

HODGE THEORY AND CYCLE THEORY OF LOCALLY SYMMETRIC SPACES

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Abstract

We discuss several results pertaining to the Hodge and cycle theories of locally symmetric spaces. The unity behind these results is motivated by a vague but fruitful analogy between locally symmetric spaces and projective varieties.

1 Introduction

Locally symmetric spaces are complete Riemannian manifolds locally modeled on certain homogeneous spaces. Their holonomy groups are typically smaller than SO_n – the holonomy group of a generic Riemannian manifold – and there are invariant tensors on the tangent space that are preserved by parallel transport. It was first observed by [Chern \[1957\]](#) that Hodge theory can be used to promote these local algebraic structures to structures that exist on the cohomology groups of locally symmetric spaces. This is very similar to what happens for compact Kähler manifolds. In fact the analogy between locally symmetric spaces and Kähler manifolds – or rather complex projective varieties – is a fruitful one in many aspects. In this report we shall discuss several instances of this analogy. We don't give proofs, we only state recent results that illustrate various items of the following dictionary.

	Projective varieties $V \subset \mathbb{P}^n$	Locally symmetric spaces $\Gamma \backslash S$
Complexity	degree	volume
Hodge theory	Hodge-Lefschetz decomposition	Matsushima's formula and (\mathfrak{g}, K) -cohomology
Cycles	Algebraic cycles (Intersections of) Hyperplane sections	Modular cycles Sums of Hecke translates of modular cycles
Cohomology	Kodaira vanishing Theorem Lefschetz hyperplane Theorem Hodge Conjecture	Vanishing theorems using spinors Automorphic Lefschetz' properties Hodge type theorems

The following result (see [Theorem 8](#) below), jointly obtained with Millson and Moeglin, shows that the right side of the above dictionary may eventually shed some light on the left (more classical) side.

Theorem. *On a projective unitary Shimura variety uniformized by the complex n -ball, any Hodge (r, r) -class with $r \in [0, n] \setminus \frac{n}{3}, \frac{2n}{3}$ is algebraic.*

Context. There has been a great deal of work on the cohomology of locally symmetric spaces. This involves methods from geometry, analysis and number theory. We note in particular that related topics have been discussed [Harris \[2014\]](#), [Venkataramana \[2010\]](#), and [Speh \[2006\]](#) in the last three ICMs. Indeed, [Harris \[2014\]](#) contains an overview of the program for analyzing cohomology of Shimura varieties developed by Langlands and Kottwitz. It aims at attaching Galois representations to the corresponding cohomology classes. Our point of view is closer to [Venkataramana \[2010\]](#) and [Speh \[2006\]](#) that discuss conjectures that naturally fit into the above dictionary. The latter has been very much influenced by former works of Oda, Venkataramana, Harris-Li discussed in [Venkataramana \[2010\]](#). We also have borrowed some expository ideas from §3 of Venkatesh Takagi lectures [Venkatesh \[2017\]](#).

2 Locally symmetric spaces

2.1 Symmetric spaces. A *symmetric space* is a Riemannian manifold whose group of symmetries contains an inversion symmetry about every point. We will be mainly concerned with symmetric spaces of non-compact type. Such a space S is associated to a connected center-free semi-simple Lie group G without compact factor. As a manifold S is the quotient G/K of G by a maximal compact subgroup $K \subset G$; it is known that all such K are conjugate inside G . One may easily verify that G preserves a Riemannian metric on S . Unless otherwise specified our symmetric spaces S will always be assumed to be of non-compact types.

For example, if $G = \mathrm{PSL}_2(\mathbb{R})$, we can take $K = \mathrm{PSO}_2$, and the associated symmetric space $S = G/K$ can be identified with the Poincaré upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and the action of G is by fractional linear transformations; it preserves the standard hyperbolic metric $|dz|^2/\mathrm{Im}(z)^2$.

If $G = \mathrm{PSL}_2(\mathbb{C})$, we can take $K = \mathrm{PSU}_2$, and the associated symmetric space S can be identified with the three-dimensional hyperbolic space \mathbb{H}^3 .

We shall be particularly concerned with cases where G is either a unitary group $\mathrm{PU}(p, q)$ or an orthogonal group $\mathrm{SO}_0(p, q)$. Thanks to special isomorphisms between low dimensional Lie groups the symmetric spaces associated to $\mathrm{PU}(1, 1)$ and $\mathrm{SO}_0(2, 1)$ are both

isometric to the Poincaré upper-half plane \mathbb{H}^2 and the symmetric space associated to $\mathrm{SO}_0(3, 1)$ is isometric to the three-dimensional hyperbolic space \mathbb{H}^3 .

Another important case to consider is that of $G = \mathrm{PSL}_n(\mathbb{R})$. Then we can take $K = \mathrm{PSO}_n$, and the symmetric space S can be identified with the space of positive definite, symmetric, real valued $n \times n$ matrices A with $\det(A) = 1$, with metric given by $\mathrm{trace}(A^{-1}dA)^2$.

2.2 Locally symmetric spaces. Locally symmetric spaces are complete Riemannian manifolds locally modeled on some symmetric space S with changes of charts given by restrictions of elements of G . It is known that all such manifolds are isometric to quotients $\Gamma \backslash S$ of S by some discrete torsion-free subgroup $\Gamma \subset G$. One may measure the complexity of a locally symmetric space by considering its volume, or equivalently the volume of a fundamental domain for the action of Γ on S . We shall be only concerned with locally symmetric spaces of finite volume; the group Γ is then a lattice in G – in many cases we shall even restrict to compact locally symmetric spaces.

By a general theorem of Borel, any symmetric space S admits a compact manifold quotient $(S\text{-manifold}) \Gamma \backslash S$. In Borel's construction Γ is a *congruence arithmetic group*. For our purpose let us define these groups as those obtained by taking a semi-simple algebraic \mathbb{Q} -group $\mathbf{H} \subset \mathrm{SL}_N$, and taking

$$(2-1) \quad \{h \in \mathbf{H}(\mathbb{Q}) : h \text{ has integral entries}\}.$$

Each such group is contained in an ambient Lie group, namely the real points of \mathbf{H} . If $\mathbf{H}(\mathbb{R})$ is isogeneous to $G \times (\text{compact})$ the projection on the first factor maps the discrete subgroup (2-1) onto a lattice Γ in G . If the compact factor in $\mathbf{H}(\mathbb{R})$ is non-trivial then Γ is necessarily co-compact in G . Finally, replacing the discrete group (2-1) by its intersection with the kernel of a reduction mod ℓ map $\mathrm{SL}_N(\mathbb{Z}) \rightarrow \mathrm{SL}_N(\mathbb{Z}/\ell\mathbb{Z})$, one can obtain a torsion-free lattice Γ . We refer to the corresponding locally symmetric spaces $\Gamma \backslash S$ as *congruence arithmetic*.

2.3 Examples. Locally symmetric spaces play a central role in geometry. Here are some important examples:

Shimura varieties. These appear in algebraic geometry as moduli spaces of certain types of Hodge structures. E.g. for all g the moduli space \mathcal{K}_g of genus g quasi-polarized K3 surfaces identifies with a locally symmetric space associated to $G = \mathrm{SO}_0(2, 19)$. Shimura varieties themselves are quasi-projective varieties. They play an important role in number theory through Langlands' program.

Complex ball quotients. By Yau’s solution to the Calabi conjecture, complex algebraic surfaces whose Chern numbers satisfy $c_1^2 = 3c_2$ are quotients of the unit ball in \mathbb{C}^2 by a torsion-free co-compact lattice in $\mathrm{PU}(2, 1)$. Most famously, this includes the classification of fake projective planes by Klingler [2003], Prasad and Yeung [2009] and Cartwright and Steger [2010]. Picard [1881], Deligne and Mostow [1986] and Thurston [1998] give many examples of ball quotients coming from natural moduli problems. Congruence arithmetic ball quotients are particular Shimura varieties.

Hyperbolic manifolds. In dimension 3, according to Thurston’s geometrization conjecture, proved by Perelman, a ‘generic’ manifold is hyperbolic. More generally, Gromov theory of δ -hyperbolic groups suggest that negative curvature is ‘quite generic.’ However, at least in dimension ≥ 5 , all known (to the author) constructions of closed manifolds that can carry a negatively curved metric are essentially obtained by rather simple surgeries on locally symmetric manifolds. These spaces therefore form a fundamental family of examples in geometry and more generally play a crucial role in geometric group theory.

Teichmüller spaces of flat unimodular metrics on tori $\mathbb{R}^n/\mathbb{Z}^n$. These are locally symmetric spaces associated to $\mathrm{PSL}_n(\mathbb{R})$. Their cohomology groups are very tightly bound to algebraic K -theory. In particular this viewpoint quite naturally leads to the famous regulator of Borel.

2.4 Notation. We have already defined $K \subset G$ and the associated Riemannian symmetric space $S = G/K$. Let \mathfrak{g} be the complexified Lie algebra of G and let G^c be a compact form of G . Let $S^c = G^c/K$ be the compact dual of S . Let θ be the Cartan involution of G fixing K and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the associated Cartan decomposition. We normalize the Riemannian metric on S^c such that multiplication by i in \mathfrak{p} becomes an isometry $T_{eK}S \rightarrow T_{eK}S^c$.

From now on $\Gamma \backslash S$ will denote a finite volume locally symmetric S -manifold. In general we try to reserve n for its real dimension, or complex dimension if S is Hermitian.

We denote by $b_k(M)$ the Betti numbers of a manifold M .

3 Hodge theory

For simplicity, in this section, we will assume that all the locally symmetric spaces $\Gamma \backslash S$ we consider are compact. This excludes some of the important examples mentioned above. However modified versions of the discussion below still apply and we will abusively ignore this issue in the rest of this document.

Being a compact manifold, the quotient $\Gamma \backslash S$ satisfies Poincaré duality. But, as mentioned in the Introduction, the Riemannian manifold $\Gamma \backslash S$ in general has a much smaller holonomy group than SO_n , and one can show that this forces $\Gamma \backslash S$ to satisfy many more constraints, see (3-5) and (3-6). These constraints can be understood in terms of *cohomological representations*, i.e. unitary representations π of G such that the relative Lie algebra cohomology $H^*(\mathfrak{g}, K; \pi)$ is non-zero (see Section 3.2 below).

Since the general setup of (\mathfrak{g}, K) -cohomology is rather forbidding we will discuss in more detail two special examples. But first, let us emphasize the analogy with projective manifolds, or rather here with Kähler manifolds.

3.1 Comparison with Kähler manifolds. Hodge theory gives a way to study the cohomology of a closed Riemannian manifold M . Indeed, each class in $H^*(M, \mathbb{C})$ has a *canonical* ‘harmonic’ representative: a differential form ω that represents this class and is of minimal L^2 norm. Equivalently the form ω is annihilated by the Hodge-Laplace operator Δ . One gets

$$(3-1) \quad \underbrace{\text{harmonic } k\text{-forms on } M}_{:= \mathcal{H}^k(M)} \xrightarrow{\cong} H^k(M, \mathbb{C}).$$

Suppose furthermore that M is an n -dimensional complex Kähler manifold. Then, its holonomy group is contained in the unitary group $U_n \subset SO_{2n}$ and there is an action of \mathbb{C}^* on each tangent space that is preserved by parallel transport. This yields an action of \mathbb{C}^* on differential forms with complex coefficients. A crucial aspect of the theory of Kähler manifolds is that this action preserves harmonic forms. It then follows from Hodge theory that \mathbb{C}^* acts on the cohomology groups and this gives rise to the Hodge decomposition.

3.2 Matsushima’s formula. Let us now come back to the case of a compact locally symmetric manifold $\Gamma \backslash S$.

Because the cotangent bundle $T^*(\Gamma \backslash S)$ is isomorphic to the bundle $\Gamma \backslash G \times_K \mathfrak{p}^* \rightarrow \Gamma \backslash G / K$, which is associated to the principal K -bundle $K \rightarrow \Gamma \backslash G \rightarrow \Gamma \backslash S$ and the adjoint representation of K in \mathfrak{p}^* , the space of differential k -forms on $\Gamma \backslash S$ can be identified with $\text{Hom}_K(\wedge^k \mathfrak{p}, C^\infty(\Gamma \backslash G))$. The corresponding complex computes the (\mathfrak{g}, K) -cohomology of $C^\infty(\Gamma \backslash G)$ – the subspace of smooth vectors in the right (quasi-)regular representation of G in $L^2(\Gamma \backslash G)$. One similarly defines the (\mathfrak{g}, K) -cohomology groups $H^\bullet(\mathfrak{g}, K; \pi)$ of any unitarizable (\mathfrak{g}, K) -module (π, V_π) . By a theorem of Harish-Chandra, the set of equivalence classes of irreducible unitarizable (\mathfrak{g}, K) -modules is naturally identified with the set of equivalence classes of irreducible unitary representations of G . In the following, we will abusively use the same notation to denote an irreducible unitary representation and its associated (\mathfrak{g}, K) -module of smooth vectors.

The decomposition of $\wedge^\bullet \mathfrak{p}^*$ into irreducible K -modules induces a decomposition of the exterior algebra $\wedge^\bullet T^*(\Gamma \backslash S) = \Gamma \backslash G \times_K \wedge^\bullet (\mathfrak{p}^*)$. This decomposition commutes with the action of the Hodge-Laplace operator, giving birth to a decomposition of the cohomology $H^\bullet(\Gamma \backslash S, \mathbb{C})$ which refines the Hodge decomposition if S is Hermitian symmetric and gives an analogous decomposition of the cohomology in the case S is not Hermitian. In both cases we will call this decomposition of $H^\bullet(\Gamma \backslash S, \mathbb{C})$ the *generalized Hodge decomposition*; it is better understood in terms of cohomological representations through Matsushima's formula:

$$(3-2) \quad H^\bullet(\Gamma \backslash S, \mathbb{C}) = \bigoplus_{\pi} m(\pi, \Gamma) H^\bullet(\mathfrak{g}, K; \pi).$$

Here the (finite) sum is over (classes of) irreducible unitary representations of G such that $H^\bullet(\mathfrak{g}, K; \pi) \neq 0$ and $m(\pi, \Gamma)$ is the (finite) multiplicity with which π occurs in the quasi-regular representation $L^2(\Gamma \backslash G)$.

Cohomological representations of G are classified in terms of the θ -stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}$. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the θ -stable Levi decomposition of \mathfrak{q} . We have $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{k} \oplus \mathfrak{u} \cap \mathfrak{p}$. Put $R = \dim(\mathfrak{u} \cap \mathfrak{p})$. The line $\wedge^R(\mathfrak{u} \cap \mathfrak{p})$ generates an irreducible representation $V(\mathfrak{q})$ of K in $\wedge^R \mathfrak{p}$.

The classification of unitary irreducible cohomological representations of G associates to each θ -stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}$ a cohomological representation $A_{\mathfrak{q}}$ characterized by the property that the only irreducible K -representation common to $\wedge^\bullet \mathfrak{p}$ and $A_{\mathfrak{q}}$ is the representation $V(\mathfrak{q})$. Moreover, every cohomological representation is an $A_{\mathfrak{q}}$, see [Vogan and Zuckerman \[1984\]](#).

Each $H^\bullet(\mathfrak{g}, K; A_{\mathfrak{q}})$ identifies with the cohomology – with degree shifted by R – of the *compact* symmetric space associated to a subgroup $L \subset G^c$ with complexified Lie algebra \mathfrak{l} . In particular, the component corresponding to the trivial representation of G in (3-2) is isomorphic to $H^\bullet(S^c, \mathbb{C})$. In the Hermitian case we recover Hirzebruch proportionality principle.

If ω belongs to $H^R(\Gamma \backslash S, \mathbb{C})$ and, under the Matsushima decomposition (3-2), lies in the component corresponding to some $A_{\mathfrak{q}}$ with $R = \dim(\mathfrak{u} \cap \mathfrak{p})$, by analogy with the notion of primitive class in the Hodge-Lefschetz decomposition, we refer to ω as a *strongly primitive* class of type $A_{\mathfrak{q}}$.

3.3 Two families of examples.

3.3.1 Compact quotients of the symmetric space associated to $\mathrm{PU}(p, q)$. Then the holonomy group is contained in $U_p \times U_q$ and S^c is the complex Grassmannian $\mathrm{Gr}_p(\mathbb{C}^{p+q})$

of p -planes in \mathbb{C}^{p+q} . We first consider the decomposition of $\wedge^\bullet \mathfrak{p}^*$ into irreducible K -modules. The symmetric space S being of Hermitian type, the exterior algebra $\wedge^\bullet \mathfrak{p}$ decomposes as:

$$(3-3) \quad \wedge^\bullet \mathfrak{p} = \wedge^\bullet \mathfrak{p}' \otimes \wedge^\bullet \mathfrak{p}''$$

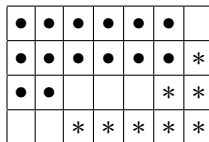
where \mathfrak{p}' and \mathfrak{p}'' respectively denote the holomorphic and anti-holomorphic tangent spaces. In the case $q = 1$ – then S is the complex ball of dimension p – it is an exercise to check that the decomposition of (3-3) into irreducible modules recovers the usual Lefschetz decomposition. But, in general, the decomposition is much finer, and it is hard to write down the full decomposition of (3-3) into irreducible modules. Indeed: as a representation of $GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ the space \mathfrak{p}' is isomorphic to $V_+ \otimes V_-^*$ where $V_+ = \mathbb{C}^p$ (resp. $V_- = \mathbb{C}^q$) is the standard representation of $GL_p(\mathbb{C})$ (resp. $GL_q(\mathbb{C})$) and the decomposition of $\wedge^\bullet \mathfrak{p}'$ is already quite complicated (see [Fulton \[1997, Equation \(19\), p. 121\]](#)):

$$(3-4) \quad \wedge^R (V_+ \otimes V_-^*) \cong \bigoplus_{\lambda \vdash R} S_\lambda(V_+) \otimes S_{\lambda^*}(V_-)^*.$$

Here we sum over all partitions of R (equivalently Young diagrams of size $|\lambda| = R$) and λ^* is the conjugate partition (or transposed Young diagram).

However, the classification of cohomological representations we just alluded to implies that very few of the irreducible submodules of $\wedge^\bullet \mathfrak{p}^*$ can occur as refined Hodge types of non-trivial cohomology classes. This is very analogous to the Kodaira vanishing theorem. The proof indeed makes a crucial use of a ‘Dirac inequality’ due to Parthasarathy, see [Borel and Wallach \[2000, Lemma II.6.11 and §II.7\]](#). The vanishing theorem thus obtained generalizes a celebrated result of [Matsushima \[1962\]](#).

The K -types $V(\mathfrak{q})$ that can occur are determined by *admissible* pairs of partitions (λ, μ) i.e. partitions λ and μ as in (3-4) and such that if λ (resp. μ) is on the top left (resp. bottom right) corner of the rectangle $p \times q$ as pictured below (with $p = 4, q = 7, \lambda = (6, 6, 2, 0)$ and $\mu = (5, 2, 1, 0)$), the complementary boxes form a disjoint union of rectangles $p_1 \times q_1 \cup \dots \cup p_r \times q_r$ (in the example below $1 \times 2 \cup 1 \times 3 \cup 1 \times 1$), see [Bergeron \[2009, Lemme 6\]](#).



We denote by $V(\lambda, \mu)$ the corresponding K -type. In particular the K -module $V(\lambda) := V(\lambda, 0)$ is isomorphic to $S_\lambda(V_+) \otimes S_{\lambda^*}(V_-)^*$. In general $V(\lambda, \mu)$ is isomorphic to the Cartan product of $V(\lambda)$ and $V(\mu)^*$. The first degree where such a K -type can occur in

the cohomology is $R = |\lambda| + |\mu|$. More precisely, it contributes to the cohomology of bi-degree $(|\lambda|, |\mu|)$ in the Hodge-Lefschetz decomposition and we have:

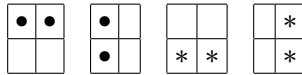
$$H^\bullet(\mathfrak{g}, K; A_{\mathfrak{q}}) \cong H^{\bullet-R}(\mathrm{Gr}_{p_1}(\mathbb{C}^{p_1+q_1}) \times \dots \times \mathrm{Gr}_{p_r}(\mathbb{C}^{p_r+q_r}), \mathbb{C}).$$

Matsushima’s formula (3-2) then strongly refines the Hodge-Lefschetz decomposition of compact quotients $\Gamma \backslash S$.

Example. Take $p = 2$ and $q = 2$. Compact quotients $\Gamma \backslash S$ are 4-dimensional complex manifolds. Their Betti numbers satisfy the relation $b_k = b_{8-k}$ because of Poincaré duality. They moreover decompose as sums $b_k = \sum_{p+q=k} h^{p,q}$ of Hodge numbers that satisfy $h^{p,q} = h^{q,p}$. But more is true: the vector (b_0, \dots, b_8) of Betti numbers of a compact quotient $\Gamma \backslash S$ is actually of the form

$$(3-5) \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 2h^{2,0} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (h^{1,1} - 1) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2(h^{3,0} + h^{2,1}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for some integer $k \geq 0$. The first vector indeed corresponds to the component of the trivial representation in Matsushima’s formula. The second term corresponds to the components of the cohomological representations $A_{\mathfrak{q}}$ with $R = 2$ that contribute either to the holomorphic or anti-holomorphic cohomology. Their associated pairs of partitions (λ, μ) are



The third term corresponds to the components of the (unique) cohomological representation $A_{\mathfrak{q}}$ with $R = 2$ that contributes to the cohomology of bi-degree $(1, 1)$. Its associated pair of partitions (λ, μ) is



And so on...

3.3.2 Compact quotients of the symmetric space associated to $\mathrm{SO}_0(p, q)$. Even though these are not Hermitian in general, Matsushima’s formula still makes sense. Considerations, similar to those in the unitary case, show that cohomological representations of G are

essentially¹ parametrized by partitions $\lambda = (\lambda_1, \dots, \lambda_p)$, with $q \geq \lambda_1 \geq \dots \geq \lambda_p \geq 0$, such that the pair (λ, λ) is admissible.

Example. Take $p = 5$ and $q = 4$. Compact quotients $\Gamma \backslash S$ are 20-dimensional real manifolds. Their Betti numbers satisfy the relation $b_k = b_{20-k}$ because of Poincaré duality. But more is true: the Betti numbers of a compact quotient $\Gamma \backslash S$ actually verify the relations

$$(3-6) \quad b_1 = b_2 = b_3 = 0, \quad b_8 \geq 2b_6 \quad \text{and} \quad b_{10} \geq 3b_6.$$

Hyperbolic manifolds of dimension n correspond to $p = n$ and $q = 1$. Then Matsushima's formula essentially gives no restrictions on the Betti numbers.²

Among the family of symmetric spaces associated to $SO_0(p, q)$, the ones where $q = 2$ are – up to exchanging the roles of p and q – the only Hermitian spaces; these are of complex dimension $n = p$. In these cases, $K \subset O_n \times O_2$ acts on $\mathfrak{p} = \mathbb{C}^n \otimes (\mathbb{C}^2)^*$ through the standard representation of O_n on \mathbb{C}^n and the standard representation of O_2 on \mathbb{C}^2 . Denote by \mathbb{C}^+ and \mathbb{C}^- the \mathbb{C} -span of the vectors $e_1 + ie_2$ and $e_1 - ie_2$ in \mathbb{C}^2 . The two lines \mathbb{C}^+ and \mathbb{C}^- are left stable by O_2 . This yields a decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ which corresponds to the decomposition given by the natural complex structure on \mathfrak{p}_0 . For each non-negative integer k the K -representation $\wedge^k \mathfrak{p} = \wedge^k (\mathfrak{p}^+ \oplus \mathfrak{p}^-)$ decomposes as the sum:

$$\wedge^k \mathfrak{p} = \bigoplus_{r+s=k} \wedge^r \mathfrak{p}^+ \otimes \wedge^s \mathfrak{p}^-.$$

The K -representations $\wedge^r \mathfrak{p}^+ \otimes \wedge^s \mathfrak{p}^-$ are not irreducible in general: there is at least a further splitting given by the Lefschetz decomposition:

$$\wedge^r \mathfrak{p}^+ \otimes \wedge^s \mathfrak{p}^- = \bigoplus_{\ell=0}^{\min(r,s)} \tau_{r-\ell, s-\ell}.$$

One can check that for $2(r+s) < n$ each K -representation $\tau_{r,s}$ is irreducible. Moreover in the range $2(r+s) < n$ only those with $r = s$ can occur as a K -type $V(\mathfrak{q})$ associated to a cohomological representation. One can moreover check that each $\tau_{r,r}$ is irreducible as long as $r < n$; it is isomorphic to some $V(\mathfrak{q})$ and corresponds to the partition $\lambda = (2r, 0_{n-r})$.³ Let us denote by $A_{r,r}$ the corresponding cohomological representation. We have:

$$H^{i,j}(\mathfrak{g}, K; A_{r,r}) = \begin{cases} \mathbb{C} & \text{if } r \leq i = j \leq n - r, 2i \neq n \\ \mathbb{C} + \mathbb{C} & \text{if } 2i = 2j = n \\ 0 & \text{otherwise.} \end{cases}$$

¹This is completely true only if both p and q are odd.

²To be precise it gives no restriction at all if n is odd and one recovers that $b_{n/2}$ is even if n is even.

³When $2(r+s) \geq n$ the partitions $(2r, 1_s, 0_{n-r-s})$ also correspond to cohomological representations.

In the particular case where n is even and $(r, r) = (0, 0)$ – so that $A_{0,0}$ is the trivial representation – we have $H^\bullet(\mathfrak{g}, K; A_{0,0}) = H^\bullet(S^c, \mathbb{C})$, where $S^c = \mathrm{SO}(n+2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$ is the complex quadric. The space $H^\bullet(S^c, \mathbb{C})$ has a basis $\{1, c_1, c_1^2, \dots, c_1^{n-1}, e\}$, where c_1 is the Chern class of the complexification of the line bundle arising from the standard representation of $\mathrm{SO}(2)$, i.e. the Kähler form on S^c , and where e is the Euler class of the vector bundle arising from the standard representation of $\mathrm{SO}(n)$.

4 Betti numbers of locally symmetric manifolds

One may wonder:

what are the Betti numbers of a random locally symmetric space ?

A classical theorem of Gromov (see [Ballmann, Gromov, and Schroeder \[1985\]](#)) bounds from above the Betti numbers of a locally symmetric space by a constant (depending only of the dimension) times its volume. It is therefore natural to investigate the growth of the Betti numbers as the volume tends to infinity. The analogous question for complex hypersurfaces in \mathbb{P}^{n+1} is classical.

4.1 Comparison with projective hypersurfaces. The fundamental projective invariant of an n -dimensional algebraic variety $V \subset \mathbb{P}^N$ is its degree d which is also equal to the volume – with respect to the standard Kähler form on \mathbb{P}^N – divided by $n!$.

In case $V \subset \mathbb{P}^{n+1}$ is an hypersurface, by standard arguments involving Lefschetz Hyperplane Theorem and Poincaré duality (see e.g. [Gayet and Welschinger \[2014, Lemma 3\]](#)), we have $b_k(V) = b_k(\mathbb{P}^n)$ for $k \neq n$. On the other hand, the Euler-Poincaré characteristic of V is equal to

$$\chi(V) = \langle c_n(T_V), [V] \rangle = \frac{1}{d} [(1-d)^{n+2} - 1] + n + 2.$$

It follows that the growth of the Betti numbers of V with respect to the degree d is given by

$$(4-1) \quad b_k(V) = \begin{cases} O(1) & \text{if } k \neq n \\ (-1)^n \chi(V) + O(1) = d^{n+1} + O(d^n) & \text{if } k = n. \end{cases}$$

4.2 Asymptotics of Betti numbers of locally symmetric manifolds. It is not obvious at all that large volume locally symmetric S -manifolds should have related topological behavior. However, one consequence of [Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet \[2017\]](#) and [Abert, Bergeron, Biringer, and Gelander \[n.d.\]](#) is the following theorem that is analogous to (4-1).

Theorem 1. *Suppose that G has property (T) and rank at least two. The growth of the Betti numbers of locally symmetric S -manifolds is given by*

$$b_k(\Gamma \backslash S) = \begin{cases} o(\text{vol}(\Gamma \backslash S)) & \text{if } k \neq \frac{1}{2} \dim S \\ \frac{\chi(S^c)}{\text{vol}(S^c)} \text{vol}(\Gamma \backslash S) + o(\text{vol}(\Gamma \backslash S)) & \text{if } k = \frac{1}{2} \dim S. \end{cases}$$

Example. Let $n \geq 3$ and let (Γ_m) be a sequence of distinct torsion-free lattices in $\text{SL}_n(\mathbb{R})$. Then for all k , we have $b_k(\Gamma_m) = o(\text{vol}(\Gamma_m \backslash \text{SL}_n(\mathbb{R})))$ as $m \rightarrow +\infty$.

Hyperbolic spaces have a rank one group of isometries and it is not hard to construct examples of large volume hyperbolic manifolds with very different topologies, see e.g. [Bergeron \[2017\]](#) for many examples. This allows in particular to construct counter-examples to the conclusion of [Theorem 1](#). However recent works of [Fraczyk \[2016\]](#) and [Fraczyk and Raimbault \[n.d.\]](#) imply that this conclusion holds for *congruence arithmetic* hyperbolic manifolds. More generally they prove:

Theorem 2. *Let S be arbitrary. The growth of the Betti numbers of congruence arithmetic S -manifolds is given by*

$$b_k(\Gamma \backslash S) = \begin{cases} o(\text{vol}(\Gamma \backslash S)) & \text{if } k \neq \frac{1}{2} \dim S \\ \frac{\chi(S^c)}{\text{vol}(S^c)} \text{vol}(\Gamma \backslash S) + o(\text{vol}(\Gamma \backslash S)) & \text{if } k = \frac{1}{2} \dim S. \end{cases}$$

Outside the middle degree it is hard to guess what should be the ‘true’ growth rate of the Betti numbers. For congruence arithmetic real hyperbolic manifolds $\Gamma \backslash \mathbb{H}^n$, associated to a fixed rational group \mathbf{H} (see [Section 2.2](#)), it was suggested by Gromov (see [Sarnak and Xue \[1991\]](#)) that

$$(4-2) \quad b_k(\Gamma \backslash \mathbb{H}^n) \ll_{\mathbf{H}, \varepsilon} \text{vol}(\Gamma \backslash \mathbb{H}^n)^{\frac{2k}{n-1} + \varepsilon}.$$

[Cossutta and Marshall \[2013\]](#) suggest – and actually prove in a quite general situation – that the best exponent is in fact $2j/n$ as long as $k \neq (n \pm 1)/2$. See [Marshall \[2014\]](#) for similar results on other classes of symmetric spaces.

Remark. In a way that is quite similar to Bismut’s proof of Demailly’s asymptotic Morse inequalities (see [Bismut \[1987\]](#) and [Demailly \[1985\]](#)) for projective varieties, the existence of an upper bound sublinear in the volume is related (see e.g. the influential [Sarnak and Xue \[1991\]](#)) to the existence of a *spectral gap* for the Hodge-Laplace operator acting on differential k -forms with $k \neq (n \pm 1)/2$:

Theorem 3. *Let k be different from $(n \pm 1)/2$. There exists a positive constant $\varepsilon = \varepsilon(n, k)$ such that for any congruence arithmetic real hyperbolic manifolds $\Gamma \backslash \mathbb{H}^n$, the first non-zero eigenvalue of the Hodge-Laplace operator of $\Gamma \backslash \mathbb{H}^n$ acting on differentiable k -forms is bounded below by ε .*

This was conjectured in [Bergeron and Clozel \[2005\]](#) and proved in [Bergeron and Clozel \[2013\]](#).

When $k = (n \pm 1)/2$ there is no spectral gap – the corresponding cohomological representation of G is ‘tempered’ – and we do not know what to expect for the growth of $b_k(\Gamma \backslash \mathbb{H}^n)$. However for particular sequences of Γ ’s, [Calegari and Emerton \[2009\]](#) were able to prove an upper bound sublinear in the volume, see also [Bergeron, Linnell, Lück, and Sauer \[2014\]](#).

4.3 Explicit computations. Matsuhima’s formula and the classification of cohomological representations imply many restrictions on the Betti numbers (e.g. in small degree some vanishing results or equality with the corresponding Betti numbers of S^c). Apart from these restrictions, explicit computations of the Betti numbers of a fixed locally symmetric space $\Gamma \backslash S$ in terms of the algebraic data defining Γ is usually a challenge. Very few cases are known. One of the first results of this type is the computation, by [J.-S. Li \[1996\]](#), of the dimension of the L^2 -cohomology space of degree g of certain congruence arithmetic quotients of the Siegel upper half space of genus g .

The proof is divided into two parts. First, relying on previous works of Howe, Jian-Shu Li proves that the cohomology is generated by certain theta series. Then he computes the dimension of the space generated by these theta series. More recently in [Bergeron, Millson, and Moeglin \[2017\]](#) and [Bergeron, Z. Li, Millson, and Moeglin \[2017\]](#) we were able to prove that a large part of the cohomology of certain locally symmetric spaces associated to $\mathrm{SO}_0(n, 2)$ is generated by certain theta series. Using previous computations by [Bruinier \[2002\]](#) of dimensions of the spaces generated by these theta series we get explicit expressions for certain Betti numbers. We prove in particular:

Theorem 4. *The rank of the Picard group of the moduli space \mathcal{K}_g , defined in [Section 2.3](#), is*

$$\frac{31g + 24}{24} - \frac{1}{4}\alpha_g - \frac{1}{6}\beta_g - \sum_{k=0}^{g-1} \left\{ \frac{k^2}{4g-4} \right\} - \#\left\{ k \mid \frac{k^2}{4g-4} \in \mathbb{Z}, 0 \leq k \leq g-1 \right\}$$

where

$$\alpha_g = \begin{cases} 0, & \text{if } g \text{ is even,} \\ \left(\frac{2g-2}{2g-3} \right) & \text{otherwise,} \end{cases} \quad \beta_g = \begin{cases} \left(\frac{g-1}{4g-5} \right) - 1, & \text{if } g \equiv 1 \pmod{3}, \\ \left(\frac{g-1}{4g-5} \right) + \left(\frac{g-1}{3} \right) & \text{otherwise,} \end{cases}$$

and $\left(\frac{a}{b} \right)$ is Jacobi symbol.

5 Cycle theory

Let M be a (closed) manifold. So far, we have computed the cohomology of M using smooth differential forms. We could as well have used currents. The resulting cohomology

groups $H^k(M, \mathbb{C})$ are the same (and similarly for the groups occurring in the Hodge-Lefschetz or Matsushima decompositions). If Z is a closed orientable submanifold of real co-dimension k , it is an integral cycle and, by Poincaré duality, it defines a class $\text{cl}(Z)$ in $H^k(M, \mathbb{C})$. The integration current on Z is closed of degree k and represents the image of $\text{cl}(Z)$ in $H^k(M, \mathbb{C})$.

By a classical theorem of Thom, any class in the rational cohomology groups $H^k(M, \mathbb{Q})$ is a rational multiple of the cycle class $\text{cl}(Z)$ of a (maybe disconnected) co-dimension k closed submanifold. When M is locally symmetric, it is natural to ask if one can restrict our choices of closed submanifold, e.g. to certain locally symmetric subspaces associated to subgroups $H \subset G$.

5.1 Comparison with projective varieties. In case M is a projective non-singular algebraic variety $V \subset \mathbb{P}^N$ over \mathbb{C} , it is natural to restrict to closed analytic subspaces $Z \subset V$, or equivalently, by Chow's theorem, to algebraic cycles. Let p be the complex co-dimension of Z in V . Two analytic subvarieties of complementary dimension meeting in isolated points have a non-negative local intersection number. Since we can find a linear subspace \mathbb{P}^{N-p} in \mathbb{P}^N meeting V in isolated points, it follows that the cycle class $\text{cl}(Z)$ is non-zero in $H^{2p}(V, \mathbb{C})$. Now the integration current on Z is closed of type (p, p) . The class $\text{cl}(Z)$ in $H^{2p}(V, \mathbb{C})$ is hence of type (p, p) . Rational (p, p) -classes are called *Hodge classes*. They form the group $\text{Hdg}^p(V, \mathbb{Q}) = H^{2p}(V, \mathbb{Q}) \cap H^{p,p}(V)$, and Hodge posed the famous:

Hodge Conjecture. On a projective non-singular algebraic variety over \mathbb{C} , any Hodge class is a rational linear combination of cycle classes $\text{cl}(Z)$ of algebraic cycles.

Hodge also proposed a further conjecture, characterizing the subspace of $H^\bullet(V, \mathbb{Q})$ spanned by the images of cohomology classes with support in a suitable closed analytic subspace of complex codimension k . Grothendieck observed that this further conjecture is false, and gave a corrected version of it in [Grothendieck \[1969\]](#).

5.2 Modular and symmetric cycle classes. Let us come back to locally symmetric manifolds $\Gamma \backslash S$. To any connected center-free semi-simple closed subgroup $H \subset G$ corresponds an embedding of the symmetric space S_H associated to H into S . If $\Gamma \cap H$ is a lattice, the inclusion $S_H \hookrightarrow S$ induces an immersion of real analytic varieties $(\Gamma \cap H) \backslash S_H \rightarrow \Gamma \backslash S$ whose associated cycle class in $H^\bullet(\Gamma \backslash S, \mathbb{C})$ we denote by C_H^Γ . We will refer to these classes as *modular classes*. When $\Gamma \backslash S$ is non-compact, C_H^Γ is sometimes compactified to give a cycle on a natural compactification of $\Gamma \backslash S$ but we won't discuss these issues here.

Examples. 1. When S is the real hyperbolic n -space \mathbb{H}^n , the modular classes in $\Gamma \backslash \mathbb{H}^n$ are the cycle classes of totally geodesic immersed submanifold of finite volume.

2. In complex ball quotients, cycle classes of finite volume quotients of sub-balls give examples of modular classes. These are the only modular classes that are cycle classes of algebraic cycles but there might be other modular classes: to the inclusion $\mathrm{SO}_0(n, 1) \subset \mathrm{PU}(n, 1)$ corresponds a totally real geodesic embedding of the real hyperbolic n -space into the complex n -ball that may projects onto a non-zero modular classes.

3. The moduli space \mathcal{K}_g of genus g quasi-polarized K3 surfaces – that identifies with a locally symmetric space associated to $G = \mathrm{SO}_0(2, 19)$ – can have arbitrarily large Picard group (see [Theorem 4](#)) and, more generally, many classes of cycles in their Chow groups. In particular there are many cycles coming from Noether-Lefschetz theory: the locus parametrizing the K3 surfaces with Picard number strictly greater than some positive integer $r \leq 19 = \dim_{\mathbb{C}} \mathcal{K}_g$ is indeed a countable union of subvarieties of co-dimension r . The cycle classes of the irreducible components of this locus are modular classes associated to subgroups $H \subset G$ isomorphic to $\mathrm{SO}_0(2, 19 - r)$. As in the case of ball quotients there are also non-algebraic modular classes.

There are a number of results on modular classes, but our current knowledge nevertheless appears to be quite poor: a large part of the literature on modular classes is only concerned in establishing the *non-vanishing* of these classes. As in the case of analytic subspaces of projective varieties, this has been addressed using the intersection numbers of these cycles, see e.g. [Millson \[1976\]](#), [Millson and Raghunathan \[1980\]](#), and [Kudla and Millson \[1990\]](#). This has also been addressed using tools coming from representation theory, see especially [Tong and Wang \[1989\]](#) and [J.-S. Li \[1992\]](#). This non-vanishing question is usually too hard to study for a given manifold; one simplifies the problem by ‘stabilizing’ it, that is to say by considering towers of finite coverings rather than a single manifold. Let us say that a modular class C_H^Γ is *virtually non-zero* if there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that the modular class $C_{H'}^{\Gamma'}$ is non-zero in $H^\bullet(\Gamma' \backslash S, \mathbb{C})$.

The following conjecture – see [Bergeron \[2006\]](#) for more details (in particular with respect to non-compact quotients $\Gamma \backslash S$) – provides a quite general answer to the question of the virtual non-vanishing of modular classes. To our knowledge this conjecture encompasses all known results. It has been (or can be) checked in most classical situations (see especially [Bergeron \[2006\]](#), [Bergeron \[2008\]](#), and [Bergeron and Clozel \[2013, 2017\]](#)). To formulate it, let us first distinguish some particular modular symbols. Say that a closed subgroup $H \subset G$ is a symmetric subgroup of G if there exists an involution τ of G such that $H = G^\tau$ is the connected component of the identity in the group of fixed points of τ . We will refer to the corresponding modular classes C_H^Γ as *symmetric modular classes*. All the modular classes from the examples above are symmetric.

Conjecture 5. *Assume for simplicity that $\Gamma \backslash S$ is compact. A symmetric modular class C_H^Γ is virtually non-zero in the strongly primitive part of the cohomology of degree $\dim S - \dim S_H$ if and only if $\text{rank}_{\mathbb{C}}(G/H) = \text{rank}_{\mathbb{C}}(K/(K \cap H))$.*

5.3 Hodge types of modular classes. Keeping in mind the analogy with projective varieties, the next step is to determine on which Hodge types of the cohomology of $\Gamma \backslash S$ modular classes can project non-trivially. Up to now it seems to have been addressed only in few particular cases. As explained in his 2002 ICM talk [Kobayashi \[2002\]](#), jointly with Oda, Kobayashi has devised a sufficient criterion for a modular class to be annihilated by a π -component in Matsushima’s decomposition (3-2). Their proof is based on a theory of discrete branching laws for unitary representations of G . The most interesting cases they can deal with are compact quotients of the symmetric space $S = \text{SO}_0(2n, 2)/\text{SO}_{2n} \times \text{SO}_2$. If $\Gamma \backslash S$ is a compact S -manifold, the contribution of the trivial representation of G to Matsushima’s formula (3-2) yields a natural injective map of cohomology groups $H^\bullet(S^c, \mathbb{C}) \subset H^\bullet(\Gamma \backslash S, \mathbb{C})$; in particular we shall see the Euler class $e \in H^{n,n}(S^c, \mathbb{C})$, defined in [Section 3.3.2](#), as an element in $H^{n,n}(\Gamma \backslash S, \mathbb{C})$. Kobayashi and Oda then prove:

Theorem 6. *The Hodge (n, n) -type component of a modular class C_H^Γ with $H \cong \text{SO}_0(2n, 1)$ is proportional to the Euler class e .*

These modular classes are cycle classes of totally real, totally geodesic submanifolds of real dimension $2n$ into $\Gamma \backslash S$ which is a Kähler manifold of complex dimension $2n$. In case $n = 1$ the space S is a product $\mathbb{H}^2 \times \mathbb{H}^2$ and the cycles derived from H are obtained by ‘partial complex conjugation’ of algebraic cycles with respect to the complex conjugation on the second factor of S . Then [Theorem 6](#) is equivalent to the well-known fact that the cycle class of a closed analytic (complex) co-dimension 1 subspace in a compact algebraic surface over \mathbb{C} has no Hodge $(2, 0) + (0, 2)$ -type components.

Beside the representation theoretic method of Kobayashi and Oda, a classical work of [Kudla and Millson \[1990\]](#) suggests another approach. Kudla and Millson indeed provide explicit dual forms to some natural modular classes in locally symmetric spaces associated to classical groups. From this, one can derive serious restrictions on the possible Hodge types to which these modular classes can contribute. Let’s describe the two main families of examples.

5.3.1 Quotients of the symmetric space associated to $\text{PU}(p, q)$. Let notations be as in [Section 3.3.1](#). Let $c_q \in H^{q,q}(S^c, \mathbb{C})$ be the top Chern class of the q -dimensional vector bundle over $S^c = \text{U}_{p+q}/\text{U}_p \times \text{U}_q$ associated to the standard representation of U_q , i.e. the q -th power of the Kähler form of S^c . Here again if $\Gamma \backslash S$ is a compact S -manifold, the contribution of the trivial representation of G to Matsushima’s formula (3-2) yields

a natural injective map of cohomology groups $H^\bullet(S^c, \mathbb{C}) \subset H^\bullet(\Gamma \backslash S, \mathbb{C})$ and we shall see the Chern class $c_q \in H^{q,q}(S^c, \mathbb{C})$ as an element in $H^{q,q}(\Gamma \backslash S, \mathbb{C})$. Wedging with c_q corresponds to applying the q -th power of the Lefschetz operator associated to the Kähler form on $\Gamma \backslash S$, and we define the subset $SH^\bullet(\Gamma \backslash S, \mathbb{C})$ of *special cohomology classes* in $H^\bullet(\Gamma \backslash S, \mathbb{C})$ by

$$(5-1) \quad SH^\bullet(\Gamma \backslash S, \mathbb{C}) = \bigoplus_{a,b=0}^p \bigoplus_{k=0}^{\min(p-a,p-b)} c_q^k H^{a \times q, b \times q}(\Gamma \backslash S, \mathbb{C}),$$

where $H^{a \times q, b \times q}(\Gamma \backslash S, \mathbb{C})$ denotes the generalized Hodge subspace of the cohomology corresponding to the pair of partitions (λ, μ) with λ an a by q rectangle and μ a b by q rectangle. As with the usual Hodge-Lefschetz decomposition, we have:

$$SH^\bullet(\Gamma \backslash S, \mathbb{C}) = \bigoplus_{a,b=0}^p SH^{aq,bq}(\Gamma \backslash S, \mathbb{C})$$

where the (usual) *primitive* part of the subspace $SH^{aq,bq}(\Gamma \backslash S, \mathbb{C})$ is exactly $H^{a \times q, b \times q}(\Gamma \backslash S, \mathbb{C})$.

Now the proof of [Bergeron, Millson, and Moeglin \[2016, Theorem 8.2\]](#) implies the following:

Proposition 7. *Let r be a non-negative integer with $r \leq p$ and let C_H^Γ be a modular class in $H^{2rq}(\Gamma \backslash S, \mathbb{C})$ with $H \cong \text{PU}(p - r, q)$. Then C_H^Γ is an algebraic class and it is contained in $\text{SHdg}^r(\Gamma \backslash S, \mathbb{Q}) := SH^{rq,rq}(\Gamma \backslash S, \mathbb{C}) \cap H^{2rq}(\Gamma \backslash S, \mathbb{Q})$.*

5.3.2 Quotients of the symmetric space associated to $\text{SO}_0(p, q)$. Similarly and with notations as in [Section 3.3.2](#), any modular class C_H^Γ , with H isomorphic to a smaller orthogonal group fixing a positive subspace, is contained in

$$(5-2) \quad SH^\bullet(\Gamma \backslash S, \mathbb{C}) = \bigoplus_{r=0}^{\lfloor p/2 \rfloor} \bigoplus_{k=0}^{p-2r} e_q^k H^{r \times q}(\Gamma \backslash S, \mathbb{C}).$$

Here e_q is zero if q is odd and is the Euler class arising from the standard representation of SO_q if q is even. We then write $\text{SHdg}^r(\Gamma \backslash S, \mathbb{Q}) = SH^{rq}(\Gamma \backslash S, \mathbb{C}) \cap H^{rq}(\Gamma \backslash S, \mathbb{Q})$.

Examples 1. If $q = 1$ the space S is the p -dimensional hyperbolic real space and the subspace $SH^\bullet(\Gamma \backslash S, \mathbb{C})$ is in fact equal to the full cohomology group $H^\bullet(\Gamma \backslash S, \mathbb{C})$.

2. If $q = 2$ the space is Hermitian and we have:

$$SH^\bullet(\Gamma \backslash S, \mathbb{C}) = \bigoplus_{r=0}^p H^{r,r}(\Gamma \backslash S, \mathbb{C}).$$

Beware that in this case the Euler class e_2 is the class of the Kähler form that we denoted c_1 in [Section 3.3.2](#).

5.4 Hodge type theorems. Modular cycle classes belong to a subspace $\text{SHdg}^\bullet(\Gamma \backslash S, \mathbb{Q})$ of the full cohomology group $H^\bullet(\Gamma \backslash S, \mathbb{C})$. Everything is therefore in place to raise a question analogous to the Hodge Conjecture:

Do modular cycle classes span the subspace $\text{SHdg}^\bullet(\Gamma \backslash S, \mathbb{Q})$?

We shall see that it is too much to hope for in general, but surprisingly enough this is close to be true in several interesting cases. Let us again consider our two main families of examples. Both cases are dealt with in joint works with Millson and Moeglin. The proofs rely heavily on Arthur's classification [Arthur \[2013\]](#) of automorphic representations of classical groups which depends on the stabilization of the trace formula for disconnected groups discussed in Waldspurger's 2014 ICM talk [Waldspurger \[2014\]](#) and recently obtained by [Mœglin and Waldspurger \[n.d.\]](#).

5.4.1 A Hodge type theorem for quotients of the symmetric space associated to $\text{PU}(p, q)$. Even in the simple case where $p = 2$ and $q = 1$ – so that S is the complex 2-ball – it was proved by [Blasius and Rogawski \[2000\]](#) that there exist compact quotients $\Gamma \backslash S$ such that the space of Hodge $(1, 1)$ -classes is not spanned by modular classes. However, vaguely stated, the main result of [Bergeron, Millson, and Moeglin \[2016\]](#) asserts that for congruence arithmetic quotients $\Gamma \backslash S$ (with arbitrary p and q 's) the special cohomology $SH^n(\Gamma \backslash S, \mathbb{C})$ is generated, for n small enough, by cup products of three types of classes:

- classes in $SH^{q,q}(\Gamma \backslash S, \mathbb{C})$;
- holomorphic and anti-holomorphic special cohomology classes, i.e. classes in $SH^{\bullet,0}(\Gamma \backslash S, \mathbb{C})$ and $SH^{0,\bullet}(\Gamma \backslash S, \mathbb{C})$;
- modular cycle classes of [Section 5.3.1](#).

5.4.2 A Hodge type theorem for quotients of the symmetric space associated to $\text{SO}_0(p, q)$. In that case, vaguely stated, the main result of [Bergeron, Millson, and Moeglin \[2017\]](#) states that as long as r is less than $\frac{1}{2}p$ and $\frac{1}{3}(p + q - 1)$, the 'primitive' subspace $H^{r \times q}(\Gamma \backslash S, \mathbb{C})$ of $SH^{r,q}(\Gamma \backslash S, \mathbb{C})$, in the decomposition (5-2), is spanned by projections of modular cycle classes.

5.5 Applications.

5.5.1 . The most striking consequences of the above mentioned 'Hodge type theorems' concern the cases where S is a complex ball or a real hyperbolic space. Indeed, in these cases $SH^\bullet(\Gamma \backslash S, \mathbb{C}) = H^\bullet(\Gamma \backslash S, \mathbb{C})$ and we obtain the two following theorems.

We first have to define more precisely the congruence arithmetic locally symmetric space we deal with: let E be either a totally real number field or a totally imaginary quadratic extension of a totally real number field. In both cases we denote by F the maximal totally real subfield of E . Now let V be a E -vector space of dimension $n + 1 \geq 3$ and let $h : V \times V \rightarrow E$ be a Hermitian form with respect to the conjugation of E/F , such that h is of signature $(n, 1)$ at one real place of F and definite at the others. Let \mathbf{H} be the semi-simple algebraic \mathbb{Q} -group obtained from the algebraic F -group $\mathrm{SU}(h)$ by restriction of scalars. To any congruence subgroup in $\mathbf{H}(\mathbb{Q})$ we attach a congruence arithmetic quotient $\Gamma \backslash S$ where S is the real hyperbolic n -space, if $E = F$, and the complex n -ball, otherwise.

Theorem 8. *Suppose that $\Gamma \backslash S$ is a closed complex n -ball quotient and let $r \in [0, n] \setminus \frac{n}{3}, \frac{2n}{3}[$. Then every Hodge class in $H^{2r}(\Gamma \backslash S, \mathbb{Q})$ is algebraic.*

Remarks. 1. Beware that here modular cycle classes *do not* span, even in co-dimension 1. One has to consider arbitrary $(1, 1)$ -classes.

2. In small degree one can even confirm Hodge's generalized conjecture in its original formulation (with \mathbb{Q} coefficients).

Theorem 9. *Suppose that $\Gamma \backslash S$ is a real hyperbolic n -manifold. Then, for all $r < n/3$, the \mathbb{Q} -vector space $H^r(\Gamma \backslash S, \mathbb{Q})$ is spanned by classes of totally geodesic cycles.*

Remarks. 1. In [Bergeron, Millson, and Moeglin \[2017\]](#) we provide strong evidence that [Theorem 9](#) should not hold above the degree $n/3$.

2. When n is even, all congruence arithmetic real hyperbolic n -manifolds are of the simple type described above. However, when n is odd, there are other types of congruence arithmetic real hyperbolic n -manifolds. These *do not* contain totally geodesic immersed co-dimension 1 submanifolds. Still, they may have a non-zero first Betti number. [Theorem 9](#) therefore cannot hold for general (congruence arithmetic) hyperbolic manifolds.

5.5.2 . When $G = \mathrm{SO}_0(n, 2)$ the space S is Hermitian and our general 'Hodge type theorem' again specializes into new cases of the Hodge conjecture. Let us emphasize the even more special case of the moduli spaces \mathcal{K}_g (in which cases we have $n = 19$): a theorem of [Oguiso \[2009\]](#) indeed implies that any curve on \mathcal{K}_g meets some of the Noether-Lefschetz (NL) divisors described in Example (3) of [Section 5.2](#). So it is natural to ask whether the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{K}_g)$ of \mathcal{K}_g with rational coefficients is spanned by NL-divisors. This was conjectured to be true by Maulik and Pandharipande, see 'Noether-Lefschetz Conjecture' [Maulik and Pandharipande \[2013, Conjecture 3\]](#). More generally, one can extend this question to higher NL-loci on \mathcal{K}_g . In [Bergeron, Z. Li, Millson, and Moeglin \[2017\]](#), we prove:

Theorem 10. *For all $g \geq 2$ and all $r \leq 4$, the cohomology group $H^{2r}(\mathcal{K}_g, \mathbb{Q})$ is spanned by NL-cycles of codimension r . In particular (taking $r = 1$), $\text{Pic}_{\mathbb{Q}}(\mathcal{K}_g) \cong H^2(\mathcal{K}_g, \mathbb{Q})$ and the Noether-Lefschetz Conjecture holds on \mathcal{K}_g for all $g \geq 2$.*

Remark. Before Bergeron, Millson, and Moeglin [2017], He and Hoffman [2012] considered another interesting special case of our general ‘Hodge type theorem,’ that of smooth Siegel modular threefolds Y where $p = 3$ and $q = 2$. They prove that $\text{Pic}(Y) \otimes \mathbb{C} \cong H^{1,1}(Y)$ is generated by Humbert surfaces.

6 Automorphic Lefschetz properties

Almost 40 years ago Oda [1981] proved that the Albanese variety of a congruence arithmetic complex ball quotient is spanned by the Hecke translates of the Jacobian of a fixed Shimura curve. This implies a version of the Lefschetz Theorem on the injection of the cohomology to Shimura curves that can arguably be considered as the starting point of the analogy between locally symmetric spaces and projective varieties that we are discussing here. Since Oda’s pioneering work, a number of criteria have been developed to determine if some Hecke translate of a given cohomology class on a locally symmetric space restricts non-trivially to a given locally symmetric subspace. Venkataramana’s ICM 2010 talk Venkataramana [2010] was devoted to this subject. We shall therefore insist on results obtained since then.

6.1 Comparison with projective varieties. Let $V \subset \mathbb{P}^N$ be a projective non-singular algebraic variety and $V \cap H$ a hyperplane section of V . Then we have the

Lefschetz Hyperplane Theorem. *The restriction map*

$$(6-1) \quad H^i(V, \mathbb{Q}) \rightarrow H^i(V \cap H, \mathbb{Q})$$

is an isomorphism for $i \leq n - 2$ and injective for $i = n - 1$.

This theorem in fact contains two quite different statements:

1. the map (6-1) is injective for $i < n$,
2. the map $H_i(V \cap H, \mathbb{Q}) \rightarrow H_i(V, \mathbb{Q})$ is injective for $i \geq n$.

6.2 Restriction to special cycles. The Lefschetz Hyperplane Theorem applies to compact quotients $\Gamma \backslash S$ of Hermitian symmetric spaces but modular classes are not ample. However, one may consider all the translates of these modular classes under Hecke operators and ask for a weaker Lefschetz property for the collection of these Hecke translates.

And a large number of such ‘weak Lefschetz properties’ indeed hold even when the symmetric space S is not Hermitian, see e.g. [Bergeron \[2006\]](#). Since the general results require a rather forbidding amount of notation we restrict our discussion to the special situation of [Section 5.5.1](#).

In this situation, to any non-degenerate, indefinite, subspace $W \subset V$ defined over E we attach a special cycle $\Gamma_W \backslash S_W \rightarrow \Gamma \backslash S$. It is of real dimension $m[E : F]$. The following theorem is our analogue of the first part of Lefschetz Hyperplane Theorem.

Theorem 11. *For every $m < n$, there exist $(m + 1)$ -dimensional subspaces W_1, \dots, W_s in V such that the restriction map*

$$(6-2) \quad H^i(\Gamma \backslash S, \mathbb{Q}) \rightarrow \bigoplus_j H^i(\Gamma_{W_j} \backslash S_{W_j}, \mathbb{Q})$$

is injective for all $i < \frac{1}{2}m[E : F]$ and for $i = \frac{1}{2}m[E : F]$ if $\Gamma \backslash S$ is closed.

The theorem can be reformulated using Hecke correspondences: to any F -rational element g of the isometry group of h we may associate a finite correspondence $(\Gamma \cap g^{-1}\Gamma g) \backslash S \rightrightarrows \Gamma \backslash S$ where the first projection is the covering projection and the second projection is induced by the multiplication by g . Write $C_g^* : H^\bullet(\Gamma \backslash S, \mathbb{Q}) \rightarrow H^\bullet(\Gamma \cap g^{-1}\Gamma g \backslash S, \mathbb{Q})$ for the induced endomorphism. [Theorem 11](#) then says that if α is a non-zero class in $H^i(\Gamma \backslash S, \mathbb{Q})$ of degree $i < \frac{1}{2} \dim_{\mathbb{R}} S_W$, then there exists a g such that $C_g^*(\alpha)$ pulls back non-trivially to $\Gamma_W \backslash S_W$.

For compact ball quotients, the theorem is due to [Oda \[1981\]](#) in degree $i = 1$, and to [Venkataramana \[2001\]](#) – confirming a conjecture of [Harris and J.-S. Li \[1998\]](#) – for all degrees. The essential point (in the case $n = m + 1$) is that a linear combination of the divisors $\Gamma_{W_i} \backslash S_{W_i} \rightarrow \Gamma \backslash S$ gives a particular ample class, the hyperplane class in the canonical projective embedding of $\Gamma \backslash S$. The Lefschetz property then follows from the hard Lefschetz theorem. For non-compact ball quotients, one can combine Venkataramana’s idea with the study of compactifications, see [Nair \[2017\]](#). It may appear quite surprising that the theorem holds for *real* hyperbolic manifolds. This is again a topological consequence of the spectral gap [Theorem 3](#). This approach gives a unified proof of [Theorem 11](#), see [Bergeron and Clozel \[2013, 2017\]](#).

6.3 Another type of Lefschetz property. As for the second part of Lefschetz Hyperplane Theorem, let us mention the following homotopical analogue of it, see [Bergeron, Haglund, and Wise \[2011\]](#).

Theorem 12. *A closed arithmetic hyperbolic manifold $\Gamma \backslash \mathbb{H}^n$ virtually retracts onto any of its co-dimension 1 modular cycle.*

In other words: if $\Lambda \backslash \mathbb{H}^{n-1} \rightarrow \Gamma \backslash \mathbb{H}^n$ is a totally geodesic immersion and if we write $\iota : \Lambda \rightarrow \Gamma$ for the corresponding (injective) morphism, there exists a finite index subgroup $\Gamma' \subset \Gamma$ and a morphism $r : \Gamma' \rightarrow \Lambda$ such that $\iota(\Lambda)$ is contained in Γ' and $r \circ \iota : \Lambda \rightarrow \Lambda$ is the identity map. In particular the induced map

$$H_i(\Lambda \backslash \mathbb{H}^{n-1}, \mathbb{Q}) \rightarrow H_i(\Gamma' \backslash \mathbb{H}^n, \mathbb{Q})$$

is injective for all $i \geq 0$.

The proof of [Theorem 12](#) is very specific to arithmetic hyperbolic manifolds that contain co-dimension 1 modular cycles; it uses that the group Γ may be ‘cubulated’ (in the sense of [Wise \[2014\]](#)). It is a very interesting open question to decide which lattices of $SO_0(n, 1)$ can be cubulated, but it is known that lattices in all other real simple Lie groups cannot. Of course, this does not prevent [Theorem 12](#) to hold for other locally symmetric spaces, but to my knowledge no other examples are known to (homotopically) retract onto a locally symmetric proper subspace except for a small finite number of beautiful examples, due to [Deraux \[2011\]](#), of complex 2-ball and 3-ball quotients that retract onto one of their totally geodesic submanifolds.

However, thanks to spectral gap properties as in [Theorem 3](#), the homological consequences of [Theorem 12](#) are more tractable in general, see [Bergeron \[2006\]](#). In the special situation of [Section 5.5.1](#) one can for example prove the following analogue of the second aspect of Lefschetz Hyperplane Theorem, see [Bergeron and Clozel \[2013\]](#).

Theorem 13. *Suppose that $\Gamma \backslash S$ is closed. Let W be a subspace of V . There exists a finite index subgroup $\Gamma' \subset \Gamma$ such that the natural map*

$$(6-3) \quad H_i(\Gamma'_W \backslash S_W, \mathbb{Q}) \rightarrow H_i(\Gamma' \backslash S, \mathbb{Q})$$

is injective for all $i \geq \frac{1}{2} \dim_{\mathbb{R}} S$.

6.4 Some refined analogies with specific projective varieties: Abelian varieties. [Theorem 9](#) is an analogue, in constant negative curvature, of the classical fact that cycle classes of totally geodesic flat sub-tori span the cohomology groups of flat tori. In fact if A is an Abelian variety, in most interesting cohomology theories $H^\bullet(A)$ is an exterior algebra on $H^1(A)$. In particular, if A is sufficiently general, the algebra of Hodge classes is generated in degree 1 and the Hodge conjecture follows.

Quite surprisingly, in small degrees, the cohomology rings of congruence arithmetic locally symmetric manifolds enjoy structural properties very analogous to those of Abelian varieties. Here again we discuss only the special situation of [Section 5.5.1](#). Suppose furthermore that $\Gamma \backslash S$ is closed and write $\text{Hdg}^\bullet(\Gamma \backslash S, \mathbb{Q}) = H^\bullet(\Gamma \backslash S, \mathbb{Q})$ if $E = F$, i.e. if S is a real hyperbolic space \mathbb{H}^n . First, the proofs of [Theorems 8](#) and [9](#) imply:

Theorem 14. *The natural morphism of algebras*

$$(6-4) \quad \wedge^\bullet \mathrm{Hdg}^1(\Gamma \backslash S, \mathbb{Q}) \rightarrow \mathrm{Hdg}^\bullet(\Gamma \backslash S, \mathbb{Q})$$

is onto in degree $< n/3$.

As opposed to what happens with Abelian varieties, the map (6-4) is not injective in general (already when $\Gamma \backslash S$ is a real hyperbolic surface). The next theorem – see [Bergeron and Clozel \[2013, 2017\]](#) – nevertheless shows that it is injective ‘up to Hecke correspondences.’

Theorem 15. *Let α and β two cohomology classes in $H^\bullet(\Gamma \backslash S, \mathbb{Q})$ of respective degrees k and ℓ with $k + \ell \leq \frac{1}{2} \dim_{\mathbb{R}} S$. Then, there exists some rational element g of the isometry group of h such that*

$$C_g^*(\alpha) \wedge \beta \neq 0 \text{ in } H^{k+\ell}(M, \mathbb{Q}).$$

For complex ball quotients [Theorem 15](#) is due to [Venkataramana \[2001\]](#). [Parthasarathy \[1982\]](#), [Clozel \[1992, 1993\]](#) and [Venkataramana \[2010\]](#) have general results of this type for other Hermitian spaces, see also [Bergeron \[2004\]](#). In [Bergeron \[2006\]](#) we consider more general non Hermitian locally symmetric spaces. Here again the key input is a spectral gap theorem.

7 Periods

Algebraic varieties admit a panoply of cohomology theories, related over \mathbb{C} by comparison isomorphisms. These give rise to different structures on the cohomology groups. Comparing two such structures leads in particular to the rich theory of periods.

When dealing with general locally symmetric manifolds we don’t have all these cohomology theories at our disposal anymore. However, using the canonical Riemannian structure on S , we can extract some numerical invariants from the cohomology, which we call ‘period matrices.’

7.1 Comparison with projective varieties. If V is a smooth projective variety defined over \mathbb{Q} , the vector space $H^k(V, \mathbb{C})$ has a natural \mathbb{Q} -structure $H_{\mathrm{dR}}^k(X/\mathbb{Q})$: choose a cover of V by Zariski affine open sets defined over \mathbb{Q} and use algebraic differential forms with coefficients in \mathbb{Q} . A comparison theorem, due to Grothendieck, gives a natural isomorphism $H_{\mathrm{dR}}^k(X/\mathbb{Q}) \otimes \mathbb{C} \cong H^k(V, \mathbb{C})$.

One calls *periods* the matrix coefficients of the comparison isomorphism

$$H_{\mathrm{dR}}^k(X/\mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\cong} H^k(V, \mathbb{C}) \otimes \mathbb{C}$$

between algebraic de Rham cohomology and singular cohomology after choosing \mathbb{Q} -bases in both groups. In general these two different \mathbb{Q} -structures are transcendent with respect to each other and periods are fundamental numerical invariants, see e.g. [Kontsevich and Zagier \[2001\]](#).

7.2 Period matrices of locally symmetric spaces. Let us come back to locally symmetric manifolds $\Gamma \backslash S$. Through the isomorphism (3-1), the Riemannian structure on S induces a positive definite quadratic form on each cohomology group $H^j(\Gamma \backslash S, \mathbb{Z})$ modulo torsion. Letting $b = b_j(\Gamma \backslash S)$, we encode the above data into a matrix

$$M = \left(\int_{\gamma_k} \omega_\ell \right)_{1 \leq k, \ell \leq b} \in \text{GL}_b(\mathbb{R})$$

where the $\gamma_k \in H_j(\Gamma \backslash S, \mathbb{Z})$ project to a basis for $H_j(\Gamma \backslash S, \mathbb{Z})$ modulo torsion and the ω_ℓ 's are an orthonormal basis for the space of harmonic j -forms on $\Gamma \backslash S$. The matrix M is well-defined up to multiplication on the left by $\text{GL}_b(\mathbb{Z})$ and on the right by an orthogonal matrix.

As an element of $\text{GL}_b(\mathbb{Z}) \backslash \text{GL}_b(\mathbb{R}) / O_b$, the matrix M is characterized by its determinant and its image in the locally symmetric space that parametrizes the space of flat b -dimensional tori of unit volume. In analogy with the classical Schottky problem, it would be interesting to analyse the locus of the latter as Γ varies. Let us restrict our attention to the apparently simpler question of bounding the determinant.

7.3 Regulators. Following [Bergeron and Venkatesh \[2013\]](#) and [Bergeron, Şengün, and Venkatesh \[2016\]](#) we call ‘degree j regulator’ the determinant of the degree j period matrix of $\Gamma \backslash S$; we denote it by $R_j(\Gamma \backslash S)$.

Note that $|R_0(\Gamma \backslash S)| = 1/\sqrt{\text{vol}(\Gamma \backslash S)}$, $|R_n(\Gamma \backslash S)| = \sqrt{\text{vol}(\Gamma \backslash S)}$, and by Poincaré duality, we have $|R_j(\Gamma \backslash S)R_{n-j}(\Gamma \backslash S)| = 1$. We propose the following:

Conjecture 16. *Fix S and j . The growth of the degree j regulators of congruence arithmetic of S -manifolds is given by*

$$\log |R_j(\Gamma \backslash S)| = o(\text{vol}(\Gamma \backslash S)).$$

In the next paragraph we relate [Conjecture 16](#) to the geometric complexity of cycles needed to generate $H_j(\Gamma \backslash S, \mathbb{R})$. ‘Hodge type theorems’ like [Theorem 9](#) suggest that the conjecture could be tractable when j is far enough from the middle degree. In general one can think of [Conjecture 16](#) as an attempt to shed little light on the mysterious cycle theory of locally symmetric spaces near the middle degree.

7.4 Back to cycles. Our reason to believe in [Conjecture 16](#) is that, roughly speaking, we expect homology classes on congruence arithmetic manifolds to be represented by cycles of low complexity. In our general situation, these cycles are not algebraic at all but one may still hope that their *topological* complexity reflects the *arithmetic* complexity of their (Langlands-)associated varieties.

In [Bergeron, Şengün, and Venkatesh \[2016\]](#) we formulate and study the following precise conjecture in a simple interesting case, namely, that of congruence arithmetic hyperbolic 3-manifolds.

Conjecture 17. *There is an absolute constant C such that, for any congruence arithmetic hyperbolic 3-manifold $\Gamma \backslash \mathbb{H}^3$, there exist immersed surfaces S_i of genus less than $\text{vol}(\Gamma \backslash \mathbb{H}^3)^C$ such that the $[S_i]$'s span $H_2(\Gamma \backslash \mathbb{H}^3, \mathbb{R})$.*

To relate [Conjecture 17](#) with $R_2(\Gamma \backslash \mathbb{H}^3)$, we study the relationship between two norms on the second homology group: the purely topological Gromov-Thurston norm and the more geometric ‘harmonic’ norm. Refining [Bergeron, Şengün, and Venkatesh \[ibid., Proposition 4.1\]](#) [Brock and Dunfield \[2017\]](#) show that these two norms are roughly proportional with explicit constants depending only on the volume and injectivity radius⁴ of $\Gamma \backslash \mathbb{H}^3$. Now, assuming [Conjecture 17](#), each $[S_j]$ has Gromov-Thurston norm – and therefore harmonic norm – which is bounded by a polynomial in $\text{vol}(\Gamma \backslash \mathbb{H}^3)$. Thus Hadamard’s inequality shows that $|R_2(\Gamma \backslash \mathbb{H}^3)| \ll \text{vol}(\Gamma \backslash \mathbb{H}^3)^{Cb_1(\Gamma \backslash \mathbb{H}^3)}$.

Remarks. 1. [Conjectures 16](#) and [17](#) are false if the manifolds are not assumed to be congruence arithmetic: [Brock and Dunfield \[ibid., Theorem 1.5\]](#) indeed construct a sequence of closed hyperbolic 3-manifolds M_n (whose injectivity radii stay bounded away from 0) with

$$\text{vol}(M_n) \rightarrow \infty, \quad b_1(M_n) = 1 \quad \text{and} \quad \limsup_n \frac{\log |R_2(M_n)|}{\text{vol}(M_n)} > 0.$$

2. Under well believed number theoretic assumptions, in [Bergeron, Şengün, and Venkatesh \[2016\]](#) we notably verify [Conjecture 17](#) when $\Gamma \backslash \mathbb{H}^3$ is a congruence cover of a Bianchi manifold with 1-dimensional cuspidal cohomology associated to a non-CM elliptic curve. In that case the proof indeed relate the complexity of the H_2 -cycle to the height of the associated elliptic curve (i.e., the minimal size of A, B so that its equation can be expressed as $y^2 = x^3 + Ax + B$). That this might be a general phenomenon was also suggested in [Calegari and Venkatesh \[2012\]](#).

⁴which is expected to be uniformly bounded away from 0 on arithmetic manifolds

8 Some regrets

Many other interesting questions could (should?) have been discussed in this survey. Hyperbolic 3-manifolds form a particularly rich and interesting family of locally symmetric manifolds. However: Matsushima’s formula gives no restriction on their cohomology and most of these manifolds do not contain totally geodesic immersed submanifolds. It may therefore appear that hyperbolic 3-manifolds are not really connected with our general story. This is not quite true: Agol [2013, 2014] proof of the celebrated ‘Virtual Haken Conjecture’ suggests considering ‘almost geodesic’ cycles rather than just geodesic ones. We haven’t addressed the rich relation between the cohomology of locally symmetric spaces and number theory. Let us simply say that our original motivation for Conjectures 16 and 17 came from the study of torsion homology and its relation with Galois representations Scholze [2014]. Finally Venkatesh’s program Venkatesh [2017] suggests fascinating relations between the period matrices of Section 7.2 and periods (in the usual sense) of automorphic forms.

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