Proceedings of the International Congress of Mathematicians
August 16–24, 1983, Warszawa

A. JOSEPH

Primitive Ideals in Enveloping Algebras

1. Introduction

1.1. A basic problem in the theory of Lie algebras is to determine up to equivalence all irreducible representations of a given Lie algebra $\mathfrak{g}$. Here we shall always assume that $\mathfrak{g}$ is finite-dimensional over a field k which is algebraically closed and of characteristic zero. For finite-dimensional representations the problem quickly reduces to the semisimple case where the results are classical. For infinite-dimensional representations even the simplest non-commutative Lie algebras admit an enormously complicated representation theory. Now, the problem is equivalent to determining up to isomorphism all simple modules of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. However, this latter viewpoint allows us to define a primitive ideal of $U(\mathfrak{g})$ to be the annihilator of a simple $U(\mathfrak{g})$ module and then to classify the set $\operatorname{Prim} U(\mathfrak{g})$ of all primitive ideals of $U(\mathfrak{g})$ which is better behaved. This procedure turns out to be rather like selecting out only the continuous representations of the corresponding Lie group and there is a remarkably similar but not quite precise correspondence between these two theories.

1.2. Dixmier [8] first drew serious attention to the study of $\operatorname{Prim} U(\mathfrak{g})$ and the early work was inspired by Kirillov's orbit method used in classifying unitary representations of real nilpotent (and subsequently solvable) Lie groups. Let G denote the algebraic adjoint group acting on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is solvable. Then, by an appropriate application of Mackey's theory of induced representations and some further ideas from ring theory and algebraic geometry, one can construct a map $f \mapsto J(f)$ of $\mathfrak{g}^*$ onto $\operatorname{Prim} U(\mathfrak{g})$ which factors to a bijection of the orbit space $\mathfrak{g}^*/G$ onto $\operatorname{Prim} U(\mathfrak{g})$. For appropriate topologies this map is known to be continuous and even a homeomorphism for $\mathfrak{g}$ nilpotent [6]. The possible bicontinuity for $\mathfrak{g}$ solvable is still unresolved and involves delicate questions pertaining to inclusion relations between primitive ideals. The main part of this work was already completed by 1974 and has been fully described in [4, 8]. For more recent results see [36].

1.3. Already in 1978, Dixmier reported [7] on the considerable progress made in the classification of Prim $U(\mathfrak{g})$ for arbitrary $\mathfrak{g}$. We are now in a position to describe a complete classification theory at least for $\mathfrak{g}$ algebraic. This consists of two parts: the semisimple case, and the reduction to the semisimple case. It is convenient to describe the two cases quite separately, as they follow a rather different philosophy. The former requires a quite new approach, whereas the latter still involves the method of induced representations; but is more subtle than the orbit method, and in particular, the relationship to $\mathfrak{g}^*/G$ becomes blurred. For example the intersection theorem of algebraic geometry carries over by bicontinuity to the nilpotent case, whereas it fails for $\mathfrak{g}$ semisimple (with respect to Gelfand–Kirillov dimension). Again, even disregarding the complication involved with the Goldie rank, an orbit in $\mathfrak{g}^*$ may give rise to more than one primitive ideal [20].

1.4. Returning to the question of describing all simple $U(\mathfrak{g})$ modules we let A_n denote the Weyl algebra of index n. It is the (non-commutative) algebra generated by the differential operators $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$ over the polynomial ring $S_n = k[x_1, x_2, \ldots, x_n]$. The representation theory of A_n is much simpler than that of an arbitrary enveloping algebra; but even this can be very complicated. Given $\mathfrak{g}$ nilpotent and $J \in \operatorname{Prim} U(\mathfrak{g})$, one has $U(\mathfrak{g})/J = A_n$ (where $2n = \dim Gf$ given $J = J(f)$). This reduces the study of simple $U(\mathfrak{g})$ modules with annihilator J to the study of simple A_n modules. In particular, S_n is a simple A_n module in an obvious fashion and leads to a simple $U(\mathfrak{g})$ module with annihilator J. Moreover, this module lifts in a suitable sense to a unitary representation of G and this construction gives rise to a bijection of $\operatorname{Prim} U(\mathfrak{g})'$ onto the unitary dual $\hat{G}$ of G. Though this is only a very simple example, it indicates the existence of a general principle, which is beginning to be more precisely formulated. Finally, for physicists, we remark that A_n is just the algebra of canonical commutation relations for n degrees of freedom and the above module was the one used by Heisenberg in describing the harmonic oscillator. There are many further correspondences and thus possibly room for application of the classification theory of Prim $U(\mathfrak{g})$ in physics.

2. The semisimple case, preliminaries

2.1. The semisimple case is far too rich even to just describe all the results here. Consequently we shall concentrate on just one aspect of the classification theory, which was formulated in ([17], 7.4), a conjecture which has now been fully resolved. For a broader outlook we refer the reader to the review given in [23] and to Jantzen's forthcoming book [15].

2.2. Assume $\mathfrak{g}$ is semisimple. Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition ([8], 1.10.14) with $R = R^+ \cup R^-$ being the corresponding decomposition of the root system R. Let $\varrho \in \mathfrak{h}^*$ denote the half-sum of the positive roots. For each $\alpha \in R$ set $\alpha^\vee := 2\alpha/(\alpha, \alpha)$ and let $s_\alpha \in \operatorname{Aut}\mathfrak{h}^*$ denote the reflection $\lambda \mapsto s_\alpha \lambda = \lambda - (\alpha^\vee, \lambda)\alpha$. The group W generated by the s_α, $\alpha \in R$, is called the *Weyl group* (for the pair $\mathfrak{g}$, $\mathfrak{h}$). It plays a fundamental role in the representation theory of $\mathfrak{g}$.

2.3. A *highest weight module* of highest weight $\lambda - \varrho \in \mathfrak{h}^*$ is any $U(\mathfrak{g})$ module generated by a vector e_λ (called a *highest weight vector*) satisfying $Xe_\lambda = 0$, $\forall X \in \mathfrak{n}^+$ and $He_\lambda = (\lambda - \varrho, H)e_\lambda$, $\forall H \in \mathfrak{h}$. For each $\lambda \in \mathfrak{h}^*$ there exists a unique universal highest weight module $M(\lambda)$ of highest weight $\lambda - \varrho$, and this admits a unique simple quotient $L(\lambda)$. We set $J(\lambda) = \operatorname{Ann} L(\lambda)$, which is a primitive ideal.

2.4. Let $Z(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$. According to Duflo ([10], II, Thm. 1) the map $\lambda \mapsto J(\lambda)$ of $\mathfrak{h}^*$ into $\operatorname{Prim} U(\mathfrak{g})$ is surjective. Composing this with the map $J \mapsto J \cap Z(\mathfrak{g})$ of $\operatorname{Prim} U(\mathfrak{g})$ into $\operatorname{Max} Z(\mathfrak{g})$ we get, by a result of Harish-Chandra ([8], 7.4) a surjection of $\mathfrak{h}^*$ onto $\operatorname{Max} Z(\mathfrak{g})$, which factors to a bijection $\mathfrak{h}^*/W \xrightarrow{\sim} \operatorname{Max} Z(\mathfrak{g})$. Consequently, $\operatorname{Prim} U(\mathfrak{g})$ is sandwiched between $\mathfrak{h}^*$ and $\mathfrak{h}^*/W$ and its classification reduces to describing the fibres of these maps.

2.5. Let $P(R) := \{\lambda \in \mathfrak{h}^* | (\alpha^\vee, \lambda) \in \mathbf{Z}, \forall \alpha \in R\}$ denote the lattice of integral weights. For expository purposes it is convenient to limit attention to this portion of $\mathfrak{h}^*$. In general similar results hold under appropriate modifications; but this is a subtle phenomenon and it is a good test of a proof if it works without this restriction. Set $P(R)^+ = \{\lambda \in P(R) | (\alpha, \lambda) \geqslant 0, \forall \alpha \in R^+\}$. It is a fundamental domain for the action of W on $P(R)$. Set $P(R)^{++} = \{\lambda \in P(R)^+ | (\alpha, \lambda) \neq 0, \forall \alpha \in R\}$, which forms the set of so-called *regular elements* of $P(R)^+$.

2.6. Fix $\lambda \in P(R)^+$ and let $\hat{\lambda}$ denote its image in $\mathfrak{h}^*/W$. Then $\operatorname{Ann}_{Z(\mathfrak{g})} M(w\lambda)$ is a maximal ideal $Z_{\hat{\lambda}}$ of $Z(\mathfrak{g})$ independent of the choice of $w \in W$. From this it is relatively easy to show that $M(w\lambda)$ has finite length with composition factors amongst the $L(y\lambda)$: $y \in W$. Let $b(w, y) := [M(w\lambda) : L(y\lambda)]$ denote the number of times $L(y\lambda)$ occurs in the composition series for

$M(w\lambda)$. These numbers are independent of the choice of $\lambda \in P(R)^{++}$ (with some well-defined degeneration for $\lambda \in P(R)^{+}$). They play a fundamental role in the representation theory of $\mathfrak{g}$, and in particular in the description of Prim $U(\mathfrak{g})$. Moreover, it was partly in an attempt to understand this role that led Kazhdan and Lusztig to formulate [27] their famous conjecture about these coefficients and which eventually led to their description in purely combinatorial terms. This work was reviewed by Jantzen in his talk at this congress [14]. It suffices to say that the $b(w, y)$ may be considered as known and, with respect to the Bruhat order, they form a matrix which is upper triangular with ones on the diagonal. It hence admits an inverse with integer coefficients which we denote by $a(w, y)$. Set

$$a(w) = \sum_{y \in W} a(w, y) y^{-1}, \quad w \in W.$$

These form a basis for the group ring $\boldsymbol{Z}W$.

2.7. Take $J \in \operatorname{Prim} U(\mathfrak{g})$. By Goldie's theorem, $U(\mathfrak{g})/J$ embeds in its ring of fractions, which is isomorphic to a matrix ring over a division algebra. The Goldie rank $\operatorname{rk}(U(\mathfrak{g})/J)$ of $U(\mathfrak{g})/J$ is defined to be the rank of this matrix ring. It is an important invariant of J. For example if $J = \operatorname{Ann} M$ for some simple finite-dimensional module M, then $\operatorname{rk} U(\mathfrak{g})/J = \dim M$ and yet $\operatorname{rk} U(\mathfrak{g})/J$ is finite even when $\dim M$ is infinite. In view of Duflo's parametrization of Prim $U(\mathfrak{g})$ discussed in 2.4 it is natural to define for each $w \in W$ the function p_w on $P(R)^{+}$ by

$$p_w(\lambda) = \operatorname{rk}\big(U(\mathfrak{g})/J(w\lambda)\big).$$

We shall see that each p_w extends to a polynomial on $\mathfrak{h}^*$. These polynomials not only lead to a classification of Prim $U(\mathfrak{g})$, but also to a remarkable connection with $\hat{W}$.

2.8. By the Poincaré–Birkhoff–Witt theorem, $U(\mathfrak{g})$ admits a filtration whose associated graded algebra identifies with the algebra of polynomial functions on $\mathfrak{g}^*$. Then for each $J \in \operatorname{Prim} U(\mathfrak{g})$ we can consider the variety $\mathfrak{V}(\operatorname{gr} J) \subset \mathfrak{g}^*$ of zeros of $\operatorname{gr} J$. We shall see that this variety is always irreducible and we describe it explicitly. (Irreducibility fails for $\mathfrak{g}$ solvable; but by a general result of Gabber [28] one always has equidimensionality). For this, let us identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form and call $X \in \mathfrak{g}$ *nilpotent* if $\operatorname{ad}_{\mathfrak{g}} X$ is a nilpotent endomorphism of $\mathfrak{g}$. The set $\mathfrak{N}$ of all nilpotent elements of $\mathfrak{g}$ is, of course, G-stable and forms a finite union of G orbits called *nilpotent orbits*. Through étale cohomology Springer [35] defined an injective map $\beta\colon \mathfrak{N}/G \to \hat{W}$. We shall see that this leads to a link with the correspondence described in 2.7, a totally unexpected and remarkable fact, which in its turn inspired a simpler version [21] of Springer's construction.

3. Semisimple case, results

3.1. Fix $w \in W$. The primitive ideals $J(w\lambda)$, $\lambda \in P(R)^+$, are pairwise distinct because their intersections with $Z(\mathfrak{g})$ are distinct maximal ideals (2.4). By the Borho–Jantzen translation principle ([5], 2.12) the inclusion relations in the fibre $\mathfrak{X}_\lambda := \{J(w\lambda)\colon w \in W\}$ over Z_λ are independent of the choice of $\lambda \in P(R)^{++}$ (with some well-defined degeneration for $\lambda \in P(R)^+$). We might therefore guess that these relations can be described purely combinatorially in terms of W. Indeed, for any subset $S \subset \boldsymbol{Z}W$ let $[S]$ denote the subset of $\{a(y)\colon y \in W\}$ which occur with non-zero coefficient in the expansion of some $s \in S$ as a linear combination of the $a(y)\colon y \in W$.

PROPOSITION. *Assume $\lambda \in P(R)^{++}$. For each pair $w,y \in W$ one has*

$$J(w\lambda) \supset J(y\lambda) \Leftrightarrow a(w) \in [a(y)W].$$

This question was formulated and the implication $\Leftarrow$ was established in [18]. The conjectured reverse implication was established by Vogan [37]. It shows that the $a(w)$ completely determine the inclusion relations between primitive ideals, which in view of Kazhdan–Lusztig's theorem can be described purely combinatorially. Lusztig [29] used this result to establish an isomorphism between a Hecke algebra derived from W and the group ring $\boldsymbol{Q}(t)W$. This was a purely combinatorial question which needed Prim $U(\mathfrak{g})$ for its solution!

3.2. Fix $w \in W$. Since ϱ is regular, one easily checks that $a(w)\varrho^m$ cannot vanish for all $m \in \boldsymbol{N}$. Let n denote the least non-negative integer with this property. Define p_w as in 2.7. Then ([19], II, 5.1, 5.5)

THEOREM. *For each $w,y \in W$, one has*
(i) $p_w = a(w)\varrho^n$, *up to a scalar.*
(ii) *Suppose $\lambda \in P(R)^{++}$. Then $p_w(\lambda) = p_y(\lambda) \Leftrightarrow J(w\lambda) = J(y\lambda)$.*

The determination of the scalars in (i) has not yet been fully completed. It is a delicate question which involves in part the construction of sufficiently many completely prime, primitive ideals (i.e. for which the quotient algebra has Goldie rank one). Apart from this, we see that the $a(w)$ determine the Goldie ranks of the primitive quotients, which by (ii) separate the elements of $\mathfrak{X}_{\hat{\lambda}}$. This separation is better than that given in 3.1, which is highly implicit; for example, it allows one to calculate card $\mathfrak{X}_{\hat{\lambda}}$ (see 3.3). One can have $p_w(\lambda) = 0$ if $\lambda \in P(R)^+$ and this exactly describes the degeneration of the fibre $\mathfrak{X}_{\hat{\lambda}}$ alluded to in 3.1.

3.3. A deep consequence of 3.1 and 3.2 is that QWp_w is a simple W module P_τ say of type $\tau \in \hat{W}$. Set $(\mathfrak{X}_\lambda)_\tau = \{J(w\lambda) | p_w \in P_\tau\}$. Note that this is actually a partition of W into so-called *double cells*. Then ([19], II, 5.5)

THEOREM. *The distinct p_y: $J(y\lambda) \in (\mathfrak{X}_{\hat{\lambda}})_\tau$ form a basis for P_τ. In particular if $\lambda \in P(R)^{++}$ then* $\operatorname{card}(\mathfrak{X}_{\hat{\lambda}})_\tau = \deg\tau$.

This result may be restated by saying that basis vectors for certain irreducible representations of W classify $\mathfrak{X}_{\hat{\lambda}}$ and hence Prim $U(\mathfrak{g})$. These bases are implicitly determined by 3.2(i); but their explicit description is still an interesting open question. Except for type A_n, not all irreducible representations of W occur as a P_τ. Those that do occur are again implicitly determined by 3.2. From this and by some case by case analysis, Barbasch and Vogan [1, 2] showed that these representations are just the special representations in the sense of Lusztig, whose definition is given in terms of the classification of irreducible representations of finite Chevalley groups. This remarkable coincidence is not yet fully understood.

3.4. Recall the definition of the map β (2.8).

THEOREM. *Take $\tau \in \hat{W}$ occurring as a P_τ. Then $\tau \in \operatorname{Im}\beta$ and for each $J \in (\mathfrak{X}_{\hat{\lambda}})_\tau$, $\mathfrak{V}(\operatorname{gr} J)$ is the Zariski closure of $\beta^{-1}(\tau)$.*

In particular, $\mathfrak{V}(\operatorname{gr} J)$ is independent of $\lambda \in P(R)^+$ and $J \in (\mathfrak{X}_\lambda)_\tau$. This was proved in ([16], 2.6). From this, by the result for induced ideals and some case by case manipulations, the above theorem was established in the present so-called integral case by Borho and Brylinski [3]. A general proof was given in [24]. It was based on Gabber's equidimensionality theorem (2.8) and a reinterpretation of Springer's construction formulated in [21] and established by Hotta [13]. The proof uses a deep separation theorem of Gabber ([25], Sect. 7). It is worth noting that not all nilpotent orbits occur in the integral fibres $\mathfrak{X}_{\hat{\lambda}}$, $\lambda \in P(R)^+$. However if one takes the union over all possible fibres $\mathfrak{X}_{\hat{\lambda}}$, $\lambda \in \mathfrak{h}^*$, then one expects to find all the nilpotent orbits. This does not seem to have yet been checked but in any case has no elegant proof.

3.5. A more delicate question than the above is to be able to associate a completely prime primitive ideal to each nilpotent orbit. This is an open problem; but it is known that one can sometimes have two ([20], Sect. 4). A further open problem is to determine the composition factors of primitive quotients considered as $U(\mathfrak{g})-U(\mathfrak{g})$ bimodules. One expects the answer to be determined by the $a(w)$, see [22].

3.6. Returning to the question raised in 1.4 we remark that the Conze embeddnig [26] of $U(\mathfrak{g})/\mathrm{Ann}M(\lambda)$ into A_n with $n = \mathrm{card}\,R^+$ makes A_n flat left $U(\mathfrak{g})/\mathrm{Ann}M(\lambda)$ module for λ regular. Combined with work of Hodges and Smith this eventually gives $\mathrm{gl\,dim}\,U(\mathfrak{g})/\mathrm{Ann}M(\lambda) \leqslant \mathrm{card}\,R$ for λ regular. Yet $\mathrm{gl\,dim}\,U(\mathfrak{g})/\mathrm{Ann}M(\lambda) = \infty$ for λ non-regular.

4. Reduction to the semisimple case

4.1. It is assumed that $\mathfrak{g} = \mathrm{Lie}(G)$ where G is a connected linear algebraic group. This is a slight restriction, which can probably be overcome without much extra difficulty. Note that G acts by automorphisms in $U(\mathfrak{g})$ and because G is connected, any ideal of $S(\mathfrak{g})$ or any two-sided ideal of $U(\mathfrak{g})$ is G stable ([4], 12.3).

4.2. Let $\mathfrak{u}$ be an ideal of $\mathfrak{g}$. If $J \in \mathrm{Prim}\,U(\mathfrak{g})$, then $J \cap U(\mathfrak{u})$ is prime ([8], 3.3.4) but it need not be primitive. It is therefore useful to be able to recognize, which prime ideals are primitive. This is provided by the following result established by Dixmier ([9], Thm. C) for $k = \boldsymbol{C}$ and by Moeglin ([30], Sec. 4) in general.

THEOREM. *The following two conditions are equivalent*
(i) $J \in \mathrm{Prim}\,U(\mathfrak{g})$.
(ii) $J \in \mathrm{Spec}\,U(\mathfrak{g})$ *and* $(\mathrm{Fract}\,U(\mathfrak{g})/J)^G$ *is reduced to scalars.*

4.3. Let $\mathfrak{u}$ be an ideal of $\mathfrak{g}$ and take $J \in \mathrm{Prim}\ U(\mathfrak{g})$. By 4.2 it easily follows that $(\mathrm{Fract}\,U(\mathfrak{u})/U(\mathfrak{u})\cap J)^G = k$. In the commutative case this would, imply the existence of a unique dense G orbit in the subvariety of $\mathfrak{u}^*$ of zeros of $J \cap U(\mathfrak{u})$. The non-commutative case was first studied by Dixmier [9] and later completely analysed by Moeglin and Rentschler [32]. One has the following

THEOREM. *Take* $J \in \mathrm{Prim}\,U(\mathfrak{g})$.
(i) ([32], (2)). *There exists* $K \in \mathrm{Prim}\,U(\mathfrak{u})$ *such that* $J \cap U(\mathfrak{u}) = \bigcap_{g \in G} gK$.

(ii) ([32], (1) bis). *Any two primitive ideals of* $U(\mathfrak{u})$ *satisfying* (i) *are conjugate under* G.

4.4. Preserve the notation and hypotheses of 4.3. Set $B = \{b \in G | bK = K\}$. This is an algebraic subgroup of G which need not be connected. Set $\mathfrak{b} = \mathrm{Lie}(B) = \{X \in \mathfrak{g} | [X, K] \subset K\}$. If I is a two-sided ideal of $U(\mathfrak{b})$ we set $\mathrm{Ind}(I, \mathfrak{b}\uparrow\mathfrak{g}) = \bigcap_{g \in G} g\big(U(\mathfrak{g})I\big)$. Again by Moeglin and Rentschler [32, 33] we have the following

THEOREM. (i) ([32], (3)) *There exists* $I \in \mathrm{Prim}\, U(\mathfrak{b})$ *such that* $I \cap U(\mathfrak{u}) = K$ *and* $J = \mathrm{Ind}(I, \mathfrak{b}\uparrow\mathfrak{g})$.

(ii) ([33]) *Any two primitive ideals of* $U(\mathfrak{b})$ *satisfying* (i) *are conjugate under* B. *In particular there are only finitely many of them.*

4.5. From now on we take $\mathfrak{u}$ to be the unipotent radical of $\mathfrak{g}$. Since $K \in \mathrm{Prim}\, U(\mathfrak{u})$, we can write $K = J(u)$ for some $u \in \mathfrak{u}^*$ (see 1.2). As a consequence of 4.3 (ii), the G orbit of u in $\mathfrak{u}^*$ is completely determined by J. Set $\mathfrak{h} = \mathrm{Stab}_{\mathfrak{g}}(u) := \{X \in \mathfrak{g} | Xu = 0\} = \{X \in \mathfrak{g} | u([X, \mathfrak{u}]) = 0\}$. One has $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$, and obviously I contains the two-sided ideal $U(\mathfrak{b})J(u)$. Set

$$L_u = \sum_{X \in \mathfrak{h} \cap \mathfrak{u}} U(\mathfrak{h})\big(X - u(X)\big)$$

which is again a two-sided ideal of $U(\mathfrak{h})$. According to Duflo ([8], 10.1.4,) we have

PROPOSITION. *There is a canonical algebra isomorphism*

$$D_u \text{ of } U(\mathfrak{b})/U(\mathfrak{b})J(u) \text{ onto } U(\mathfrak{h})/L_u \otimes_k U(\mathfrak{u})/J(u).$$

4.6. Apart from a cohomology obstruction (here it is needed that $u([\mathfrak{h}, \mathfrak{u}]) = 0$), the existence statement in 4.5 results from the fact that $\mathfrak{h}$ acts by derivations on $U(\mathfrak{u})/J(u)$, which, (1.4) being a Weyl algebra A_n, admits only inner derivations ([8], 4.6.8). Again A_n is central simple, and so D_u sets up a bijection (which we shall also denote by D_u) from the set of two-sided ideals of $U(\mathfrak{b})$ containing $U(\mathfrak{b})J(u)$ onto the set $\mathrm{Prim}_u U(\mathfrak{h})$ of two-sided ideals of $U(\mathfrak{h})$ containing L_u. This takes primes to primes and so, by 4.1, primitives to primitives. Furthermore, Blattner's criterion ([8], 5.3.6) gives

THEOREM. *If* $J_1 \in \mathrm{Prim}_u U(\mathfrak{h})$, *then* $J_0 := \mathrm{Ind}\big(D_u^{-1}(J_1), \mathfrak{b}\uparrow\mathfrak{g}\big) \in \mathrm{Prim}\; U(\mathfrak{g})$.

4.7. In the above manner the classification of Prim $U(\mathfrak{g})$ is reduced to that of Prim $U(\mathfrak{h})$ for some, usually smaller, subalgebras $\mathfrak{h}$. Indeed, for a given $J \in \operatorname{Prim} U(\mathfrak{g})$ this process gives a sequence $\mathfrak{u}_1 = \mathfrak{u}, \mathfrak{u}_2, \ldots,$ of unipotent Lie algebras, of linear forms $u_1 = u, u_2, \ldots, u_i \in \mathfrak{u}_i^*$, of algebraic groups $B_1 = B, B_2, \ldots,$ of algebraic groups $H_0 = G, H_1, \ldots,$ of primitive ideals $J_0 = J, J_1, \ldots, J_i \in \operatorname{Prim} U(\mathfrak{h}_i)$ and of primitive ideals $I_1 = I, I_2, \ldots$ $\ldots, I_i \in \operatorname{Prim} U(\mathfrak{b}_i)$, where $B_i = \operatorname{Stab}_{B_{i-1}}(J(u_i))$, $\mathfrak{b}_i = \operatorname{Lie}(B_i)$, H_i is the identity component of $\operatorname{Stab}_{H_{i-1}}(u_i)$, $\mathfrak{h}_i = \operatorname{Lie}(H_i)$, $\mathfrak{u}_{i+1}$ is the unipotent radical of $\mathfrak{h}_i$, $u_{i+1} \in \mathfrak{u}_{i+1}^*$ is in the unique H_i orbit determined by J_i (recall 4.5), I_i is in the unique B_i-orbit (4.4) determined by the pair $(J_{i-1}, J(u_i))$ and $J_i = D_{u_i}(I_i)$. The process stops when $\mathfrak{h}_t \cap \mathfrak{u}_t$ is complemented in $\mathfrak{h}_t$ by a reductive Lie algebra $\mathfrak{r}$ (Levi factor). Set $\mathfrak{n} = \sum \mathfrak{u}_i$, $\mathfrak{l} = \mathfrak{h}_t$, $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$. Observe that $\mathfrak{n}$ is the unipotent radical of $\mathfrak{p}$. From the above construction it follows that there exists $n \in \mathfrak{n}^*$ such that $n|_{\mathfrak{u}_i} = u_i$. Then the above process may be collapsed to give $J = \operatorname{Ind}(D_n^{-1}(\tilde{J}_t), \mathfrak{p} \uparrow \mathfrak{g})$ ([12], Chap. IV, Sec. 9). Here $\tilde{J}_t$ differs very slightly from J_t (just with respect to the centre of $\mathfrak{r}$). This difficulty may be overcome by replacing Ind everywhere by $\operatorname{Ind}^{\sim}$ (see [11] for definition and subtleties involved).

4.8. We now describe the parametrization of Prim $U(\mathfrak{g})$ which results from the above reduction. Take $g \in \mathfrak{g}^*$ extending $n \in \mathfrak{n}^*$. Observe that we can recover the sequence $\mathfrak{h}_0 = \mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2, \ldots,$ by setting $\mathfrak{h}_i = \operatorname{Stab}_{\mathfrak{h}_{i-1}}(g|_{\mathfrak{u}_i})$ where $\mathfrak{u}_i$ is the unipotent radical of $\mathfrak{h}_{i-1}$. Then, with $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{l}$, $\mathfrak{r}$ as above we note that $g([\mathfrak{l}, \mathfrak{n}]) = 0$. Furthermore, one checks, with respect to the form $(X, Y) \mapsto g([X, Y])$ defined by g, that $\mathfrak{p}^\perp \subset \mathfrak{h}_1$; continuing inductively we obtain $\mathfrak{p}^\perp \subset \mathfrak{l}$; so, in particular, $\mathfrak{p}$ is co-isotropic. Let $\mathfrak{g}(g)$ denote the kernel of this form and define $\mathfrak{h}(h)$ ($h := g|_{\mathfrak{h}}$) and $\mathfrak{u}(u)$ similarly. An elementary calculation gives $\mathfrak{h}(h) = \mathfrak{g}(g) + \mathfrak{u}(u)$, which applied inductively gives $\mathfrak{l}(l) \subset \mathfrak{g}(g) + \mathfrak{n}$ (where $l = g|_{\mathfrak{l}}$). Finally, assume that we have taken the extension g of n to satisfy $g(\mathfrak{r}) = 0$. One easily checks that $\mathfrak{r} \subset \mathfrak{l}(l)$ and so $\mathfrak{p} = \mathfrak{g}(g) + \mathfrak{n}$.

4.9. Given $g \in \mathfrak{g}^*$, we define (following Duflo [12], I. 8) a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ to be of strongly unipotent type with respect to g if $\mathfrak{p}$ is algebraic, co-isotropic with respect to g and $\mathfrak{p} = \mathfrak{g}(g) + \mathfrak{n}$ where $\mathfrak{n}$ is the unipotent radical of $\mathfrak{p}$. We define $g \in \mathfrak{g}^*$ to be of *Duflo type* if

(i) There exists a Levi factor $\mathfrak{r}$ of $\mathfrak{g}(g)$ satisfying $g(\mathfrak{r}) = 0$. (If this holds for one factor, it holds for all of them since they are conjugated under $G(f) := \operatorname{Stab}_G(g)$.)

(ii) There exists a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ of strongly unipotent type with respect to g.

We have seen that g in 4.8 is of Duflo type. Again if $g \in \mathfrak{g}^*$ is of Duflo type then $g|_{\mathfrak{h}}$ is of Duflo type ([12], Lemme 17). This is used to show that if $g \in \mathfrak{g}^*$ is of Duflo type then the $\mathfrak{p}$ constructed canonically in 4.8 is of strongly unipotent type with respect to g. Finally, Duflo ([12], Prop. 26 (i)) showed that if $g \in \mathfrak{g}^*$ is of Duflo type and $\mathfrak{p}$ is of strongly unipotent type with respect to g, then g is determined up to conjugation under G by its restriction to $\mathfrak{p}$.

4.10. Let $\mathfrak{g}_D^*$ denote the subset of linear forms on $\mathfrak{g}$ of Duflo type. Let $\mathfrak{p}, \mathfrak{n}, \mathfrak{l}, n$ be as defined in 4.8. Let $\mathfrak{r}_g$ be a Levi factor of $\mathfrak{g}(g)$ (we remind that they are all conjugate under $G(g)$); it is also a Levi factor of $\mathfrak{l}$. Given $P \in \operatorname{Prim} U(\mathfrak{r})$, let Q be the unique primitive ideal of $U(\mathfrak{l})$ containing

$$L_n = \sum_{X \in \mathfrak{l} \cap \mathfrak{n}} U(\mathfrak{l})\big(X - n(X)\big)$$

such that $Q \cap U(\mathfrak{r}) = P$. Set $J(g, P) = \operatorname{Ind}^{\sim}\big(D_n^{-1}(Q), \mathfrak{p}\uparrow\mathfrak{g}\big)$. Then

THEOREM. (i) *Every* $J \in \operatorname{Prim} U(\mathfrak{g})$ *takes the form* $J = J(g,P)$ *for some* $g \in \mathfrak{g}_D^*$, $P \in \operatorname{Prim} U(\mathfrak{r}_g)$.

(ii) *Given* $J(g, P) = J(g', P')$, *there exists* $s \in G$ *such that* $sg = g'$, $s\mathfrak{r}_g = \mathfrak{r}_{g'}$, $sP = P'$.

This means that the map $(g, P) \mapsto J(g, P)$ factors to a bijection of $\coprod_{g \in \mathfrak{g}_D^*} \{P \in \operatorname{Prim} U(\mathfrak{r}_g)\}/G$ onto $\operatorname{Prim} U(\mathfrak{g})$. Surjectivity is due to Duflo ([12], IV, Thm. 7) and injectivity to Moeglin and Rentschler [33]. For injectivity the crucial part was 4.4 (ii).

4.11. Retain the notation of 4.10. It is easy to see that $\operatorname{rk}\big(U(\mathfrak{r}_g)/P\big) \geqslant \operatorname{rk}\big(U(\mathfrak{g})/J(g, P)\big)$, $\forall g \in \mathfrak{g}_D^*$, $P \in \operatorname{Prim} U(\mathfrak{r}_g)$; but it is not quite obvious if equality holds. Again for each $g \in \mathfrak{g}_D^*$ the map $P \mapsto J(g, P)$ factorizes to an order isomorphism of $\big(G(g)\operatorname{Prim} U(\mathfrak{r}_g)\big)/G(g)$ onto its image in $\operatorname{Prim} U(\mathfrak{g})$. Here the order is defined by inclusion. It is not yet clear what might be the order relations with respect to the different $g \in \mathfrak{g}_D^*$, though one should expect these to be expressible, at least partly, in terms of the inclusion relations of the Zariski closures of the orbits Gg. For $\mathfrak{g}$ solvable such a result would be equivalent to the bicontinuity of the map $\mathfrak{g}^*/G \xrightarrow{\sim} \operatorname{Prim} U(\mathfrak{g})$ (recall 1.2) and this is still an open question.

Section 4 is based on a talk of Rentschler given at Oberwolfach [34] in January 1983. I should like to thank Rentschler for explaining the results he obtained in collaboration with Moeglin [32, 33] and the work of Duflo [12]. Rentschler has recently reported that the restriction on g being algebraic can be removed and has described the modifications that ensue.

References

[1] Barbasch D. and Vogan D., Primitive Ideals and Orbital Integrals in Complex Classical Groups, *Math. Ann.* **259** (1982), pp. 153–199.

[2] Barbasch D. and Vogan D., Primitive Ideals and Orbital Integrals in Complex Exceptional Groups, *J. Algebra* **80** (1983), pp. 350–382.

[3] Borho W. and Brylinski J.-L., Differential Operators on Homogeneous Spaces. I, *Invent. Math.* **69** (1982), pp. 437–476.

[4] Borho W., Gabriel P., and Rentschler R., *Primideale in Einhüllenden auflösbarer Lie Algebren,* LN 357, Springer-Verlag, Berlin 1973.

[5] Borho W. and Jantzen J.-C., Über primitive Ideale in der Einhüllenden einer halbeinfachen Lie-Algebra, *Invent. Math.* **39** (1977), pp. 1–53.

[6] Conze N., Espace des idéaux primitifs de l'algèbre enveloppante d'une algèbre de Lie nilpotente, *J. Algebra* **34** (1975), pp. 444–460.

[7] Dixmier J., Algèbres enveloppantes. In: *Proceedings of the International Congress of Mathematicians,* Helsinki, 1978, pp. 693–698.

[8] Dixmier J., *Algèbres enveloppantes,* Gauthier-Villars, Paris 1974.

[9] Dixmier J., Idéaux primitifs dans les algèbres enveloppantes, *J. Algebra* **48** (1977), pp. 86–112.

[10] Duflo M., Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple, *Ann. Math.* **105** (1977), pp. 107–120.

[11] Duflo M., Sur les idéaux induits dans les algèbres enveloppantes, *Invent. Math.* **67** (1982), pp. 385–392.

[12] Duflo M., Théorie de Mackey pour les groupes de Lie algébriques, *Acta Mathematica* (to appear) .

[13] Hotta R., *On Joseph's Construction of Weyl Group Representations,* preprint, Tohoku, 1982.

[14] Jantzen J.-C., Einhüllende Algebren halbeinfacher Lie-Algebren, *Proceedings of the International Congress of Mathematicians, Warsaw, 1983.*

[15] Jantzen J.-C., Einhüllende Algebren halbeinfacher Lie-Algebren, *Ergebnissen der Mathematik,* Springer-Verlag, 1983.

[16] Joseph A., Towards the Jantzen Conjecture. III, *Compos. Math.* **41** (1981), pp. 23–30.

[17] Joseph A., Kostant's Problem, Goldie Rank and the Gelfand–Kirillov Conjecture, *Invent. Math.* **56** (1980), pp. 191–213.

[18] Joseph A., *W- Module Structure in the Enveloping Algebra of a Semisimple Lie Algebra,* LN 728, Springer-Verlag, Berlin 1979.

[19] Joseph A., Goldie Rank in the Enveloping Algebra of a Semisimple Lie Algebra, I/II/III, *J. Algebra* **65** (1980), pp. 269–283; pp. 284–306; **73** (1981), pp. 295–326.

[20] Joseph A., *Kostant's Problem and Goldie Rank,* LN 880, Springer-Verlag, Berlin 1981.

[21] Joseph A., On the Variety of a Highest Weight Module, *J. Algebra* (to appear).

[22] Joseph A., Completion Functors in the 0 Category. In: *Non-Commutative Harmonic Analysis and Lie Groups*, Springer Lecture Notes 1020, 1983.
[23] Joseph A., On the Classification of Primitive Ideals in the Enveloping Algebra of a Semisimple Lie Algebra: *Lie Group Representations I*, Springer Lecture Notes 1024, 1983.
[24] Joseph A., On the Associated Variety of a Promitive Ideal, to appear in *J. of Algebra.*
[25] Joseph A., *Application de la theorie des anneaux aux algèbres enveloppantes,* notes de 3eme cycle, Paris, 1981.
[26] Joseph A. and Stafford T. J., to appear.
[27] Kazhdan D. A. and Lusztig G., Representations of Coxeter Groups and Hecke Algebras, *Invent. Math.* **53** (1979), pp. 165–184.
[28] Levasseur T., *Equidimensionalité de la variété caractéristique,* preprint, Paris, 1983.
[29] Lusztig G., On a Theorem of Benson and Curtis, *J. of Algebra* **71** (1981), pp. 490–498.
[30] Moeglin C., Idéaux primitifs dans les algèbres enveloppantes, *J. Math. pures et appl.* **59** (1980), pp. 265–336.
[31] Moeglin C., Idéaux bilatères des algèbres enveloppantes, *Bull. Soc. Math. France* **108** (1980), pp. 143–186.
[32] Moeglin C. and Rentschler R., Orbites d'un groupe algèbrique dans l'espace des idéaux rationnels d'une algèbre enveloppantes, *Bull. Soc. Math. France* **109** (1981), pp. 403–426.
[33] Moeglin C. and Rentschler R., *Sur la classification des idéaux primitifs des algèbres enveloppantes,* preprint, Paris, 1983.
[34] Rentschler R., *Report on Primitive Ideals in Enveloping Algebras of Lie Algebras,* talk, Oberwolfach, 1983.
[35] Springer T. A., A Construction of Representations of Weyl Groups, *Invent. Math.* **44** (1978), pp. 279–293.
[36] Tauvel P., Sur la bicontinuité de l'application de Dixmier pour les algèbres de Lie résolubles, *C. R. Acad. Sci. Paris* **294** (1982), pp. 685–687.
[37] Vogan D. A., Ordering of the Primitive Spectrum of a Semisimple Lie Algebra, *Math. Ann.* **248** (1980), pp. 195–203.