ICM 2002 · Vol. II · 583–597

On the Local Langlands Correspondence

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Abstract

The local Langlands correspondence for $GL(n)$ of a non-Archimedean local field F parametrizes irreducible admissible representations of $GL(n,F)$ in terms of representations of the Weil-Deligne group WD_F of F. The correspondence, whose existence for p-adic fields was proved in joint work of the author with R. Taylor, and then more simply by G. Henniart, is characterized by its preservation of salient properties of the two classes of representations.

The article reviews the strategies of the two proofs. Both the author's proof with Taylor and Henniart's proof are global and rely ultimately on an understanding of the ℓ-adic cohomology of a family of Shimura varieties closely related to $GL(n)$. The author's proof with Taylor provides models of the correspondence in the cohomology of deformation spaces, introduced by Drinfeld, of certain p-divisible groups with level structure.

The general local Langlands correspondence replaces $GL(n,F)$ by an arbitrary reductive group G over F, whose representations are conjecturally grouped in packets parametrized by homomorphisms from WD_F to the Langlands dual group ${}^LG$. The article describes partial results in this direction for certain classical groups G, due to Jiang-Soudry and Fargues.

The bulk of the article is devoted to motivating problems that remain open even for $GL(n)$. Foremost among them is the search for a purely local proof of the correspondence, especially the relation between the Galois-theoretic parametrization of representations of $GL(n,F)$ and the group-theoretic parametrization in terms of Bushnell-Kutzko types. Other open questions include the fine structure of the cohomological realization of the local Langlands correspondence: does the modular local Langlands correspondence of Vigneras admit a cohomological realization?

2000 Mathematics Subject Classification: 11, 14, 22.

Introduction

Compared to the absolute Galois group of a number field, e.g. $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the Galois group Γ_F of a non-archimedean local field F has a ridiculously simple structure. Modulo the inertia group I_F, there is a natural isomorphism

$$\Gamma_F/I_F \xrightarrow{\sim} Gal(\bar{k}_F/k_F),$$

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where k_F is the residue field of F. Then $Gal(\bar{k}_F/k_F)$ is topologically generated by the geometric Frobenius $Frob(x) = x^{\frac{1}{q}}$, where $q = |k_F| = p^f$ for p prime. The inertia group has a two step filtration,

$$1 \to P_F \to I_F \to \prod_{\ell \neq p} \mathbb{Z}_\ell \to 1,$$

where the wild ramification group P_F is a pro-p group.

Thus if $\sigma : \Gamma_F \to GL(n, \mathbb{C})$ is a continuous homomorphism, $n \geq 1$, then the image of σ is solvable, and $\sigma(P_F)$ is nilpotent. This is still true when σ is a finite-dimensional complex representation of the *Weil group*, the subgroup $W_F \subset \Gamma_F$ of elements whose image in $Gal(\bar{k}_F/k_F)$ is an integral power of $Frob$. Despite this simplicity, our understanding of the set of equivalence classes of n-dimensional representations of W_F is far from complete, at least when p divides n.

The reciprocity map of local class field theory:

$$F^\times \xrightarrow{\sim} W_F^{ab},$$

identifies the set $\mathcal{G}(1, F)$ of one-dimensional representations of W_F with the set $\mathcal{A}(1, F)$ of irreducible representations of $F^\times = GL(1, F)$. More than a simple bijection, this identification respects a number of salient structures, and its behavior with respect to field extensions F'/F is well understood. Moreover, it is compatible, in a straightforward way, with global class field theory, and was historically first derived as a consequence of the latter.

A simple special case of Langlands' functoriality principle is the so-called strong Artin conjecture, which identifies the Artin L-function attached to an irreducible n-dimensional representation of $Gal(\overline{\mathbb{Q}}/K)$, for a number field K, as the L-function of a cuspidal automorphic representation of $GL(n)_K$. As a local counterpart, Langlands proposed a parametrization of irreducible admissible representations of reductive groups over the local field F in terms of representations of W_F. The prototypical example is the local Langlands conjecture for $GL(n)$. By analogy with the case $n = 1$, the set of equivalence classes of irreducible admissible representations of $G_n = GL(n, F)$ is denoted $\mathcal{A}(n, F)$. By $\mathcal{G}(n, F)$ we denote the set of equivalence classes of n-dimensional representations, not of W_F but rather of the Weil-Deligne group WD_F, and only consider representations for which any lifting of $Frob$ acts semisimply. Then the general local Langlands conjecture for $GL(n)$, in its crudest form, asserts the existence of a family of bijections, as F and n vary:

$$\sigma = \sigma_{n,F} : \mathcal{A}(n, F) \xrightarrow{\sim} \mathcal{G}(n, F). \tag{0.1}$$

A normalization condition is that the central character ξ_π of $\pi \in \mathcal{A}(n, F)$ correspond to $\det \sigma(\pi)$ via local class field theory.

The first general result of this type was proved by Henniart [He1]. Early work of Bernstein and Zelevinsky reduced (0.1) to the existence of bijections

$$\sigma = \sigma_{n,F} : \mathcal{A}_0(n, F) \xrightarrow{\sim} \mathcal{G}_0(n, F), \tag{0.2}$$

where $\mathcal{G}_0(n,F)$ are the irreducible representations of W_F and $\mathcal{A}_0(n,F)$ is the supercuspidal subset of $\mathcal{A}(n,F)$. Both sides of (0.2) are homogeneous spaces under $\mathcal{A}(1,F)$, and thus under its subset $\mathcal{A}^{unr}(1,F)$ of unramified characters χ of $F^\times$: if $\pi \in \mathcal{A}_0(n,F)$ (resp. $\sigma \in \mathcal{G}_0(n,F)$), we denote by $\pi \otimes \chi$ (resp. $\sigma \otimes \chi$) the tensor product of π (resp. σ) with the one-dimensional representation $\chi \circ \det$ of G_n (resp. with the character $\sigma_{1,F}(\chi)$ of W_F). Each $\mathcal{A}^{unr}(1,F)$-orbit on either side of (0.2) has a discrete invariant, the Artin conductor $a(\pi)$, resp. $a(\sigma)$, and the sets $\mathcal{A}_0(n,F)[a]$, resp. $\mathcal{G}_0(n,F)[a]$ of orbits with given Artin conductor a are known to be finite. The main theorem of [He1] is the *numerical local Langlands correspondence*

$$|\mathcal{A}_0(n,F)[a]| = |\mathcal{G}_0(n,F)[a]|, \tag{0.3}$$

established by painstakingly counting both sides.

It has been known for some time that a family of bijections (0.2), compatible with Artin conductors and twists by $\mathcal{A}(1,F)$, is not unique. Henniart showed (the Uniqueness Theorem, [He2]) that at most one normalized bijection is compatible with contragredients and twists and satisfies the condition:

$$L(s,\pi\otimes\pi') = L(s,\sigma(\pi)\otimes\sigma(\pi')); \qquad \varepsilon(s,\pi\otimes\pi',\psi) = \varepsilon(s,\sigma(\pi)\otimes\sigma(\pi'),\psi) \tag{0.4}$$

for $\pi \in \mathcal{A}_0(n,F)$, $\pi' \in \mathcal{A}_0(n',F)$, $n' < n$. Here $\psi : F \to \mathbb{C}^\times$ is a non-trivial character. The L- and ε-factors are defined on the automorphic side in [JPS, Sh]; on the Galois side by Langlands and Deligne (cf. [D]). It is in this version that the local Langlands conjecture for $GL(n)$ has finally been established: for fields of positive characteristic in [LRS], and for p-adic fields in [HT], followed shortly thereafter by [He3] (see also [He5]).

1. Compatibility with global correspondences

As in the first proofs of local class field theory, the bijections (0.2) are constructed in [LRS,HT,He3] by local specialization of maps for certain global fields E:

$$\sigma = \sigma_{n,E} : \mathcal{A}^{good}(n,E) \hookrightarrow \mathcal{G}(n,E). \tag{1.1}$$

Here E is supposed to have a place w such that $E_w \xrightarrow{\sim} F$, $\mathcal{A}^{good}(n,E)$ is a class of cuspidal automorphic representations of $GL(n)_E$ chosen to fit the circumstances, and $\mathcal{G}(n,E)$ can be taken to be the set of equivalence classes of compatible families of n-dimensional semi-simple λ-adic representations of $Gal(\bar{E}/E)$. In particular, both sides of (1.1) as well as (0.2) are taken with ℓ-adic, rather than complex, coefficients; this does not change the problem in an essential way.

The map σ of (0.1) is particularly simple for unramified representations. An unramified $\tau \in \mathcal{G}(n,F)$ is given by an unordered n-tuple $(\chi_1,\ldots,\chi_n)$ of unramified characters of $W_F^{ab} \xrightarrow{\sim} F^\times$. Ordering the χ_i arbitrarily, we obtain a character χ of the Levi subgroup G_1^n of a Borel subgroup $B \subset G_n$. The element of $\mathcal{A}(n,F)$

corresponding to τ is then the unique subquotient $\pi(\tau) = \sigma^{-1}(\tau)$ of the normalized induced representation $Ind_B^{GL(n,F)}\chi$ containing a vector fixed by $GL(n, \mathcal{O}_F)$, $\mathcal{O}_F$ the integer ring of F. This defines a bijection, a special case of the *Satake parametrization*, between the unramified subset $\mathcal{G}^{unr}(n, F)$ and the unramified (spherical) representations $\mathcal{A}^{unr}(n, F)$ of G_n.

Fix an automorphic representation $\Pi = \otimes_v \Pi_v$ of $GL(n)_E$. The representation $\sigma_{n,E}(\Pi)$, when it exists, should have the property that

$$\sigma_{n,E}(\Pi) \mid_{W_{E_v}} = \sigma_{n,E_v}(\Pi_v) \tag{1.2}$$

for almost all v such that $\Pi_v \in \mathcal{A}^{unr}(n, E_v)$; i.e., all but finitely many v. By Chebotarev density, this determines $\sigma_{n,E}(\Pi)$ uniquely. One can then hope that

Hope 1.3. *$\sigma_{n,E}(\Pi)_{W_{E_v}}$ depends only on F and Π_v for all v,*

including $v = w$, the place of interest. Setting $\sigma_{n,F}(\Pi_v) = \sigma_{n,E}(\Pi)_{W_{E_v}}$, one then needs to show that

1.4. *For any $\pi \in \mathcal{A}_0(n, F)$ there exists $\Pi \in \mathcal{A}^{good}(n, E)$, for some E, with $\Pi_w \simeq \pi$;*

1.5. *For $\Pi \in \mathcal{A}^{good}(n, E)$, $\Pi' \in \mathcal{A}^{good}(n', E)$, the completed L-function $\Lambda(s, \sigma_{n,E}(\Pi) \otimes \sigma_{n',E}(\Pi'))$ satisfies the functional equation*

$$\Lambda(s, \sigma_{n,E}(\Pi) \otimes \sigma_{n',E}(\Pi')) = \varepsilon(s, \sigma_{n,E}(\Pi) \otimes \sigma_{n',E}(\Pi'))\Lambda(1 - s, \check{\sigma}_{n,E}(\Pi) \otimes \check{\sigma}_{n',E}(\Pi'));$$

$$\varepsilon(s, \sigma_{n,E}(\Pi) \otimes \sigma_{n',E}(\Pi')) = \prod_v \varepsilon_v(s, \sigma_{n,E}(\Pi) \otimes \sigma_{n',E}(\Pi'), \psi_v)$$

is the product of local Deligne-Langlands ε factors.

Here $\check{}$ denotes contragredient. The local additive characters ψ_v are assumed to be the local components of a continuous character of $\mathbf{A}_E/E$.

1.6. *The map $\sigma = \sigma_{n,F} : \mathcal{A}_0(n, F) \to \mathcal{G}(n, F)$*

(i) *takes values in $\mathcal{G}_0(n, F)$;*
(ii) *defines a bijection $\mathcal{A}_0(n, F) \leftrightarrow \mathcal{G}_0(n, F)$;*
(iii) *satisfies the remaining requirements of a local Langlands correspondence, especially (0.4).*

The main burden of [LRS] is the construction of a class $\mathcal{A}^{good}(n, E)$ large enough to satisfy (1.4): now a moot point, since Lafforgue has proved that all cuspidal automorphic representations of $GL(n)$ of a function field are "good" in this sense. The $\mathcal{A}^{good}(n, E)$ in [LRS] are the automorphic representations that contribute to the cohomology of an appropriate Drinfeld modular variety, constructed from scratch for the occasion, attached to the multiplicative group of a division algebra of dimension n^2 over E, unramified at the chosen w. Property (1.5) in this case follows from general results of Deligne in [D], valid only in equal characteristic. Now by (1.2), for a sufficiently large set S of places of E we have

$$\prod_{v \notin S} L(s, \Pi_v \times \Pi'_v) = \prod_{v \notin S} L(s, \sigma_{n,E_v}(\Pi_v) \otimes \sigma_{n',E_v}(\Pi'_v)), \tag{1.7}$$

where the left-hand side is the Rankin-Selberg L-function. Completing the latter to $\Lambda(s, \Pi \otimes \Pi')$ and applying [JPS] or [Sh], we find the functional equation

$$\Lambda(s, \Pi \otimes \Pi') = \prod_v \varepsilon_v(s, \Pi \otimes \Pi', \psi_v)\Lambda(1-s, \check{\Pi} \otimes \check{\Pi}'). \tag{1.8}$$

In other words, the partial L-functions, identified via (1.8), satisfy *two* functional equations (1.5) and (1.8). An argument first used by Henniart then yields (0.4), and then (1.3) and the full local Langlands conjecture follow from the Uniqueness Theorem of [He2].

When F is p-adic a class $\mathcal{A}^{CK}(n, E)$ satisfying (1.2) is implicitly defined by work of Clozel and Kottwitz [K,Cl1], provided E is a CM field. For $\mathcal{A}^{CK}(n, E)$ one can take cuspidal automorphic representations Π, cohomological at all archimedean primes, square integrable at several finite primes other than w, and such that $\check{\Pi} \simeq \Pi^c$, where c denotes conjugation of E over its maximal totally real subfield. However, the Galois-theoretic functional equation (1.5) is only available a priori when $\sigma_{n,E}(\Pi)$ is associated to a global complex representation of the Weil group of E; i.e. when $\sigma_{n,E}(\Pi)$ becomes abelian over a finite extension of E. The article [H2] showed that there were enough Π of this type in $\mathcal{A}^{CK}(n, E)$. Denoting by $\mathcal{A}^{good}(n, E)$ the set of such Π, we find that (1.4) is impossible as soon as p divides n; however, an argument in [H2], based on Brauer's theorem on induced characters and (0.3), shows that (1.4) is true "virtually," in the set of formal sums with integral coefficients of elements of $\mathcal{A}^{good}(n, E)$ for varying n. It then suffices to prove the following weak form of (1.3), which occupies the bulk of [HT]:

Theorem 1.9 [HT]. *For all $\Pi \in \mathcal{A}^{CK}(n, E)$, the semisimplification $\sigma_{n,E}(\Pi)_{W_{E_v},ss}$ of $\sigma_{n,E}(\Pi)_{W_{E_v}}$ depends only on F and Π_v for all v.*

More precisely, [HT] proves that $\sigma_{n,E}(\Pi)_{W_F,ss}$ can be calculated explicitly in the vanishing cycles of certain formal deformation spaces $M^h_{LT,F}$ defined by Drinfeld (see §2). Following [K,Cl1], the representations $\sigma_{n,E}(\Pi)$ are initially realized in the cohomology of certain Shimura varieties with canonical models over E, and (1.9) is proved by a study of their bad reduction at w. Henniart soon realized that, for $\Pi \in \mathcal{A}^{good}(n, E)$, the purely local nature of $\sigma_{n,E}(\Pi)_{W_{E_v},ss}$, and hence the definition of a map $\sigma_{n,F}$, could be derived directly from (1.5) and from the results of [He1,He2]. Though [He3] dispenses with the geometry, it is still a global argument inasmuch as it relies on [H2], which in turn depends on [K,Cl1] and [CL].the conditional base change results of [CL].

A global consequence of Theorem 1.9 is the *Generalized Ramanujan Conjecture* for the automorphic representations in $\mathcal{A}^{CK}(n, E)$: if $\Pi \in \mathcal{A}^{CK}(n, E)$ and is unitary then its local component Π_v is tempered at every finite prime v. Clozel in [Cl1] already showed this to be true for almost all unramified v. Generalizing a method developed by Lubotzky, Phillips, and Sarnak for the 2-sphere, Clozel [Cl2] uses the version of the Generalized Ramanujan Conjecture proved in [HT] to obtain effective constructions of families of equidistributed points on odd-dimensional spheres.

With (0.1) out of the way, we can propose the following improvement of (1.3):

Problem 1. Show that

$$\sigma_{n,E}(\Pi)_{W_{E_v}} \xrightarrow{\sim} \sigma_{n,E_v}(\Pi_v). \tag{1.10}$$

For $n = 2$ this was established by Carayol assuming standard conjectures on the semisimplicity of Frobenius. Theorem 1.9 shows that it holds up to semisimplification. The techniques of [HT], like the earlier work of Kottwitz treating unramified places, is based on a comparison of trace formulas, and cannot detect the difference between two representations with the same semisimplification. Assuming semisimplicity of Frobenius, the equality (1.10) follows easily from Theorem 1.9 and Deligne's conjecture, apparently inaccessible, on the purity of the monodromy weight filtration.

Compatibility with functoriality. Given cuspidal automorphic representations Π_i of $GL(n_i)_E$, for $i = 1, 2, \ldots, r$, and a homomorphism $\rho : GL(n_1) \times \cdots \times GL(n_r) \rightarrow GL(N)$ of algebraic groups, Langlands functoriality predicts the existence of an automorphic representation $\rho_*(\Pi_1 \otimes \ldots \Pi_r)$, not necessarily cuspidal, of $GL(N)_E$, such that, for almost all unramified places v of E,

$$\sigma_{N,E_v}(\rho_*(\Pi_1 \otimes \ldots \Pi_r)_v) = \rho \circ (\otimes_{i=1}^{r} \sigma_{n_i,E_v}(\Pi_{i,v})). \tag{1.11}$$

In recent years this has been proved for general number fields E in several important special cases: the tensor products $GL(2) \times GL(2) \rightarrow GL(4)$ (Ramakrishnan) and $GL(2) \times GL(3) \rightarrow GL(6)$ (Kim-Shahidi), and the symmetric powers $Sym^3 : GL(2) \rightarrow GL(4)$ (Kim-Shahidi) and $Sym^4 : GL(2) \rightarrow GL(5)$ (Kim). It has been verified in all four cases that (1.11) holds for all v.

Construction of supercuspidal representations by "backwards lifting".

The unitary representation $\pi \in \mathcal{A}_0(n, F)$ is isomorphic to its contragredient if and only if the local factor $L(s, \pi \times \pi)$ has a pole at $s = 0$, which is necessarily simple. The local factor can be decomposed as a product:

$$L(s, \pi \times \pi) = L(s, \pi, Sym^2) L(s, \pi, \wedge^2), \tag{1.12}$$

where the two terms on the right are defined for unramified π by Langlands and in general by Shahidi. Only one of the factors on the right has a pole. Using the class $\mathcal{A}^{good}(n, E)$ of automorphic representations, Henniart has shown that it is the first factor (resp. the second factor) if and only if $\sigma(\pi)$ is orthogonal (resp. symplectic);

the symplectic case only arises for n even. One thus expects that π is obtained by functorial transfer from an L-packet of a classical group G over F, via the map of L-groups ${}^LG \to GL(n,\mathbb{C})$, where ${}^LG = SO(n,\mathbb{C})$, resp. $Sp(n,\mathbb{C})$, if the first, resp. the second factor in (1.12) has a pole at $s = 0$.

In particular, when $n = 2m$ and $L(s,\pi,\wedge^2)$ has a pole at $s = 0$, π should come from an L-packet on the split group $SO(2m+1,F)$. Using a local analogue of the method of "backwards lifting," or automorphic descent, due to Ginzburg, Rallis, and Soudry, Jiang and Soudry have constructed a generic supercuspidal representation π' of $SO(2m+1,F)$ for every $\pi \in \mathcal{A}_0(n,F)$ with the indicated pole. More generally, they have obtained a complete parametrization of generic representations of split $G = SO(2m+1,F)$ in terms of Langlands parameters $WD_F \to {}^LG$ [JS]. These results should certainly extend to other classical groups.

2. Cohomological realizations of the local correspondence

The theory of the new vector implies easily that any irreducible admissible representation $\pi \in \mathcal{A}(n,F)$ has a rational model over the field of definition of its isomorphism class: the Brauer obstruction is trivial for G_n. The analogous assertion fails for representations in $\mathcal{G}(n,F)$. Thus one cannot expect the existence of a space $\mathcal{M}$, with a natural action of $G_n \times W_F$, whose cohomology of whatever sort realizes the local Langlands correspondence, as an identity of virtual representations

$$\sigma_{n,F}(\pi) = \pm[Hom_{G_n}(H_c(\mathcal{M}),\pi)] := \pm\sum_i (-1)^i Hom_{G_n}(H_c^i(\mathcal{M}),\pi). \qquad (2.1)$$

We add a third group to the picture by taking J to be an inner form of G_n, the multiplicative group of a central simple algebra D over F of dimension n^2, with Hasse invariant $\frac{r_D}{n}$. The set $\mathcal{A}(n,F)$ contains a subset $\mathcal{A}_{(2)}(n,F)$ of discrete series representations, character twists of those realized in the regular representation on $L_2(G_n)$ (modulo center). The set $\mathcal{A}(J)$ of equivalence classes of irreducible admissible representations contains an analogous subset $\mathcal{A}_{(2)}(J)$, equal to $\mathcal{A}(J)$ if D is a division algebra. The *Jacquet-Langlands correspondence* [R,DKV] is a bijection $JL : \mathcal{A}_{(2)}(G_n) \xrightarrow{\sim} \mathcal{A}_{(2)}(J)$ determined by the identity of distribution characters

$$\chi_\pi(g) = \varepsilon(J)\chi_{JL(\pi)}(j), \ \pi \in \mathcal{A}_{(2)}(G) \qquad (2.2)$$

if $\varepsilon(J) = \pm 1$ is the Kottwitz sign and g and j are elliptic regular elements with the same eigenvalues. When $r_D = 1$ there are two spaces $\tilde{\Omega}_F^n$ and $\mathcal{M}_{LT,F}^n$ with natural $G_n \times J$-actions. The former is a countable union, indexed by $\mathbb{Z}$, of copies of the profinite étale cover $\tilde{\Omega}_F^{n,0}$ of the rigid-analytic upper half space $\Omega_F^n = \mathbb{P}^{n-1}(\mathbb{C}_p) - \mathbb{P}^{n-1}(F)$, defined by Drinfeld in [D2]. The latter is the rigid generic fiber of the formal deformation space $M_{LT,F}^n$ of the one-dimensional height n formal $\mathcal{O}_F$-module with Drinfeld level structures of all degrees [D1]. A relation analogous to (2.1) was conjectured by Carayol in [C1], with $\pm = (-1)^{n-1}$:

Theorem 2.3. *For π supercuspidal*

$$\sigma^{\#}(\pi) \otimes JL(\pi) = \pm[Hom_{G_n}(H_c(\tilde{\Omega}^n_F), \pi)]$$
$$\sigma^{\#}(\pi) \otimes \pi = \pm[Hom_J(H_c(\mathcal{M}^n_{LT,F}), JL(\pi))].$$

The notation $\sigma^{\#}(\pi)$ indicates that $\sigma(\pi)$ has been twisted by an elementary factor. We use the rigid-analytic étale cohomology introduced by Berkovich [B] with coefficients in $\overline{\mathbb{Q}}_\ell$, $\ell \neq p$. For $\mathcal{M}^n_{LT,F}$ this can be interpreted as a space of vanishing cycles for the formal deformation space, viewed as a formal scheme over $Spf(\mathcal{O}_F)$. The case of $\tilde{\Omega}^n_F$ was proved in [H1], using the existence of Shimura varieties admitting rigid-analytic uniformizations by $\tilde{\Omega}^n_F$. This has recently been extended to F of equal characteristic by Hausberger [Hau]. The case of $\mathcal{M}^n_{LT,F}$, again for π supercuspidal, was initially treated by Boyer [Bo] in the equal-characteristic case. The analogous statement for F p-adic, and for any π, is the logical starting point of the proof of Theorem 1.9 in [HT].

Theorem 2.3 is extended in [HT] to general $\pi \in \mathcal{A}_{(2)}(G)$. The explicit formula for the alternating sum of the $Hom_J(H^i_c(\mathcal{M}^n_{LT,F}), JL(\pi))$ is awkward but yields a simple expression for

$$\sum_{i,j}(-1)^{i+j} Ext^j_G(Hom_J(H^i_c(\mathcal{M}^n_{LT,F}), JL(\pi)), \pi)$$
$$= \sum_{i,j,k}(-1)^{i+j+k} Ext^j_G(Ext^k_J(H^i_c(\mathcal{M}^n_{LT,F}), JL(\pi)), \pi) \quad (2.4)$$

in terms of the semisimplification of $\sigma(\pi)$. An analogous *conjectural* expression for *individual* $H^i_c(\tilde{\Omega}_F)$ has been circulating for several years and should appear in a forthcoming joint paper with Labesse. Faltings has proved [F2] that the spaces $\tilde{\Omega}_F$ and $\mathcal{M}^n_{LT,F}$ become isomorphic after p-adic completion of the latter. Thus the two questions in the following problem reduce to a single question:

Problem 2. Determine the individual representations $H^i_c(\mathcal{M})$, and the spaces $Ext^j_{G_n}(H^i_c(\tilde{\Omega}_F), \pi)$ and $Hom_J(H^i_c(\mathcal{M}^n_{LT,F}), JL(\pi))$ for all i, j, all $\pi \in \mathcal{A}(n, F)$. In particular, show that $Ext^j_{G_n}(H^i_c(\tilde{\Omega}_F), \pi)$ vanishes unless there exists $\pi' \in \mathcal{A}_{(2)}(n, F)$ such that π and π' induce the same character of the Bernstein center.

The results of [HT] imply that, for any $\pi \in \mathcal{A}_{(2)}(n, F)$, with Bernstein character β_π, the Bernstein center acts on $\sum_i (-1)^i Hom_J(H^i_c(\mathcal{M}^n_{LT,F}), JL(\pi))$ via β_π.

For π supercuspidal it is known in all cases that the spaces in Problem 2 vanish for $i \neq n-1$ (and for $j \neq 0$). This vanishing property should characterize supercuspidal π among representations in $\mathcal{A}_{(2)}(n, F)$. When π is the Steinberg representation, the $H^i_c(\Omega_F)$, as well as the corresponding Ext groups, are calculated explicitly in [SS]. The calculation in [SS] is purely local, whereas the vanishing outside the middle degree for $\pi \in \mathcal{A}_0(n, F)$ is based on properties of automorphic forms.

Problem 3. Find a purely local proof of the vanishing property for $\pi \in \mathcal{A}_0(n, F)$.

The covering group of $\tilde{\Omega}_F^{n,0}$ over Ω_F^n can be identified with the maximal compact subgroup $J^0 \subset J$. Thus $H_c^i(\tilde{\Omega}_F)$ can be written as a sum $\oplus_\tau H_c^i(\tilde{\Omega}_F)[\tau]$ of its τ-isotypic components, where τ runs over irreducible representations of J^0 or, equivalently, over inertial equivalence classes of representations of J. Closely related to Problem 3 is the following

Problem 4. Characterize $\tau \in \mathcal{A}(J)$ such that $JL^{-1}(\tau) \in \mathcal{A}_0(n, F)$. Equivalently, calculate the Jacquet functors of the G_n-spaces $H_c^i(\tilde{\Omega}_F)[\tau]$ geometrically, in terms of τ.

When n is prime $JL^{-1}(\tau) \in \mathcal{A}_0(n, F)$ if and only if $\dim \tau > 1$; when $\dim \tau = 1$ $JL^{-1}(\tau)$ is a twist of the Steinberg representation. For general n practically nothing is known.

Results of L. Fargues [Fa]. For certain classical F-groups G, Rapoport and Zink, using the deformation theory of p-divisible (Barsotti-Tate) groups, have defined pro-rigid analytic spaces $\mathcal{M}$ admitting continuous $G \times J \times W_E$ actions on their ℓ-adic cohomology, where J is an inner form of G and E, the *reflex field* of $\mathcal{M}$, is a finite extension of F [RZ]. In [R] Rapoport proposes a conjectural formula, which he attributes to Kottwitz, for the discrete series contribution to the virtual $G \times J \times W_E$-module $[H(\mathcal{M})] = \sum_i (-1)^i H_c^i(\mathcal{M}, \overline{\mathbb{Q}}_\ell)$. The pairs (G, J) considered in [RZ] include $(G_n, D^\times)$ with general Hasse invariant $\frac{r_D}{n}$, $G = J = GU(n)$, the quasi-split unitary similitude group attached to the unramified quadratic extension of F, and the symplectic similitude group $G = GSp(2n, F)$.

Theorem 2.5 (Fargues). *Suppose $F/\mathbb{Q}_p$ unramified, $(G, J) = (G_n, D^\times)$, with $(r_D, n) = 1$. For any $\pi \in \mathcal{A}_0(n, F)$ we have*

$$\sum_i (-1)^i Hom_J(H_c^i(\mathcal{M}, \overline{\mathbb{Q}}_\ell), JL(\pi))_0 = \pm \pi \otimes \wedge^? \sigma(\pi)$$

up to a simple twist. Here the subscript $_0$ denotes the G-supercuspidal part and $\wedge^?$ is a certain tensor product of exterior powers of $\sigma(\pi)$ with total weight r_D, depending on auxiliary data defining $\mathcal{M}$.

This confirms the Kottwitz-Rapoport conjectures in the case in question. For $G = J = GU(3)$ Rogawski has defined a local Langlands correspondence via base change to $GL(3)$. In that case the supercuspidal representations of G are grouped into L-packets. Fargues's techniques apply to this case as well, and he obtains a version of the Kottwitz-Rapoport conjectures, more difficult to state than Theorem 2.5 (higher Ext's are involved, and the formula is averaged over L-packets).[1] More

[1]The statement of the general Kottwitz-Rapoport conjectures in [H3] for general discrete series representations is based on a misreading of Rapoport's use of the term "discrete L parameter". The correct conjecture should involve the analogue of the alternating sum on the right-hand side of (2.4), with $JL(\pi)$ replaced by π' in the L-packet associated to π.

generally, Fargues' methods apply to classical groups attached to Shimura varieties, whenever the trace formula is known to be stable and functorial transfer from G to $GL(n)$ has been established.

In contrast to [HT], Fargues' methods are essentially rigid-analytic, and make no use of equivariant regular integral models of Shimura varieties in wildly ramified level – fortunately so, since such models are not known to exist. Heuristically, the characters of the representations of G and J on $[H(\mathcal{M})]$ can be related by applying a Lefschetz trace formula to ℓ-adic cohomology of the rigid space $\mathcal{M}$. This approach, which in principle provides no information about the W_F action, has been successfully applied to $\tilde{\Omega}^n_F$ by Faltings in [F1], and to $M^n_{LT,F}$ by Strauch [S] when $n = 2$. For higher $M^n_{LT,F}$, and for the Rapoport-Zink spaces studied by Fargues, one does not yet know how to deal with wild boundary terms in Huber's Lefschetz formula [Hu] and its higher-dimensional generalizations.

Using work of Oort and Zink on stratification of families of abelian varieties and the slope filtration for p-divisible groups, Mantovan [M] has developed another approach to the cohomology of Shimura varieties of PEL type. Closer in spirit to [HT] than to [F], [M] obtains finer results on the geometry of the special fiber and a description of the cohomology in ramified level similar to that of [F].

Cohomological realizations with torsion coefficients.

It would be convenient if the following question had an affirmative answer:

Question 5. Is $H^i_c(\tilde{\Omega}^n_F, \mathbb{Z}_\ell)$ a torsion-free $\mathbb{Z}_\ell$-module?

The global trace formula methods used in [H1] and [HT] to derive Theorem 2.3 from an analysis of the cohomology of the "simple" Shimura varieties of the title of [K] are insensitive to torsion in cohomology. When $\ell > n$ it may be possible, as in recent work of Mokrane and Tilouine, to combine ℓ-adic Hodge theory with the generalized Eichler-Shimura congruence formula, for the same "simple" Shimura varieties, to answer Question 5. For $\ell \leq n$ completely new ideas are needed.

When k is an algebraically closed field of characteristic $\ell \neq p$, Vignéras has defined a class of smooth supercuspidal representations $\mathcal{A}_{0,k}(n, F)$ of G_n with coefficients in k, and has proved that they are in bijection with the set $\mathcal{G}_{0,k}(n, F)$ of irreducible n-dimensional representations of W_F over k (see article in these Proceedings). It is natural to expect that this modular local Langlands correspondence is realized on the spaces $H^\bullet_c(\mathcal{M}, k)$, with $\mathcal{M} = \tilde{\Omega}^n_F$ or $\mathcal{M}^n_{LT,F}$.

Problem 6. Define a modular Jacquet-Langlands map $\pi \mapsto JL(\pi)$ from $\mathcal{A}_{0,k}(n, F)$ to k-representations of J, and formulate the last sentence precisely. Does the virtual W_F-module

$$(-1)^{n-1} \sum_{i,j,k} (-1)^{i+j+k} Ext^j_G(Ext^k_J(H^i_c(\mathcal{M}, k), JL(\pi)), \pi)$$

realize the modular local Langlands correspondence?

Implicit in the second question is the assumption that the modular Jacquet-Langlands map can be extended to a wider class of k-representations of G_n, perhaps including reduction (mod ℓ) of supercuspidal representations in characteristic zero. One can of course ask the same questions when $\ell = p$. In this case we can consider rigid (de Rham) cohomology, in the sense of Berthelot, as well as p-adic étale cohomology. All three groups G_n, J, and W_F have large analytic families of p-adic representations. It is not at all clear whether the p-adic cohomology of $\tilde{\Omega}^n_F$ is sufficiently rich to account for all p-adic deformations – in categories yet to be defined – of a given representation occurring in cohomology with coefficients in $\overline{\mathbb{F}}_p$.

3. Explicit parametrization of supercuspidal representations

Distribution characters.

The distribution character χ_π, a locally integrable function on the set of regular semisimple elements of $G_n = GL(n, F)$, is the fundamental analytic invariant of $\pi \in \mathcal{A}(n, F)$. For $\pi \in \mathcal{A}_{(2)}(n, F)$, $\chi_{JL(\pi)}$, related to χ_π by (2.2), extends continuously to an invariant function on $J = D^\times$ provided $(r_D, n) = 1$, which we assume. Under this hypothesis every element of J is elliptic and every elliptic regular element j is contained in a unique maximal torus $T(j)$, isomorphic to the multiplicative group of an extension K of F of degree n. Since $JL(\pi)$ is finite-dimensional, its restriction to $T(j)$ equals $\sum_\xi a_\pi(\xi)\xi$ where ξ runs over characters of $K_j^\times$ and the coefficients $a_\pi(\xi) = a_\pi(K, \xi)$ are non-negative integers, almost all zero. In this way $\pi \in \mathcal{A}_{(2)}(n, F)$ is determined by the integer-valued function $a_\pi(K, \xi)$ where K runs over degree n extensions of F and ξ over characters of $K^\times$. Invariance entails the symmetry condition $a_\pi(K', {}^\sigma\xi) = a_\pi(K, \xi)$ where $\sigma : K \xrightarrow{\sim} K'$ is an isomorphism over F; in particular, if $\sigma \in Aut_F(K)$.

Problem 7. Express $a_\pi(K, \xi)$ in terms of numerical invariants of $\sigma(\pi)$.

Of course $a_\pi(K, \xi) = 0$ unless $\xi|_{F^\times}$ coincides with the central character ξ_π of π. When $n = 2$ $a_\pi(K, \xi) \in \{0, 1\}$, and a theorem of Tunnell, completed by H. Saito, relates the nonvanishing of $a_\pi(K, \xi)$ to the local constant $\varepsilon(\frac{1}{2}, \sigma(\pi) \otimes \xi^{-1}, \psi)$. For n prime to p a conjecture of Reimann, following an earlier conjecture of Moy, expresses χ_π in terms of $\sigma(\pi)$; work in progress of Bushnell and Henniart shows that this conjecture is almost right (probably up to an unramified character of degree at two).

Parametrization via types.

A fundamental theorem of Bushnell and Kutzko asserts that every supercuspidal π can be obtained by compactly supported induction from a finite-dimensional representation τ of a subgroup $H \subset G_n$ which is compact modulo the center Z_n of G_n. The pair (H, ρ), called an *extended type*, is unique up to conjugation by G_n. The character χ_π can be obtained from (H, ρ) by a simple integral formula [BH, (A.14)].

The outstanding open problem concerning the local Langlands correspondence is undoubtedly

Problem 8. (a) Define $\sigma(\pi)$ directly in terms of (H,ρ) (and vice versa).

(b) Show directly that the definition of σ in (a) has the properties of a local Langlands correspondence.

Note that (b) presupposes a direct construction of the local Galois constants.

Problem 8 formulates the hope, often expressed, for a purely local construction of the local Langlands correspondence. Bushnell, Henniart, and Kutzko have made considerable progress toward this goal. Among other results, they have obtained:

- A formula for the conductor $a(\pi \times \pi')$, $\pi \in \mathcal{A}_0(n,F)$, $\pi' \in \mathcal{A}_0(n',F)$ [BHK];
- A purely local candidate for the base change map $\mathcal{A}(n,F) \rightarrow \mathcal{A}(n,K)$ when K/F is a tame, not necessarily Galois extension [BH, I], agreeing with Arthur-Clozel base change for K/F cyclic;
- A bijection between wildly ramified supercuspidal representations of G_{p^m} and wildly ramified[2] representations in $\mathcal{G}_0(p^m,F)$, preserving local constants [BH, II].

In each instance, the constructions and proofs are based primarily on the theory of types. A complete solution of Problem 8 remains elusive, however, absent a better understanding of the local Galois constants.

Question 9. Can the types (H,ρ) be realized in the cohomology (ℓ-adic or p-adic) of appropriate analytic subspaces of $\tilde{\Omega}^n_F$ or $\mathcal{M}^n_{LT,F}$?

Positive results for certain (H,ρ) have been announced by Genestier and Strauch, at least when $n=2$.

Acknowledgments. I thank R. Taylor, G. Henniart, and L. Fargues for their comments on earlier versions of this report.

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More or less detailed accounts of the history of the local Langlands conjecture, and of its proofs, can already be found in the literature: [Rd] and [Ku] describe the problem and the work of Bernstein and Zelevinsky, while the proofs are outlined in [C2,C3], [He4], [W], as well as the introduction to [HT].

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