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Sous la direction de **Bernhard KELLER**

Des catégories triangulées aux algèbres amassées

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Chapter 1

Introduction en français

1.1 Catégorification 2-Calabi–Yau

La notion de catégorie triangulée 2-Calabi–Yau prend son importance de la "catégorification" des algèbres amassées de S. Fomin et A. Zelevinsky au moyen de la théorie des représentations de carquois et d'algèbres de dimension finie.

Catégorifier une algèbre amassée $\mathcal{A}_Q \subset \mathbb{Q}(x_1, \ldots, n)$ associée à un carquois Q signifie se donner

- une catégorie $\mathcal{C}$ (triangulée ou de Frobenius),
- une structure amassée $\mathcal{T}$ sur $\mathcal{C}$, au sens de [BIRS],
- un objet $T \in \mathcal{T}$ dont l'algèbre des endomorphismes a pour carquois de Gabriel le carquois Q,
- un caractère d'amas envoyant T sur l'amas initial et réalisant une bijection, entre les objet dans $\mathcal{T}$ et les amas de $\mathcal{A}_Q$ d'une part, les facteurs directs indécomposables d'objets dans $\mathcal{T}$ et les variables d'amas d'autre part.

Voir l'article [Kela] pour un tour d'horizon sur ce sujet. Les deux exemples fondateurs de la catégorification d'algèbres amassées sont les catégories amassées et les catégories de modules sur les algèbres préprojectives.

A tout carquois acyclique Q est associée une catégorie amassée $\mathcal{C}_Q$, définie dans [BMR+06] et également dans [CCS06] pour le cas A_n. C'est une catégorie triangulée, par un théorème de [Kel05]. Sa structure amassée est donnée par les objets amas-basculants ; la bijection explicite ([CK06], [BCK+07]) avec les amas et variables d'amas est donnée par l'aplication de Caldero–Chapoton [CC06]. L'étude de cette application a en particulier donné lieu, dans [CK08], à deux conjectures détaillées plus loin.

Dans les catégories de modules sur les algèbres préprojectives, étudiées par C. Geiss, B. Leclerc et J. Schröer ([GLS06], [GLS05], [GLS07]...), la structure amassée est formée des objets rigides maximaux équivalents par mutation à l'objet rigide maximal initial. Un caractère d'amas est construit ([GLS07]) à l'aide des travaux de Lusztig sur les bases canoniques.

Les catégories considérées dans ces deux exemples vérifient une propriété, appelée 2-Calabi–Yau, qui s'avère primordiale pour catégorifier des algèbres amassées. Cette remarque a motivé l'étude générale des catégories triangulées 2-Calabi–Yau dans [KR07], [IY08]... Depuis, de nombreuses autres catégories 2-Calabi–Yau sont apparues en lien avec les algèbres amassées : les catégories $\underline{\mathcal{C}}_M$ de [GLSa], $\underline{\mathrm{sub}}Q_J$ de [BIRS], les catégories stables de catégories de modules de Cohen–Macaulay maximaux sur des singularités isolées ([KR07], [BIKR08]), les catégories amassées généralisées $\mathcal{C}_{Q,W}$ de [Ami]...

Cette thèse s'inscrit dans l'étude de la catégorification d'algèbres amassées par des catégories triangulées 2-Calabi–Yau. Le contenu des chapitres 4 et 6 a été publié dans [Pal08] et [Pal09] respectivement.

1.2 Deux conjectures de [CK08]

Soit Q un carquois de Dynkin et soit T un objet amas-basculant de la catégorie amassées $\mathcal{C}_Q$. On note B l'algèbre des endomorphismes de T, et $\langle\ ,\ \rangle_a$ la forme anti-symétrique associée, définie ci-dessous à la section 1.4.

Conjecture 1 ([CK08]) *La forme $\langle\ ,\ \rangle_a$ descend au groupe de Grothendieck $K_0(\operatorname{mod} B)$.*

A chaque B-module indécomposable M, on associe la fraction rationnelle

$$X_M = \sum_e \chi(\mathrm{Gr}_e(M)) \prod_i x_i^{\langle S_i, e\rangle_a - \langle S_i, M\rangle}.$$

On note $\operatorname{ind}\operatorname{mod} B$ un système de représentants des classes d'isomorphie de B-modules indécomposables.

Conjecture 2 ([CK08]) *L'ensemble $\{X_M, M \in \operatorname{ind}\operatorname{mod} B\}$ est exactement l'ensemble des variables d'amas de l'algèbre amassée $\mathcal{A}_Q$ qui n'appartiennent pas à l'amas associé à T.*

1.3 Cadre et notations

Soit k un corps algébriquement clos et soit $\mathcal{C}$ une k-catégorie triangulée Hom-finie, de Krull–Schmidt, 2-Calabi–Yau. On note $\mathcal{C}(X,Y)$ l'espace des morphismes de X vers Y, Σ le foncteur de suspension de $\mathcal{C}$, et on écrit parfois $\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$ pour $\mathcal{C}(X,\Sigma Y)$. On suppose que $\mathcal{C}$ admet un objet amas-basculant T. Cela signifie que

- l'objet T est rigide : $\mathcal{C}(T,\Sigma T)=0$,
- pour tout objet X de $\mathcal{C}$, l'égalité $\mathcal{C}(X,\Sigma T)=0$ implique $X\in\operatorname{add}T$.

Notons B l'algèbre des endomorphismes de T. Sous ces hypothèses, la catégorie $\mathcal{C}/(\Sigma T)$ est équivalente à la catégorie $\operatorname{mod}B$ des B-modules de type fini ([BMR07], [KR07], voir aussi [KZ08]). Notons F le foncteur

$$\mathcal{C}(T,?):\mathcal{C}\longrightarrow\operatorname{mod}B$$

et n le nombre de classes d'isomorphie de B-modules simples.

1.4 Forme bilinéaire antisymétrique

Pour N et N' deux B-modules de type fini, posons

$$\langle N,N'\rangle=\dim\operatorname{Hom}_B(N,N')-\dim\operatorname{Ext}^1_B(N,N')$$

et

$$\langle N,N'\rangle_a=\langle N,N'\rangle-\langle N',N\rangle.$$

Fixons deux B-modules N et N', et considérons une présentation projective $P_1\to P_0\to N\to 0$ et une coprésentation injective $0\to N\to I_0\to I_1$. On a alors deux égalités

$$\begin{aligned}\langle N,N'\rangle &= \dim\operatorname{Hom}_B(P_0,N')-\dim\operatorname{Hom}_B(P_1,N')+\delta\\ \langle N',N\rangle &= \dim\operatorname{Hom}_B(N',I_0)-\dim\operatorname{Hom}_B(N',I_1)+\delta',\end{aligned}$$

où δ et δ' sont les dimensions de certains espaces de morphismes dans $\mathcal{C}$. En utilisant la propriété de 2-Calabi–Yau, on montre que ces espaces sont duaux et on a donc l'égalité $\delta=\delta'$.

Théorème (4.11) : *L'application* $\langle\ ,\ \rangle_a$ *définit une forme bilinéaire antisymétrique sur le groupe de Grothendieck* $K_0(\operatorname{mod}B)$.

Ce théorème implique en particulier la conjecture 1. Cette simplification un peu mystérieuse des termes δ et δ' n'éclaire pas de façon satisfaisante l'étrangeté de cette forme antisymétrique définie sur le groupe de

Grothendieck $K_0(\operatorname{mod} B)$. Supposons $\mathcal{C}$ algébrique, c'est-à-dire catégorie stable d'une catégorie de Frobenius. On relie dans la section 4.3.3 la forme $\langle\ ,\ \rangle_a$ à la forme d'Euler d'une catégorie triangulée munie d'une t-structure dont le cœur est la catégorie $\operatorname{mod} B$. Cette preuve s'appuie sur une propriété de 3-Calabi–Yau démontrée dans [KR07].

1.5 Caractères amassés

Un caractère amassé (définition 4.2) sur $\mathcal{C}$, à valeurs dans un anneau commutatif A, est une application

$$\chi : \operatorname{obj} \mathcal{C} \longrightarrow A$$

telle que :

(a) si $L, M \in \mathcal{C}$ sont isomorphes, leur image coïncide, $\chi(L) = \chi(M)$,

(b) pour tous L, M dans $\mathcal{C}$, on a $\chi(L \oplus M) = \chi(L)\chi(M)$,

(c) si $\mathcal{C}(L, \Sigma M)$ est de dimension 1, alors

$$\chi(L)\chi(M) = \chi(B) + \chi(B')$$

où

$$L \to B \to M \to \Sigma L \quad \text{et} \quad M \to B' \to L \to \Sigma M$$

sont deux triangles non scindés.

Dans [CC06], P. Caldero et F. Chapoton définissent une application sur les objets de la catégorie amassée $\mathcal{C}_Q$ associée à un carquois acyclique Q, à valeurs dans $\mathbb{Q}(x_1, \ldots, x_n)$. B. Keller et P. Caldero ont montré, dans [CK08] pour les carquois de Dynkin et dans [CK06] pour les carquois acycliques, que l'application de Caldero–Chapoton est un caractère amassé. Sur la catégorie (stable) des modules sur une algèbre préprojective associée à un carquois de Dynkin, C. Geiss, B. Leclerc et J. Schröer définissent également un caractère amassé (voir [GLS07]).

Remarquons que si deux caractères amassés coïncident sur les facteurs directs indécomposables d'un objet amas-basculant T', alors ils coïncindent sur chaque rigide indécomposable "accessible depuis T'" (voir le lemme 4.18 pour un énoncé précis).

Soit $T \in \mathcal{C}$ un objet amas-basculant dont les facteurs directs indécomposables $T_1, \ldots, T_n$ sont deux à deux non isomorphes. Si $X \in \mathcal{C}$, on définit son indice et son coindice (relativement à T) de la façon suivante. Etant donnés deux triangles de Keller–Reiten

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X \quad \text{et} \quad \Sigma T_X^1 \to X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1,$$

où T_0^X, T_1^X, T_X^0 et T_X^1 sont dans $\operatorname{add} T$, l'indice de X, noté $\operatorname{ind} X$, est la classe $[FT_0^X] - [FT_1^X]$ dans $K_0(\operatorname{proj} B)$, et son coindice, noté $\operatorname{coind} X$, la classe $[FT_X^0] - [FT_X^1]$. L'indice des objets rigides est étudié dans [DK08], et utilisé pour démontrer certains résultats fortement liés à des conjectures sur les algèbres amassées.

Notons P_i, pour $i = 1, \ldots, n$, le B-module projectif indécomposable image de T_i par F, et S_i sa tête simple. L'indice et le coindice sont reliés à la forme bilinéaire antisymétrique :

Lemme (4.7) : *Soit X indécomposable dans $\mathcal{C}$. Alors*

$$\operatorname{ind} X = \begin{cases} -[P_i] \text{ si } X \simeq \Sigma T_i \\ \sum_{i=1}^n \langle FX, S_i \rangle [P_i] \text{ sinon,} \end{cases}$$

$$\operatorname{coind} X = \begin{cases} -[P_i] \text{ si } X \simeq \Sigma T_i \\ \sum_{i=1}^n \langle S_i, FX \rangle [P_i] \text{ sinon.} \end{cases}$$

Pour chaque objet M de $\mathcal{C}$, on définit, selon la formule proposée par P. Caldero et B. Keller dans [CK08], une fraction rationnelle X_M^T dans $\mathbb{Q}(x_1, \ldots, x_n)$ par :

$$X_M^T = \underline{x}^{-\operatorname{coind} M} \sum_{e \in K_0(\operatorname{mod} B)} \chi\big(\operatorname{Gr}_e(FM)\big) \prod_{i=1}^n x_i^{\langle S_i, e \rangle_a},$$

où

- $\operatorname{Gr}_e(FM)$ désigne la grassmannienne de sous-modules de vecteur dimension e de FM,
- on note $\underline{x}^a$, pour $a \in K_0(\operatorname{proj} B)$, le produit $\prod_{i=1}^n x_i^{[a:P_i]}$.

Grâce à un phénomène de dichotomie similaire à celui de la section 3.3 de [CK08], on prouve le théorème suivant.

Théorème (4.4) : *L'application $X^T : \operatorname{obj} \mathcal{C} \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$ est un caractère amassé.*

Ce théorème est valable dans un cadre relativement général. Il s'applique aux catégories amassées, comme aux catégories stables de catégories de modules sur une algèbre préprojective ou aux catégories amassées généralisées de [Ami] associées à des carquois à potentiel. Il implique, grâce à [CK06, Theorem 4] (voir aussi [BCK+07]), une forme plus générale de la conjecture 2. Supposons, dans le corollaire suivant, que $\mathcal{C}$ est la catégorie amassée associée à un carquois acyclique Q. Notons Q_T le carquois de Gabriel de l'algèbre $\mathrm{End}_{\mathcal{C}}(T)$, et $\mathcal{A}_{Q_T}$ l'algèbre amassée associée.

Corollaire (4.19) : *Le caractère amassé X^T induit une bijection entre classes d'isomorphie d'objets rigides indécomposables de $\mathcal{C}$ et variables d'amas de l'algèbre amassée $\mathcal{A}_{Q_T}$. Cette bijection envoie les objets amas-basculants de $\mathcal{C}$ sur les amas de $\mathcal{A}_{Q_T}$.*

1.6 Formule de multiplication

La première formule de multiplication démontrée est celle de [CK08], et concerne l'application de Caldero–Chapoton de la catégorie amassée associée à un carquois de Dynkin. Une formule inspirée de cette dernière est démontrée dans [GLS07] pour la catégorie des modules sur une algèbre préprojective de Dynkin. La formulation du théorème 5.1 en est l'analogue. La formule de [CK08] est généralisée dans [Huba], puis dans [XX] et [Xu].

Si ε est un morphisme dans $\mathcal{C}(L, \Sigma M)$, on note $\mathrm{mt}(\varepsilon)$ l'un quelconque des objets Y apparaissant dans un triangle $M \to Y \to L \xrightarrow{\varepsilon} \Sigma M$.

Théorème (5.1) : *Soient $L, M \in \mathcal{C}$. Si les cônes de $\mathcal{C}$ sont constructibles (voir la section 5.1.3),*

- *les fibres de l'application qui, à $\varepsilon \in \mathcal{C}(L, \Sigma M)$, associe $X^T_{\mathrm{mt}(\varepsilon)}$ sont constructibles,*

- *on a la formule*

$$\chi(\mathbb{P}\mathcal{C}(L, \Sigma M)) X^T_L X^T_M = \int_{[\varepsilon] \in \mathbb{P}\mathcal{C}(L, \Sigma M)} X^T_{\mathrm{mt}(\varepsilon)} + \int_{[\varepsilon] \in \mathbb{P}\mathcal{C}(M, \Sigma L)} X^T_{\mathrm{mt}(\varepsilon)}.$$

On montre que les hypothèses du théorème 5.1 sont vérifiées par :

- (section 5.2.4) les catégories stables de catégories de Frobenius Hom-finies, comme par exemple les catégories $\underline{\mathcal{C}}_M$ de [GLSa], dont les catégories amassées $\mathcal{C}_Q$ sont un cas particulier,

- (section 5.2.5) les catégories amassées généralisées $\mathcal{C}_{Q,W}$ de [Ami].

1.7 Groupes de Grothendieck

Certaines catégories triangulées 2-Calabi–Yau $\mathcal{C}$ n'admettant pas d'objet amas-basculant possèdent néanmoins certaines sous-catégories aux propriétés similaires. C'est le cas par exemple de

- la catégorie amassée $\mathcal{C}_{\mathcal{H}}$, où $\mathcal{H}$ est la catégorie des représentations de dimension finie sur k du carquois A_∞^∞ avec orientation linéaire [KR07],
- la catégorie D de [HJ].

Une sous-catégorie pleine $\mathcal{T}$ de $\mathcal{C}$ est dite amas-basculante ([Iya07], [KR07]) si :

- pour tous $T, T' \in \mathcal{T}$, on a $\mathrm{Ext}^1_{\mathcal{C}}(T, T') = 0$,
- pour tout $X \in \mathcal{C}$, si $\mathrm{Ext}^1_{\mathcal{C}}(X, T) = 0$ pour tout $T \in \mathcal{T}$ alors $X \in \mathcal{T}$,
- la sous-catégorie $\mathcal{T}$ est fonctoriellement finie.

Si T est un objet amas-basculant, la catégorie $\mathrm{add}\, T$ est une sous-catégories amas-basculante de $\mathcal{C}$.

On suppose désormais que $\mathcal{C}$ est algébrique, et possède une sous-catégorie amas-basculante $\mathcal{T}$. Soit $\mathcal{E}$ une catégorie de Frobenius telle que $\mathcal{C} = \underline{\mathcal{E}}$, et soit $\mathcal{M}$ la sous-catégorie pleine de $\mathcal{E}$ relevant canoniquement $\mathcal{T}$. On note $\mathcal{P}$ la sous-catégorie pleine de $\mathcal{E}$ dont les objets sont projectifs-injectifs. On note $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ la sous-catégorie pleine de la catégorie homotopique bornée $\mathcal{H}^b(\mathcal{M})$ dont les objets sont acycliques en tant que complexes d'objets de $\mathcal{E}$. En utilisant la suite exacte courte

$$0 \to \mathcal{H}^b(\mathcal{P}) \to \mathcal{D}^b(\mathcal{E}) \to \underline{\mathcal{E}} \to 0$$

de [KV87], on montre :

Lemme (6.2): *On a une suite exacte de catégories triangulées*

$$0 \to \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P}) \to \underline{\mathcal{E}} \to 0.$$

Après passage aux groupes de Grothendieck, on obtient le théorème suivant.

Théorème (6.10) : *Le groupe de Grothendieck $K_0^{\mathrm{tri}}(\underline{\mathcal{E}})$ est le quotient de $K_0^{\mathrm{add}}(\mathcal{M})$ par les relations $B_{M^*} - B_M$ où*

$$M^* \to B_M \to M \to \Sigma M^* \quad et \quad M \to B_{M^*} \to M^* \to \Sigma M$$

sont deux triangles d'échange.

Cette description de $K_0(\mathcal{C})$ permet d'améliorer un résultat de [BKL08]. Dans cet article, M. Barot, D. Kussin et H. Lenzing étudient le groupe de Grothendieck de la catégorie amassée $\mathcal{C}_A$ associée à une algèbre héréditaire ou canonique A. Ils supposent cette catégorie munie d'une structure triangulée admissible, c'est-à-dire rendant exact le foncteur de projection $\mathcal{D}^b(\operatorname{mod} A) \to \mathcal{C}_A$. Ils définissent un groupe $\overline{K}_0(\mathcal{C}_A)$ dans lequel les relations sont induites par les triangles provenant de triangles dans la catégorie dérivée. Ce groupe coïncide avec le groupe de Grothendieck usuel dans de nombreux cas :

Théorème (Barot–Kussin–Lenzing [BKL08]) : *Dans chacun des trois cas suivants, les groupes $K_0(\mathcal{C}_A)$ et $\overline{K}_0(\mathcal{C}_A)$ sont égaux.*

(i) A est une algèbre canonique dont la suite de poids $(p_1, \ldots, p_t)$ possède au moins un poids pair,

(ii) A est tubulaire,

(iii) A est héréditaire de type de représentation fini.

Sous une restriction sur la structure de catégorie triangulée de $\mathcal{C}_A$, on généralise le cas (iii) du théorème précédent.

Théorème (6.16) : *Soit A une algèbre héréditaire de dimension finie, et soit $\mathcal{C}_A$ la catégorie amassée associée. On suppose $\mathcal{C}_A$ munie de sa structure triangulée canonique définie dans* [Kel05]. *On a alors l'égalité $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$.*

1.8 Equivalences dérivées

On note $\operatorname{Mod} \mathcal{M}$ la catégorie des foncteurs k-linéaires contravariants de $\mathcal{M}$ dans la catégorie des k-espaces vectoriels. Soit $\mathcal{T}'$ une sous-catégorie amas-basculante de $\mathcal{C}$, et soit $\mathcal{M}'$ la sous-catégorie pleine de $\mathcal{E}$ correspondante.

Proposition (6.4) *Les catégories dérivées $\mathcal{D} \operatorname{Mod} \mathcal{M}$ et $\mathcal{D} \operatorname{Mod} \mathcal{M}'$ sont équivalentes.*

La preuve de cette proposition s'appuie sur un théorème de B. Keller dans [Kel94] généralisant un théorème de Rickard.

Il est montré, dans [KR07], que la catégorie $\operatorname{mod} \underline{\mathcal{M}}$ est une catégorie abélienne. On note $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ la sous-catégorie pleine de $\operatorname{per} \mathcal{M}$ formée des objets dont l'homologie est dans $\operatorname{mod} \underline{\mathcal{M}}$. La catégorie triangulée $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$ est munie d'une t-structure dont le cœur est $\operatorname{mod} \underline{\mathcal{M}}$ ([Tab07]).

Proposition (6.8) *L'équivalence dérivée de la proposition précédente entre les catégories* $\operatorname{Mod}\mathcal{M}$ *et* $\operatorname{Mod}\mathcal{M}'$ *induit une équivalence triangulée entre les catégories* $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ *et* $\operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}'$.

1.9 Formule de mutation généralisée

On note $B_{\mathcal{M}}$ la matrice anti-symétrique associée au carquois de la sous-catégorie $\mathcal{M}$ (les boucles et les 2-cycles éventuels n'interviennent pas dans cette matrice).

Lemme (6.11) *La matrice* $B_{\mathcal{M}}$ *est la matrice de la forme bilinéaire anti-symétrique*

$$\langle\ ,\ \rangle_a : K_0(\operatorname{mod}\underline{\mathcal{M}}) \times K_0(\operatorname{mod}\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

dans la base des simples.

Notons $(M_j)_{j\in J}$ et $(M'_i)_{i\in I}$ des systèmes de représentants des classes d'isomorphie d'objets indécomposables de $\mathcal{M}$ et $\mathcal{M}'$ respectivement. Pour chaque $j \in J$, fixons un triangle d'approximation de Keller–Reiten

$$\Sigma^{-1}M_j \longrightarrow \bigoplus_i \beta_{ij}M'_i \longrightarrow \bigoplus_i \alpha_{ij}M'_i \longrightarrow M_j.$$

Notons T la matrice dont les coefficients sont $t_{ij} = \alpha_{ij} - \beta_{ij}$.

Théorème (6.12) : *Les matrices* $B_{\mathcal{M}}$ *et* $B_{\mathcal{M}'}$ *sont reliées par l'égalité*

$$B_{\mathcal{M}'} = TB_{\mathcal{M}}T^{\mathrm{t}}.$$

Remarque : Pour avoir cette dernière égalité, on n'a pas supposé les sous-catégories $\mathcal{M}$ et $\mathcal{M}'$ liées par une suite de mutations.

Comme conséquence de ce théorème, on retrouve un résultat de [IY08], [BIRS] (généralisé dans [DK08]) : les objets amas-basculants d'une telle catégorie $\mathcal{C}$ ont tous le même nombre de (classes d'isomorphie de) facteurs directs indécomposables.

En suivant une méthode initiée dans [GLS06], on montre (théorème 6.14) que si $\mathcal{T}'$ est la mutation de $\mathcal{T}$ par rapport à un indécomposable M, alors la matrice $B_{\mathcal{M}'}$ est obtenue par mutation de S. Fomin et A. Zelevinsky de $B_{\mathcal{M}}$, dans la direction M. Le théorème 6.12 peut donc se voir comme une règle de mutation généralisée.

Chapter 2

Summary of results

The contents of chapter 4 have been published in [Pal08], and those of chapter 6 in [Pal09].

2.1 Notations

Let k be an algebraically closed field, and let $\mathcal{C}$ be a Hom-finite, 2-Calabi–Yau, Krull–Schmidt, triangulated k-category. We denote the space of morphisms from an object X to an object Y by $\mathcal{C}(X, Y)$, and the suspension functor of $\mathcal{C}$ by Σ. We sometimes write $\operatorname{Ext}^1_{\mathcal{C}}(X, Y)$ for $\mathcal{C}(X, \Sigma Y)$. We further assume that the category $\mathcal{C}$ admits a cluster tilting object T. Therefore,

- the object T is rigid : $\mathcal{C}(T, \Sigma T) = 0$ and
- for any object X of $\mathcal{C}$, if the space $\mathcal{C}(X, \Sigma T)$ vanishes, then X belongs to $\operatorname{add} T$.

We denote the endomorphism algebra of T by B. Under the hypotheses above, the category $\mathcal{C}/(\Sigma T)$ is equivalent to the category $\operatorname{mod} B$ of finite dimensional B modules ([BMR07], [KR07], see also [KZ08]). We denote the functor

$$\mathcal{C}(T, ?) : \mathcal{C} \longrightarrow \operatorname{mod} B$$

by F, and the number of isoclasses of simple B-modules by n.

2.2 The antisymmetric bilinear form

For any two B-modules N and N', define

$$\langle N, N' \rangle = \dim \operatorname{Hom}_B(N, N') - \dim \operatorname{Ext}^1_B(N, N')$$

and
$$\langle N, N'\rangle_a = \langle N, N'\rangle - \langle N', N\rangle.$$
Let N and N' be two B-modules. Let $P_1 \to P_0 \to N \to 0$ be a projective presentation of the module N and $0 \to N \to I_0 \to I_1$ be an injective co-presentation of N. We have the following equalities:

$$\begin{aligned}\langle N, N'\rangle &= \dim \mathrm{Hom}_B(P_0, N') - \dim \mathrm{Hom}_B(P_1, N') + \delta \\ \langle N', N\rangle &= \dim \mathrm{Hom}_B(N', I_0) - \dim \mathrm{Hom}_B(N', I_1) + \delta',\end{aligned}$$

where δ and δ' are the dimensions of some morphism spaces in $\mathcal{C}$. By the 2-Calabi–Yau property, these spaces are shown to be in duality and we have $\delta = \delta'$.

Theorem (4.11) : *The map $\langle\ ,\ \rangle_a$ induces an antisymmetric bilinear form on the Grothendieck group $K_0(\mathrm{mod}\, B)$.*

As a consequence of theorem 4.11, conjecture 1 of [CK08] holds. Assume that $\mathcal{C}$ is the stable category of a Frobenius category. We prove, in section 4.3.3, that the antisymmetric bilinear form is in fact induced by the Euler form of a triangulated category which admits a t-structure whose heart is the abelian category $\mathrm{mod}\, B$. The proof relies on a 3-Calabi–Yau property proved in [KR07].

2.3 Cluster characters

A cluster character (definition 4.2) on $\mathcal{C}$ with values in a commutative ring A is a map
$$\chi :\ \mathrm{obj}(\mathcal{C}) \longrightarrow A$$
such that

(a) for all isomorphic objects L and M, we have $\chi(L) = \chi(M)$,

(b) for all objects L and M of $\mathcal{C}$, we have $\chi(L \oplus M) = \chi(L)\chi(M)$,

(c) for all objects L and M of $\mathcal{C}$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(L, M) = 1$, we have
$$\chi(L)\chi(M) = \chi(B) + \chi(B'),$$

where B and B' are the middle terms of 'the' non-split triangles
$$L \to B \to M \to \Sigma L \ \text{ and } \ M \to B' \to L \to \Sigma M$$
with end terms L and M.

Let Q be an acyclic quiver, and denote the cluster category associated with Q by $\mathcal{C}_Q$. In [CC06], P. Caldero and F. Chapoton define a map

$$X_? : \operatorname{obj} \mathcal{C}_Q \longrightarrow \mathbb{Q}(x_1, \ldots, x_n).$$

B. Keller and P. Caldero proved that the Caldero–Chapoton map is a cluster character, in [CK08] for Dynkin quivers and in [CK06] for acyclic quivers. By considering (stable) categories of modules over preprojective algebras, C. Geiss, B. Leclerc and J. Schröer use the work of Lusztig [Lus00] in order to define a cluster character (see [GLS07]).

Note that there is a weak form of uniqueness for the cluster characters on $\mathcal{C}$ (see lemma 4.18 for a precise statement): Assume that χ_1 and χ_2 are two cluster characters on $\mathcal{C}$ with values in A, and that they coincide on the indecomposable direct summands of a cluster tilting object T'. Then, χ_1 and χ_2 coincide on each indecomposable rigid object 'reachable from T''.

Let $T \in \mathcal{C}$ be a cluster tilting object whose indecomposable direct summands $T_1, \ldots, T_n$ are pairwise non-isomorphic. For any $X \in \mathcal{C}$, the index of X (with respect to T), denoted by $\operatorname{ind} X$, is the element of $K_0(\operatorname{proj} B)$ defined as follows: Let $T_1^X \to T_0^X \to X \to \Sigma T_1^X$ be an approximation triangle of Keller–Reiten, ie. T_0^X and T_1^X belong to $\operatorname{add} T$. Then $\operatorname{ind} X$ is the class $[FT_0^X] - [FT_1^X]$. Similarly, one defines the coindex of X by $\operatorname{coind} X = [FT_X^0] - [FT_X^1]$ where T_X^0 and T_X^1 are objects of $\operatorname{add} T$ apearing in a triangle $\Sigma T_X^1 \to X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1$. The indices of rigid objects are studied in [DK08], where they are used to prove some results strongly related to conjectures on cluster algebras.

For $i = 1, \ldots, n$, denote the indecomposable projective B-module image of T_i under F by P_i, and its simple top by S_i. The next lemma shows how indices and coindices are related to the antisymmetric bilinear form.

Lemma (4.7) : *Let $X \in \mathcal{C}$ be indecomposable. Then*

$$\operatorname{ind} X = \begin{cases} -[P_i] \text{ if } X \simeq \Sigma T_i \\ \sum_{i=1}^n \langle FX, S_i \rangle [P_i] \text{ otherwise,} \end{cases}$$

$$\operatorname{coind} X = \begin{cases} -[P_i] \text{ if } X \simeq \Sigma T_i \\ \sum_{i=1}^n \langle S_i, FX \rangle [P_i] \text{ otherwise.} \end{cases}$$

For each object M of $\mathcal{C}$, define a fraction X_M^T in $\mathbb{Q}(x_1, \ldots, x_n)$, following the formula proposed in [CK08]:

$$X_M^T = \underline{x}^{-\operatorname{coind} M} \sum_{e \in K_0(\operatorname{mod} B)} \chi\big(\operatorname{Gr}_e(FM)\big) \prod_{i=1}^n x_i^{\langle S_i, e \rangle_a},$$

where

- the grassmannian of submodules of FM of dimension vector e is denoted by $\mathrm{Gr}_e(FM)$,
- for any a in $K_0(\mathrm{proj}\, B)$, the product $\prod_{i=1}^n x_i^{[a;P_i]}$ is denoted by $\underline{x}^a$.

By using a phenomenon of dichotomy similar to that of section 3.3 of [CK08], we prove the following:

Theorem (4.4) : *The map* $X^T : \mathrm{obj}\, \mathcal{C} \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$ *is a cluster character.*

This theorem holds for instance when $\mathcal{C}$ is

- the cluster category associated with an acyclic quiver,
- the stable category of modules over a preprojective algebra of Dynkin type,
- the generalized cluster category of [Ami], associated with a Jacobi-finite quiver with potential,
- the 2-Calabi–Yau reductions ([IY08], see also [CK06]) of any of the examples above...

Thanks to [CK06, Theorem 4] (see also [BCK$^+$07]), one obtains a generalization of the second conjecture of [CK08] as a corollary of theorem 4.4. Assume in the following corollary that $\mathcal{C}$ is the cluster category associated with an acyclic quiver Q. Denote the Gabriel quiver of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ by Q_T, and the corresponding cluster algebra by $\mathcal{A}_{Q_T}$.

Corollary (4.19) : *The cluster character* X^T *induces a bijection between the isoclasses of indecomposable rigid objects of* $\mathcal{C}_Q$ *and the cluster variables of the cluster algebra* $\mathcal{A}_{Q_T}$*. This bijection sends the cluster tilting objects of* $\mathcal{C}_Q$ *to the clusters of* $\mathcal{A}_{Q_T}$*.*

2.4 Multiplication formula

The first multiplication formula, concerning the Caldero–Chapoton map of the cluster category associated with a Dynkin quiver, was proved in [CK08]. Inspired from this one, another formula was proved in [GLS07], in the case of module categories over a preprojective algebra of Dynkin type. The formula of [CK08] was later generalized in [Huba], [XX] and [Xu].

For any morphism ε in $\mathcal{C}(L, \Sigma M)$, we denote any middle term Y of a triangle $M \to Y \to L \to \Sigma M$ by $\mathrm{mt}(\varepsilon)$.

Theorem (5.1) : Let $L, M \in \mathcal{C}$. Under some constructibility hypothesis (see section 5.1.3), we have

- the fibers of the map which sends $\varepsilon \in \mathcal{C}(L, \Sigma M)$ to $X^T_{\mathrm{mt}(\varepsilon)}$ are constructible, and
- we have:

$$\chi(\mathbb{P}\mathcal{C}(L, \Sigma M))X^T_L X^T_M = \int_{[\varepsilon]\in\mathbb{P}\mathcal{C}(L,\Sigma M)} X^T_{\mathrm{mt}(\varepsilon)} + \int_{[\varepsilon]\in\mathbb{P}\mathcal{C}(M,\Sigma L)} X^T_{\mathrm{mt}(\varepsilon)}.$$

We prove that this hypothesis is satisfied by

- (section 5.2.4) stable categories of Hom-finite Frobenius categories, such as the categories $\underline{\mathcal{C}}_M$ of [GLSa], some particular cases of which are the cluster categories $\mathcal{C}_Q$,
- (section 5.2.5) the generalized cluster categories $\mathcal{C}_{Q,W}$ of [Ami].

2.5 Grothendieck groups

Some triangulated categories $\mathcal{C}$, which do not admit any cluster tilting object, nevertheless have subcategories with similar properties. To name a few, this is the case of

- the cluster category $\mathcal{C}_{\mathcal{H}}$, where $\mathcal{H}$ is the category of finite dimensional representations over k of the quiver of type A_∞^∞ with linear orientation [KR07],
- the category D of [HJ].

A cluster tilting subcategory of $\mathcal{C}$ ([Iya07], [KR07]) is a full subcategory $\mathcal{T}$ such that:

- For all $T, T' \in \mathcal{T}$, we have $\mathrm{Ext}^1_{\mathcal{C}}(T, T') = 0$,
- for any X in $\mathcal{C}$, if $\mathrm{Ext}^1_{\mathcal{C}}(X, T)$ vanishes for all $T \in \mathcal{T}$, then X belongs to $\mathcal{T}$,
- the subcategory $\mathcal{T}$ is functorially finite.

Note that if T is a cluster tilting object, then the category $\operatorname{add} T$ is a cluster tilting subcategory of $\mathcal{C}$.

In all the following, we assume $\mathcal{C}$ to be algebraic, with a cluster tilting subcategory $\mathcal{T}$. Let $\mathcal{E}$ be a Frobenius category such that $\mathcal{C} = \underline{\mathcal{E}}$, and let $\mathcal{M}$ be the full subcategory of $\mathcal{E}$ which canonically lifts $\mathcal{T}$. We let $\mathcal{P}$ denote the full subcategory of $\mathcal{E}$ whose objects are projective-injective. Let $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ be the full subcategory of the bounded homotopy category $\mathcal{H}^b(\mathcal{M})$ whose objects are acyclic as complexes of objects of $\mathcal{E}$. By using the short exact sequence

$$0 \to \mathcal{H}^b(\mathcal{P}) \to \mathcal{D}^b(\mathcal{E}) \to \underline{\mathcal{E}} \to 0$$

of [KV87], we prove the following:

Lemma (6.2) : *There is a short exact sequence of triangulated categories*

$$0 \to \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \underline{\mathcal{E}} \to 0.$$

This allows us to compute the Grothendieck group of $\mathcal{C}$.

Theorem (6.10) : *The Grothendieck group $K_0^{\mathrm{tri}}(\underline{\mathcal{E}})$ is the quotient of the group $K_0^{\mathrm{add}}(\mathcal{M})$ by all relations $B_{M^*} - B_M$ where*

$$M^* \to B_M \to M \to \Sigma M^* \quad et \quad M \to B_{M^*} \to M^* \to \Sigma M$$

are exchange triangles.

This computation of $K_0(\mathcal{C})$ enables us to improve on a result of [BKL08]. In this article, M. Barot, D. Kussin and H. Lenzing study the Grothendieck group of the cluster category $\mathcal{C}_A$ associated with an algebra A which is either hereditary or canonical. They assume that the category $\mathcal{C}_A$ is endowed with an admissible triangulated structure, ie. for which the projection functor $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathcal{C}_A$ is exact. They define a group $\overline{K}_0(\mathcal{C}_A)$ by only considering the relations arising from triangles induced by the ones of $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$. This group coincides with the usual Grothendieck group in many cases:

Theorem (Barot–Kussin–Lenzing [BKL08]) *We have $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$ in each of the following three cases:*

(i) A is canonical with weight sequence $(p_1, \ldots, p_t)$ having at least one even weight.

(ii) A is tubular,

(iii) A is hereditary of finite representation type.

Under some restriction on the triangulated structure of $\mathcal{C}_A$, we have the following generalization of case (iii) of the previous theorem:

Theorem (6.16) : *Let A be a finite-dimensional hereditary algebra, and let $\mathcal{C}_A$ be the associated cluster category with its triangulated structure defined in* [Kel05]. *Then we have $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$.*

2.6 Derived equivalences

We denote the category of k-linear contravariant functors from $\mathcal{M}$ to the category of k-vector spaces by $\operatorname{Mod}\mathcal{M}$. Let $\mathcal{T}'$ be a cluster tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be the corresponding full subcategory of $\mathcal{E}$.

Proposition (6.4) : *The derived categories $\mathcal{D}\operatorname{Mod}\mathcal{M}$ and $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ are equivalent.*

The proof relies on a theorem of [Kel94] which generalizes a theorem of Rickard [Ric89].

It is proved in [KR07] that the category $\operatorname{mod}\underline{\mathcal{M}}$ is abelian. Let $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ be the full subcategory of $\operatorname{per}\mathcal{M}$ whose objects are the complexes with homologies in $\operatorname{mod}\underline{\mathcal{M}}$. The triangulated category $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ admits a t-structure whose heart is the abelian category $\operatorname{mod}\underline{\mathcal{M}}$ ([Tab07]).

Proposition (6.8) : *The derived equivalence from $\operatorname{Mod}\mathcal{M}$ to $\operatorname{Mod}\mathcal{M}'$ induces a triangle equivalence from $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ to $\operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}'$.*

2.7 The generalized mutation rule

Let $B_{\mathcal{M}}$ be the antisymmetric matrix associated with the quiver of the subcategory $\mathcal{M}$ (loops and 2-cycles do not appear in this matrix).

Lemma (6.11) : *The matrix $B_{\mathcal{M}}$ is the matrix of the antisymmetric bilinear form*

$$\langle\ ,\ \rangle_a : K_0(\operatorname{mod}\underline{\mathcal{M}}) \times K_0(\operatorname{mod}\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

in the basis given by the simple modules.

Let $(M_j)_{j\in J}$ and $(M'_i)_{i\in I}$ be systems of representatives for the isoclasses of indecomposable objects of $\mathcal{M}$ and $\mathcal{M}'$ respectively. For each $j \in J$, let

$$\Sigma^{-1}M_j \longrightarrow \bigoplus_i \beta_{ij}M'_i \longrightarrow \bigoplus_i \alpha_{ij}M'_i \longrightarrow M_j$$

be an approximation triangle of Keller–Reiten. Let T be the matrix whose coefficients are $t_{ij} = \alpha_{ij} - \beta_{ij}$ for all $(i, j) \in I \times J$.

Theorem (6.12) : *The matrices $B_{\mathcal{M}}$ and $B_{\mathcal{M}'}$ are related by the equality*

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^{\mathrm{t}}.$$

Remark : For this equality to hold, one does not have to assume the subcategories $\mathcal{M}$ and $\mathcal{M}'$ to be linked by a chain of mutations.

As a consequence of the previous theorem, we obtain a new proof of a result of [IY08], [BIRS] (generalized in [DK08]): The cluster tilting objects of such a category $\mathcal{C}$ all have the same number of (isoclasses of) indecomposable direct summands.

Inspired by a method of [GLS06], we prove in theorem 6.14 that the mutation of cluster tilting subcategories $\mathcal{T}$ is compatible with the Fomin–Zelevinsky mutation of the matrices $B_{\mathcal{M}}$. Therefore, theorem 6.14 can be thought of as a generalized mutation rule.

Chapter 3

Preliminaries

3.1 Triangulated categories

Our main references on triangulated categories are [Ver96], the book of D. Happel [Hap88] for a nice introduction and the book of A. Neeman [Nee01] for a more complete study.

3.1.1 Axioms

Let $\mathcal{C}$ be an additive category endowed with an automorphism Σ, called the suspension functor. The category $\mathcal{C}$ is triangulated if it is endowed with a class Δ of diagrams of the form $X \to Y \to Z \to \Sigma X$ which is stable under isomorphisms and which satisfies axioms (TR1) to (TR4) below. A diagram in Δ is called a triangle.

(TR1) Every morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ can be embedded into a triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$. For any object X of $\mathcal{C}$, the following diagram $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \Sigma X$ is a triangle.

(TR2) The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle if and only if the shifted diagram $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is a triangle.

(TR3) For any two triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \quad \text{and} \quad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$$

and for any morphism $(\alpha, \beta) : f \to f'$, there exists a (non unique) morphism

$\gamma : Z \to Z'$ such that

$$\begin{CD} X @>f>> Y @>g>> Z @>h>> \Sigma X \\ @V\alpha VV @V\beta VV @V\gamma VV @V\Sigma\alpha VV \\ X' @>f'>> Y' @>g'>> Z' @>h'>> \Sigma X' \end{CD}$$

is a morphism of triangles (ie. is commutative).

(TR4) (The octahedral axiom)

Let $X \xrightarrow{u} Y_1$ and $Y_1 \xrightarrow{v} Y_2$ be two composable morphisms in $\mathcal{C}$. Let there be given triangles

$$X \xrightarrow{u} Y_1 \longrightarrow Z_1 \longrightarrow \Sigma X,$$

$$Y_1 \xrightarrow{v} Y_2 \longrightarrow \Sigma X' \longrightarrow \Sigma Y_1,$$

$$X \xrightarrow{vu} Y_2 \longrightarrow Z_2 \longrightarrow \Sigma X.$$

These can be completed to a commutative diagram

$$\begin{CD}
@. X' @= X' @. \\
@. @VVV @VVV @. \\
X @>u>> Y_1 @>>> Z_1 @>>> \Sigma X \\
@| @VvVV @VVV @| \\
X @>vu>> Y_2 @>>> Z_2 @>>> \Sigma X \\
@. @VVV @VVV @. \\
@. \Sigma X' @= \Sigma X' @.
\end{CD}$$

whose rows and columns are triangles and where the following square is commutative

$$\begin{CD} Z_2 @>>> \Sigma X \\ @VVV @VVV \\ \Sigma X' @>>> \Sigma Y_1. \end{CD}$$

Remark: There are many different equivalent fomulations of the octahedral axiom, see the notes [Hubd] of A. Hubery. A nice diagram illustrating this axiom may be found in [KS94].

A functor F between two triangulated categories $(\mathcal{C}, \Sigma)$ and $(\mathcal{C}', \Sigma')$ is called a triangle functor or an exact functor if it is endowed with an invertible natural transformation $F\Sigma \to \Sigma' F$ which makes the image under F of a triangle in $\mathcal{C}$ a triangle in $\mathcal{C}'$.

3.1.2 Some properties

Let $\mathcal{C}$ be a triangulated category. The following three properties show that triangles behave almost as well as short exact sequences.

Proposition 3.1. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *be a triangle in* $\mathcal{C}$. *Then the compositions* gf *and* hg *vanish.*

Proposition 3.2. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ *be a triangle in* $\mathcal{C}$, *and let* M *be an object of* $\mathcal{C}$. *Then the following two complexes, induced by the functors represented and co-represented by* M, *are acyclic:*

$$\mathcal{C}(\Sigma X, M) \longrightarrow \mathcal{C}(Z, M) \longrightarrow \mathcal{C}(Y, M) \longrightarrow \mathcal{C}(X, M),$$

$$\mathcal{C}(M, X) \longrightarrow \mathcal{C}(M, Y) \longrightarrow \mathcal{C}(M, Z) \longrightarrow \mathcal{C}(M, \Sigma X).$$

Proposition 3.3. *Let*

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Sigma\alpha} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'
\end{array}$$

be a morphism of triangles. If two out of the three arrows α, β *and* γ *are isomorphisms, then so is the third.*

The following is a consequence of the octahedral axiom: Any commutative square

$$\begin{array}{ccc}
X & \xrightarrow{a_1} & Y_1 \\
{\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle b_1} \\
Y_2 & \xrightarrow{b_2} & Z
\end{array}$$

can be completed to a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{a_1} & Y_1 & \longrightarrow & Z_1 & \longrightarrow & \Sigma X \\
{\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle b_1} & & \downarrow & & \\
Y_2 & \xrightarrow{b_2} & Z & \longrightarrow & T_2 & \longrightarrow & \Sigma Y_2 \\
\downarrow & & \downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & T_1 & \longrightarrow & T & \longrightarrow & \Sigma Z_2 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Sigma X & & \Sigma Y_1 & & \Sigma Z_1 & &
\end{array}$$

whose rows and columns are triangles.

3.1.3 Examples

Stable categories

Exact categories have been axiomatized in [Qui73]. A small exact category is a full subcategory of an abelian category, closed under extensions. Following [GR97], we will call conflation (resp. inflation, resp. deflation) an admissible exact sequence (resp. admissible mono, resp. admissible epi). An exact category is said to have enough projectives if for any object X there is a deflation from a projective object (with respect to the exact structure) to X. Dually, it has enough injectives if any object admits an inflation to some injective object. An exact category is called a Frobenius category if it has enough projectives, enough injectives and if moreover the projectives coincide with the injectives. Let $\mathcal{E}$ be a Frobenius category. The morphisms factoring through a projective-injective object form an ideal $\mathcal{P}$ of the category $\mathcal{E}$. The stable category $\underline{\mathcal{E}}$ has the same objects as $\mathcal{E}$ but the morphisms from X to Y are given by $\underline{\mathcal{E}}(X,Y) = \mathcal{E}(X,Y)/\mathcal{P}(X,Y)$. By a result of Happel [Hap88], the stable category $\underline{\mathcal{E}}$ is triangulated. Its suspension functor is constructed in the following way: Let $X \in \mathcal{E}$, and let $X \to IX$ be any inflation to a projective-injective object. Complete it to a conflation $X \to IX \to \Sigma X$ to get the suspension of X. Let $X \xrightarrow{i} Z \xrightarrow{p} Y$ be a conflation in $\mathcal{E}$. The identity of X induces a morphism of conflations

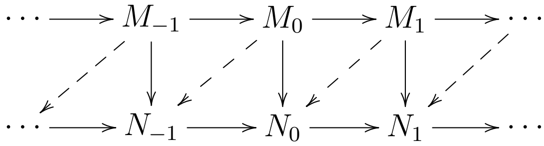
$$\begin{array}{ccccc} X & \xrightarrow{i} & Z & \xrightarrow{p} & Y \\ \| & & \downarrow & & \downarrow{\scriptstyle \varepsilon} \\ X & \longrightarrow & IX & \longrightarrow & \Sigma X. \end{array}$$

The triangles in $\underline{\mathcal{E}}$ are those isomorphic to the image in $\underline{\mathcal{E}}$ of some $(i, p, -\varepsilon)$ constructed in this way.

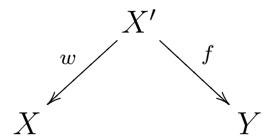
Derived categories

Our main reference on derived categories is [Ver96]. See also [Kel98] and [Noo08] for some nice lecture notes.

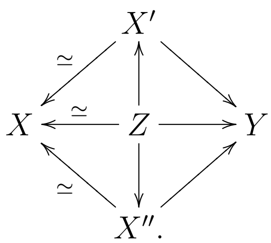
Let $\mathcal{A}$ be an abelian category. For example, $\mathcal{A}$ might be the category $\operatorname{mod} A$ of finitely generated right modules over a finite dimensional k-algebra A (where k is an algebraically closed field). Let $\mathcal{CA}$ denote the category of differential complexes over $\mathcal{A}$. Its objects are complexes of objects of $\mathcal{A}$ and its morphisms are morphisms of degree 0 which commute with the differentials. A morphism of complexes $f : M \to N$ is null-homotopic if there is a morphism $h : M \to N$ of degree -1 such that $f = dh + hd$:



If a morphism f is null-homotopic, then the induced morphism in homology vanishes. The homotopy category $\mathcal{HA}$ has the same objects as $\mathcal{CA}$, its morphisms are the classes of morphisms of complexes modulo the null-homotopic morphisms. The homotopy category is the stable category of a Frobenius category, thus it is triangulated. A morphism in $\mathcal{HA}$ is a quasi-isomorphism if it induces an isomorphism in homology. The derived category $\mathcal{DA}$ is the localization of $\mathcal{HA}$ with respect to the class of quasi-isomorphisms. For references on localization of triangulated categories see for instance [Ver96], [Nee01], [Kra]. A morphism from X to Y in $\mathcal{DA}$ is the class of a diagram



where $f \in \mathcal{HA}(X', Y)$, w is a quasi-isomorphism and two such diagrams are equivalent if there exists a commutative diagram



The derived category $\mathcal{DA}$ is a triangulated category.

3.1.4 t-structures

References on t-structures are [BBD82], [KV88] and [ATJLSS03].

Definition

Let $\mathcal{C}$ be a triangulated category. A t-structure on $\mathcal{C}$ is given by two strictly (ie. stable under isomorphisms) full subcategories $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ satisfying condition (i), (ii) and (iii) below. For any $n \in \mathbb{Z}$, denote by $\mathcal{C}^{\leq n}$ (resp. $\mathcal{C}^{\geq n}$) the subcategory $\Sigma^{-n}\mathcal{C}^{\leq 0}$ (resp. $\Sigma^{-n}\mathcal{C}^{\geq 0}$).

(i) For any $X \in \mathcal{C}^{\leq 0}$ and any $Y \in \mathcal{C}^{\geq 1}$, we have $\mathcal{C}(X,Y) = 0$.

(ii) We have the following inclusions $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 0} \supset \mathcal{C}^{\geq 1}$.

(iii) For any $X \in \mathcal{C}$, there exists a triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma A$$

where $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$.

If $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is a t-structue on $\mathcal{C}$, its heart $\mathcal{H} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is an abelian category [BBD82].

Aisles and t-structures

In [KV88], B. Keller and D. Vossieck gave a nice concise description of t-structures, via the notion of aisles. A strictly full subcategory $\mathcal{U}$ of $\mathcal{C}$ is called an aisle if

- it is stable under shifts, extensions and
- the inclusion $\mathcal{U} \to \mathcal{C}$ admits a right adjoint.

Proposition 3.4 ([KV88])**.** *A strictly full subcategory $\mathcal{U}$ is an aisle if and only if $\big(\mathcal{U}, (\Sigma\mathcal{U})^{\perp}\big)$ is a t-structure.*

Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t-structure on $\mathcal{C}$. The right adjoint to the inclusion $\mathcal{C}^{\leq n} \to \mathcal{C}$ is denoted by $\tau_{\leq n}$, and the left adjoint to $\mathcal{C}^{\geq n} \to \mathcal{C}$ by $\tau_{\geq n}$. Thus, for all $n \in \mathbb{Z}$ and all $X \in \mathcal{C}$, there is a triangle

$$\tau_{\leq n} X \longrightarrow X \longrightarrow \tau_{\geq n+1} X \longrightarrow \Sigma \tau_{\leq n} X.$$

Example

Let $\mathcal{C}$ be the bounded derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$ of a finite dimensional k-algebra A (where k is an algebraically closed field). It is the full subcategory of the derived category whose objects have homologies of finite total dimension. The strictly full subcategory $\mathcal{C}^{\leq 0}$ of complexes whose homologies vanish in positive dimensions is an aisle in $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)$. The heart of the corresponding t-structure is the abelian category $\operatorname{mod} A$. Recall that the Grothendieck group of an abelian (resp. triangulated) category is the free abelian group on the isoclasses of objects modulo the relations $[Z] = [X] + [Y]$ for each short exact sequence (resp. triangle) $X \to Z \to Y$ (resp. $X \to Z \to Y \to \Sigma X$). There is an isomorphism:

$$K_0(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} A)) \longrightarrow K_0(\operatorname{mod} A)$$

which sends a class $[X]$ to $\sum_{i \in \mathbb{Z}} (-1)^i [H^i X]$.

3.1.5 2-Calabi–Yau tilted algebras

A Hom-finite triangulated category $\mathcal{C}$ is called 2-Calabi–Yau if there are bifunctorial isomorphisms $\mathcal{C}(X, \Sigma Y) \simeq D\mathcal{C}(Y, \Sigma X)$ for any $X, Y \in \mathcal{C}$, where D denotes the duality functor $\mathrm{Hom}_k(?, k)$.

Let $\mathcal{C}$ be a (Hom-finite) 2-Calabi–Yau triangulated category. An object $T \in \mathcal{C}$ is called a cluster tilting object ([Iya07], [KR07]) if:

- It is rigid $\mathcal{C}(T, \Sigma T) = 0$ and
- for any $X \in \mathcal{C}$, if $\mathcal{C}(X, \Sigma T) = 0$ then X is in the additive closure $\mathrm{add}\, T$ of T.

An object in $\mathcal{C}$ is called basic, if its indecomposable direct summands are pairwise non isomorphic.

Theorem 3.5 ([BMR$^+$06] for cluster categories, [IY08]). *Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category. For any basic cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$ in $\mathcal{C}$ and any $i \in \{1, \ldots, n\}$, there exists a unique (up to isomorphism) indecomposable object T_i^* not isomorphic to T_i such that $(T/T_i) \oplus T_i^*$ is a cluster tilting object in $\mathcal{C}$. Moreover, there are non split triangles*

$$T_i \to B \to T_i^* \to \Sigma T_i \quad \textit{and} \quad T_i^* \to B' \to T_i \to \Sigma T_i^*$$

where $T_i \to B$ is a minimal left $\mathrm{add}\, T/T_i$-approximation and $B' \to T_i$ is a minimal right $\mathrm{add}\, T/T_i$-approximation.

Theorem 3.6 ([BMR07], [KR07], see also [KZ08]). *If T is a cluster tilting object in $\mathcal{C}$, then the functor F co-represented by T induces an equivalence of categories*

$$\mathcal{C}/(\Sigma T) \xrightarrow{\simeq} \mathrm{mod}\, \mathrm{End}_{\mathcal{C}}(T).$$

An algebra B is called 2-Calabi–Yau tilted if it is the endomorphism algebra of a cluster tilting object in a 2-Calabi–Yau triangulated category. When the triangulated category is a cluster category, the algebra B is called cluster tilted.

Let T be a cluster tilting object in a 2-Calabi–Yau triangulated category $\mathcal{C}$. We denote the corresponding 2-Calabi–Yau tilted algebra by B, and the functor $\mathcal{C}(T, ?) : \mathcal{C} \longrightarrow \mathrm{mod}\, B$ by F.

Proposition 3.7. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X$ be a triangle in $\mathcal{C}$. Then:*

- *The morphism g induces a monomorphism in the category $\mathrm{mod}\, B$ if and only if f belongs to the ideal (ΣT) of morphisms factoring through an object in $\mathrm{add}\, \Sigma T$.*

- *The morphism f induces an epimorphism in the category* $\operatorname{mod} B$ *if and only if g belongs to* (ΣT).

Moreover, if X does not have any direct summand in $\operatorname{add} \Sigma T$*, then:*

- *The B-module FX is projective if and only if X belongs to* $\operatorname{add} T$.
- *The B-module FX is injective if and only if X belongs to* $\operatorname{add} \Sigma^2 T$.

3.2 Cluster categories

3.2.1 Definition

Let Q be a finite acyclic quiver and k be an algebraically closed field. Denote by τ the Auslander–Reiten translation in the bounded derived category $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} kQ)$, and by Σ the suspension functor.

Definition 3.8 ([BMR+06], also [CCS06] for the Dynkin case A_n)**.** *The cluster category $\mathcal{C}_Q$ associated with Q is the orbit category $\mathcal{D}^b(\operatorname{mod} kQ)/\tau^{-1}\Sigma$:*

- *Its objects are the objects of $\mathcal{D}^b(\operatorname{mod} kQ)$ and*
- *for any $X, Y \in \mathcal{C}_Q$, the space of morphisms from X to Y is*

$$\mathcal{C}_Q(X,Y) = \coprod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} kQ)} \big((\tau^{-1}\Sigma)^i X, Y\big).$$

Theorem 3.9 ([Kel05])**.** *The cluster category $\mathcal{C}_Q$ is triangulated and the canonical functor $\mathcal{D}^b(\operatorname{mod} kQ) \longrightarrow \mathcal{C}_Q$ is exact.*

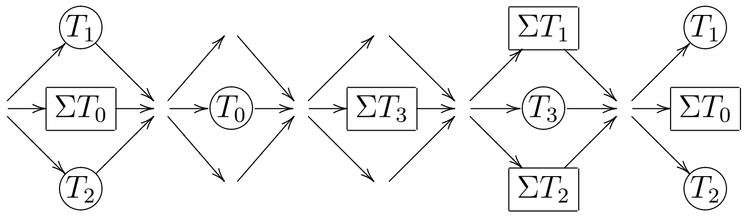
Let Q be a finite acyclic quiver. Let $\operatorname{ind}(kQ)$ be a complete set of representatives for the indecomposable kQ-modules. For each vertex i in Q, denote the corresponding indecomposable projective kQ-module by P_i.

Theorem 3.10 ([BMR+06])**.** *With the notations above:*

- *The cluster category $\mathcal{C}_Q$ is a Krull–Schmidt category.*
- *The set* $\operatorname{ind}(kQ) \amalg \{\Sigma P_i, i \in Q_0\}$ *is a complete set of representatives for the isoclasses of indecomposable objects of $\mathcal{C}_Q$.*
- *The cluster category is 2-Calabi–Yau and the image of kQ in $\mathcal{C}_Q$ is a cluster tilting object.*

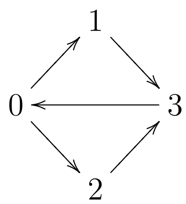
Example : The cluster category associated with the quiver D_4.

The Auslander–Reiten quiver of $\mathcal{C}_{D_4}$ is



The object $T := T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is cluster-tilting.

The quiver of $B = \mathrm{End}_{\mathcal{C}_{D_4}}(T)$ is



with the following relations: Any composition with the middle arrow vanishes, and the square is commutative.

3.2.2 Preprojective algebras

All of this section is from [GLSa]. Let k be an algebraically closed field, and let Q be a finite acyclic quiver. Its double quiver $\overline{Q}$ is obtained from Q by adding an arrow a^*, in the opposite direction, for each arrow a in Q. The preprojective algebra associated with Q is the path algebra $k\overline{Q}$ modulo the ideal generated by $\sum_{a \in Q_1} a^*a - aa^*$.

A kQ-module M is called terminal if

(i) it is preinjective,

(ii) for any indecomposable kQ-module X, if $\mathrm{Hom}_{kQ}(M, X)$ does not vanish, then X is in $\mathrm{add}\, M$ and

(iii) each indecomposable injective kQ-module is (isomorphic to) a direct summand of M.

Denote the forgetful functor $\mathrm{mod}\, \Lambda \to \mathrm{mod}\, kQ$ by π_Q, and let M be a terminal kQ-module.

Definition 3.11 ([GLSa]). *The category $\mathcal{C}_M$ is the full subcategory $\pi_Q^{-1}(\mathrm{add}\, M)$ of $\mathrm{mod}\, \Lambda$.*

Theorem 3.12 ([GLSa]). *Let M be a terminal kQ-module. Then the following hold:*

- *The category $\mathcal{C}_M$ is a Frobenius category.*
- *The stable category $\underline{\mathcal{C}}_M$ is 2-Calabi–Yau and has cluster tilting objects.*
- *If M is the direct sum $\bigoplus_{i\in Q_0} I_i \oplus \tau(I_i)$ where τ denotes the Auslander–Reiten translation of $\operatorname{mod} kQ$ and I_i the injective module associated with vertex i, then the stable category $\underline{\mathcal{C}}_M$ is triangle equivalent to the cluster category $\mathcal{C}_Q$.*

In particular, the cluster category $\mathcal{C}_Q$ is the stable category of a Hom-finite Frobenius category.

3.2.3 Tilting graph

Tilting modules

Let A be a finite dimensional k-algebra. An A-module T is a tilting module if

- its projective dimension is bounded by 1, $\operatorname{pdim} T \leq 1$,
- it is rigid, $\operatorname{Ext}^1_A(T,T) = 0$ and
- there is a short exact sequence $0 \to A \to T^0 \to T^1 \to 0$ where T^0 and T^1 are in $\operatorname{add} T$.

Let Q be an acyclic quiver. It is proved in [BMR$^+$06, Theorem 3.3], that each tilting kQ-module induces a cluster tilting object in $\mathcal{C}_Q$ and conversely that each cluster tilting object in $\mathcal{C}_Q$ comes from a tilting module over some hereditary algebra derived equivalent to kQ.

An A-module $\overline{T}$ is called an almost complete tilting module if there exists an indecomposable module T_0 such that $\overline{T} \oplus T_0$ is a tilting module. Such an indecomposable T_0 is called a complement of $\overline{T}$.

Theorem 3.13. *Let $\overline{T}$ be an almost complete tilting module. Then:*

- ([HR82], [Bon81]) *There exists a complement.*
- ([RS90], [Ung90]) *There are at most two complements, up to isomorphism.*
- ([HU89]) *There are exactly two complements if and only if $\overline{T}$ is sincere.*

Recall that a module M is sincere if each simple appears as a factor in a composition series of M, or equivalently if there are non-zero morphisms from P to M for any projective P.

Let Q be an acyclic quiver with set of vertices $\{1, \ldots, n\}$. Denote the associated cluster category by $\mathcal{C}_Q$ and the functor $\mathcal{C}_Q(kQ, ?)$ by F. Recall that F induces a projection $\mathcal{C}_Q \xrightarrow{F} \operatorname{mod} kQ$.

Remark: Let $\overline{T}$ be an almost complete tilting kQ-module. Assume that the space $\operatorname{Hom}_{kQ}(P_i, \overline{T})$ vanishes for some indecomposable projective P_i. Then we have

$$\begin{aligned} \mathcal{C}_Q(\Sigma P_i, \Sigma \overline{T}) &= \mathcal{C}_Q(P_i, \overline{T}) \\ &= \operatorname{Hom}_{kQ}(P_i, \overline{T}) \\ &\neq 0. \end{aligned}$$

Therefore $\overline{T} \oplus \Sigma P_i$ is a cluster tilting object in $\mathcal{C}_Q$.

Theorem 3.14 ([HR82], [Bon81]). *Let T be a kQ-module. Then T is tilting if and only if T is rigid and has exactly n pairwise non-isomorphic indecomposable direct factors.*

Tilting graph

Let Q be an acyclic quiver with vertices $1, \ldots, n$. The tilting graph of the cluster category $\mathcal{C}_Q$ is defined as follows:

- Its vertices are the isomorphism classes of basic cluster tilting objects.
- There is an edge joining two vertices if and only if the corresponding cluster tilting objects have all but one of their indecomposable direct summands in common.

Similarly one defines the tilting graph of $\operatorname{mod} kQ$, considering tilting modules instead of cluster tilting objects. A vertex of the tilting graph is saturated if it is n-valent. This means that it corresponds to a tilting module $T = T_1 \oplus \cdots \oplus T_n$ such that T/T_i is sincere, for all $i = 1, \ldots, n$.

Theorem 3.15 ([Ung96], Theorem 3.1). *Any connected component of the tilting graph of* $\operatorname{mod} kQ$ *contains a non-saturated vertex.*

Proposition 3.16 ([BMR+06], Proposition 3.5). *The tilting graph of $\mathcal{C}_Q$ is connected.*

This result has been recently generalized to arbitrary hereditary artin algebras by Andrew Hubery in [Hubb].

We give here a proof which slightly differs from that of [BMR+06].

Proof. Let us prove by induction on $n = |Q_0|$ that the tilting graph of $\mathcal{C}_Q$ is connected.

Let T be a basic cluster tilting object of $\mathcal{C}_Q$ which does not have any direct summand in $\mathrm{add}(\Sigma kQ)$. First note that FT is a tilting kQ-module. Indeed, it is rigid, for instance by lemma 4.8, and it has n indecomposable direct summands as required in theorem 3.14. Note also that if $\mu_i T$ does not have any direct summand in $\mathrm{add}(\Sigma kQ)$ either, then FT and $F\mu_i T$ are the two completions of the almost complete tilting module FT/FT_i.

If none of the cluster tilting objects mutation equivalent (ie. linked by a chain of mutations) to T have a direct summand in $\mathrm{add}(\Sigma kQ)$, then all the vertices of the connected component of FT in the tilting graph of $\mathrm{mod}\, kQ$ are saturated. Therefore theorem 3.15 implies that T is mutation equivalent to some cluster tilting object T' with a direct summand in $\mathrm{add}(\Sigma kQ)$, say ΣP_j. It is proved in [CK06, Theorem 5], that the subgraph of the tilting graph of $\mathcal{C}_Q$ whose vertices correspond to cluster tilting objects containing ΣP_j as a direct summand is isomorphic to the tilting graph of $\mathcal{C}_{Q'}$ for some quiver Q' with $|Q'_0| = |Q_0| - 1$. By induction, this subgraph is connected, and T' is mutation equivalent to ΣkQ.

We have thus shown that any cluster tilting object in $\mathcal{C}_Q$ is mutation equivalent to ΣkQ. Therefore the tilting graph of $\mathcal{C}_Q$ is connected. □

3.2.4 The Caldero–Chapoton map

Euler characteristic

For a variety X, $\chi(X)$ denotes its Euler–Poincaré characteristic with respect to the étale cohomology with proper support: $\chi(X) = \sum_i (-1)^i \dim H^i_c(X)$. The Euler characteristic enjoys the following properties:

- If X is an affine space, then $\chi(X) = 1$.
- If X and Y are varieties, then we have $\chi(X \amalg Y) = \chi(X) + \chi(Y)$,
- and we have $\chi(X \times Y) = \chi(X)\chi(Y)$.
- More generally, if $X \to Y$ is a surjective morphism whose fibers have constant Euler characteristic c, then $\chi(X) = c\,\chi(Y)$.

Definition

Let Q be an acyclic quiver with n vertices, and let $\mathcal{C}_Q$ be the associated cluster category. If M is a finite dimensional kQ-module and e a positive class in $K_0(\operatorname{mod} kQ)$, then $\operatorname{Gr}_e(M)$ denotes the variety of submodules of M with dimension vector e. Recall that the Euler form is defined by

$$\langle \underline{\dim} M, \underline{\dim} N \rangle = \dim \operatorname{Hom}_{kQ}(M,N) - \dim \operatorname{Ext}^1_{kQ}(M,N).$$

The Caldero–Chapoton map [CC06]

$$X_? : \operatorname{obj} \mathcal{C}_Q \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

is defined by:

- For all $M, N \in \mathcal{C}_Q$, we have $X_{M \oplus N} = X_M X_N$.
- If M is isomorphic to the shift ΣP_i of the indecomposable projective kQ-module associated with vertex i, then $X_M = x_i$.
- If M is an indecomposable kQ-module, then

$$X_M = \sum_e \chi(\operatorname{Gr}_e M) \prod_{i=1}^n x_i^{\langle S_i, e\rangle - \langle e, S_i\rangle - \langle S_i, M\rangle}.$$

Some multiplication formulae

The Caldero–Chapoton map satisfies some multiplicative properties which, in particular, categorify the mutation of cluster variables. Let Q be an acyclic quiver.

Theorem 3.17 ([CK08] for Q Dynkin, [CK06])**.** *Let L and M be two objects of $\mathcal{C}_Q$ such that $\mathcal{C}_Q(L, \Sigma M)$ has dimension 1. Fix two non-split triangles*

$$M \to B \to L \to \Sigma M \quad \textit{and} \quad L \to B' \to M \to \Sigma L.$$

Then we have $X_L X_M = X_B + X_{B'}$.

Let Q be a Dynkin quiver. For any two objects L and M of $\mathcal{C}_Q$, define $\mathcal{C}_Q(L, \Sigma M)_Y$ to be the set of the morphisms from L to ΣM which appear in a triangle $M \to Y' \to L \to \Sigma M$ where Y' is isomorphic to Y. It is shown in [CK08] that the set $\mathcal{C}_Q(L, \Sigma M)_Y$ is constructible, so that $\mathbb{P}\mathcal{C}_Q(L, \Sigma M)_Y$ has a well-defined Euler characteristic.

Theorem 3.18 ([CK08]). *Let Q be a Dynkin quiver, and let $L, M \in \mathcal{C}_Q$ be such that $\mathcal{C}_Q(L, \Sigma M)$ does not vanish. Then*

$$\chi(\mathbb{P}\mathcal{C}_Q(L, \Sigma M))X_L X_M = \sum_Y \big(\chi(\mathbb{P}\mathcal{C}_Q(L, \Sigma M)_Y) + \chi(\mathbb{P}\mathcal{C}_Q(M, \Sigma L)_Y)\big)X_Y,$$

where Y runs through the isoclasses of $\mathcal{C}_Q$.

C. Geiss, B. Lerclerc and J. Schröer proved in [GLS07] some similar formulae for an analogue of the Caldero–Chapoton map in the category of modules over a preprojective algebra.

In [Huba], A. Hubery proved a generalization of theorem 3.18, based on the existence of Hall polynomials which he proved in the affine case. Staying close to this point of vue, J. Xiao and F. Xu proved in [XX] a projective version of Green's formula and applied it to generalize the multiplication formula for acyclic cluster algebras. Another proof of this formula was found by F. Xu in [Xu] based on the 2-Calabi–Yau property, instead of Green's formula.

3.2.5 Some applications to cluster algebras

Categorification

Let Q be an acyclic quiver.

Theorem 3.19 ([CK06], see also [BCK+07]). *The Caldero–Chapoton map induces bijections between:*

- *The indecomposable rigid objects of the cluster category $\mathcal{C}_Q$ and the cluster variables of the cluster algebra $\mathcal{A}_Q$ associated with Q.*
- *The basic cluster tilting objects of $\mathcal{C}_Q$ and the clusters of $\mathcal{A}_Q$.*

The proof of this theorem relies on the categorification of the Fomin–Zelevinsky mutation given in theorem 3.17, and on the connectedness of the tilting graph of $\mathcal{C}_Q$.

The following three corollaries were conjectured in [FZ03b]. Denote the cluster algebra associated with an acyclic quiver Q by $\mathcal{A}_Q$.

Corollary 3.20 ([CK06], [BMRT07]). *Let Q be an acyclic quiver. A seed in $\mathcal{A}_Q$ is determined by its cluster.*

Indeed, if $\underline{x}$ is a cluster, the matrix of a seed containing $\underline{x}$ is determined by the endomorphism algebra of the cluster tilting object associated with $\underline{x}$ under the bijection of theorem 3.19.

Corollary 3.21 ([CK06]). *Let Q be an acyclic quiver. For any cluster variable x of $\mathcal{A}_Q$, the set of seeds whose cluster contain x form a connected subgraph of the exchange graph.*

Corollary 3.22 ([CK06]). *Let Q be an acyclic quiver. The set of seeds of $\mathcal{A}_Q$ whose matrix is acyclic form a connected subgraph of the exchange graph.*

Positivity

S. Fomin and A. Zelevinsky proved in [FZ02] that any cluster variable of a cluster algebra is a Laurent polynomial in the cluster variables appearing in any given cluster. They conjectured that the coefficients of these Laurent polynomials are non-negative.

Theorem 3.23 ([CK08] for Q Dynkin, [CR08]). *Let Q be some acyclic quiver and let $M \in \mathcal{C}_Q$ be a rigid indecomposable kQ-module. Then we have $\chi(\mathrm{Gr}_e(M)) \geq 0$ for all dimension vector e.*

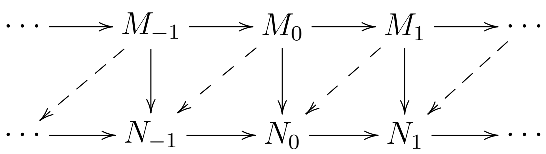
In particular, the coefficients in the Laurent expansion in the initial cluster of any cluster variable of the cluster algebra $\mathcal{A}_Q$ are non-negative.

3.2.6 Generalized cluster categories

The derived category of a dg algebra

For an introduction to derived categories of dg algebras (and of dg categories), see [Kel94] and [Kel06].

A dg algebra A is a $\mathbb{Z}$-graded k-algebra endowed with a differential $d : A \to A$ of degree $+1$ satisfying the graded Leibniz rule: For any $a \in A^p$ and any $b \in A$, $d(ab) = (da)b + (-1)^p adb$. Let A be a dg algebra. A (right) dg A-module M is a $\mathbb{Z}$-graded module over A with a differential such that: For any $m \in M^p$ and any $a \in A$, $d(ma) = (dm)a + (-1)^p mda$. The objects of the abelian category $\mathcal{C}A$ are the dg A-modules, and its morphisms are the morphisms of graded modules, of degree 0, which commute with the differentials. If A is a 'usual' algebra, A is a dg algebra concentrated in degree 0. The category $\mathcal{C}A$ is then the category of complexes of A-modules. A morphism of dg modules $f : M \to N$ is null-homotopic if there is an A-linear morphism $h : M \to N$ of degree -1 such that $f = dh + hd$:



If a morphism f is null-homotopic, then the induced morphism in homology vanishes. The homotopy category $\mathcal{H}A$ has the same objects as $\mathcal{C}A$ and its morphisms are the residue classes of morphisms in $\mathcal{C}A$ modulo null-homotopy. The homotopy category is triangulated. A quasi-isomorphism is a morphism in $\mathcal{C}A$ or in $\mathcal{H}A$ which induces an isomorphism in homology. The derived category $\mathcal{D}A$ is obtained from the homotopy category $\mathcal{H}A$ by localizing at the class of all quasi-isomorphisms. The bounded derived category $\mathcal{D}^{\mathrm{b}}A$, also denoted by $\mathcal{D}_{\mathrm{fd}}A$, is the full subcategory of $\mathcal{D}A$ of objects whose homology is of finite total dimension. The perfect derived category $\operatorname{per} A$ is the smallest full triangulated thick (ie. stable under taking direct summands) subcategory of $\mathcal{D}A$ containing A.

The Ginzburg dg algebra

The definition of the Ginzburg dg algebra comes from [Gin, section 4.2]. Let (Q, W) be a finite quiver with potential (QP for short) (see for instance [DWZ]). Define a new quiver $\widetilde{Q}$ as follows:

- The vertices of $\widetilde{Q}$ are the vertices of Q.
- Each arrow a of Q yields two arrows of $\widetilde{Q}$: An arrow, again denoted by a, which has same source and same target as the corresponding arrow of Q, and an arrow a^*, whose source is the target of a and vice versa.
- Each vertex i of Q yields a loop t_i at vertex i in $\widetilde{Q}$.

If c is a cycle of a path algebra kQ', the cyclic derivative of c associated with an arrow $a \in Q'_1$ is

$$\partial_a c = \sum_{c=vau} uv.$$

The Ginzburg dg algebra $\Gamma(Q, W)$, associated with the quiver with potential (Q, W), is the path algebra $k\widetilde{Q}$ endowed with:

- The grading obtained by requiring that any arrow a coming from an arrow in Q is of degree 0, that any arrow of the form a^* is of degree -1, and that any loop t_i is of degree -2.
- The differential defined on the generators by $\mathrm{d}a^* = \partial_a W$ for all a in Q_1, and $\mathrm{d}t_i = \sum_{t(a)=i} aa^* - \sum_{s(a)=i} a^*a$ for all i in Q_0.

The homology of the Ginzburg dg algebra in degree zero is the Jacobian algebra associated with (Q, W):

$$\mathrm{H}^0\,\Gamma(Q, W) = kQ/\langle \partial_a W, a \in Q_1 \rangle = J(Q, W).$$

The cluster category associated with a QP

Let (Q, W) be a quiver with potential, and assume that its Jacobian algebra $J(Q, W)$ is finite dimensional over k. Such a quiver with potential is called Jacobi-finite, following [Ami]. Denote by Γ the Ginzburg dg algebra associated with (Q, W).

Theorem 3.24 (Keller)**.** *Let (Q, W) be a Jacobi-finite quiver with potential. Then the bounded derived category $\mathcal{D}^b\Gamma$ is included in the perfect derived category* $\operatorname{per}\Gamma$*.*

Definition 3.25 (Amiot)**.** *The generalized cluster category $\mathcal{C}_{(Q,W)}$ associated with a Jacobi-finite quiver with potential (Q, W) is the localization of the perfect derived category* $\operatorname{per}\Gamma$ *at the subcategory $\mathcal{D}^b\Gamma$.*

Denote by T the image of Γ in $\mathcal{C}_{(Q,W)}$.

Theorem 3.26 (Amiot)**.** *Let (Q, W) be a Jacobi-finite quiver with potential. Then the triangulated category $\mathcal{C}_{(Q,W)}$ is Hom-finite, 2-Calabi–Yau and T is a cluster tilting object in $\mathcal{C}_{(Q,W)}$. Moreover, the endomorphism algebra of T $\operatorname{End}_{\mathcal{C}_{(Q,W)}}(T)$ is isomorphic to the Jacobian algebra $J(Q, W)$.*

Chapter 4

Cluster characters for 2-Calabi–Yau triangulated categories

The contents of this chapter have been published in [Pal08].

Introduction

Cluster algebras were invented and studied by S. Fomin and A. Zelevinsky in [FZ02], [FZ03a], [FZ07] and in collaboration with A. Berenstein in [BFZ05]. They are commutative algebras endowed with a distinguished set of generators called the cluster variables. These generators are gathered into overlapping sets of fixed finite cardinality, called clusters, which are defined recursively from an initial one via an operation called mutation. A cluster algebra is said to be of finite type if it only has a finite number of cluster variables. The finite type cluster algebras were classified in [FZ03a].

It was recognized in [MRZ03] that the combinatorics of cluster mutation are closely related to those of tilting theory in the representation theory of quivers and finite dimensional algebras. This discovery was the main motivation for the invention of cluster categories (in [CCS06] for the A_n-case and in [BMR$^+$06] for the general case). These are certain triangulated categories [Kel05] which, in many cases, allow one to 'categorify' cluster algebras: In the categorical setting, the cluster-tilting objects play the role of the clusters, and their indecomposable direct summands the one of the cluster variables.

In [GLS06], [GLS05], [GLS07], the authors study another setting for the categorification of cluster algebras: The module categories of preprojective algebras of Dynkin type. They succeed in categorifying a different class of cluster algebras, which also contains many cluster algebras of infinite type.

Both cluster categories and module categories of preprojective algebras of Dynkin type are 2-Calabi–Yau categories in the sense that we have bifunctorial isomorphisms

$$\operatorname{Ext}^1(X,Y) \simeq D\operatorname{Ext}^1(Y,X),$$

which are highly relevant in establishing the link with cluster algebras. This motivates the study of more general 2-Calabi–Yau categories in [KR07], [KR], [Tab07], [KZ08], [IR08], [IY08], [BIRS]. In order to show that a given 2-Calabi–Yau category 'categorifies' a given cluster algebra, a crucial point is

a) to construct an explicit map from the set of indecomposable factors of cluster-tilting objects to the set of cluster variables, and

b) to show that it is bijective.

Such a map was constructed for module categories of preprojective algebras of Dynkin type in [GLS06] using Lusztig's work [Lus00]. For cluster categories, it was defined by P. Caldero and F. Chapoton in [CC06]. More generally, for each object M of the cluster category, they defined a fraction X_M in $\mathbb{Q}(x_1,\ldots,x_n)$. The bijectivity property of the Caldero–Chapoton map was proved in [CC06] for finite type and in [CK06], cf. also [BCK$^+$07], for acyclic type.

A crucial property of the Caldero–Chapoton map is the following. For any pair of indecomposable objects L and M of $\mathcal{C}$ whose extension space $\mathcal{C}(L,\Sigma M)$ is one-dimensional, we have

$$X_L X_M = X_B + X_{B'},$$

where Σ denotes the suspension in $\mathcal{C}$ and where B and B' are the middle terms of 'the' two non-split triangles with outer terms L and M. We define, in defintion 4.2 a cluster character to be a map satisfying this multiplication formula.

This property has been proved in [CK08] in the finite case, in [GLS07] for the analogue of the Caldero–Chapoton map in the preprojective case, and in [CK06] in the acyclic case.

The main result of this chapter is a generalisation of this multiplication formula. Starting from an arbitrary cluster-tilting object T and an arbitrary 2-Calabi–Yau category $\mathcal{C}$ over an algebraically closed field (as in the setting of [KR07]), we define, for each object L of $\mathcal{C}$, a fraction X^T_L using a formula proposed in [CK08, 6.1]. We show that the map $L \mapsto X_L^T$ is a cluster character. We deduce that it has the bijectivity property in the finite and the acyclic case, which confirms conjecture 2 of [CK08]. Here, it yields a new way of expressing cluster variables as Laurent polynomials in the variables of a fixed cluster. Our theorem also applies to stable categories of preprojective algebras of Dynkin type and their Calabi–Yau reductions studied in [GLS08] and [BIRS], and to the generalized cluster categories defined in [Ami].

Let k be an algebraically closed field, and let $\mathcal{C}$ be a 2-Calabi–Yau Hom-finite triangulated k-category with a cluster-tilting object T (see section 4.1).

This chapter is organised as follows: In the first section, the notations are given and the main result is stated. In the next two sections, we investigate the exponents appearing in the definition of X_L^T. In section 4.2, we define the index and the coindex of an object of $\mathcal{C}$ and show how they are related to the exponents. Section 4.3 is devoted to the study of the antisymmetric bilinear form $\langle\ ,\ \rangle_a$ on $\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}T$. We show that this form descends to the Grothendieck group $\mathrm{K}_0(\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}T)$, confirming conjecture 1 of [CK08, 6.1]. In section 4.4, we prove that the same phenomenon of dichotomy as in [CK06, section 3] (see also [GLS07]) still holds in our setting. The results of the first sections are used in section 4.5 to prove the multiplication formula. We draw some consequences in section 4.5.2. Two examples are given in section 4.6.

4.1 Main result

Let k be an algebraically closed field, and let $\mathcal{C}$ be a k-linear triangulated category with split idempotents. Denote by Σ its suspension functor. Assume moreover that the category $\mathcal{C}$

a) is Hom-finite: For any two objects X and Y in $\mathcal{C}$, the space of morphisms $\mathcal{C}(X, Y)$ is finite-dimensional,

b) is 2-Calabi–Yau: There exist bifunctorial isomorphisms

$$\mathcal{C}(X, \Sigma Y) \simeq D\mathcal{C}(Y, \Sigma X),$$

where D denotes the duality functor $\mathsf{Hom}_k(?, k)$, and

c) admits a cluster-tilting object T, which means that

   i) $\mathcal{C}(T, \Sigma T) = 0$ and

   ii) for any X in $\mathcal{C}$, if $\mathcal{C}(X, \Sigma T) = 0$, then X belongs to the full subcategory $\mathrm{add}\, T$ formed by the direct summands of sums of copies of T.

For two objects X and Y of $\mathcal{C}$, we often write (X,Y) for the space of morphisms $\mathcal{C}(X,Y)$ and we denote its dimension by $[X,Y]$. Similarly, we write ${}^1(X,Y)$ for $\mathcal{C}(X,\Sigma Y)$ and ${}^1[X,Y]$ for its dimension. Let B be the endomorphism algebra of T in $\mathcal{C}$, and let $\operatorname{mod} B$ be the category of finite-dimensional right B-modules. As shown in [BMR07], cf. also [KR07], the functor

$$F : \mathcal{C} \longrightarrow \operatorname{mod} B \ \ , \ \ X \longmapsto \mathcal{C}(T,X),$$

induces an equivalence of categories

$$\mathcal{C}/(\Sigma T) \xrightarrow{\simeq} \operatorname{mod} B,$$

where (ΣT) denotes the ideal of morphisms of $\mathcal{C}$ which factor through a direct sum of copies of ΣT.

The following useful proposition is proved in [KR07] and [KZ08]:

Proposition 4.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$ be a triangle in $\mathcal{C}$. Then*

. *The morphism g induces a monomorphism in* $\operatorname{mod} B$ *if and only if $f \in (\Sigma T)$.*

. *The morphism f induces an epimorphism in* $\operatorname{mod} B$ *if and only if $g \in (\Sigma T)$*

Moreover, if X has no direct summands in $add \Sigma T$, then FX is projective (resp. injective) if and only if X lies in $\operatorname{add}(T)$ *(resp. in* $\operatorname{add}(\Sigma^2 T)$ *).*

Definition 4.2. *A cluster character on $\mathcal{C}$ with values in a commutative ring A is a map*

$$\chi : \ obj(\mathcal{C}) \longrightarrow A$$

such that

. *for all isomorphic objects L and M, we have $\chi(L) = \chi(M)$,*

. *for all objects L and M of $\mathcal{C}$, we have $\chi(L \oplus M) = \chi(L)\chi(M)$,*

. *for all objects L and M of $\mathcal{C}$ such that* $\dim \operatorname{Ext}^1_{\mathcal{C}}(L,M) = 1$, *we have*

$$\chi(L)\chi(M) = \chi(B) + \chi(B'),$$

where B and B' are the middle terms of 'the' non-split triangles

$$L \to B \to M \to \Sigma L \text{ and } M \to B' \to L \to \Sigma M$$

with end terms L and M.

Let N be a finite-dimensional B-module and e an element of $K_0(\operatorname{mod} B)$. We write $\operatorname{Gr}_e(N)$ for the variety of submodules N' of N whose class in $K_0(\operatorname{mod} B)$ is e. It is a closed, hence projective, subvariety of the classical Grassmannian of subspaces of N. Let $\chi(\operatorname{Gr}_e N)$ denote its Euler–Poincaré characteristic with respect to the étale cohomology with proper support. Let $K_0^{\mathrm{sp}}(\operatorname{mod} B)$ denote the 'split' Grothendieck group of $\operatorname{mod} B$, i.e. the quotient of the free abelian group on the set of isomorphism classes $[N]$ of finite-dimensional B-modules N, modulo the subgroup generated by all elements

$$[N_1 \oplus N_2] - [N_1] - [N_2].$$

We define a bilinear form

$$\langle\ ,\ \rangle : K_0^{\mathrm{sp}}(\operatorname{mod} B) \times K_0^{\mathrm{sp}}(\operatorname{mod} B) \longrightarrow \mathbb{Z}$$

by setting

$$\langle N, N' \rangle = [N, N'] \,-\, {}^1[N, N']$$

for all finite-dimensional B-modules N and N'. We define an antisymmetric bilinear form on $K_0^{\mathrm{sp}}(\operatorname{mod} B)$ by setting

$$\langle N, N' \rangle_a \;=\; \langle N, N' \rangle - \langle N', N \rangle$$

for all finite-dimensional B-modules N and N'. Let $T_1, \ldots, T_n$ be the pairwise non-isomorphic indecomposable direct summands of T and, for $i = 1, \ldots, n$, let S_i be the top of the projective B-module $P_i = FT_i$. The set of all $S_i, i = 1, \ldots, n$ is a set of representatives for the isoclasses of simple B-modules.

We need a lemma, the proof of which will be given in section 4.3.1.

Lemma 4.3. *For any $i = 1, \ldots, n$, the linear form $\langle S_i, ? \rangle_a$ on $K_0^{sp}(\operatorname{mod} B)$ induces a well-defined form*

$$\langle S_i, ? \rangle_a \;:\; K_0(\operatorname{mod} B) \to \mathbb{Z}.$$

Let $\operatorname{ind} \mathcal{C}$ be a set of representatives for the isoclasses of indecomposable objects of $\mathcal{C}$. Define, as in [CK08, 6.1], a Caldero–Chapoton map, $X_?^T$: $\operatorname{ind} \mathcal{C} \;\to\; \mathbb{Q}(x_1, \ldots, x_n)$ by

$$X_M^T = \begin{cases} x_i \text{ if } M \simeq \Sigma T_i \\ \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{\langle S_i, e \rangle_a - \langle S_i, FM \rangle} \text{ else.} \end{cases}$$

Extend it to a map $X_?^T : \mathcal{C} \to \mathbb{Q}(x_1, \ldots, x_n)$ by requiring that

$$X_{M \oplus N}^T = X_M^T X_N^T.$$

When there are no possible confusions, we often denote X_M^T by X_M. The main result of this article is the following

Theorem 4.4. *The map* $X_?^T : \mathcal{C} \to \mathbb{Q}(x_1, \ldots, x_n)$ *is a cluster character.*

We will prove the theorem in section 4.5.1, illustrate it by examples in section 4.6 and draw some consequences in section 4.5.2.

4.2 Index, coindex and Euler form

In the next two sections, our aim is to understand the exponents appearing in the definition of X_M. More precisely, for two objects L and M of $\mathcal{C}$, we want to know how the exponents in X_B depend on the choice of the middle term B of a triangle with outer terms L and M.

4.2.1 Index and coindex

Let X be an object of $\mathcal{C}$. Define its index $\operatorname{ind} X \in K_0(\operatorname{proj} B)$ as follows. There exists a triangle (see [KR1])

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X$$

with T_0^X and T_1^X in $\operatorname{add} T$. Define $\operatorname{ind} X$ to be the class $[FT_0^X] - [FT_1^X]$ in $K_0(\operatorname{proj} B)$. Similarly, define the coindex of X, denoted by $\operatorname{coind} X$, to be the class $[FT_X^0] - [FT_X^1]$ in $K_0(\operatorname{proj} B)$, where

$$X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1 \to \Sigma X$$

is a triangle in $\mathcal{C}$ with $T_X^0, T_X^1 \in \operatorname{add} T$.

Lemma 4.5. *We have the following properties:*

1. *The index and coindex are well defined.*
2. $\operatorname{ind} X = -\operatorname{coind} \Sigma X$.
3. $\operatorname{ind} T_i = [P_i]$ *and* $\operatorname{ind} \Sigma T_i = -[P_i]$ *where* $P_i = FT_i$.
4. $\operatorname{ind} X - \operatorname{coind} X$ *only depends on* $FX \in \operatorname{mod} B$.

Proof. A right $\operatorname{add} T$-approximation of an object X of $\mathcal{C}$ is a morphism $T' \xrightarrow{f} X$ with $T' \in \operatorname{add} T$ such that any morphism $T'' \to X$ with $T'' \in \operatorname{add} T$ factors through f. It is called minimal if, moreover, any morphism $T' \xrightarrow{g} T'$ such that $fg = f$ is an isomorphism. A minimal approximation is unique up to isomorphism. Assertions (2) and (3) are left to the reader.

(1) In any triangle of the form

$$T_1^X \to T_0^X \xrightarrow{f} X \to \Sigma T_1^X,$$

the morphism f is a right $\operatorname{add} T$-approximation. Therefore, any such triangle is obtained from one where f is minimal by adding a trivial triangle

$$T' \to T' \to 0 \to \Sigma T'$$

with $T' \in \operatorname{add} T$. The index is thus well-defined. Dually, one can define left approximations and show that the coindex is well-defined.
(4) Let T' be an object in $\operatorname{add} T$. Take two triangles

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X \text{ and}$$

$$X \to \Sigma^2 T_X^0 \to \Sigma^2 T_X^1 \to \Sigma X$$

with T_0^X, T_1^X, T_X^0 and T_X^1 in $\operatorname{add} T$. Then, we have two triangles

$$T_1^X \oplus T' \to T_0^X \to X \oplus \Sigma T' \to \Sigma(T_1^X \oplus T') \text{ and}$$

$$X \oplus \Sigma T' \to \Sigma^2 T_X^0 \to \Sigma^2(T_X^1 \oplus T') \to \Sigma X \oplus \Sigma^2 T'.$$

We thus have the equality:

$$\operatorname{ind}(X \oplus \Sigma T') - \operatorname{coind}(X \oplus \Sigma T') = \operatorname{ind} X - \operatorname{coind} X.$$

□

Proposition 4.6. *Let $X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{\varepsilon} \Sigma X$ be a triangle in $\mathcal{C}$. Take $C \in \mathcal{C}$ (resp. $K \in \mathcal{C}$) to be any lift of* $\operatorname{Coker} Fg$ *(resp.* $\operatorname{Ker} Ff$*). Then*

$$\begin{aligned} \operatorname{ind} Z &= \operatorname{ind} X + \operatorname{ind} Y - \operatorname{ind} C - \operatorname{ind} \Sigma^{-1} C \ \textit{and} \\ \operatorname{coind} Z &= \operatorname{coind} X + \operatorname{coind} Y - \operatorname{coind} K - \operatorname{coind} \Sigma K. \end{aligned}$$

Proof. Let us begin with the equality for the indices. First, consider the case where $FC = 0$. This means that the morphism ε belongs to the ideal (ΣT). Take two triangles

$$T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X \text{ and } \quad T_1^Y \longrightarrow T_0^Y \longrightarrow Y \longrightarrow \Sigma T_1^Y$$

in $\mathcal{C}$, where the objects T_0^X, T_1^X, T_0^Y, T_1^Y belong to the subcategory $\operatorname{add} T$. Since the morphism ε belongs to the ideal (ΣT), the following composition

$T_0^Y \to Y \xrightarrow{\varepsilon} \Sigma X$ vanishes. The morphism $T_0^Y \to Y$ thus factors through g. This gives a commutative square

$$\begin{tikzcd} T_0^X \oplus T_0^Y \arrow[r] \arrow[d] & T_0^Y \arrow[d] \\ Z \arrow[r] & Y. \end{tikzcd}$$

Fit it into a nine-diagram

$$\begin{tikzcd} T_1^X \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & T_1^Y \arrow[r] \arrow[d] & \Sigma T_1^X \\ T_0^X \arrow[r] \arrow[d] & T_0^X \oplus T_0^Y \arrow[r] \arrow[d] & T_0^Y \arrow[r, "0"] \arrow[d] \arrow[dl, dashed] \arrow[dr, "0"] & \Sigma T_0^X \\ X \arrow[r] \arrow[d] & Z \arrow[r, "g"] \arrow[d] & Y \arrow[r, "\varepsilon"] \arrow[d] & \Sigma X \\ \Sigma T_1^X & \Sigma Z' & \Sigma T_1^Y, & \end{tikzcd}$$

whose rows and columns are triangles. Since the morphism $T_1^Y \to \Sigma T_1^X$ vanishes, the triangle in the first row splits, so that we have

$$Z' \simeq T_1^X \oplus T_1^Y \text{ and } \operatorname{ind} Z = \operatorname{ind} X + \operatorname{ind} Y.$$

Now, let us prove the formula in the general case. Let $FY \xrightarrow{a} M$ be a cokernel for Fg. Since the composition $F\varepsilon\, Fg$ vanishes, the morphism $F\varepsilon$ factors through a:

$$\begin{tikzcd} FY \arrow[rr, "F\varepsilon"] \arrow[dr, "a"'] & & F\Sigma X. \\ & M \arrow[ur, dashed, "b"'] & \end{tikzcd}$$

Let $Y \xrightarrow{\alpha} C'$ be a lift of a in $\mathcal{C}$, and let β be a lift of b. The images under F of the morphisms ε and $\beta\,\alpha$ coincide, therefore the morphism $\beta\,\alpha - \varepsilon$ belongs to the ideal (ΣT). Thus there exist an object T' in $\operatorname{add} T$ and two morphisms α' and β' such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "\varepsilon"] \arrow[dr, "\left[\begin{smallmatrix}\alpha \\ \alpha'\end{smallmatrix}\right]"'] & & \Sigma X. \\ & C' \oplus \Sigma T' \arrow[ur, "{[\beta\ \beta']}"'] & \end{tikzcd}$$

Let C be the direct sum $C' \oplus \Sigma T'$.
The octahedral axiom yields a commutative diagram

$$\begin{array}{ccccccc}
 & & U & = & U & & \\
 & & \downarrow & & \downarrow & & \\
X & \longrightarrow & Z & \longrightarrow & Y & \xrightarrow{\varepsilon} & \Sigma X \\
\| & & \downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}\alpha\\ \alpha'\end{smallmatrix}\right]} & & \| \\
X & \longrightarrow & V & \xrightarrow{\gamma'} & C & \xrightarrow{[\beta\ \beta']} & \Sigma X \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\
 & & \Sigma U & = & \Sigma U, & &
\end{array}$$

whose two central rows and columns are triangles. Due to the choice of C, the morphisms γ', γ'', hence γ belong to the ideal (ΣT). We thus have the equalities:

$$\begin{aligned}
\operatorname{ind} Y &= \operatorname{ind} C + \operatorname{ind} U, \\
\operatorname{ind} X &= \operatorname{ind} V + \operatorname{ind} \Sigma^{-1} C, \\
\operatorname{ind} Z &= \operatorname{ind} V + \operatorname{ind} U,
\end{aligned}$$

giving the desired formula. Moreover, as seen in lemma 4.5 (4), the sum $\operatorname{ind} C + \operatorname{ind} \Sigma^{-1} C = \operatorname{ind} C - \operatorname{coind} C$ does not depend on the particular choice of C. Apply this formula to the triangle

$$\Sigma^{-1} X \longrightarrow \Sigma^{-1} Z \longrightarrow \Sigma^{-1} Y \longrightarrow X$$

and use lemma 4.5(2) to obtain the formula for the coindices. Remark that the long exact sequence yields the equality of $\operatorname{Coker}(-F\Sigma^{-1} g)$ and $\operatorname{Ker} Ff$. □

4.2.2 Exponents.

We now compute the index and coindex in terms of the Euler form.

Lemma 4.7. *Let $X \in \mathcal{C}$ be indecomposable. Then*

$$\operatorname{ind} X = \begin{cases} -[P_i] \text{ if } X \simeq \Sigma T_i \\ \sum_{i=1}^{n} \langle FX, S_i \rangle [P_i] \text{ else,} \end{cases}$$

$$\operatorname{coind} X = \begin{cases} -[P_i] \text{ if } X \simeq \Sigma T_i \\ \sum_{i=1}^{n} \langle S_i, FX \rangle [P_i] \text{ else.} \end{cases}$$

Proof. Let X be an indecomposable object in $\mathcal{C}$, non-isomorphic to any of the ΣT_i's. Take a triangle

$$T_1^X \xrightarrow{f} T_0^X \xrightarrow{g} X \xrightarrow{\varepsilon} \Sigma T_1^X$$

with the morphism g being a minimal right $\operatorname{add} T$-approximation, as defined in the proof of lemma 4.5. We thus get a minimal projective presentation

$$P_1^X \longrightarrow P_0^X \longrightarrow FX \longrightarrow 0$$

where $P_i^X = FT_i^X, i = 0, 1$. For any i, the differential in the complex

$$0 \longrightarrow (P_0^X, S_i) \longrightarrow (P_1^X, S_i) \longrightarrow \cdots$$

vanishes. Therefore, we have

$$\begin{aligned} [FX, S_i] &= [P_0^X, S_i] = [P_0^X : P_i], \\ {}^1[FX, S_i] &= [P_1^X, S_i] = [P_1^X : P_i], \\ \langle FX, S_i \rangle &= [\operatorname{ind} X : P_i]. \end{aligned}$$

The proof for the coindex is analogous: We use a minimal injective copresentation of FX induced by a triangle

$$X \longrightarrow \Sigma^2 T_X^0 \longrightarrow \Sigma^2 T_X^1 \longrightarrow \Sigma X.$$

□

Let us write $\underline{x}^e$ for $\prod_{i=1}^n x_i^{[e:P_i]}$ where $e \in K_0(\operatorname{proj} B)$ and $[e : P_i]$ is the ith coefficient of e in the basis $[P_1], \ldots, [P_n]$. Then, by lemma 4.7, for any indecomposable object M in $\mathcal{C}$, we have

$$X_M = \underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{<S_i, e>_a}.$$

4.3 The antisymmetric bilinear form

In this part, we give a positive answer to the first conjecture of [CK08, 6.1] and prove that the exponents in X_M are well defined. The first lemma is sufficient for this latter purpose, but is not very enlightening, whereas the second proof of theorem 4.11 gives us a better understanding of the antisymetric bilinear form. When the category $\mathcal{C}$ is algebraic, this form is, in fact, the usual Euler form on the Grothendieck group of a triangulated category together with a t-structure whose heart is the abelian category $\operatorname{mod} B$ itself.

4.3.1 The map X^T is well defined

Let us first show that any short exact sequence in $\operatorname{mod} B$ can be lifted to a triangle in $\mathcal{C}$.

Lemma 4.8. *Let $0 \to x \to y \to z \to 0$ be a short exact sequence in $\operatorname{mod} B$. Then there exists a triangle in $\mathcal{C}$*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

whose image under F is isomorphic to the given short exact sequence.

Proof. Let

$$0 \longrightarrow x \xrightarrow{i} y \xrightarrow{p} z \longrightarrow 0$$

be a short exact sequence in $\operatorname{mod} B$. Let $X \xrightarrow{f} Y$ be a lift of the monomorphism $x \xrightarrow{i} y$ in $\mathcal{C}$. Fix a triangle

$$T_1^X \longrightarrow T_0^X \longrightarrow X \longrightarrow \Sigma T_1^X$$

and form a triangle

$$X \longrightarrow Y \oplus \Sigma T_1^X \longrightarrow Z \xrightarrow{\varepsilon} \Sigma X \ .$$

The commutative left square extends to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y \oplus \Sigma T_1^X & \longrightarrow & Z & \xrightarrow{\varepsilon} & \Sigma X \\ \| & & \downarrow{\scriptstyle [0\ 1]} & & \downarrow & & \| \\ X & \longrightarrow & \Sigma T_1^X & \longrightarrow & \Sigma T_0^X & \longrightarrow & \Sigma X. \end{array}$$

so that the morphism ε lies in the ideal (ΣT). Therefore, the sequence

$$0 \longrightarrow x \xrightarrow{i} y \longrightarrow FZ \longrightarrow 0$$

is exact, and the modules FZ and z are isomorphic. □

Proof of lemma 4.3.

Let X be an object of the category $\mathcal{C}$. Using section 4.2.2 we have

$$\operatorname{coind} X - \operatorname{ind} X = \sum_{i=1}^{n} \langle S_i, FX \rangle_a \, [P_i] \ .$$

Therefore, it is sufficient to show that the form

$$\begin{array}{rcl} K_0(\operatorname{mod} B) & \longrightarrow & \mathbb{Z} \\ [FX] & \longmapsto & \operatorname{coind} X - \operatorname{ind} X \end{array}$$

is well defined. We already know that $\operatorname{coind} X - \operatorname{ind} X$ only depends on FX. Take $0 \to x \to y \to z \to 0$ to be a short exact sequence in $\operatorname{mod} B$. Lift it, as in lemma 4.8, to a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \text{ in } \mathcal{C}.$$

By proposition 4.6, we have

$$\operatorname{ind} Y - \operatorname{coind} Y = (\operatorname{ind} X + \operatorname{ind} Z) - (\operatorname{coind} X + \operatorname{coind} Z)$$

which is the required equality. □

Corollary 4.9. *The map*

$$X_?^T : \mathcal{C} \longrightarrow \mathbb{Q}(x_1, \ldots, x_n)$$

is well defined.

4.3.2 The antisymmetric bilinear form descends to the Grothendieck group

In this subsection, we prove a stronger result than in the previous one. This gives a positive answer to the first conjecture in [CK08, 6.1].

Lemma 4.10. *Let T' be any cluster-tilting object in $\mathcal{C}$. We have bifunctorial isomorphisms*

$$\mathcal{C}/_{(T')}(\Sigma^{-1}X, Y) \simeq D(T')(\Sigma^{-1}Y, X).$$

Proof. Let X and Y be two objects of $\mathcal{C}$, and let $T_1' \longrightarrow T_0' \longrightarrow X \xrightarrow{\eta} \Sigma T_1'$ be a triangle in $\mathcal{C}$, with T_0' and T_1' in $\operatorname{add} T'$. Consider the morphism

$$\begin{array}{rccc} \alpha: & (T_1', Y) & \longrightarrow & (\Sigma^{-1}X, Y) \\ & f & \longmapsto & f \circ \Sigma^{-1}\eta. \end{array}$$

We have

$$D(T')(\Sigma^{-1}X, Y) \simeq D \operatorname{Im} \alpha \simeq \operatorname{Im} D\alpha.$$

Since the category $\mathcal{C}$ is 2-Calabi–Yau, the dual of α, $D\alpha$, is isomorphic to

$$\begin{array}{rccc} \alpha': & (\Sigma^{-1}Y, X) & \longrightarrow & (\Sigma^{-1}Y, \Sigma T_1') \\ & g & \longmapsto & \eta \circ g. \end{array}$$

We thus have isomorphisms

$$\begin{array}{rcl} D(T')(\Sigma^{-1}X, Y) & \simeq & \operatorname{Im} \alpha' \\ & \simeq & (\Sigma^{-1}Y, X)/\operatorname{Ker} \alpha' \\ & \simeq & \mathcal{C}/_{(T')}(\Sigma^{-1}Y, X). \end{array}$$

□

Theorem 4.11. *The antisymmetric bilinear form* $\langle\ ,\ \rangle_a$ *descends to the Grothendieck group* $K_0(\operatorname{mod} B)$.

Proof. Let X and Y be two objects in the category $\mathcal{C}$. In order to compute $\langle FX, FY\rangle = [FX, FY] - {}^1[FX, FY]$, let us construct a projective presentation in the following way. Let

$$\Sigma^{-1}X \xrightarrow{g} T_1^X \xrightarrow{f} T_0^X \longrightarrow X$$

be a triangle in $\mathcal{C}$ with T_0^X and T_1^X being two objects in the subcategory $\operatorname{add} T$. This triangle induces an exact sequence in $\operatorname{mod} B$

$$F\Sigma^{-1}X \xrightarrow{Fg} FT_1^X \xrightarrow{Ff} FT_0^X \longrightarrow FX \longrightarrow 0,$$

where FT_0^X and FT_1^X are finite-dimensional projective B-modules. Form the complex $(*)$

$$0 \longrightarrow \operatorname{Hom}_B(FT_0^X, FY) \longrightarrow \operatorname{Hom}_B(FT_1^X, FY) \longrightarrow \operatorname{Hom}_B(F\Sigma^{-1}X, FY).$$

Since the object T is cluster-tilting in $\mathcal{C}$, there are no morphisms from any object in $\operatorname{add} T$ to any object in $\operatorname{add}\Sigma T$. The complex $(*)$ is thus isomorphic to the following one :

$$0 \longrightarrow \mathcal{C}(T_0^X, Y) \xrightarrow{f^*} \mathcal{C}(T_1^X, Y) \xrightarrow{g^*} \mathcal{C}/_{(\Sigma T)}(\Sigma^{-1}X, Y),$$

where f^* (resp. g^*) denotes the composition by f (resp. g). Therefore, we have

$$\begin{aligned} \operatorname{Hom}_B(FX, FY) &\simeq \operatorname{Ker} f^* \\ \operatorname{Ext}^1_B(FX, FY) &\simeq \operatorname{Ker} g^* / \operatorname{Im} f^*. \end{aligned}$$

We can now express the bilinear form as

$$\begin{aligned} \langle FX, FY\rangle &= \dim \operatorname{Ker} f^* - \dim \operatorname{Ker} g^* + \mathsf{rk}\, f^* \\ &= [T_0^X, Y] - [T_1^X, Y] + \mathsf{rk}\, g^*, \end{aligned}$$

with the image of the morphism g^* being the quotient by the ideal (ΣT) of the space of morphisms from $\Sigma^{-1}X$ to Y, in $\mathcal{C}$, which belong to the ideal (T):

$$\operatorname{Im} g^* = (T)/_{(\Sigma T)}(\Sigma^{-1}X, Y).$$

Similarily, using an injective copresentation given by a triangle of the form

$$X \longrightarrow \Sigma^2 T_X^0 \longrightarrow \Sigma^2 T_X^1 \xrightarrow{\beta} \Sigma X,$$

we obtain

$$\langle FY, FX\rangle = [Y, \Sigma^2 T_X^0] - [Y, \Sigma^2 T_X^1] + \mathsf{rk}\,\beta_*,$$

and $\operatorname{Im}\beta_* = (\Sigma^2 T)/_{(\Sigma T)}(Y, \Sigma X)$. By lemma 4.10, we have bifunctorial isomorphisms

$$(T)/_{(\Sigma T)}(\Sigma^{-1}X, Y) \simeq D(\Sigma T)/_{(T)}(\Sigma^{-1}Y, X) \simeq D(\Sigma^2 T)/_{(\Sigma T)}(Y, \Sigma X).$$

Therefore, we have the equality

$$\begin{aligned}\langle FX, FY\rangle_a &= [T_0^X, Y] - [T_1^X, Y] - [Y, \Sigma^2 T_X^0] + [Y, \Sigma^2 T_X^1] \\ &= [FT_0^X, FY] - [FT_1^X, FY] - [FY, F\Sigma^2 T_X^0] + [FY, F\Sigma^2 T_X^1].\end{aligned}$$

Since FT is projective and $F\Sigma^2 T$ in injective, this formula shows that $\langle\ ,\ \rangle_a$ descends to a bilinear form on the Grothendieck group $K_0(\operatorname{mod} B)$. □

4.3.3 The antisymmetric bilinear form and the Euler form.

In this subsection, assume moreover that the category $\mathcal{C}$ is algebraic, as in [KR07, section 4]: There exists a k-linear Frobenius category with split idempotents $\mathcal{E}$ whose stable category is $\mathcal{C}$. Denote by $\mathcal{M}$ the preimage, in $\mathcal{E}$, of $\operatorname{add} T$ via the canonical projection functor. The category $\mathcal{M}$ thus contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ whose objects are the projective objects in $\mathcal{E}$, and we have $\underline{\mathcal{M}} = \operatorname{add} T$. Let $\operatorname{Mod}\mathcal{M}$ be the category of $\mathcal{M}$-modules, i.e. of k-linear contravariant functors from $\mathcal{M}$ to the category of k-vector spaces. The category $\operatorname{mod}\underline{\mathcal{M}}$ of finitely presented $\underline{\mathcal{M}}$-modules is identified with the full subcategory of $\operatorname{Mod}\mathcal{M}$ of finitely presented $\mathcal{M}$-modules vanishing on $\mathcal{P}$. This last category is equivalent to the abelian category $\operatorname{mod} B$ of finitely generated B-modules. Recall that the perfect derived category $\operatorname{per}\mathcal{M}$ is the full triangulated subcategory of the derived category of $\mathcal{D}\operatorname{Mod}\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules. Define $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ to be the full subcategory of $\operatorname{per}\mathcal{M}$ whose objects X satisfy the following conditions:

. for each integer n, the finitely presented $\mathcal{M}$-module $\mathsf{H}^n X$ belongs to $\operatorname{mod}\underline{\mathcal{M}}$,

. the module $\mathsf{H}^n X$ vanishes for all but finitely many $n \in \mathbb{Z}$.

It can easily be shown that $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ is a triangulated subcategory of $\operatorname{per}\mathcal{M}$. Moreover, as shown in [Tab07], the canonical t-structure on $\mathcal{D}\operatorname{Mod}\mathcal{M}$ induces a t-structure on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$, whose heart is the abelian category $\operatorname{mod}\underline{\mathcal{M}}$.

The following lemma shows that the Euler form

$$\mathsf{K}_0\left(\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\right)\times\mathsf{K}_0\left(\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\right)\longrightarrow\mathbb{Z}$$

$$([X],[Y])\longmapsto\langle[X],[Y]\rangle=\sum_{i\in\mathbb{Z}}(-1)^i\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\left(X,\Sigma^iY\right)$$

is well defined.

Lemma 4.12. *Let X and Y belong to $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. Then the vector spaces $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\left(X,\Sigma^iY\right)$ are finite dimensional and only finitely many of them are non-zero.*

Proof. Since X belongs to $\operatorname{per}\mathcal{M}$, we may assume that it is representable: There exists M in $\mathcal{M}$ such that $X=M\hat{}$. Moreover, the module H^nY vanishes for all but finitely many $n\in\mathbb{Z}$. We thus may assume Y to be concentrated in degree 0. Therefore, the space $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\left(X,\Sigma^iY\right)=\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(M\hat{},\Sigma^i\mathsf{H}^0Y)$ vanishes for all non-zero i. For $i=0$, it equals

$$\begin{aligned}\operatorname{Hom}_{\mathcal{M}}\left(M\hat{},\mathsf{H}^0Y\right)&=\mathsf{H}^0Y(M)\\&=\operatorname{Hom}_{\underline{\mathcal{M}}}\left(\underline{\mathcal{M}}(?,M),\mathsf{H}^0Y\right).\end{aligned}$$

this last space being finite dimensional.

□

This enables us to give another proof of theorem 4.11. This proof is less general than the previous one, but is nevertheless much more enlightening. *Proof of theorem 4.11.* Let X and Y be two finitely presented $\underline{\mathcal{M}}$-modules, lying in the heart of the t-structure on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. We have:

$$\begin{aligned}\langle[X],[Y]\rangle&=\sum_{i\in\mathbb{Z}}(-1)^i\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\left(X,\Sigma^iY\right)\\&=\sum_{i=0}^{3}(-1)^i\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\left(X,\Sigma^iY\right)\qquad(4.1)\\&=\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(X,Y)-\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(X,\Sigma Y)\\&\qquad+\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(X,\Sigma^2Y)-\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(X,\Sigma^3Y)\\&=\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(X,Y)-\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(X,\Sigma Y)\qquad(4.2)\\&\qquad+\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(Y,X)-\dim\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}(Y,\Sigma X)\\&=\dim\operatorname{Hom}_{\underline{\mathcal{M}}}(X,Y)-\dim\operatorname{Ext}^1_{\underline{\mathcal{M}}}(X,Y)\\&\qquad+\dim\operatorname{Hom}_{\underline{\mathcal{M}}}(Y,X)-\dim\operatorname{Ext}^1_{\underline{\mathcal{M}}}(Y,X)\\&=\langle[X],[Y]\rangle_a\end{aligned}$$

where the classes are now taken in $K_0(\operatorname{mod} B)$. Equalities (3.1) and (3.2) are consequences of the 3-Calabi–Yau property of the category $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M}$, cf. [KR07]. □

4.4 Dichotomy

Our aim in this part is to study the coefficients appearing in the definition of X_M. In particular, we will prove that the phenomenon of dichotomy proved in [CK06] (see also [GLS07]) remains true in this more general setting.

Recall that we write $\underline{x}^e$ for $\prod_{i=1}^n x_i^{[e:P_i]}$ where $e \in K_0(\operatorname{proj} B)$ and $[e : P_i]$ is the ith coefficient of e in the basis $[P_1], \ldots, [P_n]$.

Lemma 4.13. *For any $M \in \mathcal{C}$, we have*

$$X_M = \underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{\langle S_i, e\rangle_a}.$$

Proof. We already know that this formula holds for indecomposable objects of $\mathcal{C}$, cf. section 4.2.2. Let us prove that it still holds for decomposable objects, by recursion on the number of indecomposable direct summands.

Let M and N be two objects in $\mathcal{C}$. As shown in [CC06], we have

$$\chi\left(\operatorname{Gr}_g F(M \oplus N)\right) = \sum_{e+f=g} \chi\left(\operatorname{Gr}_e FM\right) \chi\left(\operatorname{Gr}_f FN\right).$$

Therefore, we have $X_{M\oplus N} = X_M X_N =$

$$\left(\underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{<S_i,e>_a}\right)\left(\underline{x}^{-\operatorname{coind} N} \sum_f \chi(\operatorname{Gr}_e FN) \prod_{i=1}^n x_i^{<S_i,f>_a}\right)$$

$$= \underline{x}^{-(\operatorname{coind} M + \operatorname{coind} N)} \sum_g \sum_{e+f=g} \chi\left(\operatorname{Gr}_e FM\right) \chi\left(\operatorname{Gr}_f FN\right) \prod_{i=1}^n x_i^{<S_i,e+f>_a}$$

$$= \underline{x}^{-\operatorname{coind}(M\oplus N)} \sum_g \chi\left(\operatorname{Gr}_g F(M \oplus N)\right) \prod_{i=1}^n x_i^{<S_i,g>_a}$$

□

Lemma 4.14. *Let $M \xrightarrow{i} B \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$, and let $U \xrightarrow{i_U} M$ and $V \xrightarrow{i_V} L$ be two morphisms whose images under F are monomorphisms. Then the following conditions are equivalent:*

i) There exists a submodule $E \subset FB$ such that $FV = (Fp)E$ and $FU = (Fi)^{-1}E$,

ii) There exist two morphisms $e : \Sigma^{-1}V \longrightarrow U$ and $f : \Sigma^{-1}L \longrightarrow U$ such that

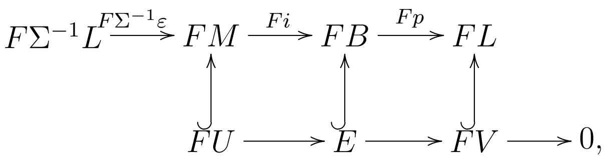
a) $(\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) = i_U e$

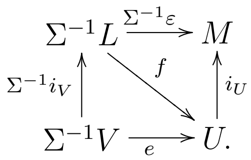
b) $e \in (T)$

c) $i_U f - \Sigma^{-1}\varepsilon \in (\Sigma T)$.

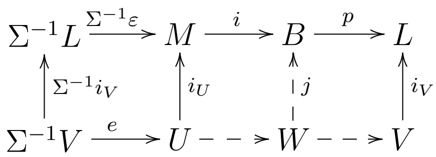
iii) Condition ii) where, moreover, $e = f\Sigma^{-1}i_V$.

The following diagrams will help the reader parse the conditions:

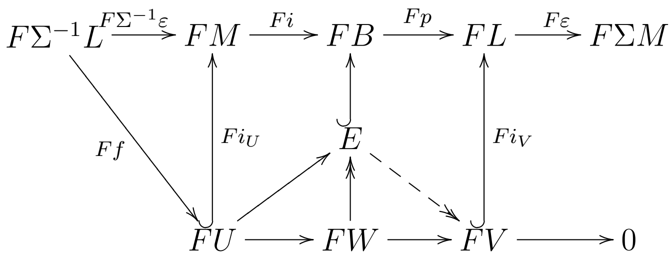




Proof. Assume condition ii) holds. Then, by a), there exists a morphism of triangles



Take E to be the image of the morphism Fj. The morphism e factors through $\operatorname{add} T$, so that we have $F\Sigma e = 0$ and the functor F induces a commutative diagram



whose rows are exact sequences. It remains to show that $FU = (Fi)^{-1}E$.

We have $FU \subset (Fi)^{-1}E$ since $(Fi)(Fi_U)$ factors through the monomorphism $E \to FB$. The existence of the morphism Ff shows, via diagram chasing, the converse inclusion.

Conversely, let $E \subset FB$ be such that $FV = (Fp)E$ and $FU = (Fi)^{-1}E$. In particular, FU contains $\operatorname{Ker} Fi = \operatorname{Im} F\Sigma^{-1}\varepsilon$ so that the morphism $F\Sigma^{-1}\varepsilon$ factors through Fi_U. This gives us the morphism f, satisfying condition c). Define the morphism e as follows. There exists a triangle

$$T_1 \longrightarrow T_0 \longrightarrow V \longrightarrow \Sigma T_1,$$

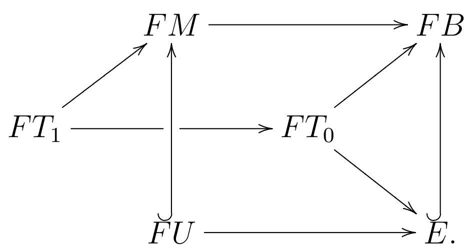
where T_1, T_0 belong to $\operatorname{add} T$. Applying the functor F to this triangle, we get an epimorphism $FT_0 \to FV$ with FT_0 projective. This epimorphism thus factors through the surjection $E \to FV$, and composing it with $E \to FB$ gives a commutative square

$$\begin{CD} FT_0 @>>> FV \\ @VVV @VVV \\ FB @>>> FL. \end{CD}$$

Since $\mathcal{C}(T, \Sigma T) = 0$, this commutative square lifts to a morphism of triangles

$$\begin{CD} \Sigma^{-1}V @>>> T_1 @>>> T_0 @>>> V \\ @VVV @VVV @VVV @VVV \\ \Sigma^{-1}L @>>> M @>>> B @>>> L. \end{CD}$$

The morphism $T_1 \to M$ thus induced, factors through the morphism $U \to M$. Indeed, we have $FU = (Fi)^{-1}E$ and the following diagram commutes :



The morphism e is then given by the composition $\Sigma^{-1}V \longrightarrow T_1 \longrightarrow U$.

Let us show that condition ii) implies condition iii). By hypothesis, we have

$$i_U e = (\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V)$$

and

$$i_U f \Sigma^{-1} i_V \equiv (\Sigma^{-1}\varepsilon)(\Sigma^{-1}i_V) \bmod (\Sigma T).$$

Therefore, the morphism $i_U\,(f\Sigma^{-1}i_V - e)$ belongs to the ideal (ΣT). The morphism Fi_U is a monomorphism, so that the morphism $h := f\Sigma^{-1}i_V - e$ lies in (ΣT). There exists a morphism $\Sigma^{-1}L \xrightarrow{l} U$ such that $h = l\Sigma^{-1}i_V$:

$$\begin{array}{ccccccc}
\Sigma^{-1}C & \xrightarrow{\in(T)} & \Sigma^{-1}V & \xrightarrow{\Sigma^{-1}i_V} & \Sigma^{-1}L & \xrightarrow{c} & C \\
& {\scriptstyle 0}\searrow & {\scriptstyle h}\downarrow{\scriptstyle \in(\Sigma T)} & \swarrow{\scriptstyle l} & & & \\
& & U & & & & .
\end{array}$$

Since the morphism $\Sigma^{-1}C \to \Sigma^{-1}V$ lies in the ideal (T), there exists a morphism of triangles

$$\begin{array}{ccccccc}
\Sigma^{-1}C & \longrightarrow & \Sigma^{-1}V & \longrightarrow & \Sigma^{-1}L & \xrightarrow{c} & C \\
\downarrow & & \| & & {\scriptstyle v}\downarrow & & \downarrow \\
T_V^1 & \xrightarrow{u} & \Sigma^{-1}V & \longrightarrow & \Sigma T_V^0 & \longrightarrow & \Sigma T_V^1 .
\end{array}$$

The composition $l\Sigma^{-1}i_V$ belongs to the ideal (ΣT), so that the composition $l(\Sigma^{-1}i_V)u$ vanishes. We thus have a morphism of triangles

$$\begin{array}{ccccccc}
T_V^1 & \xrightarrow{u} & \Sigma^{-1}V & \longrightarrow & \Sigma T_V^0 & \longrightarrow & \Sigma T_V^1 \\
\downarrow & & {\scriptstyle \Sigma^{-1}i_V}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow \\
\Sigma^{-1}C' & \longrightarrow & \Sigma^{-1}L & \xrightarrow{l} & U & \longrightarrow & C' .
\end{array}$$

Therefore, we have $(\Sigma^{-1}i_V)(l - wv) = 0$, and there exists a morphism $C \xrightarrow{l'} U$ such that $l - wv = l'c$. The morphism $l_0 = l - l'c$ thus factors through ΣT_1. Put $f_0 = f - l_0$. We have

$$f_0\Sigma^{-1}i_V = f\Sigma^{-1}i_V - l\Sigma^{-1}i_V + l'c\Sigma^{-1}i_V = e$$

and

$$\begin{aligned}
i_U f_0 &= i_U f - i_U l_0 \\
&\equiv i_U f \mod (\Sigma T) \\
&\equiv \Sigma^{-1}\varepsilon \mod (\Sigma T).
\end{aligned}$$

□

Proposition 4.15. *Let $L, M \in \mathcal{C}$ be such that* $\dim \mathcal{C}(L, \Sigma M) = 1$. *Let*

$$\Delta : M \xrightarrow{i} B \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$$

$$\textit{and } \Delta' : L \xrightarrow{i'} B' \xrightarrow{p'} M \xrightarrow{\varepsilon'} \Sigma L$$

be non-split triangles. Then conditions i) to iii) hold for the triangle Δ if and only if they do not for the triangle Δ'.

Proof. Define linear maps α from the space $\mathcal{C}(\Sigma^{-1}L, U) \oplus \mathcal{C}(\Sigma^{-1}L, M)$ to $\mathcal{C}/(T)\,(\Sigma^{-1}V, U) \oplus \mathcal{C}(\Sigma^{-1}V, M) \oplus \mathcal{C}/(\Sigma T)\,(\Sigma^{-1}L, M)$ and α' from the space $(T)(\Sigma^{-1}U, V) \oplus \mathcal{C}(\Sigma^{-1}M, V) \oplus (\Sigma T)(\Sigma^{-1}M, L)$ to $\mathcal{C}(\Sigma^{-1}U, L) \oplus \mathcal{C}(\Sigma^{-1}M, L)$ by

$$\alpha(f, \eta) = (f\Sigma^{-1}i_V, i_U f \Sigma^{-1} i_V - \eta \Sigma^{-1} i_V, i_U f - \eta)$$

and

$$\alpha'(e', f', g') = (i_V e' + g'\Sigma^{-1} i_U + i_V f' \Sigma^{-1} i_U, -g' - i_V f').$$

Since the morphism space $\mathcal{C}(L, \Sigma M)$ is one-dimensional, the morphism ε satisfies condition iii) if and only if the composition

$$\beta : \operatorname{Ker}\alpha \hookrightarrow (\Sigma^{-1}L, U) \oplus (\Sigma^{-1}L, M) \twoheadrightarrow (\Sigma^{-1}L, M)$$

does not vanish. Assume condition iii) to be false for the triangle Δ. This happens if and only if the morphism β vanishes, if and only if its dual $D\beta$ vanishes. Since the category $\mathcal{C}$ is 2-Calabi–Yau, lemma 4.10 implies that the morphism $D\beta$ is isomorphic to the morphism:

$$\beta' : (\Sigma^{-1}M, L) \hookrightarrow (\Sigma^{-1}U, L) \oplus (\Sigma^{-1}M, L) \twoheadrightarrow \operatorname{Coker}\alpha'.$$

Therefore, $\beta'(\Sigma^{-1}\varepsilon) = 0$ is equivalent to $\Sigma^{-1}\varepsilon$ being in $\operatorname{Im}\alpha'$, which is equivalent to the existence of three mophisms e', f', g' as in the diagram

$$\begin{array}{ccc} \Sigma^{-1}M & \xrightarrow{g'} & L \\ {\scriptstyle \Sigma^{-1}i_U}\uparrow & \searrow^{f'} & \uparrow{\scriptstyle i_V} \\ \Sigma^{-1}U & \xrightarrow[e']{} & V \end{array}$$

such that

$$\begin{cases} e' \in (T) \\ g' \in (\Sigma T) \\ \Sigma^{-1}\varepsilon' = i_V f' + g' \\ i_V e' = (\Sigma^{-1}\varepsilon')(\Sigma^{-1} i_U). \end{cases}$$

We have thus shown that condition iii) does not hold for the triangle Δ if and only if condition ii) holds for the triangle Δ'. □

4.5 The multiplication formula

We use sections 4.2 and 4.4 to prove the multiplication formula, and apply it to prove conjecture 2 in [CK08].

4.5.1 Proof of theorem 4.4

We use the same notations as in the statement of theorem 4.4.
Define, for any classes e, f, g in the Grothendieck group $K_0(\operatorname{mod} B)$, the following varieties

$$\begin{aligned} X_{e,f} &= \{E \subset FB \text{ s.t. } [(Fi)^{-1}E] = e \text{ and } [(Fp)E] = f\} \\ Y_{e,f} &= \{E \subset FB' \text{ s.t. } [(Fi')^{-1}E] = f \text{ and } [(Fp')E] = e\} \\ X^g_{e,f} &= X_{e,f} \cap \operatorname{Gr}_g(FB) \\ Y^g_{e,f} &= Y_{e,f} \cap \operatorname{Gr}_g(FB'). \end{aligned}$$

We thus have

$$\operatorname{Gr}_g(FB) = \coprod_{e,f} X^g_{e,f} \text{ and } \operatorname{Gr}_g(FB') = \coprod_{e,f} Y^g_{e,f}.$$

Moreover, we have

$$\begin{aligned} \chi\left(\operatorname{Gr}_e(FM) \times \operatorname{Gr}_f(FL)\right) &= \chi\left(X_{e,f} \sqcup Y_{e,f}\right) \\ &= \chi\left(X_{e,f}\right) + \chi\left(Y_{e,f}\right) \\ &= \sum_g \left(\chi\left(X^g_{e,f}\right) + \chi\left(Y^g_{e,f}\right)\right). \end{aligned}$$

where the first equality is a consequence of the dichotomy phenomenon as follows: Consider the map

$$X_{e,f} \sqcup Y_{e,f} \longrightarrow \operatorname{Gr}_e(FM) \times \operatorname{Gr}_f(FL)$$

which sends a submodule E of FB to the pair $((Fi)^{-1}E, (Fp)E)$. By proposition 4.15, it is surjective, and, as shown in [CC06], its fibers are affine spaces.

Lemma 4.16. *Let e, f and g be classes in $\mathrm{K}_0(\operatorname{mod} \operatorname{End}_{\mathcal{C}}(T))$. Assume that $X^g_{e,f}$ is non-empty. Then, we have*

$$\sum \langle S_i, g\rangle_a [P_i] - \operatorname{coind} B = \sum \langle S_i, e+f\rangle_a [P_i] - \operatorname{coind} M - \operatorname{coind} L.$$

Proof. Let E be a submodule of FB in $X^g_{e,f}$. Let $U \xrightarrow{i_U} M$ and $V \xrightarrow{i_V} L$ be two morphisms in the category $\mathcal{C}$ such that $FU \simeq (Fi)^{-1}E$, $FV \simeq (Fp)E$ and the images of i_U and i_V in $\operatorname{mod} B$ are isomorphic to the inclusions of FU in FM and FV in FL respectively. Let $K \in \mathcal{C}$ be a lift of the kernel of Fi. By proposition 4.6, the following equality holds:

$$(1) \quad \operatorname{coind} B = \operatorname{coind} M + \operatorname{coind} L - \operatorname{coind} K - \operatorname{coind}(\Sigma K).$$

By diagram chasing, the kernel of Fi is also a kernel of the induced morphism from FU to E. Therefore, in $K_0(\operatorname{mod} B)$, we have

$$(2) \quad g = e + f - [FK].$$

We have the following equalities:

$$\begin{aligned} \sum \langle S_i, FK\rangle_a [P_i] &= \operatorname{coind} K - \operatorname{ind} K \text{ (by lemma 4.7)} \\ &= \operatorname{coind} K + \operatorname{coind}(\Sigma K) \text{ (by lemma 4.5).} \end{aligned}$$

Equality (2) thus yields

$$(3) \quad \sum \langle S_i, g\rangle_a [P_i] = \sum \langle S_i, e+f\rangle_a [P_i] - \operatorname{coind} K - \operatorname{coind}(\Sigma K).$$

It only remains to sum equalities (1) and (3) to finish the proof. □

Proof of theorem 4.4.
Using lemma 4.13, we have

$$\begin{aligned} X_M X_L &= \underline{x}^{-\operatorname{coind} M - \operatorname{coind} L} \sum_{e,f} \chi(\operatorname{Gr}_e FM)\chi(\operatorname{Gr}_f FL) \prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a}, \\ X_B &= \underline{x}^{-\operatorname{coind} B} \sum_{g} \chi(\operatorname{Gr}_g FB) \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a} \text{ and} \\ X_{B'} &= \underline{x}^{-\operatorname{coind} B'} \sum_{g} \chi(\operatorname{Gr}_g FB') \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a}. \end{aligned}$$

Therefore

$$\begin{aligned} X_M X_L &= \underline{x}^{-\operatorname{coind} M - \operatorname{coind} L} \sum_{e,f} \chi\left(\operatorname{Gr}_e(FM)\right) \chi\left(\operatorname{Gr}_f(FL)\right) \prod x_i^{\langle S_i, e+f\rangle_a} \\ &= \underline{x}^{-\operatorname{coind} M - \operatorname{coind} L} \sum_{e,f,g} \left(\chi\left(X_{e,f}^g\right) + \chi\left(Y_{e,f}^g\right)\right) \prod x_i^{\langle S_i, e+f\rangle_a} \\ &= \underline{x}^{-\operatorname{coind} B} \sum_{e,f,g} \chi\left(X_{e,f}^g\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &\qquad + \underline{x}^{-\operatorname{coind} B'} \sum_{e,f,g} \chi\left(Y_{e,f}^g\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &= \underline{x}^{-\operatorname{coind} B} \sum_{g} \chi\left(\operatorname{Gr}_g(FB)\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &\qquad + \underline{x}^{-\operatorname{coind} B'} \sum_{g} \chi\left(\operatorname{Gr}_g(FB')\right) \prod x_i^{\langle S_i, g\rangle_a} \\ &= X_B + X_{B'}. \end{aligned}$$

□

4.5.2 Consequences

Let Q be a finite acyclic connected quiver, and let $\mathcal{C}$ be the cluster category associated to Q.

An object of $\mathcal{C}$ without self-extensions is called rigid. An object of $\mathcal{C}$ is called basic if its indecomposable direct summands are pairwise non-isomorphic. For a basic cluster-tilting object T of $\mathcal{C}$, let Q_T denote the quiver of End (T), and $\mathcal{A}_{Q_T}$ the associated cluster algebra.

Proposition 4.17. *A cluster character χ on $\mathcal{C}$ with values in $\mathbb{Q}(x_1, \ldots, x_n)$ which sends a basic cluster-tilting object T of $\mathcal{C}$ to a cluster of $\mathcal{A}_{Q_T}$ sends any cluster-tilting object T' of $\mathcal{C}$ to a cluster of $\mathcal{A}_{Q_T}$, and any rigid indecomposable object to a cluster variable.*

Proof. Since the tilting graph of $\mathcal{C}$ is connected, cf. [BMR+06, proposition 3.5], we can prove the first part of the proposition by recursion on the minimal number of mutations linking T' to T. Let $T'' = T''_1 \oplus \cdots \oplus T''_n$ be a basic cluster-tilting object, whose image under χ is a cluster of $\mathcal{A}_{Q_T}$. Assume that $T' = T'_1 \oplus T''_2 \oplus \cdots \oplus T''_n$ is the mutation in direction 1 of T''. Since χ is a cluster character, it satisfies the multiplication formula, and theorem 6.1 of [BMR08] shows that the mutation, in direction 1, of the cluster $(\chi(T''_1), \ldots, \chi(T''_n))$ is the cluster $(\chi(T'_1), \chi(T''_2), \ldots, \chi(T''_n))$. We have thus proved that the image under χ of any cluster-tilting object is a cluster. It is proved in [BMR+06, proposition 3.2] that any rigid indecomposable object of $\mathcal{C}$ is a direct summand of a basic cluster-tilting object. Therefore, the image under χ of any rigid indecomposable object is a cluster variable of $\mathcal{A}_{Q_T}$. □

Remark: As a corollary of the proof of proposition 4.17, a cluster character is characterised, on a set of representatives for the isoclasses of indecomposable rigid objects of $\mathcal{C}$ by the image of each direct summand of any given cluster-tilting object. In fact, using [BIRS, 1.10], this remains true in the more general context of [BIRS]: Let $\mathcal{C}$ be a Hom-finite triangulated 2-Calabi–Yau category having maximal rigid objects without loops nor strong 2-cycles. Denote by n the number of non-isomorphic indecomposable direct summands of any maximal rigid object.

Lemma 4.18. *Let χ_1 and χ_2 be two cluster characters on $\mathcal{C}$ with values in $\mathbb{Q}(x_1, \ldots, x_n)$. Assume that χ_1 and χ_2 coincide on all indecomposable direct summands of a cluster-tilting object T in $\mathcal{C}$. Then χ_1 and χ_2 coincide on all direct summands of the cluster-tilting objects in $\mathcal{C}$ which are obtained from T by a finite sequence of mutations.*

The following corollary was conjectured for the finite case in [CK08]: Let $\mathcal{C}$ be the cluster category of the finite acyclic quiver Q.

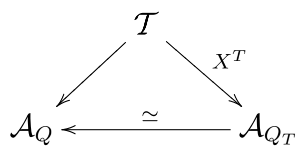
Corollary 4.19. *Let T be any basic cluster-tilting object in $\mathcal{C}$, and let Q_T denote the quiver of $End(T)$. Denote by $\mathcal{T}$ a set of representatives for the isoclasses of indecomposable rigid objects of $\mathcal{C}$. Then X^T induces a bijection from the set $\mathcal{T}$ to the set of cluster variables of the associated cluster algebra $\mathcal{A}_{Q_T}$, sending basic cluster-tilting objects to clusters.*

Proof. In view of theorem 4.4, proposition 4.17 shows that the map X^T sends rigid indecomposable objects to cluster variables and cluster-tilting objects to clusters. It remains to show that it induces a bijection. This follows from [CK06, theorem 4], where it is proved for the Caldero-Chapoton map X^{kQ}.

As in the proof of proposition 4.17, we proceed by induction on the minimal number of mutations linking T to kQ.

Let T' be a basic cluster-tilting object such that the map $X^{T'}$ induces a bijection from the set $\mathcal{T}$ to the set of cluster variables. Assume that T is the mutation in direction 1 of T'. Denote by f the canonical isomorphism from $\mathcal{A}_{Q_{T'}}$ to $\mathcal{A}_{Q_T}$. Theorem 6.1 of [BMR08] shows that the two cluster characters X^T and $f \circ X^{T'}$ coincide on the indecomposable direct summands of ΣT. Therefore, they coincide on all rigid objects and the map X^T also induces a bijection. □

Remark: We have shown that, for any basic cluster-tilting object T, we have a commutative diagram

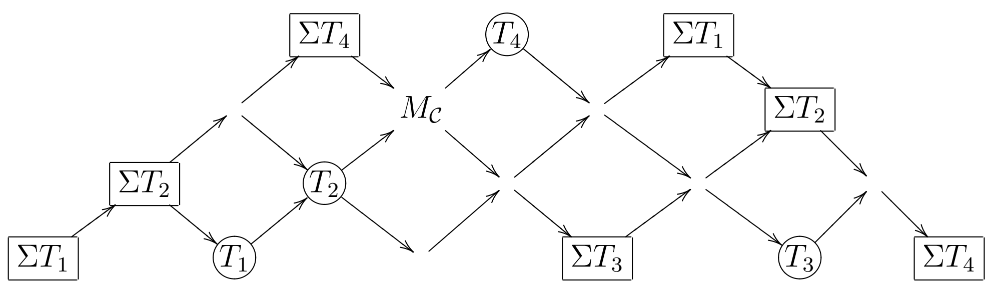


where the arrow on the left side is the Caldero–Chapoton map.

4.6 Examples

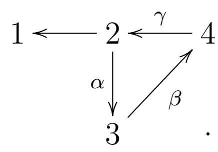
4.6.1 The cluster category $\mathcal{C}_{A_4}$

The Auslander–Reiten quiver of $\mathcal{C}_{A_4}$ is

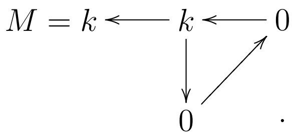


The object $T := T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is cluster-tilting. Indeed, it is obtained from the image of the kQ-projective module kQ in $\mathcal{C}_{A_4}$ by the mutation of the third vertex.

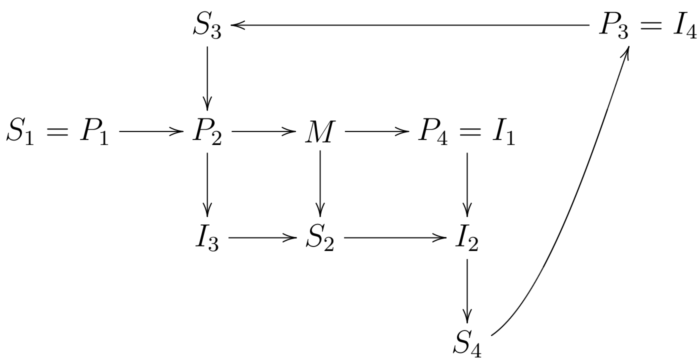
The quiver of $B = \mathrm{End}_{\mathcal{C}_{A_4}}(T)$ is



with relations $\beta\alpha = \gamma\beta = \alpha\gamma = 0$. For $i = 1, \ldots, n$, let P_i be the image of T_i in $\mathrm{mod}\, B$, let I_i be the image of $\Sigma^2 T_i$ and let S_i be the simple top of P_i. Let M be the finite-dimensional B-module given by:



The shape and the relations of the AR–quiver of B are obtained from the ones of $\mathcal{C}_{A_4}$ by deleting the vertices corresponding to the objects ΣT_i and all arrows ending to or starting from these vertices.



Let $M_{\mathcal{C}}$ be an indecomposable lift of M in $\mathcal{C}_{A_4}$. The triangles

$$T_3 \longrightarrow T_2 \longrightarrow M_{\mathcal{C}} \longrightarrow \Sigma T_3 \quad \text{and} \quad T_1 \longrightarrow T_4 \longrightarrow \Sigma^{-1} M_{\mathcal{C}} \longrightarrow \Sigma T_1$$

allows us to compute the index and coindex of $M_{\mathcal{C}}$:

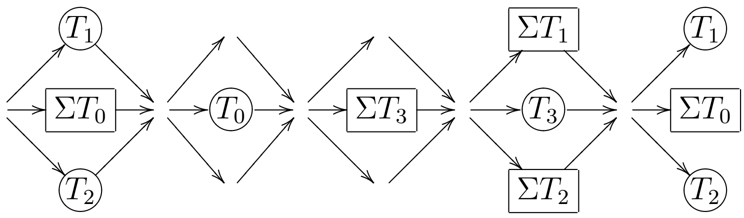
$$\begin{aligned} \mathrm{ind}\, M_{\mathcal{C}} &= [P_2] - [P_3] \\ \mathrm{coind}\, M_{\mathcal{C}} &= [P_1] - [P_4]. \end{aligned}$$

Up to isomorphism, the submodules of M are 0, the simple S_1, and M itself. We thus have

$$X_{M_{\mathcal{C}}} = \frac{x_4 x_2 + x_4 + x_3 x_1}{x_1 x_2}.$$

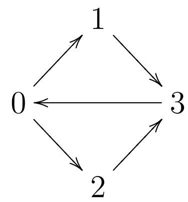
4.6.2 The cluster category $\mathcal{C}_{D_4}$

The Auslander–Reiten quiver of $\mathcal{C}_{D_4}$ is



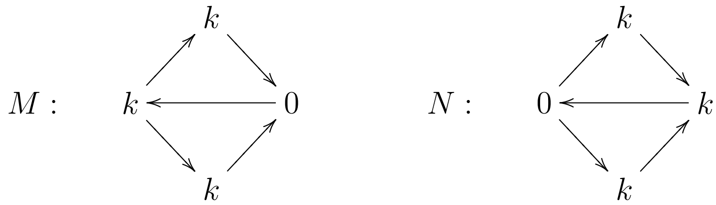
The object $T := T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is cluster-tilting.

The quiver of $B = \mathrm{End}_{\mathcal{C}_{D_4}}(T)$ is

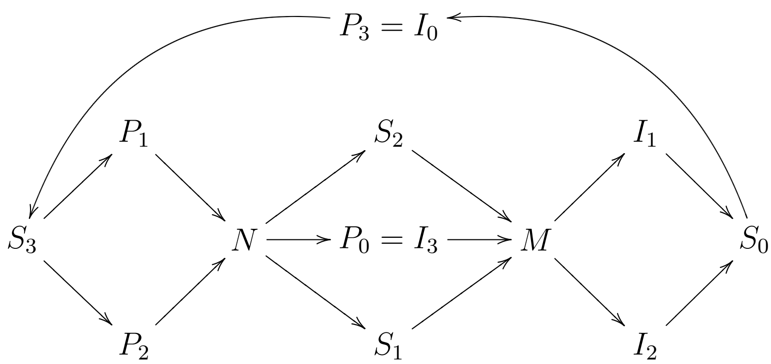


with the following relations: Any composition with the middle arrow vanishes, and the square is commutative.

For $i = 1, \ldots, n$, let P_i be the image of T_i in $\mathrm{mod}\, B$, let I_i be the image of $\Sigma^2 T_i$ and let S_i be the simple top of P_i. Let M and N be the finite-dimensional B-modules given by:



As in the previous example, one can easily compute the AR-quiver of B.



The submodules of M are, up to isomorphism, 0, S_1, S_2, $S_1 \oplus S_2$ and M. Let $M_{\mathcal{C}}$ be an indecomposable lift of M in $\mathcal{C}_{D_4}$. Either by using $\operatorname{add} T$-approximations and $\operatorname{add} \Sigma T$-approximations or by [KN02, section 5.2], one can compute the triangles

$$T_3 \longrightarrow T_0 \longrightarrow M_{\mathcal{C}} \longrightarrow \Sigma T_3 \quad \text{and} \quad T_1 \oplus T_2 \longrightarrow T_0 \longrightarrow \Sigma^{-1} M \longrightarrow \Sigma T_1 \oplus \Sigma T_2.$$

We thus have

$$\operatorname{ind} M_{\mathcal{C}} = [P_0] - [P_3], \ \operatorname{coind} M_{\mathcal{C}} = [P_1] + [P_2] - [P_0]$$

and

$$X_{M_{\mathcal{C}}} = \frac{(x_0 + x_3)^2 + x_1 x_2 x_3}{x_0 x_1 x_2}.$$

Chapter 5

Cluster characters II: A multiplication formula

Introduction

In recent years, the link between Fomin–Zelevinsky's cluster algebras [FZ02] and the representation theory of quivers and finite-dimensional algebras has been investigated intensely, cf. for example the surveys [BM06], [GLSb], [Kela]. In its most tangible form, this link is given by a map taking objects of cluster categories to elements of cluster algebras. Such a map was first constructed by P. Caldero and F. Chapoton [CC06] for cluster categories and cluster algebras associated with Dynkin quivers. The results of P. Caldero and B. Keller [CK08] yield two multiplication formulae for the Caldero–Chapoton map of cluster categories associated with Dynkin quivers. The first one categorifies the exchange relations of cluster variables and only applies to objects L and M such that $\mathrm{Ext}^1(L, M)$ is of dimension 1. The second one generalizes it to arbitrary dimensions, and yields some new relations in the associated cluster algebras. These relations very much resemble relations in dual Ringel–Hall algebras [Sch, section 5.5]. Motivated by these results, C. Geiss, B. Leclerc and J. Schröer [GLS07] proved two analogous formulae for module categories over preprojective algebras. In this latter situation, the number of isomorphism classes of indecomposable objects is usually infinite. Generalizations of the first formula were proved in [CK06] for cluster categories associated with any acyclic quiver, and later in [Pal08] for 2-Calabi–Yau triangulated categories. The first generalization of the second multiplication formula, by A. Hubery (see [Huba]), was based on the existence of Hall polynomials which he proved in the affine case [Hubc], generalizing Ringel's result [Rin90] for Dynkin quivers. Staying close to this point of view, J. Xiao and F. Xu proved in [XX] a projective version of Green's formula [Rin96] and applied it to generalize the multiplication formula for acyclic cluster algebras. Another proof of this formula was found by F. Xu in [Xu], who used the 2-Calabi–Yau property instead of Green's formula. Our aim in this chapter is to generalize the second multiplication formula to more general 2-Calabi–Yau categories for the cluster character associated with an arbitrary cluster tilting object. This in particular applies to the generalized cluster categories introduced by C. Amiot [Ami] and to stable categories of modules over a preprojective algebra.

The chapter is organized as follows: In the first section, we fix some notations and state our main result: A multiplication formula for the cluster character associated with any cluster tilting object. In section 5.2, we recall some definitions and prove the 'constructibility of kernels and cokernels' in modules categories. We apply these facts to prove that:

- If the triangulated category has constructible cones (see section 5.1.4), the sets under consideration in the multiplication formula, and in its proof, are constructible.
- Stable categories of Hom-finite Frobenius categories have constructible cones.
- Generalized cluster categories defined in [Ami] have constructible cones.

Thus, all of the 2-Calabi–Yau triangulated categories related to cluster algebras which have been introduced so far have constructible cones. Notably this holds for cluster categories associated with acyclic quivers, and for the stable categories associated with the exact subcategories of module categories over preprojective algebras constructed in [GLS08] and [BIRS]. In the last section, we prove the main theorem.

5.1 Notations and main result

Let k be the field of complex numbers. The only place where we will need more than the fact that k is an algebraically closed field is proposition 5.2 in section 5.2.1. Let $\mathcal{C}$ be a Hom-finite, 2-Calabi–Yau, Krull–Schmidt k-category which admits a cluster tilting object T. In order to prove the main theorem, a constructibility hypothesis, will be needed. This hypothesis is precisely stated in section 5.1.3 and it will always be explicitly stated when it is assumed. Stable categories of Hom-finite Frobenius categories satisfy this constructibility hypothesis, cf. section 5.2.4, so that the main theorem applies to cluster categories (thanks to the result of [GLSa] stated in the preliminary section), to stable module categories over preprojective algebras... Moreover, the main theorem applies to the generalized cluster categories of [Ami], cf. section 5.2.5.

We let B denote the endomorphism algebra of T in $\mathcal{C}$, and we let F denote the covariant functor from $\mathcal{C}$ to $\operatorname{mod} B$ co-represented by T. We denote the image in $\mathbb{Q}(x_1, \ldots, x_n)$ of an object M in $\mathcal{C}$ under the cluster character associated with T (see chapter 4) by X_M^T. Recall that it is given by the following formula: Let Q_T be the Gabriel quiver of B, and denote by $1, \ldots, n$ its vertices. For each vertex i, denote the corresponding simple (resp. projective) module by S_i (resp. P_i). Then we have

$$X_M^T = \underline{x}^{-\operatorname{coind} M} \sum_e \chi(\operatorname{Gr}_e FM) \prod_{i=1}^n x_i^{\langle S_i, e\rangle_a},$$

where $\operatorname{coind} M$ denotes the coindex of M and $\langle\ ,\ \rangle_a$ the antisymmetric bilinear form on $K_0(\operatorname{mod} B)$ (for more complete definitions, see chapter 4). For any two objects L and M in $\mathcal{C}$, and any morphism ε in $\mathcal{C}(L, \Sigma M)$, we denote any object Y appearing in a triangle of the form

$$M \longrightarrow Y \longrightarrow L \xrightarrow{\varepsilon} \Sigma M$$

by $\operatorname{mt}(\varepsilon)$ (the middle term of ε).

5.1.1 X^T-stratification

Let L and M be objects in $\mathcal{C}$. If an object Y of $\mathcal{C}$ occurs as $\operatorname{mt}(\varepsilon)$ for some morphism ε in $\mathcal{C}(L, \Sigma M)$, we let $\langle Y \rangle$ denote the set of all isomorphism classes of objects $Y' \in \mathcal{C}$ such that:

- Y' is the middle term of some morphism in $\mathcal{C}(L, \Sigma M)$,
- $\operatorname{coind} Y' = \operatorname{coind} Y$ and
- for all e in $K_0(\operatorname{mod} B)$, we have $\chi\big(\operatorname{Gr}_e(FY')\big) = \chi\big(\operatorname{Gr}_e(FY)\big)$.

The equality of classes $\langle Y \rangle = \langle Y' \rangle$ yields an equivalence relation on the 'set' of middle terms of morphisms in $\mathcal{C}(L, \Sigma M)$. Fix a set $\mathcal{Y}$ of representatives for this relation. Further, we denote the set of all ε with $\operatorname{mt}(\varepsilon) \in \langle Y \rangle$ by $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$, and the set of $\varepsilon' \in \mathcal{C}(L, \Sigma M)$ such that $X^T_{\operatorname{mt}(\varepsilon')} = X^T_{\operatorname{mt}(\varepsilon)}$ by $\langle \varepsilon \rangle$. It will be proven in section 5.2.3 that if the cylinders of the morphisms $L \to \Sigma M$ are constructible with respect to T in the sense of section 5.1.3 below, then the sets $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ are constructible, and the set $\mathcal{Y}$ is finite.

Remark that if Y' belongs to $\langle Y \rangle$, then $X^T_{Y'} = X^T_Y$. Hence the fibers of the map sending ε to $X^T_{\mathrm{mt}(\varepsilon)}$ are finite unions of sets $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$. Therefore, the sets $\langle \varepsilon \rangle$ are constructible, we have

$$\mathcal{C}(L, \Sigma M) = \coprod_{\varepsilon \in \mathcal{R}} \langle \varepsilon \rangle$$

for some finite set $\mathcal{R} \subset \mathcal{C}(L, \Sigma M)$, and

$$\mathcal{C}(L, \Sigma M) = \coprod_{Y \in \mathcal{Y}} \mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$$

is a refinement of the previous decomposition.

5.1.2 The variety $\mathrm{rep}_d\, BQ$

Let V be a finite dimensional k-vector space. We denote by $\mathrm{rep}'_B(V)$ the set of morphisms of k-algebras from B^{op} to $\mathrm{End}_k(V)$. Since B is finitely generated, the set $\mathrm{rep}'_B(V)$ is a closed subvariety of some finite product of copies of $\mathrm{End}_k(V)$.

Let Q be a finite quiver, and let $d = (d_i)_{i \in Q_0}$ be a tuple of non-negative integers. A *d-dimensional matrix representation* of Q in $\mathrm{mod}\, B$ is given by

- a right B-module structure on k^{d_i} for each vertex i of Q and
- a B-linear map $k^{d_i} \to k^{d_j}$ for each arrow $\alpha : i \to j$ of Q.

Clearly, for fixed d, the d-dimensional matrix representations of Q in $\mathrm{mod}\, B$ form an affine variety $\mathrm{rep}_d\, BQ$ on which the group $GL(d) = \prod_{i \in Q_0} GL_{d_i}(k)$ acts by changing the bases in the spaces k^{d_i}. We write $\mathrm{rep}_d\, BQ/GL(d)$ for the set of orbits.

5.1.3 Constructible cones

Let $\overrightarrow{A_4}$ be the quiver: $1 \to 2 \to 3 \to 4$. Let T, L and M be objects of $\mathcal{C}$. Let $d_{\max}$ be the 4-tuple of integers

$$(\dim FM, \dim FM + \dim FL, \dim FL, \dim F\Sigma M).$$

Let $\Phi_{L,M}$ be the map from $\mathcal{C}(L, \Sigma M)$ to

$$\coprod_{d \leq d_{\max}} \mathrm{rep}_d(B\overrightarrow{A_4})/GL(d)$$

sending a morphism ε to the orbit of the exact sequence of B-modules

$$\mathcal{C}(T,M) \xrightarrow{Fi} \mathcal{C}(T,Y) \xrightarrow{Fp} \mathcal{C}(T,L) \xrightarrow{F\varepsilon} \mathcal{C}(T,\Sigma M),$$

where $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ is a triangle in $\mathcal{C}$. The cylinders over the morphisms $L \to \Sigma M$ are *constructible with respect to* T if the map $\Phi_{L,M}$ lifts to a constructible map

$$\varphi_{L,M} : \mathcal{C}(L,\Sigma M) \longrightarrow \coprod_{d \leq d_{\max}} \operatorname{rep}_d(B\overrightarrow{A_4})$$

(see section 5.2.1). The category $\mathcal{C}$ is said to have constructible cones if this holds for arbitrary objects L,M and T.

5.1.4 Main result

Let f be a constructible function from an algebraic variety over k to any abelian group, and let C be a constructible subset of this variety. Then one defines "the integral of f on C with respect to the Euler characteristic" to be

$$\int_C f = \sum_{x \in f(C)} \chi\big(C \cap f^{-1}(x)\big)x,$$

cf. for example the introduction of [Lus97]. Our aim in this paper is to prove the following:

Theorem 5.1. *Let T be any cluster tilting object in $\mathcal{C}$. Let L and M be two objects such that the cylinders over the morphisms $L \to \Sigma M$ and $M \to \Sigma L$ are constructible with respect to T. Then we have:*

$$\chi(\mathbb{P}\mathcal{C}(L,\Sigma M))X_L^T X_M^T = \int_{[\varepsilon]\in\mathbb{P}\mathcal{C}(L,\Sigma M)} X^T_{\operatorname{mt}(\varepsilon)} + \int_{[\varepsilon]\in\mathbb{P}\mathcal{C}(M,\Sigma L)} X^T_{\operatorname{mt}(\varepsilon)},$$

where $[\varepsilon]$ denotes the class in $\mathbb{P}\mathcal{C}(L,\Sigma M)$ of a non zero morphism ε in $\mathcal{C}(L,\Sigma M)$.

The statement of the theorem is inspired from [GLS07], cf. also [XX]. We will prove it in section 5.3. Our proof is inspired from that of P. Caldero and B. Keller in [CK08]. Note that in contrast with the situation considered there, in the above formula, an infinite number of isomorphism classes of objects $\operatorname{mt}(\varepsilon)$ may appear.

5.2 Constructibility

5.2.1 Definitions

Let X be a topological space. A *locally closed subset* of X is the intersection of a closed subset with an open one. A *constructible subset* is a finite (disjoint) union of locally closed subsets. The family of constructible subsets is the smallest one containing all open (equivalently: closed) subsets of X and stable under taking finite intersections and complements. A function f from X to an abelian group is *constructible* if it is a finite $\mathbb{Z}$-linear combination of characteristic functions of constructible subsets of X. Equivalently, f is constructible if it takes a finite number of values and if its fibers are constructible subsets of X.

For an algebraic variety X, the ring of constructible functions from X to $\mathbb{Z}$ is denoted by $CF(X)$. The following proposition will be used, as in [XX], in order to prove lemma 5.5 of section 5.2.3.

Proposition 5.2. *[Dim04, Proposition 4.1.31] Associated with any morphism of complex algebraic varieties $f : X \longrightarrow Y$, there is a well-defined push-forward homomorphism $CF(f) : CF(X) \longrightarrow CF(Y)$. It is determined by the property*

$$CF(f)(1_Z)(y) = \chi(f^{-1}(y) \cap Z)$$

for any closed subvariety Z in X and any point $y \in Y$.

Let X and Y be algebraic varieties. A map $f : X \longrightarrow Y$ is said to be *constructible* if there exists a decomposition of X into a finite union of locally closed subsets $X_i, i \in I$, such that the restriction of f to each X_i is algebraic. Note that the composition of two constructible maps is constructible, and that the composition of a constructible function with a constructible map is again a constructible function.

5.2.2 Kernels and cokernels are constructible

In section 2.1 of [Xu], it is shown that the kernel and cokernel of a morphism of modules over a path algebra $\mathbb{C}Q$ are constructible. In this section, we give direct proofs in the more general case where $\mathbb{C}Q$ is replaced by a finite dimensional algebra B.

Let L and M be two finite dimensional vector spaces over the field k, of respective dimensions n and m. Let N be a linear subspace of M. Define E_N to be the set of all morphisms $f \in \mathrm{Hom}_k(L, M)$ such that $\mathrm{Im}\, f \oplus N = M$.

Lemma 5.3. *The set E_N is a locally closed subset of* $\mathrm{Hom}_k(L, M)$.

Proof. Let $(u_1, \ldots, u_n)$ be a basis of L, and let $(v_1, \ldots, v_m)$ be a basis of M whose p first vectors form a basis of N. Let r be such that $r + p = m$. Let $f : L \longrightarrow M$ be a k-linear map, and denote by $A = (a_{ij})$ its matrix in the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$. Denote by A_1 the submatrix of A formed by its first p rows and by A_2 the one formed by its last r rows. For $t \leq n$, let $P(t, n)$ be the set of all subsets of $\{1, \ldots, n\}$ of cardinality t.

The map f belongs to E_N if and only if:

a) There exists $\underline{j}$ in $P(r, n)$ such that the submatrix $(a_{ij})_{i>p, j\in\underline{j}}$ has a non-zero determinant and

b) if the last r entries of a linear combination of columns of A vanish, then the combination itself vanishes.

Condition b) is equivalent to the inclusion $\operatorname{Ker} A_2 \subseteq \operatorname{Ker} A_1$ and so to the inclusion $\operatorname{Im}(A_1^{\mathrm{t}}) \subseteq \operatorname{Im}(A_2^{\mathrm{t}})$. Therefore, condition b) can be restated as condition b'):

b') For all $i_0 \leq p$, and all $\underline{l} \in P(r+1, n)$, the determinant of the submatrix of A obtained by taking lines in $\{i_0, p+1, \ldots, m\}$ and columns in $\underline{l}$ vanishes.

Let $\Omega_{\underline{j}}$ be the set of all maps that satisfy condition a) with respect to the index set $\underline{j}$, and let F be the set of all maps that satisfy condition b'). For all $\underline{j} \in P(r, n)$, the set $\Omega_{\underline{j}}$ is an open subset of $\operatorname{Hom}_k(L, M)$ and the set F is a closed subset of $\operatorname{Hom}_k(L, M)$. Since we have the equality:

$$E_N = \Big(\bigcup_{\underline{j}\in P(r,n)} \Omega_{\underline{j}} \Big) \cap F,$$

the set E_N is locally closed in $\operatorname{Hom}_k(L, M)$. □

Let $\overrightarrow{A_2}$ be the quiver: $1 \rightarrow 2$.

Lemma 5.4. *Let B be a finite dimensional algebra, and let L and M be finitely generated B-modules of dimensions n and m respectively. The map c from $\operatorname{Hom}_B(L, M)$ to $\coprod_{d\leq m} \operatorname{rep}_{(m,d)}(B\overrightarrow{A_2})/GL(m, d)$ which sends a morphism l to the orbit of the representation $M \twoheadrightarrow \operatorname{Coker} l$ lifts to a constructible map from $\operatorname{Hom}_B(L, M)$ to $\coprod_{d\leq m} \operatorname{rep}_{(m,d)}(B\overrightarrow{A_2})$.*

Dually, the map from $\operatorname{Hom}_B(L, M)$ to $\coprod_{d\leq n} \operatorname{rep}_{(d,n)}(B\overrightarrow{A_2})/GL(d, n)$ which sends a morphism l to the orbit of the representation $\operatorname{Ker} l \rightarrowtail N$ lifts to a constructible map from $\operatorname{Hom}_B(L, M)$ to $\coprod_{d\leq n} \operatorname{rep}_{(d,n)}(B\overrightarrow{A_2})$.

Proof. Let us prove the first assertion. We keep the notations of the proof of lemma 5.3. For a subset $\underline{i}$ of $\{1,\ldots,m\}$, let $N_{\underline{i}}$ be the linear subspace of M generated by $(v_i)_{i\in\underline{i}}$. Then $\mathcal{H}\mathrm{om}_B(L,M)$ is the union of its intersections with each $E_{N_{\underline{i}}}$, for $\underline{i}\subseteq\{1,\ldots,m\}$. It is thus enough to consider the restriction of the map c to E_N, where $N \overset{i_N}{\rightarrowtail} M$ is a given linear subspace of M. Since the set E_N is the union of the locally closed subsets $\Omega_{\underline{j}}\cap F$, for $\underline{j}\in P(r,n)$, we can fix such a $\underline{j}$ and only consider the restriction of c to $\Omega_{\underline{j}}\cap F$. Let f be a morphism in $\mathrm{Hom}_B(L,M)$ and assume that f is in $\Omega_{\underline{j}}\cap F$. Then the cokernel of the k-linear map f is N and the projection p_f of M onto N along $\mathrm{Im}\, f$ is given by the $n\times p$ matrix $(1 \quad -CD^{-1})$, where C is the submatrix $(a_{ij})_{i\leq p, j\in\underline{j}}$ and D is the submatrix $(a_{ij})_{i>p, j\in\underline{j}}$. Moreover, if we denote by $\rho^M\in\mathrm{rep}'_B(M)$ the structure of B-module of M, then the structure of B-module ρ of N induced by f is given by $\rho(b) = p_f\circ\rho^M(b)\circ i_N$, for all $b\in B$. □

5.2.3 Constructibility of $\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}$

Let k, $\mathcal{C}$ and T be as in section 5.1. Recall that B denotes the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$. This algebra is the path algebra of a quiver Q_T with ideal of relations I. Recall that we denote by $1,\ldots,n$ the vertices of Q_T.

The following lemma is a particular case of [Dim04, Proposition 4.1.31], and was already stated in [XX] for hereditary algebras.

Lemma 5.5. *For any two dimension vectors e and d with $e\leq d$, the function*

$$\begin{array}{rcl}\mu_e : \mathrm{rep}_d(Q_T,I) & \longrightarrow & \mathbb{Z}\\ M & \longmapsto & \chi(\mathrm{Gr}_e\, M)\end{array}$$

is constructible.

Proof. Let $\mathrm{Gr}_e(d)$ be the closed subset of

$$\mathrm{rep}_d(Q_T,I)\times\prod_{i\in Q_0}\mathrm{Gr}_{e_i}(k^{d_i})$$

formed by those pairs (ρ,W) for which the subspaces $W_i\subseteq k^{d_i}$, $i\in Q_0$, form a subrepresentation. Apply proposition 5.2 to the first projection $f:\mathrm{Gr}_e(d)\to\mathrm{rep}_d(Q_T,I)$ and remark that $\mu_e = CF(f)(1_{\mathrm{Gr}_e(d)})$.

□

Corollary 5.6. *Let L and M be objects in $\mathcal{C}$, and let $e \leq \underline{\dim} FL + \underline{\dim} FM$ be in $K_0(\operatorname{mod} B)$. Assume that the cylinders over the morphisms $L \to \Sigma M$ are constructible. Then the function*

$$\begin{array}{rrcl} \lambda_e : & \mathcal{C}(L, \Sigma M) & \longrightarrow & \mathbb{Z} \\ & \varepsilon & \longmapsto & \chi(\operatorname{Gr}_e F \operatorname{mt}(\varepsilon)) \end{array}$$

is constructible.

Proof. By our hypothesis, the map sending $\varepsilon \in \mathcal{C}(L, \Sigma M)$ to the image of its middle term in $\coprod \operatorname{rep}_d(Q_T, I)/GL(d)$, where the union is over the dimension vectors d not greater than $\underline{\dim} FL + \underline{\dim} FM$, lifts to a constructible map from $\mathcal{C}(L, \Sigma M)$ to $\coprod \operatorname{rep}_d(Q_T, I)$. The claim therefore follows from lemma 5.5. □

Let $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$, and denote by g the class of $\operatorname{Ker} Fi$ in the Grothendieck group $K_0(\operatorname{mod} B)$.

Lemma 5.7. *We have:*

$$\operatorname{coind} Y = \operatorname{coind}(L \oplus M) - \sum_{i=1}^{n} \langle S_i, g \rangle_a [P_i].$$

Proof. Let $K \in \mathcal{C}$ lift $\operatorname{Ker} Fi$. Using respectively proposition 4.6, assertion (2) of lemma 4.5, lemma 4.7 and section 4.3, we have the following equalities:

$$\begin{array}{rcl} \operatorname{coind} Y & = & \operatorname{coind} L + \operatorname{coind} M - \operatorname{coind} K - \operatorname{coind} \Sigma K \\ & = & \operatorname{coind}(L \oplus M) + \operatorname{ind} K - \operatorname{coind} K \\ & = & \operatorname{coind}(L \oplus M) - \sum_{i=1}^{n} \langle S_i, FK \rangle_a [P_i] \\ & = & \operatorname{coind}(L \oplus M) - \sum_{i=1}^{n} \langle S_i, g \rangle_a [P_i]. \end{array}$$

□

Corollary 5.8. *Let L and M be two objects such that the cylinders over the morphisms $L \to \Sigma M$ are constructible. The map*

$$\lambda : \mathcal{C}(L, \Sigma M) \longrightarrow K_0(\operatorname{proj} B)$$

which sends ε to the coindex (or to the index) of its middle term Y is constructible.

Proof. Note that g is at most the sum of the dimension vectors of FL and FM, so that by lemma 5.7 the map λ takes a finite number of values. By our hypothesis and lemma 5.4, there exists a constructible map:

$$\mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \leq \dim FM} \operatorname{rep}'_B(k^d)$$

which lifts the map sending ε to the isomorphism class of the structure of B-module on $\operatorname{Ker} Fi$. Moreover, the map sending a module ρ in the variety $\bigcup_{d \leq \dim FM} \operatorname{rep}'_B(k^d)$ to $\sum_{i=1}^{n} \langle S_i, \rho \rangle_a [P_i]$ in $K_0(\operatorname{proj} B)$ only depends on the dimension vector of ρ and thus is constructible. Therefore, the map λ is constructible. □

Proposition 5.9. *Let $L, M \in \mathcal{C}$ be such that the cylinders over the morphisms $L \to \Sigma M$ are constructible. Then the sets $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ are constructible subsets of $\mathcal{C}(L, \Sigma M)$. Moreover, the set $\mathcal{C}(L, \Sigma M)$ is a finite disjoint union of such constructible subsets.*

Proof. Fix a triangle $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ in $\mathcal{C}$. Then $\varepsilon' \in \mathcal{C}(L, \Sigma M)$ is in $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ if and only if

- $\lambda(\varepsilon') = \lambda(\varepsilon)$ and
- For all $e \leq \underline{\dim} FY$, $\lambda_e(\varepsilon') = \lambda_e(\varepsilon)$.

Therefore, the claim follows from corollary 5.6 and corollary 5.8.

□

5.2.4 Stable categories have constructible cones

In this section, we assume moreover that $\mathcal{C}$ is the stable category of a Hom-finite, Frobenius, Krull–Schmidt category $\mathcal{E}$, which is linear over the algebraically closed field k. Our aim is to prove that such a category has constructible cones.

Let $\mathcal{P}$ denote the ideal in $\mathcal{E}$ of morphisms factoring through a projective-injective object. Let T, L and M be objects of the category $\mathcal{C}$. Fix a k-linear section s of the projection $\mathcal{E}(L, \Sigma M) \twoheadrightarrow \mathcal{C}(L, \Sigma M)$ induced by the canonical functor $\mathcal{E} \xrightarrow{\Pi} \mathcal{C}$. Fix a conflation $M \rightarrowtail IM \twoheadrightarrow \Sigma M$ in $\mathcal{E}$, with IM being projective-injective in $\mathcal{E}$, and, for any ε in $\mathcal{C}(L, \Sigma M)$, consider its pull-back via $s\varepsilon$:

$$\begin{array}{ccccc} M & \overset{\iota}{\rightarrowtail} & Y & \overset{\pi}{\twoheadrightarrow} & L \\ \| & & \downarrow & & \downarrow{\scriptstyle s\varepsilon} \\ M & \rightarrowtail & IM & \twoheadrightarrow & \Sigma M. \end{array}$$

Via Π, this diagram induces a triangle $M \xrightarrow{i} Y \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ in $\mathcal{C}$.

For any $X \in \mathcal{E}$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}(X,M) & \xrightarrow{\mathcal{E}(X,\iota)} & \mathcal{E}(X,Y) & \xrightarrow{\mathcal{E}(X,\pi)} & \mathcal{E}(X,L) \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \mathcal{E}(X,s\varepsilon)} \\ 0 & \longrightarrow & \mathcal{E}(X,M) & \longrightarrow & \mathcal{E}(X,IM) & \longrightarrow & \mathcal{E}(X,\Sigma M). \end{array}$$

Fix $X' \in \mathcal{E}$ and a morphism $X' \to X$. Let C denote the endomorphism algebra of $X' \to X$ in the category of morphisms of $\mathcal{E}$, and let $\mathcal{D}'$ denote the set of all dimension vectors $d = (d_1, d_2, d_3, d_4)$ such that $d_1 = \dim \mathcal{E}(X, M)$, $d_3 = \dim \mathcal{E}(X, L)$, $d_2 \leq d_1 + d_3$ and $d_4 = \dim \mathcal{E}(X, \Sigma M)$.

Lemma 5.10. *There exists a constructible map*

$$\mu : \mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \in \mathcal{D}'} \operatorname{rep}_d C\overrightarrow{A_4}$$

which lifts the map sending ε to the orbit of the matrix representation of $\overrightarrow{A_4}$ in $\operatorname{mod} C$ *given by* $\mathcal{E}(X,M) \xrightarrow{\mathcal{E}(X,\iota)} \mathcal{E}(X,Y) \xrightarrow{\mathcal{E}(X,\pi)} \mathcal{E}(X,L) \xrightarrow{\mathcal{E}(X,s\varepsilon)} \mathcal{E}(X,\Sigma M)$.

Proof. By definition of a pull-back, the map from the space $\mathcal{E}(X, Y)$ to $\mathcal{E}(X, IM) \oplus \mathcal{E}(X, L)$ is a kernel for the map

$$\mathcal{E}(X, IM) \oplus \mathcal{E}(X, L) \longrightarrow \mathcal{E}(X, \Sigma M).$$

Moreover, the morphism $\mathcal{E}(X,M) \xrightarrow{\mathcal{E}(X,\iota)} \mathcal{E}(X,Y)$ is a kernel for $\mathcal{E}(X, \pi)$. Therefore, lemma 5.4 in section 5.2.2 applies and such a constructible map μ exists. □

Denote by $\mathcal{D}$ the set of dimension vectors $d = (d_1, d_2, d_3, d_4)$ such that: $d_1 = \dim \mathcal{C}(T, M)$, $d_3 = \dim \mathcal{C}(T, L)$, $d_2 \leq d_1 + d_3$ and $d_4 = \dim \mathcal{C}(T, \Sigma M)$.

Proposition 5.11. *There exists a constructible map*

$$\varphi : \mathcal{C}(L, \Sigma M) \longrightarrow \coprod_{d \in \mathcal{D}} \operatorname{rep}_d B\overrightarrow{A_4}$$

which lifts the map sending ε to the orbit of the representation

$$\mathcal{C}(T,M) \xrightarrow{Fi} \mathcal{C}(T,Y) \xrightarrow{Fp} \mathcal{C}(T,L) \xrightarrow{F\varepsilon} \mathcal{C}(T,\Sigma M) .$$

Proof. Let $T \rightarrowtail IT$ be an inflation from T to a projective-injective object in $\mathcal{E}$. This inflation induces a commutative diagram $(*)$ of modules over the endomorphism algebra $\widetilde{B}$ of $T \rightarrowtail IT$ in the Frobenius category of inflations of $\mathcal{E}$:

$$(*)\qquad \begin{array}{ccccccc} \mathcal{E}(IT,M) & \longrightarrow & \mathcal{E}(IT,Y) & \longrightarrow & \mathcal{E}(IT,L) & \longrightarrow & \mathcal{E}(IT,\Sigma M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}(T,M) & \longrightarrow & \mathcal{E}(T,Y) & \longrightarrow & \mathcal{E}(T,L) & \longrightarrow & \mathcal{E}(T,\Sigma M). \end{array}$$

The map which sends ε to the orbit of the diagram $(*)$ lifts to a constructible one. This is proved by repeating the proof of lemma 5.10 for the functor

$$\mathcal{E} \longrightarrow \operatorname{mod} \widetilde{B},\ U \longmapsto \big(\mathcal{E}(IT,U) \to \mathcal{E}(T,U)\big)$$

instead of $U \mapsto \mathcal{E}(X,U)$ and using lemma 5.4 for $\widetilde{B}$.

By applying lemma 5.4 to $\widetilde{B} \otimes kA_4$, we see that the vertical cokernel of diagram $(*)$ is constructible as a $\widetilde{B} \otimes kA_4$-module. Now the claim follows because the terms of the cokernel are B-modules and B is also the stable endomorphism algebra of $T \rightarrowtail IT$ in the Frobenius category of inflations of $\mathcal{E}$. $\square$

5.2.5 Generalized cluster categories have constructible cones

Let (Q,W) be a Jacobi-finite quiver with potential W in kQ (cf. section 3.3 of [Ami]), and let Γ be the Ginzburg dg algebra associated with (Q,W) (cf. section 3.2.6). The perfect derived category $\operatorname{per}\Gamma$ is the thick subcategory of the derived category $\mathcal{D}\Gamma$ generated by Γ. The finite dimensional derived category $\mathcal{D}_{\mathrm{fd}}\Gamma$ is the full subcategory of $\mathcal{D}\Gamma$ whose objects are the dg modules whose homology is of finite total dimension. It is easy to check that an object M belongs to $\mathcal{D}_{\mathrm{fd}}\Gamma$ if and only if $\operatorname{Hom}_{\mathcal{D}\Gamma}(P,M)$ is finite dimensional for each object P of $\operatorname{per}\Gamma$.

Lemma 5.12 (Appendix of [KY]). *a) The category $\mathcal{D}_{fd}\Gamma$ is contained in* $\operatorname{per}\Gamma$.

b) An object of $\mathcal{D}\Gamma$ belongs to $\mathcal{D}_{fd}\Gamma$ if and only if it is quasi-isomorphic to a dg Γ-module of finite total dimension.

c) The category $\mathcal{D}_{fd}\Gamma$ is equivalent to the localization of the homotopy category $\mathcal{H}_{fd}\Gamma$ of right dg Γ-modules of finite total dimension with respect to its subcategory of acyclic dg modules.

Note that we stated the previous lemma under some restrictions which do not appear in the appendix of [KY]. Recall that the generalized cluster category associated with (Q, W), defined in [Ami], is the localization of the category $\operatorname{per}\Gamma$ by the full subcategory $\mathcal{D}_{\mathrm{fd}}\Gamma$.

It is proved in [Ami] that the canonical t-structure on $\mathcal{D}\Gamma$ restricts to a t-structure on $\operatorname{per}\Gamma$. We will denote this t-structure by $(\mathrm{per}^{\leq 0}, \mathrm{per}^{\geq 0})$.

Denote by $\mathcal{F}$ the full subcategory of $\operatorname{per}\Gamma$ defined by:

$$\mathcal{F} = \mathrm{per}^{\leq 0} \cap {}^{\perp}(\mathrm{per}^{\leq -2}).$$

Recall from [Ami] that the canonical functor from $\operatorname{per}\Gamma$ to $\mathcal{C}_\Gamma$ induces a k-linear equivalence from $\mathcal{F}$ to $\mathcal{C}_\Gamma$ and that the functor $\tau_{\leq -1}$ induces an equivalence from $\mathcal{F}$ to $\Sigma\mathcal{F}$.

Fix an object T in $\mathcal{C}_\Gamma$. Without loss of generality, assume that T belongs to $\mathcal{F}$. Note that the canonical cluster tilting object $\Gamma \in \mathcal{C}_\Gamma$ does belong to $\mathcal{F}$.

Lemma 5.13. *Let X be an object of $\operatorname{per}\Gamma$. If X is left orthogonal to $\mathrm{per}^{\leq -3}$, which happens for instance when X is in $\mathcal{F}$ or in $\Sigma\mathcal{F}$, then there is a functorial isomorphism*

$$\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq -1}T, X) \xrightarrow{\simeq} \mathcal{C}_\Gamma(T, X).$$

Proof. Let $X \in \operatorname{per}\Gamma$ be left orthogonal to $\mathrm{per}^{\leq -3}$. By [Ami, Proposition 2.8], we have $\mathcal{C}_\Gamma(T, X) = \varinjlim \operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, \tau_{\leq n}X)$. Moreover, for any n, we have $\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, \tau_{\leq n}X) = \operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, X)$. Let $n < -1$. The object $\tau_{[n+1,-1]}T$ is in $\mathcal{D}_{\mathrm{fd}}(\Gamma)$ and X is in $\operatorname{per}\Gamma$, so that the 3-Calabi–Yau property (see [Kel08]) implies that the space $\operatorname{Hom}_{\operatorname{per}\Gamma}(\Sigma^{-1}\tau_{[n+1,-1]}T, X)$ is isomorphic to the dual of $\operatorname{Hom}_{\operatorname{per}\Gamma}(X, \Sigma^2\tau_{[n+1,-1]}T)$. This latter vanishes since X belongs to ${}^{\perp}(\mathrm{per}^{\leq -3})$. The same argument shows that the space $\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{[n+1,-1]}T, X)$ also vanishes. Therefore applying the cohomological functor $\operatorname{Hom}_{\operatorname{per}\Gamma}(?, X)$ to the triangle

$$\Sigma^{-1}\tau_{[n+1,-1]}T \longrightarrow \tau_{\leq n}T \longrightarrow \tau_{\leq -1}T \longrightarrow \tau_{[n+1,-1]}T,$$

yields an isomorphism $\operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq n}T, X) \xrightarrow{\simeq} \operatorname{Hom}_{\operatorname{per}\Gamma}(\tau_{\leq -1}T, X)$. □

Lemma 5.14. *Let $X, Y \in \operatorname{per}\Gamma$ and assume that X belongs to ${}^{\perp}(\mathrm{per}^{\leq -3})$. Then the functor $\tau_{\geq -2}$ induces a bijection*

$$\operatorname{Hom}_{\operatorname{per}\Gamma}(X, Y) \simeq \operatorname{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{\geq -2}X, \tau_{\geq -2}Y).$$

Proof. By assumption, the object X is left orthogonal to the subcategory $\mathrm{per}^{\leq -3}$. Therefore, the space $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(X,Y)$ is isomorphic to the space $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(X,\tau_{\geq -2}Y)$, and thus to $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(\tau_{\geq -2}X,\tau_{\geq -2}Y)$. Since X and Y are perfect over Γ, their images under $\tau_{\geq -2}$ are quasi-isomorphic to dg modules of finite total dimension. □

Proposition 5.15. *Let Γ be the Ginzburg dg algebra associated with a Jacobi-finite quiver. Then the category $\mathcal{D}_{fd}(\Gamma)$ has constructible cones.*

Proof. We write $\mathfrak{n}$ for the ideal of Γ generated by the arrows of the Ginzburg quiver, and $\mathbf{p}$ for the left adjoint to the canonical functor $\mathcal{H}(\Gamma)\to\mathcal{D}(\Gamma)$. Let L, M and T be dg modules of finite total dimension. Since $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(L,\Sigma M)$ is finite dimensional, there exists a quasi-isomorphism $M\xrightarrow{w}M'$, where M' is of finite total dimension and such that any morphism $L\to\Sigma M$ may be represented by a fraction:

$$L \searrow \Sigma M' \swarrow^{\Sigma w} \Sigma M.$$

We thus obtain a surjection $\mathrm{Ext}^1_{\mathcal{H}_{\mathrm{fd}}(\Gamma)}(L,M')\twoheadrightarrow\mathrm{Ext}^1_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(L,M)$. Fix a k-linear section s of this surjection. Choose m such that $M'\mathfrak{n}^m$ and $L\mathfrak{n}^m$ vanish. Then for the cone Y of any morphism from $\Sigma^{-1}M'$ to L, we have $Y\mathfrak{n}^m=0$. For X being any one of L, M', Y we thus have isomorphisms

$$\mathcal{C}_\Gamma(T,X)\simeq\mathrm{Hom}_{\mathcal{H}(\Gamma)}(\mathbf{p}T,X)\simeq\mathrm{Hom}_{\mathcal{H}_{\mathrm{fd}}(\Gamma)}(T',X)$$

where T' denotes the finite dimensional quotient of $\mathbf{p}T$ by $(\mathbf{p}T)\mathfrak{n}^m$. The category $\mathcal{H}_{\mathrm{fd}}(\Gamma)$ is the stable category of a Hom-finite Frobenius category. By section 5.2.4, the category $\mathcal{H}_{\mathrm{fd}}(\Gamma)$ has constructible cones: There exists a constructible map $\varphi_{L,M'}$ (associated with T') as in section 5.1.3. By composing this map with the section s, we obtain a map $\varphi_{L,M}$ as required. □

Proposition 5.16. *Let Γ be the Ginzburg dg algebra associated with a Jacobi-finite quiver. Then the generalized cluster category $\mathcal{C}_\Gamma$ has constructible cones.*

Proof. Let L and M be in $\mathcal{C}_\Gamma$. Up to replacing them by isomorphic objects in $\mathcal{C}_\Gamma$, we may assume that L belongs to $\Sigma\mathcal{F}$ and M to $\mathcal{F}$. The projection then induces an isomorphism $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(L,\Sigma M)\xrightarrow{\simeq}\mathcal{C}_\Gamma(L,\Sigma M)$. Let ε be in $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(L,\Sigma M)$, and let $M\to Y\to L\xrightarrow{\varepsilon}\Sigma M$ be a triangle in $\mathrm{per}\,\Gamma$. Let us denote the sets of morphisms $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(\ ,\ )$ by $(\ ,\ )$. There is a commutative diagram

$$\begin{array}{ccccccccc}
(\tau_{\leq -1}T, \Sigma^{-1}L) & \to & (\tau_{\leq -1}T, M) & \to & (\tau_{\leq -1}T, Y) & \to & (\tau_{\leq -1}T, L) & \to & (\tau_{\leq -1}T, \Sigma M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}_\Gamma(T, \Sigma^{-1}L) & \longrightarrow & \mathcal{C}_\Gamma(T, M) & \longrightarrow & \mathcal{C}_\Gamma(T, Y) & \longrightarrow & \mathcal{C}_\Gamma(T, L) & \longrightarrow & \mathcal{C}_\Gamma(T, \Sigma M),
\end{array}$$

where the morphisms in the first two and in the last two columns are isomorphisms by lemma 5.13, and the middle one by the five lemma. Note that $\tau_{\leq -1}T$ belongs to $\Sigma\mathcal{F}$, so that, by lemma 5.14, we have isomorphisms:

$$\mathrm{Hom}_{\mathrm{per}\,\Gamma}(L, \Sigma M) \simeq \mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{\geq -2}L, \tau_{\geq -2}\Sigma M)$$

and

$$\mathcal{C}_\Gamma(T, X) \simeq \mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{[-2,-1]}T, \tau_{\geq -2}X)$$

for $X \in \{\Sigma^{-1}L, M, L, \Sigma M\}$ and thus also for X being the middle term of any triangle in $\mathrm{Ext}^1_{\mathrm{per}\,\Gamma}(L, M)$. Let $\varepsilon \in \mathcal{C}_\Gamma(L, \Sigma M)$ and let $M \to Y \to L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}_\Gamma$. Let $\overline{\varepsilon}$ be the morphism in $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{\geq -2}L, \tau_{\geq -2}\Sigma M)$ corresponding to ε and let $\tau_{\geq -2}M \to Z \to \tau_{\geq -2}L \xrightarrow{\overline{\varepsilon}} \tau_{\geq -2}\Sigma M$ be a triangle in $\mathcal{D}_{\mathrm{fd}}(\Gamma)$. Then the sequence obtained from $\Sigma^{-1}L \to M \to Y \to L \to \Sigma M$ by applying the functor $\mathcal{C}_\Gamma(T, ?)$ is isomorphic to the one obtained from $\Sigma^{-1}\tau_{\geq -2}L \to \tau_{\geq -2}M \to Z \to \tau_{\geq -2}L \to \tau_{\geq -2}\Sigma M$ by applying the functor $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fd}}(\Gamma)}(\tau_{[-2,-1]}T, ?)$. By proposition 5.15, the cylinders of the morphisms $L \to \Sigma M$ are constructible with respect to T. □

5.3 Proof of theorem 5.1

Let T be a cluster tilting object of $\mathcal{C}$. Let L and M be two objects in $\mathcal{C}$, such that the cylinders of the morphisms $L \to \Sigma M$ and $M \to \Sigma L$ are constructible with respect to T. Let ε be a morphism in $\mathcal{C}(L, \Sigma M)_{\langle Y \rangle}$ for some $Y \in \mathcal{C}$, and let $M \xrightarrow{i} Y' \xrightarrow{p} L \xrightarrow{\varepsilon} \Sigma M$ be a triangle in $\mathcal{C}$. The image of ε under $\varphi_{L,M}$ lifts the orbit of the matrix representation of $\overrightarrow{A_4}$ in $\mathrm{mod}\, B$ given by $FM \xrightarrow{Fi} FY' \xrightarrow{Fp} FL \xrightarrow{F\varepsilon} F\Sigma M$. In all of this section, we will take the liberty of denoting by Fi, Fp and FY' the image $\varphi_{L,M}(\varepsilon)$. Denote by Δ the dimension vector $\underline{\dim} FL + \underline{\dim} FM$. For any object Y in $\mathcal{C}$ and any non-negative e, f and g in $K_0(\mathrm{mod}\, B)$, let $W^Y_{LM}(e, f, g)$ be the subset of

$$\mathbb{P}\mathcal{C}(L, \Sigma M)_{\langle Y \rangle} \times \coprod_{d \leq \Delta} \prod_{i=1}^{n} \mathrm{Gr}_{g_i}(k^{d_i})$$

formed by the pairs $([\varepsilon], E)$ such that E is a submodule of FY' of dimension vector g, $\underline{\dim}(Fp)E = e$ and $\underline{\dim}(Fi)^{-1}E = f$, where FY',Fi and Fp are given by $\varphi_{L,M}(\varepsilon)$. We let

- $W_{LM}^Y(g)$ denote the union of all $W_{LM}^Y(e,f,g)$ with $e \leq \underline{\dim}FL$ and $f \leq \underline{\dim}FM$ and
- $W_{LM}^Y(e,f)$ denote the union of all $W_{LM}^Y(e,f,g)$ with $g \leq \underline{\dim}FL + \underline{\dim}FM$.

Lemma 5.17. *The sets $W_{LM}^Y(e,f,g)$ are constructible.*

Proof. Denote by Δ the dimension vector $\underline{\dim}FL + \underline{\dim}FM$, and fix a dimension vector g. Consider the map induced by $\varphi_{L,M}$ which sends a pair (ε, E) in $\mathcal{C}(L, \Sigma M)_{\langle Y\rangle} \times \coprod_{d\leq\Delta}\prod_{i\in Q_0} \mathrm{Gr}_{g_i}(k^{d_i})$ to (Fi, Fp, FY', E). By our assumption, this map (exists and) is constructible. Therefore, the subset of

$$\mathcal{C}(L, \Sigma M)_{\langle Y\rangle} \times \coprod_{d\leq\Delta}\prod_{i\in Q_0} \mathrm{Gr}_{g_i}(k^{d_i})$$

formed by the pairs (ε, E) such that E is a submodule of FY' is a constructible subset. We denote by $V_{LM}^Y(g)$ this constructible subset. We thus have a constructible function $V_{LM}^Y(g) \longrightarrow \mathbb{Z}^{2n}$ sending the pair (ε, E) to $(\underline{\dim}(Fi)^{-1}E, \underline{\dim}(Fp)E)$. This function induces a constructible function $\delta : W_{LM}^Y(g) \longrightarrow \mathbb{Z}^{2n}$, and the set $W_{LM}^Y(e,f,g)$ is the fiber of δ above (e,f). □

The fiber above the class $[\varepsilon]$ of the projection $W_{LM}^Y(g) \to \mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}$ is $\{[\varepsilon]\} \times \mathrm{Gr}_g FY'$ and thus all fibers have Euler characteristics equal to that of $\mathrm{Gr}_g FY$. Therefore we have:

$$(**) \qquad \chi\big(W_{LM}^Y(g)\big) = \chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}\big)\chi(\mathrm{Gr}_g FY).$$

Define $L(e,f)$ to be the variety $\mathbb{P}\mathcal{C}(L,\Sigma M) \times \mathrm{Gr}_e FL \times \mathrm{Gr}_f FM$. Consider the following map:

$$\begin{array}{rcl} \coprod_{Y\in\mathcal{Y}} W_{LM}^Y(e,f) & \xrightarrow{\psi} & L(e,f) \\ ([\varepsilon], E) & \longmapsto & \big([\varepsilon], (Fp)E, (Fi)^{-1}E\big). \end{array}$$

By our assumption, the map ψ is constructible.

Let $L_1(e,f)$ be the subvariety of $L(e,f)$ formed by the points in the image of ψ, and let $L_2(e,f)$ be the complement of $L_1(e,f)$ in $L(e,f)$.

We want to compute

$$\begin{aligned}\dim \mathcal{C}(L,\Sigma M)X_L X_M &= \underline{x}^{-\operatorname{coind}(L\oplus M)}\sum_{e,f}\chi(L(e,f))\prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a}\\ &= \sum_{e,f}\chi(L_1(e,f))\underline{x}^{-\operatorname{coind}(L\oplus M)}\prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a}\\ &\quad +\sum_{e,f}\chi(L_2(e,f))\underline{x}^{-\operatorname{coind}(L\oplus M)}\prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a}.\end{aligned}$$

Denote by s_1 (resp. s_2) the first term (resp. second term) in the right hand side of the last equality above.

As shown in [CC06], the fibers of ψ over $L_1(e,f)$ are affine spaces. For the convenience of the reader, we sketch a proof. Let $([\varepsilon], U, V)$ be in $L_1(e,f)$. Denote by Y the middle term of ε and by $\mathrm{Gr}_{U,V}$ the projection of the fiber $\psi^{-1}([\varepsilon], U, V)$ on the second factor $\mathrm{Gr}\, FY$. Let W be a cokernel of the injection of U in FM.

$$\begin{array}{ccccccc} W & & & & & & \\ \uparrow{\scriptstyle \pi} & & & & & & \\ FM & \xrightarrow{i} & FY & \xrightarrow{p} & FL & \longrightarrow & F\Sigma M \\ {\scriptstyle i_U}\uparrow & & \uparrow & & \uparrow{\scriptstyle i_V} & & \\ U & \longrightarrow & E & \twoheadrightarrow & V & & \end{array}$$

Lemma 5.18. (Caldero–Chapoton) *There is a bijection*

$$\mathrm{Hom}_B(V,W) \longrightarrow \mathrm{Gr}_{U,V}\,.$$

Proof. Define a free transitive action of $\mathrm{Hom}_B(V,W)$ on $\mathrm{Gr}_{U,V}$ in the following way: For any E in $\mathrm{Gr}_{U,V}$ and any g in $\mathrm{Hom}_B(V,W)$, define E_g to be the submodule of FY of elements of the form $i(m)+x$ where m belongs to FM, x belongs to E and $gpx = \pi m$. Note that E_g belongs to $\mathrm{Gr}_{U,V}$ (since the kernel of i is included in U), that $E_0 = E$ and that $(E_g)_h = E_{g+h}$. This action is free: An element $i(m)+x$ is in E if and only if m is in U. This is equivalent to the vanishing of πm, which in turn is equivalent to px belonging to the kernel of g. This action is transitive: Let E and E' be in $\mathrm{Gr}_{U,V}$. For any v in V, let $g(v)$ be $\pi(x'-x)$ where $x\in E$, $x'\in E'$ and $px = px' = v$. This defines a map $g : V \longrightarrow W$ such that $E_g = E'$. □

By lemma 5.18, we obtain the following equality between the Euler characteristics:

$$\sum_{\langle Y\rangle}\chi(W_{LM}^{Y}(e,f)) = \chi(L_1(e,f)),$$

which implies the equality

$$s_1 = \sum_{e,f,\langle Y\rangle} \chi\big(W^Y_{LM}(e,f)\big)\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f\rangle_a}.$$

If the pair $([\varepsilon], E)$ belongs to $W^Y_{LM}(e,f,g)$, then by lemma 4.16, we have

$$\sum_{i=1}^n \langle S_i, e+f\rangle_a [P_i] - \operatorname{coind}(L\oplus M) = \sum_{i=1}^n \langle S_i, g\rangle_a [P_i] - \operatorname{coind}(\operatorname{mt}(\varepsilon))$$

and $\operatorname{coind}(\operatorname{mt}(\varepsilon)) = \operatorname{coind} Y$ since the morphism ε is in $\mathcal{C}(L, \Sigma M)_{\langle Y\rangle}$. Therefore,

$$\begin{aligned}
s_1 &= \sum_{e,f,g,\langle Y\rangle} \chi\big(W^Y_{LM}(e,f,g)\big)\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^n x_i^{\langle S_i, e+f\rangle_a} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(W^Y_{LM}(e,f,g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \\
&= \sum_{g,\langle Y\rangle} \chi\big(W^Y_{LM}(g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \\
&= \sum_{\langle Y\rangle}\sum_{g} \chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}\big)\chi(\operatorname{Gr}_g FY)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^n x_i^{\langle S_i, g\rangle_a} \quad \text{by } (**) \\
&= \sum_{\langle Y\rangle} \chi\big(\mathbb{P}\mathcal{C}(L,\Sigma M)_{\langle Y\rangle}\big) X_Y.
\end{aligned}$$

Recall that since $\mathcal{C}$ is 2-Calabi–Yau, there is an isomorphism

$$\phi_{L,M} : \mathcal{C}(\Sigma^{-1}L, M) \longrightarrow D\mathcal{C}(M, \Sigma L).$$

We denote by ϕ the induced duality pairing:

$$\begin{aligned}
\phi : \mathcal{C}(\Sigma^{-1}L, M)\times\mathcal{C}(M,\Sigma L) &\longrightarrow k \\
(a,b) &\longmapsto \phi_{L,M}(a)b.
\end{aligned}$$

Let $C_{e,f}(Y,g)$ consist of all pairs $\big(([\varepsilon],U,V),([\eta],E)\big)$ in $L_2(e,f)\times W^Y_{ML}(g)$ such that $\phi(\Sigma^{-1}\varepsilon,\eta)\neq 0$, $(Fi)^{-1}E = V$ and $(Fp)E = U$, where Fi, Fp are given by $\varphi_{M,L}(\eta)$. The set $C_{e,f}(Y,g)$ is constructible, by our assumption. Let $C_{e,f}$ be the union of all $C_{e,f}(Y,g)$, where Y runs through the set of representatives $\mathcal{Y}$, and g through $K_0(\operatorname{mod} B)$. We then consider the following two projections

$$\begin{array}{ccc} C_{e,f} & \text{and} & C_{e,f}(Y,g) \\ \downarrow p_1 & & \downarrow p_2 \\ L_2(e,f) & & W^Y_{ML}(f,e,g). \end{array}$$

The aim of the next proposition is to show that the projections p_1 and p_2 are surjective, and to describe their fibers.

Let U be in $\operatorname{Gr}_e FL$, and V be in $\operatorname{Gr}_f FM$. Let $U \xrightarrow{i_U} L$ and $V \xrightarrow{i_V} M$ lift these two inclusions to the triangulated category $\mathcal{C}$. As in chapter 4, let us consider the following two morphisms: the morphism α from $\mathcal{C}(\Sigma^{-1}L,U) \oplus \mathcal{C}(\Sigma^{-1}L,M)$ to $\mathcal{C}/(T)\,(\Sigma^{-1}V,U) \oplus (\Sigma^{-1}V,M) \oplus \mathcal{C}/(\Sigma T)\,(\Sigma^{-1}L,M)$ and

$$\alpha' : (\Sigma T)(U,\Sigma V) \oplus \mathcal{C}(M,\Sigma V) \oplus (\Sigma^2 T)(M,\Sigma L) \longrightarrow \mathcal{C}(U,\Sigma L) \oplus \mathcal{C}(M,\Sigma L)$$

defined by:

$$\alpha(a,b) = (a\Sigma^{-1}i_V, i_U a\Sigma^{-1}i_V - b\Sigma^{-1}i_V, i_U a - b)$$

and

$$\alpha'(a,b,c) = \Big((\Sigma i_V)a + c\,i_U + (\Sigma i_V)b\,i_U, -c - (\Sigma i_V)b\Big).$$

Remark that the maps α and α' are dual to each other via the pairing ϕ. In the following lemma, orthogonal means orthogonal with respect to this pairing.

Proposition 5.19. [CK08, proposition 3] *With the same notations as above, the following assertions are equivalent:*

(i) The triple $([\varepsilon],U,V)$ belongs to $L_2(e,f)$.

(ii) The morphism $\Sigma^{-1}\varepsilon$ is not orthogonal to $\mathcal{C}(M,\Sigma L) \cap \operatorname{Im}\alpha'$.

(iii) There is an $\eta \in \mathcal{C}(M,\Sigma L)$ such that $\phi(\Sigma^{-1}\varepsilon,\eta) \neq 0$ and such that if

$$L \xrightarrow{i} N \xrightarrow{p} M \xrightarrow{\eta} \Sigma L$$

is a triangle in $\mathcal{C}$, then there exists $E \in \operatorname{Gr} FN$ with $(Fi)^{-1}E = V$ and $(Fp)E = U$.

Proof. Let us start with the equivalence of (i) and (ii). The same proof as that in [CK08, proposition 3] applies in this setup: Denote by p the canonical projection of $\mathcal{C}(\Sigma^{-1}L, U) \oplus \mathcal{C}(\Sigma^{-1}L, M)$ onto $\mathcal{C}(\Sigma^{-1}L, M)$. Then, by lemma 4.14, assertion (i) is equivalent to $\Sigma^{-1}\varepsilon$ not belonging to $p(\operatorname{Ker}\alpha)$. That is, the morphism $\Sigma^{-1}\varepsilon$ is not in the image of the composition:

$$q : \operatorname{Ker}\alpha \longrightarrow \mathcal{C}(\Sigma^{-1}L, U) \oplus \mathcal{C}(\Sigma^{-1}L, M) \longrightarrow \mathcal{C}(\Sigma^{-1}L, M).$$

So (i) holds if and only if $\Sigma^{-1}\varepsilon$ is not in the orthogonal of the orthogonal of the image of q. The orthogonal of the image of q is the kernel of its dual, which is given by the composition:

$$\mathcal{C}(M, \Sigma L) \longrightarrow \mathcal{C}(U, \Sigma L) \oplus \mathcal{C}(M, \Sigma L) \longrightarrow \operatorname{Coker}\alpha'.$$

Therefore assertion (i) is equivalent to the morphism $\Sigma^{-1}\varepsilon$ not being in the orthogonal of $\mathcal{C}(M, \Sigma L) \cap \operatorname{Im}\alpha'$ which proves that (i) and (ii) are equivalent.

By lemma 4.14, a morphism in $\mathcal{C}(M, \Sigma L)$ is in the image of α' if and only if it satisfies the second condition in (iii). Therefore (ii) and (iii) are equivalent. □

A variety X is called an *extension of affine spaces* in [CK08] if there is a vector space V and a surjective morphism $X \longrightarrow V$ whose fibers are affine spaces of constant dimension. Note that extensions of affine spaces have Euler characteristics equal to 1.

Proposition 5.20. [CK08, proposition 4]

a) The projection $C_{e,f} \xrightarrow{p_1} L_2(e, f)$ is surjective and its fibers are extensions of affine spaces.

b) The projection $C_{e,f}(Y, g) \xrightarrow{p_2} W^Y_{ML}(f, e, g)$ is surjective and its fibers are affine spaces.

c) If $C_{e,f}(Y, g)$ is not empty, then we have

$$\sum_{i=1}^{n} \langle S_i, e + f\rangle_a [P_i] - \operatorname{coind}(L \oplus M) = \sum_{i=1}^{n} \langle S_i, g\rangle_a [P_i] - \operatorname{coind} Y.$$

Proof. Let us first prove assertion a). The projection p_1 is surjective by the equivalence of i) and iii) in proposition 5.19. Let X be the fiber of p_1 above some $([\varepsilon], U, V)$ in $L_2(e, f)$. Let V be the set of all classes $[\eta]$ in $\mathbb{P}\big(\mathcal{C}(M, \Sigma L) \cap \operatorname{Im}\alpha'\big)$ such that $\phi(\Sigma^{-1}\varepsilon, \eta)$ does not vanish. The set V is the projectivization of the complement in $\mathcal{C}(M, \Sigma L) \cap \operatorname{Im}\alpha'$ of the hyperplane

$\operatorname{Ker}\phi(\Sigma^{-1}\varepsilon,\ )$. Hence V is a vector space. Let us consider the projection $\pi : X \longrightarrow V$. This projection is surjective by lemma 4.14. Let η represent a class in V, and let Fi, Fp be given by $\varphi_{M,L}(\eta)$. Then the fiber of π above $[\eta]$ is given by the submodules E of FY such that $(Fi)^{-1}E = V$ and $(Fp)E = U$. Lemma 5.18 thus shows that the fibers of π are affine spaces of constant dimension.

Let us prove assertion b). Let $([\eta], E)$ be in $W_{ML}^{Y}(f,e,g)$. The fiber of p_2 above $([\eta], E)$ consists of the elements of the form $\big(([\varepsilon], U, V), ([\eta], E)\big)$ where U and V are fixed submodules given by $[\eta]$ and E, and $[\varepsilon] \in \mathbb{P}\mathcal{C}(L, \Sigma M)$ is such that $\phi(\Sigma^{-1}\varepsilon, \eta)$ does not vanish. Therefore the projection p_2 is surjective and its fibers are affine spaces.

To prove assertion c), apply lemma 4.16 and remark that if Y' belongs to $\langle Y\rangle$, then Y' and Y have the same coindex. □

As a consequence, we obtain the following equalities:

$$\chi(C_{e,f}) = \chi(L_2(e,f)) \text{ and } \chi\big(C_{e,f}(Y,g)\big) = \chi\big(W_{ML}^{Y}(f,e,g)\big).$$

We are now able to compute s_2 :

$$\begin{aligned}
s_2 &= \sum_{e,f} \chi(L_2(e,f))\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a} \\
&= \sum_{e,f} \chi(C_{e,f})\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a} \text{ by 5.20 a)} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(C_{e,f}(Y,g)\big)\underline{x}^{-\operatorname{coind}(L\oplus M)} \prod_{i=1}^{n} x_i^{\langle S_i, e+f\rangle_a} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(C_{e,f}(Y,g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a} \text{ by 5.20 c)} \\
&= \sum_{e,f,g,\langle Y\rangle} \chi\big(W_{ML}^{Y}(f,e,g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a} \text{ by 5.20 b)} \\
&= \sum_{g,\langle Y\rangle} \chi\big(W_{ML}^{Y}(g)\big)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a} \\
&= \sum_{g,\langle Y\rangle} \chi\big(\mathbb{P}\mathcal{C}(M,\Sigma L)_{\langle Y\rangle}\big)\chi(\operatorname{Gr}_g FY)\underline{x}^{-\operatorname{coind} Y} \prod_{i=1}^{n} x_i^{\langle S_i, g\rangle_a} \text{ by } (**) \\
&= \sum_{\langle Y\rangle} \chi\big(\mathbb{P}\mathcal{C}(M,\Sigma L)_{\langle Y\rangle}\big) X_Y.
\end{aligned}$$

□

Chapter 6

Grothendieck group and generalized mutation rule for 2-Calabi–Yau triangulated categories

The contents of this chapter have been published in [Pal09].

Introduction

In their study [CK08] of the connections between cluster algebras (see [Zel07]) and quiver representations, P. Caldero and B. Keller conjectured that a certain antisymmetric bilinear form is well–defined on the Grothendieck group of a cluster tilted algebra associated with a finite–dimensional hereditary algebra. The conjecture is proved in 4 in the more general context of Hom-finite 2-Calabi–Yau triangulated categories. It is used in order to study the existence of a cluster character on such a category $\mathcal{C}$, by using a formula proposed by Caldero–Keller.

In the present chapter, we restrict ourselves to the case where $\mathcal{C}$ is algebraic (i.e. is the stable category of a Frobenius category). We first use this bilinear form to prove a generalized mutation rule for quivers of cluster–tilting subcategories in $\mathcal{C}$. When the cluster–tilting subcategories are related by a single mutation, this shows, via the method of [GLS06], that their quivers are related by the Fomin–Zelevinsky mutation rule. This special case was already proved in [BIRS], without assuming $\mathcal{C}$ to be algebraic.

We also compute the Grothendieck group of the triangulated category $\mathcal{C}$. In particular, this allows us to improve on results by M. Barot, D. Kussin and H. Lenzing: We compare the Grothendieck group of a cluster category $\mathcal{C}_A$ with the group $\overline{K}_0(\mathcal{C}_A)$. The latter group was defined in [BKL08] by only considering the triangles in $\mathcal{C}_A$ which are induced by those of the derived category. More precisely, we prove that those two groups are isomorphic for any cluster category associated with a finite-dimensional hereditary algebra, with its triangulated structure defined by B. Keller in [Kel05].

This chapter is organized as follows: The first section is dedicated to notation and necessary background from [FZ02], [GLS06], [KR07], [Pal08]. In section 6.2, we compute the Grothendieck group of the triangulated category $\mathcal{C}$. In section 6.3, we prove a generalized mutation rule for quivers of cluster–tilting subcategories in $\mathcal{C}$. In particular, this yields a new way to prove, under the restriction that $\mathcal{C}$ is algebraic, that the quiver of the mutation of a cluster–tilting object T is given by the Fomin–Zelevinsky mutation of the quiver of T. We finally show that $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$ for any finite-dimensional hereditary algebra A.

6.1 Notations and background

Let $\mathcal{E}$ be a Frobenius category whose idempotents split and which is linear over a given algebraically closed field k. By a result of Happel [Hap88], its stable category $\mathcal{C} = \underline{\mathcal{E}}$ is triangulated. We assume moreover, that $\mathcal{C}$ is Hom-finite, 2-Calabi–Yau and has a cluster–tilting subcategory (see section 6.1.2), and we denote by Σ its suspension functor. Note that we do not assume that $\mathcal{E}$ is Hom-finite.

We write $\mathcal{X}(\ ,\ )$, or $\mathrm{Hom}_{\mathcal{X}}(\ ,\ )$, for the morphisms in a category $\mathcal{X}$ and $\mathrm{Hom}_{\mathcal{X}}(\ ,\ )$ for the morphisms in the category of $\mathcal{X}$-modules. We also denote by $X^{\wedge}$ the projective $\mathcal{X}$-module represented by X: $X^{\wedge} = \mathcal{X}(?, X)$.

6.1.1 Fomin–Zelevinsky mutation for matrices

Let $B = (b_{ij})_{i,j\in I}$ be a finite or infinite matrix, and let k be in I. The Fomin and Zelevinsky mutation of B (see [FZ02]) in direction k is the matrix

$$\mu_k(B) = (b'_{ij})$$

defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{else.} \end{cases}$$

Note that $\mu_k\big(\mu_k(B)\big) = B$ and that if B is skew-symmetric, then so is $\mu_k(B)$.

We recall two lemmas of [GLS06], stated for infinite matrices, which will be useful in section 6.3. Note that lemma 7.2 is a restatement of [BFZ05, (3.2)]. Let $S = (s_{ij})$ be the matrix defined by

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}|-b_{ij}}{2} & \text{if } i = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

Lemma 7.1 ([GLS06, Geiss–Leclerc–Schröer]) : *Assume that B is skew-symmetric. Then, $S^2 = 1$ and the (i, j)-entry of the transpose of the matrix S is given by*

$$s^{\mathrm{t}}_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}|+b_{ij}}{2} & \text{if } j = k, \\ \delta_{ij} & \text{else.} \end{cases}$$

The matrix S yields a convenient way to describe the mutation of B in the direction k:

Lemma 7.2 ([GLS06, Geiss–Leclerc–Schröer], [BFZ05, Berenstein–Fomin–Zelevinsky]) : *Assume that B is skew-symmetric. Then we have:*

$$\mu_k(B) = S^{\mathrm{t}} B S.$$

Note that the product is well-defined since the matrix S has a finite number of non-vanishing entries in each column.

6.1.2 Cluster–tilting subcategories

A cluster–tilting subcategory (see [KR07]) of $\mathcal{C}$ is a full subcategory $\mathcal{T}$ such that

a) $\mathcal{T}$ is a linear subcategory;

b) for any object X in $\mathcal{C}$, the contravariant functor $\mathcal{C}(?, X)|_{\mathcal{T}}$ is finitely generated;

c) for any object X in $\mathcal{C}$, we have $\mathcal{C}(X, \Sigma T) = 0$ for all T in $\mathcal{T}$ if and only if X belongs to $\mathcal{T}$.

We now recall some results from [KR07], which we will use in what follows. Let $\mathcal{T}$ be a cluster–tilting subcategory of $\mathcal{C}$, and denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective–injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

The following proposition will be used implicitly, extensively in this paper.

Proposition [KR07, Keller–Reiten] :

a) *The category* $\operatorname{mod} \underline{\mathcal{M}}$ *of finitely presented* $\underline{\mathcal{M}}$*-modules is abelian.*

b) For each object $X \in \mathcal{C}$, there is a triangle

$$\Sigma^{-1}X \longrightarrow M_1^X \longrightarrow M_0^X \longrightarrow X$$

of $\mathcal{C}$, with M_0^X and M_1^X in $\underline{\mathcal{M}}$.

Recall that the perfect derived category $\operatorname{per}\mathcal{M}$ is the full triangulated subcategory of the derived category of $\mathcal{D}\operatorname{Mod}\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules.

Proposition [KR07, Keller–Reiten] :

a) For each $X \in \mathcal{E}$, there are conflations

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow 0$$

in $\mathcal{E}$, with M_0, M_1, M^0 and M^1 in $\mathcal{M}$.

b) Let Z be in $\operatorname{mod}\underline{\mathcal{M}}$. Then Z considered as an $\mathcal{M}$-module lies in the perfect derived category $\operatorname{per}\mathcal{M}$.

6.1.3 The antisymmetric bilinear form

In section 6.3, we will use the existence of the antisymmetric bilinear form $\langle\ ,\ \rangle_a$ on $K_0(\operatorname{mod}\underline{\mathcal{M}})$. We thus recall its definition from [CK08].

Let $\langle\ ,\ \rangle$ be a truncated Euler form on $\operatorname{mod}\underline{\mathcal{M}}$ defined by

$$\langle M, N\rangle = \dim \operatorname{Hom}_{\underline{\mathcal{M}}}(M,N) - \dim \operatorname{Ext}^1_{\underline{\mathcal{M}}}(M,N)$$

for any $M, N \in \operatorname{mod}\underline{\mathcal{M}}$. Define $\langle\ ,\ \rangle_a$ to be the antisymmetrization of this form:

$$\langle\ ,\ \rangle_a = \langle M, N\rangle - \langle N, M\rangle.$$

This bilinear form descends to the Grothendieck group $K_0(\operatorname{mod}\underline{\mathcal{M}})$:

Lemma [Pal08, section 3] : The antisymmetric bilinear form

$$\langle M, N\rangle_a : K_0(\operatorname{mod}\underline{\mathcal{M}}) \times K_0(\operatorname{mod}\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$$

is well-defined.

6.2 Grothendieck groups of algebraic 2-CY categories with a cluster tilting subcategory

We fix a cluster–tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and we denote by $\mathcal{M}$ its preimage in $\mathcal{E}$. In particular $\mathcal{M}$ contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective–injective objects, and we have $\underline{\mathcal{M}} = \mathcal{T}$.

We denote by $\mathcal{H}^b(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ respectively the bounded homotopy category and the bounded derived category of $\mathcal{E}$. We also denote by $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$, $\mathcal{H}^b(\mathcal{P})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ the full subcategories of $\mathcal{H}^b(\mathcal{E})$ whose objects are the $\mathcal{E}$-acyclic complexes, the complexes of projective objects in $\mathcal{E}$, the complexes of objects of $\mathcal{M}$ and the $\mathcal{E}$-acyclic complexes of objects of $\mathcal{M}$, respectively.

6.2.1 A short exact sequence of triangulated categories

Lemma 6.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be thick, full triangulated subcategories of a triangulated category $\mathcal{A}$ and let $\mathcal{B}$ be $\mathcal{A}_1 \cap \mathcal{A}_2$. Assume that for any object X in $\mathcal{A}$ there is a triangle $X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_1$ in $\mathcal{A}$, with X_1 in $\mathcal{A}_1$ and X_2 in $\mathcal{A}_2$. Then the induced functor $\mathcal{A}_1/\mathcal{B} \longrightarrow \mathcal{A}/\mathcal{A}_2$ is a triangle equivalence.*

Proof. Under these assumptions, denote by F the induced triangle functor from $\mathcal{A}_1/\mathcal{B}$ to $\mathcal{A}/\mathcal{A}_2$. We are going to show that the functor F is a full, conservative, dense functor. Since any full conservative triangle functor is fully faithful, F will then be an equivalence of categories.

We first show that F is full. Let X_1 and X'_1 be two objects in $\mathcal{A}_1$. Let f be a morphism from X_1 to X'_1 in $\mathcal{A}/\mathcal{A}_2$ and let

$$X_1 \longleftarrow Y \overset{w}{\longrightarrow} X'_1$$

be a left fraction which represents f. The morphism w is in the multiplicative system associated with $\mathcal{A}_2$ and thus yields a triangle

$$\Sigma^{-1}A_2 \to Y \xrightarrow{w} X'_1 \to A_2$$

where A_2 lies in the subcategory $\mathcal{A}_2$. Moreover, by assumption, there exists a triangle $Y_1 \to Y \to Y_2 \to \Sigma Y_1$ with Y_i in $\mathcal{A}_i$. Applying the octahedral axiom to the composition $Y_1 \to Y \to X'_1$ yields a commutative diagram whose two middle rows and columns are triangles in $\mathcal{A}$

$$\begin{array}{ccccccc}
 & & \Sigma^{-1}A_2 & = & \Sigma^{-1}A_2 & & \\
 & & \downarrow & & \downarrow & & \\
Y_1 & \longrightarrow & Y & \longrightarrow & Y_2 & \longrightarrow & \Sigma Y_1 \\
\| & & \downarrow & & \downarrow & & \| \\
Y_1 & \longrightarrow & X_1' & \longrightarrow & Z & \longrightarrow & \Sigma Y_1 \\
 & & \downarrow & & \downarrow & & \\
 & & A_2 & = & A_2 & & .
\end{array}$$

Since Y_2 and A_2 belong to $\mathcal{A}_2$, so does Z. And since X_1' and Y_1 belong to $\mathcal{A}_1$, so does Z. This implies, that Z belongs to $\mathcal{B}$. The morphism $Y_1 \to X_1'$ is in the multiplicative system of $\mathcal{A}_1$ associated with $\mathcal{B}$ and the diagram

$$\begin{array}{ccc}
 & Y_1 & \\
\swarrow & & \searrow \\
X_1 & & X_1'
\end{array}$$

is a left fraction which represents f. This implies that f is the image of a morphism in $\mathcal{A}_1/\mathcal{B}$. Therefore the functor F is full.

We now show that F is conservative. Let $X_1 \xrightarrow{f} Y_1 \to Z_1 \to \Sigma X_1$ be a triangle in $\mathcal{A}_1$. Assume that Ff is an isomorphism in $\mathcal{A}/\mathcal{A}_2$, which implies that Z_1 is an object of $\mathcal{A}_2$. Therefore, Z_1 is an object of $\mathcal{B}$ and f is an isomorphism in $\mathcal{A}_1/\mathcal{B}$.

We finally show that F is dense. Let X be an object of the category $\mathcal{A}/\mathcal{A}_2$, and let $X_1 \to X \to X_2 \to \Sigma X_1$ be a triangle in $\mathcal{A}$ with X_i in $\mathcal{A}_i$. Since X_2 belongs to $\mathcal{A}_2$, the image of the morphism $X_1 \to X$ in $\mathcal{A}/\mathcal{A}_2$ is an isomorphism. Thus X is isomorphic to the image by F of an object in $\mathcal{A}_1/\mathcal{B}$. □

As a corollary, we have the following:

Lemma 6.2. *The following sequence of triangulated categories is short exact:*

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M}) \longrightarrow \mathcal{D}^b(\mathcal{E}) \longrightarrow 0.$$

Remark: This lemma remains true if $\mathcal{C}$ is d-Calabi–Yau and $\underline{\mathcal{M}}$ is $(d-1)$-cluster–tilting, using section 5.4 of [KR07].

Proof. For any object X in $\mathcal{H}^b(\mathcal{E})$, the existence of an object M in $\mathcal{H}^b(\mathcal{M})$ and of a quasi-isomorphism w from M to X is obtained using the approximation conflations of Keller–Reiten (see section 6.1.2). Since the cone of the morphism w belongs to $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$, lemma 6.1 applies to the subcategories $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$, $\mathcal{H}^b(\mathcal{M})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{E})$ of $\mathcal{H}^b(\mathcal{E})$. $\square$

Proposition 6.3. *The following diagram is commutative with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) & \xrightarrow{i_{\mathcal{M}}} & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \longrightarrow & \underline{\mathcal{E}} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) & \longrightarrow & \mathcal{H}^b(\mathcal{M}) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow{\scriptstyle i_{\mathcal{P}}} & & \\
 & & & & \mathcal{H}^b(\mathcal{P}) & = & \mathcal{H}^b(\mathcal{P}) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (D)$$

Proof. The column on the right side has been shown to be exact in [KV87] and [Ric91]. The second row is exact by lemma 6.2. The subcategories $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ and $\mathcal{H}^b(\mathcal{P})$ of $\mathcal{H}^b(\mathcal{M})$ are left and right orthogonal to each other. This implies that the induced functors $i_{\mathcal{M}}$ and $i_{\mathcal{P}}$ are fully faithful and that taking the quotient of $\mathcal{H}^b(\mathcal{M})$ by those two subcategories either in one order or in the other gives the same category. Therefore the first row is exact. $\square$

6.2.2 Invariance under mutation

A natural question is then to which extent the diagram (D) depends on the choice of a particular cluster–tilting subcategory. Thus let $\mathcal{T}'$ be another cluster–tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in $\mathcal{E}$. Let $\operatorname{Mod}\mathcal{M}$ (resp. $\operatorname{Mod}\mathcal{M}'$) be the category of $\mathcal{M}$-modules (resp. $\mathcal{M}'$-modules), i.e. of k-linear contravariant functors from $\mathcal{M}$ (resp. $\mathcal{M}'$) to the category of k-vector spaces.

Let X be the $\mathcal{M}'$-$\mathcal{M}$-bimodule which sends the pair of objects (M', M) to the k-vector space $\mathcal{E}(M', M)$. The bimodule X induces a functor

$$F = ? \otimes_{\mathcal{M}} X : \operatorname{Mod}\mathcal{M} \longrightarrow \operatorname{Mod}\mathcal{M}'$$

denoted by T_X in [Kel94, section 6.1].

Recall that the perfect derived category $\operatorname{per}\mathcal{M}$ is the full triangulated subcategory of the derived category $\mathcal{D}\operatorname{Mod}\mathcal{M}$ generated by the finitely generated projective $\mathcal{M}$-modules.

Proposition 6.4. *The left derived functor*

$$\mathbb{L}F : \mathcal{D}\operatorname{Mod}\mathcal{M} \longrightarrow \mathcal{D}\operatorname{Mod}\mathcal{M}'$$

is an equivalence of categories.

Proof. Recall that if X is an object in a category $\mathcal{X}$, we denote by $X\hat{}$ the functor $\mathcal{X}(?,X)$ represented by X. By [Kel94, 6.1], it is enough to check the following three properties:

1. For all objects M, N of $\mathcal{M}$, the group $\operatorname{Hom}_{\mathcal{D}\operatorname{Mod}\mathcal{M}'}(\mathbb{L}FM\hat{}, \mathbb{L}FN\hat{}[n])$ vanishes for $n \neq 0$ and identifies with $\operatorname{Hom}_{\mathcal{M}}(M,N)$ for $n=0$;
2. for any object M of $\mathcal{M}$, the complex $\mathbb{L}FM\hat{}$ belongs to $\operatorname{per}\mathcal{M}'$;
3. the set $\{\mathbb{L}FM\hat{},\ M \in \mathcal{M}\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ as a triangulated category with infinite sums.

Let M be an object of $\mathcal{M}$, and let $M_1' \rightarrowtail M_0' \twoheadrightarrow M$ be a conflation in $\mathcal{E}$, with M_0' and M_1' in $\mathcal{M}'$, and whose deflation is a right $\mathcal{M}'$-approximation (cf. section 4 of [KR07]). The surjectivity of the map $(M_0')\hat{} \longrightarrow \mathcal{E}(?,M)|_{\mathcal{M}'}$ implies that the complex $P = (\cdots \to 0 \to (M_1')\hat{} \to (M_0')\hat{} \to 0 \to \cdots)$ is quasi-isomorphic to $\mathbb{L}FM\hat{} = \mathcal{E}(?,M)|_{\mathcal{M}'}$. Therefore $\mathbb{L}FM\hat{}$ belongs to the subcategory $\operatorname{per}\mathcal{M}'$ of $\mathcal{D}\operatorname{Mod}\mathcal{M}'$. Moreover, we have, for any $n \in \mathbb{Z}$ and any $N \in \mathcal{M}$, the equality

$$\operatorname{Hom}_{\mathcal{D}\operatorname{Mod}\mathcal{M}'}(\mathbb{L}FM\hat{}, \mathbb{L}FN\hat{}[n]) = \operatorname{Hom}_{\mathcal{H}^{\mathrm{b}}\operatorname{Mod}\mathcal{M}}(P, \mathcal{E}(?,N)|_{\mathcal{M}'}[n])$$

where the right-hand side vanishes for $n \neq 0, 1$. In case $n=1$ it also vanishes, since $\operatorname{Ext}^1_{\mathcal{E}}(M,N)$ vanishes. Now,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}^{\mathrm{b}}\operatorname{Mod}\mathcal{M}'}(P, \mathcal{E}(?,N)|_{\mathcal{M}'}) &\simeq \operatorname{Ker}(\mathcal{E}(M_0',N) \to \mathcal{E}(M_1',N)) \\ &\simeq \mathcal{E}(M,N).\end{aligned}$$

It only remains to be shown that the set $R = \{\mathbb{L}FM\hat{},\ M \in \mathcal{M}\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}'$. Denote by $\mathcal{R}$ the full triangulated subcategory with infinite sums of $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ generated by the set R. The set $\{(M')\hat{},\ M' \in \mathcal{M}'\}$ generates $\mathcal{D}\operatorname{Mod}\mathcal{M}'$ as a triangulated category with infinite sums. Thus it is enough to show that, for any object M' of $\mathcal{M}'$, the complex $(M')\hat{}$ concentrated in degree 0 belongs to the subcategory $\mathcal{R}$. Let M' be an object of $\mathcal{M}'$, and let $M' \overset{i}{\rightarrowtail} M_0 \overset{p}{\twoheadrightarrow} M_1$ be a conflation of $\mathcal{E}$ with M_0 and M_1 in $\mathcal{M}$. Since $\operatorname{Ext}^1_{\mathcal{E}}(?, M')|_{\mathcal{M}'}$ vanishes, we have a short exact sequence of $\mathcal{M}'$-modules

$$0 \longrightarrow \mathcal{E}(?, M')|_{\mathcal{M}'} \longrightarrow \mathcal{E}(?, M_0)|_{\mathcal{M}'} \longrightarrow \mathcal{E}(?, M_1)|_{\mathcal{M}'} \longrightarrow 0,$$

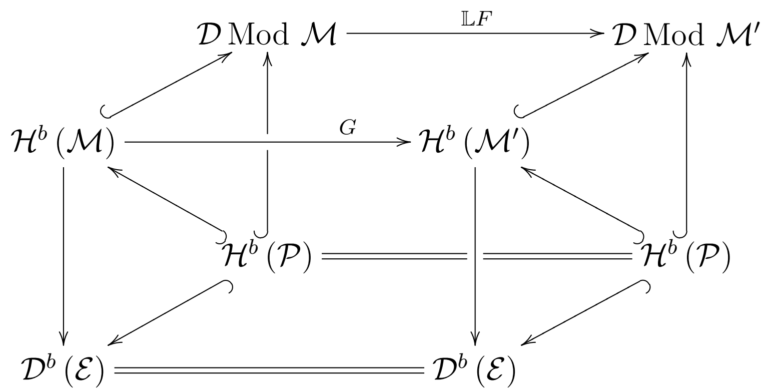
which yields the triangle

$$(M')\hat{} \longrightarrow \mathbb{L}FM_0\hat{} \longrightarrow \mathbb{L}FM_1\hat{} \longrightarrow \Sigma(M')\hat{}.$$

□

As a corollary of proposition 6.4, up to equivalence the diagram (D) does not depend on the choice of a cluster–tilting subcategory. To be more precise: The functor $\mathbb{L}F$ restricts to a functor from $\operatorname{per}\mathcal{M}$ to $\operatorname{per}\mathcal{M}'$. Let G be the functor from $\mathcal{H}^b(\mathcal{M})$ to $\mathcal{H}^b(\mathcal{M}')$ induced by this restriction via the Yoneda equivalence.

Corollary 6.5. *The following diagram is commutative*



and the functor G is an equivalence of categories.

We denote by $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ the full subcategory of $\operatorname{per}\mathcal{M}$ whose objects are the complexes with homologies in $\operatorname{mod}\underline{\mathcal{M}}$. The following lemma will allow us to compute the Grothendieck group of $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ in section 6.2.3:

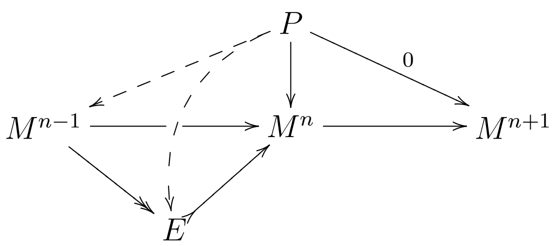
Lemma 6.6. *The canonical t-structure on the derived category $\mathcal{D}\operatorname{Mod}\mathcal{M}$ restricts to a t-structure on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$, whose heart is $\operatorname{mod}\underline{\mathcal{M}}$.*

Proof. By [KV88], it is enough to show that for any object $M^\bullet$ of $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$, its truncation $\tau_{\leq 0}M^\bullet$ in $\mathcal{D}\operatorname{Mod}\mathcal{M}$ belongs to $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. Since $M^\bullet$ is in $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$, $\tau_{\leq 0}M^\bullet$ is bounded, and is thus formed from the complexes $\mathrm{H}^i(M^\bullet)$ concentrated in one degree by taking iterated extensions. But, for any i, the $\mathcal{M}$-module $\mathrm{H}^i(M^\bullet)$ actually is an $\underline{\mathcal{M}}$-module. Therefore, by [KR07] (see section 6.1.2), it is perfect as an $\mathcal{M}$-module and it lies in $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. □

The next lemma already appears in [Tab07]. For the convenience of the reader, we include a proof.

Lemma 6.7. *The Yoneda equivalence of triangulated categories from* $\mathcal{H}^b(\mathcal{M})$ *to* $\operatorname{per}\mathcal{M}$ *induces a triangle equivalence* $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$.

Proof. We first show that the cohomology groups of an $\mathcal{E}$-acyclic bounded complex M vanish on $\mathcal{P}$. Let P be a projective object in $\mathcal{E}$ and let E be a kernel in $\mathcal{E}$ of the map $M^n \longrightarrow M^{n+1}$. Since M is $\mathcal{E}$-acyclic, such an object exists, and moreover, it is an image of the map $M^{n-1} \longrightarrow M^n$. Any map from P to M^n whose composition with $M^n \to M^{n+1}$ vanishes factors through the kernel $E \rightarrowtail M^n$. Since P is projective, this factorization factors through the deflation $M^{n-1} \twoheadrightarrow E$.



Therefore, we have $\mathrm{H}^n(M^\wedge)(P) = 0$ for all projective objects P, and $\mathrm{H}^n(M^\wedge)$ belongs to $\operatorname{mod}\underline{\mathcal{M}}$. Thus the Yoneda functor induces a fully faithful functor from $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$ to $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. To prove that it is dense, it is enough to prove that any object of the heart $\operatorname{mod}\underline{\mathcal{M}}$ of the t-structure on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ is in its essential image.

But this was proved in [KR07, section 4] (see section 6.1.2). □

Proposition 6.8. *There is a triangle equivalence of categories*

$$\operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}' \xrightarrow{\simeq} \operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$$

Proof. Since the categories $\mathcal{H}^b(\mathcal{P})$ and $\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}')$ are left–right orthogonal in $\mathcal{H}^b(\mathcal{M}')$, this is immediate from corollary 6.5 and lemma 6.7. □

6.2.3 Grothendieck groups

For a triangulated (resp. additive, resp. abelian) category $\mathcal{A}$, we denote by $K_0^{\mathrm{tri}}(\mathcal{A})$ or simply $K_0(\mathcal{A})$ (resp. $K_0^{\mathrm{add}}(\mathcal{A})$, resp. $K_0^{\mathrm{ab}}(\mathcal{A})$) its Grothendieck group (with respect to the mentioned structure of the category). For an object A in $\mathcal{A}$, we also denote by $[A]$ its class in the Grothendieck group of $\mathcal{A}$.

The short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \underline{\mathcal{E}} \longrightarrow 0$$

given by proposition 6.3 induces an exact sequence in the Grothendieck groups

$$(*) \qquad K_0\big(\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})\big) \longrightarrow K_0\big(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\big) \longrightarrow K_0\big(\underline{\mathcal{E}}\big) \longrightarrow 0.$$

Lemma 6.9. *The exact sequence* $(*)$ *is isomorphic to an exact sequence*

$$(**) \qquad K_0^{ab}\big(\operatorname{mod}\underline{\mathcal{M}}\big) \xrightarrow{\varphi} K_0^{add}\big(\underline{\mathcal{M}}\big) \longrightarrow K_0^{tri}\big(\underline{\mathcal{E}}\big) \longrightarrow 0.$$

Proof. First, note that, by [Tab07], see also lemma 6.7, we have an isomorphism between the Grothendieck groups $K_0\big(\mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})\big)$ and $K_0\big(\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\big)$. The t-structure on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ whose heart is $\operatorname{mod}\underline{\mathcal{M}}$, see lemma 6.6, in turn yields an isomorphism between the Grothendieck groups $K_0^{\mathrm{tri}}\big(\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\big)$ and $K_0^{\mathrm{ab}}\big(\operatorname{mod}\underline{\mathcal{M}}\big)$. Next, we show that the canonical additive functor α from $\underline{\mathcal{M}}$ to $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ induces an isomorphism between the Grothendieck groups $K_0^{\mathrm{add}}\big(\underline{\mathcal{M}}\big)$ and $K_0^{\mathrm{tri}}\big(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\big)$. For this, let us consider the canonical additive functor $\underline{\mathcal{M}} \xrightarrow{\beta} \mathcal{H}^b(\underline{\mathcal{M}})$ and the triangle functor γ from $\mathcal{H}^b(\mathcal{M})$ to $\mathcal{H}^b(\underline{\mathcal{M}})$. The following diagram describes the situation:

$$\begin{array}{ccc} \mathcal{H}^b(\underline{\mathcal{M}}) & \xleftarrow{\ \gamma\ } & \mathcal{H}^b(\mathcal{M}) \\ {\scriptstyle\beta}\uparrow & \nwarrow{\scriptstyle\gamma} & \downarrow \\ \underline{\mathcal{M}} & \xrightarrow{\ \alpha\ } & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \end{array}$$

The functor γ vanishes on the full subcategory $\mathcal{H}^b(\mathcal{P})$, thus inducing a triangle functor, still denoted by γ, from $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ to $\mathcal{H}^b(\underline{\mathcal{M}})$. Furthermore, the functor β induces an isomorphism at the level of Grothendieck groups, whose inverse $K_0(\beta)^{-1}$ is given by

$$\begin{array}{rcl} K_0^{\mathrm{tri}}\big(\mathcal{H}^b(\underline{\mathcal{M}})\big) & \longrightarrow & K_0^{\mathrm{add}}\big(\underline{\mathcal{M}}\big) \\ {[M]} & \longmapsto & \sum_{i\in\mathbb{Z}}(-1)^i[M^i]. \end{array}$$

As the group $K_0^{\mathrm{tri}}\big(\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})\big)$ is generated by objects concentrated in degree 0, it is straightforward to check that the morphisms $K_0(\alpha)$ and $K_0(\beta)^{-1}K_0(\gamma)$ are inverse to each other.

□

As a consequence of the exact sequence (∗∗), we have an isomorphism between $K_0^{\mathrm{tri}}(\underline{\mathcal{E}})$ and $K_0^{\mathrm{add}}(\underline{\mathcal{M}})/\operatorname{Im}\varphi$. In order to compute $K_0^{\mathrm{tri}}(\underline{\mathcal{E}})$, the map φ has to be made explicit. We first recall some results from Iyama–Yoshino [IY08] which generalize results from [BMR+06]: For any indecomposable M of $\underline{\mathcal{M}}$ not in $\mathcal{P}$, there exists M^* unique up to isomorphism such that (M, M^*) is an exchange pair, i.e.

a) M^* is an indecomposable object, not isomorphic to M and

b) the full additive subcategory of $\mathcal{C}$ generated by M^* and $\underline{\mathcal{M}}/M$ is cluster–tilting.

Moreover, there exist two (non-split) exchange triangles

$$M^* \to B_M \to M \to \Sigma M^* \text{ and } M \to B_{M^*} \to M^* \to \Sigma M$$

We may now state the following:

Theorem 6.10. *The Grothendieck group of the triangulated category $\underline{\mathcal{E}}$ is the quotient of that of the additive subcategory $\underline{\mathcal{M}}$ by all relations $[B_{M^*}] - [B_M]$:*

$$K_0^{tri}(\underline{\mathcal{E}}) \simeq K_0^{add}(\underline{\mathcal{M}})/([B_{M^*}] - [B_M])_M.$$

Proof. We denote by S_M the simple $\underline{\mathcal{M}}$-module associated to the indecomposable object M. This means that $S_M(M')$ vanishes for all indecomposable objects M' in $\underline{\mathcal{M}}$ not isomorphic to M and that $S_M(M)$ is isomorphic to k. The abelian group $K_0^{\mathrm{ab}}(\operatorname{mod}\underline{\mathcal{M}})$ is generated by all classes $[S_M]$. In view of lemma 6.9, it is sufficient to prove that the image of the class $[(S_M)^{\oplus d}]$ under φ is $[B_{M^*}] - [B_M]$, where d is the dimension of $\underline{\mathcal{E}}(M, \Sigma M^*)$. First note that the $\mathcal{M}$-module $\operatorname{Ext}^1_{\mathcal{E}}(?, M^*)|_{\mathcal{M}}$ vanishes on the projectives ; it can thus be viewed as an $\underline{\mathcal{M}}$-module, and as such, is isomorphic to $(S_M)^{\oplus d}$. After replacing B_M and $B_{M'}$ by isomorphic objects of $\underline{\mathcal{E}}$, we can assume that the exchange triangles $M^* \to B_M \to M \to \Sigma M^*$ and $M \to B_{M^*} \to M^* \to \Sigma M$ come from conflations $M^* \rightarrowtail B_M \twoheadrightarrow M$ and $M \rightarrowtail B_{M^*} \twoheadrightarrow M^*$. The spliced complex

$$(\cdots \to 0 \to M \to B_{M^*} \to B_M \to M \to 0 \to \cdots)$$

denoted by $C^\bullet$, is then an $\mathcal{E}$-acyclic complex, and it is the image of $(S_M)^{\oplus d}$ under the functor $\operatorname{mod}\underline{\mathcal{M}} \subset \operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M} \simeq \mathcal{H}^b_{\mathcal{E}-ac}(\mathcal{M})$. Indeed, we have two long exact sequences induced by the conflations above:

$$0 \to \mathcal{M}(?, M) \to \mathcal{M}(?, B_{M^*}) \to \mathcal{E}(?, M^*)|_{\mathcal{M}} \to \operatorname{Ext}^1_{\mathcal{E}}(?, M)|_{\mathcal{M}} = 0 \text{ and}$$

$\mathcal{E}(?, M^*)|_{\mathcal{M}} \rightarrowtail \mathcal{M}(?, B_M) \to \mathcal{M}(?, M) \to \mathrm{Ext}^1_{\mathcal{E}}(?, M^*)|_{\mathcal{M}} \to \mathrm{Ext}^1_{\mathcal{E}}(?, B_M)|_{\mathcal{M}}.$
Since B_M belongs to $\mathcal{M}$, the functor $\mathrm{Ext}^1_{\mathcal{E}}(?, B_M)$ vanishes on $\mathcal{M}$, and the complex:

$$(C\hat{}) : \qquad (\cdots \to 0 \to M\hat{} \to (B_{M^*})\hat{} \to (B_M)\hat{} \to M\hat{} \to 0 \to \cdots)$$

is quasi-isomorphic to $(S_M)^{\oplus d}$.

Now, in the notations of the proof of lemma 6.9, $\varphi\big(d[S_M]\big)$ is the image of the class of the $\mathcal{E}$-acyclic complex complex $C^\bullet$ under the morphism $K_0(\beta)^{-1}K_0(\gamma)$. This is $[M] - [B_M] + [B_{M^*}] - [M]$ which equals $[B_{M^*}] - [B_M]$ as claimed. □

6.3 The generalized mutation rule

Let $\mathcal{T}$ and $\mathcal{T}'$ be two cluster tilting subcategories of $\mathcal{C}$. Let Q and Q' be the quivers obtained from their Auslander–Reiten quivers by removing all loops and oriented 2-cycles.

Our aim, in this section, is to give a rule relating Q' to Q, and to prove that it generalizes the Fomin–Zelevinsky mutation rule.
Remark:

- Assume that $\mathcal{C}$ has cluster tilting objects. Then it is proved in [BIRS, Theorem I.1.6], without assuming the category $\mathcal{C}$ to be algebraic, that the Auslander–Reiten quivers of two cluster tilting objects having all but one indecomposable direct summands in common (up to isomorphism) are related by the Fomin–Zelevinsky mutation rule.

- To prove that the generalized mutation rule actually generalizes the Fomin–Zelevinsky mutation rule, we use the ideas of the section 7 of [GLS06].

6.3.1 The rule

As in section 6.2, we fix a cluster–tilting subcategory $\mathcal{T}$ of $\mathcal{C}$, and write $\mathcal{M}$ for its preimage in $\mathcal{E}$, so that $\mathcal{T} = \underline{\mathcal{M}}$. Define Q to be the quiver obtained from the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ by deleting its loops and its oriented 2-cycles. Its vertex corresponding to an indecomposable object L will also be labelled by L. We denote by a_{LN} the number of arrows from vertex L to vertex N in the quiver Q. Let $B_{\mathcal{M}}$ be the matrix whose entries are given by $b_{LN} = a_{LN} - a_{NL}$.

Let $R_{\mathcal{M}}$ be the matrix of $\langle\ ,\ \rangle_a : K_0(\mathrm{mod}\,\underline{\mathcal{M}}) \times K_0(\mathrm{mod}\,\underline{\mathcal{M}}) \longrightarrow \mathbb{Z}$ in the basis given by the classes of the simple modules.

Lemma 6.11. *The matrices $R_{\mathcal{M}}$ and $B_{\mathcal{M}}$ are equal: $R_{\mathcal{M}} = B_{\mathcal{M}}$.*

Proof. Let L and N be two non-projective indecomposable objects in $\mathcal{M}$. Then $\dim \operatorname{Hom}(S_L, S_N) - \dim \operatorname{Hom}(S_N, S_L) = 0$ and we have:

$$\langle [S_L], [S_N] \rangle_a = \dim \operatorname{Ext}^1(S_N, S_L) - \dim \operatorname{Ext}^1(S_L, S_N) = b_{L,N}.$$

□

Let $\mathcal{T}'$ be another cluster–tilting subcategory of $\mathcal{C}$, and let $\mathcal{M}'$ be its preimage in the Frobenius category $\mathcal{E}$. Let $(M'_i)_{i\in I}$ (resp. $(M_j)_{j\in J}$) be representatives for the isoclasses of non-projective indecomposable objects in $\mathcal{M}'$ (resp. $\mathcal{M}$). The equivalence of categories $\operatorname{per}_{\underline{\mathcal{M}}} \mathcal{M} \xrightarrow{\sim} \operatorname{per}_{\underline{\mathcal{M}'}} \mathcal{M}'$ of proposition 6.8 induces an isomorphism between the Grothendieck groups $K_0(\operatorname{mod} \underline{\mathcal{M}})$ and $K_0(\operatorname{mod} \underline{\mathcal{M}'})$ whose matrix, in the bases given by the classes of the simple modules, is denoted by S. The equivalence of categories $\mathcal{D} \operatorname{Mod} \mathcal{M} \xrightarrow{\sim} \mathcal{D} \operatorname{Mod} \mathcal{M}'$ restricts to the identity on $\mathcal{H}^b(\mathcal{P})$, so that it induces an equivalence $\operatorname{per} \mathcal{M} / \operatorname{per} \mathcal{P} \xrightarrow{\sim} \operatorname{per} \mathcal{M}' / \operatorname{per} \mathcal{P}$. Denote by $\operatorname{proj} \mathcal{P}$ (resp. $\operatorname{proj} \mathcal{M}$, resp. $\operatorname{proj} \mathcal{M}'$) the full subcategory of $\operatorname{mod} \mathcal{P}$ (resp. $\operatorname{Mod} \mathcal{M}$, resp. $\operatorname{Mod} \mathcal{M}'$) whose objects are the representable functors. Let T be the matrix of the induced isomorphism from $K_0(\operatorname{proj} \mathcal{M})/K_0(\operatorname{proj} \mathcal{P})$ to $K_0(\operatorname{proj} \mathcal{M}')/K_0(\operatorname{proj} \mathcal{P})$, in the bases given by the classes $[\mathcal{M}(?, M_j)]$, $j \in J$, and $[\mathcal{M}'(?, M'_i)]$, $i \in I$. The matrix T is much easier to compute than the matrix S. Its entries t_{ij} are given by the approximation triangles of Keller and Reiten in the following way: For all j, there exists a triangle of the form

$$\Sigma^{-1} M_j \longrightarrow \bigoplus_i \beta_{ij} M'_i \longrightarrow \bigoplus_i \alpha_{ij} M'_i \longrightarrow M_j.$$

Then, we have:

Theorem 6.12. *a)* (Generalized mutation rule) *The following equalities hold:*

$$t_{ij} = \alpha_{ij} - \beta_{ij}$$

and

$$B_{\mathcal{M}'} = T B_{\mathcal{M}} T^{\mathrm{t}}.$$

b) The category $\mathcal{C}$ has a cluster–tilting object if and only if all its cluster–tilting subcategories have a finite number of pairwise non-isomorphic indecomposable objects.

c) All cluster–tilting objects of $\mathcal{C}$ have the same number of indecomposable direct summands (up to isomorphism).

Note that point c) was shown in [Iya07, 5.3.3(1)] (see also [BIRS, I.1.8]) and, in a more general context, in [DK08]. Note also that, for the generalized mutation rule to hold, the cluster–tilting subcategories do not need to be related by a sequence of mutation.

Proof. Assertions b) and c) are consequences of the existence of an isomorphism between the Grothendieck groups $K_0(\operatorname{mod}\underline{\mathcal{M}})$ and $K_0(\operatorname{mod}\underline{\mathcal{M}'})$. Let us prove the equalities a). Recall from [Pal08, section 3.3], that the antisymmetric bilinear form $\langle\ ,\ \rangle_a$ on $\operatorname{mod}\underline{\mathcal{M}}$ is induced by the usual Euler form $\langle\ ,\ \rangle_E$ on $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$. The following commutative diagram

$$\begin{array}{ccc} \operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\times\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M} & \xrightarrow{\ \simeq\ } & \operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}'\times\operatorname{per}_{\underline{\mathcal{M}'}}\mathcal{M}' \\ & {\scriptstyle\langle\,,\,\rangle_E}\searrow \quad \swarrow{\scriptstyle\langle\,,\,\rangle_E} & \\ & \mathbb{Z} & \end{array},$$

thus induces a commutative diagram

$$\begin{array}{ccc} K_0(\operatorname{mod}\underline{\mathcal{M}})\times K_0(\operatorname{mod}\underline{\mathcal{M}}) & \xrightarrow{\ S\times S\ } & K_0(\operatorname{mod}\underline{\mathcal{M}'})\times K_0(\operatorname{mod}\underline{\mathcal{M}'}) \\ & {\scriptstyle\langle\,,\,\rangle_a}\searrow \quad \swarrow{\scriptstyle\langle\,,\,\rangle_a} & \\ & \mathbb{Z} & \end{array}.$$

This proves the equality $R_{\mathcal{M}} = S^{\mathrm{t}}R_{\mathcal{M}'}S$, or, by lemma 6.11,

$$(1) \qquad B_{\mathcal{M}} = S^{\mathrm{t}}B_{\mathcal{M}'}S.$$

Any object of $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}$ becomes an object of $\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}$ through the composition $\operatorname{per}_{\underline{\mathcal{M}}}\mathcal{M}\hookrightarrow\operatorname{per}\mathcal{M}\twoheadrightarrow\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}$. Let M and N be two non-projective indecomposable objects in $\mathcal{M}$. Since S_N vanishes on $\mathcal{P}$, we have

$$\begin{aligned} \operatorname{Hom}_{\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}}\big(\mathcal{M}(?,M),S_N\big) &= \operatorname{Hom}_{\operatorname{per}\mathcal{M}}\big(\mathcal{M}(?,M),S_N\big)\\ &= \operatorname{Hom}_{\operatorname{Mod}\mathcal{M}}\big(\mathcal{M}(?,M),S_N\big)\\ &= S_N(M). \end{aligned}$$

Thus $\dim\operatorname{Hom}_{\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}}\big(\mathcal{M}(?,M),S_N\big) = \delta_{MN}$, and the commutative diagram

$$\begin{array}{ccc} \operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P}\times\operatorname{per}\mathcal{M}/\operatorname{per}\mathcal{P} & \xrightarrow{\ \simeq\ } & \operatorname{per}\mathcal{M}'/\operatorname{per}\mathcal{P}\times\operatorname{per}\mathcal{M}'/\operatorname{per}\mathcal{P} \\ & {\scriptstyle R\mathcal{H}\mathrm{om}}\searrow \quad \swarrow{\scriptstyle R\mathcal{H}\mathrm{om}} & \\ & \operatorname{per}k & \end{array},$$

induces a commutative diagram

$$\begin{array}{ccc} K_0(\operatorname{proj}\mathcal{M})/K_0(\operatorname{proj}\mathcal{P})\times K_0(\operatorname{mod}\underline{\mathcal{M}}) & \xrightarrow{T\times S} & K_0(\operatorname{proj}\mathcal{M}')/K_0(\operatorname{proj}\mathcal{P})\times K_0(\operatorname{mod}\underline{\mathcal{M}'}) \\ & \searrow^{\mathrm{Id}} \quad \swarrow_{\mathrm{Id}} & \\ & \mathbb{Z} & \end{array}.$$

In other words, the matrix S is the inverse of the transpose of T:

$$(2) \qquad S = T^{\text{-t}}$$

Equalities (1) and (2) imply what was claimed, that is

$$B_{\mathcal{M}'} = TB_{\mathcal{M}}T^t.$$

Let us compute the matrix T: Let M be indecomposable non-projective in $\mathcal{M}$, and let

$$\Sigma^{-1}M \longrightarrow M_1' \longrightarrow M_0' \longrightarrow M$$

be a Keller–Reiten approximation triangle of M with respect to $\mathcal{M}'$, which we may assume to come from a conflation in $\mathcal{E}$. This conflation yields a projective resolution

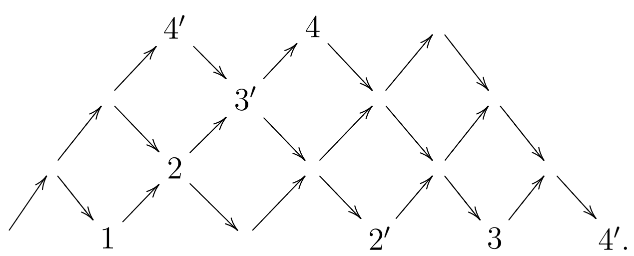
$$0 \longrightarrow (M_1')\hat{} \longrightarrow (M_0')\hat{} \longrightarrow \mathcal{E}(?, M)|_{\mathcal{M}'} \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(?, M_1')|_{\mathcal{M}'} = 0.$$

so that T sends the class of $M\hat{}$ to $[(M_0')\hat{}] - [(M_1')\hat{}]$. Therefore, t_{ij} equals $\alpha_{ij} - \beta_{ij}$. □

6.3.2 Examples

Example 1

As a first example, let $\mathcal{C}$ be the cluster category associated with the quiver of type A_4: $1 \to 2 \to 3 \to 4$. Its Auslander–Reiten quiver is the Moebius strip:



Let $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where the indecomposable M_i corresponds to the vertex labelled by i in the picture. Let also $M' = M_1' \oplus M_2' \oplus M_3' \oplus M_4'$, where $M'_1 = M_1$, and where the indecomposable M'_i corresponds to the vertex labelled by i' if $i \neq 1$. One easily computes the following Keller–Reiten approximation triangles:

$\Sigma^{-1}M_1 \longrightarrow 0 \longrightarrow M'_1 \longrightarrow M_1,$
$\Sigma^{-1}M_2 \longrightarrow M'_2 \longrightarrow M'_1 \longrightarrow M_2,$
$\Sigma^{-1}M_3 \longrightarrow M'_4 \longrightarrow 0 \longrightarrow M_4$ and
$\Sigma^{-1}M_4 \longrightarrow M'_4 \longrightarrow M'_3 \longrightarrow M_4;$

so that the matrix T is given by:

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

We also have

$$B_{M'} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let maple compute

$$T^{-1} B_{M'} T^{-t} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

which is B_M.

Example 2

Let us look at a more interesting example, where one cannot easily read the quiver of M' from the Auslander–Reiten quiver of $\mathcal{C}$. Let $\mathcal{C}$ be the cluster category associated with the quiver Q:

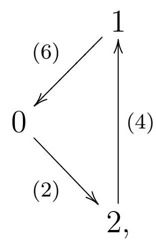
$$1 \leftleftarrows 0 \rightrightarrows 2.$$

For $i = 0, 1, 2$, let M_i be (the image in $\mathcal{C}$ of) the projective indecomposable (right) kQ-module associated with vertex i. Their dimension vectors are respectively $[1, 0, 0], [2, 1, 0]$ and $[2, 0, 1]$. Let M be the direct sum $M_0 \oplus M_1 \oplus M_2$. Let M' be the direct sum $M'_0 \oplus M'_1 \oplus M'_2$, where M'_0, M'_1 and M'_2 are (the images in $\mathcal{C}$ of) the indecomposable regular kQ-modules with dimension vectors $[1, 2, 0], [0, 1, 0]$ and $[2, 4, 1]$ respectively. As one can check, using [Kelb],

M and M' are two cluster–tilting objects of $\mathcal{C}$. Computation of Keller–Reiten approximation triangles, amounts to computing projective resolutions in mod kQ, viewed as mod $\mathrm{End}_{\mathcal{C}}(M)$. One easily computes these projective resolutions, by considering dimension vectors:

$$0 \longrightarrow 8M_0 \longrightarrow M_2 \oplus 4M_1 \longrightarrow M_2' \longrightarrow 0,$$
$$0 \longrightarrow 2M_0 \longrightarrow M_1 \longrightarrow M_1' \longrightarrow 0 \text{ and}$$
$$0 \longrightarrow 3M_0 \longrightarrow 2M_1 \longrightarrow M_0' \longrightarrow 0.$$

By applying the generalized mutation rule, one gets the following quiver



which is therefore the quiver of $\mathrm{End}_{\mathcal{C}}(M')$ since by [BMR08], there are no loops or 2-cycles in the quiver of the endomorphism algebra of a cluster–tilting object in a cluster category.

6.3.3 Back to the mutation rule

We assume in this section that the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ has no loops or 2-cycles. Under the notations of section 6.3.1, let k be in I and let (M_k, M_k') be an exchange pair (see section 6.2.3). We choose $\underline{\mathcal{M}}'$ to be the cluster–tilting subcategory of $\mathcal{C}$ obtained from $\underline{\mathcal{M}}$ by replacing M_k by M_k', so that $M_i' = M_i$ for all $i \neq k$. Recall that T is the matrix of the isomorphism $K_0(\mathrm{proj}\,\mathcal{M})/K_0(\mathrm{proj}\,\mathcal{P}) \longrightarrow K_0(\mathrm{proj}\,\mathcal{M}')/K_0(\mathrm{proj}\,\mathcal{P})$.

Lemma 6.13. *Then, the (i,j)-entry of the matrix T is given by*

$$t_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}|+b_{ij}}{2} & \text{if } j = k \\ \delta_{ij} & \text{else.} \end{cases}$$

Proof. Let us apply theorem 6.12 to compute the matrix T. For all $j \neq k$, the triangle $\Sigma^{-1}M_j \to 0 \to M_j' = M_j$ is a Keller–Reiten approximation triangle of M_j with respect to $\mathcal{M}'$. We thus have $t_{ij} = \delta_{ij}$ for all $j \neq k$. There is a triangle unique up to isomorphism

$$M_k' \longrightarrow B_{M_k} \longrightarrow M_k \longrightarrow \Sigma M_k'$$

where $B_{M_k} \longrightarrow M_k$ is a right $\underline{\mathcal{M}} \cap \underline{\mathcal{M}}'$-approximation. Since the Auslander–Reiten quiver of $\underline{\mathcal{M}}$ has no loops and no 2-cycles, B_{M_k} is isomorphic to the direct sum: $\bigoplus_{i \in I} (M'_j)^{a_{ik}}$. We thus have $t_{ik} = -\delta_{ik} + a_{ik}$, which equals $\frac{|b_{ik}|+b_{ik}}{2}$. Remark that, by lemma 7.1 of [GLS06], as stated in section 6.1.1, we have $T^2 = Id$, so that $S = T^{\mathrm{t}}$ and

$$s_{ij} = \begin{cases} -\delta_{ij} + \frac{|b_{ij}|-b_{ij}}{2} & \text{if } i = k \\ \delta_{ij} & \text{else.} \end{cases}$$

□

Theorem 6.14. *The matrix $B_{\mathcal{M}'}$ is obtained from the matrix $B_{\mathcal{M}}$ by the Fomin–Zelevinsky mutation rule in the direction M.*

Proof. By [BFZ05] (see section 6.1.1), and by lemma 6.13, we know that the mutation of the matrix $B_{\mathcal{M}}$ in direction M is given by $TB_{\mathcal{M}'}T^{\mathrm{t}}$, which is $B_{\mathcal{M}}$, by the generalized mutation rule (theorem 6.12). □

6.3.4 Cluster categories

In [BKL08], the authors study the Grothendieck group of the cluster category $\mathcal{C}_A$ associated with an algebra A which is either hereditary or canonical, endowed with any admissible triangulated structure. A triangulated structure on the category $\mathcal{C}_A$ is said to be admissible in [BKL08] if the projection functor from the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$ to $\mathcal{C}_A$ is exact (triangulated). They define a Grothendieck group $\overline{K}_0(\mathcal{C}_A)$ with respect to the triangles induced by those of $\mathcal{D}^b(\operatorname{mod} A)$, and show that it coincides with the usual Grothendieck group of the cluster category in many cases:

Theorem 6.15. [Barot–Kussin–Lenzing] *We have $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$ in each of the following three cases:*

(i) A is canonical with weight sequence $(p_1, \ldots, p_t)$ having at least one even weight.

(ii) A is tubular,

(iii) A is hereditary of finite representation type.

Under some restriction on the triangulated structure of $\mathcal{C}_A$, we have the following generalization of case (iii) of theorem 6.15:

Theorem 6.16. *Let A be a finite-dimensional hereditary algebra, and let $\mathcal{C}_A$ be the associated cluster category with its triangulated structure defined in [Kel05]. Then we have $K_0(\mathcal{C}_A) = \overline{K}_0(\mathcal{C}_A)$.*

Proof. By lemma 3.2 in [BKL08], this theorem is a corollary of the following lemma. □

Lemma 6.17. *Under the assumptions of section 6.3.1, and if moreover* $\underline{\mathcal{M}}$ *has a finite number* n *of non-isomorphic indecomposable objects, then we have an isomorphism* $K_0(\mathcal{C}) \simeq \mathbb{Z}^n / \operatorname{Im} B_{\mathcal{M}}$.

Proof. This is a restatement of theorem 6.10. □

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