n-2}(\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1)^k$ gives $\theta_n X_{n-2,n}v = X_{n-2,n}v$. This implies $\theta_n v = v$ after the argument in Section 5.3.5. □

Corollary 5.4. If $v \in T^n(V)$ such that $T'_n v = 0$ and for any $2 \leq k \leq n-1$, $\iota_k^n(\theta_k)v \neq v$, then $\theta_n v = v$.

Proof. From the lemma above, it suffices to show that $X_{n-2,n}v \neq 0$. Otherwise, take the smallest k such that $X_{k-1,n}v \neq 0$ but $X_{k,n}v = 0$. As $X_{k,n} = \iota_{k+1}^n(T'_{k+1})$, the lemma above can be applied to this case and it gives $\iota_{k+1}^n(\theta_{k+1})v = v$. This contradicts the hypothesis. □

Now let $v \in T^n(V)$ be a solution of the equation $T'_n x = X_{n-1,n}x = 0$. There are two possibilities :

1. $X_{n-2,n}v \neq 0$. As we have shown in the lemma above, $\theta_n v = v$. So $P_n v$ is a right pre-relation.
2. $X_{n-2,n}v = 0$. In this case, there exists a smallest k such that $X_{k-1,n}v \neq 0$ but $X_{k,n}v = 0$. As a corollary, we have $\iota_{k+1}^n(\theta_{k+1})v = v$.

We would like to show that only relations falling into the first case are interesting. To be more precise, if v falls into the second case, $P_n v$ can be generated by lower degree elements in the first case. This is the following lemma.

Lemma 5.8. If $v \in T^n(V)$ is an element such that $T'_n v = 0$ and $X_{n-2,n}v = 0$, then $P_n v$ is in the ideal generated by right pre-relations of lower degrees.

Proof. We apply induction on n. There is nothing to prove in the case $n = 2$.

Let $v \in T^n(V)$ such that $T'_n v = 0$ and $X_{n-2,n} v = 0$. Let k be the smallest integer such that $X_{k-1,n} v \neq 0$ but $X_{k,n} v = 0$.

After the definition of P_n, the following identity holds :

$$P_n v = (1 - \sigma_{n-1} \cdots \sigma_1) \cdots (1 - \sigma_{n-1} \cdots \sigma_{k+1}) \iota^n_{k+1}(P_{k+1}) v.$$

We write

$$v = \sum_i \sum_{\underline{j} \vdash k+1} u_i \otimes w_{i,\underline{j}},$$

where $u_i \in T^{n-k-1}(V)$ are linearly independent and $w_{i,\underline{j}} \in T^{k+1}(V) \cap k[X_{\underline{j}}]$. Then $X_{k,n}(v) = 0$ implies that

$$\sum_{\underline{j} \vdash k+1} X_{k,n} w_{i,\underline{j}} = 0,$$

where we view $w_{i,\underline{j}}$ as if in the last $k+1$ positions. As these $k[X_{\underline{j}}]$ are disjoint, for any $\underline{j}$, $X_{k,n} w_{i,\underline{j}} = 0$.

There are two cases : recall that $X_{k,n} = \iota^n_{k+1}(T'_{k+1})$,

1. $X_{k-1,n} w_{i,\underline{j}} = 0$. In this case, after applying the induction hypothesis on $w_{i,\underline{j}}$, $P_k w_{i,\underline{j}}$ is generated by right pre-relations of lower degrees. So $P_n(u_i w_{i,\underline{j}})$ is generated by right pre-relations of lower degrees after Remark 5.2.
2. $X_{k-1,n} w_{i,\underline{j}} \neq 0$, then $P_k w_{i,\underline{j}}$ is a right pre-relation of degree k and $P_n(u_i w_{i,\underline{j}})$ is generated by right pre-relations of lower degree after Remark 5.2.

As a summary, for any i and $\underline{j}$, $P_n(u_i w_{i,\underline{j}})$ is generated by right pre-relations lower degree, so is $P_n v$. □

As a conclusion, to solve the equation $T_n x = 0$ in the aim of finding defining relations, it suffices to work inside the $k[\mathfrak{B}_n]$-module $k[X_{\underline{i}}]$ such that $\theta_n(v_{\underline{i}}) = v_{\underline{i}}$.

Example 5.2. We compute pre-relations of degree 2. It should be remarked that $\mathrm{Rel}^2_r = \mathrm{Rel}^2_l$, as $P_2 = Q_2$ and $T_2 = U_2$.

In this case, it suffices to consider each $k[X_{\underline{i}}]$ where $\underline{i} = (s,t)$. The following facts are clear after Proposition 5.3 :

1. $T_2 P_2 = 1 - \theta_2$ acts as zero on Rel^2_r, so it suffices to consider the fixed points of θ_2 ;
2. $\theta_2 v_{\underline{i}} = v_{\underline{i}}$ if and only if $q_{st} q_{ts} = 1$.

These observations give the following characterization of Rel^2_r :

$$\mathrm{Rel}^2_r = \mathrm{span}\{v_s v_t - q_{st} v_t v_s |\ s \leq t \ \text{ such that } q_{ts} q_{st} = 1\}.$$

There is no redundant relations in this list and it coincides with the set of constants of degree 2.

5.5.2 Balancing left and right objects

We have remarked that in general, the left and right constants and pre-relations may not coincide. This subsection is devoted to understanding symmetries between left and right constants and pre-relations.

The following lemma is clear after Proposition 5.2.

Lemma 5.9. For any $n \geq 2$, $\Delta_n T_n = U_n \Delta_n$ and $\Delta_n P_n = Q_n \Delta_n$.

After the following result, the Garside element permutes left and right pre-relations.

Corollary 5.5. The Garside element Δ_n induces a linear isomorphism $\mathrm{Rel}_r^n \cong \mathrm{Rel}_l^n$.

Proof. As $\Delta_n^2 = \theta_n$ acts as identity on Rel_r^n, Δ_n is a linear isomorphism. It suffices to show that the image of Δ_n is contained in Rel_l^n.

Let $w \in \mathrm{Rel}_r^n$. We verify that $\Delta_n w \in \mathrm{Rel}_l^n$. The first condition holds as after the lemma above, $U_n \Delta_n w = \Delta_n T_n w = 0$ and the other point comes from the injectivity of Δ_n. If we write $w = P_n v$, then after the lemma again, $\Delta_n w = \Delta_n P_n v = Q_n \Delta_n v$ implies $\Delta_n w$ is in the image of Q_n. The invariance under θ_n is clear. □

A similar result holds when the pre-relations are replaced by constants.

Corollary 5.6. The Garside element Δ_n induces a linear isomorphism $\mathrm{Con}_r^n \cong \mathrm{Con}_l^n$.

Proof. It is clear that Δ_n sends Con_r^n to Con_l^n, then it suffices to show that Δ_n is an isomorphism.

After the decomposition (5.1) and notations therein, we can write Con_r^n and Con_l^n into direct sums of the $k[\mathfrak{B}_n]$-modules $k[X_{\underline{i}}]$ for $i \vdash n$ such that the action of θ_n is given by an invertible scalar on each summand space. So Δ_n induces a linear isomorphism

$$k[X_{\underline{i}}] \cap \mathrm{Con}_r^n \cong k[X_{\underline{i}}] \cap \mathrm{Con}_l^n$$

and then a linear isomorphism between Con_r^n and Con_l^n. □

5.6 Generalized quantum groups

5.6.1 Generalized quantum groups

In the theory of Nichols algebras, if the Nichols algebra $\mathfrak{N}(V)$ is arising from a Yetter-Drinfel'd module $V \in {}_H^H\mathcal{YD}$, we can form the bosonization $\mathfrak{N}(V)\#H$, which gives a true Hopf algebra. This construction, once applied to the Nichols algebra of diagonal type associated to the data of a symmetrizable Kac-Moody Lie algebra, gives the positive or negative part of the quantum group. But here, we would like to define them in a more direct way.

Definition 5.9. Let $A = (q_{ij})_{1 \leq i,j \leq N}$ be a braiding matrix in $M_N(k^\times)$.

1. $T^{\leq 0}(A)$ is defined as the Hopf algebra generated by F_i, $K_i^{\pm 1}$ for $i \in I$ with relations :
$$K_i F_j K_i^{-1} = q_{ij}^{-1} F_j, \;\; K_i K_i^{-1} = K_i^{-1} K_i = 1;$$

2. $T^{\geq 0}(A)$ is defined as the Hopf algebra generated by E_i, $K_i'^{\pm 1}$ for $i \in I$ with relations :
$$K_i' E_j K_i'^{-1} = q_{ij} E_j, \ \ K_i' K_i'^{-1} = K_i'^{-1} K_i' = 1,$$
3. $N^{\leq 0}(A)$ is defined by adding all right pre-relations to $T^{\leq 0}(A)$;
4. $N^{\geq 0}(A)$ is defined by adding all left pre-relations to $T^{\geq 0}(A)$.

After Theorem 5.2, pre-relations generate the defining ideals, which are, after a well-known result, radicals of a generalized Hopf pairing.

We define a generalized Hopf pairing $\varphi : T^{\geq 0}(A) \times T^{\leq 0}(A) \to k$ such that for any $i, j \in I$,
$$\varphi(E_i, F_j) = \frac{\delta_{ij}}{q - q^{-1}}, \ \ \varphi(K_i', K_j) = q_{ij},$$
$$\varphi(E_i, K_j) = \varphi(K_i', F_j) = 0.$$
Moreover, as its radicals coincide with defining ideals, it induces a non-degenerate generalized Hopf pairing $\varphi : N^{\geq 0}(A) \times N^{\leq 0}(A) \to k$.

The following quantum double construction permits us to define the generalized quantum group.

Definition 5.10 ([47])**.** Let A, B be two Hopf algebras with invertible antipodes and φ be a generalized Hopf pairing between them. The quantum double $D_\varphi(A, B)$ is defined by :

1. As a vector space, it is $A \otimes B$;
2. As a coalgebra, it is the tensor product of coalgebras A and B ;
3. As an algebra, the multiplication is given by :
$$(a \otimes b)(a' \otimes b') = \sum \varphi(S^{-1}(a'_{(1)}), b_{(1)}) \varphi(a'_{(3)}, b_{(3)}) a a'_{(2)} \otimes b_{(2)} b'.$$

Definition 5.11. The generalized quantum group $N_q(A)$ associated to the braided matrix A is defined by :
$$N_q(A) = D_\varphi(N^{\geq 0}(A), N^{\leq 0}(A))/(K_i - K_i' | \, i \in I),$$
where $(K_i - K_i' | \, i \in I)$ is the Hopf ideal generated by these $K_i - K_i'$.

We can also similarly define the Hopf algebra $T_q(A)$ by replacing $N^{\geq 0}(A)$ and $N^{\leq 0}(A)$ by $T^{\geq 0}(A)$ and $T^{\leq 0}(A)$. Then $N_q(A)$ is the quotient of $T_q(A)$ by the Hopf ideal generated by the defining ideals.

After an easy computation, the commutation relation between E_i and F_j is given by :
$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

Remark 5.5. We use the notation $N_q(A)$ instead of $U_q(A)$ as, in general, it may not related to the universal enveloping algebra associated to a Kac-Moody Lie algebra. This will be explained in Example 5.3.

5.6.2 Averaged quantum group

In the construction above, we are interested in a particular case where the braiding matrix comes from a generalized Cartan matrix.

Let $C = (c_{ij})_{1\leq i,j\leq N}$ be a generalized Cartan matrix in $M_N(\mathbb{Z})$, i.e., an integral matrix satisfying :

1. $c_{ii} = 2$;
2. For any $i \neq j$, $c_{ij} \leq 0$;
3. $c_{ij} = 0$ implies $c_{ji} = 0$.

Let q be a variable. The generalized quantum group associated to C is defined by $N_q(A)$ where $A = (q^{c_{ij}})_{1\leq i,j\leq N}$. This demands us to define $N_q(A)$ on the field $K = k(q)$, the field of rational functions of one variable on k. In fact, to the convenience of further discussion, we take $K = k(q^{\frac{1}{2}})$ since elements in our matrices may be contained in the additive group $\frac{1}{2}\mathbb{Z}$.

It is natural to ask for the specialization problem of $N_q(A)$ where A arises from a generalized Cartan matrix. But unfortunately, the specialization map is not well-defined in most cases if A is not-symmetric. We will see in Example 5.3 a counterexample.

This phenomenon motivates us to modify the definition of a generalized quantum group to respect the validity of the specialization map.

Definition 5.12. Let $C \in M_N(\mathbb{Z})$ be a generalized Cartan matrix.

1. The averaged matrix associated to C is defined by $Av(C) = (Av(C)_{ij})_{1\leq i,j\leq N}$ where $Av(C)_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$. Then $Av(C) \in M_N(\mathbb{Q})$.
2. The averaged quantum group associated to C is the generalized quantum group associated to $Av(C)$.
3. The quantum group associated to C is defined by :
   (a) If C is symmetrizable, we respect the original definition of quantum groups $U_q(C)$ as a $k(q^{\frac{1}{2}})$-algebra.
   (b) If C is non-symmetrizable, the quantum group $U_q(C)$ is defined as $N_q(Av(C))$, it is a $k(q^{\frac{1}{2}})$-algebra.

We let $N_q^{>0}(C)$ ($N_q^{<0}(C)$) denote the subalgebra of $N_q^{\geq 0}(C)$ (resp. $N_q^{\leq 0}(C)$) generated by E_i (resp. F_i) for $i \in I$. They are Nichols algebras associated to the braiding matrix $A = (q^{c_{ij}})$ (resp. $A' = (q^{-c_{ij}})$). The following result dues to N. Andruskiewitsch and H.-J. Schneider shows that passing from $N_q^{<0}(C)$ to $N_q^{<0}(Av(C))$ will not lose too many informations.

Proposition 5.6 ([6])**.** Let V and V' be two Yetter-Drinfel'd modules of diagonal type with braiding matrices $(q_{ij})_{1\leq i,j\leq N}$ and $(q'_{ij})_{1\leq i,j\leq N}$ satisfying $q_{ij}q_{ji} = q'_{ij}q'_{ji}$ for any $i, j \in I$ with respect to the bases $v_1, \cdots, v_N$ of V and $v'_1, \cdots, v'_N$ of V'. Then

1. there exists a linear isomorphism $\psi : \mathfrak{N}(V) \to \mathfrak{N}(V')$ such that for any $i \in I$, $\psi(v_i) = v'_i$;

2. this linear map ψ almost preserves the algebra structure : for any $i, j \in I$,

$$\psi(v_i v_j) = \begin{cases} q'_{ij} q_{ij}^{-1} v'_i v'_j & \text{if } i \leq j; \\ v'_i v'_j, & \text{if } i > j. \end{cases}$$

After this result, the algebraic structure on $N_q^{<0}(C)$ can be pulled-back from that on $N_q^{<0}(Av(C))$ using this isomorphism.

Remark 5.6. We let C be a non-symmetrizable Cartan matrix. After Remark 1 and Theorem 21 in [73], in order that $N_q^{<0}(C)$ to be of finite Gelfan'd-Kirillov dimension, the matrix $Av(C)$ must be in $M_N(\mathbb{Z})$. This implies that most algebras we are considering are of infinite Gelfand-Kirillov dimensions.

5.6.3 Bar involution in symmetric case

In this subsection, we suppose moreover that the braiding matrix is symmetric : that is to say, $q_{ij} = q_{ji}$ for any $i, j \in I$.

This hypothesis allows us to study the bar involution on the Nichols algebras. This operation is very fundamental in the study of quantum groups, especially for canonical (global crystal) bases.

Definition 5.13. The bar involution $- : T(V) \to T(V)$ is a k-linear automorphism given by $q \mapsto q^{-1}$ and $v_i \mapsto v_i$.

Definition 5.14. For any $i \in I$, we define $\overline{\partial_i^R}$, $\overline{\partial_i^L} \in \mathrm{End}_k(T(V))$ by the composition of bar involution and ∂_i^R, ∂_i^L.

We start from showing that bar involution descends to Nichols algebras. The following lemma is needed.

Lemma 5.10. For any $i, j \in I$, $\partial_j^R \overline{\partial_i^R} = q_{ij}^{-1} \overline{\partial_i^R} \partial_j^R$.

Proof. The following formulas hold after direct verifications : for any $i_1, \cdots, i_n \in I$,

$$\partial_j^R(v_{i_1} \cdots v_{i_n}) = q_{j,i_n} \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) v_{i_n} + v_{i_1} \cdots v_{i_{n-1}} \partial_j^R(v_{i_n}),$$

$$\overline{\partial_i^R}(v_{i_1} \cdots v_{i_n}) = q_{i,i_n}^{-1} \overline{\partial_i^R}(v_{i_1} \cdots v_{i_{n-1}}) v_{i_n} + v_{i_1} \cdots v_{i_{n-1}} \overline{\partial_i^R}(v_{i_n}).$$

Moreover, it should be remarked that it suffices to prove the lemma for monomials.

We use induction on the degree n of the monomial. The case $n = 1$ is trivial. Taking a monomial $v_{i_1} \cdots v_{i_n}$ and using formulas above, we have :

$$\begin{aligned} & \partial_j^R \overline{\partial_i^R}(v_{i_1} \cdots v_{i_n}) \\ = \; & q_{j,i_n} q_{i,i_n}^{-1} \partial_j^R \overline{\partial_i^R}(v_{i_1} \cdots v_{i_{n-1}}) v_{i_n} + q_{i,i_n}^{-1} \overline{\partial_i^R}(v_{i_1} \cdots v_{i_{n-1}}) \partial_j^R(v_{i_n}) + \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) \overline{\partial_i^R}(v_{i_n}), \end{aligned}$$

$$\begin{aligned} & \overline{\partial_i^R} \partial_j^R(v_{i_1} \cdots v_{i_n}) \\ = \; & q_{j,i_n} q_{i,i_n}^{-1} \overline{\partial_i^R} \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) v_{i_n} + q_{j,i_n} \partial_j^R(v_{i_1} \cdots v_{i_{n-1}}) \overline{\partial_i^R}(v_{i_n}) + \overline{\partial_i^R}(v_{i_1} \cdots v_{i_{n-1}}) \partial_j^R(v_{i_n}). \end{aligned}$$

Applying induction hypothesis gives

$$\begin{aligned}&(\partial_j^R\overline{\partial_i^R}-q_{ij}^{-1}\overline{\partial_i^R}\partial_j^R)(v_{i_1}\cdots v_{i_n})\\ =\ &(1-q_{ij}^{-1}q_{j,i_n})\partial_j^R(v_{i_1}\cdots v_{i_{n-1}})\overline{\partial_i^R}(v_{i_n})+(q_{i,i_n}^{-1}-q_{ij}^{-1})\overline{\partial_i^R}(v_{i_1}\cdots v_{i_{n-1}})\partial_j^R(v_{i_n}).\end{aligned}$$

Notice that if $i\neq i_n$ and $j\neq i_n$, the right hand side is zero. We separate into some cases :

1. If $i=j=i_n$, then two coefficients in the right hand side are zero.
2. If $i\neq j$, $i=i_n$, then the second term in the right hand side vanish and the first term gives zero.
3. If $i\neq j$, $j=i_n$ then the first term in the right hand side vanish and the second term gives zero.

As a summary, the identity $(\partial_j^R\overline{\partial_i^R}-q_{ij}^{-1}\overline{\partial_i^R}\partial_j^R)(v_{i_1}\cdots v_{i_n})=0$ is proved. □

Proposition 5.7. The restriction of the bar involution induces a linear isomorphism $\mathfrak{I}(V)\to\mathfrak{I}(V)$.

Proof. Since it is a graded ideal, we let $\mathfrak{I}(V)_n$ denote the set of degree n elements in $\mathfrak{I}(V)$.

We first show that $\overline{\partial_i^R}(\mathfrak{I}(V))\subset\mathfrak{I}(V)$ by induction on the degree of elements in $\mathfrak{I}(V)$. The case $n=2$ is clear as ∂_i^R annihilates $\mathfrak{I}(V)_2$. Let $v\in\mathfrak{I}(V)_n$. After Corollary 5.1, it suffices to show that for any $j\in I$, $\partial_j^R\overline{\partial_i^R}(x)\in\mathfrak{I}(V)$. From the lemma above, $\partial_j^R\overline{\partial_i^R}=q_{ij}^{-1}\overline{\partial_i^R}\partial_j^R$, where using Corollary 5.1 again, $\partial_j^R(v)$ is in $\mathfrak{I}(V)$ with lower degree so $\overline{\partial_i^R}\partial_j^R(v)\in\mathfrak{I}(V)$ after the induction hypothesis. This shows $\partial_j^R\overline{\partial_i^R}(v)\in\mathfrak{I}(V)$.

We turn to the proof of the proposition. Let $v\in\mathfrak{I}(V)_n$. Then it is clear that $\partial_i^R(\overline{v})=\overline{\overline{\partial_i^R}(v)}$. We use induction on n. The case $n=2$ is clear. The argument above shows that $\overline{\partial_i^R}(v)\in\mathfrak{I}(V)$ with lower degree, so $\overline{\overline{\partial_i^R}(v)}\in\mathfrak{I}(V)$ after the induction hypothesis. This proves that for any $i\in I$, $\partial_i^R(\overline{v})=\overline{\overline{\partial_i^R}(v)}\in\mathfrak{I}(V)$, so $\overline{v}\in\mathfrak{I}(V)$ after Corollary 5.1. □

After this proposition, the bar involution may pass to the quotient to give a k-linear isomorphism of Nichols algebras $\mathfrak{N}(V)\to\mathfrak{N}(V)$.

The rest of this subsection is devoted to show a relation between the bar involution and the action of the Garside element on the image of the Dynkin operator P_n.

Proposition 5.8. For any $v\in T^n(V)$ satisfying $\theta_n v=v$, we have $\Delta_n P_n v=(-1)^{n-1}\overline{P_n\overline{v}}$.

Proof. We start from tackling the case where $v=v_{i_1}\cdots v_{i_n}$ is a monomial. In this case, the formula to be proved can be written as $\Delta_n P_n v=(-1)^{n-1}\overline{P_n v}$.

From the definition of P_n, for $1\leq j_1<\cdots<j_s\leq n-1$, if we let

$$E_{j_1,\cdots,j_s}=(\sigma_{n-1}\cdots\sigma_{j_1})\cdots(\sigma_{n-1}\cdots\sigma_{j_s}),$$

then the Dynkin operator P_n can be written as :

$$P_n=\sum_{s=0}^{n-1}(-1)^s\sum_{1\leq j_1<\cdots j_s\leq n-1}E_{j_1,\cdots,j_s}.$$

Claim. Let $1 \leq j_1 < \cdots j_s \leq n-1$ and $1 \leq j_1' < \cdots j_t' \leq n-1$ satisfy $t+s=n-1$ and $\{j_1, \cdots, j_s, j_1' \cdots, j_t'\} = \{1, \cdots, n-1\}$. Then for any $v = v_{i_1} \cdots v_{i_n}$,

$$\Delta_n E_{j_1,\cdots,j_s} v = \overline{E_{j_1'\cdots,j_t'} v}.$$

Indeed, we do the computation in detail. To simplify notations, we define

$$Q_{i_1,\cdots,i_n}^{j_1,\cdots,j_s} = q_{j_s,j_s+1} \cdots q_{j_s,i_n} q_{j_{s-1},j_{s-1}+1} \cdots q_{j_{s-1},i_n} \cdots q_{j_1,j_1+1} \cdots q_{j_1,i_n}.$$

Then the condition $\theta_n v = v$ and the fact that the braiding matrix is symmetric give

$$Q_{i_1,\cdots,i_n}^{j_1,\cdots,j_s} Q_{i_1,\cdots,i_n}^{j_1',\cdots,j_t'} = 1. \tag{5.2}$$

With this notation,

$$\Delta_n E_{j_1,\cdots,j_s}(v_{i_1} \cdots v_{i_n}) = Q_{i_1,\cdots,i_n}^{j_1,\cdots,j_s} v_{j_1} \cdots v_{j_s} v_{j_t'} \cdots v_{j_1'},$$

$$E_{j_1',\cdots,j_t'}(v_{i_1} \cdots v_{i_n}) = Q_{i_1,\cdots,i_n}^{j_1',\cdots,j_t'} v_{j_1} \cdots v_{j_s} v_{j_t'} \cdots v_{j_1'}.$$

The equation (5.2) gives

$$Q_{i_1,\cdots,i_n}^{j_1,\cdots,j_s} = \overline{Q_{i_1,\cdots,i_n}^{j_1',\cdots,j_t'}},$$

from which the claim is proved.

Now for $v = v_{i_1} \cdots v_{i_n}$, we have

$$\begin{aligned} \Delta_n P_n v &= \sum_{s=0}^{n-1} (-1)^s \sum_{1 \leq j_1 < \cdots < j_s \leq n-1} \Delta_n E_{j_1,\cdots,j_s} v \\ &= \sum_{s=0}^{n-1} (-1)^s \sum_{1 \leq j_1' < \cdots < j_{n-1-s}' \leq n-1} \overline{E_{j_1',\cdots,j_{n-1-s}'} v} \\ &= (-1)^{n-1} \sum_{t=0}^{n-1} (-1)^t \sum_{1 \leq j_1' < \cdots < j_t' \leq n-1} \overline{E_{j_1',\cdots,j_t'} v} = (-1)^{n-1} \overline{P_n v}. \end{aligned}$$

It remains to tackle the general case : suppose that $v = \sum a_i v_i$ where v_i are monomials. Applying the formula above gives :

$$\begin{aligned} \Delta_n P_n v &= \sum a_i \Delta_n P_n v_i \\ &= (-1)^{n-1} \sum a_i \overline{P_n v_i} \\ &= (-1)^{n-1} \sum \overline{\overline{a_i} P_n v_i} = (-1)^{n-1} \overline{P_n \overline{v}}. \end{aligned}$$

□

Corollary 5.7. If $v \in T(V)$ such that $\overline{v} = v$ and $P_n v$ is a right pre-relation. Then $\overline{P_n v}$ is a left pre-relation.

For example, this holds when v is a monomial.

5.7 On the specialization problem

5.7.1 Kac-Moody Lie algebras

We start with a brief recollection on some fundamental results of Kac-Moody Lie algebras.

Let C be a generalized Cartan matrix. The Kac-Moody Lie algebra associated with C is defined by : $\mathfrak{g}(C)=\widetilde{\mathfrak{g}}(C)/\mathfrak{r}$, where $\widetilde{\mathfrak{g}}(C)$ is the Lie algebra with Chevalley generators e_i, f_i and relations with respect to some realization $(\mathfrak{h},\Pi,\Pi^\vee)$ of C and $\mathfrak{r}$ is the unique maximal ideal in $\widetilde{\mathfrak{g}}(C)$ which intersects $\mathfrak{h}$ trivially (see [43], Chapter 1 for details). Moreover, if we write

$$\widetilde{\mathfrak{g}}(C)=\widetilde{\mathfrak{n}}_-\oplus\mathfrak{h}\oplus\widetilde{\mathfrak{n}}_+$$

as vector space, we have $\mathfrak{r}=\mathfrak{r}_+\oplus\mathfrak{r}_-$ as a direct sum of ideal, where $\mathfrak{r}_+=\mathfrak{r}\cap\mathfrak{n}_+$ and $\mathfrak{r}_-=\mathfrak{r}\cap\mathfrak{n}_-$.

The following characterization of elements in $\mathfrak{r}$ is clear from Lemma 1.5 of [43].

Lemma 5.11. Let $x\in\widetilde{\mathfrak{n}}_-$. Then $x\in\mathfrak{r}_-$ if and only if for any e_i, $[e_i,x]\in\mathfrak{r}_-$. The same is true for f_i and $\mathfrak{r}_+$.

As a corollary, a part of elements in $\mathfrak{r}$ are obtained in Section 3.3 of [43] : in $\mathfrak{g}(C)$, for $i\neq j$,

$$(\mathrm{ad}e_i)^{1-a_{ij}}(e_j)=0,\ \ (\mathrm{ad}f_i)^{1-a_{ij}}(f_j)=0. \tag{5.3}$$

For this Lie algebra $\mathfrak{g}(C)$, we associate to it the universal enveloping algebra $U(\mathfrak{g}(C))$, then the relations (5.3) above hold in $U(\mathfrak{g}(C))$. But if the matrix C is not symmetric, the ideal generated by these relations may not exhaust $\mathfrak{r}$.

5.7.2 Specialization (I)

Let C be a generalized Cartan matrix and $\mathcal{A}=k[q^{\frac12},q^{-\frac12}]$. We start from considering an $\mathcal{A}$-form of $T_q(C)$.

Let $T_\mathcal{A}$ be the $\mathcal{A}$-subalgebra of $T_q(C)$ generated by E_i, F_i, $K_i^{\pm1}$ and $[K_i;0]=\frac{K_i-K_i^{-1}}{q-q^{-1}}$ for $i\in I$. Moreover, $T_\mathcal{A}$ is a Hopf algebra and is called the integral form of $T_q(C)$.

We let $(q^{\frac12}-1)$ denote the ideal in $\mathcal{A}$ generated by $q^{\frac12}-1$ and $\mathcal{A}_1=k[q^{\frac12},q^{-\frac12}]/(q^{\frac12}-1)\cong k$, where the isomorphism is given by the evaluation at 1. We define $T_1'(C)=T_q(C)\otimes_\mathcal{A}\mathcal{A}_1$ and $T_1(C)$ the quotient $T_1'(C)/(K_i-1|\,i\in I)$, where $(K_i-1|\,i\in I)$ is the Hopf ideal generated by these elements. The following facts hold :

1. There is a Hopf algebra morphism $\widetilde{\sigma}:T_q(C)\to T_1(C)\cong U(\widetilde{\mathfrak{g}}(C))$. When composed with the projection $U(\widetilde{\mathfrak{g}}(C)\to U(\mathfrak{g}(C))$, it gives a Hopf algebra morphism $\sigma:T_q(C)\to U(\mathfrak{g}(C))$, which is called the specialization map ;
2. Restrictions of the map above give the following specialization maps : $T^{<0}(C)\to U(\mathfrak{n}_-)$ and $T_q^{>0}(C)\to U(\mathfrak{n}_+)$.

To obtain a true specialization map, it is demanded that the morphism σ may pass through the quotient by defining ideals to give a Hopf algebra morphism $N_q(A)\to U(\mathfrak{g}(C))$.

Example 5.3. We consider the following non-symmetrizable generalized Cartan matrix

$$C = \begin{bmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

In the braided tensor Hopf algebra of diagonal type associated to this matrix, we want to find some particular pre-relations : it is easy to show that $\theta_4(F_3^3 F_1) = F_3^3 F_1$ and moreover

$$T_4 P_4(F_3^3 F_1) = T_4'(F_3^3 F_1) = 0.$$

This implies that $P_4(F_3^3 F_1)$ is a right pre-relation of degree 4 where :

$$P_4(F_3^3 F_1) = F_3^3 F_1 - (q^{-3} + q^{-1} + q) F_3^2 F_1 F_3 + (q^{-4} + q^{-2} + 1) F_3 F_1 F_3^2 - q^{-3} F_1 F_3^3.$$

This element is specialized to

$$[f_3, [f_3, [f_3, f_1]]] = f_3^3 f_1 - 3 f_3^2 f_1 f_3 + 3 f_3 f_1 f_3^2 - f_1 f_3^3$$

in $U(\mathfrak{n}_-)$. We show that it is not contained in $U(\mathfrak{r}_-)$ so does not give 0.

The successive adjoint actions of e_3 give :

$$[e_3, f_3^3 f_1 - 3 f_3^2 f_1 f_3 + 3 f_3 f_1 f_3^2 - f_1 f_3^3] = 3(f_3^2 f_1 - 2 f_3 f_1 f_3 + f_1 f_3^2),$$
$$[e_3, f_3^2 f_1 - 2 f_3 f_1 f_3 + f_1 f_3^2] = 4(f_3 f_1 - f_1 f_3),$$
$$[e_3, f_3 f_1 - f_1 f_3] = 3 f_1.$$

If $[f_3, [f_3, [f_3, f_1]]]$ were in $U(\mathfrak{r}_-)$, so is f_1 after Lemma 5.11. This is impossible as $\mathfrak{r}_-$ would intersect with $\mathfrak{h}$ non-trivially.

This example shows that the specialization map may not be well-defined if the matrix is not symmetric.

5.7.3 Specialization (II)

Let C be a generalized Cartan matrix and $Av(C)$ be the associated averaged matrix. To obtain a well-defined specialization map, it is needed to pass to the averaged quantum groups. So we suppose moreover that the matrix C is non-symmetrizable as the other case is well-known.

We use notations $T_q(Av(C))$, $T_q^{<0}(Av(C))$ and their quotients $U_q(C)$ and $U_q^{<0}(C)$.

Theorem 5.3. The specialization map $\sigma : T_q(Av(C)) \to U(\mathfrak{g}(C))$ passes the quotient to give a surjective map $\sigma : U_q(C) \to U(\mathfrak{g}(C))$.

The rest of this subsection is devoted to giving a proof of this theorem. We start from the following lemma :

Lemma 5.12. For any $w \in T_q^{<0}(Av(C))$ and any $i \in I$,

$$[E_i, w] = \frac{K_i \partial_i^L(w) - \partial_i^R(w) K_i^{-1}}{q - q^{-1}}.$$

This formula can be proved either by induction or by verifying directly on a monomial ; notice that the symmetry condition on the braiding matrix is essential.

Recall that $\widetilde{\sigma} : T_q(Av(C)) \to U(\widetilde{\mathfrak{g}}(C))$ is the specialization map, then we have :

Lemma 5.13. Let $w \in T_q^{<0}(Av(C))$ be an element of degree n satisfying $T_n w = 0$. Then $\widetilde{\sigma}(w) \in U(\mathfrak{r}_-)$.

Proof. We first show that $[E_i, w] \in U(\mathfrak{r}_-)$ for any $i \in I$ by induction on the degree of n. The case $n = 2$ is clear as we have computed all constants of degree 2 in Example 5.2. According to the lemma above and the fact that $T_n w = 0$, we have

$$[E_i, w] = K_i \frac{\partial_i^L(w)}{q - q^{-1}}.$$

If the right hand side is sent to zero after $\widetilde{\sigma}$, $[E_i, w] = 0 \in U(\mathfrak{r}_-)$. If not, after Lemma 5.1, $\frac{\partial_i^L(w)}{q - q^{-1}}$ is annihilated by the action of T_{n-1} and so it is specialized to an element in $U(\mathfrak{r}_-)$ according to the induction hypothesis. This proved $\widetilde{\sigma}([E_i, w]) \in U(\mathfrak{r}_-)$. As $\widetilde{\sigma}$ is an algebra morphism, $[e_i, \widetilde{\sigma}(w)] \in U(\mathfrak{r}_-)$ and then $\widetilde{\sigma}(w) \in U(\mathfrak{r}_-)$ after Lemma 5.11. □

Proof of theorem. We have proved in the lemma above that right constants are specialized to $U(\mathfrak{r}_-)$ under $\widetilde{\sigma}$. A similar argument can be applied to left constants to show that their specializations are in $U(\mathfrak{r}_+)$. So we obtain a well-defined algebra map $\sigma : U_q(C) \to U(\mathfrak{g}(C))$ and the surjectivity is clear. The theorem holds because $\mathrm{Rel}_r \subset \mathrm{Con}_r$ and $\mathrm{Rel}_l \subset \mathrm{Con}_l$. □

5.8 Application

5.8.1 General calculation

Let $A \in M_N(\mathbb{Z})$ be a generalized Cartan matrix. We consider the element $v_{\underline{i}}$ for $\underline{i} = (1^{m_1}, \cdots, N^{m_N})$:

$$\theta(v_{\underline{i}}) = q^\lambda v_{\underline{i}}$$

where

$$\lambda = \sum_{k=1}^N 2m_k(m_k - 1) - \sum_{p=1}^N \sum_{q<p} (a_{pq} + a_{qp}) m_p m_q.$$

So $v_{\underline{i}}$ is a right pre-relation only if $\lambda = 0$. To find these pre-relations, it suffices to consider the integral solutions of this quadratic form.

5.8.2 Study of the quadratic form

The calculation above motivates us to study the following quadratic forms :

$$Q(x_1, \cdots, x_n) = \sum_{i=1}^n x_i^2 - \sum_{i<j} b_{ij} x_i x_j,$$

$$S(x_1, \cdots, x_n) = \sum_{i=1}^{n} (x_i - 1)^2,$$

where $b_{ij} = a_{ij} + a_{ji}$ are non-negative integers as in the last subsection.

Let $m \leq n$ be an integer (not necessary positive) and C_m be the intersection of the following two varieties

$$Q(x_1, \cdots, x_n) = m, \quad S(x_1, \cdots, x_n) = n - m.$$

Let $E(C_m)$ be the set of integral points on C_m and $E = \bigcup_{m \leq n} E(C_m)$. Then the set of all integral solutions of $\lambda = 0$ is the same as E.

Proposition 5.9. If the quadratic form $Q(x_1, \cdots, x_n)$ is semi-positive definite, E is a finite set.

Proof. If $Q(x_1, \cdots, x_n)$ is semi-positive definite, E is a finite union of $E(C_m)$ for $0 \leq m \leq n$. For each m, as $S(x_1, \cdots, x_n) = n - m$ is compact, so is its intersection with $Q(x_1, \cdots, x_n) = m$. The finiteness of $E(C_m)$ and E is clear. $\square$

Corollary 5.8. If the quadratic form $Q(x_1, \cdots, x_n)$ is semi-positive definite, the defining ideal $\mathfrak{I}(V)$ is finitely generated.

Proof. After the proposition, there are only a finite number of $\underline{i}$ such that $k[X_{\underline{i}}]$ containing right pre-relations ; moreover, each $k[X_{\underline{i}}]$ is finite dimensional. $\square$

Chapitre 6

A Borel-Weil-Bott type theorem of quantum shuffle algebras

A main part of this chapter is contained in the pre-publication [26].

6.1 Introduction

6.1.1 History and motivations

One of the central problems of representation theory is the construction of all irreducible or indecomposable representations of a given group or (associative) algebra. In the framework of complex compact Lie groups, there are in general two systematic ways to realize the finite dimensional ones as functions on the corresponding Lie groups : the Peter-Weyl theorem and the Borel-Weil-Bott theorem. The former decomposes the algebra of square integrable functions on a compact Lie group G as a Hilbert direct sum of endomorphism rings of all finite dimensional irreducible representations ; the latter views such a representation as global sections of some equivariant line bundle $\mathcal{L}_\lambda$ over the flag variety G/B where B is a fixed Borel subgroup.

Around 1985, Drinfel'd and Jimbo constructed the quantum group $U_q(\mathfrak{g})$ as a deformation of the ordinary enveloping algebra $U(\mathfrak{g})$ associated to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. As a quotient of a Drinfel'd double, $U_q(\mathfrak{g})$ is a quasi-triangular Hopf algebra : as a consequence, the category of finite dimensional $U_q(\mathfrak{g})$-modules is braided ; this can be applied to the construction of braid group representations and then explicit solutions of the Yang-Baxter equation.

When q is not a root of unity, the similarity between representation theories of $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$ can be explained as a quantization procedure of representations. This phenomenon suggests to develop necessary tools and frameworks to generalize the Peter-Weyl and Borel-Weil-Bott theorem to quantum groups. A mock version of the former is partially achieved and clarified in a series of works due to Joseph-Letzter [41] and Caldero [11].

An analogue of Borel-Weil-Bott theorem for quantum groups is obtained in an earlier work of Anderson-Polo-Wen [1] by viewing the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ as the corresponding flag variety and a character as a line bundle over it. They finally generalized the whole theorem using techniques coming from the representation theory of algebraic groups.

In this chapter, we would like to provide another point of view on generalizing the Borel-Weil-Bott theorem to quantum groups. Compared with the approach of Anderson-Polo-Wen, our construction has following advantages :

1. The construction is functorial : no explicit coordinate and character are needed at the very beginning ;
2. We may start with a general quantum shuffle algebra (it is well-known that the negative part of a quantum group is a very particular kind of such algebras).

In general, there are four constructions of the negative part of quantum groups in the literature :

1. By generators and relations (due to Drinfel'd [21] and Jimbo [38]) ;
2. Through the Hall algebra associated to the category of representations of a quiver (due to Ringel [71], Green [32]) ;
3. As perverse sheaves over the moduli space of quiver representations (due to Lusztig [57]) ;
4. As a quantum shuffle algebra associated to an abelian group algebra and a Hopf bimodule over it (due to Rosso [73]).

We will choose the last approach and go back to the second one in the applications.

6.1.2 Known results

Apart from the work of Anderson-Polo-Wen, the Hochschild cohomology of algebras associated to quantum groups are also studied in the work of Ginzburg-Kumar [31] : they computed the Hochschild cohomology of the (strictly) negative parts of the restricted quantum groups with coefficients in regular bimodules and one-dimensional weight modules. Methods therein are motivated by those appearing in the representation theory of algebraic groups.

6.1.3 Quantum shuffle algebras

The tensor algebra $T(V)$ associated with a vector space V is a graded Hopf algebra with concatenation multiplication and shuffle comultiplication. Its graded dual $T(V)^{*,gr}$ is also a graded Hopf algebra, where the multiplication is given by the shuffle product and comultiplication is the deconcatenation. A quantum shuffle algebra is almost the latter with the replacement of the symmetric group by an associated braid group, where the braiding comes from a Yetter-Drinfel'd module structure.

To be more precise, we start with a Hopf algebra H and an H-Hopf bimodule M ; the right coinvariant space $V = M^{coR}$ admits an H-Yetter-Drinfel'd module structure and thus a braiding $\sigma : V \otimes V \to V \otimes V$. The construction of the shuffle algebra with this braiding gives the quantum shuffle algebra $S_\sigma(V)$ as a subalgebra.

6.1.4 Main results

Suppose that $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra. The main construction for the object analogous to the line bundle in our framework is given by imposing an element in the set of coinvariants M^{coR} to build another quantum shuffle algebra $S_{\widetilde{\sigma}}(W)$, which is automatically a Hopf bimodule over $S_\sigma(V)$, and the "line bundle" appears as a sub-Hopf bimodule $S_{\widetilde{\sigma}}(W)_{(1)}$ in $S_{\widetilde{\sigma}}(W)$ by considering a grading in it.

With these constructions, the main purpose of this chapter can be listed as follows :

1. Provide a non-commutative version of the line bundle over the flag variety G/B and present a Borel-Weil-Bott type theorem in this framework : this is done by considering the coHochschild homology of the quantum shuffle algebra $S_\sigma(V)$ with coefficient in the $S_\sigma(V)$-bicomodule $S_{\widetilde{\sigma}}(W)_{(1)}$ defined above and the main theorem is :
   Theorem. The coHochschild homology groups of $S_\sigma(V)$ with coefficient in the $S_\sigma(V)$-bicomodule $S_{\widetilde{\sigma}}(W)_{(1)}$ are :
   (a) If q is not a root of unity and $\lambda \in \mathcal{P}_+$, as $U_q(\mathfrak{g})$-modules :
   $$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ 0, & n \neq 0. \end{cases}$$
   (b) If $q^l = 1$ is a primitive root of unity and $\lambda \in \mathcal{P}_+^l$, as $U_q(\mathfrak{g})$-modules :
   $$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ \wedge^n(\mathfrak{n}_-), & n \geq 1. \end{cases}$$
   where $\mathfrak{n}_-$ is identified with the negative part of the Lie algebra $\mathfrak{g}$.
2. Explain how this approach can be generalized to the "line bundle" of higher degree : that is to say, there is a family of $S_\sigma(V)$-bicomodules $S_{\widetilde{\sigma}}(W)_{(n)}$ such that the coHochschild homology group of degree 0 for $S_\sigma(V)$ with coefficient in these bicomodules can be found as a sum of irreducible representations. More precisely, we will prove the following theorem in degree two :
   Theorem. Let $q \in k^*$ not be a root of unity and $\lambda \in \mathcal{P}_+$ be a dominant weight.
   (a) If for any $i \in I$, $(\lambda, \alpha_i^\vee) \neq 1$, then as $U_q(\mathfrak{g})$-modules,
   $$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)}) = \begin{cases} L(\lambda) \otimes L(\lambda) & n = 0; \\ 0, & n \neq 0. \end{cases}$$
   (b) If J is the subset of I containing those $j \in I$ such that $(\lambda, \alpha_j^\vee) = 1$, then as $U_q(\mathfrak{g})$-modules,
   $$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)}) = \begin{cases} (L(\lambda) \otimes L(\lambda)) / \bigoplus_{j \in J} L(2\lambda - \alpha_j) & n = 0; \\ 0, & n \neq 0. \end{cases}$$
3. The construction of the $S_\sigma(V)$-bicomodules $S_{\widetilde{\sigma}}(W)_{(n)}$ above provides a systematical way to construct inductively both the negative part of a quantum group

and the PBW basis. This can be interpreted as an inductive construction of the composition algebra associated to a quiver. Moreover, we explain how to construct the composition algebra of a framed quiver (more precisely, the version of Crawley-Boevey [14]) starting from the unframed one. This inductive construction has different initial data from that in [73] as the braiding here is of diagonal type but of quantum group type.

6.1.5 Constitution of this chapter

The organization of this chapter is as follows :

We start in Section 6.2 with a recollection on quantum shuffle algebras and constructions around them. In Section 6.3 we explain how to recover quantum groups as quantum shuffle algebras and composition algebras. Section 6.4 is devoted to giving the main construction of the $S_\sigma(V)$-Hopf bimodule $S_{\tilde{\sigma}}(W)_{(n)}$ and a theorem of Rosso. CoHochschild homology is recalled in Section 6.5, moreover we discuss the module and comodule structures on these groups therein. Section 6.6, as a main part of this chapter, calculates the coHochschild homology groups, which gives an analogue of Borel-Weil-Bott theorem. As a continuation, we consider the case of degree two in Section 6.7 and obtain a similar result. Finally, in Section 6.8 and 6.9, we will discuss some applications of our construction such as an inductive construction of composition algebras associated with quivers.

6.2 Recollections on quantum shuffle algebras

We fix a field k of characteristic 0 in this chapter. All algebras, modules, vector spaces and tensor products are over k if not specified.

In this section, we recall the construction of quantum shuffle algebras given in [73]. For the basic notion of Hopf algebras, see [81].

6.2.1 Symmetric groups and braid groups

We fix some integer $n \geq 1$. Let $\mathfrak{S}_n$ denote the symmetric group acting on an alphabet with n elements, say, $\{1, 2, \cdots, n\}$. For an integer $0 \leq k \leq n$, we let $\mathfrak{S}_{k,n-k}$ denote the set of $(k, n-k)$-shuffles in $\mathfrak{S}_n$ defined by

$$\mathfrak{S}_{k,n-k} = \{\omega \in \mathfrak{S}_n |\ \omega^{-1}(1) < \cdots < \omega^{-1}(k),\ \omega^{-1}(k+1) < \cdots < \omega^{-1}(n)\}.$$

Moreover, once $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ is viewed as a subgroup of $\mathfrak{S}_n$, the multiplication gives a bijection

$$(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \times \mathfrak{S}_{k,n-k} \xrightarrow{\sim} \mathfrak{S}_n$$

which induces a decomposition of $\mathfrak{S}_n$.

The braid group $\mathfrak{B}_n$ is generated by $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ and relations :

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| \geq 2, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, \cdots, n-2.$$

For $1 \leq i \leq n-1$, we let s_i denote the transposition $(i, i+1)$ in $\mathfrak{S}_n$. Because $\mathfrak{S}_n$ can be viewed as imposing relations $\sigma_i^2 = 1$ in $\mathfrak{B}_n$, there exists a canonical projection $\pi_n : \mathfrak{B}_n \to \mathfrak{S}_n$ by sending σ_i to s_i.

This projection admits a section in the level of sets : it is a map $T : \mathfrak{S}_n \to \mathfrak{B}_n$ sending a reduced expression $\omega = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n$ to $T_\omega = \sigma_{i_1} \cdots \sigma_{i_k} \in \mathfrak{B}_n$. This map T is called the Matsumoto section.

6.2.2 Hopf bimodules and coinvariants

Let H be a Hopf algebra with invertible antipode and M be a vector space.

Definition 6.1. M is called an H-Hopf bimodule if it satisfies :

1. M is an H-bimodule ;
2. M is an H-bicomodule with structure maps $\delta_L : M \to H \otimes M$ and $\delta_R : M \to M \otimes H$;
3. These two structures are compatible : the maps δ_L and δ_R are bimodule morphisms, where the bimodule structures on $H \otimes M$ and $M \otimes H$ are given by the tensor product.

If M is an H-Hopf bimodule, it is at the same time a left H-Hopf module and a right H-Hopf module (for the definition of Hopf modules, see [81]).

One of the most important structures for Hopf modules is the set of coinvariants as it gives a parametrization of blocks in such modules. The set of left coinvariants M^{coL} and right coinvariants M^{coR} are defined by :

$$M^{coL} = \{m \in M |\ \delta_L(m) = 1 \otimes m\}, \quad M^{coR} = \{m \in M |\ \delta_R(m) = m \otimes 1\}.$$

Proposition 6.1 ([81])**.** Let M be a right H-Hopf module. Then there exists an isomorphism of right H-Hopf modules : $M \cong M^{coR} \otimes H$, where the right hand side admits the trivial right Hopf module structure. Moreover, maps in two directions are given by :

$$M \to M^{coR} \otimes H, \quad m \mapsto \sum P(m_{(0)}) \otimes m_{(1)}, \quad M^{coR} \otimes H \to M, \quad m \otimes h \mapsto mh,$$

where $m \in M$, $h \in H$, $\delta_R(m) = \sum m_{(0)} \otimes m_{(1)}$ and $P : M \to M^{coR}$ is defined by :

$$P(m) = \sum m_{(0)} S(m_{(1)}).$$

We have an analogous result for left H-Hopf modules.

Now we concentrate on the set of right coinvariants M^{coR} : it admits some left structures.

1. As δ_L and δ_R are compatible, M^{coR} is a left subcomodule of M.
2. Once we defined the left H-module structure on M by the adjoint action, say $h.m = \sum h_{(1)} m S(h_{(2)})$ for $h \in H$ and $m \in M$, M^{coR} is a left H-module.

But these will not give M^{coR} a left H-Hopf module structure as the adjoint action is not in general a left comodule morphism. This difference raises up a structure of great interest called Yetter-Drinfel'd module.

6.2.3 Yetter-Drinfel'd modules

Let H be a Hopf algebra. A vector space V is called a (left) H-Yetter-Drinfel'd module if it is simultaneously an H-module and an H-comodule satisfying the Yetter-Drinfel'd compatibility condition : for any $h \in H$ and $v \in V$,

$$\sum h_{(1)}v_{(-1)} \otimes h_{(2)}.v_{(0)} = \sum (h_{(1)}.v)_{(-1)}h_{(2)} \otimes (h_{(1)}.v)_{(0)},$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and $\rho(v) = \sum v_{(-1)} \otimes v_{(0)}$ are Sweedler notations for coproducts and comodule structure maps.

Morphisms between two H-Yetter-Drinfel'd modules are linear maps preserving H-module and H-comodule structures.

The compatibility condition between left module and comodule structures on M^{coR} in the last subsection can be explained in the framework of Yetter-Drinfel'd module.

Proposition 6.2 ([73]). Let H be a Hopf algebra. There exists an equivalence of category between the category of H-Hopf bimodules and the category of H-Yetter-Drinfel'd modules which sends a Hopf bimodule M to the set of its right coinvariants M^{coR}.

6.2.4 Quantum double construction

Let A and B be two Hopf algebras with invertible antipodes. A generalized Hopf pairing between A and B is a bilinear form $\varphi : A \times B \to \mathbb{C}$ satisfying :

1. For any $a \in A$, $b, b' \in B$, $\varphi(a, bb') = \sum \varphi(a_{(1)}, b)\varphi(a_{(2)}, b')$;
2. For any $a, a' \in A$, $b \in B$, $\varphi(aa', b) = \sum \varphi(a, b_{(2)})\varphi(a', b_{(1)})$;
3. For any $a \in A$, $b \in B$, $\varphi(a, 1) = \varepsilon(a)$, $\varphi(1, b) = \varepsilon(b)$.

If φ is a generalized Hopf pairing between A and B, the quantum double $D_\varphi(A, B)$ is defined by :

1. As a vector space, it is $A \otimes B$;
2. As a coalgebra, it is the tensor product of coalgebras A and B ;
3. As an algebra, the multiplication is given by :

$$(a \otimes b)(a' \otimes b') = \sum \varphi(S^{-1}(a'_{(1)}), b_{(1)})\varphi(a'_{(3)}, b_{(3)})aa'_{(2)} \otimes b_{(2)}b'.$$

If H is a finite dimensional Hopf algebra, it is well-known that there exists an equivalence between the category of H-Yetter-Drinfel'd modules and the category of modules over the quantum double $D_\varphi(H) = D_\varphi(H, H^*)$ where the generalized Hopf pairing is given by the duality between H and H^*. The following result is a consequence of Proposition 6.2.

Corollary 6.1. Let H be a finite dimensional Hopf algebra. There exists an equivalence of category between the category of H-Hopf bimodules and the category of modules over $D_\varphi(H)$ which sends M to the set of its right coinvariants M^{coR}.

Remark 6.1. It should be pointed out that the corollary above holds if H is a graded Hopf algebra with finite dimensional components and H^* is its graded dual.

6.2.5 Tensor product structures

In this subsection, we consider tensor product structures on two categories mentioned above.

Let M and N be two Hopf bimodules. We define an H-bimodule and an H-bicomodule structure on $M \otimes_H N$ as follows :

1. The bimodule structure comes from the left module structure on M and right module structure on N ;
2. The bicomodule structure comes from the one on the tensor product of two bicomodules.

Lemma 6.1. The module and comodule structures above are well-defined. With these structures, $M \otimes_H N$ is an H-Hopf bimodule.

Proof. The only problem occurs on the well-definedness of the comodule structures. We concentrate on the case of left comodule : at first, it is clear that there is a linear map $\widetilde{\delta_L} : M \otimes N \to H \otimes M \otimes_H N$; it suffices to show that it passes through the quotient to give $\delta_L : M \otimes_H N \to H \otimes M \otimes_H N$. This last point can be obtained by a simple verification. □

The following proposition implies that the equivalence of category in the last two subsections preserves tensor product structures.

Proposition 6.3 ([73]). Let M and N be two Hopf bimodules. Then as H-Yetter-Drinfel'd modules, we have :

$$(M \otimes_H N)^{coR} \cong M^{coR} \otimes N^{coR}.$$

If moreover H is of finite dimensional, the isomorphism above preserves $D_\varphi(H)$-module structures.

6.2.6 Constructions of braiding

In the category of H-Hopf bimodules, Woronowicz introduced a braiding structure which is explained in [73]. In this subsection, we discuss the relation between braidings appearing in these three categories.

1. Let M and N be two H-Hopf bimodules. Then there exists a unique H-Hopf bimodule isomorphism $\sigma : M \otimes_H N \to N \otimes_H M$ such that for any $\omega \in M^{coL}$ and $\eta \in M^{coR}$, $\sigma(\omega \otimes \eta) = \eta \otimes \omega$. Moreover, σ satisfies the braid equation. So the category of H-Hopf modules is a braided tensor category.
2. Let V and W be two H-Yetter-Drinfel'd modules. We define an isomorphism of Yetter-Drinfel'd modules $\sigma : V \otimes W \to W \otimes V$ by

$$\sigma(v \otimes w) = \sum v_{(-1)}.w \otimes v_{(0)}.$$

   Then σ satisfies the braid equation and so the category of H-Yetter-Drinfel'd modules is a braided tensor category.

3. If H is a finite dimensional Hopf algebra, the quantum double $D_\varphi(H)$ is a quasi-triangular Hopf algebra and so the category of modules over $D_\varphi(H)$ admits a braided tensor structure where the braiding is given by the action of the R-matrix in $D_\varphi(H)$.

Theorem 6.1 ([73]). The functor sending M to M^{coR} is an equivalence of braided tensor category between the category of H-Hopf bimodules and of H-Yetter-Drinfel'd modules. If moreover H is of finite dimensional, the above two categories are equivalent to the braided tensor category formed by $D_\varphi(H)$-modules.

6.2.7 Tensor algebra and its dual

Let H be a Hopf algebra and M be an H-Hopf bimodule. We have constructed a braiding on the set of its right coinvariants M^{coR}, which, from now on, will be denoted by V for short.

This construction gives a representation of braid group $\mathfrak{B}_n$ on $V^{\otimes n}$ by sending σ_i to $id^{\otimes(i-1)} \otimes \sigma \otimes id^{(n-i-1)}$.

We consider the tensor space

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

of V and write $(v_1, \cdots, v_n)$ for the pure tensor $v_1 \otimes \cdots \otimes v_n$ where $v_1, \cdots, v_n \in V$. It is well known that there is a braided Hopf algebra structure (for a definition, see [6]) on $T(V)$ defined by :

1. The algebra structure is given by the concatenation ;
2. The coalgebra structure is graded and is given by : for $v_1, \cdots, v_n \in V$, the $V^{\otimes p} \otimes V^{\otimes(n-p)}$-component of $\Delta((v_1, \cdots, v_n))$ is the shuffle action

$$\sum_{\sigma \in \mathfrak{S}_{p,n-p}} T_\sigma((v_1, \cdots, v_n)),$$

where T_σ is the image of σ under the Matsumoto section.

If the graded dual of $T(V)$ is under consideration, we have a dual algebra and a dual coalgebra structure on it :

1. The algebra structure is graded and is defined by : for $v_1, \cdots, v_n \in V$,

$$(v_1, \cdots, v_p) * (v_{p+1}, \cdots, v_n) = \sum_{\sigma \in \mathfrak{S}_{p,n-p}} T_\sigma((v_1, \cdots, v_n));$$

2. The coalgebra structure is given by the deconcatenation :

$$\Delta((v_1, \cdots, v_n)) = (v_1, \cdots, v_n) \otimes 1 + \sum_{p=1}^{n-1} (v_1, \cdots, v_p) \otimes (v_{p+1}, \cdots, v_n) + 1 \otimes (v_1, \cdots, v_n).$$

As shown in Proposition 9 of [73], $T(V)$, with structures defined above, is a braided Hopf algebra. We let $T_\sigma(V)$ denote it.

6.2.8 Cotensor product

The cotensor product over a coalgebra C is a dual version of the tensor product over some fixed algebra A. We recall the definition of cotensor product in this subsection, more information can be found in [18], [67] and [73].

Let C be a coalgebra and M, N be two C-bicomodules. The cotensor product of M and N is a C-bicomodule defined as follows : we consider two linear maps $\delta_R \otimes id_N$, $id_M \otimes \delta_L : M \otimes N \to M \otimes C \otimes N$; the cotensor product of M and N, which is denoted by $M \Box_C N$, is the equalizer of $\delta_R \otimes id_N$ and $id_M \otimes \delta_L$.

6.2.9 Quantum shuffle algebras and their bosonizations

In this subsection, we recall the definition of the quantum shuffle algebra given in [73] and [74]. Notations in previous subsections will be adopted.

We start from considering the linear map $V \to T_\sigma(V)$ given by the identity map. From the universal property of $T(V)$ as an algebra, we obtain a graded algebra morphism $\pi : T(V) \to T_\sigma(V)$.

Definition 6.2. The image of the graded algebra morphism π is called the quantum shuffle algebra and will be denoted by $S_\sigma(V)$; it is a subalgebra of $T_\sigma(V)$.

For any integer $n \geq 1$, we define two elements

$$S_n = \sum_{\sigma \in \mathfrak{S}_n} \sigma \in k[\mathfrak{S}_n], \quad \Sigma_n = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \in k[\mathfrak{B}_n].$$

The element Σ_n, once acting on $V^{\otimes n}$, is called a symmetrization operator. The following lemma hides between lines of [73].

Lemma 6.2 ([73])**.** When restricted to $V^{\otimes n}$, the map $\pi : T(V) \to T_\sigma(V)$ is given by the symmetrization operator Σ_n.

This gives the following isomorphism of braided Hopf algebras

$$\overline{\pi} : T(V) \Big/ \bigoplus_{n=2}^{\infty} \ker \Sigma_n \cong S_\sigma(V).$$

Remark 6.2. Up to a symmetrization map, the Nichols algebra defined in [67] and [6] is isomorphic to the quantum symmetric algebra as braided Hopf algebra.

At last, we describe the bosonization of quantum shuffle algebras, following [73].

In fact, instead of considering only the set of right coinvariants, we can start with the H-Hopf bimodule M and consider the cotensor Hopf algebra

$$T_H^{\Box}(M) = H \oplus \left(\bigoplus_{n=1}^{\infty} M^{\Box_H n} \right).$$

We let $S_H(M)$ denote the sub-Hopf algebra of $T_H^{\Box}(M)$ generated by H and M. It is an H-Hopf bimodule and the set of its right coinvariants is isomorphic to $S_\sigma(V)$ as an algebra ; moreover, as an algebra, $S_H(M)$ is isomorphic to the crossed product of H and $S_\sigma(V)$.

This $S_H(M)$ is called the bosonization of $S_\sigma(V)$ by the Hopf algebra H.

6.2.10 Pairings between quantum shuffle algebras

We explain the construction of graded non-degenerate Hopf pairings on $S_H(M)$ and $S_\sigma(V)$ in this subsection, after [13].

Let H be a graded Hopf algebra and $\psi_0 : H \times H \to k$ be a non-degenerate Hopf pairing. Let $\psi_1 : V \times V \to k$ be a non-degenerate bilinear form such that

$$\psi_1(h.v, w) = \sum \psi_0(h, w_{(-1)})\psi_1(v, w_{(0)}), \quad \psi_1(v, h.w) = \sum \psi_0(v_{(-1)}, h)\psi_1(v_{(0)}, w)$$

for any $h \in H$ and $v, w \in V$.

After [13], there exists a unique non-degenerate graded Hopf pairing $\psi : S_H(M) \times S_H(M) \to k$ extending ψ_0 and ψ_1. Once restricted to $S_\sigma(V)$, we obtain a non-degenerate graded Hopf pairing $\psi : S_\sigma(V) \times S_\sigma(V) \to k$. As ψ is graded, for any $m, n \in \mathbb{Z}_{\geq 0}$ such that $m \neq n$, we have :

$$\psi(S_\sigma^m(V), S_\sigma^n(V)) = 0$$

and the restriction of ψ on $S_\sigma^n(V) \times S_\sigma^n(V)$ is non-degenerate.

Remark 6.3. Such a pairing ψ always exists when the quantum shuffle algebra is of diagonal type (we refer to Section 6.3.1 for a definition).

6.2.11 Remarks on braided structures

We have mentioned above that $S_\sigma(V)$ is not a Hopf algebra but a braided Hopf algebra : that is to say, the product and coproduct structures are compatible up to the braiding coming from the Yetter-Drinfel'd structure. For example, for $x, y \in S_\sigma(V)$,

$$\Delta(xy) = \sum x_{(1)}((x_{(2)})_{(-1)}.y_{(1)}) \otimes (x_{(2)})_{(0)}y_{(2)}.$$

If the braiding under consideration is of diagonal type (for example, see Section 6.3.1 below and this is the case will be studied in this chapter), $(x_{(2)})_{(-1)}$ is given by a constant $\chi(x_{(2)}, y_{(1)})$ depending only on the degree of $x_{(2)}$ and $y_{(1)}$ if they are homogeneous. In this case, the term $(x_{(2)})_{(0)}$ is nothing but $x_{(2)}$, so the formula above reads

$$\Delta(xy) = \sum \chi(x_{(2)}, y_{(1)})x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}.$$

In this case, χ satisfies following properties : for homogeneous elements $a, b, c \in S_\sigma(V)$,

$$\chi(ab, c) = \chi(a, c)\chi(b, c), \quad \chi(a, bc) = \chi(a, b)\chi(a, c).$$

In the following argument, we will adopt this χ notation and sometimes suppress the prefix "braided" for short.

6.3 Construction of quantum groups

6.3.1 Construction of quantum groups

In this subsection, we recall the construction of the strictly negative part of a quantized enveloping algebra (quantum group) as a quantum shuffle algebra.

Let $H = k[G]$ be the group algebra of a finitely generated abelian group G. After the classification of finite rank $\mathbb{Z}$-modules, there exists an integer $r \geq 0$ and some positive integers $l_1, \cdots, l_s$ such that

$$G \cong \mathbb{Z}^r \times \left(\prod_{i=1}^{s} \mathbb{Z}/l_i \right).$$

We suppose that $n = r + s$ is the rank of G. From the argument before Lemma 14 in [73], if $V = M^{coR}$ is the set of right coinvariants of an H-Hopf bimodule M and $(F_1, \cdots, F_n)$ is a basis of the vector space V, then the braiding on V can be characterized by a square matrix of n^2 numbers q_{ij}, $1 \leq i, j \leq n$, which is called the braiding matrix.

To be more concrete, if we let $K_1, \cdots, K_n$ denote a free $\mathbb{Z}$-basis of G, then the left H-comodule structure on V is given by $\delta_L(F_i) = K_i^{-1} \otimes F_i$ and the left H-module structure is determined by $K_i.F_j = q_{ij}^{-1} F_j$. With this construction, the braiding is characterized by

$$\sigma(F_i \otimes F_j) = K_i^{-1}.F_j \otimes F_i = q_{ij} F_j \otimes F_i.$$

In particular, if the braiding matrix comes from some data in Lie theory, for example, a symmetrizable Cartan matrix, the quantum shuffle algebra constructed above is of great interest.

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix and $(d_1, \cdots, d_n)$ be positive integers such that the matrix $A = (d_i c_{ij}) = (a_{ij})$ is symmetric. We choose $q \neq 0, \pm 1$ be an element in k and define the braiding matrix $(q_{ij}) \in M_n(\mathbb{C})$ by $q_{ij} = q^{a_{ij}}$.

For each symmetrizable generalized Cartan matrix C, we can associate to it a Kac-Moody Lie algebra $\mathfrak{g}(C)$. After Drinfel'd and Jimbo, there exists a corresponding quantized enveloping algebra $U_q(\mathfrak{g}(C))$ defined by generators and relations. The following theorem permits us to give a functorial construction of the (strictly) negative part of such algebras.

Theorem 6.2 ([73]). After the construction above, we have :

1. Let $G = \mathbb{Z}^n$ and q not be a root of unity. Then the quantum shuffle algebra $S_\sigma(V)$ is isomorphic, as a braided Hopf algebra, to the strictly negative part of $U_q(\mathfrak{g}(C))$. Moreover, the bosonization $S_H(M)$ is isomorphic, as a Hopf algebra, to the negative part of $U_q(\mathfrak{g}(C))$.
2. We fix a positive integer $l \geq 3$ and a primitive l-th root of unity q. Let $G = (\mathbb{Z}/l)^n$. Then the quantum shuffle algebra $S_\sigma(V)$ is isomorphic to the strictly negative part of the restricted quantized enveloping algebra $u_q(\mathfrak{g}(C))$. Moreover, the bosonization $S_H(M)$ is isomorphic, as a Hopf algebra, to the quotient of the negative part of $u_q(\mathfrak{g}(C))$ by the Hopf ideal generated by $K_i^l - 1$, $i = 1, \cdots, n$.

To obtain the whole quantum group, it suffices to double the bosonization $S_H(M)$ using the quantum double construction given in Section 6.2.4 then identify two copies of H. The book [47] can be served as a good reference for this construction.

The isomorphism given in Section 6.2.9

$$\overline{\pi} : T(V) \Big/ \bigoplus_{n=2}^{\infty} \ker \Sigma_n \cong S_\sigma(V),$$

implies that the kernel $\bigoplus_{n=2}^{\infty} \ker \Sigma_n$ is indeed generated by quantized Serre relations in $U_q(\mathfrak{g})$.

6.3.2 Composition algebras

This subsection is devoted to recalling another point of view for quantum groups from the representation theory of quivers.

A family of important examples of the generalized symmetrizable Cartan matrices comes from quivers, which can be viewed as the Dynkin graphs associated to such matrices.

Let $Q = (I, F)$ be an unoriented quiver with n vertices and with no edge loops. A matrix $C \in M_n(\mathbb{Z})$ can be associated to it when we set $-c_{ij}$ the number of edges between vertices i and j and $c_{ii} = 2$. Thus we obtain a Kac-Moody Lie algebra $\mathfrak{g}(Q)$ and then a quantized enveloping algebra $U_q(\mathfrak{g}(Q))$. We let $U_q^{<0}(\mathfrak{g}(Q))$ denote the strict negative part of $U_q(\mathfrak{g}(Q))$.

Once the base field $k = \mathbb{F}_q$ is taken to be a finite field of q elements, we can consider the Hall algebra H_Q associated to the category of representations of the quiver Q, named $\mathcal{C}_Q$. H_Q is a braided Hopf algebra where the braiding comes from the multiplicative Euler form on weights of elements to be exchanged, where the Euler form is defined on the Grothendieck group of $\mathcal{C}_Q$, which can be viewed as the set of weights of representations. Details can be found in the introductory notes [80].

The composition algebra C_Q is defined as a sub-braided Hopf algebra of H_Q generated as an algebra by all isomorphism classes of simple objects in $\mathcal{C}_Q$.

We remark that in H_Q and C_Q, the cardinal q of the finite field can be treated as a formal parameter or a non-zero number which is not a root of unity.

Theorem 6.3 (Ringel, Green)**.** There exists an isomorphism of braided Hopf algebras

$$U_q^{<0}(\mathfrak{g}(Q)) \cong C_Q$$

given by sending F_i to the irreducible representation of Q associated to the vertex i.

6.4 Main Construction and Rosso's theorem

6.4.1 Data

We preserve assumptions in Section 6.3.1 and fix the following notations :

1. $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of the generalized Cartan matrix C and W is the Weyl group.
2. $(\cdot, \cdot)$ is a W-invariant bilinear form on $\mathbb{Q}\Pi$ such that $(\alpha_i, \alpha_j) = a_{ij}$.
3. $\mathcal{P} = \{\lambda \in \mathfrak{h}^* | \ (\lambda, \alpha_i) \in \mathbb{Z}, \forall i = 1, \cdots, n\}$ is that weight lattice and $\mathcal{P}_+ \subset \mathcal{P}$ is the set of dominant weights in $\mathfrak{g}$.
4. For $\lambda \in \mathcal{P}$, $L(\lambda)$ is the unique (up to isomorphism) irreducible representation of $\mathfrak{g}$ with highest weight λ.

If q is a primitive root of unity such that $q^l = 1$ for some odd $l \geq 3$, we let $\mathcal{P}^l$ denote the set $\{\lambda \in \mathcal{P} | \, |(\lambda, \alpha_i)| < l\}$ and $\mathcal{P}^l_+ = \mathcal{P}^l \cap \mathcal{P}_+$.

For a weight $\lambda \in \mathcal{P}$ or $\mathcal{P}^l$, we let K denote the commutative Hopf algebra generated by H together with group-like elements $K_\lambda^{\pm 1}$ such that $K_\lambda K_\lambda^{-1} = K_\lambda^{-1} K_\lambda = 1$ and let W denote the vector space generated by V and v_λ. We dispose the following structures on W.

1. W is a left K-comodule with structure map $\delta_L(v_\lambda) = K_\lambda^{-1} \otimes v_\lambda$, when restricted to V, δ_L is the H-comodule structural map of V ;
2. W is a left K-module with the module structure given by :
$$K_\lambda.F_i = q^{(\lambda,\alpha_i)} F_i, \;\; K_\lambda.v_\lambda = q^{-2} v_\lambda \text{ and } K_i.v_\lambda = q^{(\lambda,\alpha_i)} v_\lambda.$$
The other actions come from those in V.

As a vector space, we set $N = W \otimes K$; it admits an K-Hopf bimodule structure with the following definitions :

1. Right module and comodule structures are trivial : i.e., they come from the regular right K-module and K-comodule structures on K ;
2. Left module and comodule structures come from the tensor product.

Then it is clear that when this structure is under consideration, $N^{coR} = W$.

Starting with this K-Hopf bimodule N, we can construct the corresponding quantum shuffle algebra $S_{\widetilde{\sigma}}(W)$ and its bosonization $S_K(N)$. An easy computation gives the following formula for the braiding $\widetilde{\sigma}$ on $W \otimes W$: when restricted to $V \otimes V$, it coincides with σ ;

$$\widetilde{\sigma}(F_i \otimes v_\lambda) = q^{-(\lambda,\alpha_i)} v_\lambda \otimes F_i, \quad \widetilde{\sigma}(v_\lambda \otimes v_\lambda) = q^2 v_\lambda \otimes v_\lambda \quad \widetilde{\sigma}(v_\lambda \otimes F_i) = q^{-(\lambda,\alpha_i)} F_i \otimes v_\lambda.$$

On $S_{\widetilde{\sigma}}(W)$, we introduce a degree structure by letting $\deg(F_i) = 0$ and $\deg(v_\lambda) = 1$. We let $S_{\widetilde{\sigma}}(W)_{(k)}$ denote the subspace of $S_{\widetilde{\sigma}}(W)$ consisting of elements of degree k. For any k, $S_{\widetilde{\sigma}}(W)_{(k)}$ does not admit an algebra structure.

As $V \subset W$ is a K-submodule and $H \subset K$ is a sub-Hopf algebra, V is a sub-K-Yetter-Drinfel'd module of W. It is then clear that (V, σ) is a sub-braided vector space of $(W, \widetilde{\sigma})$ and $S_\sigma(V)$ is a sub-braided Hopf algebra of $S_{\widetilde{\sigma}}(W)$ in the category of K-Yetter-Drinfel'd modules.

6.4.2 Hopf bimodule structure on $S_{\widetilde{\sigma}}(W)$

With the gradation defined in the end of last subsection, $S_{\widetilde{\sigma}}(W)$ is a graded braided Hopf algebra with $S_{\widetilde{\sigma}}(W)_{(0)} = S_\sigma(V)$.

The projection $p : S_{\widetilde{\sigma}}(W) \to S_\sigma(V)$ onto degree 0 and the embedding $i : S_\sigma(V) \to S_{\widetilde{\sigma}}(W)$ into degree 0 are both braided Hopf algebra morphisms. This gives $S_{\widetilde{\sigma}}(W)$ a braided-$S_\sigma(V)$-Hopf bimodule structure : left and right comodule structural maps are $(p \otimes \mathrm{id})\Delta$ and $(\mathrm{id} \otimes p)\Delta$; left and right module structure are induced by $i : S_\sigma(V) \to S_{\widetilde{\sigma}}(W)$.

As elements in $S_\sigma(V)$ are of degree 0, for each $k \in \mathbb{N}$, $S_{\widetilde{\sigma}}(W)_{(k)}$ inherits a braided Hopf bimodule structure. To simplify notations, we let M_k denote $S_{\widetilde{\sigma}}(W)_{(k)}$ and M denote $S_{\widetilde{\sigma}}(W)$.

6.4.3 A theorem of Rosso

In this subsection, we explain a theorem due to M. Rosso which computes the coinvariant space M_1^{coR}.

As M and each M_k for $k \in \mathbb{N}$ are right braided $S_\sigma(V)$-Hopf modules, we let M^{coR} and M_k^{coR} denote the set of their right coinvariants, respectively. The braided version of the structural theorem of Hopf modules can be then applied to give the following isomorphisms of right braided Hopf modules :

$$S_{\widetilde{\sigma}}(W) \cong M^{coR} \otimes S_\sigma(V), \quad S_{\widetilde{\sigma}}(W)_{(k)} \cong M_k^{coR} \otimes S_\sigma(V).$$

We discuss the module and comodule structures on these sets of coinvariants :

1. As we have explained in Section 6.2.2, for any $k \in \mathbb{N}$, M^{coR} and M_k^{coR} admit adjoint $S_\sigma(V)$-module structures.
2. M^{coR} and M_k^{coR} are all $S_\sigma(V)$-left comodules : they are induced by the left comodule structures on M and M_k, respectively.
3. The module and comodule structures on M^{coR} and M_k^{coR} are compatible in the sense of Yetter-Drinfel'd. Thus both of them are $S_\sigma(V)$-Yetter-Drinfel'd modules.

In fact, we can use the bosonization procedure to avoid all prefixes "braided". As both $S_\sigma(V)$ and $S_{\widetilde{\sigma}}(W)$ are in the category of K-Yetter-Drinfel'd modules, the bosonization with K gives two Hopf algebras $S_K(M)$ and $S_K(N)$. If the Hopf algebra K is designated to be of degree 0, the projection onto degree 0 and the embedding into degree 0 endow $S_K(N)$ and all $S_K(N)_{(k)}$ for $k \in \mathbb{N}$ Hopf bimodule structures over $S_K(M)$. The structural theorem of right Hopf modules can be then applied to give

$$S_K(N) \cong M^{coR} \otimes S_K(M), \quad S_K(N)_{(k)} \cong M_k^{coR} \otimes S_K(M).$$

Then M^{coR} and M_k^{coR} are in the category of $S_K(M)$-Yetter-Drinfel'd modules. After Theorem 2.1, they admit $D_\varphi(S_K(M))$-module structures. Moreover, as $D_\varphi(S_H(M))$ is a sub-Hopf algebra of $D_\varphi(S_K(M))$, they admit $D_\varphi(S_H(M))$-structures.

If the generalized Hopf pairing is carefully chosen (for example, we take the pairing in the definition of the quantum group as a double), the H-action and the dual of H-coaction with respect to the pairing coincide. As a consequence of Theorem 6.2, M^{coR} and M_k^{coR} for any $k \in \mathbb{N}$ admit $U_q(\mathfrak{g})$-module structures.

The following theorem is due to Rosso [75].

Theorem 6.4 (Rosso)**.** M_1^{coR} is an irreducible $U_q(\mathfrak{g})$-module of highest weight λ, so it is isomorphic to $L(\lambda)$.

6.5 Coalgebra homology and module structures

6.5.1 Hochschild homology of an algebra

Let A be an associative algebra and M be an A-bimodule.

The Hochschild homology of A with coefficient in M is defined as the homology of the following complex $(C_\bullet(A,M),d)$, where

$$C_n(A,M) = M \otimes A^{\otimes n},$$

and the differential $d : C_n(A,M) \to C_{n-1}(A,M)$ is given by : for $a_1, \cdots, a_n \in A$ and $m \in M$,

$$\begin{aligned} d(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

We denote $H_n(A,M) = H_n(C_\bullet(A,M),d)$, the n-th homology group of the complex $(C_\bullet(A,M),d)$.

6.5.2 coHochschild homology of a coalgebra

The coHochschild homology of a coalgebra is a dual version of the construction of the Hochschild homology of an algebra and was first studied by P. Cartier.

Let C be a coalgebra and N be a C-bicomodule. We recall the coHochschild homology of C with coefficient in a C-bicomodule N given in [18] as follows :

We let $R^n(C,N)$ denote $N \otimes C^{\otimes n}$ and the differential $\delta : N \otimes C^{\otimes n} \to N \otimes C^{\otimes(n+1)}$ is given by :

$$\begin{aligned} \delta(n \otimes c_1 \otimes \cdots \otimes c_n) &= \delta_R(n) \otimes c_1 \otimes \cdots \otimes c_n \\ &\quad + \sum_{i=1}^{n} (-1)^i n \otimes c_1 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n \\ &\quad + (-1)^{n+1} \sum n_{(0)} \otimes c_1 \otimes \cdots \otimes c_n \otimes n_{(-1)}, \end{aligned}$$

where δ_L, δ_R are C-bicomodule structure maps and for $n \in N$, $\delta_L(n) = \sum n_{(-1)} \otimes n_{(0)}$.

The coHochschild homology of C with coefficient in a C-bicomodule N is defined by

$$\mathrm{Hoch}^i(C,N) = H^i(R^\bullet(C,N),\delta),$$

where $H^i(R^\bullet(C,N),\delta)$ is the i-th cohomology group of the complex $(R^\bullet(C,N),\delta)$.

6.5.3 Module and comodule structures on coHochschild homology

In this subsection, we discuss how module and comodule structures on N induce such structures on the coHochschild homology groups.

At first, we suppose that C and D are two coalgebras. Comodule structures which we will work with are defined by :

1. N is a C-bicomodule such that the left C-comodule structure is trivial.

2. N admits a (D,C)-comodule structure ; that is to say, N admits a left D-comodule structure compatible with its right C-comodule structure).
3. The coalgebra C admits a left trivial D-comodule structure.

Then for any integer $i \geq 0$, $N \otimes C^{\otimes i}$ admits a left D-comodule structure given by the tensor product if structures above are under consideration.

Proposition 6.4. For any integer $i \geq 0$, $\mathrm{Hoch}^i(C,N)$ inherits a D-comodule structure.

Proof. It suffices to show that for any $i \geq 0$,

$$d^i : N \otimes C^{\otimes i} \to N \otimes C^{\otimes (i+1)}$$

is a D-comodule morphism. It suffices to show the commutativity of the following diagram :

$$\begin{array}{ccc} N \otimes C^{\otimes i} & \xrightarrow{d^i} & N \otimes C^{\otimes (i+1)} \\ \downarrow \delta_L^D & & \downarrow \delta_L^D \\ D \otimes N \otimes C^{\otimes i} & \xrightarrow{id_D \otimes d^i} & D \otimes N \otimes C^{\otimes (i+1)}. \end{array}$$

We take an element $n \otimes c^1 \otimes \cdots \otimes c^i \in N \otimes C^{\otimes i}$, then

$$\begin{aligned} & (id_D \otimes d^i)(\delta_L^D(n \otimes c^1 \otimes \cdots \otimes c^i)) \\ = & \sum n_{(-1)} \otimes n_{(0)} \otimes n_{(1)} \otimes c^1 \otimes \cdots \otimes c^i \\ & + \sum_{p=1}^{i} (-1)^p \sum n_{(-1)} \otimes n_{(0)} \otimes c^1 \otimes \cdots \otimes c^p_{(1)} \otimes c^p_{(2)} \otimes \cdots \otimes c^i \\ & + (-1)^{i+1} \sum n_{(-1)} \otimes n_{(0)} \otimes c^1 \otimes \cdots \otimes c^i \otimes 1 \\ = & \delta_L^D(d^i(n \otimes c^1 \otimes \cdots \otimes c^i)). \end{aligned}$$

As a consequence, δ_L^D induces $\delta_L^D : \mathrm{Hoch}^i(C,N) \to D \otimes \mathrm{Hoch}^i(C,N)$, which gives a D-comodule structure on $\mathrm{Hoch}^i(C,N)$. □

Moreover, we consider the module structure on the coHochschild homology groups. At this time, we suppose that following data are given :

1. C is a trivial D-bimodule given by the counit ε ;
2. M is a C-bicomodule where the left C-comodule is trivial ;
3. M is a D-bimodule, then it is a left adjoint D-module ;
4. the right C-comodule structural map on M is a D-bimodule morphism.

Then for any integer $i \geq 0$, $M \otimes C^{\otimes i}$ admits a D-module structure given by : it acts on M by the adjoint action and acts trivially on the other components.

Proposition 6.5. For any integer $i \geq 0$, $\mathrm{Hoch}^i(C,M)$ inherits a D-module structure.

Proof. As in the last proposition, it suffices to show that for any $i \geq 0$, $d^i : M \otimes C^{\otimes i} \to M \otimes C^{\otimes(i+1)}$ is a morphism of D-module, that is to say, the following diagram commutes :

$$\begin{array}{ccc} D \otimes M \otimes C^{\otimes i} & \xrightarrow{\mathrm{id}_D \otimes d^i} & D \otimes M \otimes C^{\otimes(i+1)} \\ \downarrow a & & \downarrow a \\ M \otimes C^{\otimes i} & \xrightarrow{d^i} & M \otimes C^{\otimes(i+1)}, \end{array}$$

where a is the left adjoint D-module structural map.

We take an element $d \otimes m \otimes c^1 \otimes \cdots \otimes c^i \in D \otimes M \otimes C^{\otimes i}$, then

$$\begin{aligned} & a \circ (\mathrm{id} \otimes d^i)(d \otimes m \otimes c^1 \otimes \cdots \otimes c^i) \\ = & \sum d_{(1)} m_{(0)} S(d_{(2)}) \otimes m_{(1)} \otimes c^1 \otimes \cdots \otimes c^i \\ & + \sum_{p=1}^{i} (-1)^p d_{(1)} m S(d_{(2)}) \otimes c^1 \otimes \cdots \otimes c^p_{(1)} \otimes c^p_{(2)} \otimes \cdots \otimes c^i \\ & + (-1)^{i+1} \sum d_{(1)} m S(d_{(2)}) \otimes c^1 \otimes \cdots \otimes c^i \otimes 1. \end{aligned}$$

On the other side,

$$\begin{aligned} & d^i \circ a(d \otimes m \otimes c^1 \otimes \cdots \otimes c^i) \\ = & d^i \left(\sum d_{(1)} m S(d_{(2)}) \otimes c^1 \otimes \cdots \otimes c^i \right) \\ = & a \circ (\mathrm{id} \otimes d^i)(d \otimes m \otimes c^1 \otimes \cdots \otimes c^i). \end{aligned}$$

□

As a summary, for two Hopf algebras C, D and a vector space M satisfying the following conditions :

1. C is a trivial D-Hopf bimodule ;
2. M is a C-bicomodule where the left comodule structure is trivial ;
3. M is a left D-comodule such that M is a (D, C)-bicomodule ;
4. M is a D-bimodule such that the right C-comodule structural map is a D-bimodule morphism.

Then for any $i \in \mathbb{N}$, $\mathrm{Hoch}^i(C, M)$ inherits an adjoint D-module structure and a D-comodule structure from the corresponding ones on M.

6.6 A Borel-Weil-Bott type theorem

In this section, we compute the coHochschild homology of $S_\sigma(V)$ with coefficient in the bicomodule $S_{\widetilde{\sigma}}(W)_{(1)}$ to obtain a Borel-Weil-Bott type theorem. These can be viewed as an analogue of the flag variety and an equivariant line bundle over it respectively. We explain them in the following correspondence :

1. The quantum shuffle algebra $S_\sigma(V)$ can be viewed as an analogue of a non-commutative object corresponds to the flag variety G/B.

2. The $S_\sigma(V)$-Hopf bimodule generated by one vector (for example, v_λ) is an analogous of the equivariant line bundle $\mathcal{L}(\lambda)$ generated by a weight λ vector over G/B.
3. The set of coinvariants in $S_{\tilde{\sigma}}(W)$ is an analogue of the set of global invariants (more precisely, the set of global sections) on $\mathcal{L}(\lambda)$.

6.6.1 Main construction

For this cohomological purpose, we need to do a little change for the $S_\sigma(V)$-module and comodule structures on $S_{\tilde{\sigma}}(W)$.

1. The right $S_\sigma(V)$-module structure is given by the multiplication in $S_{\tilde{\sigma}}(W)$.
2. The left $S_\sigma(V)$-comodule structure on $S_{\tilde{\sigma}}(W)$ is defined by :

$$\delta_L : S_{\tilde{\sigma}}(W) \to S_\sigma(V) \otimes S_{\tilde{\sigma}}(W), \quad F_i \mapsto 1 \otimes F_i, \; v_\lambda \mapsto 1 \otimes v_\lambda.$$

3. The right $S_\sigma(V)$-comodule structure on $S_{\tilde{\sigma}}(W)$ is given by :

$$\delta_R : S_{\tilde{\sigma}}(W) \to S_{\tilde{\sigma}}(W) \otimes S_\sigma(V), \quad F_i \mapsto F_i \otimes 1 + 1 \otimes F_i, \; v_\lambda \mapsto v_\lambda \otimes 1.$$

That is to say, we trivialize the left comodule structure and make right structures being untouched.

Lemma 6.3. With structures defined above, both $S_{\tilde{\sigma}}(W)$ and $S_{\tilde{\sigma}}(W)_{(1)}$ are right $S_\sigma(V)$-Hopf modules and $S_{\tilde{\sigma}}(W)$-bicomodules.

It should be pointed out that as we do not touch the right Hopf module structure, the set of right coinvariants will be the same as the original case. That is to say, if we let M_1 denote $S_{\tilde{\sigma}}(W)_{(1)}$ with the above module and comodule structures, then as vector space, $M_1^{coR} \cong L(\lambda)$.

6.6.2 Calculation of Hoch^0

Now we proceed to calculate the degree 0 coHochschild homology of $S_\sigma(V)$ as a coalgebra with coefficient in the $S_\sigma(V)$-bicomodule $S_{\tilde{\sigma}}(W)_{(1)}$. We point out that this will only use the $S_\sigma(V)$-bicomodule structure on M_1.

It is better to start with a general framework. This will be useful to explain the set of coinvariants as some "global sections".

Let C be a coalgebra and M be a right C-Hopf module. We give M a trivial left C-comodule structure by defining $\delta_L(m) = 1 \otimes m$. Then M is a C-bicomodule.

Proposition 6.6. With assumptions above, $\mathrm{Hoch}^0(C, M) = M^{coR}$.

Proof. We compute $\mathrm{Hoch}^0(C, M)$. From definition, this is given by the kernel of

$$d^0 : M \to M \otimes C, \quad m \mapsto \sum m_{(0)} \otimes m_{(-1)} - \sum m_{(0)} \otimes m_{(1)},$$

where $\delta_L(m) = \sum m_{(-1)} \otimes m_{(0)}$ and $\delta_R(m) = \sum m_{(0)} \otimes m_{(1)}$.

The trivialization of the left comodule structure implies that $d^0(m) = 0$ if and only if $\sum m_{(0)} \otimes m_{(1)} = m \otimes 1$, which is equivalent to $m \in M^{coR}$. □

According this proposition, $\mathrm{Hoch}^0(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) \cong L(\lambda)$ as vector space. Now we will endow them with module and comodule structures using machineries built in Section 6.5.3.

To make notations more transparent, we denote $C = D = S_\sigma(V)$. The $S_\sigma(V)$-structures given at the beginning of this section on $S_{\widetilde{\sigma}}(W)$ are treated as C-module and comodule structures.

Now we define the D-module and comodule structures on $S_{\widetilde{\sigma}}(W)$:

1. The left $S_\sigma(V)$-comodule structure is given by :

$$\delta_L : S_{\widetilde{\sigma}}(W) \to S_\sigma(V) \otimes S_{\widetilde{\sigma}}(W), \quad F_i \mapsto F_i \otimes 1 + 1 \otimes F_i, \; v_\lambda \mapsto 1 \otimes v_\lambda.$$

2. The left $S_\sigma(V)$-module structure is given by the adjoint action.

Then $S_{\widetilde{\sigma}}(W)_{(1)}$ inherits these structures.

It is clear that these C, D-module and comodule structures on M satisfy the hypothesis before Proposition 6.4 and 6.5. So according to these propositions, for any i, the homology group $\mathrm{Hoch}^i(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)})$ admits a D-comodule structure and a D-module structure given by the adjoint action. These module and comodule structures satisfy the Yetter-Drinfel'd compatibility condition, thus all homology groups $\mathrm{Hoch}^i(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)})$ admit $U_q(\mathfrak{g})$-module structures.

Corollary 6.2. Let $\lambda \in \mathcal{P}$ (if $q^l = 1$ is a primitive root of unity, $\lambda \in \mathcal{P}^l$). As $U_q(\mathfrak{g})$-modules, we have $\mathrm{Hoch}^0(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) \cong L(\lambda)$.

The following part of this section is devoted to calculating the higher coHochschild homology groups in the generic and root of unity cases respectively.

6.6.3 Duality between Hochschild and coHochschild homologies

We start with a general setting : suppose that

$$C_\bullet : \cdots \xrightarrow{d} C_n \xrightarrow{d} \cdots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} M,$$

$$C'_\bullet : \cdots \xleftarrow{\delta} C_n \xleftarrow{\delta} \cdots \xleftarrow{\delta} C_2 \xleftarrow{\delta} C_1 \xleftarrow{\delta} M$$

are two complexes of finite dimensional vector spaces where M is in degree 0 and C_i in degree i such that

1. for each $i = 1, 2, \cdots$, there exists a bilinear form $\varphi_i : C_i \times C_i \to k$;
2. there is a bilinear form $\varphi_0 : M \times M \to k$;
3. differentials d and δ are adjoint to each other with respect to these pairings.

We let $H_\bullet(M)$ (resp. $H^\bullet(M)$) denote the homology group of the complex $C_\bullet$ (resp. $C'_\bullet$). As these complexes have differentials which are adjoint to each other, the bilinear forms $\varphi_0, \varphi_1, \cdots, \varphi_n, \cdots$ induce a bilinear form $\overline{\varphi} : H_n(M) \times H^n(M) \to k$ for any $n \geq 0$.

If moreover these bilinear forms are non-degenerate, we have the following duality result whose proof is direct.

Proposition 6.7. Let $\varphi_0, \varphi_1, \cdots, \varphi_n, \cdots$ be non-degenerate bilinear forms. Then $\overline{\varphi} : H_n(M) \times H^n(M) \to k$ is non-degenerate.

6.6.4 Application to quantum shuffle algebra

We fix a non-degenerate graded Hopf pairing on $S_{\tilde{\sigma}}(W)$ as explained in Section 6.2.10. This subsection is devoted to giving a proof the following result :

Proposition 6.8. For any $n \geq 0$, the bilinear form

$$\varphi : H_n(S_\sigma(V), S_{\tilde{\sigma}}(W)_{(1)}) \times \mathrm{Hoch}^n(S_\sigma(V), S_{\tilde{\sigma}}(W)_{(1)}) \to k$$

is non-degenerate.

We consider the Bar complex of $S_{\tilde{\sigma}}(W)_{(1)}$

$$S_{\tilde{\sigma}}(W)_{(1)} \otimes S_\sigma(V)^{\otimes n} \xrightarrow{d} \cdots \xrightarrow{d} S_{\tilde{\sigma}}(W)_{(1)} \otimes S_\sigma(V) \xrightarrow{d} S_{\tilde{\sigma}}(W)_{(1)}$$

and the coBar complex

$$S_{\tilde{\sigma}}(W)_{(1)} \xrightarrow{\delta} S_{\tilde{\sigma}}(W)_{(1)} \otimes S_\sigma(V) \xrightarrow{\delta} \cdots \xrightarrow{\delta} S_{\tilde{\sigma}}(W)_{(1)} \otimes S_\sigma(V)^{\otimes n} \xrightarrow{\delta} \cdots .$$

The $S_\sigma(V)$-bimodule and bicomodule structures on $S_{\tilde{\sigma}}(W)_{(1)}$ are given by :

1. The left $S_\sigma(V)$-module structure on $S_{\tilde{\sigma}}(W)_{(1)}$ is given by multiplication and the right module structure is given by the augmentation map ε ;
2. The left $S_\sigma(V)$-comodule structure on $S_{\tilde{\sigma}}(W)_{(1)}$ is trivial and the right comodule structure is deduced from the restriction of the comultiplication in $S_{\tilde{\sigma}}(W)_{(1)}$.

With these definitions, $S_\sigma(V)$-bimodule and bicomodule structures on $S_{\tilde{\sigma}}(W)_{(1)}$ are in duality and differentials d and δ are adjoint to each other with respect to the pairing.

Moreover, both $S_\sigma(V)$ and $S_{\tilde{\sigma}}(W)_{(1)}$ have gradations induced by the cotensor coalgebra :

$$S_\sigma(V) = \bigoplus_{n=0}^{\infty} S_\sigma^n(V), \quad S_{\tilde{\sigma}}(W)_{(1)} = \bigoplus_{n=0}^{\infty} S_{\tilde{\sigma}}^n(W)_{(1)},$$

where $S_\sigma^n(V)$ and $S_{\tilde{\sigma}}^n(W)_{(1)}$ are linearly generated by monomials of length n in the quantum shuffle product.

Then the Bar and coBar complexes admit gradations induced by those on $S_\sigma(V)$ and $S_{\tilde{\sigma}}(W)_{(1)}$: for example, elements of degree p in $S_{\tilde{\sigma}}(W)_{(1)} \otimes S_\sigma(V)^{\otimes n}$ are formed by :

$$\bigoplus_{i_0+\cdots+i_n=p} S_{\tilde{\sigma}}^{i_0}(W)_{(1)} \otimes S_\sigma^{i_1}(V) \otimes \cdots \otimes S_\sigma^{i_n}(V).$$

From the definition of differentials d and δ, they both preserve this gradation on the Bar and coBar complexes and give gradations on the Hochschild and coHochschild homology groups ; we let $H_n(S_\sigma(V), S_{\tilde{\sigma}}(W)_{(1)})_t$ and $\mathrm{Hoch}^n(S_\sigma(V), S_{\tilde{\sigma}}(W)_{(1)})_t$ denote sets of homology classes of degree t with homology degree n.

Fixing some degree t, there is a subcomplex $C_\bullet \to S_{\tilde{\sigma}}^t(W)_{(1)}$ defined by :

$$C_r = \bigoplus_{i_0+\cdots+i_r=t} S_{\tilde{\sigma}}^{i_0}(W)_{(1)} \otimes S_\sigma^{i_1}(V) \otimes \cdots \otimes S_\sigma^{i_r}(V);$$

these C_r are finite dimensional.

This subcomplex satisfies conditions in Proposition 6.7, so the Hopf pairing induces an isomorphism of vector space : for any $n, t = 0, 1, \cdots$,

$$H_n(S_\sigma(V), S_{\tilde{\sigma}}(W)_{(1)})_t \cong \mathrm{Hoch}^n(S_\sigma(V), S_{\tilde{\sigma}}(W)_{(1)})_t.$$

6.6.5 De Concini-Kac filtration

Let S be a commutative totally ordered semi-group and A be an S-filtered algebra with unit, that is to say : $A = \bigcup_{s\in S} A_s$ such that

1. for any $s \in S$, A_s is a subspace of A ;
2. for any $s < s' \in S$, $A_s \subset A_{s'}$;
3. for any $s, s' \in S$, $A_s \cdot A_{s'} \subset A_{s+s'}$.

The graded algebra associated to this filtration is denoted by $grA = \bigoplus_{s\in S} gr_s A$, where $gr_s A = A_s / \sum_{s'<s} A_{s'}$.

Let M be a free left A-module with generating set M_0. A filtration of A induces a filtration on M by defining $M_s = A_s \cdot M_0$. Then $\{M_s\}_{s\in S}$ forms a filtration on M which is compatible with the A-module structure : for any $s, s' \in S$, $A_s.M_{s'} \subset M_{s+s'}$. We let grM denote the associated graded vector space $\bigoplus_{s\in S} gr_s M$, where $gr_s M = M_s / \sum_{s'<s} M_{s'}$. Then grM is a left grA-module.

From now on and until the end of this chapter, $\mathfrak{g}$ is assumed to be a finite dimensional semi-simple Lie algebra.

We study the De Concini-Kac filtration in the rest of this subsection.

We fix an expression of the longest element w_0 in the Weyl group W, which permits us to construct a PBW basis of $S_\sigma(V)$ using Lyndon words. Let $\Delta_+ = \{\beta_1, \cdots, \beta_N\}$ be the set of positive roots associated to the Cartan matrix C. Once an expression of w_0 is fixed, we obtain a total ordering on Δ_+ : for example, we may suppose that $\beta_1 > \beta_2 > \cdots > \beta_N$.

For each $\beta_i \in \Delta_+$, there is a PBW root vector $F_{\beta_i} \in S_\sigma(V)$ such that the set

$$\{F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} |\ (i_1, \cdots, i_N) \in \mathbb{N}^N\}$$

forms a linear basis of $S_\sigma(V)$ which can be identified with the lattice $\mathbb{N}^N$. We equip $\mathbb{N}^N$ with its lexicographical ordering, then it is a totally ordered commutative semi-group. For a monomial $F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N}$, we define its degree

$$d(F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N}) = (i_1, \cdots, i_N) \in \mathbb{N}^N.$$

If $\underline{i} = (i_1, \cdots, i_N)$, the notation $F^{\underline{i}} = F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N}$ will be adopted.

Lemma 6.4 (Levendorskii-Soibelman, Kirillov-Reshetikhin)**.** For any $\beta_i < \beta_j$, we have :

$$F_{\beta_j} F_{\beta_i} - q^{(\beta_i, \beta_j)} F_{\beta_i} F_{\beta_j} = \sum_{\underline{k}\in\mathbb{N}^N} \alpha_{\underline{k}} F^{\underline{k}},$$

where $\alpha_{\underline{k}} \in k$ and $\alpha_{\underline{k}} \neq 0$ unless $d(F^{\underline{k}}) < d(F_{\beta_i} F_{\beta_j})$.

We define an $\mathbb{N}^N$-filtration on $S_\sigma(V)$ by : for $\underline{i} \in \mathbb{N}^N$, $S_\sigma(V)_{\underline{i}}$ is the linear subspace of $S_\sigma(V)$ generated by monomials $F^{\underline{k}}$ such that $d(F^{\underline{k}}) \leq \underline{i}$.

Proposition 6.9 (De Concini-Kac, [16])**.**

1. $\{S_\sigma(V)_{\underline{i}} |\ \underline{i} \in \mathbb{N}^N\}$ forms an $\mathbb{N}^N$-filtration of $S_\sigma(V)$.

2. The associated graded algebra $grS_\sigma(V)$ is generated by homogeneous generators $\{F_{\beta_i}|\ i = 1, \cdots, N\}$ and relations :

$$F_{\beta_j} F_{\beta_i} = q^{(\beta_i,\beta_j)} F_{\beta_i} F_{\beta_j}, \quad \text{for } \beta_i < \beta_j.$$

That is to say, $grS_\sigma(V)$ is a kind of "multi-parameter quantum plane"; it is an integral algebra, i.e., has no nontrivial zero-divisors.

Now we turn to study the induced $\mathbb{N}^N$-filtration on the left $S_\sigma(V)$-module $S_{\widetilde{\sigma}}(W)_{(1)}$.

As we have seen before, there is an isomorphism of vector space

$$S_{\widetilde{\sigma}}(W)_{(1)} \cong L(\lambda) \otimes S_\sigma(V).$$

Let $v_1, \cdots, v_r$ be a linear basis of $L(\lambda)$, where $r = dimL(\lambda)$. Then a linear basis of $S_{\widetilde{\sigma}}(W)_{(1)}$ is given by

$$F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r},$$

where $\underline{i} = (i_1, \cdots, i_N) \in \mathbb{N}^N$ and $\underline{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_r) \in \{0,1\}^r$ such that $|\varepsilon| = \sum_{i=1}^r \varepsilon_i = 1$.

The $S_\sigma(V)$-module structure on $S_{\widetilde{\sigma}}(W)_{(1)}$ is given by the left multiplication in $S_\sigma(V)$, so the induced $\mathbb{N}^N$-filtration on $M = S_{\widetilde{\sigma}}(W)_{(1)}$ is as follows : for any $\underline{s} \in \mathbb{N}^N$,

$$M_{\underline{s}} = \{F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r} |\ \underline{i} \leq \underline{s}\}.$$

We let $grM = grS_{\widetilde{\sigma}}(W)_{(1)}$ denote the associated $grS_\sigma(V)$-module.

We have an explicit description of grM according to Theorem 6.4 :

$$grM = \bigoplus_{\underline{i} \in \mathbb{N}^N, k=1,\cdots,r} F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} \otimes v_k,$$

where the $grS_\sigma(V)$-module structure is given by : for $F_{\beta_t} \in grS_\sigma(V)$ and $F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} \otimes v_k \in grM$,

$$F_{\beta_t} \cdot F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} \otimes v_k = \prod_{s=1}^{t-1} q^{-i_s(\beta_s,\beta_t)} F_{\beta_1}^{i_1} \cdots F_{\beta_t}^{i_t+1} \cdots F_{\beta_N}^{i_N} \otimes v_k.$$

The $S_\sigma(V)$-bimodule structure on $S_{\widetilde{\sigma}}(W)_{(1)}$ gives $grS_{\widetilde{\sigma}}(W)_{(1)}$ a $grS_\sigma(V)$-bimodule structure.

This construction still works when $q^l = 1$ is a root of unity, we refer to [31] for complete statements.

6.6.6 Hochschild homology of graded algebra : generic case

This section is devoted to computing the Hochschild homology group $H_\bullet(grS_\sigma(V), grS_{\widetilde{\sigma}}(W)_{(1)})$ with the bimodule structure defined above. To simplify the notation, we let $G_\sigma(V)$ and $G_{\widetilde{\sigma}}(W)_{(1)}$ denote $grS_\sigma(V)$ and $grS_{\widetilde{\sigma}}(W)_{(1)}$ respectively.

The main theorem of this section is :

Theorem 6.5. Let $\lambda \in \mathcal{P}_+$ and q not be a root of unity. Then the Hochschild homology group of $G_\sigma(V)$ with coefficient in $G_{\widetilde{\sigma}}(W)_{(1)}$ is given by :

$$H_n(G_\sigma(V), G_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ 0, & n \neq 0. \end{cases}$$

The main idea of the proof is to use the Koszul resolution of the Koszul algebra $G_\sigma(V)$, then apply an analogue of the homotopy defined by M. Wambst in [84].

We let $\Lambda_q(V)$ denote the graded algebra generated by homogeneous generators $F_{\beta_1}, \cdots, F_{\beta_N}$ of degree 1 and relations

1. for any $\beta_i < \beta_j$, $F_{\beta_j}F_{\beta_i} + q^{(\beta_i,\beta_j)}F_{\beta_i}F_{\beta_j} = 0$;
2. for any $i = 1, \cdots, N$, $F_{\beta_i}^2 = 0$.

Then $\Lambda_q(V) = \bigoplus_{k=0}^N \Lambda_q^k(V)$, where $\Lambda_q^k(V)$ is generated as a vector space by $F_{\beta_{i_1}} \wedge \cdots \wedge F_{\beta_{i_k}}$ for $i_1 < \cdots < i_k$. This $\Lambda_q(V)$ is the Koszul dual of $G_\sigma(V)$.

According Theorem 5.3 in [68], $G_\sigma(V)$ is a homogeneous Koszul algebra because it is obviously a PBW algebra (see Section 5.1 of [68] for a definition). Then as a $G_\sigma(V)$-bimodule, there is a Koszul complex starting from $G_{\widetilde{\sigma}}(W)_{(1)}$:

$$\cdots \longrightarrow G_{\widetilde{\sigma}}(W)_{(1)} \otimes \Lambda_q^k(V) \xrightarrow{d} \cdots \xrightarrow{d} G_{\widetilde{\sigma}}(W)_{(1)} \otimes \Lambda_q^1(V) \xrightarrow{d} G_{\widetilde{\sigma}}(W)_{(1)}.$$

We write down the differential d explicitly : for $\underline{i} \in \mathbb{N}^N$ and $\underline{\varepsilon} \in \{0,1\}^r$ with $|\underline{\varepsilon}| = 1$, we denote

$$F^{(\underline{i},\underline{\varepsilon})} = F_{\beta_1}^{i_1} \cdots F_{\beta_N}^{i_N} v_1^{\varepsilon_1} \cdots v_r^{\varepsilon_r},$$

then

$$d(F^{(\underline{i},\underline{\varepsilon})} \otimes F_{\beta_{i_1}} \wedge \cdots \wedge F_{\beta_{i_n}}) = \sum_{k=1}^n (-1)^{k-1} \prod_{s=k+1}^n Q_{i_k i_s} F_{\beta_{i_k}} F^{(\underline{i},\underline{\varepsilon})} \otimes F_{\beta_1} \wedge \cdots \wedge \widehat{F_{\beta_{i_k}}} \wedge \cdots \wedge F_{\beta_{i_n}},$$

where $Q_{i_k i_s} = q^{(\beta_{i_k}, \beta_{i_s})}$ and the hat notation hides the corresponding term. It should be pointed out that d is well-defined because the right $G_\sigma(V)$-module on $G_{\widetilde{\sigma}}(W)_{(1)}$ is trivial.

To simplify notations, we let Q_{ij} denote the number such that $F_{\beta_i}F_{\beta_j} = Q_{ij}F_{\beta_j}F_{\beta_i}$ for any positive roots β_i, β_j. Then relations in $G_\sigma(V)$ imply that $Q_{ij} = Q_{ji}^{-1}$ for any $i \neq j$ and $Q_{ii} = 1$.

Now we mimic the definition of the homotopy as in [84], Section 6 to show that the Koszul complex above is acyclic.

For any $\alpha = (\underline{i}, \underline{\varepsilon}) \in \mathbb{N}^N \times \{0,1\}^r$ as above with $|\underline{\varepsilon}| = 1$ and any $\beta \in \{0,1\}^N$, we define for any $i = 1, \cdots, N$,

$$\Omega(\alpha, \beta, i) = \begin{cases} 0 & \beta_i = 0; \\ \varepsilon(\beta, i) \displaystyle\prod_{s=i+1}^N Q_{is}^{\beta_s} \prod_{p=1}^{i-1} Q_{pi}^{-i_p}, & \beta_i \neq 0, \end{cases}$$

where $\varepsilon(\beta, i) = (-1)^{\sum_{s=1}^{i-1} \beta_s}$.

For any $i = 1, \cdots, N$, we let $[i]$ denote the element in $\mathbb{N}^N \times \{0,1\}^r$ or in $\{0,1\}^N$ such that its i-th component is 1 and the others are zero. Then the differential in the Koszul complex can be written as :

$$d(F^\alpha \otimes F^\beta) = \sum_{i=1}^N \Omega(\alpha, \beta, i) F^{\alpha+[i]} \otimes F^{\beta-[i]}.$$

With notations above, we define

$$\omega(\alpha,\beta,i)=\begin{cases}0 & \beta_i=1 \text{ or } \alpha_i=0;\\ \Omega(\alpha-[i],\beta+[i],i)^{-1}, & \text{if not.}\end{cases}$$

We define a map $h: G_{\widetilde{\sigma}}(W)_{(1)}\otimes\Lambda_q^n(V)\to G_{\widetilde{\sigma}}(W)_{(1)}\otimes\Lambda_q^{n+1}(V)$ by :

$$h(F^\alpha\otimes F^\beta)=\frac{1}{||\alpha+\beta||}\sum_{i=1}^N\omega(\alpha,\beta,i)F^{\alpha-[i]}\otimes F^{\beta+[i]},$$

where $||\alpha+\beta||=Card(\{i=1,\cdots,N|\ (\alpha+\beta)_i\neq 0\})$ (here we extend β by 0 to an element in $\mathbb{N}^N\times\{0,1\}^r\subset\mathbb{N}^{N+r}$ and $(\alpha+\beta)_i$ is the i-th component of $\alpha+\beta$).

Lemma 6.5. *$hd+dh=1$, i.e., h is a homotopy.*

A similar argument as in the proof of Theorem 6.1 of [84] can be applied to our case to prove this lemma. We provide at the end of this subsection some details of this verification as a modified version will be applied to the root of unity case. Thus the complex $G_{\widetilde{\sigma}}(W)_{(1)}\otimes\Lambda_q^\bullet(V)$ is acyclic.

Thus it suffices to compute the degree 0 homology group. This can be directly calculated as follows : the degree 0 homology group is $G_{\widetilde{\sigma}}(W)_{(1)}/\mathrm{im}d$ where

$$d: G_{\widetilde{\sigma}}(W)_{(1)}\otimes V\to G_{\widetilde{\sigma}}(W)_{(1)}$$

is given by :

$$d(F^\alpha\otimes F_{\beta_i})=F_{\beta_i}F^\alpha.$$

Recall that we have a linear basis $F^{(\underline{i},\underline{\varepsilon})}$ for $G_{\widetilde{\sigma}}(W)_{(1)}$ where $\underline{i}\in\mathbb{N}^N$ and $\underline{\varepsilon}\in\{0,1\}^r$ satisfying $|\underline{\varepsilon}|=1$. So after going to the quotient, the surviving elements are those $F^{(\underline{i},\underline{\varepsilon})}$ with $\underline{i}=(0,\cdots,0)$. Thus $H_0(G_\sigma(V),G_{\widetilde{\sigma}}(W)_{(1)})=L(\lambda)$ as vector space. □

Proof of Lemma. It suffices to verify that h is a homotopy. After the formula of d and h,

$$dh(F^\alpha\otimes F^\beta)=\frac{1}{||\alpha+\beta||}\sum_{i=1}^N\sum_{j=1}^N\omega(\alpha,\beta,i)\Omega(\alpha-[i],\beta+[i],j)F^{\alpha-[i]+[j]}\otimes F^{\beta+[i]-[j]},$$

$$hd(F^\alpha\otimes F^\beta)=\frac{1}{||\alpha+\beta||}\sum_{i=1}^N\sum_{j=1}^N\Omega(\alpha,\beta,j)\omega(\alpha+[j],\beta-[j],i)F^{\alpha-[i]+[j]}\otimes F^{\beta+[i]-[j]}.$$

We want to show that for $i\neq j$,

$$\omega(\alpha,\beta,i)\Omega(\alpha-[i],\beta+[i],j)+\Omega(\alpha,\beta,j)\omega(\alpha+[j],\beta-[j],i)=0.$$

If $i<j$, from the definition of Ω and ω, we have :

$$\begin{aligned}&\varepsilon(\beta,i)\varepsilon(\beta+[i],j)\left(\prod_{s=j+1}^N Q_{js}^{\beta_s}\prod_{p=1}^{j-1}Q_{pj}^{-i_p}\right)\left(\prod_{s=i+1}^N Q_{is}^{\beta_s}Q_{ij}^{-1}\prod_{p=1}^{i-1}Q_{pi}^{-i_p}\right)^{-1}\\ +\ &\varepsilon(\beta,j)\varepsilon(\beta-[j],i)\left(\prod_{s=j+1}^N Q_{js}^{\beta_s}\prod_{p=1}^{j-1}Q_{pj}^{-i_p}Q_{ij}\right)\left(\prod_{s=i+1}^N Q_{is}^{\beta_s}\prod_{p=1}^{i-1}Q_{pi}^{-i_p}\right)^{-1}=0\end{aligned}$$

Then it suffices to show that if $i < j$,

$$\varepsilon(\beta, i)\varepsilon(\beta + [i], j) + \varepsilon(\beta, j)\varepsilon(\beta - [j], i) = 0,$$

but this is clear from definition.

The case $i > j$ can be similarly obtained.

Now it suffices to prove that

$$\sum_{i=1}^{N} (\omega(\alpha, \beta, i)\Omega(\alpha - [i], \beta + [i], i) + \Omega(\alpha, \beta, i)\omega(\alpha + [i], \beta - [i], i)) = ||\alpha + \beta||.$$

Notice that each multiplication of ω and Ω is either 0 or 1. So we separate it into four cases :

1. $\alpha_i = 0$, $\beta_i = 0$; in this case, the i-summand is 0 ;
2. $\alpha_i \neq 0$, $\beta_i = 0$; in this case, the i-summand is 1 ;
3. $\alpha_i = 0$, $\beta_i = 1$; in this case, the i-summand is 1 ;
4. $\alpha_i \neq 0$, $\beta_i = 1$; in this case, the i-summand is 1.

Thus $\omega(\alpha, \beta, i)\Omega(\alpha - [i], \beta + [i], i) + \Omega(\alpha, \beta, i)\omega(\alpha + [i], \beta - [i], i) = 1$ if and only if $(\alpha + \beta)_i \neq 0$, from which we proved the above identity. □

6.6.7 Hochschild homology of graded algebra : root of unity case

We let $q^l = 1$ be a primitive l-th root of unity in this subsection.

Theorem 6.6. Let $\lambda \in \mathcal{P}_+^l$. The Hochschild homology groups of $G_\sigma(V)$ with coefficients in $G_{\widetilde{\sigma}}(W)_{(1)}$ are given by :

$$H_n(G_\sigma(V), G_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ \wedge^n(\mathfrak{n}_-), & n \neq 1. \end{cases}$$

where $\mathfrak{n}_-$ is identified with the negative part of the Lie algebra $\mathfrak{g}$.

The rest of this subsection is devoted to proving this theorem.

We use the Koszul complex as in the generic case :

$$K_\bullet = \cdots \longrightarrow G_{\widetilde{\sigma}}(W)_{(1)} \otimes \Lambda_q^k(V) \xrightarrow{d} \cdots \xrightarrow{d} G_{\widetilde{\sigma}}(W)_{(1)} \otimes \Lambda_q^1(V) \xrightarrow{d} G_{\widetilde{\sigma}}(W)_{(1)}.$$

We consider the following subcomplex of $K_\bullet$:

$$S_\bullet = \cdots \longrightarrow \bigoplus_{1 \leq i_1 < \cdots < i_k \leq N} kF_{\beta_{i_1}}^{l-1} \cdots F_{\beta_{i_k}}^{l-1} \otimes F_{\beta_{i_1}} \wedge \cdots \wedge F_{\beta_{i_k}} \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{s=1}^{N} kF_{\beta_{i_s}}^{l-1} \otimes F_{\beta_{i_s}} \xrightarrow{d} 0 .$$

From the definition of the differential and the fact that for any $1 \leq i \leq n$, $F_{\beta_i}^l = 0$, each term in $S_\bullet$ has no pre-image under d in $K_\bullet$. Thus we obtain a complement $R_\bullet$ of $S_\bullet$ in $K_\bullet$ such that as complexes,

$$K_\bullet = S_\bullet \oplus R_\bullet.$$

It is clear that in $S_\bullet$, all differentials are zero, so identifying $F_{\beta_{i_1}}^{l-1}\cdots F_{\beta_{i_k}}^{l-1}\otimes F_{\beta_{i_1}}\wedge\cdots\wedge F_{\beta_{i_k}}$ with $F_{\beta_{i_1}}\wedge\cdots\wedge F_{\beta_{i_k}}$ gives a bijection : for $k\geq 1$,

$$H_k(S_\bullet)\xrightarrow{\sim}\wedge^k(\mathfrak{n}_-)\ .$$

Now we proceed to show that the complex $R_\bullet$ is acyclic with $H_0(R_\bullet)=L(\lambda)$ by applying a modification of the homotopy defined in the generic case.

We explain the modifications :

1. The definition of $\Omega(\alpha,\beta,j)$ for $j=1,\cdots,N$, $\alpha=(\underline{i},\underline{\varepsilon})\in(\mathbb{Z}/l)^N\times\{0,1\}^r$, $\beta\in\{0,1\}^N$ with $|\underline{\varepsilon}|=1$:
$$\Omega(\alpha,\beta,j)=\begin{cases}0 & \beta_j=0 \text{ or } \underline{i}_j=l-1;\\ \varepsilon(\beta,j)\displaystyle\prod_{s=j+1}^{N}Q_{js}^{\beta_s}\prod_{p=1}^{j-1}Q_{pi}^{-i_p}, & \beta_j\neq 0 \text{ and } \underline{i}_j\neq l-1,\end{cases}$$
2. The definition of $\omega(\alpha,\beta,j)$ need not to be changed.
3. In the last step of the proof, there are 6 cases to be considered :
   (a) $\alpha_i=0$, $\beta_i=0$, then the i-summand is 0 ;
   (b) $\alpha_i=0$, $\beta_i=1$, then the i-summand is 1 ;
   (c) $\alpha_i=l-1$, $\beta_i=0$, then the i-summand is 1 ;
   (d) $\alpha_i=l-1$, $\beta_i=1$, then the i-summand is 0 ;
   (e) $\alpha_i\neq 0,l-1$, $\beta_i=0$, then the i-summand is 1 ;
   (f) $\alpha_i\neq 0,l-1$, $\beta_i=1$, then the i-summand is 1.

   So the i-summand is 0 if and only if $(\alpha+\beta)_i=0$ in $(\mathbb{Z}/l)^N\times\{0,1\}^r$. Moreover, in the complex $R_\bullet$, there does not exist a term $F^\alpha\otimes F^\beta$ such that for any i, $(\alpha+\beta)_i=0$ (as such terms are all contained in $S_\bullet$), which implies that $||\alpha+\beta||\neq 0$. So a homotopy from $R_\bullet$ to itself can be defined similarly as in the generic case, which shows that $R_\bullet$ is acyclic.

Finally, the result concerned with $H_0(R_\bullet)$ is similar with the generic case, which finishes the proof.

6.6.8 Main results

The theorem in the last subsection permits us to compute the Hochschild homology groups $H_\bullet(S_\sigma(V),S_{\widetilde{\sigma}}(W)_{(1)})$ by an argument of spectral sequence.

Theorem 6.7. The Hochschild homology groups of $S_\sigma(V)$ with coefficient in the $S_\sigma(V)$-bimodule $S_{\widetilde{\sigma}}(W)_{(1)}$ are :

1. If q is not a root of unity and $\lambda\in\mathcal{P}_+$, we have :
$$H_n(S_\sigma(V),S_{\widetilde{\sigma}}(W)_{(1)})=\begin{cases}L(\lambda) & n=0;\\ 0, & n\neq 0.\end{cases}$$

2. If $q^l = 1$ is a primitive root of unity and $\lambda \in \mathcal{P}_+^l$, we have :

$$H_n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) = \begin{cases} L(\lambda) & n = 0; \\ \wedge^n(\mathfrak{n}_-), & n \geq 1. \end{cases}$$

where $\mathfrak{n}_-$ is identified with the negative part of the Lie algebra $\mathfrak{g}$.

To pass from the graded case to the general case, it suffices to apply the following lemma due to May [62], Theorem 3.

Lemma 6.6 (May spectral sequence)**.** Let A be a filtered algebra with unit such that its filtration is exhaustive, M be a filtered A-module where the filtration is induced from that of A. Then there exists a convergent spectral sequence

$$E^2_{p,q} = H_{p+q}(grA, grM) \Longrightarrow H_\bullet(A, M).$$

In our context, from theorems in last sections, the spectral sequence collapses at E^2-term : it is clear in the generic case ; in the root of unity case, this holds as all differentials are zero in the E^2-sheet. This gives the desired isomorphism of homology groups :

$$H_n(G_\sigma(V), G_{\widetilde{\sigma}}(W)_{(1)}) \xrightarrow{\sim} H_n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(1)}) \ ,$$

which finishes the proof.

As a corollary, the first theorem announced in the introduction comes from Proposition 6.8.

6.7 On the study of coinvariants of degree 2

General assumption : from now on until the end of this chapter, we suppose that q is not a root of unity and $\lambda \in \mathcal{P}_+$ is a dominant weight.

In this section, as a continuation of Theorem 6.4, we will study the set of coinvariants of degree 2. We keep the notations of last sections.

We fix an integer $n \geq 1$. Let M_n denote the subspace of $S_{\widetilde{\sigma}}(W)$ containing elements of degree n with the degree structure defined by $deg(F_i) = 0$ and $deg(v_\lambda) = 1$. The same argument as in last sections shows that $M_n = (S_{\widetilde{\sigma}}(W))_{(n)}$ is an $S_\sigma(V)$-sub-Hopf bimodule of $S_{\widetilde{\sigma}}(W)$. Then we can consider the set of right coinvariants M_n^{coR} in M_n and from the structure theorem of Hopf bimodules,

$$(S_{\widetilde{\sigma}}(W))_{(n)} = M_n \cong M_n^{coR} \otimes S_\sigma(V).$$

6.7.1 Basic construction

In the following part of this section, we will concentrate on the case $n = 2$ to give an explicit description of M_2^{coR}.

The main tool for tackling this case is the following construction.

At first, we consider the following commutative diagram :

$$\begin{array}{ccc} T(W)_{(1)} \otimes_{T(V)} T(W)_{(1)} & \xrightarrow{S_1} & S_{\tilde{\sigma}}(W)_{(1)} \otimes_{S_\sigma(V)} S_{\tilde{\sigma}}(W)_{(1)} \\ \cong \downarrow m & & \downarrow m \\ T(W)_{(2)} & \xrightarrow{S_2} & S_{\tilde{\sigma}}(W)_{(2)}. \end{array}$$

We start by explaining morphisms appearing in this diagram. Let $\Sigma_n = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \in k[\mathfrak{B}_n]$ denote the symmetrization operator, it acts linearly on $V^{\otimes n}$. The map

$$S_1 = \bigoplus_{n,m=0}^{\infty} \Sigma_n \otimes \Sigma_m$$

is given by the symmetrization on both components. It is well-defined because as explained in Section 6.2.9, the symmetrization map $T(W) \to S_{\tilde{\sigma}}(W)$ is a morphism of algebra. The morphism $S_2 = \bigoplus_{n=0}^{\infty} \Sigma_n$ is just the symmetrization map. The horizontal morphisms are given by symmetrization, so both of them are surjection. Two vertical morphisms are given by multiplications and so the left one is an isomorphism. It permits us to identify elements in $T(W)_{(1)} \otimes_{T(V)} T(W)_{(1)}$ and $T(W)_{(2)}$. We will denote $S_2 \circ m$ by S_2 for short.

Lemma 6.7. We have $\ker S_1 \subset \ker S_2$, so the right vertical map m is surjective.

Proof. This comes from a general observation. For three integers $n, n_1, n_2 \geq 0$ satisfying $n = n_1 + n_2$, as shown in Section 6.2.1, we can decompose the symmetric group $\mathfrak{S}_n$ as

$$\mathfrak{S}_n = (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}) \circ \mathfrak{S}_{n_1,n_2}.$$

Moreover, this decomposition can be lifted to the braid group $\mathfrak{B}_n$ by the Matsumoto section. So for an element x in $W^{\otimes n}$, if

$$\sum_{\sigma \in \mathfrak{S}_{n_1,n_2}} T_\sigma(x) = 0,$$

the total symmetrization

$$\sum_{\omega \in \mathfrak{S}_n} T_\omega(x) = \sum_{\tau \in \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}} T_\tau \left(\sum_{\sigma \in \mathfrak{S}_{n_1,n_2}} T_\sigma(x) \right) = 0.$$

From the definition of S_1 and S_2, it is now clear that $\ker S_1 \subset \ker S_2$. Thus we obtain a linear surjection

$$S_{\tilde{\sigma}}(W)_{(1)} \otimes_{S_\sigma(V)} S_{\tilde{\sigma}}(W)_{(1)} \to S_{\tilde{\sigma}}(W)_{(2)}$$

given by the multiplication. □

Now we consider the inclusion map

$$S_{\tilde{\sigma}}(W)_{(1)} \to S_{\tilde{\sigma}}(W)$$

which is an $S_\sigma(V)$-bimodule morphism. Thanks to the universal property of the tensor algebra (see, for example, Proposition 1.4.1 of [67]), it can be lifted to a morphism of algebra

$$T_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)}) \to S_{\widetilde{\sigma}}(W)$$

given by the multiplication.

A similar argument as in the lemma above can be applied to show the following corollary :

Corollary 6.3. The multiplication map $T_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)}) \to S_{\widetilde{\sigma}}(W)$ is surjective.

That is to say, as an $S_\sigma(V)$-bimodule, $S_{\widetilde{\sigma}}(W)$ is generated by $S_{\widetilde{\sigma}}(W)_{(1)}$.

We proceed to consider the $S_\sigma(V)$-structures of this morphism.

1. $T_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)})$ is an $S_\sigma(V)$-bicomodule : for each $n \in \mathbb{N}$, as a tensor product, $T^n(S_{\widetilde{\sigma}}(W)_{(1)})$ is an $S_\sigma(V)$-bicomodule. Combined with the canonical projection, we obtain linear maps

$$\delta_L : T^n(S_{\widetilde{\sigma}}(W)_{(1)}) \to S_\sigma(V) \otimes T^n_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)}),$$

$$\delta_R : T^n(S_{\widetilde{\sigma}}(W)_{(1)}) \to T^n_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)}) \otimes S_\sigma(V)$$

which make $T^n_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)})$ an $S_\sigma(V)$-bicomodule, according to Lemma 6.1.

2. $T_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)})$ is an $S_\sigma(V)$-bimodule as each $T^n_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)})$ is.
3. These two structures on $T_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)})$ make it into an $S_\sigma(V)$-Hopf bimodule.
4. $S_{\widetilde{\sigma}}(W)$ has its ordinary $S_\sigma(V)$-bimodule and bicomodule structures as in the beginning of Section 6.4.2.
5. As right $S_\sigma(V)$-Hopf modules, there are isomorphisms

$$M_n^{coR} \otimes S_\sigma(V) \cong S_{\widetilde{\sigma}}(W)_{(n)}, \; M^{coR} \otimes S_\sigma(V) \cong S_{\widetilde{\sigma}}(W)$$

given by multiplications.

Combining these observations and constructions, we have : as left $S_\sigma(V)$-modules and comodules,

$$T_{S_\sigma(V)}(S_{\widetilde{\sigma}}(W)_{(1)}) \cong T(M_1^{coR}) \otimes S_\sigma(V),$$

$$S_{\widetilde{\sigma}}(W) \cong \bigoplus_{n=0}^{\infty} M_n^{coR} \otimes S_\sigma(V).$$

These M_n^{coR} and $T(M_1^{coR})$ are $S_\sigma(V)$-Yetter Drinfel'd modules as explained in Section 6.2.2. Once $id \otimes \varepsilon$ is applied to both sides, we obtain a surjection of $S_\sigma(V)$-Yetter-Drinfel'd modules $T(M_1^{coR}) \to \bigoplus_{n=0}^{\infty} M_n^{coR}$ given by the multiplication. The following corollary is just a particular case :

Corollary 6.4. The multiplication gives an $S_\sigma(V)$-Yetter-Drinfel'd module surjection

$$m : M_1^{coR} \otimes M_1^{coR} \to M_2^{coR}.$$

6.7.2 Study of M_2^{coR} : non-critical case

The non-critical and critical case are separated in view of studying new Serre relations appearing when passing from degree 1 to degree 2.

Let λ be a dominant weight. We call λ non-critical in degree 2 if there does not exist $i \in I$ such that $(\lambda, \alpha_i^\vee) = 1$. If the degree is under consideration, we will call λ non-critical.

We start from a general remark. Let $C' = (c'_{i,j})_{(n+1)\times(n+1)}$be the generalized Cartan matrix obtained from C by adding a last row and a last column : its elements are : for $1 \leq i,j \leq n$, $c'_{i,j} = c_{i,j}$; $c'_{n+1,n+1} = 2$ and for $1 \leq i \leq n$, $c'_{n+1,i} = c'_{i,n+1} = -(\lambda, \alpha_i^\vee)$.

Thanks to Theorem 6.2, $S_{\widetilde{\sigma}}(W) \cong U_q^-(\mathfrak{g}(C'))$ as braided Hopf algebra.

From the definition of the quantized enveloping algebra, $U_q^-(\mathfrak{g}(C'))$, as an algebra, is freely generated by $F_1, \cdots, F_n$ and v_λ with relations :

$$ad(F_i)^{1-a_{ij}}(F_j) = 0, \quad i \neq j = 1, \cdots, n;$$

$$ad(v_\lambda)^{1+(\lambda,\alpha_i^\vee)}(F_i) = 0, \quad ad(F_i)^{1+(\lambda,\alpha_i^\vee)}(v_\lambda) = 0, \quad \text{for } i = 1, \cdots, n.$$

The following theorem determines the set of coinvariants of degree 2 in the non-critical case.

Theorem 6.8. Suppose that for any $i \in I$, $(\lambda, \alpha_i^\vee) \neq 1$. Then the multiplication map gives an isomorphism of left $S_\sigma(V)$-Yetter-Drinfel'd modules

$$L(\lambda) \otimes L(\lambda) \xrightarrow{\sim} M_2^{coR}.$$

Proof. It suffices to show that in this case, the surjection $m : S_{\widetilde{\sigma}}(W)_{(1)} \otimes_{S_\sigma(V)} S_{\widetilde{\sigma}}(W)_{(1)} \to S_{\widetilde{\sigma}}(W)_{(2)}$ is an isomorphism.

The following lemma comes from basic linear algebra.

Lemma 6.8. Let U, V, W be three vector spaces and $f : U \to V$, $g : V \to W$ be two linear surjections. We denote $h = g \circ f$. Then h is surjective, $\ker f \subset \ker h$ and $\ker g = \ker h / \ker f$.

From this lemma, to prove that m is an injection, it suffices to show that $\ker(S_2 \circ m) = \ker S_1$.

We consider the difference between $\ker S_1$ and $\ker S_2$. From the general remark before the theorem and the fact that $\ker S_1$ and $\ker S_2$ are generated by quantized Serre relations, it suffices to compare the quantized Serre relations appearing in $S_{\widetilde{\sigma}}(W)_{(1)} \otimes_{S_\sigma(V)} S_{\widetilde{\sigma}}(W)_{(1)}$ and $S_{\widetilde{\sigma}}(W)_{(2)}$.

It is clear that the difference may contain only relations $ad(v_\lambda)^{1+(\lambda,\alpha_i^\vee)}(F_i) = 0$. But for a monomial in $S_{\widetilde{\sigma}}(W)_{(2)}$, v_λ appears twice, so the only possible relation in the difference of the kernel is given by $ad(v_\lambda)^2(F_i) = 0$ for some $i = 1, \cdots, n$.

As we are in the non-critical case : for any $i = 1, \cdots, n$, $(\lambda, \alpha_i^\vee) \neq 1$. This forbids such relations and thus there is no difference between $\ker S_1$ and $\ker S_2$. $\square$

6.7.3 Study of M_2^{coR} : critical case

In this subsection, we will study the critical case, that is to say, there exists some $i \in I$ such that $(\lambda, \alpha_i^\vee) = 1$.

We start from dealing with the case that there is only one $i \in I$ such that $(\lambda, \alpha_i^\vee) = 1$.

Theorem 6.9. Let $i \in I$ be the only element such that $(\lambda, \alpha_i^\vee) = 1$. We let $L(2\lambda - \alpha_i)$ denote the sub-Yetter-Drinfel'd module of $L(\lambda) \otimes L(\lambda)$ generated by

$$P_i = v_\lambda \otimes ad(v_\lambda)(F_i) - q^{2+(\lambda,\alpha_i^\vee)} ad(v_\lambda)(F_i) \otimes v_\lambda,$$

which is irreducible as a $U_q(\mathfrak{g})$-module. Then the multiplication map gives an isomorphism of Yetter-Drinfel'd modules

$$(L(\lambda) \otimes L(\lambda)) / L(2\lambda - \alpha_i) \xrightarrow{\sim} M_2^{coR}.$$

Proof. As in the proof above, we start from considering the difference between $\ker S_2$ and $\ker S_1$. Because $(\lambda, \alpha_i^\vee) = 1$, the same argument as in the last theorem shows that this difference is generated by the element $v_\lambda^2 F_i$ in $T(W)_{(2)}$, that is to say, the relation $ad(v_\lambda)^2(F_i) = 0$ in $S_{\widetilde{\sigma}}(W)_{(2)}$. (We may calculate this directly, or adopt the method given in [25]. For example, see the example in Section 6.3 therein.)

In $S_{\widetilde{\sigma}}(W)_{(1)} \otimes_{S_\sigma(V)} S_{\widetilde{\sigma}}(W)_{(1)}$, the element $v_\lambda^2 F_i$ corresponds to

$$P_i = v_\lambda \otimes ad(v_\lambda)(F_i) - q^{2-(\lambda,\alpha_i^\vee)} ad(v_\lambda)(F_i) \otimes v_\lambda.$$

This element is not zero as $(\lambda, \alpha_i^\vee) = 1$ and $q \neq \pm 1$ imply that $ad(v_\lambda)(F_i) \neq 0$. Moreover, the element P_i is also in $M_1^{coR} \otimes M_1^{coR}$ and the multiplication $m : M_1^{coR} \otimes M_1^{coR} \to M_2^{coR}$ maps P_i to 0.

As a summary, we have shown that the kernel of $m : M_1^{coR} \otimes M_1^{coR} \to M_2^{coR}$ is generated by P_i as an $S_\sigma(V)$-Yetter-Drinfel'd module because $v_\lambda^2 F_i$ is the only relation in $\ker S_2 / \ker S_1$.

Moreover, the weight of P_i is $2\lambda - \alpha_i$, which is a dominant weight after the dominance of λ and $(\lambda, \alpha_i^\vee) = 1$. Then the $S_\sigma(V)$-Yetter-Drinfel'd module generated by P_i is isomorphic to the irreducible representation $L(2\lambda - \alpha_i)$ of $U_q(\mathfrak{g})$. Thus we obtained an isomorphism of $U_q(\mathfrak{g})$-module

$$(L(\lambda) \otimes L(\lambda)) / L(2\lambda - \alpha_i) \xrightarrow{\sim} M_2^{coR}.$$

□

Remark 6.4. We should point out that, as $v_\lambda^2 \in M_2^{coR}$, there is always a copy of the highest weight representation $L(2\lambda)$ in M_2^{coR}.

Corollary 6.5. Let $J \subset I$ be the subset of I containing elements $j \in I$ satisfying $(\lambda, \alpha_j^\vee) = 1$. Then we have an isomorphism of $S_\sigma(V)$-Yetter-Drinfel'd modules

$$(L(\lambda) \otimes L(\lambda)) \Big/ \bigoplus_{j \in J} L(2\lambda - \alpha_j) \xrightarrow{\sim} M_2^{coR}.$$

where $L(2\lambda - \alpha_j)$ is the sub-Yetter-Drinfel'd module of $L(\lambda) \otimes L(\lambda)$ generated by P_j as defined in Theorem 6.9, which is irreducible as $U_q(\mathfrak{g})$-module.

Proof. In this case, the difference between $\ker S_2$ and $\ker S_1$ is generated by $\{v_\lambda^2 F_j |\ j \in J\} \subset T(W)_{(2)}$. Moreover, because these $L(2\lambda - \alpha_j)$ are irreducible as $U_q(\mathfrak{g})$-modules, they intersect trivially. Thus the same argument as in the theorem above shows that the kernel of $m : M_1^{coR} \otimes M_1^{coR} \to M_2^{coR}$ is generated by P_j, which are defined in the above theorem, as an $S_\sigma(V)$-Yetter-Drinfel'd module. This gives the corollary. □

6.7.4 Homological interpretation

Results obtained in this section can be interpreted in the framework of coHochschild homology as follows :

Theorem 6.10. Let $q \in k^*$ not be a root of unity and $\lambda \in \mathcal{P}_+$ be a dominant weight.

1. If for any $i \in I$, $(\lambda, \alpha_i^\vee) \neq 1$, then as $U_q(\mathfrak{g})$-modules,

$$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)}) = \begin{cases} L(\lambda) \otimes L(\lambda) & n = 0; \\ 0, & n \neq 0. \end{cases}$$

2. If J is the subset of I containing those $j \in I$ satisfying $(\lambda, \alpha_j^\vee) = 1$, then as $U_q(\mathfrak{g})$-modules,

$$\mathrm{Hoch}^n(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)}) = \begin{cases} (L(\lambda) \otimes L(\lambda))/ \bigoplus_{j \in J} L(2\lambda - \alpha_j) & n = 0; \\ 0, & n \neq 0. \end{cases}$$

Proof. Results on $\mathrm{Hoch}^0(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)})$ come from Proposition 6.6, Theorem 6.9 and Corollary 6.5. For the vanishing of higher homology groups, arguments from Section 6.6.4 to Section 6.6.6 on $S_{\widetilde{\sigma}}(W)_{(1)}$ can be applied similarly to $S_{\widetilde{\sigma}}(W)_{(2)}$ as we have already known $\mathrm{Hoch}^0(S_\sigma(V), S_{\widetilde{\sigma}}(W)_{(2)})$. □

6.8 Examples and PBW basis

This section is devoted to providing some examples for the construction above. Moreover, we explain how to give an inductive construction of PBW basis in the type A.

6.8.1 Examples

Theorems in proceeding sections give us some interesting examples for the construction of quantum algebras.

Example 6.1. As a warm-up example, we consider the construction of the strictly negative part of $U_q(sl_4)$ from the same part of $U_q(sl_3)$.

We adopt notations in [9], Planche I. For this construction, we suppose that $V = \{F_1, F_2\}$ with a braiding determined by the Cartan matrix ; we choose $\lambda = \varpi_2$ be the dominant weight such that $(\lambda, \alpha_1^\vee) = 0$ and $(\lambda, \alpha_2^\vee) = 1$.

Let W be the vector space generated by F_1, F_2 and v_λ and $\widetilde{\sigma}$ as in the construction.

Then $S_{\widetilde{\sigma}}(W)$ is isomorphic, as a braided Hopf algebra, to the strictly negative part of $U_q(sl_4)$.

Thus if the strictly negative part of $U_q(sl_4)$ is viewed as a Hopf bimodule over the same part of $U_q(sl_3)$, we obtain from theorems above that

$$S_{\widetilde{\sigma}}(W)_{(1)} \cong L(\varpi_2) \otimes S_\sigma(V), \quad S_{\widetilde{\sigma}}(W)_{(2)} \cong L(2\varpi_2) \otimes S_\sigma(V).$$

The second isomorphism can be proved as follows : at first, from the dimension formula (see, for example, [36]),

$$dimL(\varpi_2) = 3, \quad dimL(2\varpi_2) = 6, \quad dim(2\varpi_2 - \alpha_2) = 3.$$

Then Theorem 6.9 gives the isomorphism above by comparing dimensions.

We turn to a general construction for the type A.

Example 6.2. For and integer $n \geq 2$, the method in the example above can be used to construct $U_q(sl_{n+1})$ from $U_q(sl_n)$. It is different from the one given in [73] : braidings here are of diagonal type but in [73], they are much more complicated (for example, in type A, they are of Hecke type ; in general, they are of quantum group type).

In $U_q(sl_n)$, we choose the dominant weight $\lambda = \varpi_{n-1}$. Then the construction above gives the strictly negative part of $U_q(sl_{n+1})$. Moreover, the same argument as in the example above gives :

$$S_{\widetilde{\sigma}}(W)_{(1)} \cong L(\varpi_{n-1}) \otimes S_\sigma(V), \quad S_{\widetilde{\sigma}}(W)_{(2)} \cong L(2\varpi_{n-1}) \otimes S_\sigma(V).$$

In the case of type A_n, when $\lambda = \varpi_{n-1}$, we have a uniform description for all degree p coinvariants.

Theorem 6.11. We keep hypothesis in the above example. For any integer $p \geq 0$, we have an isomorphism of $S_\sigma(V)$-Yetter-Drinfel'd modules :

$$S_{\widetilde{\sigma}}(W)_{(p)} \cong L(p\varpi_{n-1}) \otimes S_\sigma(V).$$

Thus if we let M denote $S_{\widetilde{\sigma}}(W)$, then

$$M^{coR} \cong \bigoplus_{p=0}^{\infty} L(p\varpi_{n-1}).$$

Proof. It suffices to notice that

$$L((p-1)\varpi_{n-1}) \otimes L(\varpi_{n-1}) \cong L(p\varpi_{n-1}) \oplus L(p\varpi_{n-1} - \alpha_{n-1}).$$

□

We turn to the type D.

Example 6.3. Before going to the construction, we remark that the Planche IV of [9] has a typo not contained in the errata : in the third line of (II), the sum should be over $i \leq k < j$.

In this example, we construct the negative part of $U_q(so_8)$ from that of $U_q(sl_4)$. For this sake, we choose $\lambda = \varpi_2$. Again from the dimension formula, we obtain that

$$dim L(\varpi_2) = 6, \quad dim L(2\varpi_2) = 20, \quad dim L(2\varpi_2 - \alpha_2) = 15.$$

Thus $dim M_2^{coR} = 21$ and it must contain a unique copy of $L(2\varpi_2)$, whose dimension is 20. So we obtain that

$$M_2^{coR} \cong L(2\varpi_2) \oplus L(0).$$

As a byproduct, this construction gives a decomposition

$$L(\varpi_2) \otimes L(\varpi_2) \cong L(2\varpi_2) \oplus L(2\varpi_2 - \alpha_2) \oplus L(0).$$

Remark 6.5. For a general $n \geq 2$, we can construct $U_q(so_{2n})$ from $U_q(sl_n)$ by choosing $\lambda = \varpi_{n-2}$.

More interesting examples would be obtained when more complicated representations are chosen.

Example 6.4. The strictly positive part of $U_q(\widehat{sl_2})$.

This can be obtained from the construction above starting from the strictly negative part of $U_q(sl_2)$ by choosing $\lambda = \alpha = 2\varpi_1$, which is the highest weight of the adjoint representation of sl_2.

In this example, $(\lambda, \alpha^\vee) = 2$, so it falls into the non-critical case, from which we obtain an isomorphism of $U_q(sl_2)$-modules given by multiplication

$$L(\alpha) \otimes L(\alpha) \cong M_2^{coR}.$$

Then the latter has dimension 9.

In [15], a PBW basis of $U_q(\widehat{sl}_2)$ is constructed : the M_2^{coR} part above corresponds to 9 basis elements

$$E_{\delta+\alpha_2}, \quad E_{2\delta}, \quad E_{2\delta+\alpha_1}, \quad E_\delta E_{\delta+\alpha_1}, \quad E_\delta E_{\alpha_2}, \quad E_{\delta+\alpha_1} E_{\alpha_2}, \quad E_\delta^2, \quad E_{\alpha_2}^2, \quad E_{\delta+\alpha_1}^2,$$

where notations in [15] are adopted.

As $U_q(sl_2)$-modules,

$$L(\alpha) \otimes L(\alpha) \cong L(4\varpi_1) \oplus L(2\varpi_1) \oplus L(0);$$

this corresponds to a decomposition of these 9 PBW basis elements under the adjoint action of E_{α_1}.

Remark 6.6. For general cases, although types of the braiding are different, choices in [73], Section 4 can be adopted directly to our case. In the non-simply laced case, we should enlarge the torus part as shown in [73].

6.8.2 PBW basis : type A case

This construction can be used to give an inductive construction of PBW basis for the strictly negative part when $\mathfrak{g}$ is a finite dimensional simple Lie algebra. In this subsection, we give an example for the type A case.

We recall that in the last section of [73], there is an inductive construction of PBW basis. That construction comes from the following fact : the braiding in the type A is of Hecke type, so the quantum shuffle algebra can be factorized through an exterior algebra, which gives a PBW type basis.

The construction we will give in type A is different from the one in [73] : in the whole construction, the braiding we are considering is of diagonal type but of quantum group type, which is much simpler.

We use $S_\sigma(V)$ for the strictly negative part of $U_q(sl_n)$ and choose $\lambda = \varpi_{n-1}$ be the fundamental representation, then $S_{\widetilde{\sigma}}(W)$ is isomorphic to the strictly negative part of $U_q(sl_{n+1})$. Suppose that we have constructed a PBW basis for $S_\sigma(V)$.

From Theorem 6.11, we have

$$\bigoplus_{p=0}^{\infty} (L(p\varpi_{n-1}) \otimes S_\sigma(V)) \xrightarrow{\sim} S_{\widetilde{\sigma}}(W),$$

where $L(p\varpi_{n-1})$ is the irreducible $U_q(sl_n)$-module and the isomorphism is given by the multiplication. We have the following observations :

1. All elements in $L(p\varpi_{n-1})$ are of the form $ad(F)(v_\lambda^p)$ for some $F \in S_\sigma(V)$.
2. Any $F \in S_\sigma(V)$ can be written as a linear combination of ordered product of the PBW basis elements in $S_\sigma(V)$.
3. Let H be a Hopf algebra and M be an H-bimodule algebra. For $x \in H$ and $a, b \in M$, the adjoint action has the property :
$$ad(x)(ab) = \sum ad(x_{(1)})(a)ad(x_{(2)})(b).$$
4. As $ad(F_{n-1})^2(v_\lambda) = ad(v_\lambda)^2(F_{n-1}) = 0$, elements in $L(\varpi_{n-1})$ are q-commutative when they are multiplied in $S_{\widetilde{\sigma}}(W)$ (see [47], Proposition 3.2, page 39). (To obtain this, we do not need to know the information of the Hecke type braiding.)

So to obtain a linear basis of $L(p\varpi_{n-1})$, it suffices make all elements of $S_\sigma(V)$ act in an adjoint way on v_λ^p and from (3) above, it can be written as an ordered product of a linear basis in $L(\varpi_{n-1})$. As basis elements of $L(\varpi_{n-1})$ are q-commutative, if we let $v_1, \cdots, v_n$ denote a basis of $L(\varpi_{n-1})$ and fix some order (for example, $v_1 > \cdots > v_n$) on these basis elements, any element in M^{coR} can be written as an ordered product $v_1^{s_1} \cdots v_n^{s_n}$. We want to show that $\{v_1^{s_1} \cdots v_n^{s_n} | \ s_1, \cdots, s_n \geq 0\}$ forms a linear basis of M^{coR}.

To show this, we notice that in each v_i, there exists only one copy of v_λ. Thus it suffices to show that degree p elements in this basis form a basis of $L(p\varpi_{n-1})$.

At first, all degree p elements in $\{v_1^{s_1} \cdots v_n^{s_n} | \ s_1, \cdots, s_n \geq 0\}$ are contained in $L(p\varpi_{n-1})$; moreover, these elements will generate $L(p\varpi_{n-1})$ as a vector space. We count degree p elements in the above set, there are $\binom{n+p-1}{p}$, which is exactly the dimension of $L(p\varpi_{n-1})$.

Thus we obtain that elements in $\{v_1^{s_1}\cdots v_n^{s_n}|\ s_1,\cdots,s_n\geq 0\}$ are linearly independent, so a linear basis of M^{coR}.

6.9 Inductive construction of quivers and composition algebras

This section is devoted to explaining the results above in a graphical setting.

6.9.1 Quivers associated to quantum shuffle algebras

Let $S_\sigma(V)$ be a quantum shuffle algebra with a basis $\{F_1,\cdots,F_n\}$. Moreover, we suppose that

1. The braiding is given by a braiding matrix with diagonal q^2, the other entries are given by negative integer powers of q ;
2. The braiding σ is symmetric, that is to say, if for any $v,w\in V$, $\sigma(v\otimes w)=\alpha w\otimes v$, then $\sigma(w\otimes v)=\alpha v\otimes w$.

Starting from $S_\sigma(V)$ with these properties, we can associate with it an unoriented quiver $Q=(I,F)$ constructed by :

1. The set of vertices I is indexed by the basis $\{F_1,\cdots,F_n\}$, we let i denote the vertex corresponding to F_i ;
2. For $i\neq j\in I$, if $\sigma(F_i\otimes F_j)=q^{a_{ij}}F_j\otimes F_i$, there are $-a_{ij}$ unoriented edges in F connecting vertices i and j.

Moreover, we can choose a total order on the basis elements $\{F_1,\cdots,F_n\}$ of V (for example, $F_1\leq F_2\leq\cdots\leq F_n$). This will give an orientation on Q by changing an edge to an arrow from the smaller vertex to the bigger one.

6.9.2 Inductive construction of quiver

As we are restricted to the finite dimensional case, we will discuss the construction of new quivers from one of type A, D, or E.

Let Q be an oriented quiver of type A, D, or E. We let $\mathfrak{g}(Q)$ denote the corresponding simple Lie algebra associated to the underlying graph of Q (it means that we forget the orientation).

From the quiver Q, we can construct a quantum shuffle algebra $S_\sigma(V)$ with Cartan matrix A the adjacent matrix of Q with diagonal elements given by 2.

Let λ be a dominant weight in the weight lattice of $\mathfrak{g}(Q)$. Then as in Section 6.4, we form a quantum shuffle algebra $S_{\widetilde{\sigma}}(W)$ by adding v_λ.

The following proposition is clear from the construction.

Proposition 6.10. The unoriented quiver Q associated with $S_{\widetilde{\sigma}}(W)$ can be constructed by adding a vertex $n+1$ and $(\lambda,\alpha_i^\vee)$ edges between vertex i and $n+1$.

We can define an order on W by letting v_λ the maximal element. From now on, when we are talking about the orientation of this new quiver, it is always of this form.

Example 6.5. (1). We take Q the quiver of type A_n and $\lambda = \varpi_1 + \varpi_n$ (for the type A_1, $\lambda = 2\varpi_1$). The construction above will give us the extended Dynkin quiver $A_n^{(1)}$. And then $S_{\widetilde{\sigma}}(W)$ is the negative part of the affine quantum group $U_q(\widehat{sl_{n+1}})$.
(2). We take Q the quiver of type A_1 and $\lambda = \frac{3\alpha}{2}$. This will give us the oriented quiver

$$\bullet^1 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \bullet^2,$$

which is of wild type.

Some other type of examples associated with framed quivers will be given in the next subsection.

6.9.3 Applications

In this subsection, we will discuss the construction of some family of quiver associated with the framed quiver after Nakajima.

For an oriented quiver $Q = (I, F)$ with $|I| = n$, Nakajima associated with it a framed quiver $Q^f = (I^f, F^f)$ constructed by :

1. Taking another copy of I, denoted by I', and a bijection $I \to I'$ by $i \mapsto i'$. We define $I^f = I \sqcup I'$.
2. For any map $i \mapsto i'$ above, add an arrow from i to i' in F. Thus we obtain F^f.

A dimension vector of a representation of Q^f is a pair $(\mathbf{v}, \mathbf{w})$ where $\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{N}^I$ and $\mathbf{w} = (w_1, \cdots, w_n) \in \mathbb{N}^{I'}$. Once this dimension vector is fixed, in [14], Crawley-Boevey constructed another quiver $Q^{\mathbf{w}}$ whose representations are in bijective correspondence with representations of the framed quiver.

This new quiver $Q^{\mathbf{w}} = (I \cup \{\infty\}, F^{\mathbf{w}})$ associated with $\mathbf{w}$ is defined as follows :

1. We add a new vertex ∞.
2. For any $i \in I$, we add w_i arrows from i to ∞ in the set of arrows F, which gives $F^{\mathbf{w}}$.

Thus there is a bijection between representations of Q^f with dimension vector $(\mathbf{v}, \mathbf{w})$ and those of $Q^{\mathbf{w}}$ of dimension vector $(\mathbf{v}, 1)$.

Now we explain how to construct $Q^{\mathbf{w}}$ in our framework. We suppose that the original quiver Q is of type A, D, E as we have concentrated on these cases at the beginning.

Let $\mathfrak{g} = \mathfrak{g}(Q)$ be the Lie algebra associated with the quiver Q. For the fixed dimension vector $\mathbf{w} = (w_1, \cdots, w_n)$, we choose the weight λ in the weight lattice of $\mathfrak{g}$ such that $(\lambda, \alpha_i^\vee) = w_i$. As components of a dimension vector are non-negative, λ is dominant. Then we construct the quantum symmetric algebra $S_\sigma(V)$ corresponding to $\mathfrak{g}$ and $S_{\widetilde{\sigma}}(W)$ by adding a highest weight vector v_λ into $S_\sigma(V)$ as we did in previous sections.

It is clear from the construction above that the quiver associated to $S_{\widetilde{\sigma}}(W)$ is exactly $Q^{\mathbf{w}}$.

The following proposition comes from Theorem 6.3.

Proposition 6.11. Let $C_{Q^{\mathbf{w}}}$ denote the composition algebra associated to the quiver $Q^{\mathbf{w}}$. Then as braided Hopf algebras, we have an isomorphism :

$$S_{\widetilde{\sigma}}(W) \xrightarrow{\sim} C_{Q^{\mathbf{w}}}.$$

Chapitre 7

Dedekind η-function and quantum groups

Results in this chapter are pre-published in [28]

7.1 Introduction

7.1.1 History

The partition function $p(n)$ of a positive integer n and its numerous variants have a long history in combinatorics and number theory. A natural method to study these functions defined on the set of integers is considering their generating functions (for example : $\psi(x) = \sum_{n\geq 0} p(n)x^n$) to study their analytical properties, the algebraic equation they satisfy or the (quasi-)symmetries under group actions and so on.

In the case of partition function, although $p(n)$ augments rapidly and it is not possible to express $\psi(x)$ in a compact form, its inverse $\psi(x)^{-1}$, which seems to be more complicated, simplify the story by the formula

$$\psi(x)^{-1} = \prod_{n\geq 1}(1-x^n).$$

This inverse of $\psi(x)$, denoted by $\varphi(x)$, is a very standard mathematical object as many modular forms can be constructed starting from $\varphi(x)$. For example, $\eta(x) = x^{\frac{1}{24}}\varphi(x)$ is the Dedekind η-function and $\Delta(x) = \eta(x)^{24}$ is a modular form of weight 12 whose expansion into power series of x gives the famous Ramanujan's τ-function as coefficients.

Some other powers of $\varphi(x)$ are studied by Euler, Jacobi and some other people in various domains of mathematics such as combinatorics, number theory, θ-functions, index theorems and so on.

As an example, Euler showed a relation between $\varphi(x)$ and the pentagon numbers

$$\varphi(x) = \sum_{n\in\mathbb{Z}}(-1)^n x^{\frac{3n^2-n}{2}}$$

and then Jacobi deduced the expression of $\varphi(x)^3$ by triangle numbers in his study of elliptic functions :

$$\varphi(x)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{n(n+1)}{2}}.$$

7.1.2 Work of MacDonald and Kostant

These formulas containing certain powers of the Dedekind η-function are largely generalized in the work of I. MacDonald [59] by explaining them as particular cases of the Weyl denominator formulas associated to some affine root systems. As an example, the Jacobi identity above can be interpreted through combinatorial informations arising from the affine root system of type A_1.

To be more precise, for any reduced root system on a finite dimensional real vector space V with the standard bilinear form $(\cdot,\cdot)$, we can associate to it a complex Lie algebra $\mathfrak{g}$. The following formula (formula (0.5) in [59]) is obtained as the specialization of the Weyl denominator formula :

$$\eta(x)^d = \sum_{\mu\in M} d(\mu)x^{(\mu+\rho,\mu+\rho)/2g}, \tag{7.1}$$

where M is some set contained in the set of dominant integral weights, $d = \dim\mathfrak{g}$, $d(\mu)$ is the dimension of the irreducible representation of $\mathfrak{g}$ of highest weight μ, $g = \frac{1}{2}((\phi+\rho,\phi+\rho)-(\rho,\rho))$, ϕ is the highest root of $\mathfrak{g}$ and ρ is half of the sum of positive roots.

When the root system in the MacDonald's identity (7.1) arises from a complex compact simple Lie group G which is moreover simply connected, B. Kostant [50] made the set M precise by connecting it with the trace of a Coxeter element c in the Weyl group W acting on the subspace of weight zero $V_1(\lambda)_0$ in the irreducible representation $V_1(\lambda)$ associated to a dominant integral weight $\lambda \in \mathcal{P}_+$. If the Lie group is simply laced (i.e., type A,D,E), the formula due to Kostant reads :

$$\eta(x)^{\dim G} = \sum_{\lambda\in\mathcal{P}_+} \mathrm{Tr}(c, V_1(\lambda)_0)\dim V_1(\lambda)x^{(\lambda+\rho,\lambda+\rho)}. \tag{7.2}$$

A similar result holding for general G can be found in [50] (see Theorem 7.2).

These formulas have various explications by using different tools in Lie theory, a summary of corresponding results can be found in a Bourbaki seminar talk [17] by M. Demazure.

7.1.3 Quantum groups and representations

Quantum groups (quantized enveloping algebras) appear in the middle of eighties after the work of Drinfel'd and Jimbo in the aim of finding solutions of the Yang-Baxter equation ; it can be looked as deformations (of Hopf algebras) of classical enveloping algebras associated to symmetrizable Kac-Moody Lie algebras.

This quantization procedure deforms not only the enveloping algebras themselves but also structures related to them : integrable representations, Weyl groups and so on.

Moreover, some new structures and tools appear only after this process : R-matrices, canonical (crystal) bases, integral forms, specialization to roots of unity and so on.

The appearance of the parameter q in the quantum groups enriches the internal structure of the enveloping algebra as the latter can be recovered from the former by specializing q to 1. For example, this one-dimensional freedom allows us to separate some kinds of knots or links in labeling different crossings by this parameter.

7.1.4 Quantum Weyl group

The Weyl group associated to a finite dimensional simple Lie algebra reveals its internal symmetries by permuting root spaces. Moreover, it acts simultaneously on the integrable representations which makes it possible to give a classification of them.

In the quantization procedure mentioned above, when algebras and their representations are deformed, it is natural to study the behaviors of automorphism groups under this procedure : for instance, the Weyl group.

The quantization of Weyl groups acting on integrable representations is archived after the work of Kirillov-Reshetikhin [49] and Levendorski-Soibelman [52] in the aim of giving an explicit formula of the R-matrix. As this procedure arises from the deformation of the Poisson-Lie group structure [52], it is compatible with the whole quantization picture. To be more precise, the action of Weyl groups on the integrable representations are deformed in a way preserving the Coxeter commutation relations but increasing the order of generators which results a lift of the Weyl groups to Artin braid groups.

This tool is essential in the study of quantum groups : constructions of PBW basis and R-matrices are direct from the action of quantum Weyl groups. Moreover, the conjugation by the quantum Weyl group gives an action of the Artin braid group on the quantized enveloping algebras, which is shown in [49], [52] and [76] to agree with Lusztig's automorphisms T_i.

7.1.5 Main results

The main objective of this chapter is to prove identities in the spirit of formula (7.1) and (7.2) in the framework of quantum groups. It is surprising that powers of Dedekind η-function can be expressed as a trace of an explicit operator on the quantum coordinate algebra, which gives compact forms of identities cited above.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $U_q(\mathfrak{g})$ the associated quantum group over $\mathbb{C}(q)$. The Artin braid group $\mathfrak{B}_\mathfrak{g}$ associated to the Weyl group W of $\mathfrak{g}$ acts on the irreducible representation $V(\lambda)$ of $U_q(\mathfrak{g})$ with dominant weight $\lambda \in \mathcal{P}_+$. Let $\{\sigma_1, \cdots, \sigma_l\}$ be the set of generators of $\mathfrak{B}_\mathfrak{g}$, $\Pi = \sigma_1 \cdots \sigma_l$ be a Coxeter element and h be the Coxeter number of the Weyl group. Then $\Pi \otimes \mathrm{id}$ acts on the quantum coordinate algebra $\mathbb{C}_q[G] = \bigoplus_{\lambda \in \mathcal{P}_+} V(\lambda) \otimes V(\lambda)^*$ componentwise and we obtain finally

Theorem. The following identity holds :

$$\mathrm{Tr}(\Pi \otimes \mathrm{id}, \mathbb{C}_q[G]) = \left(\prod_{i=1}^{l} \varphi(q^{(\alpha_i, \alpha_i)}) \right)^{h+1}.$$

These formulas serve as new interpretations of results due to Macdonald and Kostant in the classical case.

7.1.6 Constitution of this chapter

After giving recollections on quantum groups and quantum Weyl groups in Section 7.2 and 7.3, we explain the relation between quantum Weyl groups and R-matrices in Section 7.4. Section 7.5 is devoted to computing the action of the centre of the Artin braid group on the irreducible representations of Lie algebra $\mathfrak{g}$. This will lead to the main theorem in Section 7.6.

7.2 Quantum groups

This section is devoted to giving some recollections on different definitions of quantum groups.

7.2.1 Notations

We fix notations for Lie algebras and their representations.

1. $\mathfrak{g}$ is a finite dimensional simple Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$. We let $l = \dim\mathfrak{h}$ denote the rank of $\mathfrak{g}$ and I be the index set $\{1, \cdots, l\}$.
2. $\Phi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the Killing form given by $\Phi(x, y) = \mathrm{Tr}(\mathrm{ad}x\mathrm{ad}y)$ for $x, y \in \mathfrak{g}$. Its restriction on $\mathfrak{h}$ is non-degenerate and then induces a bilinear form on $\mathfrak{h}^*$ which is also denoted by Φ.
3. Δ $(\Delta_+) \subset \mathfrak{h}^*$ is the set of (positive) roots of $\mathfrak{g}$.
4. $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ is the set of simple roots of $\mathfrak{g}$.
5. $C = (c_{ij})_{l\times l}$, where $c_{ij} = 2\Phi(\alpha_i, \alpha_j)/\Phi(\alpha_i, \alpha_i)$, is the Cartan matrix of $\mathfrak{g}$.
6. W is the Weyl group of $\mathfrak{g}$ generated by simple reflections $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ for $i \in I$ where $s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$.
7. A Coxeter element is a product of all simple reflections, Coxeter elements are conjugate in W.
8. $D = \mathrm{diag}(d_1, \cdots, d_l)$ is the diagonal matrix with integers d_i relatively prime such that $A = DC$ is a symmetric matrix.
9. $(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{Q}$ is the normalized bilinear form on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = a_{ij}$.
10. $\mathcal{Q} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_l$ is the root lattice and $\mathcal{Q}_+ = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_l$.
11. $\{\varpi_1, \cdots, \varpi_l\}$ is the set of fundamental weights in $\mathfrak{h}^*$ such that $(\varpi_i, \alpha_j) = \delta_{ij}$ where δ_{ij} is the Kronecker notation.
12. $\mathcal{P} = \mathbb{Z}\varpi_1 + \cdots + \mathbb{Z}\varpi_l$ is the weight lattice and $\mathcal{P}_+ = \mathbb{N}\varpi_1 + \cdots + \mathbb{N}\varpi_l$ is the set of dominant integral weights.
13. For $\lambda \in \mathcal{P}_+$, $V_1(\lambda)$ is the finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ.

7.2.2 Definition

From now on, we suppose that q is a variable and $q_i = q^{d_i}$. The q-numbers are defined by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{i=1}^{n} [i]_q.$$

Definition 7.1. The quantized enveloping algebra (quantum group) $U_q(\mathfrak{g})$ is the associative $\mathbb{C}(q)$-algebra with unit generated by E_i, F_i, K_i, K_i^{-1} for $i \in I$ and relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q_i^{c_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{c_{ij}}F_j, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

for $i \neq j \in I$,

$$\sum_{r=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-c_{ij}-r}E_jE_i^r = 0, \quad \sum_{r=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-c_{ij}-r}F_jF_i^r = 0.$$

There exists a unique Hopf algebra structure on $U_q(\mathfrak{g})$: for $i \in I$,

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$\varepsilon(K_i^{\pm 1}) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0,$$

$$S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i, \quad S(K_i) = K_i^{-1}, \quad S(K_i^{-1}) = K_i.$$

Remark 7.1. This definition works for any symmetrizable Kac-Moody Lie algebras, our restriction is determined by the nature of the problem.

For $\lambda \in \mathcal{P}_+$, we let $V(\lambda)$ denote the finite dimensional irreducible representation of $U_q(\mathfrak{g})$ of highest weight λ and type 1.

The following normalized generators will also be used

$$E_i^{(n)} = \frac{E_i^n}{[n]_q!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]_q!}.$$

7.2.3 An $\hbar$-adic version

There is an $\hbar$-adic version of the quantum group which is the original definition of Drinfel'd. As we will switch between these two versions of a quantum group several times in the later discussion, the definition is recalled in this subsection.

Let $\hbar$ be a variable and $\mathbb{C}[[\hbar]]$ be the ring of formal series in the parameter $\hbar$.

Definition 7.2. The $\hbar$-adic version of a quantized enveloping algebra $U_\hbar(\mathfrak{g})$ is the associative algebra with unit over $\mathbb{C}[[\hbar]]$, generated by X_i^+, X_i^-, H_i for $i \in I$ and relations

$$[H_i, H_j] = 0, \quad [H_i, X_j^+] = a_{ij}X_i^+, \quad [H_i, X_j^-] = -a_{ij}X_j^-,$$

$$[X_i^+, X_j^-] = \delta_{ij}\frac{e^{\hbar d_i H_i} - e^{-\hbar d_i H_i}}{e^{\hbar d_i} - e^{-\hbar d_i}},$$

for $i \neq j \in I$,

$$\sum_{r=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} (X_i^{\pm})^{1-c_{ij}-r} X_j^{\pm} (X_i^{\pm})^r = 0.$$

There exists a unique Hopf algebra structure on $U_\hbar(\mathfrak{g})$:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^+) = X_i^+ \otimes e^{\hbar d_i H_i} + 1 \otimes X_i^+,$$

$$\Delta(X_i^-) = X_i^- \otimes 1 + e^{-\hbar d_i H_i} \otimes X_i^-,$$

$$\varepsilon(X_i^+) = 0, \quad \varepsilon(X_i^-) = 0, \quad \varepsilon(H_i) = 0,$$

$$S(X_i^+) = -X_i^+ e^{-\hbar d_i H_i}, \quad S(X_i^-) = -e^{\hbar d_i H_i} X_i^-, \quad S(H_i) = -H_i.$$

Remark 7.2. 1. To obtain $U_q(\mathfrak{g})$ inside $U_\hbar(\mathfrak{g})$, it suffices to take $q = e^\hbar$ and $K_i = e^{\hbar d_i H_i}$.

2. The advantage of working in the $\hbar$-adic framework is that the R-matrix can be well defined when a completion of the tensor product is properly chosen. But as a disadvantage, we could not specialize $U_\hbar(\mathfrak{g})$ to any complex number except 0.

7.2.4 Specialization

Let $U(\mathfrak{g})$ be the enveloping algebra associated to $\mathfrak{g}$ with generators e_i, f_i, h_i for $i \in I$.

It should be remarked that $U_q(\mathfrak{g})$ has a $\mathbb{Z}[q, q^{-1}]$-form which is called an integral form (for example, see Chapter 9 in [12] for details). This integral form allows us to specialize $U_q(\mathfrak{g})$ to any non-zero complex number. We let $\lim_{q\to\alpha} U_q(\mathfrak{g})$ denote the specialized Hopf algebra. It is well known that $\lim_{q\to 1} U_q(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})$.

To be more precise, under the specialization $q \to 1$, K_i is sent to 1 and $[K_i; 0] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ gives h_i for $i \in I$.

Moreover, finite dimensional representations of $U_q(\mathfrak{g})$ can be specialized : for example, when q tends to 1, the representation $V(\lambda)$ will be specialized to $V_1(\lambda)$.

7.3 Quantum Weyl groups

This section is devoted to giving a summary for the definition of quantum Weyl groups in the ordinary and $\hbar$-adic cases. Then we discuss their relations and specializations.

7.3.1 Braid groups associated to Weyl groups

We start from considering some specific elements in the Artin braid group associated to a Weyl group.

For $i, j \in I$, if the product $c_{ij}c_{ji} = 0, 1, 2, 3, 4$, we let $m_{ij} = 2, 3, 4, 6, \infty$ respectively.

Definition 7.3. The Artin braid group $\mathfrak{B}_\mathfrak{g}$ associated to the Weyl group W of $\mathfrak{g}$ is a group generated by $\sigma_1, \cdots, \sigma_l$ and relations

$$\sigma_i\sigma_j\cdots\sigma_i\sigma_j = \sigma_j\sigma_i\cdots\sigma_j\sigma_i,$$

where lengths of words in both sides are m_{ij}.

For example, if the Lie algebra $\mathfrak{g}$ is of type A_l, then $m_{ij}=3$ if $|i-j|=1$, otherwise $m_{ij}=2$. The Artin group $\mathfrak{B}_\mathfrak{g}$ is the usual braid group $\mathfrak{B}_{l+1}$.

We let $\Pi = \sigma_1\cdots\sigma_l$ be a product of generators in $\mathfrak{B}_\mathfrak{g}$ and call it a Coxeter element. We let h denote the Coxeter number of the Weyl group W. The following proposition explains some properties concerning the Coxeter element Π.

For an element w in the Weyl group W with reduced expression $w = s_{i_1}\cdots s_{i_t}$, we let $T(w)=\sigma_{i_1}\cdots\sigma_{i_t}$ be the element in $\mathfrak{B}_\mathfrak{g}$. It is well-known that $T(w)$ is independent of the reduced expression. Let w_0 be the longest element in W. We call $\Delta = T(w_0)$ the Garside element in $\mathfrak{B}_\mathfrak{g}$.

Proposition 7.1 ([10], Lemma 5.8 and Satz 7.1)**.**

1. Let Δ be the Garside element in $\mathfrak{B}_\mathfrak{g}$. Then $\Pi^h = \Delta^2$.
2. If $\mathfrak{g}$ is not isomorphic to sl_2, the centre $Z(\mathfrak{B}_\mathfrak{g})$ of $\mathfrak{B}_\mathfrak{g}$ is generated by Δ^2.
3. If $\mathfrak{g}\cong sl_2$, the centre $Z(\mathfrak{B}_\mathfrak{g})$ is generated by $\sigma_1 = \Delta$.

7.3.2 Braid group acting on representations

We define the q-exponential function by

$$\exp_q(x) = \sum_{k=0}^{\infty}\frac{1}{[k]_q!}q^{\frac{k(k-1)}{2}}x^k.$$

The objective of this section is to recall an action of Artin braid group associated to the Weyl group of $\mathfrak{g}$ on the integrable modules of $U_q(\mathfrak{g})$, following [49], [52] and [76]. We start from the sl_2 case.

For $i\in I$, we let $U_q(\mathfrak{g})_i$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by E_i, F_i and $K_i^{\pm 1}$. It inherits a Hopf algebra structure from $U_q(\mathfrak{g})$. Moreover, as a Hopf algebra, $U_q(\mathfrak{g})_i$ is isomorphic to $U_{q_i}(sl_2)$.

Let $V(n)$ denote the $(n+1)$-dimensional irreducible representation of $U_q(\mathfrak{g})_i$ of type 1. For $i\in I$, we define an endomorphism $S_i\in \mathrm{End}(V(n))$ by

$$S_i = \exp_{q_i^{-1}}(q_i^{-1}E_iK_i^{-1})\exp_{q_i^{-1}}(-F_i)\exp_{q_i^{-1}}(q_iE_iK_i)q_i^{H_i(H_i+1)/2},$$

where $q_i^{H_i(H_i+1)/2}$ sends $v\in V(n)$ to $q_i^{m(m+1)/2}$ if $K_i.v = q_i^m.v$. This operator $S_i\in \mathrm{End}(V(n))$ is well-defined as both E_i and F_i act nilpotently on $V(n)$.

We want to obtain an explicit form for the action of S_i. If a basis $v_0,\cdots,v_n$ of $V(n)$ is chosen in such a way that

$$E_i.v_0 = 0,\quad F_i^{(k)}.v_0 = v_k,\quad K_i.v_0 = q_i^n v_0,$$

we have the following result.

Lemma 7.1 ([76])**.** The action of S_i on $V(n)$ is given by

$$S_i.v_k = (-1)^{n-k} q_i^{(n-k)(k+1)} v_{n-k}.$$

As a direct corollary, $S_i \in \mathrm{End}(V(n))$ is an automorphism.

We turn to the general case. Let M be an integrable $U_q(\mathfrak{g})$-module. As M is a direct sum of irreducible $U_q(\mathfrak{g})_i$-modules, $S_i \in \mathrm{End}(M)$ is well-defined. As these S_i are invertible, we could consider the group generated by $\{S_i |\ i \in I\}$ in $\mathrm{End}(M)$. Using relations between S_i and Lusztig's automorphism T_i which will be recalled later, Saito proved the following result :

Proposition 7.2 ([76])**.** Let M be an integrable $U_q(\mathfrak{g})$-module. The assignment $\sigma_i \mapsto S_i$ extends to a group homomorphism between the the Artin group $\mathfrak{B}_\mathfrak{g}$ and the subgroup of $Aut(M)$ generated by $\{S_i |\ i \in I\}$.

7.3.3 Lusztig's automorphism

There is another action of the Artin braid group on $U_q(\mathfrak{g})$ constructed by Lusztig, see [58] for details.

Definition 7.4. There exist algebra automorphisms $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i, \quad T_i(K_j) = K_j K_i^{-a_{ij}}$$

and for any $i \neq j$,

$$T_i(E_j) = \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)},$$

$$T_i(F_j) = \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)}.$$

These T_i are called Lusztig's automorphisms.

It is proved by Lusztig that these T_i satisfy the relations in the Artin braid group associated to the Weyl group of $\mathfrak{g}$, which gives the following proposition :

Proposition 7.3 ([58])**.** The assignment $\sigma_i \mapsto T_i$ extends to a group homomorphism between the Artin braid group $\mathfrak{B}_\mathfrak{g}$ and the subgroup of $Aut(U_q(\mathfrak{g}))$ generated by $\{T_i |\ i \in I\}$.

The quantum group $U_q(\mathfrak{g})$ acts on the integrable module M and the two braid group actions above are closely related in the following way :

Proposition 7.4 ([76])**.** For any $x \in U_q(\mathfrak{g})$ and any integrable $U_q(\mathfrak{g})$-module M, $T_i(x) = S_i x S_i^{-1}$ in $\mathrm{End}(M)$.

After Proposition 7.2 and 7.3, we can define a Coxeter element Π in both $\mathrm{Aut}(M)$ and $\mathrm{Aut}(U_q(\mathfrak{g}))$ by $S_1 \cdots S_l$ and $T_1 \cdots T_l$ respectively. Results in Proposition 7.1 hold for these elements in both of the automorphism groups.

7.3.4 Quantum Weyl group : $\hbar$-adic version

We start from the sl_2 case as above.

Let $U_\hbar(sl_2)$ be the $\hbar$-adic quantized enveloping algebra associated to sl_2. Let $V(n)$ be the irreducible representation of $U_\hbar(sl_2)$ of rank $n+1$. It is a free $\mathbb{C}[[\hbar]]$-module with basis $u_0, \cdots, u_n$ such that

$$H.u_k = (n-2k)u_k, \quad X^+.u_k = [n-k+1]_q u_{k-1}, \quad X^-.u_k = [k+1]_q u_{k+1}.$$

For $r, s = 0, \cdots, n$, we define linear functions $C_{r,s}^{(n)} : U_\hbar(sl_2) \to \mathbb{C}[[\hbar]]$ by : for any $x \in U_\hbar(sl_2)$,

$$x.u_s = \sum_{r=0}^{n} C_{r,s}^{(n)}(x)u_r.$$

Let $\mathcal{F}_\hbar(sl_2) \subset U_\hbar(sl_2)^*$ denote its $\mathbb{C}[[\hbar]]$-subalgebra generated by the set $\{C_{r,s}^{(n)} | \ n \in \mathbb{N},\ 0 \leq r, s \leq n\}$. We define a linear form $w_\hbar \in \mathcal{F}_\hbar(sl_2)^*$ by :

$$w_\hbar(C_{r,s}^{(n)}) = \begin{cases} (-1)^r e^{\frac{1}{4}\hbar n^2 + \hbar r} & \text{if } r+s=n, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that there is a natural embedding $U_\hbar(sl_2) \to \mathcal{F}_\hbar(sl_2)^*$, we let $\widetilde{U_\hbar}(sl_2)$ denote the subalgebra of $\mathcal{F}_\hbar(sl_2)^*$ generated by $U_\hbar(sl_2)$ and $w_\hbar$ and call it the quantum Weyl group of $U_\hbar(sl_2)$. The commutation relations between $w_\hbar$ and elements in $U_\hbar(sl_2)$ are given in Section 8.2 of [12].

There is another element $\widetilde{w}_\hbar$ defined by

$$\widetilde{w}_\hbar = w_\hbar \exp\left(-\frac{\hbar}{4}H^2\right) = \exp\left(-\frac{\hbar}{4}H^2\right) w_\hbar \in \mathcal{F}_\hbar(sl_2)^*.$$

We will see later that $\widetilde{w}_\hbar$ is closely related to the Artin braid group action defined above.

We turn to the general case where $\mathfrak{g}$ is a simple Lie algebra. We notice that $U_\hbar(\mathfrak{g})$ is generated by $U_\hbar(\mathfrak{g})_i$, where $U_\hbar(\mathfrak{g})_i$ is the sub-Hopf algebra of $U_\hbar(\mathfrak{g})$ generated by X_i^+, X_i^- and H_i which is moreover isomorphic to $U_{d_i\hbar}(sl_2)$ as a Hopf algebra.

The inclusion $U_\hbar(\mathfrak{g})_i \to U_\hbar(\mathfrak{g})$ induces a projection $\mathcal{F}_\hbar(\mathfrak{g}) \to \mathcal{F}_\hbar(\mathfrak{g})_i$ given by the restriction where $\mathcal{F}_\hbar(\mathfrak{g})$ is the Hopf algebra generated by matrix elements of all finite dimensional irreducible representations of $U_\hbar(g)$ of type 1, and then induces an inclusion $\mathcal{F}_\hbar(\mathfrak{g})_i^* \to \mathcal{F}_\hbar(\mathfrak{g})^*$.

As $\mathcal{F}_\hbar(\mathfrak{g})_i^*$ is isomorphic to $\mathcal{F}_{d_i\hbar}(sl_2)^*$, we can pull back $w_\hbar \in \mathcal{F}_{d_i\hbar}(sl_2)^*$ to obtain an element $w_{\hbar,i} \in \mathcal{F}_\hbar(\mathfrak{g})_i^*$. We define

$$\widetilde{w}_{\hbar,i} = w_{\hbar,i} \exp\left(-\frac{\hbar d_i}{4}H_i^2\right) = \exp\left(-\frac{\hbar d_i}{4}H_i^2\right) w_{\hbar,i} \in \mathcal{F}_\hbar(\mathfrak{g})_i^*.$$

Definition 7.5. The quantum Weyl group $\widetilde{U_\hbar}(\mathfrak{g})$ associated to $U_\hbar(\mathfrak{g})$ is the subalgebra of $\mathcal{F}_\hbar(\mathfrak{g})^*$ generated by $U_\hbar(\mathfrak{g})$ and $w_{\hbar,i}$ for $i \in I$.

In fact, the quantum Weyl group is a Hopf algebra, after the following proposition.

Proposition 7.5 ([49],[52]). With the following definition

$$\Delta(w_{\hbar,i}) = R_i^{-1}(w_{\hbar,i} \otimes w_{\hbar,i}), \quad \varepsilon(w_{\hbar,i}) = 1, \quad S(w_{\hbar,i}) = w_{\hbar,i} e^{\hbar d_i H_i},$$

where R_i^{-1} is the inverse of the R-matrix associated $U_\hbar(\mathfrak{g})_i$ (see Section 7.4.2 for a detailed discussion), the quantum Weyl group $\widetilde{U_\hbar}(\mathfrak{g})$ is a Hopf algebra.

Remark 7.3. In the classical case, as can be viewed as a group algebra, the Weyl group itself is a Hopf algebra. But this is not the case when everything is quantized : according to the proposition above, these $w_{\hbar,i}$ are not group-like and the antipode does not give the inverse. That is why it is needed to put the $U_\hbar(\mathfrak{g})$ part in the definition of the quantum Weyl group to retain the Hopf algebra structure.

The following result gives a relation between S_i, T_i and $w_{\hbar,i}$.

Proposition 7.6. After the identification $q = e^\hbar$ and $K_i = e^{\hbar d_i H_i}$, for any integrable $U_q(\mathfrak{g})$-module M, $\widetilde{w}_{\hbar,i}$ and S_i coincide as elements in $\mathrm{End}(M)$.

Proof. It suffices to consider the sl_2 case and representation $V(n)$, which is the case we can compute.

Choosing a basis of $V(n)$ as in Section 7.3.2, we want to compute the action of $w_\hbar$ on v_i. As $V(m)$ is a finite dimensional left $U_\hbar(sl_2)$-module, it inherits a right $\mathcal{F}_\hbar(sl_2)$-comodule structure and then a left $\mathcal{F}_\hbar(sl_2)^*$-module structure ; after an easy computation, we have

$$w_\hbar.v_i = \sum_{j=0}^{n} w_\hbar(C_{j,i}^{(n)})v_j = (-1)^{n-i} e^{\hbar(\frac{1}{4}n^2+n-i)} v_{n-i}.$$

As $\exp\left(-\frac{\hbar}{4}H^2\right)$ acts as a scalar $e^{-\frac{\hbar}{4}(n-2i)^2}$ on v_i, the action of $\widetilde{w}_\hbar$ on v_i is given by

$$\widetilde{w}_\hbar.v_i = (-1)^{n-i} e^{\hbar(n-i)(i+1)} v_{n-i},$$

which coincides with the action of S_i in Lemma 7.1. □

As a direct consequence, for any $x \in U_q(\mathfrak{g})$,

$$\widetilde{w}_{\hbar,i} x \widetilde{w}_{\hbar,i}^{-1} = T_i(x)$$

as endomorphisms in $\mathrm{End}(M)$, where in the left hand side, we consider x as an element in $U_\hbar(\mathfrak{g})$.

7.3.5 Specialization

For $\lambda \in \mathcal{P}_+$, as $V(\lambda)$ is an integrable $U_q(\mathfrak{g})$-module, $S_i \in \mathrm{End}(V(\lambda))$. Once the variable q is specialized to 1, $V(\lambda)$ goes to $V_1(\lambda)$ and S_i is sent to

$$s_i = \exp(e_i)\exp(-f_i)\exp(e_i) \in \mathrm{End}(V_1(\lambda)),$$

which coincides with the action of the simple reflection s_i in the Weyl group W on the integrable representations of $U(\mathfrak{g})$.

7.4 R-matrix

One of the remarkable properties of quantum groups comes from the quasi-triangularity.

7.4.1 Definition and construction

Definition 7.6. Let H be a Hopf algebra and $R \in H \otimes H$ be an invertible element. The pair (H, R) is called a quasi-triangular Hopf algebra (QTHA) if

1. For any $x \in H$, $\Delta^{op}(x)R = R\Delta(x)$.
2. $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$, $(\mathrm{id} \otimes \Delta) = R_{13}R_{12}$, where for $R = \sum s_i \otimes t_i$, $R_{13} = \sum s_i \otimes 1 \otimes t_i$, $R_{12} = \sum s_i \otimes t_i \otimes 1$, $R_{23} = \sum 1 \otimes s_i \otimes t_i$.

If this is the case, we call R an R-matrix.

If (H, R) is a quasi-triangular Hopf algebra, the R-matrix will satisfy the famous Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

which endows a braid structure on the category of H-modules.

The following well-known theorem due to Drinfel'd gives one of the advantages of quantum groups.

Theorem 7.1. There exists $R \in U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$ such that $(U_\hbar(\mathfrak{g}), R)$ is a QTHA.

Remark 7.4. In fact, the R-matrix of $U_\hbar(\mathfrak{g})$ exists only in a completion of $U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$, see [46], Chapter XVII for a detailed discussion on this problem.

7.4.2 Construction of R-matrix

The aim of this subsection is to recall an explicit construction of the R-matrix, which will be used in the proof of the main theorem.

We start from the construction of PBW basis elements in $U_q(\mathfrak{g})$.

Let $w \in W$ be an element in the Weyl group of $\mathfrak{g}$ with a reduced expression $w = s_{i_1} \cdots s_{i_t}$. Lusztig defined an automorphism $T_w \in \mathrm{Aut}(U_q(\mathfrak{g}))$ by $T_w = T_{i_1} \cdots T_{i_t}$ and showed that it is independent of the reduced expression chosen in the very beginning.

Now let $w_0 \in W$ be the longest element in the Weyl group. We fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ where N is the cardinal of the set of positive roots in $\mathfrak{g}$. We denote

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \cdots, \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N});$$

the set $\{\beta_1, \cdots, \beta_N\}$ coincides with the set of positive roots. The root vectors in $U_q(\mathfrak{g})$ are defined by : for $r = 1, \cdots, N$,

$$E_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(E_{i_r}), \quad F_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(F_{i_r}).$$

Then the set

$$\{E_{\beta_1}^{r_1} \cdots E_{\beta_N}^{r_N} K_1^{s_1} \cdots K_l^{s_l} F_{\beta_1}^{t_1} \cdots F_{\beta_N}^{t_N} |\ r_1, \cdots, r_N, t_1, \cdots, t_N \in \mathbb{N}, s_1, \cdots, s_l \in \mathbb{Z}\}$$

forms a linear basis of $U_q(\mathfrak{g})$.

For the $\hbar$-adic version of quantized enveloping algebras, the same construction can be applied to construct $X^+_{\beta_i}$ and $X^-_{\beta_i}$ through replacing E_{i_r} by $X^+_{i_r}$ and F_{i_r} by $X^-_{i_r}$ in the definition of Lusztig's automorphisms T_i and the root vectors E_{β_r}, F_{β_r}.

The R-matrix of $U_\hbar(\mathfrak{g})$ is given by

$$R_\hbar = \exp\left(\hbar\sum_{i,j} B_{ij}H_i\otimes H_j\right)\overrightarrow{\prod_{\beta\in\Delta_+}}\exp_{q_\beta}\left((1-q_\beta^{-2})X^+_\beta\otimes X^-_\beta\right)$$

where $B=(B_{ij})$ is the inverse of the Cartan matrix C and $q_\beta = e^{\hbar d_\beta}$ ($d_\beta = d_i$ if the positive root β is conjugate to α_i under the action of Weyl group). Moreover, the product is taken in the order on the set of positive roots induced by the fixed decomposition of w_0.

For example, in the case of $U_\hbar(sl_2)$, the R-matrix can be written explicitly as :

$$R_\hbar = \left(\sum_{m=0}^\infty\left(\frac{\hbar}{2}\right)^m\frac{1}{m!}H^m\otimes H^m\right)\left(\sum_{n=0}^\infty\frac{(1-q^{-2})^n}{[n]_q!}q^{n(n+1)/2}(X^+)^n\otimes(X^-)^n\right).$$

7.4.3 Drinfel'd element u

Let $R=\sum s_i\otimes t_i$ be the R-matrix of the $\hbar$-adic quantized enveloping algebra $U_\hbar(\mathfrak{g})$.

Drinfeld defined an invertible element $u=\sum S(t_i)s_i\in U_\hbar(\mathfrak{g})$ in [22] such that the square of the antipode S^2 coincides with the adjoint action of u. As S^2 is given by the adjoint action of $K_{2\rho}=e^{\hbar H_{2\rho}}$, where $H_{2\rho}=H_{\beta_1}+\cdots+H_{\beta_N}$ ($H_{\beta_i}=H_{\alpha_{j_1}}+\cdots+H_{\alpha_{j_t}}$ if $\beta_i=\alpha_{j_1}+\cdots+\alpha_{j_t}$) using notations in the last section, the element $K_{-2\rho}u=uK_{-2\rho}$ is in the centre of $U_\hbar(\mathfrak{g})$.

As $K_{-2\rho}u$ is in the centre of $U_\hbar(\mathfrak{g})$, it acts on $V(\lambda)$ as a scalar after Schur lemma. The following lemma is well-known :

Lemma 7.2. The central element $K_{-2\rho}u$ acts as multiplication by $q^{-(\lambda,\lambda+2\rho)}$ on $V(\lambda)$.

We let $V(\lambda)_0$ denote the subspace of weight 0 in $V(\lambda)$. As $K_{-2\rho}$ acts as identity on $V(\lambda)_0$, u acts as a scalar $q^{-(\lambda,\lambda+2\rho)}$ on it.

For example, using the explicit expression of the R-matrix, we have the following formula of u in the case of sl_2 : (we write E for X^+ and F for X^-) :

$$u=\sum_{n=0}^\infty\frac{(1-q^{-2})^n}{[n]_q!}q^{n(n+1)/2}S(F^n)\left(\sum_{m=0}^\infty\left(-\frac{\hbar}{2}\right)^m\frac{1}{m!}H^{2m}\right)E^n.$$

As $\mathfrak{B}_\mathfrak{g}$ is an Artin braid group, the square of $w_{\hbar,i}$ will not give identity in general. In fact, it is closely related to the Drinfel'd element, as will be explained in the following result. It should be remarked that it is slightly different from Proposition 8.2.4 in [12].

Proposition 7.7 ([49])**.** For any $j\in I$, let $u_{\alpha_j}=\sum S(t_i)s_i$, where $R_{\alpha_j}=\sum s_i\otimes t_i$ is the R-matrix of $U_\hbar(\mathfrak{g})_j$. Then

$$w_{\hbar,j}^2 = u_{\alpha_j}^{-1}\exp\left(\hbar d_jH_j\right)\varepsilon_j,$$

where $\varepsilon_j\in\mathcal{F}_\hbar(sl_2)_j^*$ is defined by $\varepsilon_j(C^{(n)}_{r,s})=(-1)^n$.

Proof. It suffices to show this in the case of sl_2. We adopt notations in Section 7.3.2, then for $0 \leq k \leq n$, from the computation in Proposition 7.6,

$$\begin{aligned} w_\hbar^2.v_k &= w_\hbar\left((-1)^{n-k}e^{\hbar\left(\frac{1}{4}n^2+n-k\right)}v_{n-k}\right) \\ &= (-1)^n e^{\hbar\left(\frac{1}{2}n^2+n\right)}v_k. \end{aligned}$$

On the other side, after Lemma 7.2,

$$u^{-1}K\varepsilon.v_k = (-1)^n q^{\frac{1}{2}n^2+n}v_k.$$

□

7.5 Action of central element

The main target of this section is to compute the action of $Z(\mathfrak{B}_\mathfrak{g})$ on $V(\lambda)$.

7.5.1 Action on extremal vectors

The Artin braid group $\mathfrak{B}_\mathfrak{g}$ acts on $V(\lambda)$. From Proposition 7.1, let $\theta = \Delta^2 = \Pi^h$ be the generator of the centre $Z(\mathfrak{B}_\mathfrak{g})$; we compute the action of θ on the highest weight vector v_λ in this subsection.

Let $w_0 = s_{i_1}\cdots s_{i_N}$ be the fixed reduced expression of w_0 as in Section 7.4.2, then we have

Lemma 7.3. The generator of $Z(\mathfrak{B}_\mathfrak{g})$ can be written as

$$\theta = \sigma_{i_1}\cdots\sigma_{i_N}\sigma_{i_N}\cdots\sigma_{i_1}.$$

Proof. As both sides have the same length, it suffices to show that the right hand side is in the centre of $\mathfrak{B}_\mathfrak{g}$. Since the Garside element Δ has the property $\Delta\sigma_i = \sigma_{l-i}\Delta$, for any $1 \leq t \leq l$,

$$\sigma_t\sigma_{i_1}\cdots\sigma_{i_N}\sigma_{i_N}\cdots\sigma_{i_1} = \sigma_{i_1}\cdots\sigma_{i_N}\sigma_{l-t}\sigma_{i_N}\cdots\sigma_{i_1} = \sigma_{i_1}\cdots\sigma_{i_N}\sigma_{i_N}\cdots\sigma_{i_1}\sigma_t.$$

□

Return to our situation, when acting on $V(\lambda)$, the central element θ in $\mathfrak{B}_\mathfrak{g}$ has the following expression after Proposition 7.4 :

$$\begin{aligned} \theta &= S_{i_1}\cdots S_{i_{N-1}}S_{i_N}^2S_{i_{N-1}}\cdots S_{i_1} \\ &= S_{i_1}\cdots S_{i_{N-2}}T_{i_{N-1}}(S_{i_N}^2)S_{i_{N-1}}^2S_{i_{N-2}}\cdots S_{i_1} \\ &= T_{i_1}\cdots T_{i_{N-1}}(S_{i_N}^2)S_{i_1}\cdots S_{i_{N-1}}S_{i_{N-1}}\cdots S_{i_1} \\ &= T_{i_1}\cdots T_{i_{N-1}}(S_{i_N}^2)T_{i_1}\cdots T_{i_{N-2}}(S_{i_{N-1}}^2)\cdots T_{i_1}(S_{i_2}^2)S_{i_1}^2. \end{aligned}$$

We start from computing $S_{i_1}^2.v_\lambda$. Combining Proposition 7.6 and 7.7, for any $k = 1,\cdots,l$,

$$S_k^2 = \widetilde{w}_{\hbar,k}^2 = w_{\hbar,k}^2\exp\left(-\frac{\hbar}{2}d_kH_k^2\right) = u_k^{-1}\exp\left(\hbar d_kH_k\right)\exp\left(-\frac{\hbar}{2}d_kH_k^2\right)\varepsilon_k.$$

When $S_{i_1}^2$ is acted on v_λ, ε_{i_1} gives a constant $c_1 \in \{\pm 1\}$ and $\exp(\hbar d_{i_1} H_{i_1})$ gives $q^{(\lambda,\alpha_{i_1})}$. We compute the action of

$$u_{i_1} = \sum_{n=0}^{\infty} Q(n) S(F_{i_1}^n) \left(\sum_{m=0}^{\infty} \left(\frac{\hbar d_{i_1}}{2} \right)^m \frac{1}{m!} S(H_{i_1}^m) H_{i_1}^m \right) E_{i_1}^n$$

on the highest weight vector where $Q(n)$ is a rational function in q such that $Q(0) = 1$: it is clear that only the middle part containing H_{i_1} contributes, which is given by :

$$\sum_{m=0}^{\infty} \left(-\frac{\hbar d_{i_1}}{2} \right)^m \frac{1}{m!} H_{i_1}^{2m}.v_\lambda = e^{-\frac{\hbar}{2} d_{i_1} (\lambda,\alpha_{i_1})^2}.$$

As a consequence, $u_{i_1}^{-1}$ acts as a scalar $\exp\left(-\frac{\hbar}{2} d_{i_1} (\lambda, \alpha_{i_1})^2\right)$ on v_λ. As

$$\exp\left(\frac{\hbar}{2} d_{i_1} H_{i_1}^2 \right).v_\lambda = e^{-\frac{\hbar}{2} d_{i_1} (\lambda,\alpha_{i_1})^2},$$

we finally obtain that

$$S_{i_1}^2.v_\lambda = c_1 q^{(\lambda,\alpha_1)} v_\lambda.$$

We turn to consider the action of a general term $T_{i_1} \cdots T_{i_{k-1}}(S_{i_k}^2)$ on v_λ. The same argument as above can be applied here : we let

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}),$$

then

$$T_{i_1} \cdots T_{i_{k-1}}(S_{i_k}^2) = T_{i_1} \cdots T_{i_{k-1}} \left(u_{i_k}^{-1} \exp(\hbar d_{i_k} H_{i_k}) \exp\left(-\frac{\hbar}{2} d_{i_k} H_{i_k}^2 \right) \varepsilon_{i_k} \right).$$

Since $T_{i_1} \cdots T_{i_{k-1}}$ is an algebra morphism, we compute each part in the right hand side :

$$T_{i_1} \cdots T_{i_{k-1}} \left(\exp\left(\hbar d_{i_k} H_{i_k} \right) \right) = K_{\beta_k},$$

$$T_{i_1} \cdots T_{i_{k-1}} \left(\exp\left(-\frac{\hbar}{2} d_{i_k} H_{i_k}^2 \right) \right) = \exp\left(-\frac{\hbar}{2} d_{i_k} H_{\beta_k}^2 \right),$$

and

$$\begin{aligned} & T_{i_1} \cdots T_{i_{k-1}}(u_{i_k}) \\ = \; & T_{i_1} \cdots T_{i_{k-1}} \left(\sum_{n=0}^{\infty} (*) S(F_{i_k}^n) \left(\sum_{m=0}^{\infty} \left(-\frac{\hbar d_{i_k}}{2} \right)^m \frac{1}{m!} H_{i_k}^{2m} \right) \right) T_{i_1} \cdots T_{i_{k-1}} \left(E_{i_k}^n \right) \\ = \; & T_{i_1} \cdots T_{i_{k-1}} \left(\sum_{n=0}^{\infty} (*) S(F_{i_k}^n) \left(\sum_{m=0}^{\infty} \left(-\frac{\hbar d_{i_k}}{2} \right)^m \frac{1}{m!} H_{i_k}^{2m} \right) \right) E_{\beta_k}^n. \end{aligned}$$

When acting on the highest weight vector, the only part which contributes is the middle one consisting of H_{i_k}, so it suffices to compute

$$T_{i_1} \cdots T_{i_{k-1}} \left(\sum_{m=0}^{\infty} \left(-\frac{\hbar d_{i_k}}{2} \right)^m \frac{1}{m!} H_{i_k}^{2m} \right) = \exp\left(-\frac{\hbar}{2} d_{i_k} H_{\beta_k}^2 \right),$$

so $T_{i_1}\cdots T_{i_{k-1}}(u_{i_k}^{-1})$ acts as $\exp\left(\frac{\hbar}{2}d_{i_k}H_{\beta_k}^2\right)$ on v_λ.

Combining the computation above, we have : there exists a constant $c_k\in\{\pm1\}$ such that

$$T_{i_1}\cdots T_{i_{k-1}}(S_{i_k}^2).v_\lambda = c_k q^{(\lambda,\beta_k)}v_\lambda.$$

As $\beta_1,\cdots,\beta_N$ run over the set of positive roots, we have proved the following result :

Proposition 7.8. There exists a constant $c\in\{\pm1\}$ such that

$$\theta.v_\lambda = cq^{(\lambda,2\rho)}v_\lambda.$$

As $\theta\in Z(\mathfrak{B}_\mathfrak{g})$, it acts by the same constant on each $\mathfrak{B}_\mathfrak{g}$-orbit in $V(\lambda)$.

7.5.2 Central automorphism action

Let $U_q^{\geq 0}(\mathfrak{g})$, (reps. $U_q^{\leq 0}(\mathfrak{g}); U_q^{<0}(\mathfrak{g})$) denote the sub-Hopf algebra of $U_q(\mathfrak{g})$ generated by $E_i, K_i^{\pm1}$ (resp. $F_i, K_i^{\pm1}$; F_i). We compute the action of $T_{w_0}^2$ on PBW basis of $U_q^{\leq 0}(\mathfrak{g})$ in this subsection. It is known that T_{w_0} permutes $U_q^{\geq 0}(\mathfrak{g})$ and $U_q^{\leq 0}(\mathfrak{g})$, so $T_{w_0}^2$ is an automorphism of $U_q^{\leq 0}(\mathfrak{g})$ and $U_q^{\geq 0}(\mathfrak{g})$.

For $i\in I$, we let $\hat{i}$ denote the index satisfying $w_0(\alpha_i)=\alpha_{\hat{i}}$.

Lemma 7.4. For $i\in I$, the following identities hold :

$$T_{w_0}^2(E_i) = q^{(\alpha_i,\alpha_i)}K_i^{-2}E_i,\quad T_{w_0}^2(F_i) = q^{(\alpha_i,\alpha_i)}K_i^{2}F_i,\quad T_{w_0}^2(K_i)=K_i.$$

Proof. A similar computation as in [56] Section 5.7 gives

$$T_{w_0}(E_i) = -F_{\hat{i}}K_{\hat{i}},\quad T_{w_0}(F_i) = -K_{\hat{i}}^{-1}E_{\hat{i}},\quad T_{w_0}(K_i) = K_{\hat{i}}^{-1}.$$

Then the lemma is clear as $w_0(\alpha_{\hat{i}})=\alpha_i$. □

We turn to consider the action of $T_{w_0}^2$ on a root vector $F_{\beta_k}=T_{i_1}\cdots T_{i_{k-1}}(F_{i_k})$. As $T_{w_0}^2\in Z(\mathfrak{B}_\mathfrak{g})$ (here $\mathfrak{B}_\mathfrak{g}$ is the Artin braid group generated by $\{T_i|\ i\in I\}$),

$$\begin{aligned} T_{w_0}^2 T_{i_1}\cdots T_{i_{k-1}}(F_{i_k}) &= T_{i_1}\cdots T_{i_{k-1}}T_{w_0}^2(F_{i_k}) \\ &= T_{i_1}\cdots T_{i_{k-1}}\left(q^{(\alpha_{i_k},\alpha_{i_k})}K_{i_k}^2F_{i_k}\right) \\ &= q^{(\beta_k,\beta_k)}K_{\beta_k}^2F_{\beta_k}, \end{aligned}$$

where $q^{(\alpha_{i_k},\alpha_{i_k})}=q^{(\beta_k,\beta_k)}$ as the bilinear form is invariant under the action of Weyl group.

So in general, we have

$$\begin{aligned} T_{w_0}^2(F_{\beta_{j_1}}\cdots F_{\beta_{j_t}}) &= q^{\sum_{k=1}^t(\beta_{j_k},\beta_{j_k})}K_{\beta_{j_1}}^2F_{\beta_{j_1}}\cdots K_{\beta_{j_t}}^2F_{\beta_{j_t}} \\ &= q^{\sum_{k=1}^t(\beta_{j_k},\beta_{j_k})+\sum_{i<k}2(\beta_{j_i},\beta_{j_k})}K_{\beta_{j_1}}^2\cdots K_{\beta_{j_t}}^2F_{\beta_{j_1}}\cdots F_{\beta_{j_t}} \\ &= q^{(\beta,\beta)}K_\beta^2F_{\beta_{j_1}}\cdots F_{\beta_{j_t}}, \end{aligned}$$

where $\beta=\beta_{j_1}+\cdots+\beta_{j_t}$. These calculations give the following

Proposition 7.9. Let $x_\beta\in U_q^{<0}(\mathfrak{g})_{-\beta}$. Then

$$T_{w_0}^2(x_\beta) = q^{(\beta,\beta)}K_\beta^2x_\beta.$$

7.5.3 Central element action on Weyl group orbits

Proposition 7.10. There exists a constant $c \in \{\pm 1\}$ such that for any non-zero vector $v \in V(\lambda)_0$ of weight 0,

$$\theta.v = cq^{(\lambda,\lambda+2\rho)}v.$$

Proof. It should be remarked that if λ is not in the root lattice $\mathcal{Q}$, there will be no vector of weight 0. So if $v \in V(\lambda)_0$ is a non-zero vector, $\lambda \in \mathcal{Q}_+$ and there exists $x \in U_q^{<0}(\mathfrak{g})_{-\lambda}$ such that $v = x.v_\lambda$.

After Proposition 7.8 and 7.9, we have the following computation :

$$\begin{aligned} \theta.v &= S_{i_1}\cdots S_{i_N}S_{i_N}\cdots S_{i_1}x.v_\lambda \\ &= T_{i_1}\cdots T_{i_N}T_{i_N}\cdots T_{i_1}(x)\theta.v_\lambda \\ &= T_{w_0}^2(x)\theta.v_\lambda \\ &= q^{(\lambda,\lambda)}K_\lambda x\theta.v_\lambda \\ &= cq^{(\lambda,\lambda+2\rho)}v, \end{aligned}$$

where $c \in \{\pm 1\}$ comes from Proposition 7.8 by evaluating on the highest weight vector, so it does not depend on the choice of $v \in V(\lambda)_0$. □

As a consequence, θ acts on $V(\lambda)_0$ as a constant $cq^{(\lambda,\lambda+2\rho)}$.

Moreover, this method can be applied to compute the action of the central element on each $\mathfrak{B}_\mathfrak{g}$-orbit. For example, if $\mathfrak{g} = sl_3$ and $\lambda = \alpha_1 + \alpha_2$, then $V(\lambda)$ is the adjoint representation of dimension 8. θ acts as q^4 on the outer cycle and q^6 on the inner one (it is the zero-weight space in this case).

7.5.4 Trace of Coxeter element

We compute the trace of the Coxeter element when it acts on $V(\lambda)$. The following observation simplifies the computation.

Let $\mathrm{wt}(V(\lambda))$ denote the set of weights appearing in $V(\lambda)$. Then as

$$S_i(V(\lambda)_\mu) \subset V(\lambda)_{\mu-(\mu,\alpha_i)\alpha_i}$$

after the definition of S_i, the action of Artin braid group $\mathfrak{B}_\mathfrak{g}$ on $\mathrm{wt}(V(\lambda))$ is identical with that of the Weyl group W. The action of Π is the same as the Coxeter element $c = s_1\cdots s_l \in W$ which has no fixed point in $\mathrm{wt}(V(\lambda))\backslash\{0\}$.

A standard proof of the statement above can be found in [9], Chapitre V, n° 6.2.

As an immediate consequence of this observation, we have

$$\mathrm{Tr}(\Pi, V(\lambda)) = \mathrm{Tr}(\Pi, V(\lambda)_0).$$

Moreover, generators of $\mathfrak{B}_\mathfrak{g}$ preserve the zero-weight space $V(\lambda)_0$, so we can also look $\mathfrak{B}_\mathfrak{g}$ as a subgroup of $\mathrm{Aut}(V(\lambda)_0)$.

Notice that $\Pi^h = \Delta^2 = \theta$, so after Proposition 7.10, Π^h acts as a scalar $cq^{(\lambda,\lambda+2\rho)}$ on $V(\lambda)_0$ for some $c \in \{\pm 1\}$. If we let Λ denote the set of roots of the equation $x^h = c$ in $\mathbb{C}$, then the eigenvalues of Π belong to the set

$$\{y.q^{\frac{(\lambda,\lambda+2\rho)}{h}} \mid y \in \Lambda\}$$

and the trace $\mathrm{Tr}(\Pi, V(\lambda)_0)$ is given by $\delta q^{\frac{(\lambda,\lambda+2\rho)}{h}}$ for some $\delta \in \mathbb{C}$. As a summary, we have proved that

Proposition 7.11. There exists a constant $\delta \in \mathbb{C}$ which does not depend on q such that

$$\mathrm{Tr}(\Pi, V(\lambda)) = \delta q^{\frac{(\lambda,\lambda+2\rho)}{h}}.$$

We will determine this constant in the next section and see that it is in fact non-zero.

7.6 Main theorem

7.6.1 Dedekind η-function

We give a recollection on η-function in this subsection.

For a positive integer n, we let $p(n)$ denote the partition number of n and $p(0) = 1$. Let $\psi(x) = \sum_{n\geq 0} p(n)x^n$ be their generating function and $\varphi(x) = \psi(x)^{-1}$. Then $\varphi(x)$ has a simple expression

$$\varphi(x) = \prod_{n=1}^{\infty}(1 - x^n).$$

This $\varphi(x)$ is the simplest hypergeometric series and is closely related to the theory of modular forms.

Let $\eta(x) = x^{\frac{1}{24}}\varphi(x)$. Then $\eta^{24}(x)$ is a modular form of weight 12 and $\eta(x)$ is called the Dedekind η-function. The expansion of $\eta(x)^{24}$ into power series contains Ramanujan's τ-function as coefficients.

7.6.2 A theorem due to Kostant

We preserve notations for a simple Lie algebra $\mathfrak{g}$ given in Section 7.2.1.

Let $V_1(\lambda)$ be the irreducible representation of $\mathfrak{g}$ of highest weight λ and $V_1(\lambda)_0$ be the subspace of $V_1(\lambda)$ of weight 0. Then the Weyl group W acts on $V_1(\lambda)$ and therefore on $V_1(\lambda)_0$. We let

$$\tau_\lambda : W \to \mathrm{Aut}(V_1(\lambda)_0)$$

denote this representation. Let $c = s_1 \cdots s_l$ be a Coxeter element in W, $c(\lambda) = \Phi(\lambda + \rho, \lambda + \rho) - \Phi(\rho, \rho)$ where Φ is the Killing form on $\mathfrak{h}^*$. Let h be the Coxeter number of W.

Theorem 7.2. [[50]] The following identity holds :

$$\left(\prod_{i=1}^{l} \varphi(x^{h\Phi(\alpha_i,\alpha_i)})\right)^{h+1} = \sum_{\lambda\in\mathcal{P}_+} \mathrm{Tr}(c, V_1(\lambda)_0)\mathrm{dim}V_1(\lambda)x^{c(\lambda)}.$$

In particular, if $\mathfrak{g}$ is simply laced (i.e. of type A,D,E), the identity above has the form

$$\varphi(x)^{dim\mathfrak{g}} = \sum_{\lambda\in\mathcal{P}_+} \mathrm{Tr}(c, V_1(\lambda)_0)\mathrm{dim}V_1(\lambda)x^{c(\lambda)}.$$

Moreover, $\mathrm{Tr}(c, V_1(\lambda)_0) \in \{-1, 0, 1\}$.

To simplify the notation, we let $\varepsilon(\lambda)$ denote $\mathrm{Tr}(c, V_1(\lambda)_0)$.

Remark 7.5. If $\mathfrak{g} = sl_2$, this theorem gives the famous Jacobi identity

$$\varphi(x)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{\frac{n(n+1)}{2}}.$$

Some discussions on particular cases of Theorem 7.2 can be found in [17] and [50].

7.6.3 Main result

We give an explanation of the identity in Theorem 7.2 in the framework of quantum groups.

Coxeter numbers and Killing forms

The Killing form on $\mathfrak{h}^*$ is proportional to the inner product on the root system, i.e., there exists a constant $k \in \mathbb{C}^*$ such that for any $x, y \in \mathfrak{h}^*$,

$$k\Phi(x, y) = (x, y).$$

For any simple Lie algebra $\mathfrak{g}$, we define a constant $r_{\mathfrak{g}} = \frac{k}{h}$. The following table gives the explicit values of $r_{\mathfrak{g}}$ where the values of h and k are taken from [9].

$\mathfrak{g}$	h	k	$r_{\mathfrak{g}}$
A_l	$l+1$	$2(l+1)$	2
$B_l (l \geq 2)$	$2l$	$4l-2$	$\frac{2l-1}{l}$
$C_l (l \geq 2)$	$2l$	$4l+4$	$\frac{2l+2}{l}$
$D_l (l \geq 3)$	$2l-2$	$4l-4$	2
E_6	12	24	2
E_7	18	36	2
E_8	30	60	2
F_4	12	18	3/2
G_2	6	24	4

Main theorem

Let

$$\mathbb{C}_q[G] = \bigoplus_{\lambda \in \mathcal{P}_+} \mathrm{End}(V(\lambda)) = \bigoplus_{\lambda \in \mathcal{P}_+} V(\lambda) \otimes V(\lambda)^*$$

be the quantum coordinate algebra which can be viewed as a deformation of the algebra of regular functions of a semi-simple algebraic group G.

It is clear that there is a canonical embedding

$$\bigoplus_{\lambda \in \mathcal{P}_+} \mathrm{End}(V(\lambda)) \otimes \mathrm{End}(V(\lambda)^*) \to \mathrm{End}(\mathbb{C}_q[G]).$$

Keeping notations in previous sections, we state the main theorem of this chapter.

Theorem 7.3. Let Π be the Coxeter element in the Artin braid group $\mathfrak{B}_{\mathfrak{g}}$ and $V(\lambda)_0$ be the zero-weight space in $V(\lambda)$ for $\lambda \in \mathcal{P}_+$. We denote $c(\lambda) = \Phi(\lambda+\rho, \lambda+\rho) - \Phi(\rho, \rho)$.

1. We have
$$\mathrm{Tr}(\Pi, V(\lambda)) = \mathrm{Tr}(\Pi, V(\lambda)_0) = \varepsilon(\lambda) q^{r_{\mathfrak{g}} c(\lambda)}.$$
2. The following identity holds
$$\mathrm{Tr}(\Pi \otimes \mathrm{id}, \mathbb{C}_q[G]) = \left(\prod_{i=1}^{l} \varphi(q^{(\alpha_i, \alpha_i)}) \right)^{h+1},$$
where we look $\Pi \otimes \mathrm{id}$ as in $\mathrm{End}(\mathbb{C}_q[G])$ through the embedding above.
3. In particular, if $\mathfrak{g}$ is simply laced, i.e., of type A, D, E, then
$$\mathrm{Tr}(\Pi \otimes \mathrm{id}, \mathbb{C}_q[G]) = \varphi(q^2)^{dim \mathfrak{g}}.$$

Proof. We start from proving (2) and (3) by supposing (1) holds.

The point (2) comes from the following computation using (1) and Theorem 7.2 :

$$\begin{aligned}
\mathrm{Tr}(\Pi \otimes \mathrm{id}, \mathbb{C}_q[G]) &= \mathrm{Tr} \left(\Pi \otimes \mathrm{id}, \bigoplus_{\lambda \in \mathcal{P}_+} V(\lambda) \otimes V(\lambda)^* \right) \\
&= \sum_{\lambda \in \mathcal{P}_+} \mathrm{Tr} \left(\Pi \otimes \mathrm{id}, V(\lambda) \otimes V(\lambda)^* \right) \\
&= \sum_{\lambda \in \mathcal{P}_+} \dim_{\mathbb{C}(q)} V(\lambda) \mathrm{Tr}(\Pi, V(\lambda)) \\
&= \sum_{\lambda \in \mathcal{P}_+} \varepsilon(\lambda) \dim_{\mathbb{C}(q)} V(\lambda) q^{r_{\mathfrak{g}} c(\lambda)} \\
&= \left(\prod_{i=1}^{l} \varphi(q^{r_{\mathfrak{g}} h \Phi(\alpha_i, \alpha_i)}) \right)^{h+1} \\
&= \left(\prod_{i=1}^{l} \varphi(q^{(\alpha_i, \alpha_i)}) \right)^{h+1}.
\end{aligned}$$

To show the point (3), it suffices to notice that in the simply laced case, $(\alpha_i, \alpha_i) = 2$ and $l(h+1) = \dim \mathfrak{g}$.

Now we proceed to prove (1).

After Proposition 7.11, there exists some constant $\delta \in \mathbb{C}$ such that

$$\mathrm{Tr}(\Pi, V(\lambda)) = \delta q^{\frac{(\lambda, \lambda+2\rho)}{h}} = \delta q^{r_{\mathfrak{g}} c(\lambda)}.$$

To determine this constant, we consider the specialization of Π and $V(\lambda)$. As remarked in Section 7.3.5, when q is specialized to 1, the automorphism $S_i \in \mathrm{End}(V(\lambda))$ goes to $s_i \in \mathrm{End}(V_1(\lambda))$ in the Weyl group W and therefore Π is specialized to the Coxeter element $c \in W$. In the formula above, the left hand side has limit $\mathrm{Tr}(c, V_1(\lambda))$ when q tends to 1. On the other hand, as $\delta \in \mathbb{C}$, the right hand side has limit δ, from which $\delta = \varepsilon(\lambda)$ and the theorem is proved. □

Variations

As a variant of the above identity, we can form the following series

$$\varphi(q,t) = \sum_{\lambda \in \mathcal{P}_+} \mathrm{Tr}(\Pi, V(\lambda)) \dim_{\mathbb{C}(q)} V(\lambda) t^{c(\lambda)}$$

as in the Theorem 7.2. Then it is not difficult to show that

$$\varphi(q,t) = \left(\prod_{i=1}^{l} \varphi\left((q^k t^h)^{\Phi(\alpha_i,\alpha_i)} \right) \right)^{h+1}.$$

When $\mathfrak{g}$ is simply laced, the identity above gives

$$\varphi(q,t) = \varphi(q^2 t)^{\dim \mathfrak{g}} = \left(\prod_{n \geq 1} (1 - q^{2n} t^n) \right)^{\dim \mathfrak{g}}$$

Example : $U_q(sl_2)$ case

When $\mathfrak{g} = sl_2$, results in Theorem 7.3 can be directly verified.

In this case, there is only one generator S in the Artin braid group. Let $V(n)$ be the irreducible representation of dimension $n+1$ of $U_q(sl_2)$ of type 1 with a basis chosen as in Section 7.3.2. The Coxeter element in this case is given by S.

If n is odd, there is no zero-weight space in $V(n)$, in this case, $\mathrm{Tr}(S, V(n)) = 0$.

If n is even, the zero-weight space in $V(n)$ is of dimension 1 which is generated by v_m, where $n = 2m$.

The action of S on v_m is given by

$$S.v_m = (-1)^m q^{(n-m)(m+1)} v_m = (-1)^m q^{m(m+1)} v_m.$$

From which the trace $\mathrm{Tr}(S \otimes \mathrm{id}, \mathbb{C}_q[G])$:

$$\mathrm{Tr}(S \otimes \mathrm{id}, \mathbb{C}_q[G]) = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)},$$

which coincides with $\varphi(q^2)^3$, after Jacobi's identity.

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