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Banach KK-theory and the Baum-Connes Conjecture

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Abstract

The report below describes the applications of Banach KK-theory to a conjecture of P. Baum and A. Connes about the K-theory of group C^*-algebras, and a new proof of the classification by Harish-Chandra, the construction by Parthasarathy and the exhaustion by Atiyah and Schmid of the discrete series representations of connected semi-simple Lie groups.

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This report is intended to be very elementary. In the first part we outline the main results in Banach KK-theory and the applications to the Baum-Connes conjecture. In the second part we show how the Baum-Connes conjecture for connected semi-simple Lie groups can be applied to recover the classification of the discrete series representations.

1. Banach KK-theory and the Baum-Connes conjecture

There are many surveys on Kasparov's KK-theory and the Baum-Connes conjecture (see [4, 48, 49, 21, 27, 13, 54]) and on Banach KK-theory ([49, 38]).

1.1. Generalized Fredholm modules

We wish to define A-linear Fredholm operators (where A is a Banach algebra), with an index in $K_0(A)$. If $A = \mathbb{C}$, this index should be the usual index of $\mathbb{C}$-linear Fredholm operators in $K_0(\mathbb{C}) = \mathbb{Z}$.

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We define a Banach algebra as a (non necessarily unital) $\mathbb{C}$-algebra A that is complete for a norm $||.||$ satisfying $||ab|| \leq ||a|| ||b||$ for any $a, b \in A$. If A and B are Banach algebras a morphism $\theta : A \to B$ is an algebra morphism such that $||\theta(a)|| \leq ||a||$ for any $a \in A$.

K_0 and K_1 are two covariant functors from the category of Banach algebras to the category of abelian groups. If X is a locally compact space and $C_0(X)$ the algebra of continuous functions vanishing at infinity, $K_0(C_0(X))$ and $K_1(C_0(X))$ are the Atiyah-Hirzebruch K-theory groups. For technical reasons we shall restrict ourselves to unital Banach algebras in this subsection.

Let A be a unital Banach algebra.

A right A-module E is finitely generated projective if and only if it is a direct summand in A^n for some integer n. The set of isomorphism classes of right finitely generated projective A-modules is a semigroup because the direct sum of two right finitely generated projective A-modules is a right finitely generated projective A-module. Then $K_0(A)$ is the universal group associated to this semigroup (*i.e.* the group of formal differences of elements of the semigroup). If $\theta : A \to B$ is a morphism of unital Banach algebras, and E is a right finitely generated projective A-module then $E \otimes_A B$ is a right finitely generated projective B-module and this defines $\theta_* : K_0(A) \to K_0(B)$.

There is another definition of $K_0(A)$ for which the functoriality is even more obvious : $K_0(A)$ is the quotient of the free abelian group generated by all idempotents p in $M_k(A)$ for some integer k, by the relations $\left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right] = [p] + [q]$ for any idempotents $p \in M_k(A)$ and $q \in M_l(A)$ and $[p] = [q]$ if p, q are idempotents of $M_k(A)$ and are connected by a path of idempotents in $M_k(A)$ and $[0] = 0$ where 0 is the idempotent 0 in $M_k(A)$. The link with the former definition is that any idempotent $p \in M_k(A)$ acts on the left on A^k as a projector P and $\mathrm{Im}P$ is a right finitely generated projective A-module (it is a direct summand in the right A-module A^k).

The following construction was performed for C^*-algebras by Mischenko and Kasparov, in connection with the Novikov conjecture ([43, 28]). We adapt it to Banach algebras.

A right Banach A-module is a Banach space (with a given norm $||.||_E$) equipped with a right action of A such that $1 \in A$ acts by identity and $||xa||_E \leq ||x||_E ||a||_A$ for any $x \in E$ and $a \in A$. Let E and F be right Banach A-modules. A morphism $u : E \to F$ of right Banach A-modules is a continuous $\mathbb{C}$-linear map such that $u(xa) = u(x)a$ for any $x \in E$ and $a \in A$. The space $\mathcal{L}_A(E, F)$ of such morphisms is a Banach space with norm $||u|| = \sup_{x \in E, ||x||_E = 1} ||u(x)||_F$. A morphism $u \in \mathcal{L}_A(E, F)$ is said to be "A-rank one" if $u = w \circ v$ with $v \in \mathcal{L}_A(E, A)$ and $w \in \mathcal{L}_A(A, F)$. The space $\mathcal{K}_A(E, F)$ of A-compact morphisms is the closed vector span of A-rank one morphisms in $\mathcal{L}_A(E, F)$. If $E = F$, $\mathcal{L}_A(E) = \mathcal{L}_A(E, E)$ is a Banach algebra and $K_A(E) = K_A(E, E)$ is a closed ideal in it.

Definition 1.1.1 *A Fredholm module over A is the data of a $\mathbb{Z}/2$ graded right Banach A-module E and an odd morphism $T \in \mathcal{L}_A(E)$ such that $T^2 - \mathrm{Id}_E \in \mathcal{K}_A(E)$.*

In other words $E = E_0 \oplus E_1$, $T = \begin{pmatrix} 0 & v \\ u & 0 \end{pmatrix}$ and $u \in \mathcal{L}_A(E_0, E_1)$ and $v \in$

$\mathcal{L}_A(E_1, E_0)$ satisfy $vu - \mathrm{Id}_{E_0} \in \mathcal{K}_A(E_0)$ and $uv - \mathrm{Id}_{E_1} \in \mathcal{K}_A(E_1)$.

If (E, T) is a Fredholm module over A and $\theta : A \to B$ a unital morphism then $(E \otimes_A B, T \otimes 1)$ is a Fredholm module over B (here $E \otimes_A B$ is the completion of $E \otimes_A^{\mathrm{alg}} B$ for the maximal Banach norm such that $\|x \otimes b\| \leq \|x\|_E \|b\|_B$ for $x \in E$ and $b \in B$).

Let $A[0,1]$ be the Banach algebra of continuous functions from $[0,1]$ to A with the norm $\|f\| = \sup_{t \in [0,1]} \|f(t)\|_A$ and $\theta_0, \theta_1 : A[0,1] \to A$ the evaluations at 0 and 1. Two Fredholm modules on A are said to be homotopic if they are the images by θ_0 and θ_1 of a Fredholm module over $A[0,1]$.

Theorem 1.1.2 *There is a functorial bijection between $K_0(A)$ and the set of homotopy classes of Fredholm modules over A, for any unital Banach algebra A.*

Let (E_0, E_1, u, v) be a Fredholm module over A. Its index, *i.e.* the corresponding element in $K_0(A)$, is constructed as follows. It is possible to find $n \in \mathbb{N}$ and $w \in \mathcal{K}_A(A^n, E_1)$ such that $(u, w) \in \mathcal{L}_A(E_0 \oplus A^n, E_1)$ is surjective. Its kernel is then finitely generated projective and the index is the formal difference of $\mathrm{Ker}((u, w))$ and A^n.

An ungraded Fredholm module over A is the data of a (ungraded) right Banach module E over A, and $T \in \mathcal{L}_A(E)$ such that $T^2 - \mathrm{Id}_E \in \mathcal{K}_A(E)$. There is a functorial bijection between $K_1(A)$ and the set of homotopy classes of ungraded Fredholm modules.

For a non-unital algebra A, $K_0(A) = \mathrm{Ker}(K_0(\tilde{A}) \to K_0(\mathbb{C}) = \mathbb{Z})$ and $K_1(A) = K_1(\tilde{A})$ where $\tilde{A} = A \oplus \mathbb{C}1$. In particular every idempotent in $M_k(A)$ gives a class in $K_0(A)$ but in general not all classes in $K_0(A)$ are obtained in this way. The definition of a Fredholm module should be slightly modified for non-unital Banach algebras, but the theorem 1.1.2 remains true.

1.2. Statement of the Baum-Connes conjecture

Let G be a second countable, locally compact group. We fix a left-invariant Haar measure dg on G. Denote by $C_c(G)$ the convolution algebra of complex-valued continuous compactly supported functions on G. The convolution of $f, f' \in C_c(G)$ is given by $f * f'(g) = \int_G f(h) f'(h^{-1}g)dh$ for any $g \in G$.

When G is discrete and dg is the counting measure, $C_c(G)$ is also denoted by $\mathbb{C}G$ and if e_g denotes the delta function at $g \in G$ (equal to 1 at g and 0 elsewhere), $(e_g)_{g \in G}$ is a basis of $\mathbb{C}G$ and the convolution product is given by $e_g e_{g'} = e_{gg'}$.

The completion of $C_c(G)$ for the norm $\|f\|_{L^1} = \int_G |f(g)| dg$ is a Banach algebra and is denoted by $L^1(G)$.

For any $f \in C_c(G)$ let $\lambda(f)$ be the operator $f' \mapsto f * f'$ on $L^2(G)$. The completion of $C_c(G)$ by the operator norm $\|f\|_{\mathrm{red}} = \|\lambda(f)\|_{\mathcal{L}_{\mathbb{C}}(L^2(G))}$ is called the reduced C^*-algebra of G and denoted by $C^*_{\mathrm{red}}(G)$. If G is discrete $(e_{g'})_{g' \in G}$ is an orthonormal basis of $L^2(G)$ and $\lambda(e_g) : e_{g'} \mapsto e_{gg'}$.

For any $f \in C_c(G)$, $\|f\|_{L^1} \geq \|f\|_{\mathrm{red}}$ and $L^1(G)$ is a dense subalgebra of $C^*_{\mathrm{red}}(G)$. We denote by $i : L^1(G) \to C^*_{\mathrm{red}}(G)$ the inclusion.

Assume now that M is a smooth compact manifold, and $\tilde{M}$ a Galois covering of M with group G (if $\tilde{M}$ is simply connected, $G = \pi_1(M)$). Let E_0 and E_1 be two smooth hermitian finite-dimensional vector bundles over M and u an order 0 elliptic pseudo-differential operator from $L^2(M, E_0)$ to $L^2(M, E_1)$. Since u is elliptic there is an order 0 pseudo-differential operator $v : L^2(M, E_1) \to L^2(M, E_0)$ such that $\mathrm{Id}_{L^2(M,E_0)} - vu$ and $\mathrm{Id}_{L^2(M,E_1)} - uv$ have order ≤ -1 and therefore are compact. Let $\mathcal{E}$ be the quotient of $\tilde{M} \times C^*_{\mathrm{red}}(G)$ by the diagonal action of G (G acting on $C^*_{\mathrm{red}}(G)$ by left translations) : $\mathcal{E}$ is a flat bundle of right $C^*_{\mathrm{red}}(G)$-modules over M, whose fibers are isomorphic to $C^*_{\mathrm{red}}(G)$. Then $L^2(M, E_0 \otimes \mathcal{E})$ and $L^2(M, E_1 \otimes \mathcal{E})$ are right Banach (in fact Hilbert) modules over $C^*_{\mathrm{red}}(G)$ and it is possible to lift u and v to $\tilde{u}$ and $\tilde{v}$ so that $(L^2(M, E_0 \otimes \mathcal{E}), L^2(M, E_1 \otimes \mathcal{E}), \tilde{u}, \tilde{v})$ is a Fredholm module over $C^*_{\mathrm{red}}(G)$, whose index lies in $K_0(C^*_{\mathrm{red}}(G))$ and the index does not depend on the choice of the liftings.

The operator u represents a "K-homology class" in $K_0(M)$, and using the classifying map $M \to BG$, it defines an element of $K_{0,c}(BG)$, the K-homology with compact support of the classifying space BG. For any discrete group G we can define a morphism of abelian groups $K_{*,c}(BG) \to K_*(C^*_{\mathrm{red}}(G))$ ($* = 0, 1$). This morphism is the Baum-Connes assembly map when G is discrete and torsion free. When G is not discrete or has torsion, the index construction can be performed starting from a proper action of G (instead of the free and proper action of G on $\tilde{M}$ in the last paragraph), and therefore we have to introduce the space $\underline{E}G$ that classifies the proper actions of G. Using Kasparov equivariant KK-theory, the G-equivariant K-homology $K^G_*(\underline{E}G)$ with G-compact support ($* = 0, 1$) may be defined, and there is an assembly map

$$\mu_{\mathrm{red}} : K^G_*(\underline{E}G) \to K_*(C^*_{\mathrm{red}}(G)).$$

In the same way we can define $\mu_{L^1} : K^G_*(\underline{E}G) \to K_*(L^1(G))$ and $\mu_{\mathrm{red}} = i_* \circ \mu_{L^1}$.

Baum-Connes conjecture [3, 4] : *If G is a second countable, locally compact group then the assembly map $\mu_{\mathrm{red}} : K^G_*(\underline{E}G) \to K_*(C^*_{\mathrm{red}}(G))$ is an isomorphism.*

Bost conjectured : If G is a second countable, locally compact group (and has reasonable geometric properties) then the assembly map $\mu_{L^1} : K^G_*(\underline{E}G) \to K_*(L^1(G))$ is an isomorphism.

In many cases $K^G_*(\underline{E}G)$ can be computed. For instance if G is a discrete torsion free subgroup of a reductive Lie group H and K is a maximal compact subgroup of H, then a possible $\underline{E}G$ is H/K and $K^G_*(\underline{E}G)$ is the K-homology with compact support of $G \backslash H/K$. This group may be computed thanks to Mayer Vietoris sequences. See part 2 for the case where G is a Lie group.

1.3. KK-theory

For any C^*-algebras A and B, Kasparov [28, 31] defined an abelian group $KK(A, B)$, covariant in B and contravariant in A. There is a product $KK(A, B) \otimes$

$KK(B,C) \to KK(A,C)$. Moreover $KK(\mathbb{C},A) = K_0(A)$ and therefore the product gives a morphism $KK(A,B) \to \mathrm{Hom}(K_0(A),K_0(B))$. The definition of $KK(A,B)$ is like definition 1.3.1 below, but with Hilbert modules instead of Banach modules.

For any Banach algebras A and B, we define [37, 49] an abelian group $KK^{\mathrm{ban}}(A,B)$, covariant in B and contravariant in A. There is no product, but a morphism $KK^{\mathrm{ban}}(A,B) \to \mathrm{Hom}(K_0(A),K_0(B))$. Assume that B is unital (otherwise the definition has to be slightly modified).

Definition 1.3.1 *$E^{\mathrm{ban}}(A,B)$ is the set of isomorphism classes of data (E,π,T), where E is a $\mathbb{Z}/2\mathbb{Z}$-graded right Banach module, $\pi : A \to \mathcal{L}_B(E)$ is a morphism of Banach algebras and takes values in even operators, and $T \in \mathcal{L}_B(E)$ is odd and satisfies $a(T^2 - \mathrm{Id}_E) \in K_B(E)$ and $aT - Ta \in K_B(E)$ for any $a \in A$.*

Then $KK^{\mathrm{ban}}(A,B)$ is the set of homotopy classes in $E^{\mathrm{ban}}(A,B)$, where the homotopy relation is defined using $E^{\mathrm{ban}}(A,B[0,1])$.
Remark : $E^{\mathrm{ban}}(\mathbb{C},B)$ is the set of isomorphism classes of Fredholm modules over B and $KK^{\mathrm{ban}}(\mathbb{C},B) = K_0(B)$.

If p is an idempotent in A, and $(E,\pi,T) \in E^{\mathrm{ban}}(A,B)$, the image of $[p] \in K_0(A)$ by the image of $[(E,\pi,T)] \in KK^{\mathrm{ban}}(A,B)$ in $\mathrm{Hom}(K_0(A),K_0(B))$ is defined to be the index of the Fredholm module over B equal to $(\mathrm{Im}\pi(p), \pi(p)T\pi(p))$. When p is an idempotent in $M_k(A)$, we use the image of p by $M_k(A) \to \mathcal{L}_B(E^k)$. This is enough to define the morphism $KK^{\mathrm{ban}}(A,B) \to \mathrm{Hom}(K_0(A),K_0(B))$, when A is unital.

The same definition with ungraded modules gives $KK_1^{\mathrm{ban}}(A,B)$, and, with the notation $KK = KK_0$, we have a morphism $KK_i^{\mathrm{ban}}(A,B) \to \mathrm{Hom}(K_j(A),K_{i+j}(B))$, where all the indices are modulo 2.

1.4. Status of injectivity and the element γ

The injectivity of the Baum-Connes map μ_{red} (and therefore of μ_{L^1}) is known for the following very large classes of groups :

a) groups acting continuously properly isometrically on a complete simply connected riemannian manifold with controlled non-positive sectional curvature, and in particular closed subgroups of reductive Lie groups ([29, 31]),

b) groups acting continuously properly isometrically on an affine building and in particular closed subgroups of reductive p-adic groups ([32]),

c) groups acting continuously properly isometrically on a discrete metric space with good properies at infinity (weakly geodesic, uniformly locally finite, and "bolic" [33, 34]), and in particular hyperbolic groups (*i.e.* word-hyperbolic in the sense of Gromov),

d) groups acting continuously amenably on a compact space ([22]).

In the cases a),b),c) above, the proof of injectivity provides an explicit idempotent endomorphism on $K_*(C^*_{\mathrm{red}}(G))$ whose image is the image of μ_{red} (and the same for μ_{L^1}). In case d), J.-L. Tu has also constructed such an endomorphism, but in a less explicit way.

To state this we need to understand a baby case of Kasparov's equivariant KK-groups. Let G be a second countable, locally compact group. We denote by $E_G(\mathbb{C},\mathbb{C})$ the set of isomorphism classes of triples (H,π,T) where H is a $\mathbb{Z}/2$-graded Hilbert space, π a unitary representation of G on H (such that for any $x \in H$, $g \mapsto gx$ is continuous from G to H) and T an odd operator on H such $T^2 - \mathrm{Id}_H$ is compact and $\pi(g)T\pi(g^{-1}) - T$ is compact and depends norm continuously on $g \in G$. Then $KK_G(\mathbb{C},\mathbb{C})$ is the quotient of $E_G(\mathbb{C},\mathbb{C})$ by homotopy. Kasparov proved that $KK_G(\mathbb{C},\mathbb{C})$ has a ring structure (using direct sum for the addition and tensor products together with a quite difficult construction for the multiplication).

If π is a unitary representation of G on a Hilbert space H_0 and $H_1 = 0$ then $(H,\pi,0) \in E_G(\mathbb{C},\mathbb{C})$ if and only if H_0 has finite dimension. If moreover $H_0 = \mathbb{C}$ and π is the trivial representation of G, the class of $(H,\pi,0)$ is the unit of $KK_G(\mathbb{C},\mathbb{C})$ and is denoted by 1. If G is compact the classes of $(H,\pi,0)$ with $H_1 = 0$ (and $\dim H_0 < +\infty$) generate $KK_G(\mathbb{C},\mathbb{C})$ and $KK_G(\mathbb{C},\mathbb{C})$ is equal to the representation ring of G.

The important fact is that there is a "descent morphism"

$$j_{\mathrm{red}} : KK_G(\mathbb{C},\mathbb{C}) \to \mathrm{End}(K_*(C^*_{\mathrm{red}}(G))).$$

In fact it is a ring homomorphism and $j_{\mathrm{red}}(1) = \mathrm{Id}_{K_*(C^*_{\mathrm{red}}(G))}$. It is defined as the composite of two maps $KK_G(\mathbb{C},\mathbb{C}) \to KK(C^*_{\mathrm{red}}(G), C^*_{\mathrm{red}}(G)) \to \mathrm{End}(K_*(C^*_{\mathrm{red}}(G)))$. The construction of j_{red} is due to Kasparov. The construction of j_{L^1} to be explained below is an adaptation of it.

The following extremely important theorem also contains earlier works of Mishchenko and Solovjev.

Theorem 1.4.1 *(Kasparov, Kasparov-Skandalis [31, 32, 33, 34]) If G belongs to one of the classes a),b),c) above, the geometric conditions in a),b) or c) allow to construct an idempotent element $\gamma \in KK_G(\mathbb{C},\mathbb{C})$ such that μ_{red} is injective and its image is equal to the image of the idempotent $j_{\mathrm{red}}(\gamma) \in \mathrm{End}(K_*(C^*_{\mathrm{red}}(G)))$.*

1.5. Homotopies between γ and 1

We assume that G belongs to one of the classes a),b),c). Then the injectivity of μ_{red} is known and the surjectivity is equivalent to the equality $j_{\mathrm{red}}(\gamma) = \mathrm{Id} \in \mathrm{End}(K_*(C^*_{\mathrm{red}}(G)))$.

Theorem 1.5.1 *We have $\gamma = 1$ in $KK_G(\mathbb{C},\mathbb{C})$ if*

1. *G is a free group (Cuntz, [14]) or a closed subgroup of $SO(n,1)$ (Kasparov, [30]) or of $SU(n,1)$ (Julg-Kasparov, [25]) or of $SL_2(\mathbb{F})$ with $\mathbb{F}$ a local non-archimedian field (Julg-Valette, [24]),*
2. *G acts isometrically and properly on a Hilbert space (Higson-Kasparov [20, 27]).*

In fact the second case contains the first one.

If G has property (T) and is not compact, $\gamma \neq 1$ in $KK_G(\mathbb{C},\mathbb{C})$: it is impossible to deform 1 to γ in $E_G(\mathbb{C},\mathbb{C})$ because the trivial representation is isolated among unitary representations of G if G has property (T) and γ can be represented by (H,π,T) such that H has no invariant vector (and even H is tempered). All simple real or p-adic groups of rank ≥ 2, and $Sp(n,1)$ and $F_{4(-20)}$, and all their lattices, have property (T) (see [19]).

It is then natural to broaden the class of representations in order to break the isolation of the trivial one. In [26] Julg proposed to use uniformly bounded representations on Hilbert spaces (to solve the case of $Sp(n,1)$).

For any non compact group G the trivial representation is not isolated among isometric representations in Banach spaces (think of the left regular representation on $L^p(G)$, p going to infinity).

Definition 1.5.2 *Let $E_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$ be the set of isomorphism classes of triples (E,π,T) with E a $\mathbb{Z}/2$-graded Banach space endowed with an isometric representation of G (such that $g \mapsto gx$ is continuous from G to E for any $x \in E$), $T \in \mathcal{L}_{\mathbb{C}}(E)$ an odd operator such that $T^2 - \mathrm{Id}_E$ belongs to $\mathcal{K}_{\mathbb{C}}(E)$ and $\pi(g)T\pi(g^{-1}) - T$ belongs to $\mathcal{K}_{\mathbb{C}}(E)$ and depends norm continuously on $g \in G$.*

Then $KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$ is defined as the quotient of $E_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$ by homotopy.

Since any unitary representation of G on a Hilbert space H is an isometric representation on the Banach space H, there is a natural morphism of abelian groups $KK_G(\mathbb{C},\mathbb{C}) \to KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$.

To state our main theorem, we need to look at slightly smaller classes than a) and c) above. We call these new classes a') and c'). They are morally the same, and in particular they respectively contain all closed subgroups of reductive Lie groups, and all hyperbolic groups (for general hyperbolic groups see [42], and [37] for a slightly different approach).

Theorem 1.5.3 *[37, 49] For any group G in the classes a'), b), or c'), we have $\gamma = 1$ in $KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$.*

In fact the statement is slightly incorrect, we should allow representations with a slow growth, but this adds no real difficulty. The proof of this theorem is quite technical. Let me just indicate some ingredients involved. If G is in class a') then G acts continuously isometrically properly on a complete simply connected riemmannian manifold X with controlled non-positive sectional curvature, and X is contractible (through geodesics) and the de Rham cohomology of X (without support) is $\mathbb{C}$ in degree 0 and 0 in other degrees. It is possible to put norms on the spaces of differential forms (on which G acts) and to build a parametrix for the de Rham operator (in the spirit of the Poincaré lemma) in order to obtain a resolution of the trivial representation, and in our language an element of $E_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$ equal to 1 in $KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$. The norms we use are essentially Sobolev L^∞ norms. Then it is possible to conjugate the operators by an exponential of the distance to a fixed point in X and then to deform these norms to Hilbert norms (through L^p norms, $p \in [2,+\infty]$) and to reach γ.

If G belongs to class b) the de Rham complex is replaced by the simplicial homology complex (with L^1 norms) on the building. If G belongs to class c') a Rips complex plays the same role as the building in b).

It is not possible to apply directly this theorem to the Baum-Connes conjecture because there is no obvious descent map $KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C}) \to \mathrm{End}(K_*(C^*_{\mathrm{red}}(G)))$, and in the next subsection we shall see the difficulties encountered and the way one bypasses them in a few cases.

On the other hand, we may apply this theorem to Bost conjecture, because there is descent map $j_{L^1} : KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C}) \to KK^{\mathrm{ban}}(L^1(G), L^1(G))$.

We explain it when G is discrete. Let $(E,\pi,T) \in E_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C})$. We denote by $L^1(G,E)$ the completion of $E \otimes \mathbb{C}G$ for the norm $\|\sum_{g\in G} x(g)\otimes e_g\| = \sum_{g\in G}\|x(g)\|_E$. Then $L^1(G,E)$ is a right Banach $L^1(G)$-module by the formula $(x\otimes e_g)e_{g'} = x\otimes e_{gg'}$ and there is a Banach algebra morphism $\hat\pi : L^1(G) \to \mathcal{L}_{L^1(G)}(L^1(G,E))$ by the formula $\hat\pi(e_{g'})(x\otimes e_g) = \pi(g')(x)\otimes e_{g'g}$. Then $(L^1(G,E),\hat\pi, T\otimes 1) \in E^{\mathrm{ban}}(L^1(G), L^1(G))$ gives the desired class in $KK^{\mathrm{ban}}(L^1(G), L^1(G))$.

This and section 1.3 imply the Bost conjecture in many cases.

Theorem 1.5.4 *For any group G in the classes a'), b) or c'), $\mu_{L^1} : K_*^G(\underline{E}G) \to K_*(L^1(G))$ is an isomorphism.*

1.6. Unconditional completions

Let G be a second countable, locally compact group. Let $\mathcal{A}(G)$ be a Banach algebra containing $C_c(G)$ as a dense subalgebra. We write $\mathcal{A}(G)$ instead of $\mathcal{A}$ for notational convenience. We ask for a necessary and sufficient condition such that there is a "natural" descent map $j_{\mathcal{A}} : KK_G^{\mathrm{ban}}(\mathbb{C},\mathbb{C}) \to KK^{\mathrm{ban}}(\mathcal{A}(G),\mathcal{A}(G))$.

In order to simplify the argument below, we will assume G to be discrete.

Let E be a Banach space with an isometric representation of G. Then $E\otimes\mathbb{C}G$ has a right $\mathbb{C}G$-module structure given by $(x\otimes e_g)e_{g'} = x\otimes e_{gg'}$ and there is a morphism $\hat\pi : \mathbb{C}G \to \mathrm{End}_{\mathbb{C}G}(E\otimes\mathbb{C}G)$ given by the formula $\hat\pi(e_{g'})(x\otimes e_g) = \pi(g')(x)\otimes e_{g'g}$. We look for a completion $\mathcal{A}(G,E)$ of $E\otimes\mathbb{C}G$ by a Banach norm such that $\mathcal{A}(G,E)$ is a right Banach $\mathcal{A}(G)$-module and $\hat\pi$ extends to a morphism of Banach algebras $\hat\pi : \mathcal{A}(G) \to \mathcal{L}_{\mathcal{A}(G)}(\mathcal{A}(G,E))$.

In order to have enough $\mathcal{A}(G)$-rank one operators, it is quite natural to assume that the norm on $\mathcal{A}(G,E)$ satisfies : for any $x\in E$ and $\xi\in\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})$, if we denote by $R_x : \mathbb{C}G \to E\otimes\mathbb{C}G$ the map $e_g \mapsto x\otimes e_g$ and by $S_\xi : E\otimes\mathbb{C}G \to \mathbb{C}G$ the map $y\otimes e_g \mapsto \xi(y)e_g$, we have $\|R_x(f)\|_{\mathcal{A}(G,E)} \le \|x\|_E\|f\|_{\mathcal{A}(G)}$ for any $f\in\mathbb{C}G$ and $\|S_\xi(\omega)\|_{\mathcal{A}(G)} \le \|\xi\|_{\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})}\|\omega\|_{\mathcal{A}(G,E)}$ for any $\omega\in E\otimes\mathbb{C}G$. Now fix $x\in E$ and $\xi\in\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})$ and denote by 1 the unit in G. For any $f=\sum_{g\in G} f(g)e_g \in \mathbb{C}G$, $S_\xi(\hat\pi(f)(R_x(e_1)))$ is $\sum_{g\in G}\xi(\pi(g)(x))f(g)e_g$ in $\mathbb{C}G$. For any function c on G, we define the Schur multiplication by c to be the pointwise product $\mathbb{C}G\to\mathbb{C}G$, $\sum_{g\in G} f(g)e_g \mapsto \sum_{g\in G} c(g)f(g)e_g$.

In this way we obtain the following necessary condition : for any $x\in E$ and $\xi\in\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})$ the Schur multiplication by the matrix coefficient $g\mapsto\xi(\pi(g)(x))$ is bounded from $\mathcal{A}(G)$ to itself and its norm (in $\mathcal{L}_{\mathbb{C}}(\mathcal{A}(G))$) is less than $\|x\|_E\|\xi\|_{\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})}$. But for any L^∞-function c on G we can find an isometric representation π of G on a Banach space E and $x \in E$ and $\xi \in \mathcal{L}_{\mathbb{C}}(E,\mathbb{C})$ such that $\|x\|_E\|\xi\|_{\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})} = \|c\|_{L^\infty}$ and $c(g) = \xi(\pi(g)x)$ for any $g \in G$ (take $E = L^1(G)$, $x = \delta_1$, $\xi = c$). Therefore a necessary condition is that $\mathcal{A}(G)$ is an unconditional completion in the following sense.

Definition 1.6.1 *A Banach algebra $\mathcal{A}(G)$ (with a given norm $\|.\|_{\mathcal{A}(G)}$) containing $C_c(G)$ as a dense subalgebra is called an unconditional completion if the norm $\|f\|_{\mathcal{A}(G)}$ of $f \in C_c(G)$ only depends on $g \mapsto |f(g)|$, $G \to \mathbb{R}_+$.*

Remark that $L^1(G)$ is an unconditional completion of $C_c(G)$ but $C^*_{\text{red}}(G)$ is not.

In fact this condition is also sufficient to construct the descent map. For the sake of simplicity, we still assume that G is discrete. If $\mathcal{A}(G)$ is an unconditional completion of $\mathbb{C}G$, and (E,π,T) is in $E^{\text{ban}}_G(\mathbb{C},\mathbb{C})$, we define $\mathcal{A}(G,E)$ as the completion of $E \otimes \mathbb{C}G$ for the norm $\|\sum_{g\in G} x(g) \otimes e_g\| = \|\sum_{g\in G} \|x(g)\|_E \; e_g\|_{\mathcal{A}(G)}$ and $\mathcal{A}(G,E)$ is a right Banach module over $\mathcal{A}(G)$ and there is a morphism $\tilde{\pi} : \mathcal{A}(G) \to \mathcal{L}_{\mathcal{A}(G)}(\mathcal{A}(G,E))$, and $(\mathcal{A}(G,E),\tilde{\pi},T\otimes 1) \in E^{\text{ban}}(\mathcal{A}(G),\mathcal{A}(G))$.

In this way, for any unconditional completion $\mathcal{A}(G)$ of $\mathbb{C}G$, we have a descent map $j^{\text{ban}}_{\mathcal{A}} : KK^{\text{ban}}_G(\mathbb{C},\mathbb{C}) \to KK^{\text{ban}}(\mathcal{A}(G),\mathcal{A}(G)) \to \text{End}(K_*(\mathcal{A}(G)))$. We can also define an assembly map $\mu_{\mathcal{A}} : K^G_*(\underline{E}G) \to K_*(\mathcal{A}(G))$. If $\mathcal{A}(G)$ is an involutive subalgebra of $C^*_{\text{red}}(G)$, and $i : \mathcal{A}(G) \to C^*_{\text{red}}(G)$ denotes the inclusion, $\mu_{\text{red}} = i_* \circ \mu_{\mathcal{A}}$.

Theorem 1.6.2 *([37]) For any group G in the classes a'), b) or c'), and for any unconditional completion $\mathcal{A}(G)$ of $C_c(G)$, $\mu_{\mathcal{A}} : K^G_*(\underline{E}G) \to K_*(\mathcal{A}(G))$ is an isomorphism.*

Let A, B be Banach algebras and $i : A \to B$ an injective morphism of Banach algebras. We say that A is stable under holomorphic functional calculus in B if any element of A has the same spectrum in A and in B. If A is dense and stable under holomorphic functional calculus in B then $i_* : K_*(A) \to K_*(B)$ is an isomorphism (see the appendix of [6]).

Corollary 1.6.3 *For any group G in the classes a'), b) or c'), if $C_c(G)$ admits an unconditional completion $\mathcal{A}(G)$ which is an involutive subalgebra of $C^*_{\text{red}}(G)$ and is stable under holomorphic functional calculus in $C^*_{\text{red}}(G)$, then $\mu_{\text{red}} : K^G_*(\underline{E}G) \to K_*(C^*_{\text{red}}(G))$ is an isomorphism.*

This condition is fulfilled for

a) hyperbolic groups,

b) cocompact lattices in a product of a finite number of groups among Lie or p-adic groups of rank one, $SL_3(\mathbb{F})$ with $\mathbb{F}$ a local field (even $\mathbb{H}$) and $E_{6(-26)}$,

c) reductive Lie groups and reductive groups over non-archimedean local fields.

In case c), $\mathcal{A}(G)$ is a variant of the Schwartz algebra of the group ([37]). In this case the Baum-Connes conjecture was already known for linear connected reductive groups (Wassermann [55]) and for the p-adic GL_n (Baum, Higson, Plymen [5]). In case a),b) this result is based on a property first introduced by Haagerup for the free group and called (RD) (for rapid decay) by Jolissaint ([23]). In case a),b) G has property (RD) : this is due to Haagerup for free groups ([16]), Jolissant for "geometric hyperbolic groups", de la Harpe for general hyperbolic groups ([18]), Ramagge, Robertson and Steger for SL_3 of a non-archimedian local field ([47]), the author for $SL_3(\mathbb{R})$ and $SL_3(\mathbb{C})$ ([39]), Chatterji for $SL_3(\mathbb{H})$ and $E_{6(-26)}$ ([10]), and the remark that it holds for products is due to Ramagge, Robertson and Steger ([47]) in a particular case, and independantly to Chatterji ([10]) and Talbi ([50]) in general. A discrete group G has property (RD) if there is a lenght function $\ell : G \to \mathbb{R}_+$ (*i.e.* a function satisfying $\ell(g^{-1}) = \ell(g)$ and $\ell(gh) \leq \ell(g) + \ell(h)$ for any $g, h \in G$) such that for $s \in \mathbb{R}_+$ big enough, the completion $H^s(G)$ of $\mathbb{C}G$ for the norm $\| \sum f(g)e_g\|_{H^s(G)} = \| \sum(1+\ell(g))^s f(g)e_g\|_{L^2(G)}$ is contained in $C^*_{\mathrm{red}}(G)$. Then, for s big enough, $H^s(G)$ is a Banach algebra and an involutive subalgebra of $C^*_{\mathrm{red}}(G)$ and is dense and stable under holomorphic functional calculus ([23, 39]); it is obvious that $H^s(G)$ is an unconditional completion of $\mathbb{C}G$.

As a consequence of this result the Baum-Connes conjecture has been proven for all almost connected groups by Chabert, Echterhoff and Nest ([9]).

1.7. Trying to push the method further

In order to prove new cases of the surjectivity of the Baum-Connes map (when the injectivity is proven and the γ element exists) we should look for a dense subalgebra $\mathcal{A}(G)$ of $C^*_{\mathrm{red}}(G)$ that is stable under holomorphic functional calculus and a homotopy between γ and 1 through (perhaps special kind of) elements of $E^{\mathrm{ban}}_G(\mathbb{C}, \mathbb{C})$ which all give a map $K_*(\mathcal{A}(G)) \to K_*(C^*_{\mathrm{red}}(G))$ by the descent construction. Thanks to the discussion in subsection 1. a necessary condition for this is that for any (E, π, T) in the homotopy between γ and 1, for any $x \in E$ and $\xi \in \mathcal{L}_{\mathbb{C}}(E, \mathbb{C})$, the Schur multiplication by the matrix coefficient $g \mapsto \xi(\pi(g)(x))$ is bounded from $\mathcal{A}(G)$ to $C^*_{\mathrm{red}}(G)$ and has norm $\leq \|x\|_E \|\xi\|_{\mathcal{L}_{\mathbb{C}}(E,\mathbb{C})}$. So we should first look for a homotopy between γ and 1 such that the fewest possible matrix coefficients appear. For groups acting properly on buildings, this homotopy can be shown to exist. The problem for general discrete groups properly acting on buildings is to find a subalgebra $\mathcal{A}(G)$ of $C^*_{\mathrm{red}}(G)$ that is stable under holomorphic functional calculus and satisfies the condition with respect to these matrix coefficients. The first step (the crucial one I think) should be to find a subalgebra $\mathcal{A}(G)$ of $C^*_{\mathrm{red}}(G)$ that is stable under holomorphic functional calculus and satisfies the following condition : there is a integer n, a distance d on the building and a point x_0 on the building such that the Schur product by the characteristic function of $\{g \in G, d(x_0, gx_0) \leq r\}$ from $\mathcal{A}(G)$ to $C^*_{\mathrm{red}}(G)$ has norm less than $(1+r)^n$, for any $r \in \mathbb{R}_+$.

1.8. The Baum-Connes conjecture with coefficients

Let G be a second countable, locally compact group and A a G-Banach algebra (*i.e.* a Banach algebra on which G acts continuously by isometric automorphisms $g : a \mapsto g(a)$). The space $C_c(G, A)$ of A-valued continuous compactly supported functions on G is endowed with the following convolution product : $f * f'(g) = \int_G f(h)h(f'(h^{-1}g))dh$ and the completion $L^1(G, A)$ of $C_c(G, A)$ for the norm $\|f\| =$

$\int_G \|f(g)\|_A dg$ is a Banach algebra. More generally for any unconditional completion $\mathcal{A}(G)$, we define $\mathcal{A}(G,A)$ to be the completion of $C_c(G,A)$ for the norm $\|f\|_{\mathcal{A}(G,A)} = \big\|g \mapsto \|f(g)\|_A\big\|_{\mathcal{A}(G)}$.

For any G-Banach algebras A and B, we define in [37] an abelian group $KK_G^{\mathrm{ban}}(A,B)$. This is a contravariant functor in A and a covariant functor in B. When $G=1$ this is equal to $KK^{\mathrm{ban}}(A,B)$. For any unconditional completion $\mathcal{A}(G)$ of $C_c(G)$, there is descent morphism $KK_G^{\mathrm{ban}}(A,B) \to KK^{\mathrm{ban}}(\mathcal{A}(G,A),\mathcal{A}(G,B))$.

These constructions are adaptations of the classical constructions for C^*-algebras : for any G-C^*-algebra A (*i.e.* G acts continuously by C^*-algebras automorphisms on A) we have a natural C^*-algebra $C^*_{\mathrm{red}}(G,A)$ containing $L^1(G,A)$ as a dense subalgebra. If B is another G-C^*-algebra, Kasparov defined an abelian group $KK_G(A,B)$. This is a contravariant functor in A and a covariant functor in B. When $G=1$ this is equal to $KK(A,B)$. There is an associative and distributive product $KK_G(A,B)\otimes KK_G(B,C) \to KK_G(A,C)$ and a descent morphism $KK_G(A,B) \to KK(C^*_{\mathrm{red}}(G,A), C^*_{\mathrm{red}}(G,B))$.

Let $K_*^G(\underline{E}G,A)$, $*=0,1$, be the inductive limit over G-invariant G-compact subsets Z of $\underline{E}G$ of $KK_{G,*}(C_0(Z),A)$. Then the assembly map

$$\mu_{\mathrm{red},A} : K_*^G(\underline{E}G,A) \to K_*(C^*_{\mathrm{red}}(G,A))$$

is defined in [4] and similar maps $\mu_{L^1,A}$, and more generally $\mu_{\mathcal{A},A}$ for any unconditional completion $\mathcal{A}(G)$, can be defined.

The Baum-Connes conjecture "with coefficients" claims that $\mu_{\mathrm{red},A}$ is an isomorphism and the Bost conjecture "with coefficients" claims that $\mu_{L^1,A}$ is an isomorphism. Theorems 1.4.1, 1.5.4, 1.6.2 are still true with arbitrary coefficients.

The surjectivity of the Baum-Connes conjecture with coefficients has been counter-exampled recently (Higson, Lafforgue, Ozawa, Skandalis, Yu) using a random group constructed by Gromov ([15]) but Bost conjecture with coefficients still stands. If the Baum-Connes conjecture with coefficients is true for a group, it is true also for all its closed subgroups; the Baum-Connes conjecture with coefficients is also stable under various kinds of extensions (Chabert [7], Chabert-Echterhoff [8], Oyono [44], and Tu [51]).

Kasparov's equivariant KK-theory was generalized to groupoids by Le Gall [31, 40, 41] and this generalized KK-theory was applied by Tu in [52, 53] to the bijectivity of the Baum-Connes map for amenable groupoids and the injectivity for (the holonomy groupoids of) hyperbolic foliations. It is possible to generalize also Banach KK-theory and unconditional completions. In this way we obtain the Baum-Connes conjecture for any hyperbolic group, with coefficients in any commutative C^*-algebra, and also for foliations with compact basis, admitting a (strictly) negatively curved longitudinal riemannian metric, and such that the holonomy groupoid is Hausdorff and has simply connected fibers (not yet published).

2. Discrete series representations of connected semi-simple Lie groups

In this part we examine how the Baum-Connes conjecture for a connected semi-simple Lie group with finite center can be used to establish the construction of the discrete series by Dirac induction ([17, 45, 1]). That this is morally true is known from the beginning of the conjecture (see for instance [12]). In the proof we shall introduce 3 ingredients : these are classical facts stated here without proof. Parts of the argument apply to more general groups (not connected, not semi-simple).

This work owes its existence to Paul Baum. He asked me to study the problem and we discussed a lot.

2.1. Dirac operators

Let G be a Lie group, with a finite number of connected components, and K a maximal compact subgroup. We assume that there exists a G-invariant orientation on G/K. For the sake of simplicity, we assume that G/K admits a G-invariant spin structure (it is true anyway for a two fold covering of G). More precisely let $\mathfrak{p}$ be a complementary subspace for the Lie algebra $\mathfrak{k}$ of K in the Lie algebra $\mathfrak{g}$ of G. We choose $\mathfrak{p}$ such that it is invariant for the adjoint action of K and we endow it with a K-invariant euclidian metric. The above assumption means that the homomorphism $K \to \mathrm{SO}(\mathfrak{p})$ lifts to $\mathrm{Spin}(\mathfrak{p})$. We denote by S the associated spin representation of K. If $\dim(G/K)$ is even, S is $\mathbb{Z}/2\mathbb{Z}$-graded. We write $i = \dim(G/K)$ [2].

We denote by $R(K)$ the (complex) representation ring of K and for any finite dimensional representation V of K we denote by $[V]$ its class in $R(K)$.

Let V be a finite dimensional representation of K. Let E_V be the right Banach (in fact Hilbert) module over $C^*_{\mathrm{red}}(G)$ ($\mathbb{Z}/2\mathbb{Z}$-graded if $i = 0$ [2]) whose elements are the K-invariant elements in $V^* \otimes S^* \otimes C^*_{\mathrm{red}}(G)$, where K acts by left translations on $C^*_{\mathrm{red}}(G)$. Let D_V be the unbounded $C^*_{\mathrm{red}}(G)$-linear operator on E_V equal to $\sum 1 \otimes c(p_i) \otimes p_i$, where the sum is over i, (p_i) is an orthonormal basis of $\mathfrak{p}$, p_i denotes also the associated right invariant vector field on G, and $c(p_i)$ is the Clifford multiplication by p_i. Let $T_V = \frac{D_V}{\sqrt{1+D_V^2}}$. Then we define $[d_V] \in K_i(C^*_{\mathrm{red}}(G))$ to be the class of the Fredholm module (E_V, T_V) over $C^*_{\mathrm{red}}(G)$.

In other words, E_V is the completion of the space of smooth compactly supported sections of the bundle on $K\backslash G$ associated to the representation $V^* \otimes S^*$ of K, for the norm $\|w\| = \sup_{f \in L^2(G), \|f\|_{L^2(G)}=1} \|w * f\|_{L^2((V^* \otimes S^*)\times_K G)}$, and D_V is the Dirac operator, twisted by V^*.

Connes-Kasparov conjecture. *The group morphism $\mu_{\mathrm{red}} : R(K) \to K_i(C^*_{\mathrm{red}}(G))$ defined by $[V] \mapsto [d_V]$ is an isomomorphism, and $K_{i+1}(C^*_{\mathrm{red}}(G)) = 0$.*

This is a special case of the Baum-Connes conjecture because we may take $\underline{E}G = G/K$ and thus $K_i^G(\underline{E}G) = R(K)$ and $K_{i+1}^G(\underline{E}G) = 0$. It was checked for G connected reductive linear in [55] and the Baum-Connes conjecture was proved for any reductive group in [37] (see c) of the corollary 1.6.3 above).

The following lemma has been suggested to me by Francois Pierrot. Assume that i is even. Let moreover H be a unitary tempered admissible representation of G. This implies that we have a C^*-homomorphism $C^*_{\mathrm{red}}(G) \to \mathcal{K}(H)$. For any element $x \in K_0(C^*_{\mathrm{red}}(G))$ we denote by $\langle H, x\rangle \in \mathbb{Z}$ the image of x by $K_0(C^*_{\mathrm{red}}(G)) \to K_0(\mathcal{K}(H)) = \mathbb{Z}$. If x is the class of an idempotent $p \in C^*_{\mathrm{red}}(G)$, the image of p in

$\mathcal{K}(H)$ is a finite rank projector, whose rank is $\langle H, x\rangle$.

Lemma 2.1.1 *We have* $\langle H, [d_V]\rangle = \dim(V^* \otimes S^* \otimes H)^K$.

2.2. Dual-Dirac operators

From now on we assume that G is a connected semi-simple Lie group with finite center and we still assume that G/K has a G-invariant spin structure. Kasparov has constructed an element $\eta \in \mathrm{Hom}(K_i(C^*_{\mathrm{red}}(G)), R(K))$ (coming from an element of $KK_i(C^*_{\mathrm{red}}(G), C^*_{\mathrm{red}}(K))$, itself coming from an element of $KK_{G,i}(\mathbb{C}, C_0(G/K)))$. Kasparov has shown that $\eta \circ \mu_{\mathrm{red}} = \mathrm{Id}_{R(K)}$ [29, 31].

Here is the detail of the construction. The G-invariant riemannian structure on G/K given by the chosen K-invariant euclidian metric on $\mathfrak{p}$ has non-positive curvature. Let ρ be the distance to the origin and $\xi = d(\sqrt{1+\rho^2})$. Let V be a finite dimensional complex representation of K, endowed with an invariant hermitian metric. Let H_V be the space of L^2 sections of the hermitian G-equivariant fibre bundle on G/K associated to the representation of K on $S \otimes V$ and let $c_{\xi,V}$ be the Clifford multiplication by ξ. In other words H_V is the subspace of K-invariant vectors in $L^2(G) \otimes S \otimes V$, where K acts by right translations $L^2(G)$, and $c_{\xi,V}$ is the restriction to this subspace of the tensor product of the Clifford multiplication by ξ on $L^2(G) \otimes S$ with Id_V. Left translation by G on G/K or on $L^2(G)$ gives rise to a (C^*-)morphism $\pi_V : C^*_{\mathrm{red}}(G) \to \mathcal{L}_{\mathbb{C}}(H_V)$ and $(H_V, \pi_V, c_{\xi,V})$ defines $\eta_V \in KK_i^{\mathrm{ban}}(C^*_{\mathrm{red}}(G), \mathbb{C})$ (in fact in $KK_i(C^*_{\mathrm{red}}(G), \mathbb{C})$). We denote by $[\eta_V] \in \mathrm{Hom}(K_i(C^*_{\mathrm{red}}(G)), \mathbb{Z})$ the associated map, and $\eta = \sum_V [\eta_V][V] \in \mathrm{Hom}(K_i(C^*_{\mathrm{red}}(G)), R(K))$, where the sum is over the irreducible representations of K.

Since the Connes-Kasparov conjecture is true, $\mu_{\mathrm{red}} : R(K) \to K_i(C^*_{\mathrm{red}}(G))$ and $\eta : K_i(C^*_{\mathrm{red}}(G)) \to R(K)$ are inverse of each other and $K_{i+1}(C^*_{\mathrm{red}}(G)) = 0$.

Let H be a discrete series representation of G, *i.e.* an irreducible unitary representation with a positive mass in the Plancherel measure. We recall that this is equivalent to the fact that some (whence all) matrix coefficient $c_x(g) = \langle x, \pi(g)x\rangle$, $x \in H$, $\|x\| = 1$, is square-integrable. Then $\|c_x\|^2_{L^2(G)}$ is independant of x, and its inverse is the formal degree d_H of H, which is also the mass of H in the Plancherel measure. We introduce a first ingredient.

Ingredient 1. All discrete series representations of G are isolated in the tempered dual.

In other words, all matrix coefficients belong to $C^*_{\mathrm{red}}(G)$. In fact a standard asymptotic expansion argument shows that for any K-finite vector $x \in H$, c_x belongs to the Schwartz algebra ([17], II, corollary 1 page 77).

Therefore there exists an idempotent $p \in C^*_{\mathrm{red}}(G)$ such that the image in $L^2(G)$ of the image of p by the left regular representation is H^* as a representation of G on the right. In fact we can take $p = d_H \overline{c_x}$ for any $x \in H$, $\|x\| = 1$, where $\overline{c_x}(g) = \overline{c_x(g)}$. The class of p in $K_0(C^*_{\mathrm{red}}(G))$ only depends on H and we denote it by $[H]$. It is easy to see that $i : \oplus_H \mathbb{Z} \to K_0(C^*_{\mathrm{red}}(G))$, $(n_H)_H \mapsto \sum_H n_H[H]$, where the sums are over the discrete series representations of G, is an injection. Indeed, if H and H' are discrete series representations of G, $\langle H', [H]\rangle = 1$ if $H = H'$ and 0 otherwise.

As a corollary we see that if $i = 1$ [2], G has no discrete series representations. From now on we assume $i = 0$ [2].

The first part of the following lemma was suggested to me by Georges Skandalis. Let H be a discrete series representation of G. We write $\eta([H]) = \sum_V n_V[V]$ in $R(K)$ where the sum is finite and over the irreducible representations of K (in the notation above, $n_V = [\eta_V]([H])$).

Lemma 2.2.1 *If V is an irreducible representation of K, $n_V = \dim(H^* \otimes S \otimes V)^K$ and therefore $n_V = \langle H, [d_V]\rangle$.*

We have $1 = \langle H, [H]\rangle = \langle H, \mu_{\mathrm{red}} \circ \eta([H])\rangle = \sum_V n_V \langle H, [d_V]\rangle = \sum_V n_V^2$. Therefore one of the n_V is ± 1 and the others are 0.

Alternatively we can consider the morphisms

$$\oplus_V \mathbb{Z}[V] = R(K) \stackrel{\mu_{\mathrm{red}}}{\to} K_0(C^*_{\mathrm{red}}(G)) \stackrel{\pi}{\to} \prod_H \mathbb{Z} \text{ where } \pi(x) = (\langle H, x\rangle)_H$$

$$\text{and} \qquad \oplus_H \mathbb{Z} \stackrel{i}{\to} K_0(C^*_{\mathrm{red}}(G)) \stackrel{\eta}{\to} R(K) = \oplus_V \mathbb{Z}[V]$$

where the sums are over the irreducible representations V of K and the discrete series representations H of G. Their product $\pi \circ \mu_{\mathrm{red}} \circ \eta \circ i = \pi \circ i$ is equal to the inclusion of $\oplus_H \mathbb{Z}$ in $\prod_H \mathbb{Z}$ and their matrices in the base $([V])_V$ and the canonical base of $\oplus_H \mathbb{Z}$ are transpose of each other. Therefore each column of the matrix of $\eta \circ i$ contains exactly one non-zero coefficient, which is equal to ± 1. A posteriori, π takes its values in $\oplus_H \mathbb{Z}$.

Corollary 2.2.2 *The discrete series representations of G are in bijection with a subset of the set of isomorphism classes of irreducible representations of K. The irreducible representation V of K associated to a discrete series representation H is such that $V = \pm(H \otimes S^*)$ as a formal combination of irreducible representations of K, and H occurs in the kernel of the twisted Dirac operator D_V.*

Corollary 2.2.3 *If $\operatorname{rank} G \neq \operatorname{rank} K$, G has no discrete series.*

In this case S^* is 0 in $R(K)$ (Barbasch and Moscovici [2] (1.2.5) page 156) : this was indicated to me by Henri Moscovici.

2.3. A trace formula

From now on we assume that $\operatorname{rank} G = \operatorname{rank} K$. Let T a maximal torus in K (therefore also in G). Choose a Weyl chamber for the root system of $\mathfrak{g}$ and choose the Weyl chamber of the root system of $\mathfrak{k}$ containing it. Let V be an irreducible representation of K, μ its highest wheight, and $\lambda = \mu + \rho_K$ where ρ_K is the half sum of the positive roots of $\mathfrak{k}$.

We recall that the unbounded trace $\mathrm{Tr} : C^*_{\mathrm{red}}(G) \to \mathbb{R}, f \mapsto f(1)$ gives rise to a group morphism $K_0(C^*_{\mathrm{red}}(G)) \to \mathbb{R}$. When H is a discrete series representation of G, $\mathrm{Tr}([H])$ is the value at 1 of $p = d_H \overline{c_x}$ for some $x \in H$, $\|x\| = 1$, and therefore it is the formal degree d_H of H and is > 0.

Ingredient 2. $\mathrm{Tr}([d_V]) = \prod_{\alpha\in\Psi} \frac{(\lambda,\alpha)}{(\rho,\alpha)}$, where Ψ is the set of simple roots of the chosen positive root system in $\mathfrak{g}$, and ρ is the half sum of the positive roots of this system.

In this formula is used a right normalization of the Haar measure (if G is linear it is the one for which the maximal compact subgroup of the complexification of G has measure 1). This formula is proven in [11] by a heat equation method, and in [1] by Atiyah's L^2-index theorem.

Corollary 2.3.1 *If λ is singular for $\mathfrak{g}$, $[V]$ does not correspond to a discrete series representation of G.*

Ingredient 3. For any $x \in K_0(C^*_{\mathrm{red}}(G))$ such that $\mathrm{Tr}(x) \neq 0$, there is a discrete series representation H such that $\langle H, x\rangle \neq 0$.

By the Plancherel formula, if $\hat{G}$ is the tempered spectrum of G, $\mathrm{Tr}(x) = \int_{\hat{G}} \langle H, x\rangle \, dH$. We have to prove that, for almost all H outside the discrete series, $\langle H, x\rangle = 0$. There are several possible arguments :

- almost all H outside the discrete series are induced from a parabolic subgroup and belong to a family of representations indexed by some $\mathbb{R}^p$, but $\langle H', x\rangle$ is constant when H' varies in this family and goes to 0 when H' goes to infinity,
- write $x = [d_V]$ for some V, then the H outside the discrete series with $\langle H, x\rangle \neq 0$ have measure 0 by [1] p15 (3.19), p50 (9.8) and p51 (9.12) or by [11] p318-320.

Corollary 2.3.2 *If λ is not singular for $\mathfrak{g}$, $[V]$ does correspond to a discrete series representation, whose formal degree is* $\left| \prod_{\alpha\in\Psi} \frac{(\lambda,\alpha)}{(\rho,\alpha)} \right|$.

We have recovered some results proved in [17], [45] and [1].

References

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