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SOME QUESTIONS AROUND QUASI-PERIODIC DYNAMICS

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Abstract

We propose in these notes a list of some old and new questions related to quasi-periodic dynamics. A main aspect of quasi-periodic dynamics is the crucial influence of arithmetics on the dynamical features, with a strong duality in general between Diophantine and Liouville behavior. We will discuss rigidity and stability in Diophantine dynamics as well as their absence in Liouville ones. Beyond this classical dichotomy between the Diophantine and the Liouville worlds, we discuss some unified approaches and some phenomena that are valid in both worlds. Our focus is mainly on low dimensional dynamics such as circle diffeomorphisms, disc dynamics, quasi-periodic cocycles, or surface flows, as well as finite dimensional Hamiltonian systems.

In an opposite direction, the study of the dynamical properties of some diagonal and unipotent actions on the space of lattices can be applied to arithmetics, namely to the theory of Diophantine approximations. We will mention in the last section some problems related to that topic.

The field of quasi-periodic dynamics is very extensive and has a wide range of interactions with other mathematical domains. The list of questions we propose is naturally far from exhaustive and our choice was often motivated by our research involvements.

1 Arithmetic conditions

A vector $\alpha \in \mathbb{R}^d$ is *non-resonant* if it has rationally independent coordinates: for all $(k_1 \dots, k_d) \in \mathbb{Z}^d$, the identity $\sum_{i=1}^d k_i\alpha_i = 0$ implies $k_i = 0$ for $i = 1, \dots, d$; otherwise, it is called *resonant*.

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For $\gamma, \sigma > 0$, we define the set $DC_d(\gamma,\sigma) \subset \mathbb{R}^d$ of *diophantine* vectors with *exponent* σ and *constant* γ as the set of $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ such that

$$\forall (k_1, \ldots, k_d) \in \mathbb{Z}^d,\ |\sum_{i=1}^d k_i \alpha_i| \geq \frac{\gamma}{(\sum_{i=1}^d |k_i|)^\sigma}; \tag{1-1}$$

we then set $DC_d(\sigma) = \bigcup_{\gamma>0} DC(\gamma,\sigma)$, $DC_d = \bigcup_{\sigma>0} DC(\sigma)$. For each fixed $\sigma > d$ and γ small enough the set $DC(\gamma,\sigma)$ has positive Lebesgue measure in the unit ball of $\mathbb{R}^d$ and the Lebesgue measure of its complement goes to zero as γ goes to zero. Thus the sets $DC(\sigma), \sigma > d$ and DC_d have full Lebesgue measure in $\mathbb{R}^d$. The set DC_d is the set of *Diophantine* vectors of $\mathbb{R}^d$ while its complement in the set of non-resonant vectors is called the set of *Liouville* vectors.

For a translation vectors of $\mathbb{T}^d$ defined as $\alpha + \mathbb{Z}^d$, $\alpha \in \mathbb{R}^d$, we say that it is resonant, Diophantine or Liouville, if the $\mathbb{R}^{d+1}$ vector $(1,\alpha)$ is resonant, Diophantine or Liouville respectively.

2 Diffeomorphisms of the circle and the torus

For $k \in \mathbb{N} \cup \{\infty, \omega\}$ we define $\mathrm{Diff}_0^k(\mathbb{T}^d)$ as the set of orientation preserving homeomorphisms of $\mathbb{T}^d$ of class C^k together with their inverse. To any $f \in \mathrm{Diff}_0^0(\mathbb{T}^d)$ one can associate its *rotation set* $\rho(f) := \{\int_{\mathbb{T}} (\bar{f} - id) d\mu, \mu \in \mathfrak{M}(f)\} \mod \mathbb{Z}^d$ where $\bar{f} : \mathbb{R}^d \to \mathbb{R}^d$ is a lift of f and $\mathfrak{M}(f)$ is the set of all f-invariant probability measures on $\mathbb{T}^d$. Let $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ be the translation $x \mapsto x + \alpha$, $\mathfrak{F}_\alpha^k(\mathbb{T}^d) = \{f \in \mathrm{Diff}_0^k(\mathbb{T}^d),\ \rho(f) = \{\alpha\}\}$, $\mathcal{O}_\alpha^k(\mathbb{T}^d) = \{h \circ T_\alpha \circ h^{-1}, h \in \mathrm{Diff}_0^k(\mathbb{T}^d)\}$. We say that $f \in \mathrm{Diff}_0^\infty(\mathbb{T}^d)$ is *almost reducible* if there exists a sequence $(h_n)_{n\in\mathbb{N}} \in (\mathrm{Diff}_0^\infty(\mathbb{T}^d))^{\mathbb{N}}$ such that $h_n \circ f \circ h_n^{-1}$ converges in the C^∞-topology to T_α. When $d = 1$, $\rho(f)$ is reduced to a single element and we denote by $\rho(f)$ this element. By Denjoy Theorem, any $f \in \mathrm{Diff}_0^k(\mathbb{T})$ with $k \geq 2$, is conjugated by an orientation preserving homeomorphism to T_α. If furthermore α is Diophantine and $k = \infty$ then by Herman-Yoccoz theorem Herman [1979], Yoccoz [1984] this conjugacy is smooth which amounts to $\mathfrak{F}_\alpha^\infty(\mathbb{T}) = \mathcal{O}_\alpha^\infty(\mathbb{T})$. It is of course natural to try to extend this result to the higher dimensional situation where f is an orientation preserving diffeomorphism of the d-dimensional torus $\mathbb{T}^d$. Unfortunately, no Denjoy theorem is available in this situation and the only reasonable question to ask for is the following

Question 1. *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a smooth diffeomorphism of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ which is topologically conjugate to a translation $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$, $x \mapsto x + \alpha$ with α Diophantine. Is the conjugacy smooth?*

Notice that when $d = 2$, even if α is Diophantine, $\mathcal{F}_\alpha^\infty(\mathbb{T}^2)$ is not equal to $\mathcal{O}_\alpha^\infty(\mathbb{T}^2)$ or $\overline{\mathcal{O}_\alpha^\infty(\mathbb{T}^d)}$ as is shown by taking projectivization of cocyles in $SW^\infty(\mathbb{T}, SL(2,\mathbb{R}))$: such cocycles have a uniquely defined rotation number, that can be chosen Diophantine, and at the same time can have positive Lyapunov exponents (which prevents the projective action to be conjugated to a translation) (cf. Herman [1983]). Analogously, by taking projectivization of cocycles in $SW^\omega(\mathbb{T}, SL(2,\mathbb{R}))$ and using Avila's theory characterizing sub-critical/critical cocycles and the Almost Reducibility Conjecture (see Section 5) one can show that there exist elements of $\overline{\mathcal{O}_\alpha^\infty(\mathbb{T}^d)}$ which are not C^∞- almost reducible and, even if $\alpha \in \mathbb{T}^d$ is Diophantine, that the set $\mathcal{O}_\alpha^\infty(\mathbb{T}^d)$ is not closed.

In a similar vein

Question 2. *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a smooth diffeomorphism which is topologically conjugate to the translation with α non-resonant. Is it C^∞-accumulated by elements of $\mathcal{O}_\alpha^\infty(\mathbb{T}^d)$? Is it C^∞-almost reducible?*

When $d = 1$ the first and the second part of the preceding question have a positive answer. Yoccoz proved Yoccoz [1995b] that $\mathcal{F}_\alpha^\infty(\mathbb{T}) = \overline{\mathcal{O}_\alpha^\infty(\mathbb{T})}$ and it is proved in Avila and Krikorian [n.d.(b)] that any smooth orientation preserving diffeomorphism of the circle is C^∞-almost reducible. The proof of this result uses renormalization techniques which at the present time doesn't seem to extend to the higher dimensional case. Still the situation in the semi-local case might be more accessible.

Question 3. *Same questions as in Questions 1 and 2 in the semi-local case that is for f in some neighborhood of the set of rotations, independent of α.*

If one assumes α to be Diophantine and f to be in a neighborhood of T_α that *depends* on α the answer to Question 1 is positive; this can be proved by standard KAM techniques.

3 Pseudo-rotations of the disc

A C^k ($k \in \mathbb{N} \cup \{\infty, \omega\}$) *pseudo rotation* of the disk $\mathbb{D} = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$ is a C^k orientation and *area preserving* diffeomorphism of the disk $\mathbb{D}$ that fixes the origin, leaves invariant the boundary $\partial\mathbb{D}$ of the disk and with *no other periodic point than the origin*. Like in the case of circle diffeomorphisms one can define for such pseudo-rotation a unique rotation number around the origin which is invariant by conjugation (see for example Franks [1988b, Corollary 2.6] or Franks [1988a, Theorem 3.3]). Anosov and Katok [1970] constructed in 1970, *via* approximation by periodic dynamics, *ergodic* (for the area measure) and infinitely differentiable pseudo-rotations of the disk, providing thus the first examples of pseudo-rotations which are not topologically conjugate to rigid rotations. By a theorem of Franks and Handel [2012] a *transitive* area and orientation preserving diffeomorphism of the disk fixing the origin and leaving invariant the boundary of the disk must be a pseudo-rotation.

3.1 Birkhoff rigidity conjecture.

A famous question on pseudo-rotations attributed to Birkhoff is the following.

Question 4. *Is a real analytic pseudo-rotation of angle α analytically conjugated to the rotation R_α of angle α on the disc?*

Addressing this question should involve the artihmetics of α. On one hand, Rüssmann Rüssmann [1967] proved the following alternative for a Diophantine (in fact of Brjuno type is sufficient) elliptic fixed point of a real analytic area preserving surface diffeomorphism f: either the point is surrounded by a positive measure set of invariant circles with different Diophantine frequencies, or the map f is locally conjugate to a rotation in the neighborhood of the fixed point. On the other hand, when the real analytic category is relaxed to infinite differentiability, Anosov-Katok construction provides many counter-examples to the preceding question (for Liouville α's). We can thus divide the preceding question into two questions

Question 5. *Can one construct Anosov-Katok examples (viz. ergodic pseudo-rotations) in the real analytic category? If possible, can one impose the rotation number to be any non-Brjuno number?*

Question 6 (Reducibility)**.** *Is it true that every C^k, $k = \infty, \omega$, pseudo-rotation of the disk with diophantine rotation number α C^k-conjugated to a rigid rotation by angle α?*

Notice that in the smooth category the answer to Question 5 is positive: for the first part this is the existence of Anosov-Katok ergodic, even weak mixing, pseudo-rotations and for the second part one can prove that for any Liouville number α, there exists weak mixing pseudo-rotations as well as examples that are isomorphic to the rotation of frequency α on the circle Fayad and Saprykina [2005] and Fayad, Saprykina, and Windsor [2007]. Together with Herman's last geometric theorem, this gives in the C^∞-case a complete dichotomy between Diophantine and Liouville behavior.

Let $\mathcal{F}_\alpha^\infty$ be the set of C^∞ pseudo-rotations with rotation number α and $\mathcal{O}_\alpha^\infty$ be the set of $h \circ R_\alpha \circ h^{-1}$ where h is a C^∞ area and orientation preserving map of the disk fixing 0 and leaving invariant the boundary of the disk. A weaker question in the smooth case is :

Question 7. *For α diophantine is $\mathcal{O}_\alpha^\infty$ closed for the C^∞-topology?*

In fact, a more general question than Question 6 is the following:

Question 8 (Almost reducibility). *Is any C^k-pseudo-rotation $k = \infty, \omega$, f of the disk with irrational rotation number α almost reducible: there exists a sequence of area preserving smooth map h_n such that $h_n \circ f \circ h_n^{-1}$ converges in the C^k topology to R_α (in the analytic case this convergence should occur on a fixed complex neighborhood of the disk)?*

Question 6 has a positive answer in the local case (Rüssmann for $k = \omega$, Herman, Fayad and Krikorian [2009a] for $k = \infty$) that is when f is in some C^k-neighborhood of R_α (the size of this neighborhood *depending* on the *arithmetics* of α). Thus, a positive answer to Question 8 would imply a positive answer to Question 6. When $k = \infty$, Question 8 (hence Question 6) has a positive answer in the semi-local case Avila and Krikorian [n.d.(b)] that is with the extra assumption that for some k and ε *independent* of α, the C^k-norm of $Df - id$ is less than ε. In this situation one also has $\mathfrak{F}_\alpha^\infty \cap W \subset \overline{\mathcal{O}_\alpha}^\infty$, where W is a neighborhood for the C^∞-topology of the set of rigid rotations. The proof of the result of Avila and Krikorian [ibid.] is based on renormalization techniques and on the fact (proved in Avila, Fayad, Calvez, Xu, and Zhang [2015]) that if one has a control on the C^1-norm of a pseudo-rotation f, the displacement $\max_{\mathbb{D}} \|f - id\|$ polynomially compares with the rotation number of f. Such a control is in general not true for diffeomorphisms of the circle. It is thus natural to ask:

Question 9. *Describe the set of smooth diffeomorphisms of the circle that are obtained as the restriction on $\mathbb{D}$ of the dynamics of pseudo-rotations?*

3.2 Rigidity times, mixing and entropy.

A diffeomorphism of class C^k, $k \in \mathbb{N} \cup \{\infty\}$, is said to admit C^k *rigidity times* (or for short is C^k-rigid) if there exists a sequence q_n such that f^{q_n} converges to the Identity map in the C^k topology. If we just know that the latter holds in a fixed neighborhood of some point p, we say that f is C^k *locally rigid* at p. All the smooth examples on the disc or the sphere obtained by the Anosov-Katok method are C^∞*-rigid* by construction. Obviously, rigidity or local rigidity precludes mixing. Hence, the following natural question was raised in Fayad and Katok [2004] in connection with the smooth realization problem and the Anosov-Katok construction method.

Question 10. *Is it true that a smooth area preserving diffeomorphism of the disc with zero metric entropy is not mixing?*

In the case of zero topological entropy, and in light of Franks and Handel result, the question becomes

Question 11. *Is it true that a smooth pseudo-rotation is not mixing?*

Bramham [2015] proved that this is true if the rotation number is sufficiently Liouville; indeed he proves in that case the existence of C^0-rigidity times. It was shown in Avila, Fayad, Calvez, Xu, and Zhang [2015] that real analytic pseudo-rotations (with no restriction on the rotation number) are never topologically mixing. By a combination of KAM results and control of recurrence for pseudo-rotations with Liouville rotation numbers, it is actually shown that real analytic pseudo-rotations are C^∞ locally rigid near their center.

Note that the following is not known, except in C^1 regularity where a positive answer is given by Bochi [2002].

Question 12. *Does there exist a smooth area preserving disc diffeomorphism that has zero metric entropy and positive topological entropy?*

The following question was raised by Bramham in Bramham [2015].

Question 13. *Does every C^k pseudo-rotation f admit C^0 rigidity times? The question can be asked for any $k \geq 1$, $k = \infty$ or $k = \omega$.*

In the case $k = \omega$ or $\rho(f)$ Diophantine and $k = \infty$, the latter question becomes an intermediate question relative to the Birkhoff-Herman problem on the conjugability of f to the rigid disc rotation of angle $\rho(f)$. In Avila, Fayad, Calvez, Xu, and Zhang [2015] it was shown that for every irrational α, if an analytic pseudo-rotation of angle α is sufficiently close to R_α then it admits C^∞-rigidity times.

Question 14. *Given a fixed analyticity strip, does there exist $\epsilon > 0$ such that if a real analytic pseudo-rotation is ϵ close to the rotation on the given analyticity strip, then it is rigid?*

An *a priori* control on the growth of $\|Df^m\|$ for a pseudo-rotation is sufficient to deduce the existence of rigidity times for larger classes of rotation numbers. If for example a polynomial bound holds on the growth of $\|Df^m\|$ for a smooth pseudo-rotation, then the existence of C^∞ rigidity times would follow for any Liouville rotation number (see Avila, Fayad, Calvez, Xu, and Zhang [ibid.]). In the case of a circle diffeomorphism f a gap in the growth of these norms is known to hold between exponential growth in the case f has a hyperbolic periodic point or a growth bounded by $O(m^2)$ if not Polterovich and Sodin [2004]. Does a similar dichotomy hold for area preserving disc diffeomorphisms?

Question 15. *Is there any polynomial bound on the growth of the derivatives of a pseudo-rotation? Does every C^∞ pseudo-rotation with Liouville rotation number admit C^0 (or even C^∞) rigidity times?*

With Herman's last geometric theorem, a positive answer to the second part of Question 15 would imply that smooth pseudo-rotations, and therefore area preserving smooth diffeomorphisms of the disc with zero topological entropy are never topologically mixing.

In the proof of absence of mixing of an analytic pseudo-rotation, Avila, Fayad, Calvez, Xu, and Zhang [2015] uses an *a priori* bound on the growth of the derivatives of the iterates of a pseudo-rotation that is obtained *via* an effective finite information version of the Katok closing lemma for an area preserving surface diffeomorphism f. This effective result provides a positive gap in the possible growth of the derivatives of f between exponential and sub-exponential.

In Fayad and Zhang [2017], an explicit finite information condition is obtained for area preserving C^2 surface diffeomorphisms, that guarantees positive topological entropy.

Question 16. *Find a finite information condition on the complexity growth of an area preserving C^2 surface diffeomorphism that insures positive metric entropy.*

Finally, inspired by Rüssmann and Herman's last geometric theorem on one hand, and the Liouville pseudo-rotations rigidity on the other, we ask the following

Question 17. *Can a smooth area preserving diffeomorphism of a surface that has an irrational elliptic fixed point be topologically mixing? Can it have an orbit that converges to the fixed point?*

4 Hamiltonian systems

A C^2 function $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ such that $DH(0) = 0$ defines on a neighborhood of 0 a hamiltonian vector field $X_H(x, y) = (\partial_y H(x, y), -\partial_x H(x, y))$ and its flow ϕ_H^t is a flow of symplectic diffeomorphisms preserving the origin. We shall assume that $0 \in \mathbb{R}^{2n}$ is an elliptic equilibrium point with H of the following form

$$H(x, y) = \sum_{j=1}^{d} \omega_j (x_j^2 + y_j^2)/2 + O_3(x, y), \tag{4-2}$$

where the frequency vector ω is *non-resonant*.

Alternatively we may take H a C^2 function defined on $\mathbb{T}^d \times \mathbb{R}^d$ and consider its Hamiltonian flow $X_H(\theta, r) = (\partial_r H(\theta, r), -\partial_\theta H(\theta, r))$. If

$$H(\theta, r) = \langle \omega_0, r \rangle + \mathcal{O}(r^2) \tag{4-3}$$

then the torus $\mathbb{T}^d \times \{0\}$ is invariant under the Hamiltonian flow and the induced dynamics on this torus is the translation $\phi_H^t : \theta \mapsto \theta + t\omega_0$. Moreover this torus is Lagrangian with respect to the canonical symplectic form $d\theta \wedge dr$ on $\mathbb{T}^d \times \mathbb{R}^d$. When ω is Diophantine we say that this torus is a KAM torus.

The stability of an equilibrium or of an invariant quasi-periodic torus by a Hamiltonian flow can be studied from three points of view. The usual topological or Lyapunov stability, the stability in a measure theoretic or probabilistic sense which can be addressed by KAM theory (Kolmogorov, Arnold, Moser), or the effective stability in which one is interested in quantitative stability in time.

4.1 Topological stability. Arnold conjectured that apart from two cases, the case of a sign-definite quadratic part, and generically for $d = 2$, an elliptic equilibrium point is generically unstable.

Conjecture 4.1 (Arnold)**.** *An elliptic equilibrium point of a generic analytic Hamiltonian system is Lyapounov unstable, provided $n \geq 3$ and the quadratic part of the Hamiltonian function at the equilibrium point is not sign-definite.*

Despite a rich literature and a wealth of results in the C^∞ smoothness (to give a list of contributions would exceed the scope of this presentation), this conjecture is wide open in the real analytic category, to such an extent that under our standing assumptions (real-analyticity of the Hamiltonian and a non-resonance condition on the frequency vector) not a single example of instability is known.

Question 18. *Give examples of an analytic Hamiltonian that have a non-resonant elliptic equilibrium (or a non-resonant Lagrangian quasi-periodic torus) that is Lyapunov unstable.*

Question 19. *Give examples of an analytic Hamiltonian that have a non-resonant elliptic equilibrium (or a non-resonant Lagrangian quasi-periodic torus) that attracts an orbit (distinct from the equilibrium or the torus itself).*

In Fayad, Marco, and Sauzin [n.d.] an example is given of a Gevrey regular Hamiltonien on $\mathbb{R}^6$ that has a non-resonant fixed point at the origin and that has an orbit distinct from the origin that converges to it in the future. In Kaloshin and Saprykina [2012] and Guardia and Kaloshin [2014], Arnold diffusion methods are used to yield in particular orbits that have α-limit or ω-limit sets that are non-resonant invariant Lagrangian tori instead of a single non-resonant fixed point.

Following Perez-Marco we ask:

Question 20. *Is it true that a smooth Hamiltonian flow with a non-resonant elliptic equilibrium isolated from periodic points has a hedgehog (a totally invariant compact connected set containing the origin)?*

Regarding the additional stability features of elliptic fixed points in the case of two degrees of freedom, we ask the following

Question 21. *Is the iso-energetic twist condition the optimal condition for Lyapunov stability of an irrational elliptic equilibrium in two degrees of freedom?*

A smooth example of an irrational equilibrium was constructed by F. Trujillo that satisfies the Kolmogorov non degeneracy condition in $d = 2$ degrees of freedom and that has diffusing orbits in some special energy levels.

4.2 Beyond the classical KAM theory. An equilibrium (or an invariant torus) of a Hamiltonian system is said to be KAM stable if it is accumulated by a positive measure of invariant KAM tori, and if the set of these tori has density one in the neighborhood of the equilibrium (or the invariant torus).

4.2.1 Weak transversality conditions. In classical KAM theory, an elliptic fixed point is shown to be KAM-stable under the hypothesis that the frequency vector at the fixed point is non-resonant (or just sufficiently non-resonant) and that the Hamiltonian is sufficiently smooth and satisfies a generic non degeneracy condition of its Hessian matrix at the fixed point. Further development of the theory allowed to relax the non degeneracy condition. In Eliasson, Fayad, and Krikorian [2013] KAM-stability was established for non-resonant elliptic fixed points under the (most general) Rüssmann transversality condition on the Birkhoff normal form of the Hamiltonian. Similar results were obtained for Diophantine invariant tori in Eliasson, Fayad, and Krikorian [2015].

4.2.2 Absence of transversality conditions.

Conjecture 4.2. *[Herman] Prove that an elliptic equilibrium with a diophantine frequency or a KAM torus of an analytic Hamiltonian is accumulated by a set of positive measure of KAM tori.*

Clearly, one can of course ask whether KAM stability also holds.

Conjecture 4.2 was was made by M. Herman in his ICM98 lecture (in the context of symplectomorphisms). The conjecture is known to be true in two degrees of freedom Rüssmann [1967], but remains open in general. It is shown in Eliasson, Fayad, and Krikorian [2015] that an analytic invariant torus $\mathcal{T}_0$ with Diophantine frequency ω_0 is never isolated due to the following alternative. If the Birkhoff normal form of the Hamiltonian at $\mathcal{T}_0$ satisfies a Rüssmann transversality condition, the torus $\mathcal{T}_0$ is accumulated by KAM tori of positive total measure. If the Birkhoff normal form is degenerate, there exists a subvariety of dimension at least $d+1$ that is foliated by analytic invariant tori with frequency ω_0.

For Liouville frequencies, one does not expect the conjecture to hold.

Question 22. *Give an example of an analytic Hamiltonian that has a non-resonant (Liouville) elliptic equilibrium that is not is accumulated by a set of positive measure of KAM tori.*

In the C^∞ category (or Gevrey), counter-examples to stability with positive probability can be obtained: in 2 or more degrees of freedom for Liouville frequencies; and in 3 or more degrees of freedom for any frequency vector (Eliasson, Fayad, and Krikorian [ibid.] for $d \geq 4$ and Fayad and Saprykina [2005] for $d \geq 3$). In the remaining case of

Diophantine equilibrium with $d = 2$, Herman proved stability with positive probability without any twist condition (see Fayad and Krikorian [2009a]).

4.3 Effective stability. Combining KAM theory, Nekhoroshev theory and estimates of Normal Birkhoff forms, it was proven in Bounemoura, Fayad, and Niederman [2017] that generically, both in a topological and measure-theoretical sense, an invariant Lagrangian Diophantine torus of a Hamiltonian system is doubly exponentially stable in the sense that nearby solutions remain close to the torus for an interval of time which is doubly exponentially large with respect to the inverse of the distance to the torus. It is proven there also that for an arbitrary small perturbation of a generic integrable Hamiltonian system, there is a set of almost full positive Lebesgue measure of KAM tori which are doubly exponentially stable. These results hold true for real-analytic but more generally for Gevrey smooth systems. Similar results for elliptic equilibria are obtained in Bounemoura, Fayad, and Niederman [2015].

Question 23. *Give examples of analytic or Geverey differentiable Hamiltonians that have a Diophantine elliptic equilibrium with positive definite twist, that is not more than doubly-exponentially stable in time. Show that this is generic.*

Question 24. *Give an example of an analytic Hamiltonian that has a non-resonant elliptic equilibrium with positive definite twist that is not more than exponentially stable in time.*

Question 25. *Give an example of an analytic Hamiltonian that has a Diophantine elliptic equilibrium that is not more than exponentially stable in time.*

4.4 On invariant tori of convex Hamiltonians.

4.4.1 The "last invariant curve" of annulus twist maps. A classic topic in Hamiltonian systems is that of the regularity of the invariant curves of annulus twist maps. A celebrated result of Birkhoff states that such curves (if they are not homotopic to a point) must be Lipschitz. Numerical evidence seems to indicate that invariant curves are always at least C^1. After Mather and Arnaud we ask the following.

Question 26. *Give an example of a C^r, $r \in [2, \infty) \cup \{\omega\}$, annulus twist map that has an invariant C^0 but not C^1 curve with minimal restricted dynamics.*

In Avila and Fayad [n.d.], a C^1 example is constructed, and Arnaud [2011] gives a C^1 example with an invariant C^0 but not C^1 curve having Denjoy type restricted dynamics.

Due to a result proved by Herman the problem can be reduced to finding a minimal circle homeomorphism f such that $f + f^{-1}$ is C^r but f is only C^0.

Question 27. *Give an example of a C^r, $r \in [2, \infty) \cup \{\omega\}$, annulus twist map that has an invariant C^r curve that is not accumulated by other invariant curves.*

4.4.2 On the destruction of all tori.

Given the Hamiltonian $H = \frac{1}{2}\sum r_i^2$ on $\mathbb{T}^d \times \mathbb{R}^d$.

Question 28. *What is the maximum of r for which it is possible to perturb H so that the perturbed flow has no invariant Lagrangian torus that is the graph of a C^1 function.*

By Herman, $r \geq d + 2 - \epsilon$, $\forall \epsilon > 0$. We also know that $r \leq 2d$ (see Pöschel [1982]). In Cheng and L. Wang [2013], given any frequency ω, a $C^{2d-\epsilon}$ perturbation of H is given that has no invariant Lagrangian torus with as unique rotation frequency vector ω.

4.5 Birkhoff Normal Forms.

Let $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ be a real analytic hamiltonian function admitting 0 as an elliptic non-resonant fixed point. One can always *formally* conjugate H to an *integrable* hamiltonian: there exist a *formal* (exact) symplectic germ of diffeomorphism g tangent to the identity and a *formal* series $N \in \mathbb{R}[[r_1, \ldots, r_d]]$ such that $g_* X_H = X_B$ where $B(x, y) = N(x_1^2 + y_1^2, \ldots, x_d^2 + y_d^2)$. This B is unique and is called the *Birkhoff Normal Form* (BNF). This formal object is an invariant of C^k-conjugations ($k = \infty, \omega$). Birkhoff Normal Forms can be defined for C^k ($k = \infty, \omega$) symplectic diffeomorphisms admitting an invariant elliptic fixed point or even (in the case of symplectic diffeomorphisms or hamiltonian flows) in a neighborhood of an invariant KAM torus (the frequency must be then diophantine). Siegel [1954] proved that in general the conjugating transformation could not be convergent and Eliasson asked whether the Birkhoff Normal Form itself could be convergent. In the real analytic setting Pérez-Marco [2003] proved that for any given non-resonant quadratic part one has the following dichotomy: either the BNF always converges or it generically diverges. Gong [2012] provided an example of divergent BNF with Liouville frequencies. In Krikorian [n.d.] it is proved that the BNF of a real analytic symplectic diffeomorphism admitting a diophantine elliptic fixed point (with torsion) is generally divergent.

Question 29. *Let H be a real analytic Hamiltonian admitting the origin as a diophantine elliptic fixed point and assume that its Birkhoff Normal Form defines a real analytic function. Is H real analytically conjugated to its Birkhoff Normal Form on a neighborhood of the origin?*

5 Dynamics of quasi-periodic cocycles

Let G be a Lie group (possibly infinite dimensional). A quasi-periodic cocycle of class C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$ is a map $(\alpha, A) : \mathbb{T}^d \times G \to \mathbb{T}^d \times G$ of the form $(\alpha, A) : (x, y) \mapsto (x +$

$\alpha, A(x)y)$ where $\alpha \in \mathbb{T}^d$ (we assume α to be non-resonant) and $A : \mathbb{T}^d \to G$ is of class C^k. We denote the set of such cocycles (α, A) by $SW^k(\mathbb{T}^d, G)$ (or $SW^k_\alpha(\mathbb{T}^d, G)$). The iterates $(\alpha, A)^n$ of (α, A) are of the form $(n\alpha, A^{(n)})$ where (for $n \geq 1$) $A^{(n)}$ is the fibered product $A^{(n)}(\cdot) = A(\cdot+(n-1)\alpha)\cdots A(\cdot+\alpha)A(\cdot)$. Two cocycles (α, A_1) and (α, A_2) are said to be C^l-*conjugated* if there exists a map $B : \mathbb{T}^d \to G$ (or $B : \mathbb{R}^d/N\mathbb{Z}^d \to G$ for some $N \in \mathbb{N}^*$) of class C^l such that $(\alpha, A_2) = (0, B)\circ(\alpha, A_1)\circ(0, B)^{-1}$ or equivalently $A_2 = B(\cdot+\alpha)A_1 B(\cdot)^{-1}$. The cocycle (α, A) is said to be *reducible* if it is conjugated to a constant cocycle and, when H is a subgroup of G, H-*reducible* if it is conjugated to an H-valued (not necessarily constant) cocycle. We say that the cocycle is *linear* when the group G is a group of matrices.

5.1 The case $G = SL(2,\mathbb{R})$. Quasi-periodic $SL(2,\mathbb{R})$-valued cocycles play an important role in the theory of *quasi-periodic Schrödinger operators on* $\mathbb{Z}$ of the form $H_x : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, $H_x : (u_n)_{n\in\mathbb{Z}} \mapsto (u_{n+1} + u_{n-1} + V(x + n\alpha)u_n)_{n\in\mathbb{Z}}$; indeed, the (generalized) eigenvalue equation $H_x u = Eu$ leads naturally to studying the dynamics of a family of $SL(2,\mathbb{R})$-valued quasi-periodic cocycles depending on E, the so-called *Schrödinger cocycles*. Many spectral objects or quantities – such as, resolvent sets (complement of the spectrum), spectral measures, density of states, speed of decay of Green functions... – of the family of operators H_x, $x \in \mathbb{T}^d$, can be related to dynamical notions or invariants for the associated family of Schrödinger cocycles – namely (in that order), uniform hyperbolicity, m-functions, fibered rotation number, Lyapunov exponents... We refer to Eliasson [1998], You [2018] for more details on this topic.

There are two important quantities associated to $SL(2\,\mathbb{R})$-valued quasi-periodic cocycles which are invariant by conjugation[1]: the Lyapunov exponent $L(\alpha, A)$ which measure the exponential speed of growth of the iterates of the cocycle (α, A) and the fibered rotation number $\rho(\alpha, A)$ which measures the average speed of rotation of non-zero vectors in the plane under iteration of the cocycle. It is of course tempting to try and classify $SL(2,\mathbb{R})$-cocycles according to these two invariants.

The case of real analytic cocycles with one frequency is particularly well understood. In that situation, following A. Avila [2015], one can associate to any cocycle $(\alpha, A) \in SW^\omega(\mathbb{T}, SL(2,\mathbb{R}))$ a natural family $(\alpha, A_\varepsilon) \in SW^\omega(\mathbb{T}, SL(2,\mathbb{C}))$ (ε in some neighborhood of 0) with $A_\varepsilon(\cdot) = A(\cdot+\varepsilon\sqrt{-1})$. The function $\varepsilon \mapsto L(\alpha, A_\varepsilon)$ plays a very important role in the theory; Avila proved that it is an even convex continuous piecewise affine map with *quantized* slopes in $2\pi\mathbb{Z}$ (this is the phenomenon of "quantization of acceleration") and that the complex cocycle (α, A_ε) is *uniformly hyperbolic* if and only ε is not a break point of $\varepsilon \mapsto L(\alpha, A_\varepsilon)$. This analysis leads to the notions of *critical*, *supercritical* and *subcritical* cocycles, where this last term refers to the fact that the function $\varepsilon \mapsto L(\alpha, A_\varepsilon)$

[1] for the rotation number one has to assume the conjugating map to be homotopic to the identity

is zero on an neighborhood of $\varepsilon = 0$. A cocycle $(\alpha, A) \in SW^{\omega}(\mathbb{T}, SL(2,\mathbb{R}))$ (homotopic to the identity) can thus have four distinct possible behaviors if one adds to the three preceding ones uniform hyperbolicity. Moreover, the quantization of acceleration allows to *predict* the possible transitions between these four regimes and to draw consequences on the spectrum of Schrödinger operators such as for example the possibility of co-existence of absolutely continuous or pure point spectrum for some type of potentials (cf. Avila [ibid.] and for other examples Bjerklöv and Krikorian [n.d.]). The most striking *global* result on the dynamics of these cocycles is certainly the “Almost reducibility conjecture” proved by Avila Avila [2010], Avila [n.d.] which asserts that any *subcritical* cocycle in $SW^{\omega}(\mathbb{T}, SL(2,\mathbb{R}))$ is *almost-reducible* (in the analytic category, on a fixed complex neighborhood of the real axis). By Hou and You [2012], You and Zhou [2013] in the real analytic semi-local situation (viz. when A is close to a constant, this closeness being independent of α) a cocycle (α, A) is either uniformly hyperbolic or subcritical.

In the C^{∞} category, or for many-frequencies systems, our understanding of the dynamics of cocycles is much less complete. There are important reducibility or almost-reducibility results (Dinaburg and J. G. Sinaĭ [1975], Eliasson [1992], Krikorian [1999a], Krikorian [1999b], Krikorian [2001], Avila and Krikorian [2006],Puig [2004], Puig [2006], Fayad and Krikorian [2009b], Avila, Fayad, and Krikorian [2011], Hou and You [2012], You and Zhou [2013], Avila and Krikorian [2015]...) but they often involve diophantine conditions and/or are of perturbative nature. Moreover, the semi-local version of the Almost reducibility conjecture has no reasonable equivalent in the smooth (or even Gevrey) setting Avila and Krikorian [n.d.(c)]. Still, one can ask:

Question 30. *Is the semi-local version of the Almost reducibility conjecture true for cocyles in quasi-analytic classes?*

Let’s say that a cocycle is *stable* if it is not accumulated by non-uniformly hyperbolic systems (with the same frequency vector on the base). Having in mind Avila’s classification one can ask:

Question 31. *Is every stable cocycle in $SW^{k}(\mathbb{T}^{d}, SL(2,\mathbb{R}))$, $k = \infty, \omega$, almost-reducible?*

5.2 The symplectic case. Cocycles in $SW^{k}(\mathbb{T}^{d}, Sp(2n,\mathbb{R}))$ are of interst when one tries to understand the dynamics of a symplectic diffeomorphism in the neighborhood of an invariant torus (they appear as linearized dynamics) or in the study of quasi-periodic Schrödinger operators on strips $\mathbb{Z} \times \{1, \ldots, n\}$. For such cocycles one can define $2n$ Lyapunov exponents (symmetric with respect to 0) and one fibered Maslov index which plays the role of a fibered rotation number (cf. Xu [2016] and the references there).

We denote by $SO(2,\mathbb{R})$ the set of symplectic rotations $R_t = \begin{pmatrix} (\cos t) I_n & -(\sin t) I_n \\ (\sin t) I_n & (\cos t) I_n \end{pmatrix}$.

Question 32. *Let $(\alpha, A) \in SW^{\infty}(\mathbb{T}, Sp(2n,\mathbb{R}))$ homotopic (resp. non homotopic) to the identity where $\alpha \in \mathbb{T}$ is (recurrent) diophantine. Is it true that for Lebesgue almost all $t \in \mathbb{R}$ the following dichotomy holds: either the cocycle $(\alpha, R_t A)$ is C^{∞}-reducible (resp. $SO(2,\mathbb{R})$-reducible) or its upper Lyapunov exponent is positive?*

When $n = 1$ the answer is positive (Avila and Krikorian [2006] for the case homotopic to the identity, Avila and Krikorian [2015] for the case non-homotopic to the identity). The proof of this result is based on a renormalization procedure which works when the cocycle has some mild boundedness property and on a reduction to this case based on Kotani theory. In the case $n \geq 2$ such a Kotani theory was developed by Xu in Xu [2016], Xu [2015]. Following the same strategy as in Avila and Krikorian [2006] one should be then reduced to studying cocycles with values in the maximal compact subgroup of $Sp(2n,\mathbb{R})$. Unfortunately, one cannot conclude like in the case $n = 1$ since no reasonable *a priori* notion of fibered rotation number can be defined for cocycles with values in non-abelian compact groups (they can be defined *a posteriori* once one knows the cocycle is reducible; see Karaliolios [2017], Karaliolios [2016] for related results).

5.3 The case $G = \mathrm{Diff}_0^{\infty}(\mathbb{T})$.

A cocycle $(\alpha, A) \in SW(\mathbb{T}^d, SL(2,\mathbb{R}))$ naturally produces a *projective cocycle* $(\alpha, \bar{A}) \in SW(\mathbb{T}^d, \mathrm{Hom}(\mathbb{S}^1))$ where $\mathrm{Hom}(\mathbb{S}^1)$ is the group of homographies acting on $\mathbb{S}^1$; namely $\bar{A}(x) \cdot v = (A(x)v)/\|A(x)v\|$. It is thus natural to look at the more general case where the underlying group is the group of orientation preserving diffeomorphisms of the circle. In that case one can still define a fibered rotation number Herman [1983]. For the topological aspects of the theory of such quasi-periodically forced circle diffeomorphisms see Bjerklöv and Jäger [2009].

Question 33 (Non-linear Eliasson Theorem)**.** *Let $\alpha \in \mathbb{T}^d$ be a fixed diophantine vector and $G = \mathrm{Diff}_0^{\infty}(\mathbb{R}/\mathbb{Z})$. Does there exist k_0, ε_0 depending only on α such that for any $(\alpha, A) \in SW^{\infty}(\mathbb{T}^d, \mathrm{Diff}_0^{\infty}(\mathbb{R}/\mathbb{Z}))$ of the form $(\alpha, A)(x, y) = (x + \alpha, y + \beta + f(x, y))$ with $\|f\|_{C^{k_0}} \leq \varepsilon_0$ and $\rho(\alpha, A)$ diophantine, the cocycle (α, A) is C^{∞}-reducible?*

When $G = \mathrm{Hom}(\mathbb{S}^1)$ the answer is positive and is (the C^{∞}-version of) a theorem of Eliasson [1992] which has many consequences in the theory of quasi-periodic Schrödinger operators. If one allows ε_0 to depend on ρ then the result is true and is essentially a (generalization of a) theorem by Arnold. Its proof is classical KAM theory. In Krikorian, J. Wang, You, and Zhou [n.d.] a result of rotations-reducibility is proved where ε_0 depends on ρ but with considerably weaker assumption on α than KAM theory usually allows (compare with Avila, Fayad, and Krikorian [2011], Fayad and Krikorian [2009b] for stronger results in the case of linear cocycles).

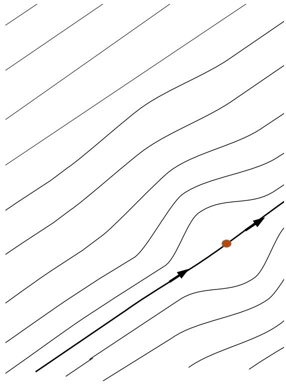

Fig 1. Degenerate saddle acting as a stopping point

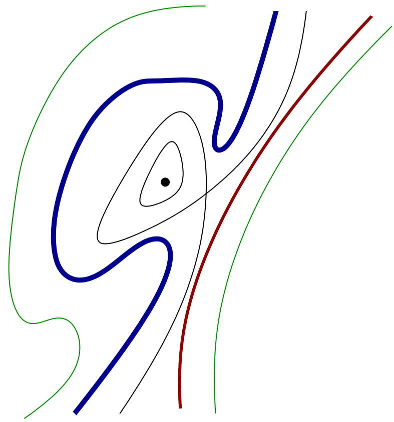

Fig 2. Non-degenerate saddle that causes asymmetry

6 Mixing surface flows

6.1 Spectral type. Area preserving surface flows provide the lowest dimensional setting in which it is interesting to study conservative systems. Such flows are sometimes called multi-valued Hamiltonian flows to emphasize their relation with solid state physics that was pointed out by Novikov [1982]. Via Poincaré sections, these flows are related to special flows above circle rotations or more generally above IETs (Interval exchange transformations). One can thus view them as time changes of translation flows on surfaces.

Katok and then Kochergin showed the absence of mixing of area preserving flows on the two torus if they do not have singularities Katok [1975] and Kočergin [1972].

The simplest mixing examples are those with one (degenerate) singularity on the two torus produced by Kochergin in the 1970s Kočergin [1975]. Kochergin flows are time changes of linear flows on the two torus with an irrational slope and with a rest point (see Figure 1).

Multi-valued Hamiltonian flows on higher genus surfaces can also be mixing (or mixing on an open ergodic component) in the presence of non-degenerate saddle type singularities that have some asymmetry (see Figure 2). Such flows are called Arnol'd flows and their mixing property, conjectured by Arnol'd in Arnold [1991], was obtained by Y. G. Sinaĭ and Khanin [1992] and in more generality by Kochergin [2003, 2004]. Note that Ulcigrai proved in Ulcigrai [2011] that area preserving flows with non-degenerate saddle singularities are generically not mixing (due to symmetry in the saddles).

Question 34. *Study the spectral type and spectral multiplicity of mixing flows on surfaces.*

By spectral type of a flow $\{T^t\}$ we mean the spectral type of the associated Koopman operator $U_t : L^2(M, \mu) : f \to f \circ T^t$.

It was proved in Fayad, Forni, and Kanigowski [2016] that Kochergin flows with a sufficiently strong power like singularity have for almost every slope a maximal spectral type that is equivalent to Lebesgue measure. The study of the spectral multiplicity of these flows is interesting in its relation to the Banach problem on the existence of a dynamical system with simple Lebesgue spectrum. It is probable however that the spectral multiplicity of Kochergin flows is infinite. Mixing reparametrizations of linear flows with simple spectrum were obtained in Fayad [2005] and it would be interesting to study their maximal spectral type following Fayad, Forni, and Kanigowski [2016].

Question 35. *Is it true that Arnol'd mixing flows have in general a purely singular spectral type?*

Arnol'd conjectured a power-like decay of correlation in the non-degenerate asymmetric case, but the decay is more likely to be logarithmic, at least between general regular observables or characteristic functions of regular sets such as balls or squares. Even a lower bound on the decay of correlations is not sufficient to preclude absolute continuity of the maximal spectral type. However, an approach based on slowly coalescent periodic approximations as in Fayad [2006] may be explored in the aim of proving that the spectrum is purely singular.

6.2 Spectral type of related systems.

Question 36. *Prove that all IET have a purely singular maximal spectral type.*

It is known that almost every IET, namely those that are not of constant type, are rigid. It follows that their maximal spectral type is purely singular. For the remaining IETs, partial rigidity was proven by Katok and used to show the absence of mixing, but proving that the spectral type is purely singular appears to be more delicate.

Question 37. *Prove that on $\mathbb{T}^3$ there exists a real analytic strictly positive reparametrization of a minimal translation flow that has a Lebesgue maximal spectral type.*

The difference with the Kochergin flows is that such flows would also be uniquely ergodic. Mixing real analytic reparametrizations of linear flows on $\mathbb{T}^3$ were obtained in Fayad [2002].

6.3 Multiple mixing. The question of multiple mixing for mixing systems is one of the oldest unsolved questions of ergodic theory.

Question 38. *Are all mixing surface flows mixing of all orders?*

Arnold and Kochergin mixing conservative flows on surfaces stand as the main and almost only natural class of mixing transformations for which higher order mixing has not been established nor disproved in full generality. Under suitable arithmetic conditions on their unique rotation vector, of full Lebesgue measure in the first case and of full Hausdorff dimension in the second, it was shown in Fayad and Kanigowski [2016] that these flows are mixing of any order, Kanigowski, Kuaaga-Przymus, and Ulcigrai [n.d.] for flows on higher genus surfaces).

7 Ergodic theory of diagonal actions on the space of lattices and applications to metric Diophantine approximation

The Diophantine properties of linear forms of one or several variables evaluated at integer points are intimately related to the divergence rates of some orbits under some diagonal actions in the space of (linear or affine) lattices of $\mathbb{R}^n$. This link is due to what can be called the Dani correspondence principle between the small values of the linear forms on one hand and the visits to the cusp of certain orbits of certain diagonal actions on the space of lattices (affine lattices in the case of inhomogeneous linear forms). The ergodic study of diagonal and unipotent actions on the space of lattices provides indeed an efficient substitute to the continued fraction algorithm that played a crucial role in the rich metric theory of Diophantine approximations in dimension 1. There is a number of important contributions to number theory related to this principle and to progress in the theory of homogeneous actions for example the surveys Dani [1994], Hasselblatt and Katok [2002], Einsiedler and Lindenstrauss [2006], Eskin [2010], and Marklof [2006, 2007]). We mention here a list of questions related to the statistical properties of Kronecker sequences that can be approached using this same principle. More details and questions can be found in Dolgopyat and Fayad [2015].

7.1 Kronecker sequences. A quantitative measure of uniform distribution of Kronecker sequences is given by the discrepancy function: for a set $\mathcal{C} \subset \mathbb{T}^d$ let

$$D(\alpha, x, \mathcal{C}, N) = \sum_{n=0}^{N-1} \mathbf{1}_{\mathcal{C}}(x + k\alpha) - N\,\text{volume}(\mathcal{C})$$

where $(\alpha, x) \in \mathbb{T}^d \times \mathbb{T}^d$ and $\mathbf{1}_{\mathcal{C}}$ is the characteristic function of the set $\mathcal{C}$.

Uniform distribution of the sequence $x + k\alpha$ on $\mathbb{T}^d$ is equivalent to the fact that, for regular sets $\mathcal{C}$, $D(\alpha, x, \mathcal{C}, N)/N \to 0$ as $N \to \infty$. A step further is the study of the rate of convergence to 0 of $D(\alpha, x, \mathcal{C}, N)/N$.

Already for $d = 1$, it is clear that if $\alpha \in \mathbb{T} - \mathbb{Q}$ is fixed, the discrepancy $D(\alpha, x, \mathcal{C}, N)$ displays an oscillatory behavior according to the position of N with respect to the denominators of the best rational approximations of α. A great deal of work in Diophantine approximation has been done on giving upper and lower bounds to the oscillations of the discrepancy function (as a function of N) in relation with the arithmetic properties of $\alpha \in \mathbb{T}^d$.

In particular, let

$$\overline{D}(\alpha, N) = \sup_{\Omega \in \mathbb{B}} D(\alpha, 0, \Omega, N)$$

where the supremum is taken over all sets Ω in some natural class of sets $\mathbb{B}$, for example balls or boxes.

The case of (straight) boxes was extensively studied, and properties of the sequence $\overline{D}(\alpha, N)$ were obtained with a special emphasis on their relations with the Diophantine approximation properties of α. In particular, Beck [1994] proves that when $\mathbb{B}$ is the set of straight boxes in $\mathbb{T}^d$ then for arbitrary positive increasing function $\phi(n)$

$$\sum_n \frac{1}{\phi(n)} < \infty \Longleftrightarrow \frac{\overline{D}(\alpha, N)}{(\ln N)^d \phi(\ln \ln N)} \text{ is bounded for almost every } \alpha \in \mathbb{T}^d. \tag{7-4}$$

In dimension $d = 1$, this result is the content of Khinchine theorems obtained in the early 1920's, and it follows easily from well-known results from the metrical theory of continued fractions (see for example the introduction of Beck [ibid.]). The higher dimensional case is significantly more difficult and many questions that are relatively easy to settle in dimension 1 remain open. We mention some here and refer to Beck [1994] and Kuipers and Niederreiter [1974] for others.

Question 39. *Is it true that* $\limsup \frac{\overline{D}(\alpha,N)}{\ln^d N} > 0$ **for all** $\alpha \in \mathbb{T}^d$*?*

Question 40. *Is it true that there exists* α *such that* $\limsup \frac{\overline{D}(\alpha,N)}{\ln^d N} < +\infty$*?*

The above questions and results can be asked for balls and more general convex sets.

Question 41. *Is it true that for any* $\epsilon > 0$*, for almost every* $\alpha \in \mathbb{T}^d$ *and for any convex set* $\mathcal{C}$ *in* $\mathbb{T}^d$

$$\frac{D(\alpha, 0, \mathcal{C}, N)}{N^{\frac{d-1}{2d}+\epsilon}}$$

is bounded?

The bound in (7-4) focuses on how bad can the discrepancy become along a subsequence of N, for a fixed α in a full measure set. In a sense, it deals with the worst case scenario and do not capture the oscillations of the discrepancy.

Another point of view is to let $(\alpha, x) \in \mathbb{T}^d \times \mathbb{T}^d$ be random and have limit laws that hold for *all* N. By random we mean distributed according to a smooth density on the tori. For $d = 1$, this was done by Kesten who proved in the 1960s that the discrepancies of the number of visits of the Kronecker sequence to an interval, normalized by $\rho \ln N$ (where ρ depends on the interval but is constant if the length of the interval is irrational) converges to a Cauchy distribution.

One can ask whether Kesten's convergence remains valid for a fixed x. Another question is what happens in higher dimension? In particular :

Question 42. *Is it true that there exists $\rho > 0$, such that when $\mathcal{C}$ is a generic box in $\mathbb{T}^d$ and α is uniformly distributed on $\mathbb{T}^d$, then $\frac{D(\alpha,0,\mathcal{C},N)}{\rho(\ln N)^d}$ converges in distribution to the Cauchy law?*

In Dolgopyat and Fayad [2012] this was proved when x and the box $\mathcal{C}$ are also random (a shape is randomized by applying small deformations distributed according to a smooth measure on the space of isometries). It was shown in Dolgopyat and Fayad [2014] that in the case of a strictly convex shape $\mathcal{C} \subset \mathbb{T}^d$ one has $\frac{D(\alpha,x,r\mathcal{C},N)}{r^{\frac{d-1}{2}} N^{\frac{d-1}{2d}}}$ converges in distribution to a non standard law when $(\alpha, x) \in \mathbb{T}^d \times \mathbb{T}^d$ and $r > 0$ are random. The convex set $r\mathcal{C}$ is the rescaled set from $\mathcal{C}$ by factor r around some fixed point inside $\mathcal{C}$.

A *semialgebraic* set $\mathcal{C}$ in $\mathbb{T}^d$ is a set defined by a finite number of algebraic inequalities. This includes a diverse collection of sets such as balls, cubes, cylinders, simplexes etc. Following Dolgopyat and Fayad [ibid.] we ask

Question 43. *Assume $\mathcal{C}$ is semialgebraic. Does there exist a sequence $a_N = a_N(\mathcal{C})$ such that $\frac{D(\alpha,x,\mathcal{C},N)}{a_N}$ converges in distribution when $(x, \alpha) \in \times\mathbb{T}^d \times \mathbb{T}^d$ are random.*

One can study the fluctuations of the ergodic sums above toral translations for functions other from characteristic functions. The following is interesting for its connection with number theory as well as with the ergodic theory of some natural classes of dynamical systems such as surface flows.

Question 44. *Study the behavior of the ergodic sums $\sum_{n_1=1}^{N} A(x + n\alpha)$ for functions A that are smooth except for a finite number of singularities.*

The fluctuations can be studied for fixed α or x, as well as for random values. One should then try to classify the fluctuations according to the type of the singularities : power, fractional power, logarithmic (we refer to Marklof [2007] and Dolgopyat and Fayad [2015] for more details and questions).

7.2 Higher dimensional actions. Replacing the $\mathbb{Z}$ action by translation with $\mathbb{Z}^k$ actions (see we get following Dolgopyat and Fayad [2015]

Question 45. *Study the ergodic sums* $\sum_{j=1}^{m}\sum_{n_j=1}^{N} A(x + \sum_{j=1}^{m}\alpha_j n_j)$, *with* $(x, \alpha_1, \ldots, \alpha_m) \in (\mathbb{T}^d)^{m+1}$.

In the case where $A = \chi_I - |I|$ and χ_I the indicator of an interval we get the following possible extension of Kesten's theorem to the statistical behavior of linear forms.

Question 46. *Show that as* $x \in \mathbb{T}$ *and* $\alpha \in \mathbb{T}^m$ *are random*

$$\frac{1}{\rho(\ln N)^d}\sum_{j=1}^{m}\sum_{n_j=1}^{N} A(x + \sum_{j=1}^{m}\alpha_j n_j)$$

converges in distribution to a Cauchy law for some $\rho > 0$.

One can also investigate analogues of the Shrinking Targets Theorems of Dolgopyat, Fayad, and Vinogradov [2017] for $\mathbb{Z}^k$ actions.

Question 47. *Let* $l, \hat{l} : \mathbb{R}^d \to \mathbb{R}$, *be linear forms with random coefficients,* $Q : \mathbb{R}^d \to \mathbb{R}$ *be a positive definite quadratic form. Investigate limit theorems, after adequate renormalization, for the number of solutions to*

(a) $\{l(n)\}Q(n) \le c, |n| \le N$;
(b) $\{l(n)\}|\hat{l}(n)| \le c, |n| \le N$;
(c) $|l(n)Q(n)| \le c, |n| \le N$;
(d) $|l(n)\hat{l}(n)| < c, |n| \le N$.

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