

Action Minimizing Solutions of the Newtonian n -body Problem: From Homology to Symmetry

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(A la mémoire de Nicole Desolneux)

Abstract

An action minimizing path between two given configurations, spatial or planar, of the n -body problem is always a true – collision-free – solution. Based on a remarkable idea of Christian Marchal, this theorem implies the existence of new “simple” symmetric periodic solutions, among which the Eight for 3 bodies, the Hip-Hop for 4 bodies and their generalizations.

2000 Mathematics Subject Classification: 70F07, 70F10, 70F16, 34C14, 34C25.

Keywords and Phrases: n -body problem, Action, Symmetry.

0. Introduction

Finding periodic geodesics on a riemannian manifold as length minimizers in a fixed non-trivial homology or homotopy class is commonplace lore. Advocated by Poincaré [P] as early as 1896, the search for periodic solutions of a given period T of the n -body problem as action minimizers in a fixed non-trivial homology or homotopy class is rendered difficult by the possible existence of collisions due to the relative weakness of the newtonian potential: the action of a solution stays finite even when some of the bodies are colliding. Very few results are available: among them Gordon’s characterization of Kepler solutions [G] for 2 bodies in \mathbb{R}^2 , Venturelli’s characterization of Lagrange equilateral solutions [V1] for 3 bodies in \mathbb{R}^3 , Arioli, Gazzola and Terracini’s characterization of retrograde Hill’s orbits [AGT]

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for the restricted 3-body problem in \mathbb{R}^2 . In particular, no truly new solution of the n -body problem was found in this way; indeed, these results confirm the view that the action-minimizing periodic solutions are the “simplest” ones in their class.

The action minimization method has recently been given a new impetus by the replacement of the topological constraints by symmetry ones. This idea was first introduced by the Italian school [C-Z][DGM][SeT] as another mean of forcing coercivity of the problem, i.e. forbidding a minimizer to be “at infinity”. The bodies were forced to occupy, after half a period, a position symmetrical of the original one with respect to the center of mass of the system. It is proved in [CD] that in a space of even dimension, say \mathbb{R}^2 , the minimizers in this symmetry class include relative equilibrium solutions (i.e. solutions which are “rigid body like”); moreover all minimizers are of this form provided a certain “finiteness” hypothesis is verified (see [C3]). Such relative equilibria can occur only for the so called *central configurations* [AC], the most famous of which is Lagrange equilateral triangle.

Recently, a new type of symmetry was considered, which originates in the invariance of the Lagrangian under permutations of equal masses. This has led to the discovery of a whole world of new solutions in the case when all the bodies have the same mass. The most surprising ones are the “choreographies” whose name, given by Carles Simó, fits the beautiful figures they display on the screen in animated computer experiments ([CGMS],[S2]). Referring to my survey article [C3] for a bibliography and a description of the few cases in which existence proofs are available (the Hip-Hop [CV] for 4 bodies in \mathbb{R}^3 , the Eight [CM] for 3 bodies in \mathbb{R}^2 , Chen’s solutions [Ch] for 4 bodies in \mathbb{R}^2), I mainly address here a powerful theorem which solves completely the collision problem for the fixed ends problem in the case of arbitrary masses. This is pertinent because, as we shall see, it allows one to prove the existence of collision-free minimizers under well chosen symmetry constraints. This theorem is the result of the efforts of Richard Montgomery, Susanna Terracini, Andrea Venturelli [V2], and, for the last – fundamental – stone, Christian Marchal [M2] [M3]. I present here a complete proof and, in particular, a simplified version of Marchal’s remarkable idea, which avoids numerical computations. I discuss also new applications to minimization under symmetry constraints and open problems.

Notations. By a *configuration* of n bodies in an euclidean space $(E, \langle \rangle)$ we understand an n -tuple $x = (\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n) \in E^n$. The *configuration space* of the n -body problem is the quotient of the set of configurations by the action of translations (see [AC]). It may be identified as in [C3] with the set \mathcal{X} of configurations whose center of mass $\vec{r}_G = (\sum_{i=1}^n m_i)^{-1} \sum_{i=1}^n m_i \vec{r}_i$ is at the origin. It is endowed with the “mass scalar product” $(\vec{r}_1, \dots, \vec{r}_n) \cdot (\vec{s}_1, \dots, \vec{s}_n) = \sum_{i=1}^n m_i \langle (\vec{r}_i - \vec{r}_G), (\vec{s}_i - \vec{s}_G) \rangle$. The *non-collision* configurations – the ones such that no two bodies \vec{r}_i coincide – form an open dense subset $\hat{\mathcal{X}}$ of \mathcal{X} . The functions $I = x \cdot x$, $J = x \cdot y$, $K = y \cdot y$, defined on the *phase space* $\hat{\mathcal{X}} \times \mathcal{X}$ (whose elements are noted (x, y)) are the basic isometry-invariants of the n -body problem. They are respectively the *moment of inertia* of the configuration with respect to its center of mass, its time derivative

and twice the *kinetic energy* in a galilean frame which fixes the center of mass. The *potential function* (opposite of the potential energy), the *Hamiltonian* (=total energy) and the *Lagrangian* are respectively defined by

$$U = \sum_{i < j} m_i m_j \|\vec{r}_i - \vec{r}_j\|^{-1}, \quad H = \frac{1}{2}K - U, \quad L = \frac{1}{2}K + U.$$

In terms of the gradient ∇ for the mass metric, the equations of the n -body problem,

$$m_i \ddot{r}_i(t) = \sum_{j \neq i} m_i m_j \frac{\vec{r}_j(t) - \vec{r}_i(t)}{\|\vec{r}_j(t) - \vec{r}_i(t)\|^3}, \quad i = 1, \dots, n,$$

can be written $\ddot{x} = \nabla U(x)$. They are the Euler-Lagrange equations of the action, which to a path $x(t)$ associates the real number

$$\mathcal{A}_T(x(t)) = \int_0^T L(x(t), \dot{x}(t)) dt.$$

Remark. In the perturbations, we shall not bother about fixing the center of mass because replacing $K = \sum_{i=1}^n m_i \|\vec{v}_i - \vec{v}_G\|^2$ by $\sum m_i \|\vec{v}_i\|^2$ only increases the action.

1. The fixed-ends problem

Question. Given two configurations, – possibly with collisions – of n point masses in \mathbb{R}^3 (resp. \mathbb{R}^2) and a positive real number T , does there exist a solution of the Newtonian n -body problem which connects them in the time T ?

A natural way of looking for a solution is to seek for a minimizer of the action $\mathcal{A}_T(x)$ over the space $\Lambda_0^T(x_i, x_f)$ of paths $x(t)$ in the configuration space $\hat{\mathcal{X}}$ which start at time 0 in the configuration x_i and end at time T in the configuration x_f . For the integral to be defined, it is natural to work in the Sobolev space of paths which are square integrable together with their first derivative in the sense of distributions.

The main problem, already mentioned by Poincaré in 1896 (see [P] where he introduces the method in a slightly different context), is that a minimum could well be such that, for a non-empty set of instants (necessarily of measure zero), the system undergoes a collision of two or more bodies, which prevents it from being a true solution (see [C3]). At an isolated collision time, the renormalized configuration is known to be approaching the set of central configurations (the ones which admit homothetic motion [C2]) but very little is understood of these configurations for more than 3 bodies. Continuous families of such configurations could exist (the “finiteness problem”) and even if they didn’t, there would be no guarantee that at collision the renormalized configuration has a limit : it might have one only modulo rotations (the “infinite spin problem”). Nevertheless, we prove the

Theorem. *A minimizer of the action in $\Lambda_0^T(x_i, x_f)$ is collision-free on the whole open interval $]0, T[$. Hence, the answer to the Question is yes, both in \mathbb{R}^3 and \mathbb{R}^2 .*

In the next paragraph, Marchal's idea to prove that isolated collisions do not occur in a minimizer is explained on the Kepler problem. If the finiteness problem is supposed to be solved, it works in the same way for the general n -body problem (surprisingly, the infinite spin problem is irrelevant). We then address the finiteness problem with Terracini's technique of *blow up*, which reduces the problem of isolated collisions to the case of parabolic homothetic solutions; finally we show, following Montgomery and Venturelli, that accumulation of collisions do not occur in a minimizer provided no subclusters collide. The theorem then follows by induction on the number of bodies involved in a collision.

Remark. A similar assertion, based on numerical experiments, was made by Tiancheng Ouyang in Guanajuato (Hamsys, march 2001) but no proof appeared.

2. The Kepler problem as a model for the study of isolated collisions

The case of two bodies contains already many ingredients of the general situation. As is well-known, the 2-body problem is equivalent to the problem of a particle attracted to a fixed center 0, the so-called Kepler problem (or 1-fixed center problem). We call *collision-ejection* a solution in which the particle follows a straight line segment from its initial position \vec{r}_i to the attracting center and (possibly) another straight line segment from the attracting center to its final position \vec{r}_f .

A test assertion. *A collision-ejection solution of the Kepler problem does not minimize the action in the Sobolev space $\Lambda_0^T(\vec{r}_i, \vec{r}_f)$ of paths joining \vec{r}_i to \vec{r}_f .*

At least four proofs may be given of the truth of this assertion but only the fourth one using Marchal's idea is robust enough to lead to complete generalization. In the first one, we use the explicit knowledge of the solutions of the 2-body problem [A1] to identify the minimizers with the "direct" arcs of solution, not going "around" the attracting center (this arc is uniquely determined provided \vec{r}_i , 0 and \vec{r}_f do not lie on a line in this order). In the second one, we find a "simple" path without collision (straight line, circle, uniform motion) which has lower action. In the third one, supposing that a minimizer $\vec{r}(t)$ has a collision with the fixed center at time 0, we find a local deformation $\vec{r}_\epsilon(t) = \vec{r}(t) + \epsilon\varphi(t)\vec{s}$, which has lower action and no collision. Such deformations were used by many people, including Susanna Terracini, Gianfausto Dell'Antonio, Richard Montgomery and Christian Marchal. If we chose, with Montgomery, $\varphi(t) = 1$ if $0 \leq t \leq \epsilon^{\frac{3}{2}}$, $\varphi(t) = \epsilon^{-1}(\epsilon^{\frac{3}{2}} + \epsilon - t)$ if $\epsilon^{\frac{3}{2}} \leq t \leq \epsilon^{\frac{3}{2}} + \epsilon$ and $\varphi(t) = 0$ if $t \geq \epsilon^{\frac{3}{2}} + \epsilon$, the gain in action is $c\sqrt{\epsilon}(1 + O(\sqrt{\epsilon}\log(1/\sqrt{\epsilon})))$ provided the unit vector \vec{s} is well chosen. We come to the fourth proof, for which we must distinguish two cases according to the dimension of the ambient space.

(i) The case of \mathbb{R}^3 . Let $t \mapsto \vec{r}(t)$ be a collision-ejection solution of the Kepler problem, $\ddot{\vec{r}}(t) = -\vec{r}(t)/|\vec{r}(t)|^3$, such that $\vec{r}(-T') = \vec{r}_i$, $\vec{r}(0) = 0$, $\vec{r}(T) = \vec{r}_f$, $T, T' >$

0. We consider the following family of continuous deformations of $\vec{r}(t)$, parametrized by an element \vec{s} of the unit sphere S^2 in \mathbb{R}^3 : if $R'(t) = (1 + \frac{t}{T'})\rho$ and $R(t) = (1 - \frac{t}{T})\rho$,

$$\vec{r}_{\vec{s}}(t) = \vec{r}(t) + R'(t)\vec{s} \text{ if } -T' \leq t \leq 0, \quad \vec{r}_{\vec{s}}(t) = \vec{r}(t) + R(t)\vec{s} \text{ if } 0 \leq t \leq T.$$

It is a simplification of Marchal's original choice but the idea is the same: to show that the action \mathcal{A} of $\vec{r}(t)$ is strictly bigger than the average $\mathcal{A}_m = \int_{S^2} \mathcal{A}(\vec{r}_{\vec{s}}(t)) d\sigma$, where $d\sigma$ denotes the normalized area form, that is the unique rotation invariant probability measure on S^2 . This will imply the existence of at least one direction \vec{s} for which $\vec{r}_{\vec{s}}(t)$ has lower action than $\vec{r}(t)$ (because the set of good \vec{s} has positive measure we could choose \vec{s} so that $\vec{r}_{\vec{s}}$ is collision-free but this is irrelevant).

The linearity of the integral and the similar behaviour of ejection and collision allow to replace in the proof $\vec{r}(t)$ and $\vec{r}_{\vec{s}}(t)$ by their restrictions to the interval $[0, T]$. Moreover, it follows from the "blow-up" method (see 3.2) that it is enough to consider a parabolic solution $\vec{r}(t)$, that is $\vec{r}(t) = \gamma t^{\frac{2}{3}} \vec{c}$, with $\gamma = (9/2)^{\frac{1}{3}}$ if $|\vec{c}| = 1$.

By Fubini theorem applied to the positive integrand,

$$\begin{aligned} \mathcal{A}_m &= \int_{S^2} d\sigma \int_0^T \left(\frac{|\dot{\vec{r}}_{\vec{s}}|^2}{2} + \frac{1}{|\vec{r}_{\vec{s}}|} \right) dt = \int_0^T dt \int_{S^2} \left(\frac{|\dot{\vec{r}}_{\vec{s}}|^2}{2} + \frac{1}{|\vec{r}_{\vec{s}}|} \right) d\sigma, \quad \text{and} \\ \mathcal{A}_m - \mathcal{A} &= \int_0^T dt \left[\frac{\dot{R}(t)^2}{2} + \int_{S^2} \dot{R}(t) \vec{s} \cdot \vec{r}(t) d\sigma \right] + \int_0^T dt \left[\int_{S^2} \frac{d\sigma}{|\vec{r}_{\vec{s}}(t)|} - \frac{1}{|\vec{r}(t)|} \right]. \end{aligned}$$

The first integral reduces to $\frac{1}{2} \int_0^T \dot{R}(t)^2 dt = \rho^2/2T$ because of the antisymmetry in \vec{s} of the scalar product. The second is the difference in potential resulting from the replacement of the particle $\vec{r}(t)$ by a homogeneous hollow sphere of the same mass and increasing radius $R(t)$. Because of the harmonicity of Newton potential in \mathbb{R}^3 , the potential $U_0(\vec{r}, R) := \int \frac{d\sigma}{|\vec{r} - R\vec{s}|} = \int \frac{d\sigma}{|\vec{r} + R\vec{s}|}$ of a homogeneous hollow sphere of radius R is

$$U_0(\vec{r}, R) = \frac{1}{R} \quad \text{if } |\vec{r}| \leq R, \quad U_0(\vec{r}, R) = \frac{1}{|\vec{r}|} \quad \text{if } |\vec{r}| \geq R.$$

If 0 enters this sphere at time t_0 , $|\vec{r}(t_0)| = R(t_0)$, i.e. $\rho = \gamma t_0^{\frac{2}{3}} + O(t_0^{\frac{5}{3}})$, and

$$\mathcal{A} - \mathcal{A}_m = \frac{\rho^2}{2T} + \int_0^{t_0} \left[\frac{1}{R(t)} - \frac{1}{|\vec{r}(t)|} \right] dt = -\frac{2}{\gamma} t_0^{\frac{1}{3}} + O(t_0^{\frac{4}{3}}) \leq 0 \text{ if } \rho, \text{ hence } t_0, \text{ is small.}$$

(ii) **The case of \mathbb{R}^2 .** The Newtonian potential is not harmonic in \mathbb{R}^2 and this makes things somewhat more complicated. Marchal proposes to replace the sphere by a disk of radius R endowed with the projection $\sigma(\theta, x) = 1/(2\pi R\sqrt{R^2 - x^2})$ (in polar coordinates) of the uniform density on the sphere of the same radius. The potential function $U_0(\vec{r}, R)$ of such a disk (total mass 1) may be recovered from the

general computation done, via complex function theory, for a thin elliptic plate with a given density which is constant on homothetic ellipses (see [B] and [Ma]):

$$U_0(\vec{r}, R) = \frac{\pi}{2R} \quad \text{if } |\vec{r}| \leq R, \quad U_0(\vec{r}, R) = \frac{1}{R} \arcsin\left(\frac{R}{|\vec{r}|}\right) \quad \text{if } |\vec{r}| \geq R.$$

It does not coincide any more, but asymptotically, with Newton's potential $1/|\vec{r}|$ of the center of mass outside the disk but it is still constant in the interior and the proof works as well as in the spatial case: as $\arcsin(x) \leq x + (\frac{\pi}{2} - 1)x^3$ for $x \geq 0$, the difference in actions between the mean of the modified actions when \vec{s} belongs to the unit disk and the original becomes

$$\begin{aligned} A_m - \mathcal{A} &= \frac{\rho^2}{2T} + \int_0^{t_0} \left[\frac{\pi}{2R(t)} - \frac{1}{|\vec{r}(t)|} \right] dt + \int_{t_0}^T \left[\frac{1}{R(t)} \arcsin\left(\frac{R(t)}{|\vec{r}(t)|}\right) - \frac{1}{|\vec{r}(t)|} \right] dt \\ &\leq \frac{\rho^2}{2T} + \left[-\frac{\pi T}{2\rho} \log\left(1 - \frac{t}{T}\right) - \frac{3}{\gamma} t^{\frac{1}{3}} \right]_0^{t_0} + \int_0^{t_0} \left(\frac{\pi}{2} - 1 \right) \rho^2 \left(1 - \frac{t}{T}\right)^2 \frac{1}{\gamma^3 t^2} dt \\ &= \frac{\gamma^2 t_0^{\frac{4}{3}}}{2T} (1 + O(t_0)) + \left(\frac{\pi}{2} - 3 \right) \frac{1}{\gamma} t_0^{\frac{1}{3}} (1 + O(t_0)) + \left(\frac{\pi}{2} - 1 \right) \frac{1}{\gamma} t_0^{\frac{1}{3}} + O\left(t_0^{\frac{4}{3}} \log\left(\frac{1}{t_0}\right)\right) \\ &= (\pi - 4) \frac{1}{\gamma} t_0^{\frac{1}{3}} + O\left(t_0^{\frac{4}{3}} \log\left(\frac{1}{t_0}\right)\right) \leq 0 \quad \text{for } \rho, \text{ hence } t_0, \text{ small.} \end{aligned}$$

3. Proof of the theorem

3.1 The induction. We define the following statements about a minimizer $x(t)$:

(I_p) If a collision of p bodies occurs in $x(t)$ for $t \in]0, T[$, it is isolated.

(II_p) No collision of $m \leq p$ bodies occurs in $x(t)$ for $t \in]0, T[$.

In 3.2 we prove that (I_p) implies that no collision of p bodies occurs in $]0, T[$, hence that (II_p) and (I_{p+1}) imply (II_{p+1}). In 3.3 we prove that (II_p) implies (I_{p+1}). As (II_1) is empty, it implies (I_2), hence (II_2), etc... up to (II_n) which is the conclusion. If k -collisions are present in x_i or x_f but not j -collisions for $j < k$, the induction proves that $j < k$ -collisions are absent. The next step proves that k -collisions, including the ones at the ends, are isolated and everything goes through.

Remark. The induction may succeed because a p -body collision cannot be a limit of q -body collisions with $q > p$. Still, accumulation of collisions involving bodies in different clusters could *a priori* occur, e.g. a sequence of double collisions 23, 12, 34, 23, 13, 24, 23, ... converging to a quadruple collision 1234, or even a converging sequence of such sequences. Induction on the number of bodies in the clusters fortunately avoids having to deal with such problems.

3.2 Elimination of isolated collisions.

3.2.1 The blow-up technique. This technique was introduced by S. Terracini and developped in the thesis of A. Venturelli [V2]. It is based on the homogeneity of the potential (compare [C2]). It allows proving the

Proposition. *If a minimizer $x(t)$ of the fixed ends problem for n -bodies possesses an isolated collision of $p \leq n$ bodies, there is a parabolic (i.e. zero energy) homothetic collision-ejection solution $\bar{x}(t)$ of the p -body problem which is also a minimizer of the fixed ends problem.*

Proof. To keep the exposition as simple as possible, I describe the case of a total collision. In the general case of partial (and possibly simultaneous) collisions, everything goes through in the same way because the blow up sends all bodies not concerned by the collision to infinity (for more details, see [V2]).

Assuming that the collision occurs at $t = 0$, we define $x^\lambda(t) = \lambda^{-\frac{2}{3}}x(\lambda t)$ for $\lambda > 0$. If $x(t)$ is a solution of the n -body problem, so is $x^\lambda(t)$. Moreover, for any path $x(t)$ in $\Lambda_{T_1}^{T_2}(x_i, x_f)$, the path $x^\lambda(t)$ belongs to $\Lambda_{T_1}^{T_2}(x^\lambda(T_1), x^\lambda(T_2))$ and its action is equal to $\lambda^{-\frac{1}{3}}$ times the action of the restriction of $x(t)$ to the interval $[\lambda T_1, \lambda T_2]$. Hence, if $x(t)$ is action minimizer in $\Lambda_{T_1}^{T_2}(x_i, x_f)$, so is x^λ in $\Lambda_{T_1}^{T_2}(x^\lambda(T_1), x^\lambda(T_2))$. Now, Sundman's estimates recalled above imply that, $\{x^\lambda, 0 < \lambda < \lambda_0\}$ is bounded in $H^1([T_1, T_2], \mathcal{X})$, hence weakly compact, so that there exists a sequence $\lambda_n \rightarrow 0$ such that x^{λ_n} converges weakly (and hence uniformly) in $H^1([0, T], \mathcal{X})$ to a solution \bar{x} . One shows that \bar{x} is made of a parabolic homothetic collision solution followed by a parabolic homothetic ejection solution (the two central configurations involved are a priori distinct). Moreover, it follows from the weak lower semi-continuity of the action that \bar{x} is a minimizer in $\Lambda_{T_1}^{T_2}(\bar{x}(T_1), \bar{x}(T_2))$ (see [V2]).

3.2.2 The mean perturbed action. We shall deal only with the case of \mathbb{R}^3 and refer the reader to the Kepler case for the modifications needed in the case of \mathbb{R}^2 . Thanks to “blow up”, we may suppose that our minimizer $x(t)$ is a parabolic homothetic collision-ejection solution $x(t) = (\vec{r}_1(t), \dots, \vec{r}_p(t)) = x_0|t|^{\frac{2}{3}}$ of the p -body problem. As in the Kepler case, we may restrict to the ejection part, corresponding to $t \in [0, T]$. One studies deformations of $x(t)$ of the form

$$x_s^k(t) = (\vec{r}_1(t), \dots, \vec{r}_k(t) + R(t)\vec{s}, \dots, \vec{r}_p(t)),$$

where, as before, $R(t) = (1 - \frac{t}{T})\rho$ with ρ a small positive real number and \vec{s} belongs to the unit sphere. The same computation as in the Kepler case leads to an average action \mathcal{A}_m^k such that

$$\mathcal{A}_m^k - \mathcal{A} \leq \frac{m_k}{2} \frac{\rho^2}{T} + \sum_{j \neq k, j \leq p} m_j m_k \int_0^{t_{jk}} \left[\frac{1}{R(t)} - \frac{1}{r_{jk}(t)} \right] dt,$$

where $r_{jk} = |\vec{r}_k - \vec{r}_j|$ and t_{jk} is defined by $r_{jk}(t) = R(t)$ (the inequality sign comes from the fact that the deformations do not keep the center of mass fixed).

As $r_{jk}(t) = c_{jk}t^{\frac{2}{3}}$, one concludes as in the Kepler case that $\mathcal{A}_m^k - \mathcal{A} < 0$.

Remark. We could have dispensed with “blow up” in case similitude classes of central configurations were isolated but certainly not otherwise. This is because, the best control Sundman's theory may give us on the asymptotic behaviour of

the colliding bodies is that their moment of inertia I_c with respect to their center of mass and their potential U_c are respectively equivalent to $I_0 t^{\frac{4}{3}}$ and $U_0 t^{-\frac{2}{3}}$ (see [C2]). This implies the existence, for $1 \leq j < k \leq p$, of $0 < a_{jk} \leq b_{jk}$ such that for t small enough, one has $a_{jk} t^{\frac{2}{3}} \leq r_{jk}(t) \leq b_{jk} t^{\frac{2}{3}}$. It follows that

$$\mathcal{A}_m^k - \mathcal{A} \leq \frac{m_k}{2T} b_{jk}^2 t^{\frac{4}{3}} + O(t^{\frac{7}{3}}) - \sum_{j \neq k, j \leq p} m_j m_k \left(\left[-\frac{1}{a_{jk}} + \frac{3}{b_{jk}} \right] t^{\frac{1}{3}} + o(t^{\frac{1}{3}}) \right).$$

If similitude classes of central configurations are isolated, there is a limit shape and we may take a_{jk} and b_{jk} as close as we wish. Otherwise we cannot conclude.

3.3 The elimination of non-isolated collisions. It remains to prove that (II_p) implies (I_{p+1}) . We use energy considerations, an idea which goes back to R. Montgomery and was further developed in Venturelli's thesis [V2].

Proposition. *Let $x(t)$ be a minimizer of the fixed ends problem. If $x(t)$ has no p -body collisions for $p < p_0$, collisions of p_0 bodies are isolated.*

Sketch of proof. I shall give the proof in the case of a total collision (i.e. $p_0 = n$) and then explain what has to be changed in the general case.

(i) Using the behavior of the action under reparametrization, let us prove that the energy stays constant along a minimizer, whatever be the collisions. For this let us consider variations $x_\epsilon(t)$ of the form $x_\epsilon(t) = x(\varphi_\epsilon(t))$ where $t \mapsto \tau = \varphi_\epsilon(t)$ is a differentiable family of diffeomorphisms of $[0, T]$ starting from $\varphi_0(t) \equiv t$:

$$\mathcal{A}(x_\epsilon) = \int_0^T \left(\frac{\|\dot{x}_\epsilon(t)\|^2}{2} + U(x_\epsilon(t)) \right) dt = \int_0^T \left(\frac{1}{\lambda_\epsilon(\tau)} \frac{\|\dot{x}(\tau)\|^2}{2} + \lambda_\epsilon(\tau) U(x(\tau)) \right) d\tau,$$

where $\lambda_\epsilon = dt/d\tau = 1/\dot{\varphi}_\epsilon(\varphi_\epsilon^{-1}(\tau))$. The derivative at $\epsilon = 0$ of $a(\epsilon) = \mathcal{A}(x_\epsilon)$ is

$$\frac{da}{d\epsilon}(0) = \int_0^T \left(\frac{\|\dot{x}(\tau)\|^2}{2} - U(x(\tau)) \right) \delta\lambda(\tau) d\tau = \int_0^T H(x(\tau), \dot{x}(\tau)) \delta\lambda(\tau) d\tau,$$

where $\delta\lambda(\tau) = \frac{d\lambda_\epsilon(\tau)}{d\epsilon}|_{\epsilon=0}$. As the variations $\delta\lambda$ satisfy the constraint $\int_0^T \delta\lambda(\tau) d\tau = 0$, which comes from the fact that $\int_0^T \lambda_\epsilon(\tau) d\tau = T$, we get that there exists a real constant c such $H(x(\tau), \dot{x}(\tau)) = c$ wherever it is defined.

(ii) Let t_0 be an instant at which total collisions accumulate. Let us choose two sequences (a_n) and (b_n) of instants of total collision which converge to t_0 and are such that no total collision occurs in the open intervals $]a_n, b_n[$. The moment of inertia I of the system with respect to its center of mass is equal to zero at each of the instants a_n or b_n and hence has at least one maximum ξ_n in the interval $]a_n, b_n[$. As no partial collision occurs, the motion is regular in each of these intervals and at each such maximum, the second time-derivative $\ddot{I}(\xi_n)$ has to be non positive. But the value $U(\xi_n)$ of the potential function tends to $+\infty$ as $n \rightarrow +\infty$, while the

energy H stays constant. One then deduces from the Lagrange -Jacobi relation $\ddot{I} = 4H + 2U$ that $\ddot{I}(\xi_n) \rightarrow +\infty$, which is a contradiction.

In the general case, when μ is some cluster not containing all the bodies, the energy H_μ of μ is no more constant but one can get from a refinement of the same proof that it is still an absolutely continuous function of time as long as no collision occurs between a body of the cluster and a body of the complementary cluster (see [V2]). This implies that H_μ stays locally bounded and allows the argument of (ii) to work because, by hypothesis, no partial collision occurs in the cluster.

4. Periodic solutions

4.1 Homological or homotopical constraints. Going back to the 1896 Note of Poincaré already alluded to, the idea of constructing periodic solutions of the n -body problem as the “simplest” (action minimizing) ones in a given homology or homotopy class of the configuration space is very natural if one compares to the construction of periodic geodesics as minimizing the length in a non trivial homology or homotopy class. As already noticed by Poincaré, this works beautifully in the so-called “strong force problem”, corresponding to a potential in $1/r^2$ or stronger, where each collision path has infinite action [CGMS]. Unfortunately, in the Newtonian case, most of the time minimizers have collisions and hence are not true periodic solutions [M]. This is already true in the planar Kepler problem: it follows from Gordon’s work [G] (see also [C3]) that the only minimizers of the action among the loops of a fixed period T whose index in the punctured plane is different from $0, \pm 1$, is an ejection-collision one ! (for an analogue result in the planar three-body problem, see [V1]).

In such cases, solving the fixed ends problem is of no use. Among the cases where minimizers in a fixed homology or homotopy class have no collision are

1) Gordon’s theorem for the planar Kepler problem when one fixes the index to ± 1 (resp. when one insists only on the index being different from 0): a minimizer is any elliptic solution of the given period.

2) Venturelli’s generalization [V1] of Gordon’s theorem to the planar three-body problem with homology class fixed in such a way that along a period, each side of the triangle makes exactly one complete turn in the same direction: a minimizer is any elliptic homographic motion of the equilateral triangle, of the given period.

4.2 Symmetry constraints. In order to find “new” solutions as action minimizers, another type of constraints on the loops must be introduced, which somewhat allows using fixed ends type results. We ask the loops to be invariant under the action of a finite group G . An invariant loop is completely defined by its restriction to an interval of time on which G induces no constraint. The restriction to such a “fundamental domain” of a minimizer among G -invariant loops is a minimizer of the fixed ends problems between its extremities. This leads to a new collision problem: a minimizer could well have a collision at the initial or final instant.

(i) Choreographies. We first show, following Andrea Venturelli, the

Theorem. *A minimizer among n -choreographies has no collision.*

Recall that the choreographies are fixed loops under the action of the group $\mathbb{Z}/n\mathbb{Z}$ whose generator cyclically permutes the bodies after one n -th of the period (see [CGMS]); hence a fundamental domain can be chosen as any time interval of length T/n . If there were collisions at the ends, one would get a contradiction with the theorem by just shifting the fundamental domain to the right or to the left. One can prove (using [CD]) that the regular n -gon minimizes in all cases where it minimizes $\tilde{U} = I^{\frac{1}{2}}U$. But this is no more true for $n \geq 6$. So, what is the min ?

(ii) Generalized Hip-Hops. This works also for the “italian” (anti)symmetry:

Theorem. *A minimizer among loops $x(t)$ in \mathbb{R}^3 such that $x(t+T/2) = -x(t)$ has no collision. Moreover, it is never a planar solution.*

The last assertion comes from the fact that a relative equilibrium $x(t)$ whose configuration x_0 minimizes $\tilde{U} = I^{\frac{1}{2}}U$ is always a minimizer among the planar (anti)symmetric loops ([CD] and [C3]). But, applied to a variation $z(t) = z_0 \cos \frac{2\pi t}{T}$ normal to the plane of $x(t)$, the Hessian of the action is easily seen [C4] to be

$$d^2 \mathcal{A}(x(t))(z(t), z(t)) = I_0^{-\frac{1}{2}} d^2 \tilde{U}(x_0)(z_0, z_0) \int_0^T \cos^2 \frac{2\pi t}{T} dt,$$

where $I_0 = x_0 \cdot x_0$. Now, results of Pacella and Moeckel [Mo1] say that one can always choose z_0 such that $d^2 \tilde{U}(x_0)(z_0, z_0) < 0$. Hence, a relative equilibrium ceases being a minimizer in \mathbb{R}^3 . This ends the proof because other possible minimizers of the planar problem would have the same action as a relative equilibrium (thanks to A. Venturelli for this remark). In reference to [CV], I propose to call *generalized Hip-Hops* these minimizers. They are the best approximations I can think of in \mathbb{R}^3 to the non-existing relative equilibria of non-planar central configurations (recall [AC] that such relative equilibria exist in \mathbb{R}^4).

(iii) Eights with less symmetry. As another example, we prove the existence of solutions “of the Eight type” but with less symmetry than the full dihedral group $D_6 = \{s, \sigma | s^6 = 1, \sigma^2 = 1, s\sigma = \sigma s^{-1}\}$ (see [C3]). We consider the subgroups $\mathbb{Z}/6\mathbb{Z} = \{s\}$ and $D_3 = \{s^2, \sigma\}$.

Theorem. *A minimizer among $\mathbb{Z}/6\mathbb{Z}$ -invariant loops has no collision. The same is true for a minimizer among D_3 -invariant loops.*

Instead of minimizing the action over one twelfth of the period between an Euler configuration at time 0 and an isosceles one at time $T/12$ (see [CM]), one minimizes only over one sixth of the period: in the first case from an isosceles configuration at time t_0 to a symmetric one at time $t_0 + T/6$, in the second one from an Euler configuration at time 0 to another one at time $T/6$. Venturelli’s trick of translating the fundamental domain works in the first case where t_0 is arbitrary (a

translation of time transforms a minimizer into a minimizer) but not in the second one where, as for the initial D_6 -action, an Euler configuration can only occur at times which are integer multiples of $T/6$. To prove the absence of collisions at the initial and the final instant in the second case, we notice that such a collision is necessarily a triple (i.e. total) collision. If this happens, the action of the path is greater than the one of a homothetic ejection solution of equilateral type, a path which is not of the required type, but this is irrelevant here. The conclusion follows because the action of this last path is itself greater than the one of one sixth of the “equipotential model” (see [C3],[CM]). If a minimizer among $\mathbb{Z}/6\mathbb{Z}$ or D_3 symmetric loops possesses the whole D_6 symmetry of the Eight is unknown.

(iv) The P_{12} family. Marchal discovered the P_{12} family, which continues the Eight solution in three-space up to Lagrange equilateral solution, through choreographies in a rotating frame [M1]. It is parametrized by an angle u between 0 and $\frac{\pi}{6}$: the solution labeled by u minimizes the action in fixed time $T/12$ between configurations which are symmetric with respect to a line Δ through the origin which contains body 0 and configurations which are symmetric with respect to a plane P through the origin which contains body 2 and makes angle u with Δ . We shall think of Δ as being horizontal and of P as being vertical (Figure 1).

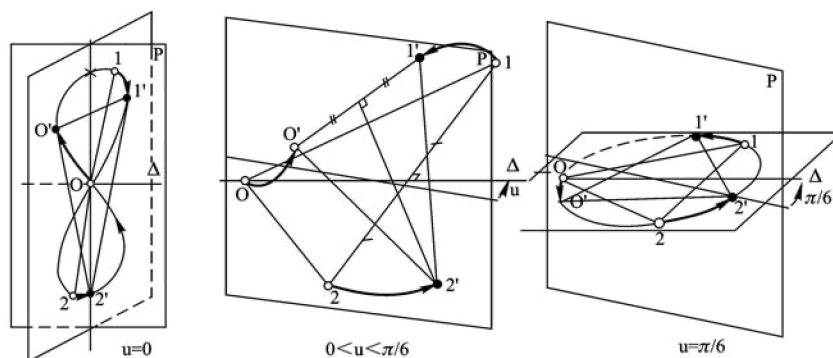


Figure 1 (fixed frame)

For $u = 0$, one gets the Eight in the vertical plane orthogonal to Δ (and hence to P); for $u = \frac{\pi}{6}$, one gets Lagrange solution in the horizontal plane (containing Δ and orthogonal to P). For $\frac{\pi}{6} \leq u \leq \frac{\pi}{3}$, the minimizer is a horizontal Lagrange solution whose size increases to infinity and action decreases to 0. The x^4 -type bifurcation of the minimizer at $u = \pi/6$ was analyzed by Marchal. In a frame rotating around the vertical axis of an angle $-u$ in time $T/12$, one gets a family of D_6 -symmetric choreographies of period T between the Eight and twice Lagrange (figure 2).

The relevant action of D_6 on the configuration space of three bodies in \mathbb{R}^3 is a direct generalization of the one which leaves the Eight invariant. It is defined as

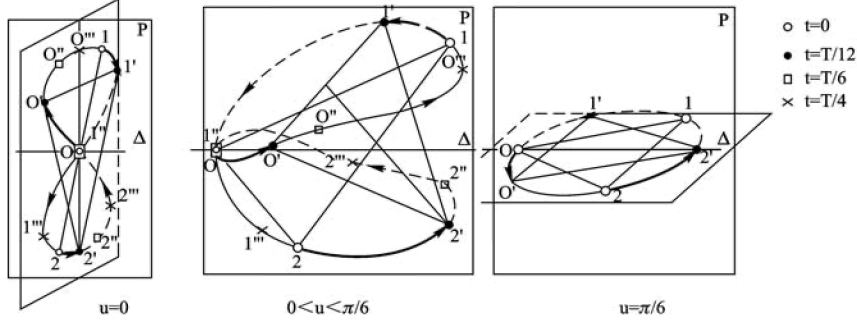


Figure 2 (rotating rame)

follows (the notations are the ones of [C3]):

$$\begin{aligned}\alpha(s)(\vec{r}_0, \vec{r}_1, \vec{r}_2) &= (\Sigma \vec{r}_2, \Sigma \vec{r}_0, \Sigma \vec{r}_1), & \beta(s)(t) &= t + T/6, \\ \alpha(\sigma)(\vec{r}_0, \vec{r}_1, \vec{r}_2) &= (\Delta \vec{r}_0, \Delta \vec{r}_2, \Delta \vec{r}_1), & \beta(\sigma)(t) &= -t,\end{aligned}$$

where Σ (resp. Δ) denotes the symmetry with respect to the horizontal plane (resp. to the line Δ).

Thanks to the fact that a minimizer of the fixed ends problem has no collision, the proof boils down to proving that a minimizing path has no collisions at its ends.

(i) Getting rid of triple collisions is easy: one notices firstly that the action of any path undergoing a triple collision is bigger than the action $A_3 = \frac{9}{2}(\frac{\pi^2}{3})^{\frac{1}{3}}T^{\frac{1}{3}}$ (notation of [CM]) of the homothetic solution of the equilateral triangle which goes in the same amount of time T from collision to zero velocity, secondly that this last action is bigger than the one $A_L = \frac{9}{2 \times 3^{\frac{1}{3}}}(\frac{\pi}{3} - u)^{\frac{2}{3}}T^{\frac{1}{3}}$ of a horizontal Lagrange solution which rotates an angle $\frac{\pi}{3} - u$ during the given amount of time. In fact, this last action is even smaller than the action $A_2 = (3\sqrt{2})^{-\frac{2}{3}}A_3$ used in [CM] as long as $u > (\sqrt{2} - 1)\frac{\pi}{6}$, in which case, the absence of any kind of collision in a minimizer follows.

(ii) For double collisions, which we have not yet discarded if $u \leq (\sqrt{2} - 1)\frac{\pi}{6}$, we provide a local deformation of the path which eliminates the collision and lowers the action. The two cases (collision at initial or final time) are similar, the only difference being the replacement of the symmetry with respect to the line Δ by the symmetry with respect to the plane P in the constraints imposed to the perturbation direction \vec{v} . Supposing that a minimizer $x(t) = (\vec{r}_0(t), \vec{r}_1(t), \vec{r}_2(t))$ has a collision between bodies 1 and 2 at the initial time 0, we deform it into $x_\epsilon(t) = x(t) + \epsilon\varphi(t)(0, \vec{v}, -\vec{v})$ where $\varphi(t)$ is chosen as in the third proof for the Kepler case, Using Sundman's estimates on the behavior of the bodies near a double collision, one shows that for a good choice of \vec{v} , the action again decreases by $c\sqrt{\epsilon}(1 + O(\sqrt{\epsilon}\log(1/\sqrt{\epsilon})))$.

Remarks. 1) Our argument works for one value of u at a time. As no uniqueness is proved, neither is continuity with u of the family. Such continuity would imply the existence among the family of spatial 3-body choreographies in the fixed frame.

2) A *direct* proof of the existence of the P_{12} family should stem from the following remark: in [CM], the existence of the Eight is deduced from the fact that the action A_2 of a 2-body solution going from collision to zero velocity is (slightly) greater than the action a of an equipotential path for which $I = I_0$ is constant as well as $K = K_0 = U = U_0$. For the Lagrange solution also, $I = I_{\frac{\pi}{6}}, K = K_{\frac{\pi}{6}} = U = U_{\frac{\pi}{6}}$ are constant and $I_{\frac{\pi}{6}} > I_0, K_{\frac{\pi}{6}} = U_{\frac{\pi}{6}} < K_0 = U_0$. In particular, the action of the Lagrange solution is smaller than the one of the equipotential model. When u increases, one should be able to construct a path with the right end conditions whose action is a decreasing function of u and hence is smaller than A_2 .

3) The first continuation of the Eight into a family of rotating planar choreographies was given by Michel Hénon [CGMS] who used the same program as in [H]. A third family should exist, rotating around an axis orthogonal to the first two.

5. Related results and open problems

Two global questions seem to be out of reach at the moment: unicity and possible extra symmetries of minimizers.

As an example of the first, numerical evidence by Simo suggests unicity of the Eight but in [CM] we do not even prove that each lobe is convex, only that it is star-shaped (the problem is near the crossing point). This is nevertheless enough to imply that the braid it defines in space time $\mathbb{R}^2 \times \mathbb{R}/T\mathbb{Z}$ (equivalent to the homotopy class in the configuration space) is the “Borromean rings”, the signature of a truly triple interaction (also noticed in [Ber] in a different context).

As an example of the second one, we do not know if the $\mathbb{Z}/4\mathbb{Z}$ -symmetry and the “brake” property of the Hip-Hop solution [CV] follow automatically from minimizing the action among loops such that $x(t+T/2) = -x(t)$ (compare 4.2 (ii)). One is tempted to compare this problem to the celebrated result of Alain Albouy [A2] which states the existence of some symmetry in any central configuration of 4 equal masses (and implies that there is only a finite number of them). But there is Moeckel’s numerical example [Mo2] of a central configuration of eight equal masses without any symmetry. And according to [SW], there exists such an example minimizing \tilde{U} for $n = 46$. For more on symmetry, see [V2].

Identifying minimizers, even when one knows that they are collision-free, is usually too difficult a problem (see 4.2 (i) and (ii)). Understanding their stability properties may sometimes be attempted theoretically [Ar],[O], or numerically [S1].

Another type of questions is connected with minimization with mixed constraints: symmetry and homology or homotopy. One can ask, for example, if the Eight is a minimizing choreography in its homology class $(0,0,0)$ (each side of the triangle has zero total rotation). An interesting example of mixed conditions may be found in [V2] where generalizations of the Hip-Hop lead to spatial choreographies of 4 equal masses. But, as for most choreographies, no proof was found of the existence of Gerver’s “supereight” with four equal masses [CGMS], [C3].

I am indebted to Christian Marchal, Richard Montgomery, David Sauzin, Susanna Terracini and Andrea Venturelli for many illuminating discussions and comments.

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