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Inversion Formula and Invariant Differential Operators on Solvable Lie Groups

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In this article, I show some applications of Kirillov's method of orbits to the harmonic analysis of bi-invariant differential operators on Lie groups. For instance, we obtain the Plancherel formula for an exponential solvable group, and the fact that on such a group a bi-invariant differential operator has a fundamental solution. We prove that, on a solvable group, a bi-invariant differential operator is locally solvable. This is a joint work with Mustapha Raïs.

1. Notations. We consider a connected Lie group G with Lie algebra $\boldsymbol{g}$. We choose a Haar measure dX on $\boldsymbol{g}$ and denote by dg the associated left Haar measure on G. We define a positive function j on $\boldsymbol{g}$ by $j(X)^2 = d(\exp X)/dX$. We denote by Δ the modular function on G: $\Delta(g) = d(hg)/dh$. We denote by $U(\boldsymbol{g})$ the complex enveloping algebra of $\boldsymbol{g}$, by $Z(\boldsymbol{g})$ its center, by $Z'(\boldsymbol{g})$ the set of semi-invariants in $U(\boldsymbol{g})$. We denote by $S(\boldsymbol{g})$ the complex symmetric algebra of $\boldsymbol{g}$, by $I(\boldsymbol{g})$ the subalgebra of invariants.

Let $\boldsymbol{g}^*$ be the dual space of $\boldsymbol{g}$. It is identified to the dual group of $\boldsymbol{g}$ by the pairing $\exp(i\langle f, X\rangle)$. If m is a bounded measure on $\boldsymbol{g}$, we put $\hat{m}(f) = \langle m, e^{if}\rangle$. If n is a bounded measure on $\boldsymbol{g}^*$, we put $\hat{n}(X) = \langle n, e^{iX}\rangle$. We denote by df the dual Haar measure on $\boldsymbol{g}^*$.

Let $f \in \boldsymbol{g}^*$. We denote by G_f the stabilizer of f in G, and by $\boldsymbol{g}_f$ its Lie algebra. Then $\boldsymbol{g}_f$ is the kernel of the 2-form B_f on $\boldsymbol{g}$ defined by f. Let $\Omega \subset \boldsymbol{g}^*$ be a G-orbit. We denote by β_Ω the canonical invariant measure on Ω, normalized as in [1, p. 20].

2. Definition of the mapping a. We fix a connected G-invariant open neighborhood V of 0 in $\boldsymbol{g}$ such that exp is a diffeomorphism of V onto $W = \exp(V)$. Let ϕ be a distribution on W. We define a distribution $a(\phi)$ on V by the formula

$$\langle a(\varphi), j\varphi \circ \exp\rangle = \langle \phi, \varphi\rangle \quad \text{for all } \varphi \in C_c^\infty(W).$$

When G is semisimple (and V completely invariant), Harish-Chandra [5] proved the following result:

(1) $a(u * \phi) = a(u) * a(\phi)$ for all $u \in Z(\mathfrak{g})$ and all central distributions ϕ on W.

This paper would be greatly simplified if we knew this result for all Lie groups. Suppose G is solvable. Then I proved in [3] that (1) is true if ϕ is in $Z(\mathfrak{g})$. We shall prove below that (1) is also true for some distributions associated with orbits of G in $\mathfrak{g}^*$.

3. Definition of some distributions. We fix a G-orbit Ω in $\mathfrak{g}^*$ of maximal dimension, a character χ of G with values in $\boldsymbol{C}^*$, and a function ψ on Ω such that $\psi(g^{-1}f) = \chi(g)\psi(f)$ for all $f \in \Omega$ and $g \in G$. We shall say that ψ has weight χ. We assume that the following is verified:

There exist a positive N and a norm $\|\cdot\|$ on $\mathfrak{g}^*$ such that

(2) $$\int_\Omega (1 + \|f\|)^{-N} |\psi(f)| \, \beta_\Omega(df) < \infty.$$

For $\varphi \in C_c(W)$, we put

(3) $$\langle \phi(\Omega, \psi), \varphi\rangle = \int_\Omega (j\varphi \circ \exp)^\wedge (f)\psi(f)\beta_\Omega(df).$$

Thus, $\phi(\Omega, \psi)$ is a distribution on W, and $a(\phi(\Omega, \psi))$ is the restriction to V of the Fourier transform $(\psi\beta_\Omega)^\sim$ of $\psi\beta_\Omega$. Under the adjoint representation, $\phi(\Omega, \psi)$ is semi-invariant with weight $\Delta\chi$.

4. The distributions $\phi(\Omega, \varphi)$ on solvable groups. We assume that G is a solvable connected simply connected Lie group. We consider an integral G-orbit $\Omega \subset \mathfrak{g}^*$: This means there exists a unitary character η of G_f (f is some point of Ω) whose differential is the restriction of if to $\mathfrak{g}_f$. Fix such a character η. We denote by $\pi_{\Omega,\eta}$ the irreducible unitary class of representations of G associated to these data by Auslander and Kostant, with the normalization of [1, p. 217].

Let χ be a character of G with values in $(0, \infty)$, and ψ a positive function of weight χ on Ω. There is a canonical way to associate to ψ a selfadjoint positive (in general unbounded) operator A_ψ in the space of $\pi_{\Omega,\eta}$ such that

$$\pi_{\Omega,\eta}(g)A_\psi\pi_{\Omega,\eta}(g^{-1}) = \chi(g)A_\psi \quad \text{for all } g \in G.$$

Suppose moreover that (2) is verified and that Ω has maximal dimension. Then, if $\varphi \in C_c^\infty(G)$, the operator $A_\psi^{1/2}\pi_{\Omega,\eta}(\varphi)A_\psi^{1/2}$ can be extended to a trace class operator, and we have

(4) $$\operatorname{tr}(A_\psi^{1/2}\pi_{\Omega,\eta}(\varphi)A_\psi^{1/2}) = \langle \phi(\Omega, \psi), \varphi\rangle \quad \text{for all } \varphi \in C_c^\infty(W).$$

(This is an extension of [1, Chapter IX].)

Let $u \in Z(\mathfrak{g})$. Then $a(u)^\wedge$ is an invariant polynomial on $\mathfrak{g}^*$. We denote by $a(u)^\wedge(\Omega)$ its constant value on Ω. We have

(5) $$\pi_{\Omega,\eta}(u) = a(u)^\wedge(\Omega)\mathrm{Id}.$$

If ϕ is a distribution on G, put $d\check{\phi}(g) = d\phi(g^{-1})$. Formulas (4) and (5) imply

$$\check{u} * \phi(\Omega, \psi) = a(u)^\wedge(\Omega)\phi(\Omega, \psi) \tag{6}$$

which is equivalent to

$$a(\check{u} * \phi(\Omega, \psi)) = a(\check{u}) * a(\phi(\Omega, \psi)). \tag{7}$$

When (2) is verified, but Ω not integral, there is probably a formula analogous to (4) involving instead of $\pi_{\Omega,\eta}$ one of the factorial normal representations constructed by Pukańszky [7].

5. Harmonic analysis of the distributions $u^{(s)}$ $(u \in Z(\boldsymbol{g}))$. We do not suppose for the moment that G is solvable. We suppose there is a G-invariant subset $\boldsymbol{g}_r^*$ of $\boldsymbol{g}^*$ which is the union of locally closed orbits of maximal dimension, and such that $\boldsymbol{g}^* - \boldsymbol{g}_r^*$ is of Lebesgue measure 0. Choose a nonzero rational function θ' on $\boldsymbol{g}^*$, semi-invariant with weight Δ^{-1} (cf. [2]) and put $\theta = |\theta'|$. The measure $\theta(f)^{-1}df$ on $\boldsymbol{g}^*$ is G-invariant. The quotient space $X = \boldsymbol{g}_r^*/G$ is a standard Borel space. There exists a positive Borel measure m_θ on X such that

$$\int_{\boldsymbol{g}^*} h(f)\, df = \int_X m_\theta(d\Omega) \int_\Omega h(f)\theta(f)\beta_\Omega(df) \tag{8}$$

for all positive Borel functions on $\boldsymbol{g}^*$.

Apply (8) to the function $h(f) = (1 + \|f\|)^{-N}$, where $N = 1 + \dim \boldsymbol{g}$. Fubini's theorem shows that we have

$$\int_\Omega (1 + \|f\|)^{-N}\theta(f)\beta_\Omega(df) < \infty, \tag{9}$$

for almost all $\Omega \in X$. Then, the distribution $\phi(\Omega, \theta)$ is defined on W for almost all $\Omega \in X$. Notice that these distributions are central. We shall use them to decompose other interesting central distributions on W.

Let $p \in S(\boldsymbol{g})$ be such that $\hat{p}$ is a positive function on $\boldsymbol{g}^*$. For $s \in \boldsymbol{C}$ with $\mathrm{Re}(s) \geqq 0$, we denote by p^s the tempered distribution on $\boldsymbol{g}$ whose Fourier transform is $\hat{p}^s$. Then, by the Atiyah-Bernstein theorem, p^s can be extended to a meromorphic function of s defined in $\boldsymbol{C}$ with values in the space of tempered distributions. Let $u \in Z(g)$ such that $a(u)^\wedge$ is positive. We shall denote by $u^{(s)}$ the distribution on W such that $a(u^{(s)}) = a(u)^s$. Thus $u^{(s)}$ is a meromorphic function of $s \in \boldsymbol{C}$ with values in the space of central distributions on W.

From (8) and (3), we get the following formula: Let $u \in Z(\boldsymbol{g})$ such that $a(u)^\wedge$ is positive. Suppose $\mathrm{Re}(s) \geqq 0$. Let $\varphi \in C_c(W)$. We have

$$\langle u^{(s)}, \varphi\rangle = \int_X m_\theta(d\,\Omega)a(\check{u})^\wedge(\Omega)^s\langle\phi(\Omega, \theta), \varphi\rangle. \tag{10}$$

When $u = \delta$, (10) gives a local inversion formula:

$$\varphi(1) = \int_X m_\theta(d\Omega)\langle\phi(\Omega, \theta), \varphi\rangle. \tag{11}$$

6. Solvability of bi-invariant differential operators on solvable Lie groups. In this section, G is a simply connected solvable Lie group.

We have the following result. Let $u \in Z(\mathfrak{g})$ be such that $a(u)^\wedge$ is positive, and let $v \in Z(\mathfrak{g})$. We have the equality of meromorphic functions:

$$a(v * u^{(s)}) = a(v) * a(u^{(s)}). \tag{12}$$

To prove (12), we imbed G in a simply connected group $\tilde{G}$ corresponding to an algebraic envelope $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$, and remark that we can assume $G = \tilde{G}$. Then all orbits of G in $\mathfrak{g}^*$ are integral and locally closed. Then (12) follows from (7) and (10).

It follows from (12) that the constant term of the Laurent expansion of $u^{(s)}$ at $s = -1$ is a distribution E on W which verifies $u * E = E * u = \delta$ (the Dirac mass at 1). From this, it is easy to obtain the following theorem.

THEOREM 1. *Let G be a connected solvable Lie group. Keep the notations of* §§1 *and* 2. *Let $u \in Z'(\mathfrak{g})$. There exists a distribution E on W such that $u * E = E * u = \delta$, and such that, for all $v \in Z'(\mathfrak{g})$, $a(v * E) = a(v) * a(E)$.*

When G is an exponential solvable group, we may choose $W = G$. In this case u has a fundamental solution defined in all of G. For G simply connected nilpotent, this is due to Raïs [**8**].

We do not know any example of a simply connected solvable group G, and $u \in Z'(\mathfrak{g})$, without a global fundamental solution.

7. The Plancherel formula for an exponential solvable group. In this section, G is an exponential solvable group. In this case, it is known that all orbits in $\mathfrak{g}^*$ are locally closed and simply connected. To each orbit is associated one unitary irreducible class π_Ω of representations of G. Choose the function θ on $\mathfrak{g}^*$ as in §5. Denote by $A_{\theta,\Omega}$ the positive selfadjoint operator in the space of π_Ω associated with the restriction of θ to Ω. Let $\mathfrak{g}_r^*$ be the union of orbits of maximal dimension in $\mathfrak{g}^*$. Define X and m_θ as in §5. From (11) and (4), we get the following inversion formula.

For almost all $\Omega \in X$, the operator $A_{\theta,\Omega}^{1/2}\, \pi_\Omega(\varphi) A_{\theta,\Omega}^{1/2}$ extends to a trace class operator for all $\varphi \in C_c^\infty(G)$, and we have

$$\varphi(1) = \int_X m_\theta(d\Omega)\ \operatorname{tr}(A_{\theta,\Omega}^{1/2}\pi_\Omega(\varphi)A_{\theta,\Omega}^{1/2}). \tag{13}$$

For each $\Omega \in X$, realize π_Ω in some Hilbert space H_Ω, and denote by $L_2(H_\Omega)$ the Hilbert space of Hilbert-Schmidt operators on H_Ω. From π_Ω, we get an irreducible representation of $G \times G$ in $L_2(H_\Omega)$.

THEOREM 2. *Let $\varphi \in L_1(G) \cap L_2(G)$. For almost all $\Omega \in X$, the operator $\pi_\Omega(\varphi)A_{\theta,\Omega}^{1/2}$ extends to an element $[\pi_\Omega(\varphi)A_{\theta,\Omega}^{1/2}]$ of $L_2(H_\Omega)$. The mapping $\varphi \to \{[\pi_\Omega(\varphi)A_{\theta,\Omega}^{1/2}]\}_{\Omega\in X}$ extends to an isometry U of $L_2(G)$ onto $\int_X L_2(H_\Omega)m_\theta(d\Omega)$. The isometry U intertwines the representation of $G \times G$ in $L_2(G)$ (the double regular representation) and the representation of $G \times G$ in $\int_X L_2(H_\Omega)m_\theta(d\Omega)$.*

Theorem 2 is the Plancherel theorem for G. If G is unimodular, it is a well-known application of formula (13).

8. Square-integrable representations of exponential solvable groups. In this section, G is an exponential solvable group with center Z. We choose a left Haar measure on G/Z. Let π be a unitary irreducible representation of G in a Hilbert space H. It is square-integrable (mod Z) if and only if it occurs discretely in $L_2(G, \eta)$, where η is the restriction of π to Z, and $L_2(G, \eta)$ the space of the representation induced by η to G. Suppose π is square-integrable (mod Z). There is on H a positive selfadjoint operator K, called the formal degree, which verifies $\pi(g)K\pi(g)^{-1} = \Delta(g)^{-1}K$ for all $g \in G$, and such that

$$\int_{G/Z} |(k, \pi(g)h)|^2 \, dg = \|k\|^2 \, \|K^{-1/2}h\|^2$$

for all $k \in H$ and $h \in \operatorname{dom} K^{-1/2}$ (cf. [4]).

The extension of Theorem 2 to $L_2(G, \eta)$ gives the following result (which, if G is nilpotent, is due to Moore and Wolf [6]):

Let $\Omega \subset \boldsymbol{g}^*$. The representation π_Ω is square-integrable (mod Z) if and only if $G_f = Z$ for all $f \in \Omega$. Let $e_1, \cdots, e_{2d}$ be a basis of $\boldsymbol{g}/\boldsymbol{z}$ such that the unit cube has volume 1. If $f \in \boldsymbol{g}^*$, we denote by $\operatorname{discr}(f)$ the discriminant of the 2-form on $\boldsymbol{g}/\boldsymbol{z}$ deduced from B_f, relative to the basis $e_1, \cdots, e_{2d}$. Suppose π_Ω is square-integrable (mod Z). If $f \in \Omega$, put $\psi(f) = (2\pi)^{-d}|\operatorname{discr}(f)|$. Then the formal degree of π_Ω is the operator A_ψ (cf. §4).

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