

Einstein 4-manifolds and singularities

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Abstract. In this note we report on recent progress on the desingularization of real Einstein 4-manifolds. A new type of obstruction is introduced, with applications to the compactification of the moduli space of Einstein metrics, and to the correspondence between conformal metrics in dimension d and asymptotically hyperbolic Einstein metrics in dimension $d + 1$.

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1. The Einstein equation

The Einstein equation reads

$$\mathrm{Ric}(g) = \Lambda g,$$

where g is a metric on a manifold M^n (in local coordinates $g = \sum g_{ij} dx^i dx^j$), Λ is a real number called the cosmological constant, and $\mathrm{Ric}(g) = \sum R_{ij} dx^i dx^j$ is the Ricci tensor of the metric g .

Of course the equation comes from general relativity, in which case the manifold is 4-dimensional and the metric g is Lorentzian, that is of signature (1,3). Here, we consider the case a Riemannian Einstein metric (positive signature), which has deep connections with geometry and topology, as is illustrated by the situation in low dimension that we now review briefly.

In dimension 2, a metric is Einstein if it has constant curvature equal to λ ; there always exists an Einstein metric on a compact Riemann surface (this is equivalent to the uniformization theorem), and one has the dichotomy

- $\Lambda > 0$: M is a sphere;
- $\Lambda = 0$: M is a torus;
- $\Lambda < 0$: M is a surface of genus $g \geq 2$.

In dimension 3, again a metric is Einstein if it has constant curvature equal to 2Λ , so we have a similar dichotomy between spherical, flat or hyperbolic geometry according to the sign of Λ . The question of understanding which compact 3-manifolds carry an Einstein metric is completely understood, and deeply connected with the topology: the case $\Lambda > 0$ is that of a 3-sphere (and its finite quotients), and it is related to the Poincaré conjecture (proved by Perelman) saying that a compact simply connected 3-manifold is a 3-sphere—this can be

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phrased by saying that it carries a constant curvature metric, and the Einstein equation plays an important role in the proof through its heat flow, the Ricci flow

$$\frac{dg}{dt} = -2\text{Ric}(g).$$

The Ricci flow has been more generally used by Perelman to prove Thurston's geometrization conjecture, according to which any compact 3-manifold is decomposed into pieces, each of which carries one of eight homogeneous geometries (including the three constant curvature ones).

In higher dimension, it is no more true that an Einstein metric has constant curvature: the Ricci tensor is just a part of the Riemannian curvature, which contains another component (the Weyl curvature). The questions of existence and uniqueness are far from being solved. In dimension at least 5, there is no known obstruction to the existence of Einstein metrics. In dimension 4, the situation is more interesting: there is a strong relation between Einstein metrics and topology: this can be illustrated by the Hitchin-Thorpe inequality between the Euler characteristic χ and the signature τ of a compact Einstein 4-manifold:

$$2\chi(M) \geq 3|\tau(M)|. \quad (1.1)$$

This gives a topological restriction on the manifold M . More subtle obstructions are based on Gromov's idea of minimal volume (Besson-Courtois-Gallot) or on Seiberg-Witten theory (LeBrun), see the nice survey of LeBrun [16] and the references there.

The Riemannian Einstein equation is a nonlinear elliptic equation (transversely to the action of the group of diffeomorphisms), and the linearization L is a selfadjoint operator, see for example [5]. This means that one cannot extract much information on the deformations of a solution:

- either $\ker L = 0$, then the solution is rigid;
- either $\ker L \neq 0$, then there are infinitesimal deformations, but there is a space $\text{coker } L = \ker L$ of the same dimension of obstructions, so one cannot say anything in general on the local structure of the deformation space. For example there is no known bound on the dimension of the moduli space of Einstein metrics on a given manifold.

Except in the case of special structures (Kähler or other special holonomies like quaternion-Kähler, hyper-Kähler, etc.) there is no general method to produce Einstein metrics (running the Ricci flow is of course a method, but it remains very difficult to analyze in higher dimension). Things are better for Einstein metrics on manifolds with boundary: it turns out that there exists a natural boundary problem for Einstein metrics, on which some general features of Einstein metrics can be tested, and which has its own geometric interest, in relation with conformal geometry. We now explain these ideas which originated in the work of Fefferman and Graham [11].

So let now (M, g) be a manifold with boundary $\partial M = X$, and choose on M a defining function x of X , so that $x > 0$ in the interior of M , and vanishes at first order over X . Given a metric γ on X , we consider metrics g in the interior of M such that, when $x \rightarrow 0$,

$$g \sim \frac{dx^2 + \gamma}{x^2}. \quad (1.2)$$

This behaviour depends only on the conformal class $[\gamma]$ of γ : indeed, if γ is transformed into $\varphi^2\gamma$, then for $\tilde{x} = \varphi x$ one has $\frac{dx^2 + \gamma}{x^2} = \frac{d\tilde{x}^2 + \varphi^2\gamma}{\tilde{x}^2} + \text{l.o.t.}$ The conformal metric $[\gamma]$ is called the conformal infinity of g .

For example, if g is the hyperbolic metric on the ball, then the conformal infinity is the standard conformal metric on the boundary sphere. More generally, the behaviour (1.2) implies that the sectional curvature of g goes to -1 when $x \rightarrow 0$, hence the name of these metrics, which are called asymptotically hyperbolic (AH).

Dirichlet problem at infinity. Given a conformal class $[\gamma]$ on X , find an AH Einstein metric g in M such that the conformal infinity of g is $[\gamma]$.

The motivation of the original work of Fefferman and Graham is the study of conformal geometry through the corresponding Einstein metrics. The idea is that the formal development of g near the boundary captures invariant conformal properties of γ . This perspective was very fruitful, see [6, 12]. The correspondance received a lot of attention because it underlies a physical correspondance, the AdS/CFT correspondance [17, 24].

The global problem is well-behaved: when the metric g is non degenerate (meaning that the linearization of the problem has trivial L^2 kernel, which often happens), then one can fill a small deformation of $[\gamma]$ by a small deformation of g . This was first observed by Graham and Lee [13]. Important ideas to solve the problem were introduced by Anderson (see later in the text), but the main difficulty remains to analyze the compactness problem: is the map which associates to the Einstein metric g its conformal infinity $[\gamma]$ proper? it is clear that such a property, together with the nice local deformation property, enables to solve the Dirichlet problem by a continuity method.

2. Compactness

So we now pass to compactness problems. We specialize to dimension 4. There is a very good compactness result on Einstein metrics, which was obtained by Anderson [1] and by Bando, Kasue and Nakajima [3].

Theorem 2.1. *Suppose (M_i, g_i) is a sequence of compact Einstein 4-manifolds, with cosmological constant ± 1 or 0, satisfying the following hypothesis:*

- (1) *the diameter of (M_i, g_i) is bounded above;*
- (2) *the volume of (M_i, g_i) is bounded from below;*
- (3) *the L^2 norm of the curvature, $\int_{M_i} |R(g_i)|^2 d\text{vol}(g_i)$, is bounded above.*

Then a subsequence (M_i, g_i) converges for the Gromov-Hausdorff distance to a 4-orbifold (M_0, g_0) with isolated orbifold singularities. The convergence is C^∞ outside the singularities.

Moreover, for each singularity, there is a rescaling $\frac{g_i}{t_i}$ with $t_i \rightarrow 0$ such that $(M_i, \frac{g_i}{t_i})$ converges to a noncompact Ricci flat 4-manifold which is Asymptotically Locally Euclidean (ALE), that is it has one end and this end is asymptotic to the flat metric on \mathbb{R}^4/Γ for some finite subgroup $\Gamma \subset SO_4$.

There has been a lot of progress recently to understand the limits of Einstein manifolds in higher dimension, see the article by Naber in the same volume [18].

The first hypothesis of the theorem guarantees that there is no cusp formation; the second hypothesis that there is no collapsing on a lower dimensional space; the third hypothesis is topological, because for an Einstein metric g on a compact 4-manifold M , one has $\frac{1}{8\pi^2} \int_M |R(g)|^2 d\text{vol}(g) = \chi(M)$.

The ALE spaces which appear at the limit are the “bubbles” of the problem. This notion of bubble appears similarly in a lot of geometric problems (pseudo-holomorphic disks, instantons, harmonic maps, etc.). Similarly to these problems, another bubble can appear where a singularity forms, and one gets a tree of bubbles: the smooth ALE space mentioned in the statement is the deepest bubble.

A basic problem in understanding the possible limits of Einstein 4-manifolds is to classify the possible bubbles, that is the Ricci flat ALE 4-manifolds. There is a well-known family of hyper-Kähler (hence Ricci flat) ALE 4-manifolds (also called gravitational instantons), constructed by Kronheimer [14], who also classified all hyper-Kähler ALE 4-manifolds [15]. The finite subgroups of SO_4 which appear are the finite subgroups of SU_2 . Also some cyclic subgroups of SO_4 which are not contained into a SU_2 appear as finite quotients of Kronheimer’s ALE spaces. It is an old open important question whether all simply connected Ricci flat ALE 4-manifolds are hyper-Kähler (and therefore one of Kronheimer’s spaces). Nakajima [19] proved that if one adds the condition that the manifold is spin for a spin structure which is also ALE in some sense, then the answer is yes.

The simplest example of a Ricci flat ALE space is the Eguchi-Hanson space [10]: topologically it is T^*S^2 . The Eguchi-Hanson metric g_{EH} is asymptotic to the flat metric on $\mathbb{R}^4/\mathbb{Z}_2$. Actually T^*S^2 with the zero section removed is diffeomorphic to $(\mathbb{R}^4 \setminus \{0\})/\mathbb{Z}_2$; from the complex geometry point of view it is a desingularization of the A_1 singularity $\mathbb{C}^2/\mathbb{Z}_2$; all Kronheimer’s spaces are deformations of desingularizations of the Kleinian singularities, that is of \mathbb{C}^2/Γ for Γ a finite subgroup of SU_2 . In this way one gets a short list of singularities (A_k , D_k , E_6 , E_7 and E_8).

The kind of degeneration described in theorem 2.1 does occur. Actually Kähler geometry provides lots of examples. The first one [20, 23] was the singular Kummer surface (M_0, g_0) , with

$$M_0 = \mathbb{T}^4/\mathbb{Z}_2,$$

a quotient of the 4-torus by an involution with 16 singular points of type $\mathbb{C}^2/\mathbb{Z}_2$, and g_0 is the flat metric. Then there is a family (M_t, g_t) of smooth K3 surfaces with their Ricci flat metrics g_t (coming from Yau’s solution of the Calabi conjecture), which degenerate to (M_0, g_0) exactly in the way described by the theorem. Moreover, one can describe quite concretely the behaviour of g_t when $t \rightarrow 0$. Consider the two following regions in M_0 and in the Eguchi-Hanson space:

1. near a singular point $p_0 \in M_0$, note r the radius from p_0 , the region

$$A_t = \{t^{\frac{1}{4}} \leq r \leq 2t^{\frac{1}{4}}\};$$

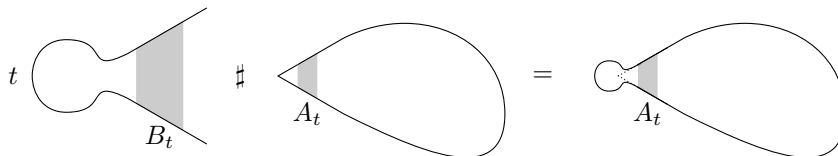
2. at the end of the metric g_{EH} , which is asymptotic to the cone $\mathbb{C}^2/\mathbb{Z}_2$, the region (where R is the radius near infinity)

$$B_t = \{t^{-\frac{1}{4}} \leq R \leq 2t^{-\frac{1}{4}}\}.$$

(Actually g_{EH} is close to the flat cone metric by a factor $O(R^{-4})$).

The homothety h_t of factor \sqrt{t} identifies B_t with A_t , and sends the metric tg_{EH} to a metric which is very close to g_0 when $t \rightarrow 0$. So we can construct a new manifold M with a new metric $g_0 \# tg_{EH}$ by gluing at each singular point the region $(\{r \geq t^{\frac{1}{4}}\}, g_0)$ in M_0 with the region $(\{R \leq 2t^{-\frac{1}{4}}\}, tg_{EH})$ in the Eguchi-Hanson space, identifying A_t and B_t by h_t and

interpolating between the (very close) metrics g_0 and tg_{EH} on A_t . The process is illustrated by the figure below.



The metric $g_0 \# tg_{\text{EH}}$ does not satisfy any more the Einstein equation in the damage area, but one can prove that it is indeed a very good approximation of g_t . In particular it illustrates well the behaviour of g_t when $t \rightarrow 0$: on one hand, $g_0 \# tg_{\text{EH}} \rightarrow g_0$, on the other hand $\frac{1}{t}(g_0 \# tg_{\text{EH}}) \rightarrow g_{\text{EH}}$.

The compactness theorem 2.1 says basically that all the limits arise in this way, but, as mentioned before, there is no classification of the possible Ricci flat ALE spaces at the limit. In the sequel, we will see that it is not true that any 4-orbifold with a singular Einstein metric can be approximated by smooth Einstein metrics in a similar way. This leads to new restrictions on the compactification of the moduli space of Einstein metrics.

3. Desingularization

It is a fundamental algebraic fact that the 2-forms in dimension 4 decompose into selfdual and anti-selfdual 2-forms:

$$\Omega^2 = \Omega_+ \oplus \Omega_- . \quad (3.1)$$

The Riemannian curvature tensor can be seen as a symmetric endomorphism of Ω^2 . Therefore it decomposes on (3.1), and the various components are

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_+ & \text{Ric}_0 \\ \text{Ric}_0 & \mathbf{R}_- \end{pmatrix} . \quad (3.2)$$

Moreover, \mathbf{R}_\pm decompose into a scalar part and a trace free part, which can be identified with the Weyl tensor W :

$$\mathbf{R}_\pm = \frac{\text{Scal}}{12} \pm W_\pm . \quad (3.3)$$

We now start from an Einstein 4-orbifold (M_0, g_0) , which can be compact or AH. We consider only the case of the simplest singularity $\mathbb{R}^4/\mathbb{Z}_2$. Let p_0 be a singular point (so of type $\mathbb{R}^4/\mathbb{Z}_2$). To simplify the statements, we assume that there is only one point, but the results are unchanged if there are several ones.

The following says that there is an obstruction to the existence of a sequence of metrics which desingularize g_0 :

Theorem 3.1 ([8]). *Suppose that a sequence of Einstein manifolds (M_i, g_i) converges to a non degenerate (M_0, g_0) , in such a way that g_i is close to $g_0 \# t_i g_{\text{EH}}$ for a sequence of real numbers $t_i \rightarrow 0$. Then*

$$\det \mathbf{R}_+^{g_0}(p_0) = 0 . \quad (3.4)$$

Here ‘close’ in the theorem refers to some weighted $C^{1,\alpha}$ Hölder norm.

In particular, the spaces of constant curvature $\mathbf{R}_{\pm} = \frac{\text{Scal}}{12}$, so if the curvature is nonzero then (3.4) cannot be satisfied. It follows that spherical or hyperbolic orbifolds cannot be limits of Einstein manifolds as in the theorem:

Corollary 3.2 ([8]). *Suppose (M_0^4, g_0) satisfies the same hypothesis and has constant curvature ± 1 . Then (M_0, g_0) is not the limit of a sequence of Einstein manifolds (M_i, g_i) as in theorem 3.1.*

For example, the corollary applies to the round metric on S^4/\mathbb{Z}_2 , where the action of \mathbb{Z}_2 has two fixed points (the two poles), or to $\mathbb{B}^4/\mathbb{Z}_2$, the quotient of the hyperbolic 4-ball by \mathbb{Z}_2 ; it was already known that there is no U_2 -invariant desingularization (U_2 -invariant Einstein metrics in dimension 4 are explicitly understood).

Of course the corollary is still a partial result: a stronger result would be: if $(M_i, g_i) \rightarrow (M_0, g_0)$ in the Gromov-Hausdorff sense, and at each singularity a rescaling $(M_i, \frac{g_i}{t_i})$ converges Gromov-Hausdorff to the Eguchi-Hanson space, then the obstruction (3.4) is satisfied, and in particular the limit cannot be spherical or hyperbolic. This statement requires to strengthen the convergence of the metric to get the hypothesis of the theorem. Nevertheless we believe that theorem 3.1 already exhibits a new type of restriction on the Einstein metrics which can appear in the compactification of the moduli space of Einstein metrics.

One may also ask the question for the other singularities: for the other Kleinian singularities and their finite quotients, the answer is that the obstruction (3.4) still holds, together with other obstructions: actually the number of scalar obstructions equals the b_2 of the corresponding ALE space (work in preparation). So the corollary should remain true for these singularities. The case of other singularities depend on the question mentioned above of the classification of all Ricci flat ALE spaces.

Now pass to some more precise remarks about theorem 3.1. First, note that if the Eguchi-Hanson space is glued with the opposite orientation (which results in a different topological space) then the condition (3.4) becomes $\det \mathbf{R}_-^{g_0}(p_0) = 0$ (this is clear since the Einstein equation does not depend on the orientation).

Also note that in the Kähler case, choosing a basis $(\omega_1, \omega_2, \omega_3)$ of Ω_+ such that ω_1 is the Kähler form, one has

$$\mathbf{R}_+ = \begin{pmatrix} \frac{\text{Scal}}{4} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad (3.5)$$

so the condition (3.4) is automatically satisfied. Indeed it is well known that there is no such obstruction in the Kähler case.

When one considers the gluing $g_0 \# t g_{\text{EH}}$, there is an ambiguity which gives a gauge parameter: indeed one can apply an element $u \in SO_4/\mathbb{Z}_2$ when identifying the parts A_t and B_t of the cone $\mathbb{R}^4/\mathbb{Z}_2$ (applying an orientation reversing element of O_4 amounts to changing the orientation of the Eguchi-Hanson space and was considered just above). It turns out that the isometry group of g_{EH} is $\text{Isom}(g_{\text{EH}}) = (U_2/\pm 1) \rtimes \mathbb{Z}_2$ (where $U_2 \subset SO_4$ is the standard unitary subgroup, and the \mathbb{Z}_2 is generated by $(z_1, z_2) \mapsto (-\bar{z}_2, \bar{z}_1)$, inducing the antipodal map on S^2). Taking u in $\text{Isom}(g_{\text{EH}})$ does not change $g_0 \# t g_{\text{EH}}$, so the remaining parameter is in $SO_4/(U_2 \rtimes \mathbb{Z}_2) = \mathbf{P}\Omega_+(\mathbb{R}^4)$.

This means that the ambiguity u can be interpreted as a real line in $\Omega_+(\mathbb{R}^4)$. This is related to the obstruction (3.4): note u_i the gauge parameter used for $g_0 \# t_i g_{\text{EH}}$, then one can add to the statement of the theorem the fact that the directions in $\Omega_+(\mathbb{R}^4)$ corresponding to the limits of the gauge parameters u_i must be in the kernel of $\mathbf{R}_+^{g_0}(p_0)$. (This also fits with

the Kähler picture, since when $\text{Scal} \neq 0$, this condition implies that the complex structure of the orbifold must be glued with a complex structure of the Eguchi-Hanson space which is orthogonal to that of $T^*\mathbb{C}P^1$, and in particular does not admit a holomorphic sphere; but indeed a Kähler-Einstein metric with $\text{Scal} \neq 0$ can not admit a holomorphic sphere of self intersection -2).

In particular, if $\text{rk } \mathbf{R}_+^{g_0}(p_0) = 1$, then the kernel of $\mathbf{R}_+^{g_0}(p_0)$ gives a direction in $\Omega_+(\mathbb{R}^4)$, so a gauge parameter $u = \lim u_i$. This is of importance in the reverse construction, that we now describe.

To see if the condition (3.4) is the only local obstruction to the desingularization, it is important to produce an Einstein desingularization (M, g_t) from the singular (M, g_0) . It turns out that this is not possible in general on a compact manifold, because, as mentioned earlier, there are always global obstructions to deformation which make the problem untractable. Fortunately, the problem becomes much better in the AH setting:

Theorem 3.3 ([8]). *Suppose that (M_0, g_0) is a non degenerate AH Einstein orbifold, with a singularity of type $\mathbb{R}^4/\mathbb{Z}_2$ at the point p_0 . If g_0 satisfies the condition (3.4), then there exists a family of AH Einstein metrics g_t on a topological desingularization M such that (M_0, g_0) is the limit of (M, g_t) when $t \rightarrow 0$.*

Again the theorem is still valid if there are several singular points: the topological desingularization M is obtained by replacing each singular point by a sphere of self intersection -2 .

An important fact to note in the theorem is that the conformal infinity γ_t induced by g_t on ∂M depends on t , and converges to the conformal infinity γ_0 of g_0 on $\partial M_0 = \partial M$: it is this flexibility which enables to solve the problem in the AH case.

There is an explicit family [4, 21], called the AdS-Taub-Bolt family, of U_2 invariant metrics on T^*S^2 , which converge to an orbifold metric on $\mathbb{B}^4/\mathbb{Z}_2$. The limit is not the hyperbolic metric (this is impossible by corollary 3.2), but a \mathbb{Z}_2 quotient of a selfdual Einstein metric on \mathbb{B}^4 , which is a member of a 1-parameter family found by Pedersen [22]; more precisely, it is the unique member of this family which satisfies the obstruction (3.4).

4. Degree theory and wall crossing

We now consider the AH setting, and study the consequences of theorem 3.3 on the Dirichlet problem at infinity stated in section 1.

Let (M_0, g_0) be an AH Einstein 4-orbifold, with conformal infinity $[\gamma_0]$ on the boundary $\partial M_0 = X$. Again for simplicity, suppose that we have only one singular point. We still restrict to the simplest singularity A_1 , and we ask g_0 to be non degenerate (remind this means that the L^2 kernel of the linearization vanishes). This implies that, given a small deformation γ of γ_0 , there exists a deformation g_0^γ of g_0 , which is an AH Einstein orbifold with conformal infinity γ .

We also suppose that condition (3.4) holds for g_0 . Then, inside the space \mathcal{C} of all conformal metrics on X , we can consider, at least near γ_0 , the space of conformal metrics on X such that the corresponding orbifold Einstein metric also satisfies (3.4):

$$\mathcal{C}_0 = \{\gamma \in \mathcal{C}, \det \mathbf{R}_+^{g_0^\gamma}(p_0) = 0\}. \quad (4.1)$$

Therefore, all the metrics g_0^γ with $\gamma \in \mathcal{C}_0$ can be desingularized by theorem 3.3, leading to AH Einstein metrics g_t^γ ($t > 0$) on the topological desingularization M of M_0 .

Theorem 4.1 ([8, 9]). *Suppose that $\text{rk } \mathbf{R}_+^{g_0}(p_0) = 2$ (this is a way to say that the vanishing of $\det \mathbf{R}_+^{g_0}(p_0)$ is non degenerate). Then*

- (1) *The set \mathcal{C}_0 is a smooth hypersurface of \mathcal{C} near γ_0 .*
- (2) *For γ near \mathcal{C}_0 , all the desingularized Einstein metrics have their conformal infinity on the side of \mathcal{C}_0 determined by*

$$\det \mathbf{R}_+^{g_0^\gamma}(p_0) > 0. \quad (4.2)$$

This result means that \mathcal{C}_0 is a ‘wall’ for the Dirichlet problem at infinity on M : for a conformal infinity on the side (4.2) of the wall, there is an AH Einstein metric with this conformal infinity (one of the metrics g_t^γ); when the conformal infinity goes to the wall, the Einstein metric degenerates, and disappears on the other side.

This is better understood in the setting of the degree theory proposed by Anderson [2] for the Dirichlet problem at infinity. The idea is the following: let \mathcal{M} be the space of all AH Einstein metrics on M , and consider the map

$$\Phi : \mathcal{M} \longrightarrow \mathcal{C} \quad (4.3)$$

defined by: $\Phi(g)$ is the conformal infinity of g . Anderson proved that, in a suitable Banach topology, if $\pi_1(M, X) = 0$, the map Φ is Fredholm of index 0. If there exists some open set $U \subset \mathcal{C}$ over which the map Φ is proper, then Sard-Smale theory gives a well-defined notion of degree of Φ which counts the number of preimages of an element of U . A priori, the degree is only defined in \mathbb{Z}_2 , but there is a way to count the solutions with sign (the sign is the number of negative eigenvalues of the linearization) and to define a degree with values in \mathbb{Z} . In some cases, one may hope to calculate the degree at some special points of U , and if it does not vanish, this implies that the map Φ is surjective over U .

It turns out that the properness of the map (4.3) is a difficult problem, which is far from being solved in general. The paper [2] is written under the following assumptions:

1. $\dim M = 4$: this is to be able to use the strong compactness results for Einstein metrics in dimension 4;
2. $U = \{\gamma \text{ on } X, \text{Scal}^\gamma > 0\}$: this is used to avoid cusp formation in the limits, and is also natural from the point of view of the physicists; it replaces the hypothesis on the volume in theorem 2.1; there are also counterexamples to properness with flat conformal infinities;
3. the map $H_2(X, \mathbb{k}) \rightarrow H_2(M, \mathbb{k})$ is surjective for any field \mathbb{k} : this is to avoid the degeneration to an Einstein orbifold, because in that case some 2-homology must exist in the interior of M (for example, the 2-sphere in the case of the degeneration to a $\mathbb{R}^4/\mathbb{Z}_2$ singularity).

The general case to consider in dimension 4 is when one relaxes the third hypothesis. Here, theorem 4.1 gives insight on what to expect. We are far from being able to prove something here, but the following speculations may help to understand the meaning of the theorem.

It is clear that in this general case, the map Φ is not proper: indeed we have explicit examples of orbifold degenerations of AH Einstein metrics. But we at least understand what

is happening when there is a degeneration of M to an orbifold M_0 with an A_1 singularity, obtained by contracting a 2-sphere of self intersection -2 : the number of preimages of Φ changes when one goes through the wall \mathcal{C}_0 defined by (4.1), in the precise way given by theorem 4.1. In this way, theorem 4.1 can be interpreted as a wall crossing formula calculating the jump of the degree on M when one goes across \mathcal{C}_0 .

Of course in general there are several (-2) spheres which can be contracted, so they give rise to several walls in \mathcal{C} : one can hope to have Φ proper in the regions delimited by these walls, and jumping across the walls like in theorem 4.1.

Now, all this is for A_1 singularities, so what is happening for the other singularities? the other Kleinian singularities are obtained by contracting a number of (-2) spheres, say k , and indeed one expects to obtain k obstructions to desingularization: so it seems that the generic case is that of A_1 singularities, the other Kleinian singularities being obtained when k walls intersect in a certain way; so the wall crossing formula for the A_1 case might be sufficient. The case of finite quotients of Kleinian singularities is also similar.

To transform these speculations into a proof, one would need to prove the properness of the map Φ outside the walls obtained from the various possible orbifolds obtained from M : in particular, this requires the classification of the Ricci flat ALE spaces, mentioned in section 2.1, and a better understanding of the behaviour of a degenerating Einstein metric.

5. Some ideas of the proofs

The beginning of the proof builds on usual ideas in ‘gluing problems’ appearing in geometric analysis. For small t we have a metric $g_0 \# t g_{\text{EH}}$ which is an approximate solution of the Einstein equation, which is better and better when $t \rightarrow 0$, and one wants to deform it into a true solution if t is small enough. In general, this is possible if the two pieces (here g_0 and g_{EH}) are not obstructed for the deformation theory of the Einstein problem. The point is that this is never true for g_{EH} (or more generally for any ALE space), because g_{EH} comes in a 1-parameter family given by scaling. More precisely, the linearization of the Einstein equation on the Eguchi-Hanson space (or more generally any hyper-Kähler space) is

$$L = d_-^* d_- : \Omega_- \Omega_+ \longrightarrow \Omega_- \Omega_+, \quad (5.1)$$

where one uses the identification $\Omega_- (\mathbb{R}^4) \Omega_+ (\mathbb{R}^4) = \text{Sym}_0^2 (\mathbb{R}^4)$ given by $u \otimes v \mapsto u \circ v$. (The operator on the trace part is just the usual Laplacian). On a hyper-Kähler manifold, the bundle Ω_+ is a flat trivial bundle: $\Omega_+ = \mathbb{R}^3$. So the operator L is identified with the Laplacian $d_-^* d_-$ acting on $\Omega_- \otimes \mathbb{R}^3$, and its L^2 -kernel is therefore the L^2 cohomology of the Eguchi-Hanson space:

$$\ker_{L^2} L = L^2 H^2 \otimes \mathbb{R}^3 \simeq \mathbb{R}^3. \quad (5.2)$$

Indeed the L^2 cohomology of Eguchi-Hanson is generated by the Poincaré dual of the 2-sphere. Let choose a basis (o_1, o_2, o_3) of this obstruction space. Then usual techniques enable to deform $g_0 \# t g_{\text{EH}}$ into a (basically unique) solution of the Einstein equation modulo these obstructions:

$$\text{Ric}(g_t) - \Lambda g_t = \sum_1^3 \lambda_i(t) o_i. \quad (5.3)$$

(This is not the exact equation to be solved because one must respect the Bianchi identity, but it gives the idea). The problem becomes to analyse the functions $\lambda_i(t)$ and their possible vanishing.

The way to do this is to refine the approximate metric $g_0 \# t g_{\text{EH}}$: if one has an approximation to a better order of a solution of (5.3), then g_t will be closer to this new approximation and this can give the first terms of the development of $\lambda_i(t)$.

The idea here is to refine the ALE metric g_{EH} into a metric h_t before gluing it to g_0 : the metric h_t is a perturbation of g_{EH} which should satisfy the equation

$$\text{Ric}(h_t) = t\Lambda h_t \quad (5.4)$$

instead of $\text{Ric}(g_{\text{EH}}) = 0$ (so that $\text{Ric}(th_t) = \Lambda(th_t)$); and it should match better $\frac{g_0}{t}$ near infinity: denote euc the standard Euclidean metric, then near p_0 , in normal coordinates, one has

$$g_0 = \text{euc} + g_2 + O(r^4), \quad (5.5)$$

where g_2 is an order 2 term:

$$g_2 = \sum a_{ijkl} x^i x^j dx^k dx^l. \quad (5.6)$$

We can ask h_t to match these order 2 terms in the following way: when we perform the homothety h_t , we transfer the coordinates x^i near 0 into the coordinates $X^i = t^{-\frac{1}{2}} x^i$ near infinity on Eguchi-Hanson, so

$$\frac{g_2}{t} = t \sum a_{ijkl} X^i X^j dX^k dX^l. \quad (5.7)$$

So it is natural to look for a first order deformation $h_t = g_{\text{EH}} + th$ which satisfies at infinity

$$h \sim \sum a_{ijkl} X^i X^j dX^k dX^l \quad (5.8)$$

while (5.4) becomes

$$Lh = \Lambda g_{\text{EH}}. \quad (5.9)$$

The first order deformation $g_{\text{EH}} + th$ is not a metric on the whole Eguchi-Hanson space, since the perturbation h blows up at infinity. Nevertheless it will define a metric on the region which is considered in the gluing, that is $R \leq 2t^{-\frac{1}{4}}$.

Now it turns out that the system (5.8) (5.9) is obstructed and has no solution in general, because of the cokernel of L (which equals its kernel). Actually, instead of (5.9), one can only solve

$$Lh = \Lambda g_{\text{EH}} + \sum_1^3 \lambda_i o_i, \quad (5.10)$$

where the real numbers λ_i are also unknown.

At the end, the system (5.8) (5.10) has a solution (h, λ_i) , and the λ_i depends only on the second order terms g_2 of g_0 at p_0 , that is on the curvature of g_0 at p_0 . There are some arguments using in particular the invariance of the system to calculate precisely the λ_i and one finds (up to a constant)

$$\lambda_i = \langle \mathbf{R}_+^{g_0}(p_0) \omega_1, \omega_i \rangle, \quad (5.11)$$

where (ω_i) is an orthonormal basis of Ω_+ . Then, using the approximate metric $g_0 \sharp t(g_{\text{EH}} + th)$, one can show that the coefficient $\lambda_i(t)$ appearing in (5.3) has the expansion

$$\lambda_i(t) = t\lambda_i + O(t^2). \quad (5.12)$$

In particular, the vanishing of $\lambda_i(t)$ forces $\lambda_i = 0$, which by (5.11) means

$$\mathbf{R}_+^{g_0}(p_0)\omega_1 = 0. \quad (5.13)$$

Therefore $\mathbf{R}_+^{g_0}(p_0)$ has a kernel; using the gauge freedom, one can reduce this condition to $\det \mathbf{R}_+^{g_0}(p_0) = 0$, which proves theorem 3.1.

For the desingularization itself (theorem 3.3), the work is far from being finished, since from the hypothesis $\mathbf{R}_+^{g_0}(p_0)\omega_1 = 0$ we have killed only the first term in the development of $\lambda_i(t)$. Here one uses the fact that (M_0, g_0) is an AH Einstein manifold, which gives the flexibility to vary g_0 varying its conformal infinity γ_0 . In particular, one considers the map $F = (F_1, F_2, F_3) : \mathcal{C} \rightarrow \mathbb{R}^3$ defined by

$$\gamma \longmapsto (\lambda_1(g_0^\gamma), \lambda_2(g_0^\gamma), \lambda_3(g_0^\gamma)), \quad (5.14)$$

where g_0^γ is the Einstein orbifold metric on M_0 with conformal infinity γ , and the λ_i are defined by (5.11). Then one proves that the map F is submersive at γ_0 : despite the fact that the space \mathcal{C} is infinite dimensional, this is not an obvious fact, and the proof relies in particular on a unique continuation theorem proved in [7]. This means that there exist directions γ_i in the space of conformal structures, such that

$$d_{\gamma_0} F_i(\gamma_j) = \delta_{ij}. \quad (5.15)$$

Consider now the metric g_t and the functions $\lambda_i(t)$ in (5.3) as depending also of the conformal infinity γ , and note this dependence as $g_t(\gamma)$, $\lambda_i(t, \gamma)$. From equations (5.12) and (5.15) it is now immediate that there exist functions $a_i(t) = O(t)$ such that

$$\lambda_i(t, \gamma_0 + \sum_1^3 a_j(t)\gamma_j) = 0, \quad (5.16)$$

which means that the metric $g_t(\gamma_0 + \sum_1^3 a_j(t)\gamma_j)$ is the expected solution of the Einstein equation.

Proving theorem 4.1 requires substantially new arguments. The first step is to refine the previous arguments.

If $\text{rk } \mathbf{R}_+^{g_0}(p_0) = 2$, one can show that actually $\lambda_2(t)$ and $\lambda_3(t)$ can be killed just by varying the gauge parameter, so there is no need to deform the conformal infinity in the directions γ_2 and γ_3 , the direction γ_1 is sufficient. So one can obtain a solution $g_t(\gamma_0 + a_1(t)\gamma_1)$.

Moreover one can construct a more refined deformation of g_{EH} which matches even better g_0 at infinity before gluing, by obtaining the coincidence not only of the terms of order 2, but also the terms of order 4; the whole construction is then refined to obtain a better expansion of $\lambda_1(t)$:

$$\lambda_1(t) = t\lambda_1 + t^2\mu_1 + O(t^3), \quad (5.17)$$

where μ_1 is some a priori non explicit number, obtained when finding the second order terms of the solution modulo obstructions of the equation (5.4). Then it is clear that the function $a_1(t)$ such that $g_t(\gamma_0 + a_1(t)\gamma_1)$ is the expected Einstein metric satisfies

$$a_1(t) \sim -\mu_1 t \quad (5.18)$$

when $t \rightarrow 0$. If μ_1 has a sign, then all the solutions are exactly on the side of \mathcal{C}_0 determined by the direction $-\mu_1\gamma_1$ at γ_0 .

Calculating μ_1 is difficult. Up to now, the analysis used essentially the linearization the Einstein equation (and some global properties). But calculating μ_1 involves understanding the second order terms of the equation, in order to find the second order terms of the solution of (5.4). We will not give any detail here, see [9], except to say that the hyper-Kähler nature of g_{EH} helps a lot to get insight on these second order terms and on the calculation of μ_1 . From this theorem 4.1 is deduced.

Finally let us say that the proofs of both theorems do not rely on the precise form of the Eguchi-Hanson metric, but more on the fact that the Eguchi-Hanson space has a one dimensional L^2 -cohomology and has a Hamiltonian circle action which rotates the other complex structures. There are lots of other spaces with the same geometric properties, if one allows orbifold singularities inside. For example, the A_k singularity $\mathbb{C}^2/\mathbb{Z}_{k+1}$ has a partial desingularization satisfying the same properties, but with an orbifold point with a A_{k-1} singularity. Using the same techniques as above, one can calculate an expansion for $\det \mathbf{R}_+$ at the singular point and find an obstruction to continuing the desingularization. So it seems that an inductive process can be started, leading to k obstructions to desingularization. Unfortunately this process can not be carried out so easily, because the non degeneracy hypothesis seems difficult to prove for the partial desingularizations. Nevertheless the author believes he is able to overcome this technical problem using some refined analysis.

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