Solving Pseudo-Differential Equations

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Abstract

In 1957, Hans Lewy constructed a counterexample showing that very simple and natural differential equations can fail to have local solutions. A geometric interpretation and a generalization of this counterexample were given in 1960 by L.Hörmander. In the early seventies, L.Nirenberg and F.Treves proposed a geometric condition on the principal symbol, the so-called condition (ψ), and provided strong arguments suggesting that it should be equivalent to local solvability. The necessity of condition (ψ) for solvability of pseudo-differential equations was proved by L.Hörmander in 1981. The sufficiency of condition (ψ) for solvability of differential equations was proved by R.Beals and C.Fefferman in 1973. For differential equations in any dimension and for pseudo-differential equations in two dimensions, it was shown more precisely that (ψ) implies solvability with a loss of one derivative with respect to the elliptic case: for instance, for a complex vector field X satisfying (ψ), $f \in L^2_{\text{loc}}$, the equation $Xu = f$ has a solution $u \in L^2_{\text{loc}}$.

In 1994, it was proved by N.L. that condition (ψ) does not imply solvability with loss of one derivative for pseudo-differential equations, contradicting repeated claims by several authors. However in 1996, N.Dencker proved that these counterexamples were indeed solvable, but with a loss of two derivatives. We shall explore the structure of this phenomenon from both sides: on the one hand, there are first-order pseudo-differential equations satisfying condition (ψ) such that no L^2_{loc} solution can be found with some source in L^2_{loc}. On the other hand, we shall see that, for these examples, there exists a solution in the Sobolev space H^{-1}_{loc}.

The sufficiency of condition (ψ) for solvability of pseudo-differential equations in three or more dimensions is still an open problem. In 2001, N.Dencker announced that he has proved that condition (ψ) implies solvability (with a loss of two derivatives), settling the Nirenberg-Treves conjecture. Although his paper contains several bright and new ideas, it is the opinion of the author of these lines that a number of points in his article need clarification.

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1. From Hans Lewy to Nirenberg-Treves' condition (ψ)

Year 1957.

The Hans Lewy operator L_0, introduced in [20], is the following complex vector field in $\mathbb{R}^3$

$$L_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3}. \tag{1.1}$$

There exists $f \in C^\infty$ such that the equation

$$L_0 u = f \tag{1.2}$$

has no distribution solution, even locally. This discovery came as a great shock for several reasons. First of all, L_0 has a very simple expression and is natural as the Cauchy-Riemann operator on the boundary of the pseudo-convex domain

$$\{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + 2\operatorname{Im} z_2 < 0\}.$$

Moreover L_0 is a non-vanishing vector field so that no pathological behaviour related to multiple characteristics is to be expected. In the fifties, it was certainly the conventional wisdom that any "reasonable" operator should be locally solvable, and obviously (1.1) was indeed very reasonable so the conclusion was that, once more, the CW should be revisited. One of the questions posed by such a counterexample was to find some geometric explanation for this phenomenon.

1960.

This was done in 1960 by L.Hörmander in [7] who proved that if p is the symbol of a differential operator such that, at some point (x, ξ) in the cotangent bundle,

$$p(x, \xi) = 0 \quad \text{and} \quad \{\operatorname{Re} p, \operatorname{Im} p\}(x, \xi) > 0, \tag{1.3}$$

then the operator P with principal symbol p is not locally solvable at x; in fact, there exists $f \in C^\infty$ such that, for any neighborhood V of x the equation $Pu = f$ has no solution $u \in \mathcal{D}'(V)$. Of course, in the case of differential operators, the sign > 0 in (1.3) can be replaced by $\neq 0$ since the Poisson bracket $\{\operatorname{Re} p, \operatorname{Im} p\}$ is then an homogeneous polynomial with odd degree in the variable ξ. Nevertheless, it appeared later (in [8]) that the same statement is true for pseudo-differential operators, so we keep it that way. Since the symbol of $-iL_0$ is $\xi_1 - x_2\xi_3 + i(\xi_2 + x_1\xi_3)$, and the Poisson bracket $\{\xi_1 - x_2\xi_3, \xi_2 + x_1\xi_3\} = 2\xi_3$, the assumption (1.3) is fulfilled for L_0 at any point x in the base and the nonsolvability property follows. This gives a necessary condition for local solvability of pseudo-differential equations: a locally solvable operator P with principal symbol p should satisfy

$$\{\operatorname{Re} p, \operatorname{Im} p\}(x, \xi) \leq 0 \quad \text{at} \quad p(x, \xi) = 0. \tag{1.4}$$

Naturally, condition (1.4) is far from being sufficient for solvability (see e.g. the nonsolvable M_3 below in (1.5)). After the papers [20], [7], the curiosity of the

mathematical community was aroused in search of a geometric condition on the principal symbol, characterizing local solvability of principal type operators. It is important to note that for principal type operators with a real principal symbol, such as a non-vanishing real vector field, or the wave equation, local solvability was known after the 1955 paper of L.Hörmander in [6]. In fact these results extend quite easily to the pseudo-differential real principal type case. As shown by the Hans Lewy counterexample and the necessary condition (1.4), the matters are quite different for complex-valued symbols.

1963.

It is certainly helpful to look now at some simple models. For $t, x \in \mathbb{R}$, with the usual notations

$$D_t = -i\partial_t, \quad (\widehat{|D_x|u})(\xi) = |\xi|\hat{u}(\xi),$$

where $\hat{u}$ is the x-Fourier transform of u, $l \in \mathbb{N}$, let us consider the operators defined by

$$M_l = D_t + it^l D_x, \quad N_l = D_t + it^l|D_x|. \tag{1.5}$$

It is indeed rather easy to prove that, for $k \in \mathbb{N}$, $M_{2k}, N_{2k}, N^*_{2k+1}$ are solvable whereas M_{2k+1}, N_{2k+1} are nonsolvable. In particular, the operators M_1, N_1 satisfy (1.3). On the other hand, the operator $N^*_1 = D_t - it|D_x|$ is indeed solvable since its adjoint operator N_1 verifies the a priori estimate

$$T\|N_1u\|_{L^2(\mathbb{R}^2)} \geq \|u\|_{L^2(\mathbb{R}^2)},$$

for a smooth compactly supported u vanishing for $|t| \geq T/2$. No such estimate is satisfied by $N^*_1 u$ since its x-Fourier transform is

$$-i\partial_t v - it|\xi|v = (-i)(\partial_t v + t|\xi|v),$$

where v is the x-Fourier transform of u. A solution of $N^*_1 u = 0$ is thus given by the inverse Fourier transform of $e^{-t^2|\xi|/2}$, ruining solvability for the operator N_1. A complete study of solvability properties of the models M_l was done in [23] by L.Nirenberg and F.Treves, who also provided a sufficient condition of solvability for vector fields; the analytic-hypoellipticity properties of these operators were also studied in a paper by S.Mizohata [21].

1971.

The ODE-like examples (1.5) led L.Nirenberg and F.Treves in [24–25–26] to formulate a conjecture and to prove it in a number of cases, providing strong grounds in its favour. To explain this, let us look simply at the operator

$$L = D_t + iq(t, x, D_x), \tag{1.6}$$

where q is a real-valued first-order symbol. The symbol of L is thus $\tau + iq(t, x, \xi)$. The bicharacteristic curves of the real part are oriented straight lines with direction $\partial/\partial t$; now we examine the variations of the imaginary part $q(t, x, \xi)$ along these lines. It amounts only to check the functions $t \mapsto q(t, x, \xi)$ for fixed (x, ξ). The

good cases in (1.5) (when solvability holds) are $t^{2k}\xi, -t^{2k+1}|\xi|$: when t increases these functions do not change sign from $-$ to $+$. The bad cases are $t^{2k+1}|\xi|$: when t increases these functions do change sign from $-$ to $+$; in particular, the nonsolvable case (1.3), tackled in [8], corresponds to a change of sign of $\operatorname{Im} p$ from $-$ to $+$ at a simple zero. The general formulation of condition (ψ) for a principal type operator with principal symbol p is as follows: for all $z \in \mathbb{C}$, $\operatorname{Im}(zp)$ does not change sign from $-$ to $+$ along the oriented bicharacteristic curves of $\operatorname{Re}(zp)$. It is a remarkable and non-trivial fact that this condition is invariant by multiplication by an elliptic factor as well as by composition with an homogeneous canonical transformation. The *Nirenberg-Treves conjecture*, proved in several cases in [24–25–26], such as for differential operators with analytic coefficients, states that, *for a principal type pseudo-differential equation, condition (ψ) is equivalent to local solvability.*

The paper [25] introduced a radically new method of proof of energy estimates for the adjoint operator L^* based on a factorization of q in (1.6): whenever

$$q(t,x,\xi) = a(t,x,\xi)b(x,\xi) \tag{1.7}$$

with $a \leq 0$ of order 0 and b of order 1, then the operator L in (1.6) is locally solvable. Looking simply at the ODE

$$D_t + ia(t,x,\xi)b(x,\xi) = (-i)\big(\partial_t - a(t,x,\xi)b(x,\xi)\big), \tag{1.8}$$

it is clear that in the region $\{b(x,\xi) \geq 0\}$, the forward Cauchy problem for (1.8) is well posed, whereas in $\{b(x,\xi) \leq 0\}$, well-posedness holds for the backward Cauchy problem. This remark led L.Nirenberg and F.Treves to use as a multiplier in the energy method the sign of the operator with symbol b. They were also able to provide the proper commutator estimates to handle the remainder terms generated by this operator-theoretic method. Although a factorization (1.7) can be obtained for differential operators with analytic regularity satisfying condition (ψ), such a factorization is not true in the C^∞ case. Incidentally, one should note that for differential operators, condition (ψ) is equivalent to ruling out any change of sign of $\operatorname{Im} p$ along the bicharacteristics of $\operatorname{Re} p$ (the latter condition is called condition (P)); this fact is due to the identity $p(x,-\xi) = (-1)^m p(x,\xi)$, valid for an homogeneous polynomial of degree m in the variable ξ.

Using the Malgrange-Weierstrass theorem on normal forms of complex-valued non-degenerate C^∞ functions and the Egorov theorem on quantization of homogeneous canonical transformations, there is no loss of generality considering only first order operators of type (1.6). The expression of condition (ψ) for L is then very simple since it reads

$$q(t,x,\xi) < 0 \quad \text{and} \quad s > t \Longrightarrow q(s,x,\xi) \leq 0. \tag{1.9}$$

Note that the expression of condition (P) for L is simply $q(t,x,\xi)q(s,x,\xi) \geq 0$. Much later in 1988, N.Lerner [14] proved the sufficiency of condition (ψ) for local solvability of pseudo-differential equations in two dimensions and as well for the classical oblique-derivative problem [15]. The method of proof of these results

is based upon a factorization analogous to (1.7) but where $b(x,\xi)$ is replaced by $\beta(t,x)|\xi|$ and β is a smooth function such that $t \mapsto \beta(t,x)$ does not change sign from $+$ to $-$ when t increases. Then a properly defined sign of $\beta(t,x)$ appears as a non-decreasing operator and the Nirenberg-Treves energy method can be adapted to this situation.

1973.

At this date, R.Beals and C.Fefferman [1] took as a starting point the previous results of L.Nirenberg and F.Treves and, removing the analyticity assumption, they were able to prove the sufficiency of condition (P) for local solvability, obtaining thus the sufficiency of condition (ψ) for local solvability of differential equations. The key ingredient was a drastically new vision of the pseudo-differential calculus, defined to obtain the factorization (1.7) in regions of the phase space much smaller than cones or semi-classical "boxes" $\{(x,\xi),|x| \le 1,|\xi| \le h^{-1}\}$. Considering the family $\{q(t,x,\xi)\}_{t\in[-1,1]}$ of classical homogeneous symbols of order 1, they define, via a Calderón-Zygmund decomposition, a pseudo-differential calculus depending on the family $\{q(t,\cdot)\}$, in which all these symbols are first order but also such that, at some level t_0, some ellipticity property of $q(t_0,\cdot)$ or $\nabla_{x,\xi}q(t_0,\cdot)$ is satisfied. Condition (P) then implies easily a factorization of type (1.7) and the Nirenberg-Treves energy method can be used. It is interesting to notice that some versions of these new pseudo-differential calculi were used later on for the proof of the Fefferman-Phong inequality [5]. In fact, the proof of R.Beals and C.Fefferman marked the day when microlocal analysis stopped being only homogeneous or semi-classical, thanks to methods of harmonic analysis such as Calderón-Zygmund decomposition made compatible with the Heisenberg uncertainty principle.

1978.

Going back to solvability problems, the existence of C^∞ solutions for C^∞ sources was proved by L.Hörmander in [9] for pseudo-differential equations satisfying condition (P). For such an operator P of order m, satisfying also a non-trapping condition, a semi-global existence theorem was proved, with a loss of $1+\epsilon$ derivatives, with $\epsilon > 0$. Following an idea given by R.D.Moyer [22] for a result in two dimensions, L.Hörmander proved in [10] that condition (ψ) is necessary for local solvability: assuming that condition (ψ) is not satisfied for a principal type operator P, he was able to construct approximate non-trivial solutions u for the adjoint equation $P^*u = 0$, which implies that P is not solvable. Although the construction is elementary for the model operators N_{2k+1} in (1.5) (as sketched above for N_1 in our 1963 section), the multidimensional proof is rather involved and based upon a geometrical optics method adapted to the complex case. The details can be found in the proof of theorem 26.4.7′ of [11].

We refer the reader to the paper [13] for a more detailed historical overview of this problem. On the other hand, it is clear that our interest is focused on solvability in the C^∞ category. Let us nevertheless recall that the sufficiency of condition (ψ) in the analytic category (for microdifferential operators acting on microfunctions) was proved by J.-M.Trépreau [27] (see also [12], chapter vii).

2. Counting the loss of derivatives

Condition (ψ) does not imply solvability with loss of one derivative.

Let us consider a principal-type pseudo-differential operator L of order m. We shall say that L is locally solvable with a loss of μ derivatives whenever the equation $Lu = f$ has a local solution u in the Sobolev space $H^{s+m-\mu}$ for a source f in H^s. Note that the loss is zero if and only if L is elliptic. Since for the simplest principal type equation $\partial/\partial x_1$, the loss of derivatives is 1, we shall consider that 1 is the "ordinary" loss of derivatives. When L satisfies condition (P) (e.g. if L is a differential operator satisfying condition (ψ)), or when L satisfies condition (ψ) in two dimensions, the estimates

$$C\|L^*u\|_{H^s} \geq \|u\|_{H^{s+m-1}}, \tag{2.1}$$

valid for smooth compactly supported u with a small enough support, imply local solvability with loss of 1 derivative, the ordinary loss referred to above. For many years, repeated claims were made that condition (ψ) for L implies (2.1), that is solvability with loss of 1 derivative. It turned out that these claims were wrong, as shown in [16] by the following result (see also section 6 in the survey [13]).

Theorem 2.1. *There exists a principal type first-order pseudo-differential operator L in three dimensions, satisfying condition (ψ), a sequence u_k of C_c^∞ functions with* $\operatorname{supp} u_k \subset \{x \in \mathbb{R}^3, |x| \leq 1/k\}$ *such that*

$$\|u_k\|_{L^2(\mathbb{R}^3)} = 1, \quad \lim_{k \to +\infty} \|L^*u_k\|_{L^2(\mathbb{R}^3)} = 0. \tag{2.2}$$

As a consequence, for this L, there exists $f \in L^2$ such that the equation $Lu = f$ has no local solution u in L^2. We shall now briefly examine some of the main features of this counterexample, leaving aside the technicalities which can be found in the papers quoted above. Let us try, with $(t, x, y) \in \mathbb{R}^3$,

$$L = D_t - ia(t)\big(D_x + H(t)V(x)|D_y|\big), \tag{2.3}$$

with $H = \mathbf{1}_{\mathbb{R}_+}$, $C^\infty(\mathbb{R}) \ni V \geq 0$, $C^\infty(\mathbb{R}) \ni a \geq 0$ flat at 0. Since the function $q(t, x, y, \xi, \eta) = -a(t)\big(\xi + H(t)V(x)|\eta|\big)$ satisfies (1.9) as the product of the non-positive function $-a(t)$ by the non-decreasing function $t \mapsto \xi + H(t)V(x)|\eta|$, the operator L satisfies condition (ψ). To simplify the exposition, let us assume that $a \equiv 1$, which introduces a rather unimportant singularity in the t-variable, let us replace $|D_y|$ by a positive (large) parameter Λ, which allows us to work now only with the two real variables t, x and let us set $W = \Lambda V$. We are looking for a non-trivial solution $u(t, x)$ of $L^*u = 0$, which means then

$$\partial_t u = \begin{cases} D_x u, & \text{for } t < 0, \\ \big(D_x + W(x)\big)u, & \text{for } t > 0. \end{cases}$$

The operator $D_x + W$ is unitarily equivalent to D_x: with $A'(x) = W(x)$, we have $D_x + W(x) = e^{-iA(x)} D_x e^{iA(x)}$, so that the negative eigenspace of the operator

$D_x + W(x)$ is $\{v \in L^2(\mathbb{R}), \text{supp}\, \widehat{e^{iA}v} \subset \mathbb{R}_-\}$. Since we want u to decay when $t \to \pm\infty$, we need to choose $v_1, v_2 \in L^2(\mathbb{R})$, such that

$$u(t,x) = \begin{cases} e^{tD_x}v_1, & \text{supp}\, \widehat{v_1} \subset \mathbb{R}_+ & \text{for } t < 0, \\ e^{t(D_x+W)}v_2, & \text{supp}\, \widehat{e^{iA}v_2} \subset \mathbb{R}_- & \text{for } t > 0. \end{cases} \tag{2.4}$$

We shall not be able to choose $v_1 = v_2$ in (2.4), so we could only hope for L^*u to be small if $\|v_2 - v_1\|_{L^2(\mathbb{R})}$ is small. Thus this counterexample is likely to work if the unit spheres of the vector spaces

$$E_1^+ = \{v \in L^2(\mathbb{R}), \text{supp}\, \widehat{v} \subset \mathbb{R}_+\} \quad \text{and} \quad E_2^- = \{v \in L^2(\mathbb{R}), \text{supp}\, \widehat{e^{iA}v} \subset \mathbb{R}_-\}$$

are close. Note that since $W \geq 0$, we get $E_1^+ \cap E_2^- = \{0\}$: in fact, with $L^2(\mathbb{R})$ scalar products, we have

$$v \in E_1^+ \cap E_2^- \Longrightarrow 0 \overset{v \in E_1^+}{\leq} \langle Dv, v \rangle \overset{0 \leq W}{\leq} \langle (D+W)v, v \rangle \overset{v \in E_2^-}{\leq} 0 \Longrightarrow \langle Dv, v \rangle = 0$$

which gives $v = 0$ since $v \in E_1^+$. Nevertheless, the "angle" between E_1^+ and E_2^- could be small for a careful choice of a positive W. It turns out that $W_0(x) = \pi\delta_0(x)$ is such a choice. Of course, several problems remain such as regularize W_0 in such a way that it becomes a first-order semi-classical symbol, redo the same construction with a smooth function a flat at 0 and various other things.

Anyhow, these difficulties eventually turn out to be only technical, and *in fine*, the actual reason for which theorem 2.1 is true is simply that the positive eigenspace of D_x (i.e. $L^2(\mathbb{R})$ functions whose Fourier transform is supported in $\mathbb{R}_+$) could be arbitrarily close to the negative eigenspace of $D_x + W(x)$ for some non-negative W, triggering nonsolvability in L^2 for the three-dimensional model operator

$$D_t - ia(t)\big(D_x + \mathbf{1}_{\mathbb{R}_+}(t)W(x)|D_y|\big), \tag{2.5}$$

where a is some non-negative function, flat at 0. This phenomenon is called the "drift" in [16] and could not occur for differential operators or for pseudo-differential operators in two dimensions. A more geometric point of view is that for a principal type symbol p, satisfying condition (ψ), one may have bicharacteristics of $\text{Re}\, p$ which stay in the set $\{\text{Im}\, p = 0\}$. This can even occur for operators satisfying condition (P). However condition (P) ensures that the nearby bicharacteristics of $\text{Re}\, p$ stay either in $\{\text{Im}\, p \geq 0\}$ or in $\{\text{Im}\, p \leq 0\}$. This is no longer the case when condition (ψ) holds, although the bicharacteristics are not allowed to pass from $\{\text{Im}\, p < 0\}$ to $\{\text{Im}\, p > 0\}$. The situation of having a bicharacteristic of $\text{Re}\, p$ staying in $\{\text{Im}\, p = 0\}$ will generically trigger the drift phenomenon mentioned above when condition (P) does not hold. So the counterexamples to solvability with loss of one derivative are in fact very close to operators satisfying condition (P).

A related remark is that the ODE-like solvable models in (1.5) do not catch the generality allowed by condition (ψ). Even for subelliptic operators, whose tranposed are of course locally solvable, it is known that other model operators than M_{2k}, N_l can occur. In particular the three-dimensional models $D_t + it^{2k}(D_x + t^{2l+1}x^{2m}|D_y|)$, where k, l, m are non-negative integers are indeed subelliptic and are not reducible to (1.5) (see chapter 27 in [11] and the remark before corollary 27.2.4 there).

Solvability with loss of two derivatives.

Although theorem 2.1 demonstrates that condition (ψ) does not imply solvability with loss of one derivative, the counterexamples constructed in this theorem are indeed solvable, but with a loss of two derivatives, as proven by N.Dencker in 1996 [2]. The same author gave a generalization of his results in [3] and later on, analogous results were given in [17].

A measurable function $p(t, x, \xi)$ defined on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ will be called in the next theorem a symbol of order m whenever, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$

$$\sup_{(t,x,\xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n} |(\partial_x^\alpha \partial_\xi^\beta p)(t, x, \xi)| (1 + |\xi|)^{-m+|\beta|} < +\infty. \tag{2.6}$$

Theorem 2.2. *Let $a(t, x, \xi)$ be a non-positive symbol of order 0, $b(t, x, \xi)$ be a real-valued symbol of order 1 such that $\partial_t b \geq 0$, and $r(t, x, \xi)$ be a (complex-valued) symbol of order 0. Then the operator*

$$L = D_t + ia(t, x, D_x)b(t, x, D_x) + r(t, x, D_x) \tag{2.7}$$

is locally solvable with a loss of two derivatives. Since the counterexamples constructed in theorem 2.1 are in fact of type (2.7), they are locally solvable with a loss of two derivatives.

In fact, for all points in $\mathbb{R}^{n+1}$, there exists a neighborhood V, a positive constant C such that, for all $u \in C_c^\infty(V)$

$$C\|L^* u\|_{H^0} \geq \|u\|_{H^{-1}}. \tag{2.8}$$

This estimate actually represents a loss of two derivatives for the first-order L; the estimate with loss of 0 derivative would be $\|L^* u\|_{H^0} \gtrsim \|u\|_{H^1}$, the estimate with loss of one derivative would be $\|L^* u\|_{H^0} \gtrsim \|u\|_{H^0}$, and both are false, the first because L^* is not elliptic, the second from theorem 2.1. The proof of theorem 2.2 is essentially based upon the energy method which boils down to compute for all $T \in \mathbb{R}$

$$\text{Re}\langle L^* u, iBu + iH(t - T)u\rangle_{L^2(\mathbb{R}^{n+1})}$$

where $B = b(t, x, D_x)$. Some complications occur in the proof from the rather weak assumption $\partial_t b \geq 0$ and also from the lower order terms. Anyhow, the correct multiplier is essentially given by $b(t, x, D_x)$. Theorem 2.2 can be proved for much more general classes of pseudo-differential operators than those given by (2.6). As a consequence, it can be extended naturally to contain the solvability result under condition (P) (but with a loss of two derivatives, see e.g. theorem 3.4 in [17]).

Miscellaneous results.

Let us mention that the operator (1.6) is solvable with a loss of one derivative (the ordinary loss) if condition (ψ) is satisfied (i.e. (1.9)) as well as the extra condition

$$|\partial_x q(t, x, \xi)|^2 |\xi|^{-1} + |\partial_\xi q(t, x, \xi)|^2 |\xi| \leq C |\partial_t q(t, x, \xi)| \quad \text{when} \quad q(t, x, \xi) = 0.$$

This result is proved in [18] and shows that "transversal" changes of sign do not generate difficulties. Solvability with loss of one derivative is also true for operators satisfying condition (ψ) such that the changes of sign take place on a Lagrangean manifold, e.g. operators (1.6) such that the sign of $q(t, x, \xi)$ does not depend on ξ, i.e. $q(t, x, \xi)q(t, x, \eta) \geq 0$ for all (t, x, ξ, η). This result is proved in section 8 of [13] which provides a generalization of [15] where the standard oblique-derivative problem was tackled. On the other hand, it was proved in [19] that for a first-order pseudo-differential operator L satisfying condition (ψ), there exists a L^2 bounded perturbation R such that $L + R$ is locally solvable with loss of two derivatives.

3. Conclusion and perspectives

The following facts are known for principal type pseudo-differential operators.

F1. Local solvability implies (ψ).

F2. For differential operators and in two dimensions, (ψ) implies local solvability.

F3. (ψ) does not imply local solvability with loss of one derivative.

F4. The known counterexamples in (F3) are solvable with loss of two derivatives.

The following questions are open.

Q1. Is (ψ) sufficient for local solvability in three or more dimensions?

Q2. If the answer to Q1 is yes, what is the loss of derivatives?

Q3. In addition to (ψ), which condition should be required to get local solvability with loss of one derivative?

Q4. Is analyticity of the principal symbol and condition (ψ) sufficient for local solvability?

The most important question is with no doubt Q1, since, with F1, it would settle the Nirenberg-Treves conjecture. From F3, it appears that the possible loss in Q2 should be > 1. In 2001, N.Dencker announced in [4] a positive answer to Q1, with answer 2 in Q2. His paper contains several new and interesting ideas; however, the author of this report was not able to understand thoroughly his article.

The Nirenberg-Treves conjecture is an important question of analysis, connecting a geometric (classical) property of symbols (Hamiltonians) to a priori inequalities for the quantized operators. The conventional wisdom on this problem turned out to be painfully wrong in the past, requiring the most careful examination of future claims.

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