# Séminaire de théorie des nombres

## Le 22 février 2010 à 14h

# Rational values of the Riemann zeta function

## Exposé de David Masser (University of Basle)

**Résumé :** It is classical that the values

$$\zeta(0) = -\frac{1}{2},\ \zeta(-1) = -\frac{1}{12},\ \zeta(-2) = 0,\ \zeta(-3) = \frac{1}{120},$$
$$\zeta(-4) = 0, \ldots,\ \zeta(-11) = \frac{691}{32760}, \ldots$$

at non-positive integers are all rational. By contrast the values

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90}, \ldots,\ \zeta(12) = \frac{691\pi^{12}}{638512875}, \ldots$$

are all irrational thanks to the transcendence of $\pi$. Apéry proved in 1978 that $\zeta(3)$ is irrational, but we still do not know that $\zeta(5)$ is irrational. However Ball and Rivoal proved in 2001 that the number of irrationals among $\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(2n+1)$ is at least $c \log n$ for some $c > 0$ independent of $n$. Even less is known about $\zeta(x)$ at rational $x$, say with $2 < x < 3$. We sketch a proof that the number of these $x$ with denominator at most $n$, such that $\zeta(x)$ is rational also with denominator at most $n$, is at most $C(\frac{\log n}{\log \log n})^2$ for some $C$ also independent of $n \geq 3$.