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G. M. HENKIN

Tangent Cauchy–Riemann Equations and the Yang–Mills, Higgs and Dirac Fields

The first part of the paper contains a survey of conditions for the local and global solvability of the tangent Cauchy–Riemann equations on q-concave CR-manifolds.

In the second part results are presented concerning the representation (by the Radon–Penrose type transformation) of the classical Yang–Mills, Higgs and Dirac fields as solutions of the Cauchy–Riemann equations on 1-concave submanifolds of twistor (or supertwistor) space.

1. Cauchy–Riemann equations on q-concave CR-manifolds

1.1. $\bar\partial_\tau$-closed forms and their local approximation by $\bar\partial_\tau$-exact forms. Let X be an n-dimensional complex manifold, let E be a holomorphic vector bundle over X. We denote by $\bar\partial$ a Cauchy–Riemann operator annihilating all holomorphic sections of the fibre bundle E over X. Let L be a real, closed submanifold of X of co-dimension k which can be represented in each coordinate neighbourhood $\Omega \subset X$ in the form:

$$L \cap \Omega = \{z \in \Omega \colon \varrho_1(z) = \ldots = \varrho_k(z) = 0\}, \tag{1.1}$$

where $\{\varrho_\nu\}$ are smooth real-valued functions in the domain $\Omega \subset X$ satisfying the condition $\bar\partial\varrho_1 \wedge \ldots \wedge \bar\partial\varrho_k \neq 0$ on $L \cap \Omega$.

For a fixed point $p \in L$ the complex tangent space $T_p^c(L)$ has a complex dimension $n-k$ and in local coordinates $z = (z_1, \ldots, z_n)$ is determined by the equations

$$T_p^c(L) = \left\{\zeta \in C^n \colon \sum_{j=1}^{n} \frac{\partial \varrho_\nu}{\partial z_j}(p)\,\zeta_j = 0, \quad \nu = 1, 2, \ldots, k\right\}.$$

Such a manifold is called a (generic) CR-manifold.

Let us denote by $C_{0,q}^{(s)}(L, E)$, $0 \leqslant q \leqslant n-k$, $s \geqslant 0$, the space of differential forms of type $(0, q)$ on M with E-valued $C^{(s)}$-smooth coefficients.

If $f \in C_{0,q-1}(L, E)$ and $g \in C_{0,q}(L, E)$ are such that for any compactly supported form $\varphi \in C_{n,n-k-q}^{(\infty)}(X, E^*)$, where E^* is the fibre bundle dual to E, we have

$$\int_L g \wedge \varphi = (-1)^q \int_L f \wedge \bar{\partial}\varphi,$$

then, by definition, we shall write

$$\bar{\partial}_\tau f = g, \tag{1.2}$$

where $\bar{\partial}_\tau$ is the tangent Cauchy–Riemann operator.

A necessary condition for (1.2) to be locally solvable is, first of all, the condition $\bar{\partial}_\tau g = 0$ on L. The forms (resp. the functions) satisfying this condition are called CR-forms (resp. CR-functions).

If the manifold L and the form g are real-analytic then the condition $\bar{\partial}_\tau g = 0$ is also sufficient for the local solvability of (1.2). In this case the dimensions of the domains in M where (1.2) is solvable depend not only on the manifold L but also on real-analytic properties of the CR-form g (see [43]).

If either the form g or the manifold M is not real-analytic, then, generally speaking, the condition $\bar{\partial}_\tau g = 0$ is no longer sufficient for the local solvability of (1.2) (Hans Lewy's effect).

In this case, however, an important general result concerning the approximate local solvability of equation (1.2) is proved by M. Baouendi and F. Treves.

THEOREM 1.1 (F. Treves [45]). *Let L be a smooth CR-submanifold (of co-dimension k) in a complex manifold X. Then for any point $p \in L$ there exists a neighbourhood Ω_p such that every CR-form $g \in C_{0,r}^{(s)}(L)$, $r = 0, 1, \ldots$ $\ldots, n-k$, can be approximated on $L \cap \Omega_p$ by $\bar{\partial}$-closed forms from $C_{0,r}^{(\infty)}(\Omega_p)$ as exactly as desired in $C^{(s)}$-topology. If, moreover, $r > 0$ then these forms are $\bar{\partial}_\tau$-exact on $L \cap \Omega_p$.*

For hypersurfaces this result was proved earlier (see [1, 13]).

1.2. q-concave CR-manifolds and the local exactness of CR-forms of type $(0, r)$ for $r < q$ and $r > n-k-q$. The study of conditions for the exact (and not merely approximate) local and global solvability of the equation (1.2), initiated by H. Lewy in his classical work, has been carried sufficiently far for the case where L is a hypersurface in X (see J. J. Kohn [25],

J. Kohn, H. Rossi [26], A. Andreotti, C. D. Hill [1], G. M. Henkin [13], A. Bogges [6]).

In recent years the results of these works have been generalized to the case of CR-manifolds of arbitrary co-dimension. The formulations of the main results use E. Levi's form of a manifold L. The last one is given by the equality

$$L_{p,\lambda}(L)(\zeta) = \sum_{\nu=1}^{k} \sum_{\alpha,\beta=1}^{n} \lambda_\nu \left(\frac{\partial^2 \varrho_\nu}{\partial z_\alpha \partial \bar{z}_\beta}(p)\zeta_\alpha \bar{\zeta}_\beta\right),$$

where $p \in L$, $\zeta \in T_p^c(L)$, $\lambda \in R^k$.

The manifold L is said to be q-concave (resp. weak q-concave) at the point $p \in L$, if for all $\lambda \in R^k \setminus \{0\}$ the form $L_{p,\lambda}(L)$ has on $T_p^c(L)$ at most q negative (resp. q nonpositive) eigenvalues.

The basic theorems on the local solvability of the equation (1.2) result from the following general proposition on the $\bar{\partial}$-closed extension of CR-forms into a neighbourhood of a generic CR-manifold.

THEOREM 1.2 (G. M. Henkin [16]). *If the CR-manifold of the form* (1.1) *is q-concave then there exists a neighbourhood X' of the manifold L, such that for all r satisfying condition $0 \leqslant r < q$ or $n-k-q < r \leqslant n-k$, and for any CR-form $f \in C_{0,r}^{(s)}(L, E)$, $s \geqslant 0$, there exists a $\bar{\partial}$-closed form $F \in C_{0,r}^{(s-1/2-\varepsilon)}(X', E)$ such that $F|_L = f$ and $F \wedge \partial\varrho_1 \wedge \ldots \wedge \bar{\partial}\varrho_k \in C^{(s)}(\Omega \cap X')$ for every coordinate neighbourhood Ω on X.*

The method used in the proof of Theorem 1.2 actually gives an explicit integral formula for F in terms of f.

In the case of CR-functions Theorem 1.2 confirms a conjecture due to I. Naruki [35], where the respective statement concerning CR-functions was proved for "standard" CR-manifolds, i.e. for manifolds of the type

$$L = \{(z, w) \in C^k \times C^{n-k}\colon \operatorname{Im} z_\nu = F_\nu(w, \bar{w}),\ \nu = 1, 2, \ldots, k\},$$

where $\{F_\nu\}$ are Hermitian forms on C^{n-k}.

For hypersurfaces and $s = \infty$ Theorem 1.2 was first obtained in a paper by A. Andreotti and C. D. Hill [1].

From Theorem 1.2 the following facts can be deduced concerning the local solvability of the equation (1.2).

THEOREM 1.2a ([16]). *Under the conditions of Theorem* 1.2 *for any point $p \in L$ and any sufficiently small neighbourhood Ω_p of the point p, for all r: $1 \leqslant r < q$, and for any CR-form $g \in C_{0,r}^{(s)}(L, E)$, there exists a form $f \in C_{0,r-1}^{(s+1/2-\varepsilon)}(L \cap \Omega_p, E)$ satisfying on $L \cap \Omega_p$ the equality $\bar{\partial}_\tau f = g$.*

Theorem 1.2a was regarded in the literature as a plausible conjecture (see [2, 44]). Earlier similar results had been obtained (for $s = \infty$) in the work of F. Treves [44] for "tubelike" CR-manifolds, i.e., for manifolds of the type

$$L = \{z \in C^n\colon \operatorname{Im} z_\nu = \varphi_\nu(\operatorname{Im} z_{k+1}, \ldots, \operatorname{Im} z_n),\ \nu = 1, 2, \ldots, k\},$$

and also for "standard" manifolds, in the work of H. Rossi, M. Vergne [40]. Moreover, in the work of M. Sato, T. Kawai, M. Kashivara [41], the microlocal variant of Theorem 1.2a was established.

THEOREM 1.2b ([16]). *Under the conditions of Theorem 1.2, for any Stein domain $\Omega \subset X$, for all r, $n-k-q < r \leqslant n-k$, and for any CR-form $g \in C_{0,r}^{(s)}(L\cap\Omega, E)$, there exists a form $f \in C_{0,r-1}^{(s+1/2-\varepsilon)}(L\cap\Omega, E)$ satisfying on $L\cap\Omega$ the equality $\bar{\partial}_\tau f = g$.*

For $s = \infty$ Theorem 1.2b was first obtained in a very important paper by I. Naruki [34].

1.3. A criterion for the local solvability of "nonsolvable" tangent Cauchy-Riemann equations in q-concave manifolds. The requirement of q-concavity of the CR-manifold L in the formulation of Theorem 1.2 would be an adequate condition provided that the following result explaining H. Lewy's effect were true.

THEOREM 1.3 (A. Andreotti, G. Fredricks, M. Nacinovich [2]). *If for a CR-manifold of type (1.1), for some $p \in L$ and $\lambda \in R^k \setminus \{0\}$, the form $L_{p,\lambda}(L)$ is not degenerate on $T_p^c(L)$ and has q negative and $n-k-q$ positive eigenvalues, then for any sufficiently small neighbourhood U of the point p there exists a CR-form f from $C_{0,q}^{(\infty)}(L\cap U, E)$ which is not $\bar{\partial}_\tau$-exact on $L\cap U$.*

A microlocal variant of Theorem 1.3 was obtained earlier in a paper by M. Sato, T. Kawai, M. Kashivara [41].

For hypersurfaces Theorem 1.3 was obtained earlier in a paper by A. Andreotti and C. Hill [1].

Theorem 1.3 shows the necessity of complementary conditions for the solvability of the equation (1.2) when g is a CR-form of type $(0, q)$ on a q-concave CR-manifold. A criterion for the local solvability of (1.2) in this case was obtained in [16]. This criterion we shall now formulate for real-analytic CR-manifolds only, in the form of a criterion of extendability of the CR-form g to a $\bar{\partial}$-closed form $\tilde{g}$ in a neighbourhood of L.

THEOREM 1.4 ([16]). *Let L be a real-analytic q-concave CR-manifold of type* (1.1). *A CR-form $g \in C_{0,q}(L, E)$ can be extended to a $\bar{\partial}$-closed form $\tilde{g}$ in some neighbourhood of L if and only if, given any $p \in L$, there exists a neighbourhood Ω_p such that the $(0, q)$-form described by:*

$$Kf(z) = \int_{\zeta \in \Omega_p \cap L} f(\zeta) K_q(\zeta, z)$$

is real-analytic on $L \cap \Omega_p$, where $K_q(\zeta, z)$ is a suitable singular kernel of the Cauchy–Fantappe type which is a real-analytic CR-form of type $(n, n-k-q)$ with respect to the variable $\zeta \in (\Omega_p \cap L) \setminus \{z\}$.

For hypersurfaces a corresponding criterion was established earlier (see [13]).

Theorem 1.4 is a basis for the proof of the following more subtle solvability criterion, which we shall formulate as the "edge of wedge" theorem or CR-forms.

THEOREM 1.4a (R. A. Ayrapetian, G. M. Henkin [4]). *Let a q-concave fgeneric CR-manifold L of co-dimension k in X belong to CR-manifolds L_j, $j = 1, 2, \ldots, k$ of co-dimension $k-1$ such that for all $p \in L$ the tangent spaces $T_p(L)$ belong to the linear hull of the complex tangent spaces $\{T_p^c(L_j)\}$. Then the CR-form g from $C_{0,q}(L)$ admits a $\bar{\partial}$-closed extension into a neighbourhood of the manifold L if the form g is a trace on L of some CR-form $\tilde{g} \in C_{0,q}(\bigcup_{j=1}^{k} L_j)$.*

For $q = 0$ Theorem 1.4a is a generalization of a number of results extending the classical theorems of S. N. Bernstein on separate analyticity and of N. N. Bogolubov on the "edge of wedge" (see [5], [46]).

It is also Theorem 1.4 on which is based the proof of the following result concerning the solvability of the equation (1.2) on a q-concave manifold for $(0, q)$-forms with a (sufficiently) compact support.

THEOREM 1.4b (G. M. Henkin [19]). *Given a q-concave CR-manifold L with $q \geqslant 1$, for any point $p \in L$, any sufficiently small pseudoconvex neighbourhood Ω_p of this point, any r: $1 \leqslant r \leqslant q$, and any CR-form $g \in C_{0,r}^{(s)}(L, E)$ with a support in $\Omega_p \cap L$, there exists a form $f \in C_{0,r-1}^{(s+1/2-\varepsilon)}(L, E)$ with a support in $\Omega_p \cap L$ satisfying on L the equation $\bar{\partial}_\tau f = g$.*

For $r < q$ (and for $s = \infty$) Theorem 1.4b was obtained by I. Naruki [34]. Theorem 1.4b seems to be a new one even for hypersurfaces. In particular it implies that for a CR-function on any 1-concave CR-manifold L the

Hartoggs–Bochner effect holds. This generalizes N. Sibony's earlier statement [42] concerning the validity of the local maximum principle for CR-functions on a 1-concave CR-manifold.

1.4. Conditions for a global solvability of the tangent Cauchy–Riemann equations. Let us denote by $A_{0,q}^{(s)}(L, E)$ the subspace in $C_{0,q}^{(s)}(L, E)$ composed of $\bar\partial_\tau$-closed forms, and by $B_{0,q}^{(s)}(L, E)$ the subspace in $C_{0,q}^{(s)}(L, E)$ composed of forms $\bar\partial_\tau f$, where $f \in C_{0,q-1}^{(s)}(L, E)$. We shall consider the space of $\bar\partial_\tau$-cohomologies

$$H_{(s)}^q(L, E) = A_{0,q}^{(s)}(L, E)/B_{0,q}^{(s)}(L, E).$$

Theorems 1.2 and 1.4 enable us to establish a criterion for the global solvability of equation (1.2) together with a criterion for the finitely dimensionality of the cohomology space $H_{(s)}^{(q)}(L, E)$.

THEOREM 1.5 ([16]). *Under the conditions of Theorem 1.2 for any r: $1 \leqslant r \leqslant q$ (resp. $n-k-q < r \leqslant n-k$) and any pseudo-concave (resp. pseudo-convex) domain $\Omega \Subset X$, a necessary and sufficient condition for a CR-form f from $A_{0,r}^{(s)}(L\cap\Omega, E)$ to be $\bar\partial_\tau$-exact on $L\cap\Omega$, and to belong to the space $B_{0,r}^{(s)}(L\cap\Omega, E)$ is that $\int\limits_L f\wedge\varphi = 0$ for any form $\varphi \in A_{n,n-k-r}^{(\infty)}(X, E^*)$ with a support in the domain Ω. Furthermore, for all $r < q$ (resp. $r > n-k-q$), the spaces $H_{(s)}^r(L\cap\Omega, E)$ are finitely dimensional.*

In the case where L is a compact hypersurface in X, the statement of Theorem 1.5 has been well known (see J. Kohn, H. Rossi [25], [26]).

With regard to the subjects considered in the second part of the paper we shall now take into consideration smooth $(0,1)$-forms θ on L with values in $\operatorname{End} E$. Such a form we shall call $\bar\partial_\tau$-exact if

$$K^{-1}\bar\partial_\tau K = \theta \quad \text{on } L, \tag{1.3}$$

where K is a smooth function with values in non-degenerate endomorphisms E.

A necessary condition for the local solvability of (1.3) is now the equality:

$$\bar\partial_\tau\theta + \theta\wedge\theta = 0. \tag{1.4}$$

The space of the smooth (of the class C^∞) forms θ satisfying (1.4) and considered up to the transformation of the form

$$\theta \sim \tilde\theta = K^{-1}\bar\partial_\tau K + K^{-1}\theta K$$

we shall further denote by $H^{0,1}(L, \mathrm{GL}(E))$.

Any element θ from $H^{0,1}(L, \mathrm{GL}(E))$ defines a CR-fibre bundle E_θ over L which is topologically equivalent to the fibre bundle E. Namely, those smooth sections h of the fibre bundle E which are annihilated by the operator $\bar{\partial}_\tau + \theta$,

$$\bar{\partial}_\tau h + \theta h = 0,$$

we shall call CR-sections of the fibre bundle E_θ.

In contradistinction to holomorphic fibre bundles over X, a CR-fibre bundle over L cannot always be given by transition matrix-functions. For instance, from Theorem 1.2 it follows that on a 1-concave CR-manifold the CR-fibre bundle E can be given by means of CR-matrix-functions of pass on some cover if and only if E_θ is a trace on L of some holomorphic fibre bundle in a neighbourhood of L. As above, using the Cauchy–Riemann operator $\bar{\partial}_\tau + \theta$, cohomology spaces $H^q_{(s)}(L, E_\theta)$ are introduced. For a q-concave CR-manifold L, for any $\theta \in H^q_{(s)}(L, \mathrm{GL}(E))$ and any r: $1 \leqslant r < q$ or $r > n-k-q$, the local results of Theorems 1.2b and 1.4b and the global result of Theorem 1.5 are still valid.

2. Yang–Mills, Higgs and Dirac fields as solutions of Cauchy–Riemann equations

R. Penrose ([36], [37]) has proposed a promising program of a reconstruction of the foundations of relativistic physics, which would result in a transformation of relativistic physics into a part of analytic geometry in the space of complex light lines (the theory of twistors).

We shall present a number of results developing Penrose's program. These results prove that the theory of the classical Yang–Mills, Higgs and Dirac fields on Minkowski spaces can be transformed into the theory of Cauchy–Riemann tangent equations on a 1-concave submanifold of the twistor (supertwistor) space. Some of the results of the first part of the paper acquire here a "physical" interpretation.

2.1. The spaces of complex and real zero lines. Let CM_0 be a complex Minkowski space, i.e., a four-dimensional complex space with spinor coordinates $u = \{u_{AB'}, A = 0, 1;\ B' = 0', 1'\}$ and with a metric $\det(du_{AB'})$. Then a real Minkowski space M_0 is formed of those points $u \in CM_0$ for which the matrix $u_{AB'}$ is Hermitian.

Moreover, let T_+ and T_- be two reciprocally dual four-dimensional complex spaces (of twistors and dual twistors) with coordinates $(z^A, z_{B'})$

and $(w_A, w^{B'})$, and with the bilinear form $\langle z \cdot w\rangle = z^A w_A - z_{A'} w^{A'}$; then the coordinates in T_+ and T_- are the homogeneous coordinates of the corresponding points in the three-dimensional projective spaces $\mathscr{L}_\pm = P(T_\pm)$.

The equation $\langle z \cdot w\rangle = 0$ distinguishes a five-dimensional hypersurface $\mathscr{L}$ on $\mathscr{L}_+ \otimes \mathscr{L}_-$. The points of this surface parametrize complex zero-lines in the Minkowski space; given a fixed point $(z, w) \in \mathscr{L}$, there exists a corresponding zero-line in CM_0, given by the equations:

$$z_{B'} = u_{AB'} z^A, \quad w_A = u_{AB'} w^{B'}. \tag{2.1}$$

Conversely, for a fixed point $u \in CM_0$, the equations (2.1) considered as conditions for (z, w), generate two reciprocally orthogonal two-dimensional subspaces in T_+ and T_-, which, on the other hand, determine on $\mathscr{L}$ a quadric $\mathscr{L}(u) = \mathscr{L}_+(u) \otimes \mathscr{L}_-(u)$.

The foregoing correspondence enables us to identify the compactifiable (and complexifiable) Minkowski space CM with the manifold of all two-dimensional subspaces in T_+ (or in T_-). Corresponding tautological two-dimensional fibre bundles over CM, denoted by $S_\pm$, are called spinor fibre bundles.

In virtue of (2.1), over CM_0 there are natural trivializations of these fibre bundles; z^A are the coordinates in the fibre $S_+(u)$, $w^{A'}$ are the coordinates in the fibre $S_-(u)$, $u \in CM_0$.

Given a domain U in a compactifiable real Minkowski space M, we shall denote by $L(U)$, $L_+(U)$, $L_-(U)$ the real submanifolds in the complex manifolds $\mathscr{L}, \mathscr{L}_\pm$, of the form:

$$L(U) = \bigcup_{u \in U} \mathscr{L}(u), \quad L_\pm(U) = \bigcup_{u \in U} \mathscr{L}_\pm(u).$$

Let us put $L = L(M)$, $L_\pm = L_\pm(M)$. Each of the manifolds L_+ and L_- parametrizes the (real) light rays on the real Minkowski space M, whereas the manoifold L parametrizes pairs of intersecting world lines on M (or, in other words, the complex zero-lines in CM, passing through M). We have

$$L_+ = \{z \in \mathscr{L}_+ \colon \operatorname{Im}(z^0 \bar{z}_{0'} + z^1 \bar{z}_{1'}) = 0\},$$
$$L_- = \{w \in \mathscr{L}_- \colon \operatorname{Im}(w_0 \overline{w}^{0'} + w_1 \overline{w}^{1'}) = 0\},$$
$$L = \{(z, w) \in \mathscr{L} \colon z \in L_+, w \in L^-\}.$$

The manifolds $L_\pm$ are real hypersurfaces in the spaces $\mathscr{L}_\pm$, whose Levi forms are non-degenerate at any point $p \in L_\pm$ and have one positive and one negative eigenvalue on $T_p^c(L_\pm)$, i.e., $L_\pm$ are strongly 1-concave manifolds.

The manifold L, of co-dimension 2 in the complex manifold $\mathscr{L}$, has singularities on a fully real submanifold

$$S = \{(z, w) \in L \colon (w_A, w^{B'}) = (\bar{z}^{A'}, \bar{z}_B)\}.$$

For all $p \in L \setminus S$ we have $\dim T_p^c(L) = 3$, and for any $\lambda \in R^2 \setminus \{0\}$ the Levi form $L_{p,\lambda}(L)$ has on $T_p^c(L)$ one positive, one negative, and one zero eigenvalue, i.e., L is a 1-concave and, at the same time, weak 2-concave CR-manifold outside S.

Further, let E be a trivial n-dimensional fibre bundle. We shall also assume that the domain $U \subset M$ is such that its intersections with all light rays are connected and simply connected. Further, let us also denote by $H^{0,1}(L(U), \mathrm{GL}(n, C))$ (resp. $H^{0,1}(L_\pm(U), \mathrm{GL}(n, C))$) a set of all CR-fibre bundles E_θ topologically equivalent to E and moreover analytically equivalent to E on each quadric $L(u)$, $u \in U$. The last statement means that the $(0, 1)$-form indexing the fibre bundle E_θ can be represented on each quadric $L(u)$

$$\theta|_{L(u)} = K_u^{-1} \bar{\partial} K_u, \tag{2.2}$$

where the function K_u with values in $\mathrm{GL}(n, C)$ smoothly depends on the parameter $u \in U$.

For one-dimensional topologically trivial fibre bundles for instance, the condition (2.2) holds automatically.

In the sequel we shall put $E_\theta(m, k) = E_\theta \otimes \mathcal{O}(m, k)$, where $\mathcal{O}(m, k)$ is a one-dimensional fibre bundle over $\mathscr{L}_+ \otimes \mathscr{L}_-$, the holomorphic sections of which are holomorphic functions on $T_+ \otimes T_-$ of homogeneity (m, k) with respect to variables $(z, w) \in T_+ \otimes T_-$.

Theorems 1.2, 1.3, 1.5 applied to the CR-manifolds $L(U)$, $L_+(U)$, $L_-(U)$ enable us to state, first, that the spaces of CR-fibre bundles $H^{0,1}(L(U), \mathrm{GL}(n, C))$ and $H^{0,1}(L_\pm(U), \mathrm{GL}(n, C))$ are non-trivial (and infinitely dimensional) and, secondly, that among spaces of cohomologies with coefficients in the fibre bundles $E_\theta(m, k)$ there are in general no other non-trivial (infinitely dimensional) spaces but the spaces $H^1(L_\pm(U), E_\theta(m))$, $H^1(L(U), E_\theta(m, k))$ and $H^2(L(U), E_\theta(m, k))$.

R. Penrose's transformation enables us to identify, in a surprisingly natural way, the elements of these spaces as cohomologies, and also the fibre bundles E_θ as physical fields on Minkowski spaces.

2.2. The Penrose transformation and criteria of solvability of the tangent Cauchy–Riemann equations on $L(U)$. Let a CR-fibre bundle E_θ be indexed by the elements $\theta \in H^{0,1}(L(U), \mathrm{GL}(n, C))$, and let $K = K(u, z^A, w^{B'})$ be a function satisfying (2.2). Let $d_1 = z^A w^{B'} \partial/\partial u^{AB'}$ be an operator of differentiation along light lines. (1.2) implies that the function $(d_1 K)K^{-1}$ is holomorphic on each quadric $L(u)$, $u \in U$, and therefore can be represented in the form

$$(d_1 K)K^{-1} = z^A w^{B'} a_{AB'}(u), \tag{2.3}$$

where $\{a_{AB'}\}$ are smooth functions of $u \in U$.

We shall consider a 1-form $a = a_{AB'} du^{AB'}$ in the domain $U \subset M$. It is determined by the given construction up to the gauge $a \sim \tilde{a} = b^{-1} db + b^{-1} ab$, where b is a $\mathrm{GL}(n, C)$-valued smooth function. Thus the form a determines a $\mathrm{GL}(n, C)$-connection ∇_a in the fibre bundle E over $U \subset M$. The correspondence $\theta \mapsto \mathscr{P}\theta = \nabla_a$ will be called the Penrose transform of the form θ.

For the elements of cohomology spaces with the coefficients in the fibre bundles $E_\theta(m, k)$ the definition of the Penrose transforms depends rather strongly on the numbers m and k. In the cases which are most interesting for us these definitions are the following: If a fixed function K_u satisfies (2.2), or, in other words, a form a satisfies (2.3), then, for $\Psi_+ \in H^1(L(U), E_\theta(-1, 0))$, $\varphi_+ \in H^1(L(U), E_\theta(-2, 0))$, and $\Omega \in H^1(L(U), \mathrm{End}\, E_\theta(-1, -1))$ we shall put

$$\psi^{A'} = \mathscr{P}\Psi_+ = \int\limits_{L_+(u)} K \frac{\partial}{\partial z_{A'}} \Psi_+ \wedge z^A dz_A, \qquad \Phi_+ = \mathscr{P}\varphi_+ = \int\limits_{L_+(u)} K\varphi \wedge z^A dz_A, \tag{2.4}$$

$$\omega = \mathscr{P}\Omega = \int\limits_{L_+(u)} K\left(w^{A'} \frac{\partial}{\partial z^{A'}} \Omega\right) K^{-1} \wedge z^A dz_A. \tag{2.4^1}$$

For $F_+ \in H^2(L(U), E_\theta(-3, -1))$, $G_+ \in H^2(L(U), E_\theta(-3, -2))$ and $J \in H^2(L(U), \mathrm{End}\, E_\theta(-3, -3))$ we shall put

$$f_+ = \mathscr{P}F_+ = \int\limits_{L(u)} K z^A \frac{\partial F_+}{\partial w^A} \wedge z^B dz_B \wedge w^{B'} dw_{B'},$$

$$g_A = \mathscr{P}G_+ = \int\limits_{L(u)} z_A K G_+ \wedge z^B dz_B \wedge w^{B'} dw_{B'}, \tag{2.5}$$

$$j = \mathscr{P}J = j_A^{B'} du^{A0'} \wedge du^{A1'} \wedge du_{AB'}, \tag{2.5^1}$$

where

$$j_A^{B'} = \iint\limits_{L(u)} KJK^{-1} z_A w^{B'} \wedge z^B dz_B \wedge w^{A'} dw_{A'}.$$

The Penrose transforms for cohomologies with coefficients in $E_\theta(0, -1)$, $E_\theta(0, -2)$, $E_\theta(-1, -3)$, $E_\theta(-2, -3)$ are defined analogously. Let us note that the fields $\psi^{A'}$, φ_+, etc. are determined by the given construction up to the gauges $\psi^{A'} \sim \tilde{\psi}^{A'} = b\psi^{A'}$; $\varphi_+ \sim \tilde{\varphi}_+ = b\varphi_+$ etc., where b is a gauge function for the connection a.

The following result sums up (and to a certain extent generalizes) a number of statements given in [47], [9], [12], [49], [24], [30], [14], [20], [21], [7].

THEOREM 2.1. *The Penrose transformation of the form* (2.3) *establishes a canonical isomorphism between the space of the CR-fibre bundles* $\theta \in H^{0,1}(L(U), \mathrm{GL}(n, C))$ *and the space of all smooth connections* $\nabla_a = \mathscr{P}\theta$ *in the fibre bundle* E *over* U. *Moreover, the CR-fibre bundles from* $H^{0,1}(L_\pm(U), \mathrm{GL}(n, C))$ *are transformed into self-dual* (*resp. anti-self-dual*) *connections.*

Subsequently, for fixed $\theta \in H^{0,1}(L(U), \mathrm{GL}(n, C))$, *the Penrose transformations of the form* (2.4), (2.5) *establish an isomorphism of the cohomology spaces* $H^1(L(U), E_\theta(-1, 0))$, $H^1(L(U), E_\theta(-2, 0))$, $H^2(L(U), E_\theta(-3, -2))$, $H^2(L(U), E_\theta(-3, -1))$ *with the spaces of smooth sections over* U *of the fibre bundles* $E\otimes S_-\otimes \Lambda^2 S_+$; $E\otimes\Lambda^2 S_+$; $E\otimes S_+\otimes\Lambda^2 S_+\otimes\Lambda^2 S_-$; $E\otimes(\Lambda^2 S_+)^2\otimes\Lambda^2 S_-$, *respectively. If, moreover,* $\theta \in H^{0,1}(L_+(U), \mathrm{GL}(n, C))$ *then the elements of the spaces* $H^1(L_+(U), E_\theta(-1, 0))$ *and* $H^1(L_+(U), E_\theta(-2, 0))$ *are transformed into the solutions* $\psi_{A'}$ *and* φ_+ *of the Weil–Dirac equation* $\nabla_a^{AA'}\psi_{A'} = 0$ *and the d'Alembert equation* $\Box_a\varphi_+ = 0$, *respectively, in the self-dual field* $a = \mathscr{P}\theta$.

Finally, the correspondence of (2.4^1) *and* (2.5^1) *realizes the isomorphisms of the spaces* $H^1(L(U), \mathrm{End}\,E_\theta(-1, -1))$ *and* $H^2(L(U), \mathrm{End}\,E_\theta(-3, -3))$ *with, respectively, the space of smooth sections of the fibre bundle* $\mathrm{End}\,E\otimes\Lambda^2 S_+\otimes\Lambda^2 S_-$ *and the space of smooth 3-forms* j *on* U *with values in* $\mathrm{End}\,E$, *satisfying the equation* $dj + [a, j] = 0$.

From Theorems 2.1 and 1.2, as a simple corollary, follows

THEOREM 2.1a. *Let the* (0, 1)*-forms* Θ, $\Phi_\pm$, $\Psi_\pm$ *represent elements of the respective spaces of one-dimensional cohomologies on* $L(U)$ (*or* $L_\pm(U)$). *Then a necessary and sufficient condition for the Cauchy–Riemann equations of the forms* $a^{-1}\bar{\partial}_\tau a = \theta$; $\bar{\partial}_\tau\beta_\pm + \theta\beta_\pm = \Phi_\pm$; $\bar{\partial}_\tau\gamma_\pm + \theta\gamma_\pm = \Psi_\pm$ *to be solvable* (*resp. locally solvable in a neighbourhood of any point*) *is that the Penrose* *transforms $\mathscr{P}\theta$, $\mathscr{P}\Phi_{\pm}$, $\mathscr{P}\Psi_{\pm}$ of these forms be equal to zero (resp. real-analytic) on U.*

Let us note that a following criterion for the $\bar\partial_\tau$-exactness of the forms $\Phi_{\pm}$, $\Psi_{\pm}$ in a neighbourhood of a fixed point $p \in L_{\pm}(U)$ was formulated in [17] as a development of Penrose's ideas [38]: the singular spectrum of the forms $\mathscr{P}\Phi_{\pm}$, $\mathscr{P}\Psi_{\pm}$ must not contain zero-bi-characteristics corresponding to light rays $l_{\pm}(\tau)$.

2.3. The Maxwell–Yang–Mills, Weil–Dirac and Klein–Gordon equations as the Cauchy–Riemann equations. Let $L^{(j)}(U)$ denote the j-th infinitesimal neighbourhood of the manifold $L(U) \subset L_+(U)\times L_-(U)$. We shall denote by $H^{0,1}(L^{(j)}(U),\ \mathrm{GL}(n, C))$ the subspace of CR-fibre bundles in $H^{0,1}(L(U),\ \mathrm{GL}(n, C))$ indexed by smooth $(0,1)$-forms θ on $L_+(U)\times\times L_-(U)$ which, first, satisfy on $L^{(j)}(U)$ the Cauchy–Riemann–Cartan equation of the form $\bar\partial\theta+\theta\wedge\theta = X_{j+1}\langle z\cdot w\rangle^{j+1}$ where X_{j+1} is a smooth $(0,2)$-form on $L(U)$ representing some element of the space $H^2(L(U),\ \mathrm{End}\,E_\theta(-j-1,\ -j-1))$, and, secondly, are considered up to the $\bar\partial$-gauge of the form

$$\theta \sim \tilde\theta = K^{-1}\bar\partial K+K^{-1}\theta K+O(\langle z\cdot w\rangle^{j+1}).$$

We shall denote by $H^q(L^{(j)}(U),\ E_\theta)$, where $\theta \in H^{0,1}(L^{(j)}(U),\ \mathrm{GL}(n, C))$, a subspace in $H^q(L(U),\ E_\theta)$ given by smooth E-valued $(0,q)$-forms ω on $L_+\times L_-$ satisfying on $L^{(j)}(U)$ the Cauchy–Riemann equation $\bar\partial_\tau\omega+\theta\wedge\omega = Y_{j+1}\langle z\cdot w\rangle^{j+1}$ and considered up to the $\bar\partial$-gauge $\omega\sim\tilde\omega = \omega+\bar\partial_\tau a+\theta\wedge a+O(\langle z\cdot w\rangle^{j+1})$.

It was proved in [14], [20] that any element of the space $H^{0,1}(L(U),\ \mathrm{GL}(n, C))$ can be extended (in only one way) to an element of the space $H^{0,1}(L^{(2)}(U),\ \mathrm{GL}(n, C))$ and can be represented by the form θ on $L_+(U)\times\times L_-(U)$ satisfying the relation

$$\bar\partial_\tau\theta+\theta\wedge\theta = J\langle z\cdot w\rangle^3. \tag{2.6}$$

Moreover, the elements of the space $H^{0,1}(L^{(1)}(U),\ \mathrm{GL}(n, C))$ can be indexed by the forms $\tilde\theta$

$$\tilde\theta = \theta+\Omega\langle z\cdot w\rangle, \tag{2.7}$$

where θ satisfies (2.6) and $\Omega\in H^1(L(U),\ \mathrm{End}\,E_\theta(-1,\ -1))$.

Furthermore (see [20], [21]), for fixed θ satisfying (2.6), the elements Ψ_+, Φ_+, Ω of the spaces $H^1(L(U),\ E_\theta(-1,0))$, $H^1(L(U),\ E_\theta(-2,0))$ and $H^1(L(U),\ \mathrm{End}\,E_\theta(-1,\ -1))$ with a suitable gauge satisfy the relations of the form

$$\begin{aligned}\bar{\partial}\Psi_+ + \theta\wedge\Psi_+ &= G_+\langle z\cdot w\rangle^2,\\ \bar{\partial}\Phi_+ + \theta\wedge\Phi_+ &= F_+\langle z\cdot w\rangle.\end{aligned}\tag{2.8}$$

The Cauchy-Riemann equations of the forms (2.6), (2.8) proved to be equivalent (on a twistor space) to the Maxwell–Yang–Mills, Weil–Dirac and Klein–Gordon equations, respectively.

THEOREM 2.2 (G. M. Henkin, Yu. Manin [14], [20], [30], [32]). *In order that the forms θ and J; Ψ_+ and G_+; Φ_+ and F_+ satisfy the equations* (2.6) *and* (2.8) *it is necessary and sufficient that their Penrose transforms of the forms* (2.3)–(2.5) *satisfy the equations*

$$d*f+[a,*f] = \frac{3}{\pi^2}j,\tag{2.6^1}$$

$$\nabla_{AA'}\psi^{A'} = \frac{1}{2\pi^2}g_A,\qquad \Box\varphi_+ = \frac{1}{\pi i}f_+,\tag{2.8^1}$$

respectively, where $f = da+a\wedge a$ is the curvature form of the connection a, and $*$ is the Hodge operator corresponding to the Minkowski metric,

$$\nabla_{AA'} = \frac{\partial}{\partial u^{AA'}}+a^{AA'},\qquad \Box = \nabla^{AB'}\nabla_{AB'}.$$

From Theorems 2.2 and 2.1, as a corollary, we obtain the following result.

THEOREM 2.3 (E. Witten [50], T. Isenberg, Ph. Yasskin, P. Green [24], G. M. Henkin, Yu. I. Manin [14], [20], [32]). *The Penrose transformation establishes a canonical isomorphism between*:

(a) *the space of fibre bundles $H^{0,1}(L^{(3)}(U), \mathrm{GL}(n,C))$ and the space of all smooth $\mathrm{GL}(n,C)$-connections in E satisfying the Yang–Mills equation*

$$d*f+[a,*f] = 0,$$

(b) *the space $H^1(L^{(2)}(U), E_0)$, where $\theta\in H^{0,1}(L^{(2)}(U), \mathrm{GL}(n,C))$ and the space of the smooth solutions on U of the Weil–Dirac equation*:

$$\nabla_{AA'}\psi^{A'} = 0,$$

(c) *the space $H^1(L^{(1)}(U), E_{\tilde{\theta}})$, where $\tilde{\theta} = (\theta+\Omega\langle z\cdot w\rangle)\in H^{0,1}(L^{(1)}(U), \mathrm{GL}(n,C))$ and the space of solutions of the Klein–Gordon equation*

$$\Box\varphi_+ + \omega\varphi_+ = 0,$$

where $\omega = \mathscr{P}\Omega$.

Theorem 2.4a (for analytic connections a) was obtained by E. Witten [50] and by J. Isenberg and P. Green [24]. Theorems 2.4b and 2.4c (also for analytic fields a, $\psi_{A'}$, φ_+) were obtained by Yu. I. Manin and the present author [20], [32]. A generalization to the case of non-analytic fields a, $\psi_{A'}$ and φ_+ was obtained in [14], [15].

2.4. Super-symmetrical Yang–Mills equations and the tangent Cauchy–Riemann equations on the space of super-light rays. Physicists deal mostly with interacting Yang–Mills, Dirac or Klein–Gordon (Higgs) fields, and not with free ones. Super-symmetrical interactions of these fields (so called N-super-symmetrical Yang–Mills fields, $N = 1, 2, 3, 4$) are especially popular at present. Not entering into details, let us note that besides the connection field a (the Yang–Mills field) these equations contain: for $N = 1$, two spinor fields ψ_A and $\psi_{A'}$ (the Dirac fields); for $N = 2$, two scalar fields φ_+ and φ_- (the Higgs fields) and four spinor fields $\psi_A, \psi_{A'}, \chi_A, \chi_{A'}$; for $N = 3, 4$, six scalar fields and eight spinor fields. Moreover, the spinor fields take values in $\operatorname{End} E \otimes S_\mp \otimes \Lambda^2 S_\pm \otimes \Lambda_1$ and the scalar fields take values in $\operatorname{End} E \otimes \Lambda^2 S_\pm \otimes \Lambda_0$, where Λ_0, Λ_1 are the subspaces of respectively, even and odd elements of the Grassmann algebra Λ. The equations of motion in the 2-super-symmetrical Yang–Mills theory take the form

$$\begin{gathered}
\Box\, \varphi_\pm + \{\psi_{A^\pm}, \chi^{A^\pm}\} \pm \tfrac{1}{2}\left[[\varphi_+, \varphi_-], \varphi_\pm\right] = 0,\\
\nabla_{A^-A^+} \psi^{A^\pm} \pm \tfrac{1}{2}[\varphi_\pm, \psi_{A^\mp}] = 0,\\
\nabla_{A^-A^+} \chi^{A^\pm} \pm \tfrac{1}{2}[\varphi_\pm, \chi_{A^\mp}] = 0,\\
2\nabla_{A^-A^+} f^{A^+B^+} + \{\psi_{A^-}, \chi^{B^+}\} + \\
+ \{\chi_{A^-}, \psi^{B^+}\} + \tfrac{1}{2}\left([\nabla^{B^+}_{A^-}\varphi_+, \varphi_-] - [\varphi_+, \nabla^{B^+}_{A^-}\varphi_-]\right) = 0,
\end{gathered} \tag{2.9}$$

where the symbol $A^\pm$ denotes A' and A, respectively; $\nabla_{AA'} = \dfrac{\partial}{\partial u^{AA'}} + [a^{AA'}, \cdot]$; $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ are the symbols of the commutator and the anti-commutator.

On the basis of Theorem 2.1, for any smooth fields a, $\psi_+, \psi_{A'}, \chi_A, \chi_{A'}, \varphi_\pm$ defined on $U \subset M$, one can find uniquely determined (up to $\bar\partial_\tau$-exact forms) smooth forms $\theta, \Psi_\pm, X_\pm, \Phi_\pm, \Omega$, defined on $L(U)$ with values, respectively, in $\operatorname{End} E \otimes \mathcal{O}(k, l) \otimes \Lambda_i$, where $i = 0, 1$; $(k, l) = (0, 0)$, $(-1, 0)$, $(0, -1)$, $(-2, 0)$, $(0, -2)$, $(-1, -1)$, such that, first, the Cauchy–Riemann relations of the form

$$\bar\partial_\tau \theta + \theta \wedge \theta = 0 \quad \text{on } L^{(2)}(U),$$

$$\bar\partial_\tau \Psi_\pm + [\theta, \Psi_\pm] = 0, \quad \bar\partial_\tau X_\pm + [\theta, X_\pm] = 0 \quad \text{on } L^{(1)}(U), \tag{2.10}$$

$$\bar\partial_\tau \Phi_\pm + [\theta, \Phi_\pm] = \{\Psi_\pm, X_\pm\},$$

$$\bar\partial_\tau \Omega + [\theta, \Omega] = 2\{\Psi_+, X_-\} + 2\{X_+, \Psi_-\} \quad \text{on } L(U)$$

are satisfied and, secondly, we have

$$\mathscr{P}\theta = a, \quad \mathscr{P}\Psi_\pm = \psi_{A\pm}, \quad \mathscr{P}X_\pm = \chi_{A\pm}, \tag{2.11}$$

$$\tilde{\mathscr{P}}\Phi_\pm = \varphi_\pm, \quad \tilde{\mathscr{P}}\Omega = [\varphi_+, \varphi_-],$$

where $\mathscr{P}$ is the suitable modification of $\mathscr{P}$, accounting non-closeness of forms $\Phi_\pm$ and Ω.

On the basis of [15], [32] the following result was obtained in [18].

THEOREM 2.4a ([18]). *In order that the fields a, $\psi_{A\pm}$ $\chi_{A\pm}$, $\varphi_\pm$ on U satisfy the super-symmetric Yang–Mills equations* (2.9) *it is necessary and sufficient that the fields θ, $\Psi_\pm$, $X_\pm$, $\Phi_\pm$, Ω corresponding to them in virtue of the relations* (2.10), (2.11), *satisfy the Cauchy–Riemann equations of the form:*

$$\bar\partial \Phi_\pm + [\theta, \Phi_\pm] + \{\Psi_\pm, X_\pm\} - \tfrac{1}{2}[\Omega, \Phi_\pm]\langle z\cdot w\rangle = 0,$$

$$\bar\partial \Omega + [\theta, \Omega] + 2\{\Psi_+, X_-\} + 2\{X_+, \Psi_-\} + {} $$

$$+ [\Phi_+, \Phi_-]\langle z\cdot w\rangle = 0 \quad \text{on } L^{(1)}(U),$$

$$\left.\begin{aligned} \bar\partial \Psi_\pm + [\theta, \Psi_\pm] + \tfrac{1}{2}[\Phi_\pm, \Psi_\mp]\langle z\cdot w\rangle = 0,\\ \bar\partial X_\pm + [\theta, X_\pm] + \tfrac{1}{2}[\Phi_\pm, X_\mp]\langle z\cdot w\rangle = 0,\end{aligned}\right\} \tag{2.12}$$

$$\text{on } L^{(2)}(U),$$

$$\bar\partial \theta + \theta\wedge\theta + \tfrac{1}{6}(\bar\partial\Omega + [\theta, \Omega])\langle z\cdot w\rangle + \tfrac{1}{3}(\{\Psi_+, X_-\} + \{X_+, \Psi_-\})\langle z\cdot w\rangle + {}$$

$$+ \tfrac{1}{6}[\Phi_+, \Phi_-]\langle z\cdot w\rangle^2$$

$$\text{on } L^{(3)}(U).$$

At first sight the relations (2.12) seem to be as complicated as the equations (2.9). However, the equations (2.12), and not (2.9), are the ones which have a clear geometrical meaning.

In fact, following A. Ferber [10] and E. Witten [50], let us consider first the projective spaces of super-twistors $\mathscr{L}^N_+$ (resp. dual super-twistors $\mathscr{L}^N_-$) with four even coordinates z^A, $z_{B'}$ (resp. w_A, $w^{B'}$) and N odd coordinates ζ^k (resp. η^k), and secondly a super-manifold $\mathscr{L}_N(U) = \{(z, \zeta;$

$w, \eta) \in \mathscr{L}_+^N \times \mathscr{L}_-^N$: $\langle z \cdot w \rangle = \zeta^k \eta_k$, $(z, w) \in L(U)\}$, parametrizing the analogues of light rays on the Minkowski super-space.

We shall examine the CR-submanifolds of the form

$$L_+^N = \{(z, \zeta) \in \mathscr{L}_+^N\colon \mathrm{Im}(z^0 \bar{z}_{0'} + z^1 \bar{z}_{1'}) = \zeta^k \bar{\zeta}_k\},$$

$$L_-^N = \{(w, \eta) \in \mathscr{L}_-^N\colon \mathrm{Im}(w_0 \bar{w}^{0'} + w_1 \bar{w}^{1'}) = \eta^k \bar{\eta}_k\},$$

$$L_N(U) = (L_+^N \times L_-^N) \cap \mathscr{L}_N(U)$$

on these super-manifolds.

We shall denote by $L_N^{(j)}(U)$ the j-th infinitesimal neighbourhood of the manifold $L_N(U) \subset L_+^N \times L_-^N$. Now, we shall examine on the CR-manifold $L_+^2 \times L_-^2$ the $(0, 1)$-form Θ given by

$$\Theta = \theta + \Psi_+ \zeta_1 + X_+ \zeta_2 + \Psi_- \eta_2 + X_- \eta_1 + \\ + \Phi_+ \zeta_1 \zeta_2 + \Phi_- \eta_1 \eta_2 + \Omega(\zeta_1 \eta_1 + \zeta_2 \eta_2), \tag{2.13}$$

where the forms $\theta, \Psi_\pm, X_\pm, \Phi_\pm$ and Ω satisfy the relations (2.10). In virtue of (2.10) we have $\bar{\partial}_\tau \Theta + \Theta \wedge \Theta = 0$ on $L(U)$, i.e., the form Θ defines a CR-fibre bundle over $L_2(U)$ which is trivial over any quadric $L(u)$, $u \in U$.

THEOREM 2.4b ([18]). *In order that the form Θ given by* (2.13) *be gauge-equivalent to a form satisfying the equation*

$$\bar{\partial}_\tau \Theta + \Theta \wedge \Theta = O\big((\langle z \cdot w \rangle - \zeta^k \eta_k)^2\big) \tag{2.14}$$

it is necessary and sufficient that the components $(\theta, \Psi_\pm, X_\pm, \Phi_\pm, \Omega)$ of the form Θ satisfy the Cauchy–Riemann equations of the form (2.12).

The equality (2.14) means that the form Θ defines a CR-fibre bundle over $L_2^{(1)}(U)$.

From Theorems 2.4a and 2.4b and their analogues for other super-symmetrical Yang–Mills theories results the following

THEOREM 2.4 (E. Witten [50], G. M. Henkin [18]). *The Penrose transformation establishes an isomorphism of the space of CR-fibre bundles over $L_N^{(3-N)}$, $N = 1, 2, 3$, trivial on all quadrics $L(u)$, $u \in U$, with the space of* (*smooth*) *solutions of the N-symmetrical system of Yang–Mills equations.*

For $N = 3$ (and for holomorphic fields and fibre bundles) this result was obtained in a paper by E. Witten [50]. More exactly, in [50] the equations of motion of the 3-supersymmetrical Yang–Mills system were reinterpreted as the conditions for the integrability of a connection along super-light rays in the Minkowski super-space. An extension of Witten's result to the case of the remaining $N = 1, 2, 4$ was obtained in [18].

For the formulation of this result in the case of $N = 4$ it is necessary to employ a natural extension of the well-known twistor transformation (see [8]) $\mathscr{T}$ establishing a canonical isomorphism between the spaces $H^1(L_+(U), \mathcal{O}(2s-2, 0))$ and $H^1(L_-(U), \mathcal{O}(0, -2s-2))$, $s = 0, \frac{1}{2}, 1$, onto the CR-fibre bundles over $L_4(U)$.

THEOREM 2.5 ([18]). *The Penrose transformation establishes an isomorphism between the space of CR-fibre bundles over $L_4(U)$ invariant with respect to the twistor transformation $\mathscr{T}$ and trivial on all quadrics $L(u)$, $u \in U$, and the space of all (smooth) solutions on U of the 4-supersymmetrical Yang–Mills system.*

For $N = 3, 4$, recently, A. A. Rosly [39] has discovered another interpretation of the Yang–Mills relations as integrability conditions along some tangent subspaces (of purely odd dimension) of the Minkowski super-space.

In comparison with Theorem 2.4, Theorems 2.4a and 2.4b from [18] contain additional information — a twistor interpretation of all fields and equations entering into the super-symmetrical Yang–Mills system. Let us note that these results give development of a paper by Yu. I. Manin [31], where the cohomological component analysis is given on $\mathscr{L}_3(U)$ of the 3-super-symmetrical Yang–Mills equations on U.

Interpretations in terms of Cauchy–Riemann equations over twistors of some other classical (not super-symmetrical) interactions between the Yang–Mills–Higgs and Dirac fields are obtained in papers [15], [18], [29].

We have touched here only part of the works dealing with a twistor interpretation of gauge fields on a plane Minkowski space. The problem of establishing the twistor theory for non-plane Minkowski spaces was discussed in a very impressive paper by R. Penrose [36]. Further developments of this work were obtained by C. Le Brun [27], [28], T. Isenberg, Ph. Yasskin [24], Yu. I. Manin, I. Penkov [33].

The twistor theory yields, as we know great results in establishing exact (and physically interesting) solutions of the self-dual Einstein and the Yang–Mills equations (see M. F. Atiyah [3], R. Penrose [36], R. Ward [48] and others). One can hope that the twistor interpretation of non-self-dual equations can also lead, to some new interesting solutions. The first non-trivial investigations of this kind are contained in papers by P. Forgacs, Z. Horvath, L. Palla [11] and Yu. I. Manin [31].

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