

Dynamics of Renormalization Operators

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Abstract

It is a remarkable characteristic of some classes of low-dimensional dynamical systems that their long time behavior at a short spatial scale is described by an induced dynamical system in the same class. The *renormalization operator* that relates the original and the induced transformations can then be iterated, and a basic theme is that certain features (such as hyperbolicity, or the existence of an attractor) of the resulting “dynamics in parameter space” impact the behavior of the underlying systems. Classical illustrations of this mechanism include the Feigenbaum-Coulet-Tresser universality in the cascade of period doubling bifurcations for unimodal maps and Herman’s Theorem on linearizability of circle diffeomorphisms. We will discuss some recent applications of the renormalization approach, focusing on what it reveals about the dynamics at typical parameter values.

Mathematics Subject Classification (2010). Primary 37E20.

Keywords. Renormalization, low-dimensional dynamics, circle diffeomorphisms, one-frequency cocycles, unimodal maps.

1. Introduction

The concept of renormalization arises in many forms through mathematics and physics. Our aim here is to discuss its incarnation as a tool in the analysis of certain classes of dynamical systems. More particularly, we will be interested in situations where renormalization gives rise to a non-trivial dynamical system in parameter space.

Inducing is a common technique to try to understand the dynamics of a map f (possibly partially defined) on some space X , restricted to a certain region $Y \subset X$. An inducing procedure gives rise to a new map g on Y which at each point coincides with some iterate of f , i.e., $g(x) = f^{n(x)}(x)$ for some positive

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integer $n(x)$, at each $x \in Y$ for which g is defined. The most usual choice of inducing procedure (and essentially the only one we will need to consider) is to take g simply as the first return map, so that $n(x)$ is the smallest positive integer such that $f^{n(x)}(x) \in Y$. Naturally, this induced map may look quite different from the original one.

It is a remarkable characteristic of certain classes of dynamical systems that an inducing procedure can be defined which produces maps in the same class. An example, to which we will come back to later, is the map $f(x) = 3.5x(1-x)$ on $X = [0, 1]$. The second iterate of f can be seen to restrict to a self-map of a subinterval Y around the critical point $1/2$. Both f and $g = f^2 : Y \rightarrow Y$ belong to the class of *unimodal maps* of an interval, whose distinguishing feature is the presence of a single turning point.

When an inducing procedure can be defined, acting on a certain class of dynamical systems, it can be of course iterated, which will produce a sequence of induced maps on successively smaller regions of space. A renormalization operator is defined by considering the induced dynamics after a suitable coordinate change (just affine rescaling in all situations we will consider), so that all dynamics considered occur at a fixed spatial scale. This allows the renormalization operator to have interesting dynamics in itself, e.g., it might admit a fixed point.

The actual implementation of the renormalization technique is naturally quite dependent of the systems at hand, so most of this paper will be dedicated to describing how it is applied in a few specific situations. We will focus on how features of the renormalization dynamics have concrete repercussions on the behavior or renormalized systems, and how this leads to the solution of very natural problems.

The variations in the implementation of renormalization should not mask the several underlying common themes in the cases of successful application of the renormalization approach:

1. The renormalizable dynamics is usually low-dimensional. This can be thought of as a conformality issue: in large dimensions, the distinct intrinsic scales of the different directions may be rather difficult to account for.
2. Renormalizable dynamical systems are not chaotic, i.e., iteration does not produce too much complexity. This is because each unit of time, after renormalization, represents several units of time of the original dynamics. So if the Lyapunov exponent $\lim_{n \rightarrow \infty} \frac{1}{n} \ln |Df^n(x)|$, which measures the exponential rate of growth of the derivative, is positive, then it will increase under renormalization. A similar consideration applies to entropy. It is thus clear that in these situations the successive renormalizations must diverge, and no interesting renormalization dynamics can take place.
3. Renormalization of non-linear dynamical systems takes place in an infinite dimensional functional space, so identifying a renormalization attractor

plays a crucial role: it basically constrains the possibilities of the small scale behavior of the original dynamics.

4. Contrary to the renormalizable dynamics, the renormalization attractor tends to display hyperbolicity: thus renormalization acts very chaotically. A lot of the effectiveness of the renormalization approach is indeed due to the fact that moderate disorder is usually more complicated to analyze than large disorder (which, for instance, can bring into play very effective probabilistic techniques).

While our focus here will be on nonlinear maps, renormalization can also be a useful concept in the absence of nonlinearity. One example is given by interval exchange transformations, i.e., bijections of an interval I with a finite “singular set” and which restrict to translations on each interval not intersecting the singular set. Once the size of the singular set is fixed, the renormalization dynamics takes place in a finite (but large if the singular set is large) dimensional parameter space, and is related to the Teichmüller flow in moduli spaces of Abelian differentials [M], [V1], [V2]. In this case, the chaotic properties of the renormalization dynamics lead to a particularly precise stochastic modeling, and plays a key role in the description of the behavior of typical maps (see the survey [A4] and references therein). Here we will only discuss the very particular case where the singular set consists of exactly one point: in this case the interval exchange transformation gives (after gluing the extremes of the interval) a rigid rotation of the circle.

The case of rigid rotations is interesting for us since some natural classes of nonlinear dynamics can be considered as nonlinear deformations of it. Here, renormalization can be used as a way to reduce the amount of nonlinearity: in terms of the dynamics of the renormalization operator, this corresponds to showing that the finite dimensional subset of linear systems is an attractor. The analysis of the renormalization dynamics is of course much simplified by the fact that we already know from the beginning what is the “candidate attractor”, and the only problem is in establishing that it indeed attracts orbits. However, even in this simple situation, we will be able to identify an important theme, which is the key role of *a priori bounds*, or precompactness of renormalization orbits (which usually takes the form of a rough estimate on the nonlinearity). In other words, before worrying about convergence to an attractor, we should establish non-divergence.

If the nonlinearity is too large, renormalization can not hope to decrease it, and a central problem is then the construction of the attractor itself. We will discuss a recently developed approach to convergence of renormalization in such a setting, in which the attractor is produced naturally by “iteration in parameter space” (given suitable *a priori* bounds).

1.1. Outline of the remaining of the paper. The sections in this paper are arranged roughly according to “increasing nonlinearity”.

We start by quickly going through the case of rigid rotations in §2, as a preamble to addressing circle diffeomorphisms in §3. Our focus will be on Herman's celebrated work on linearization. Essentially, renormalization admits a global attractor corresponding to the locus of rigid rotations, and this allows one to obtain global results by reducing to the "local case" of nearly linear systems.

We next consider the setting of one-frequency cocycles, where one "adds nonlinearity" to rigid rotations through a projective extension in §4. Here a "linear attractor" still exists, but it is no longer a global one, and understanding the nature of the obstruction to convergence has important repercussions.

We then discuss a bit about the role of renormalization in the analysis of the boundary of the basin of attraction of the linear attractor §5. For one-frequency cocycles, this regards the (still poorly understood) "onset of divergence" of renormalization, while for circle diffeomorphisms one just allows for some degeneration, in the form of critical points of inflection type.

This is followed by a much more detailed treatment of the renormalization theory of unimodal maps, with a critical point of turning type, in §6, which was first developed in connection with the Feigenbaum-Couillet-Tresser universality phenomenon. A key issue we will explore is the need to construct the renormalization attractor using the renormalization dynamics itself.

Acknowledgements

I am grateful to Amie Wilkinson, Carlos Matheus and David Damanik for several suggestions.

2. Rigid Rotations

In this section, we will consider translations on \mathbb{R}/\mathbb{Z} , that is, $f(x) = x + \alpha$ where we may assume that $0 \leq \alpha < 1$. In this case, one can define a renormalization operator based on the classical continued fraction algorithm, and hence to the *Gauss map* $G(x) = \{x^{-1}\} = x - [x^{-1}]$ (where $\{\cdot\}$ and $[\cdot]$ denote, respectively, the fractional and the integer parts of a real number) as follows. Let us assume for definiteness that α is irrational, so α has an infinite continued fraction expansion

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}, \quad (1)$$

with a_i positive integers. Consider also the continued fraction approximants p_n/q_n , given inductively by the formulas $p_0 = 0$, $q_0 = 1$, $p_1 = 1$, $q_1 = a_1$, and for $n \geq 2$, $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$. We recall that p_n/q_n approximate α from alternate sides, so that $\beta_n = (-1)^n (q_n \alpha - p_n) > 0$. Then

$\alpha_n = \beta_n/\beta_{n-1}$ are irrational numbers in $(0, 1)$ obtained by applying successively the Gauss map: $\alpha_n = G^n(\alpha)$.

The first return map to $[0, \alpha) = [0, \beta_0)$ has the form $f'(x) = x + (a+1)\alpha - 1$ for $x \in [0, \beta_1)$ and $f'(x) = x + a\alpha - 1$ on $[\beta_1, \beta_0)$. This discontinuous map on an interval can be seen as a continuous map on the circle by “gluing the extremal points” 0 and α , via the translation $x \mapsto x + \alpha$. Since the gluing map is a translation, the “new” circle has an Euclidean structure and can be identified with the original one of \mathbb{R}/\mathbb{Z} (this encodes the rescaling part of the renormalization procedure). It is easy to see that in the new coordinates the first return map is again a rigid translation by $\pm\alpha_1$, the sign depending on whether the identification does or does not reverse orientation. Here it will be most convenient to take an identification that reverses orientation, so that the renormalization of $x \mapsto x + \alpha$ is $x \mapsto x + \alpha_1$, so that the renormalization operator acting on rigid irrational translations is just the Gauss map $\alpha \mapsto G(\alpha)$ acting on the parameter space $(0, 1) \cap \mathbb{Q}$.

The Gauss map is of course a classical example of a chaotic dynamical system [Man]. It preserves the probability measure $d\mu = \frac{1}{\ln 2} \frac{dx}{1+x}$, with respect to which it has a positive Lyapunov exponent. The strong mixing properties of the Gauss map have of course many applications in the analysis of the distribution of continued fraction coefficients.

3. Diffeomorphisms of the Circle

The rigid rotations of the circle we discussed in the previous section form a finite dimensional subset in the infinite dimensional space of orientation preserving smooth diffeomorphisms of the circle. To what extent do the dynamics of nonlinear diffeomorphisms behave as a linear one?

The answer to this question begins with the combinatorial theory of Poincaré [MS]. Any orientation preserving homeomorphism of the circle f has a well defined rotation number $\rho(f)$ (defined up to an integer), which captures the speed in which orbits “go around the circle”. This is most easily defined as the reduction modulo 1 of the translation number $\lim(F^n(x) - x)/n$ of an arbitrary lift $F : \mathbb{R} \rightarrow \mathbb{R}$ of f (capturing this time the drift of F -orbits), which is readily seen to exist.¹ Notice that for a rigid rotation $f : x \mapsto x + \alpha$ we have $\rho(f) = \alpha$. For an arbitrary homeomorphism, we have:

1. f has a periodic orbit (of period q) if and only if $\rho(f)$ is rational (of the form p/q with $(p, q) = 1$). In this case, every f -orbit is asymptotic to a periodic orbit.

¹Letting m_n and M_n be the minimum and maximum of $F^n(x) - x$ for $x \in \mathbb{R}/\mathbb{Z}$ (or which is the same, for $x \in [0, 1]$, since $F^n(x+k) = F^n(x) + k$ for each $k \in \mathbb{Z}$), we see that $0 \leq M_n - m_n \leq 1$. Since m_n is superadditive and M_n is subadditive, the limits of m_n/n and M_n/n must exist and coincide.

2. If $\rho(f)$ is irrational then the orbits of f have the same combinatorial structure of the orbits of the translation $x \mapsto x + \rho(f)$: for each n , the cyclic order of $(f^k(x))_{k=0}^{n-1}$ is the same as that of $(k\rho(f))_{k=0}^{n-1}$.

We will from now on restrict our attention to the most interesting case when $\rho(f)$ is irrational. Let \mathcal{I} stand for the set of diffeomorphisms of the circle with irrational rotation number. In this case, it emerges from the combinatorial description of the orbits that there is a semi-conjugacy to the linear model, i.e., a continuous surjective map $h : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ satisfying $h(f(x)) = h(x) + \rho(f)$ (h is essentially unique, the only freedom available being postcomposition with arbitrary rigid rotations). The natural question is whether the orbit structure is the same also from the topological point of view: is f actually conjugated to the linear model, i.e., is h in fact a homeomorphism? This is answered quite satisfactorily by Denjoy's topological theory. At the level of homeomorphisms, it is easy to find counterexamples: one can blow up an orbit of a rigid rotation with irrational rotation number to create so-called *wandering intervals* (an interval which is disjoint of all positive iterates but does not lie in the basin of attraction of a periodic orbit). Carrying out this construction more carefully, one gets C^1 Denjoy counterexamples, but Denjoy proved that there are no C^2 Denjoy counterexamples: every C^2 diffeomorphism with irrational rotation number is topologically conjugated to a rigid rotation [MS].²

3.1. Renormalization dynamics. Recall that if f is a rigid rotation, the n -th renormalization of an irrational rotation of the circle $f : x \mapsto x + \alpha$ can be obtained by taking the first return map to an interval $[x, f^{q_n-1}(x)]$ with endpoints identified. We would like to extend this definition to an arbitrary smooth diffeomorphism with rotation number α , but we must be careful with the gluing procedure: just gluing with a translation (which generates a circle with Euclidean structure) is not natural here and will in general not produce a diffeomorphism, but only a homeomorphism. The natural way to glue is to use the dynamics itself, i.e., the map f^{q_n-1} , to generate a "smooth circle", on which the first return map indeed acts smoothly.

Unfortunately there is no canonical way to identify the smooth circle with the canonical one (\mathbb{R}/\mathbb{Z}), so this procedure does not really yield a renormalization operator acting on \mathcal{I} . This issue can be resolved by considering \mathbb{Z}^2 -actions as the basic object to be renormalized. Without going into details of this definition, we shall say that the renormalizations become more and more linear if after rescaling (by an affine map $[x, f^{q_n-1}(x)] \rightarrow [0, 1]$), both the gluing map and (each of the two smooth branches of) the first return map converge to translations (say, in the C^∞ -topology if one is dealing with smooth maps).

A simple feature of the renormalization dynamics is that since the combinatorics of the renormalized map only depend on the combinatorics of the orbits of the original one, it is clear that the rotation number transforms as for the

²See also [Y1] and [DKN] for more recent results on absence of wandering intervals.

renormalization of rigid translations, i.e., via the Gauss map. Thus renormalization can be seen as fibering over the Gauss map, and if global convergence of renormalization is established the fibers will thus be identified with *stable manifolds*. We shall see similar situations later, where the existence of a good “candidate stable manifold” will turn out to be central to the analysis of convergence in some more nonlinear situations.

Let us describe the parts of the strategy in the proof of convergence of renormalization (assuming sufficient smoothness) which are perhaps most significant in getting an idea of *why* global convergence takes place.

The first step in most proofs of convergence of renormalization involves the proof on non-divergence (in the form of establishing suitable *a priori bounds*). For circle diffeomorphisms, the crucial such bound comes from the Denjoy-Koksma inequality. It gives an estimate on distortion which implies, in particular, that Df^{q_n} is bounded for all n (this already prevents the existence of wandering intervals, and hence gives Denjoy’s Theorem on topological linearizability). It was a remarkable discovery of Herman [H1] that iteration always leads to cancellations of high order derivatives of f^{q_n} , and thus to global convergence of renormalization. After subsequent work of Yoccoz [Y2], this mechanism was understood in terms of the chain rule for the Schwarzian derivative,

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2} \left(\frac{D^2 f}{Df} \right)^2, \quad (2)$$

which gives

$$Sf^n = \sum_{k=0}^{n-1} (Sf \circ f^k)(Df^k)^2. \quad (3)$$

The control of distortion coming from the Denjoy-Koksma inequality gives

$$Df^k(x) \sim C \frac{l_n(f^k(x))}{l_n(x)}, \quad 0 \leq k \leq q_n - 1, \quad (4)$$

where $l_n(y)$ is the length of the interval $[y, f^{q_n}(y)]$. This allows one to control the term $(Df^k)^2$: indeed the intervals $(f^k(x), f^{k+q_n}(x))$ are disjoint for $0 \leq k \leq q_n - 1$, so that $\sum_{k=0}^{q_n-1} l_n(f^k(x)) \leq 1$ and

$$|Sf^{q_n}(x)| \leq C \max_{0 \leq k \leq q_n-1} \frac{l_n(f^k(x))}{l_n(x)^2}, \quad (5)$$

Since the Schwarzian derivative has order 2, rescaling kills the large term $1/l_n(x)^2$. Using that $\lim_{n \rightarrow \infty} \sup_y l_n(y) = 0$ (by Denjoy’s Theorem giving topological conjugacy with irrational rotations), one gets that, after rescaling, the Schwarzian derivative of both the gluing map and the first return map is indeed going to 0.

Convergence to a linear attractor can be immediately used as a ways of “global to local” reduction. We will now discuss the most famous example of such an application.

3.2. Linearization. Let us continue our discussion of how the dynamics of circle diffeomorphisms resemble that of rigid rotations, assuming enough regularity to guarantee that f is topologically linearizable. The next step is to ask whether the *local geometry* of the orbit structure is also the same. For instance, given three nearby points in the same orbit, are the ratios between distances close to those for the rigid rotation? This (properly quantified) property is actually equivalent to C^1 -linearizability, that is, to h being a C^1 diffeomorphism.

It is easy to see that no condition on the regularity of f will be sufficient to guarantee C^1 -linearizability. Indeed, if f is any nonlinear diffeomorphism of the circle whose lifts extend holomorphically to an entire map $\mathbb{C} \rightarrow \mathbb{C}$ there exists $\theta \in \mathbb{R}$ such that $f_\theta : x \mapsto f(x) + \theta$ has irrational rotation number but is not C^1 -linearizable. This can be seen as follows:

1. $\theta \mapsto \rho(f_\theta)$ is a continuous non-decreasing map $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ of degree 1,
2. It follows that $\rho(f_\theta) \in \mathbb{Q}/\mathbb{Z}$ for a dense countable subset of the closure K_f of $\{\theta \in \mathbb{R}/\mathbb{Z}, \rho(f_\theta) \in \mathbb{R} \setminus \mathbb{Q}\}$.
3. If θ is such that $\rho(f_\theta) = p/q$, then every orbit of f_θ is asymptotic to one of *finitely many* periodic orbits.³ In particular,

$$\inf_{n \geq 1} \inf_{x \in \mathbb{R}/\mathbb{Z}} Df_\theta^n(x) = 0 \quad (6)$$

for any such θ .

4. A Baire category argument shows that (6) holds in fact for generic $\theta \in K_f$, which implies that f_θ is not C^1 -conjugate to a rigid translation. (Note that for generic $\theta \in K(f_\theta)$, we do have $\rho(f_\theta) \notin \mathbb{Q}/\mathbb{Z}$.)

What we wanted to highlight by giving the above argument is that in it one clearly sees that a source of trouble to C^1 -linearizability comes from “contagion” from rational rotation numbers. It turns out that positive results can be obtained if, besides regularity, one assumes that the rotation number is badly approximable by rational numbers.

3.3. The KAM Theorem. Let us consider first the *local* version of the linearizability problem, where one restricts considerations to circle diffeomorphisms close to linear. It can be attacked by a fast iteration scheme (KAM, after Kolmogorov, Arnold and Moser), first introduced by Kolmogorov in the treatment of a considerably more complicated conjugacy problem [Kol]. We will restrict ourselves to give an idea of the setup. Let us assume that we can write $f : x \mapsto x + \rho(f) + \epsilon v(x)$, with v regular and ϵ small, and let us try to solve for

³Here we use that $f_\theta^q(x) = x$ has at most finitely many solutions, which follows from the hypothesis on the holomorphic extension of the lift.

some regular conjugacy close to the identity, $h : x \mapsto x + \epsilon w(x)$ between f and some $x \mapsto x + \beta$. Writing the conjugacy equation, one gets

$$x + \rho(f) + \epsilon v(x) + \epsilon w(f(x)) = x + \epsilon w(x) + \rho(f), \quad (7)$$

i.e.

$$v(x) = w(x) - w(f(x)). \quad (8)$$

Since f is close to the translation by $\rho(f)$, it is reasonable to approximate (8) by the *cohomological equation* $v(x) = w(x) - w(x + \rho(f))$. To solve it we must assume that v has average 0 (integrate both sides), in which case a smooth solution w always exists provided v is smooth and $\rho(f)$ is *Diophantine* in the sense that rational approximations can be only polynomially good (in terms of the denominators of the continued fraction approximations, this gives $\ln q_{n+1} = O(\ln q_n)$), as can be seen by considering the Fourier series expansion. Since $f(x)$ is assumed to have rotation number exactly $\rho(f)$, it can be shown that the average of v is close to 0, so following this procedure we get an approximate solution of (8). With such a solution in hand, we can obtain an approximate conjugacy between f and the rigid translation (in this one step, we only manage to conjugate f with another nonlinear map, but which is closer to the linear model). Iterating this process again, we should obtain a sequence of conjugacies h_n between f and maps with decreasing nonlinearity, the desired conjugacy appearing only as the limit of the h_n .

We are of course skipping the core of the argument here, which is that there is loss of regularity which is apparent when solving the cohomological equation. The full treatment was given by Arnold [Ar] in the case where f is analytic (the obtained conjugacy is analytic as well in this case), the smooth case is due to Moser, see, e.g., [H1].

3.4. The Herman-Yoccoz Theorem. While the hypothesis that f be close to a rigid rotation is obviously important in the argument above, Arnold advanced the daring conjecture that his linearizability theorem should also hold in general. This later became the Herman-Yoccoz Theorem [Y2]:

Theorem 1. *Let f be a smooth (respectively, analytic) orientation preserving diffeomorphism of the circle with Diophantine rotation number. Then f is smoothly (respectively, analytically) conjugated to a rigid rotation.*

A weaker version of this theorem was first proved by Herman [H1], assuming a stricter (but still full measure) condition on the rotation number. Following the lucid account of Sullivan [S3], we will focus on this version since it is the one that illustrates most transparently the importance of convergence of renormalization (more precise results can be associated with an estimate on the *rate* of convergence), taking only a few lines. Indeed, let f be a smooth diffeomorphism with Diophantine rotation number. Its renormalizations are becoming closer and closer to rigid rotations. Assume first that the rotation

number of f is fixed by the Gauss map (for instance, it is the golden mean). Then it is clear that at some point the renormalizations belong to the “domain of convergence of the KAM algorithm”, so the renormalization will be linearizable. It follows f itself is linearizable: Since linearizability concerns the local geometry of orbits (c.f. the beginning of §3.2), it must be invariant under renormalization. In general the rotation number does change under renormalization, and while the Diophantine class is invariant under the Gauss map, the “Diophantineness” (measured in the quantification of the Diophantine condition $\ln q_{n+1} = O(\ln q_n)$) may degenerate at each step, and with it the size of the region where the KAM algorithm works. But at least for almost every rotation number, there will be infinitely many times for which the renormalized rotation numbers satisfy a fixed Diophantine condition (e.g., $\ln q_{n+1} \leq 10 \ln q_n$): this is immediate from the ergodicity of the Gauss map. For such rotation numbers, we do not need to worry about trying to hit a moving target (comparing the speed of convergence of renormalization with the possible decrease in range of the KAM method), thus global linearizability follows.

Remark 3.1. As Sullivan notes in [S3], Herman did not use the renormalization language, though his work fitted perfectly into it. The full renormalization formalism was implemented in this context by Khanin-Sinai [SK].

4. One-frequency Cocycles

We now consider a situation where renormalization presents a finite-dimensional local attracting set (again corresponding to setting the nonlinearity to zero) but which clearly can not be a global attractor. It is the precise understanding of the obstructions to convergence of renormalization that plays an important role in establishing a global theory.

4.1. The local character of linearizability in two dimensions. A few years after establishing the global nature of linearizability of diffeomorphisms of the circle satisfying suitable arithmetic conditions, Herman wrote another seminal paper [H2]. According to the title, it is about both “a method to minorate Lyapunov exponents” and “some examples showing the local character of the Arnold-Moser Theorem in dimension 2”.

The examples discussed by Herman are analytic diffeomorphisms of \mathbb{T}^2 that are isotopic to the identity, fiber over a rigid irrational rotation, and act projectively in the second coordinate. They can be written as a skew-product, or *cocycle*, $(\alpha, A) : (x, w) \mapsto (x + \alpha, A(x) \cdot w)$ where $A : \mathbb{R}/\mathbb{Z} \rightarrow \text{SL}(2, \mathbb{R})$ is an analytic map homotopic to a constant. The iterates of a cocycle have the form $(\alpha, A)^n = (n\alpha, A_n)$ with $A_n(x) = A(x + (n-1)\alpha) \cdots A(x)$. A class of particular interest consists of one-frequency Schrödinger cocycles, where

$$A = A^{(E-v)} = \begin{pmatrix} E-v & -1 \\ 1 & 0 \end{pmatrix}, \quad (9)$$

with v an analytic map $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ and E some real constant. Schrödinger cocycles are relevant to the analysis of one-frequency Schrödinger operators $H = H_{\alpha,v}$. These are bounded self-adjoint operators on $\ell^2(\mathbb{Z})$ of the form

$$(Hu)_n = u_{n+1} + u_{n-1} + v(n\alpha)u_n, \quad (10)$$

since a formal solution of $Hu = Eu$ satisfies $\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = A_n(0) \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}$.

Just as for diffeomorphisms of the circle, one can define a *rotation vector* (as the reduction modulo 1 of the drift in \mathbb{R}^2 of a lift). The first coordinate of the rotation vector is obviously α , while the second is called the *fibered rotation number*. For Schrödinger cocycles, there is a beautiful reinterpretation [AS] of the fibered rotation number of $(\alpha, A^{(E-v)})$, as $1 - N(E)$ where N is the *integrated density of states* of the operator $H_{\alpha,v}$, which gives the limiting proportion of eigenvalues of restrictions of $H_{\alpha,v}$ (to intervals of increasing length) that lie in $(-\infty, E]$. In particular, for fixed v , any rotation vector (α, β) can be realized by choosing E appropriately.

In [H2], Herman discusses how the Arnold-Moser (KAM) Theorem gives a local linearization theorem in this setting: If the rotation vector satisfies a Diophantine condition then analytic linearizability holds, provided A is sufficiently close to a constant. (The use of KAM methods in connection with quasiperiodic Schrödinger operators was pioneered by Dinaburg-Sinai [DS].) On the other hand, [H2] also introduces Herman's famous "subharmonicity method" to minorate the Lyapunov exponent

$$L = \lim \frac{1}{n} \int \ln \|A_n(x)\| dx. \quad (11)$$

For Schrödinger cocycles, it implies that if v is a non-constant trigonometric polynomial $\sum_{|k| \leq m} a_k e^{2\pi i k x}$ with $|a_m| > 1$ then $L > 0$.

The positivity of the Lyapunov exponent is incompatible with even topological linearizability, since it implies in particular that the dynamics of (α, A) is not distal (if $\sup \|A_n(x)\| = \infty$ then there exist $y \neq y'$ such that $\inf d(A_n(x) \cdot y, A_n(x) \cdot y') = 0$). Thus by choosing v and E appropriately one obtains a non-linearizable cocycle which nevertheless has a Diophantine rotation vector.

Remark 4.1. Even near constants, there are *uniformly hyperbolic* cocycles, for which $\|A_n(x)\|$ grows exponentially fast uniformly on x , and in particular have positive Lyapunov exponents. The locus of uniformly hyperbolic cocycles is open and quite simple to analyze, much like the complement of the closure of circle diffeomorphisms with irrational rotation number. The examples constructed by Herman have a rather different nature, since the rotation vector of a uniformly hyperbolic cocycle is linearly dependent over the rationals. Cocycles with a positive Lyapunov exponent but which are not uniformly hyperbolic are called *nonuniformly hyperbolic*.

4.2. The basin of the renormalization attractor. Just as in the case of circle diffeomorphisms, one can try to define a renormalization operator acting on cocycles by considering the first return map to the annulus $[x_0, x_0 + q_n \alpha] \times \mathbb{R}/\mathbb{Z}$, where we identify the boundary circles via $(x, y) \mapsto (x + q_n \alpha, A_{q_n}(x) \cdot y)$. We will again omit the details of the formalized definition in terms of \mathbb{Z}^2 -actions.

As usual, if the Lyapunov exponent is positive then renormalization magnifies it, so the renormalization orbits can not converge to any attractor (recall the second theme listed in the introduction). Starting with a cocycle with Diophantine rotation vector which is sufficiently close to linear, so that the KAM Theorem applies, the successive renormalizations become increasingly linear. Thus the locus of linear cocycles behaves as a local, but not global (since it misses the Herman's examples), attractor for cocycles with Diophantine rotation vectors.⁴

What is in fact the basin of the renormalization attractor? Naturally, it is contained in the locus of zero Lyapunov exponents. Since the basin of a local attractor is by nature open, and the locus of zero Lyapunov exponents is closed (this is a deep result of Goldstein-Schlag [GoSc] and Bourgain-Jitomirskaya [BJ]), the inclusion is in fact strict. In [AK1], [AK2], it is shown that there is, however, equality “modulo 0”. For simplicity, we state the result for Schrödinger cocycles:

Theorem 2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $v : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ be analytic. Then for almost every $E \in \mathbb{R}$, if the Lyapunov exponent of $(\alpha, A^{(E-v)})$ is zero then the successive renormalizations of $(\alpha, A^{(E-v)})$ become increasingly linear.*

A much more detailed analysis of the “critical set” separating converging and diverging orbits of the renormalization operator has been carried out more recently as a part of a program to produce a global theory of one-frequency Schrödinger operators [A1], [A2], [A3]. It shows that (for fixed Diophantine α), the critical set is not only of zero measure, but it has zero measure *inside a codimension one subset*. This more precise description is important because the analysis of a single Schrödinger operator depends on a one-parameter family of cocycles: it allows us to make statements about *every energy E* in the spectrum of *almost every potential*.

5. Hitting the Limits of Linear Attractors

In the analysis of one-frequency cocycles, it is clear that the renormalization dynamics is not going to be governed by a nice attractor once the nonlinearity is

⁴The analysis can be extended considerably beyond Diophantine rotation vectors, but the arguments are not as simple as just applying the KAM Theorem.

so large that the Lyapunov exponent becomes positive.⁵ A more subtle problem concerns the renormalization of critical cocycles, at the onset of nonuniform hyperbolicity (see Remark 4.1). Their renormalizations can no longer converge to linear cocycles, but they could still be governed by an attractor. One reason to hope for it is the way renormalization acts on the Lyapunov exponent of complexifications: for critical cocycles one has, for $\epsilon > 0$ small,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|A_n(x + \epsilon i)\| = 2\pi\omega\epsilon, \quad (12)$$

where ω is a positive integer called the acceleration (this “quantization” property was only recently discovered, in [A1]). This simple dependence behaves perfectly under renormalization, so that a renormalized critical cocycle is a critical cocycle with the same acceleration. Thus the acceleration measures an irreducible amount of nonlinearity of critical cocycles (since cocycles close to a constant must have zero acceleration), which contrary to a positive Lyapunov exponent does not grow with renormalization.

However, since it is known that if the matrix products $A_n(x)$ remain bounded for all times, then renormalization must converge to the linear attractor [AK2], it seems unrealistic to expect for renormalization to converge in the traditional sense. Maybe it might be necessary to modify the definition of the renormalization operator, perhaps by introducing nonlinear changes of coordinates? Let us note that a very different kind of renormalization mechanism [HS] has been previously considered in the analysis of some features of criticality, in the particular case of the critical Almost Mathieu Operator (with potential $v(x) = 2 \cos 2\pi x$). This especially symmetric (under so-called Aubry duality [GJS]) model has the remarkable property that the associated cocycles are critical for all energies in the spectrum, and because of (numerically) observed self-similarity in the spectrum, it is very tempting to imagine that there is a renormalization attractor somewhere in the picture. The situation here may be related to the (even less understood) breakdown of KAM behavior in area-preserving maps (discussed, e.g., in [McK]).

A similar (but much more well understood) situation concerns the case of analytic circle maps. Diffeomorphisms of the circle form an open set where renormalization acts quite nicely, but what about the *critical circle maps* in its boundary? Those are still homeomorphisms, and so have a well defined rotation number, but the critical points introduce an irreducible (conserved under renormalization) amount of nonlinearity. There is a well-developed renormalization theory in this case, particularly about the main component of the boundary of diffeomorphisms, consisting of critical circle maps with a single critical

⁵It might be still possible to obtain results describing the asymptotics of the diverging renormalization orbits, but currently there is nothing more than interesting heuristics in this direction.

point: as it turns out, there exists a renormalization attractor, and this lies behind fundamental rigidity results (see [FM1], [FM2], [Ya1], [Ya2], [KT]).⁶

If one goes beyond critical circle maps, one starts dealing with non-invertible maps of the circle. We will however go in a slightly different direction, and discuss next non-invertible maps of the interval, focusing on the particular class for which much of the renormalization theory was developed.

6. Analytic Unimodal Maps

Let $f : I \rightarrow I$ be an analytic unimodal map. Thus f has a unique critical point, which is of turning type (maximum or minimum) and located in $\text{int}I$. By an affine change of coordinates, we may normalize it so that the critical point is at the origin and $f(x) = f(0) + x^d + O(x^{d+1})$ for some even integer $d \geq 2$, called the degree. Basic examples of analytic unimodal maps are given by the (appropriate restrictions of) unicritical polynomials $x \mapsto x^d + c$ (for the suitable range of $c \in \mathbb{R}$ for which an invariant interval exists). The precise domain of definition of a unimodal map is not of too much importance, since it only concerns trivial aspects of the dynamics.

A unimodal map is called renormalizable if there is an interval $I' \subset I$ around 0 and an integer $n > 1$ such that $f^n(I') \subset I'$ but $f^j(I') \cap \text{int}I' = \emptyset$ for $1 \leq j \leq n - 1$. Then $f' = f^n : I' \rightarrow I'$ is again unimodal. The set of possible values of n form a finite or infinite sequence $n_1 < n_2 < \dots$, such that $n_j | n_k$ for $j < k$. The normalization of (the appropriate restriction) of f^{n_j} is called the j -th renormalization. The renormalization operator R takes each renormalizable map f to its first renormalization Rf , and the j -th renormalization is obtained by iterating it j -times. If $R^j f$ is renormalizable for every $j \in \mathbb{N}$, f is called infinitely renormalizable.

The renormalization period of f is $n = n_1$, while the renormalization combinatorics of f is the permutation of $\pi : \{0, \dots, n - 1\} \rightarrow \{0, \dots, n - 1\}$ such that $\pi(j) < \pi(k)$ if and only if $f^j(0) < f^k(0)$. All integers $n \geq 2$ do arise as the renormalization periods of some unimodal map. The renormalization combinatorics is not, in general, determined by the period. We let Σ be the countable set of all possible renormalization combinatorics.

The existence of a critical point has the important consequence that all renormalizations have an “irreducible nonlinearity”. While in the situations considered in §3 and §4 we could readily define an invariant set which was a candidate to be a renormalization attractor, proving any kind of convergence

⁶Particularly Khanin-Teplinsky show (using exponential convergence of renormalization) that for analytic circle homeomorphisms with a single critical point of fixed odd degree $d \geq 3$, any two maps with the same irrational rotation number must be C^1 -conjugate. This is in stark contrast with the situation for circle diffeomorphisms, as no kind of Diophantine condition is necessary.

of renormalization for unimodal maps will involve constructing the attractor in the process.

Important aspects of the dynamics of unimodal maps are impacted by the degree, and most especially by whether $d = 2$ (the quadratic case) or $d > 2$ (the higher degree case). The ultimate source of this difference lies in a specific “decay of geometry” property valid in the quadratic case but not in the higher degree case, which diminishes the importance of nonlinearity in small scales *before the first renormalization*. This impacts, in particular, the analysis of attractors of the unimodal dynamics: in the quadratic case, Milnor’s notion of topological and measure-theoretical attractor coincide [L1],⁷ but this is not true, in general, in sufficiently high degree [BKNS].

6.1. Feigenbaum-Coulet-Tresser phenomenon. Renormalization of unimodal maps is most well known for its role in the understanding of universality in the period doubling bifurcation. Considering, say, the quadratic family $p_c(x) = x^2 + c$, which define unimodal maps for $c \in [-2, 1/4]$, one sees that for c close to $1/4$, the iterates of the critical point are asymptotic to a fixed point. This persists as one decreases the parameter c , until a moment c_0 at which the so-called saddle-node bifurcation takes place. Just below it, the fixed point becomes repelling, but a nearby period 2 cycle emerges, which still attracts the critical orbit. This again persists until another moment c_1 , where another saddle-node bifurcation takes place and a period 4-cycle emerges. Proceeding in this way, one defines the sequence of period-doubling bifurcation moments c_k (at which a 2^k -cycle gives birth to a 2^{k+1} -cycle). The remarkable fact is that c_k converges at a geometric rate, so that

$$\frac{c_k - c_{k+1}}{c_{k+1} - c_{k+2}} \rightarrow 4.669\dots \quad (13)$$

(this limit is called the Feigenbaum constant). But the big surprise is that if one considers another family of analytic unimodal maps f_c with quadratic critical point (say, close to the quadratic one, to avoid transversality issues), one gets a very different sequence of bifurcation moments \tilde{c}_k , but which still converge geometrically *with the same rate*. The Feigenbaum constant is a universal quantitative feature of the cascade of period doubling bifurcations for unimodal maps with a quadratic critical point. For fixed higher degree d , the same phenomenon occurs (with a “Feigenbaum constant” associated to each d).

Dynamics of the renormalization operator comes into play because the limiting parameter of the cascade of period doubling bifurcations corresponds to an infinitely renormalizable unimodal map f , with $n_j = 2^j$. According to

⁷By definition, an attractor should have a large *basin* (of points which are asymptotic to the attractor). If largeness is understood in terms of Baire category one gets the topological notion, while if it is understood in terms of Lebesgue measure one gets the measure-theoretical one.

the Renormalization Conjectures, advanced by Feigenbaum and Coullet-Tresser ([F], [TC]), the renormalizations $R^n f$ should converge to a universal (for each fixed degree) unimodal map f_* , a solution of the Feigenbaum-Cvitanovic equation $f_*^2(\lambda x) = \lambda f_*(x)$. Moreover, in some suitable functional space, the derivative of renormalization at f_* should be hyperbolic, and its spectrum outside the unit disk should consist of a single simple eigenvalue: In other words, f_* should be an hyperbolic fixed point with one-dimensional unstable direction. One can show that the Renormalization Conjectures imply that the cascade of period doubling bifurcations undergone by a generic (i.e., satisfying a transversality condition) family does indeed converge geometrically at a rate given precisely by the value of the eigenvalue of DRf_* which lies outside the unit disk.

There is a long history to the Renormalization Conjectures, which were initially addressed in a formal computer assisted proof of Lanford [La] (dealing with the existence and hyperbolicity of a renormalization fixed point in the quadratic case), see [L4] and references therein.

6.2. Role in the measure-theoretical analysis of parameters. While beautiful, the theory of the period doubling bifurcation only concerns the most ordered part of the dynamics of unimodal maps. Through the whole cascading process, one only faces dynamics displaying attracting periodic orbits, and only at the limit of the cascade one gets something more complicated (the attractor is no longer a periodic orbit, but the suitable limit of period 2^k -orbits, i.e., a Cantor set restricted to which the dynamics is conjugate to translation by one in the ring of 2-adic integers).

On the other side of the parameter space ($c = -2$ for the quadratic family), one gets a very different situation. The map $x \mapsto x^2 - 2$, also called the Ulam-Neumann map, possesses an invariant probability measure which is equivalent to the restriction of Lebesgue measure to $[-2, 2]$. This measure is ergodic and so Lebesgue almost every orbit is equidistributed with respect to it.

The Ulam-Neumann map shows that unimodal dynamics is consistent with chaos (the invariant measure has a positive Lyapunov exponent), but looks quite unstable. Indeed, Lyubich [L2] and Graczyk-Swiatek [GS] proved that in the quadratic family there exists an open and dense set of parameters corresponding to *regular* unimodal maps (for which the critical point is asymptotic to an attracting periodic orbit). However Jakobson [J] showed that there is a positive measure set of parameters c (near -2) corresponding to *stochastic* unimodal maps (with an absolutely continuous invariant probability measure with positive Lyapunov exponent). Thus while only regular behavior is “topologically robust”, both regular and stochastic behaviors are “measure-theoretically robust”. Such results extend to more general analytic unimodal maps, the density of regular behavior being however much harder in higher degree [KSS].

With these preliminaries, we can now present the main result on the measure-theoretic dynamics of unicritical polynomials (in the quadratic case, it is due to Lyubich [L5]).

Theorem 3 ([AL1], [AL2]). *Almost every unicritical polynomial $x^d + c$ is either regular or stochastic.*

What about infinitely renormalizable maps? Those are neither regular nor stochastic, so to get to Theorem 3 one must show in particular that infinitely renormalizable parameters correspond to a zero Lebesgue measure set of parameters.⁸ While the explanation of the Feigenbaum-Coulet-Tresser phenomenon lies in understanding the dynamics of the renormalization operator of period 2 (governed by a single hyperbolic fixed point), here we will need to understand the full renormalization dynamics, incorporating all renormalization combinatorics.

It follows from the density of regular parameters that the set of infinitely renormalizable parameters in the unicritical family (with d fixed) is homeomorphic to the set of irrational numbers in $(0, 1)$. Indeed, the combinatorics of successive renormalization behaves much like the digits in the continued fraction expansion of an irrational number: Any sequence of renormalization combinatorics is realized by a unique parameter value. This hints to the fact that “along the direction of the unicritical families” the dynamics of renormalization should resemble to some extent the shift on $\mathbb{N}^{\mathbb{N}}$.

If instead of specifying the full renormalization combinatorics one merely specify the the combinatorics of the first n renormalizations, one obtains an interval (or renormalization window) of parameters. The idea of the measure-theoretic analysis of infinitely renormalizable parameters is that the renormalization window is a distorted copy of the full parameter space. Corresponding, e.g., to the tame end of the parameter space consisting of regular dynamics, one finds accordingly a region of regular parameters inside the renormalization window. If we can control the distortion involved in the renormalization process, we will conclude that there are “definite gaps” in arbitrarily small scales around any infinitely renormalizable parameter. Thus the set of infinitely renormalizable parameters has no Lebesgue density point, and must thus have zero Lebesgue measure.

The control of the dynamics of renormalization needed in the argument lies behind a deep generalization of the Renormalization Conjectures. A program in this direction was initially advanced by Sullivan [S1] in the case of bounded combinatorics, in the sense that one restricts considerations to infinitely renormalizable maps f such that the renormalization periods of $R^k f$ is bounded (independently of k) by some fixed (but arbitrary) constant. In this setting, Sullivan [S2] (see also [MS]) constructed a global renormalization attractor (homeomorphic to the Cantor set $F^{\mathbb{Z}}$ for a finite part $F \subset \Sigma$), McMullen [McM] proved exponential convergence to the attractor, and Lyubich proved that the renormalization attractor is hyperbolic (a Smale horseshoe) with one-dimensional unstable direction [L4]. The hyperbolicity of the full renormalization operator

⁸Of course, the proof of Theorem 3 involves a substantial understanding of non-infinitely renormalizable dynamics [MN], [L3], [AKLS], [ALS], which we will not go through here.

was proved by Lyubich in the quadratic case [L5]. In this tour de force, the analysis of exponential contraction depends on special fine geometry features of the complex dynamics of quadratic polynomials [L2].

We should note that it is quite important to choose an appropriate functional setting to study the dynamics of the renormalization operator. Following Douady-Hubbard [DH], it is natural to consider the action of renormalization in spaces of *polynomial-like germs*: These may be thought of as obtained from unicritical polynomials by suitable *hybrid* deformations of the complex structure of the Riemann sphere (by Douady-Hubbard's Straightening Theorem). In this setting, the hybrid classes provide natural candidate stable manifolds of renormalization, being easily seen to be forward invariant under renormalization. Establishing that the hybrid classes are actually stable manifolds is a crucial step in the construction of the renormalization attractor.

6.2.1. Convergence of renormalization. One central point of [AL1] is that convergence along the candidate stable manifolds can be derived from *beau a priori bounds* (a concept introduced by Sullivan). This is a rough geometric control that is known to hold in general and translates to *universal precompactness* of the renormalization orbits, by exploiting the global dynamics of the renormalization operator. While it is beyond the point of this paper to discuss how the necessary a priori bounds (due to [LS] and [LY]) are obtained, we can give some ideas about how they lead to convergence.

The candidate stable manifolds can be endowed with a complex structure, which is respected by renormalization. It is important to note that we only get this complex structure by allowing deformations which are not real symmetric, and hence do not correspond to actual unimodal maps, and the beau a priori bounds only concern, in principle, the real-symmetric deformations.

The hybrid classes are all equivalent to a same functional space \mathcal{E} , hence the action of the renormalization operator along the family of all hybrid classes of infinitely renormalizable maps corresponds to the action of a certain "renormalization groupoid" \mathcal{R} acting holomorphically on \mathcal{E} . Naturally, \mathcal{R} respects the real trace $\mathcal{E}^{\mathbb{R}} \subset \mathcal{E}$ corresponding to legitimate unimodal deformations.

Using a version of the Schwarz Lemma, one obtains non-expansion of the renormalization groupoid, which together with the beau a priori bounds in $\mathcal{E}^{\mathbb{R}}$ implies that \mathcal{R} is *almost periodic*. An abstract analysis of almost periodic groupoids shows that either the renormalization groupoid is uniformly contracting or the lack of contraction is detected by a non-constant holomorphic idempotent P in its limit set $\omega(\mathcal{R})$.

We want to show that any holomorphic idempotent in $\omega(\mathcal{R})$ is non-constant. By holomorphicity, it is enough to show non-constancy along $\mathcal{E}^{\mathbb{R}}$. The beau a priori bounds imply that $P(\mathcal{E}^{\mathbb{R}})$ is a compact set, and since P is a sufficiently regular idempotent, it must be a manifold. As expected from a deformation space, $\mathcal{E}^{\mathbb{R}}$ turns out to be contractible, so its image by an idempotent is contractible as well. Since the only contractible compact manifold is a point, we

conclude that $P|\mathcal{E}^{\mathbb{R}}$ must be indeed constant. This implies, by contradiction, that the renormalization groupoid is uniformly contracting, as desired.

Remark 6.1. The argument above uses only a few properties of the renormalization groupoid (holomorphicity, real-symmetry, and appropriate precompactness along $\mathcal{E}^{\mathbb{R}}$), and can be used to establish uniform contraction of any other groupoid with those properties. In particular, finer geometric properties of infinitely renormalizable maps (that tend to be quite dependent on the combinatorics and degree) can play no role. In previous, more restricted, approaches, contraction was always ultimately obtained as a consequence of such less robust features.

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