On the Size of Quotients by Definable Equivalence Relations

ALAIN LOUVEAU

CNRS, Equipe d'Analyse
Université Paris 6
Tour 46-4e, 4 Place Jussieu
F-75252 Paris Cedex 05, France

Introduction

The study of simply definable equivalence relations, and in particular of Borel and analytic ones, on Polish spaces, has attracted a lot of attention in descriptive set theory since 1970, when the first fundamental result in the subject, Silver's theorem, was proved.

One motivation has certainly been the conjecture of Vaught in model theory, one of the oldest still open problems in mathematical logic, which can be interpreted as a question about certain analytic equivalence relations. Another motivation is that these objects are extremely common in many different fields of mathematics, and the answers to basic facts about them are certainly desirable. A third motivation is that the progress made on the tools and techniques of what is now called "effective" descriptive set theory, which is instrumental in many results in the subject, made it possible to hope for nontrivial results.

Since the 1970s, the study of these equivalence relations has been developed in many directions. This article will only consider one central question: Given a Polish space X, and a simply definable equivalence relation E on X, what can be said about the size of the quotient set X/E of equivalence classes?

In the rest of this paper, X will always denote a Polish space, i.e. a topological space homeomorphic to a separable complete metric space. And as all uncountable Polish spaces are Borel isomorphic, and we will work up to Borel isomorphism anyway, one can think of X as being the Cantor space $C = \{0,1\}^{\mathbb{N}}$. (One could also consider more general domains, like analytic or coanalytic ones, but simple manipulations usually allow us to reduce the questions to Polish domains.) It will be understood that X is the domain of an equivalence relation E. And as E is a subset of the Polish space X^2, definability properties of E make sense in that space. This paper will focus on the Borel equivalence relations and on the analytic ones. (A set is *analytic* in a Polish space if it is a continuous image of a Borel set. It is *coanalytic* if its complement is analytic.) What about the "size" of the quotient X/E? The usual set-theoretic notion is that of cardinality, and it is historically the first that has been studied, again in relation with Vaught's conjecture. In the first section of this paper, I will present the dichotomy results

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of Silver (which imply that for Borel E, the quotient is either countable or of cardinality the continuum) and Burgess (that for analytic E, and regardless of the exact value of the continuum, the only other possibility is $\aleph_1$). I will also relate these results to Vaught's conjecture.

In a second section, I will introduce and study another notion of "size" for quotients, more in the spirit of descriptive set theory, where the one-to-one maps between quotients that compare their size are now assumed to be "definable" (i.e. to come from definable maps on the domains). This leads to a partial ordering between equivalence relations, first introduced by Friedman and Stanley, for which it is still possible to get dichotomy results in the Borel case (Harrington-Kechris-Louveau) and in the analytic case (Becker, Hjorth-Kechris), which I will discuss, together with some other properties of this ordering.

This article is not a research paper (in particular, no proofs are given, and it is a bit of a pity, for proofs convey often more of the flavor of the subject than the results). It is also not a survey paper (even for the narrow problem it considers, this would need much more space), but a mere introduction to the subject, with a chosen sample of results (many important works, especially about the Vaught conjecture and its generalization, the topological Vaught conjecture, are not mentioned). For the reader interested in the subject, there should soon be available a monograph, by Kechris ([K]), called "Lectures on definable group actions and equivalence relations", that will contain the material presented here, and much, much more.

1 Cardinality of quotients

The following result of Silver gives the possible cardinalities of quotients X/E, when E is a Borel, or even a coanalytic, equivalence relation on a Polish space X.

THEOREM (SILVER [SI]). *Let E be a coanalytic equivalence relation on the Polish space X. Then either*

(i) X/E is countable

or

(ii) There is in X a copy of the Cantor space C consisting of pairwise E-inequivalent elements (so in particular X/E has cardinality the continuum).

A subset of X that picks at most one point in each equivalence class of E is called a *partial transversal*, and a *transversal* if in addition it meets all classes. So Silver's theorem asserts that if X/E is uncountable, it admits a partial transversal that is a perfect set homeomorphic to C. One then says that E has *perfectly many* classes. The question of the existence of well-behaved (total) transversals will be discussed in the next section.

Silver's result is reminiscent of the classical perfect set theorem for analytic sets — which indeed can be viewed as a particular case of it. But it is much harder to prove. Silver's original proof is both difficult and metamathematical. A later simpler proof by Harrington (unpublished) had a considerable influence on the subject because it brought into it the "effective" techniques borrowed from the theory of definability on the integers, hence based ultimately on computability theory, that are now central to the subject.

Silver's result was the starting point of a series of investigations about the number of classes of definable equivalence relations. The next result takes care of the analytic case.

THEOREM (BURGESS [B]). *Let E be an analytic equivalence relation on the Polish space X. Then either*
(i) X/E has cardinality at most $\aleph_1$
or
(ii) E has perfectly many classes.

This result says that, compared to the Borel case, the analytic case adds at most one possibility, namely that of $\aleph_1$ many, but not perfectly many, equivalence classes. Such analytic equivalence relations do exist: it is easy to give the set LO of linear orders with domain $\mathbb{N}$ a Polish topology. For this topology, the following relation between linear orders is analytic: two orders are equivalent if they are well-orders of the same length, or if both are not well-orders. This relation does not have perfectly many classes, and its quotient space is isomorphic to the set ω_1 of countable ordinals.

After the results of Silver and Burgess, investigations have been pursued in two main directions. The first one concerns even more complicated definable equivalence relations. For this line of research, see the papers of Harrington-Sami [HSa] and Harrington-Shelah [HSh].

Another line of research is concerned with the possibility of extending Silver's dichotomy result to interesting subclasses of analytic equivalence relations, in particular in relation with Vaught's conjecture in model theory.

Vaught's original conjecture is the statement that any first-order theory (in a countable language) has either countably many or else continuum-many nonisomorphic countable models.

As in the case of linear orders, it is possible to equip the set of models of a given first-order theory that have domain $\mathbb{N}$ with a natural Polish topology, for which the isomorphism relation becomes analytic, but not always Borel. So Silver's result does not apply to this situation, and the result of Burgess is inconclusive.

On the other hand, the isomorphism equivalence relations that arise in model theory are not arbitrary analytic equivalence relations. They are a particular case of the following situation: Suppose we are given a Polish group G (i.e. a topological group that is a Polish space), and a Borel action $\alpha : G \times X \to X$ of G on X. One can then consider the associated orbit equivalence relation E_α on X defined by

$$xE_\alpha y \leftrightarrow \exists g \in G \ \alpha(g,x) = y$$

(the model theory case corresponds to the natural action of the symmetric group $S_\infty(\mathbb{N})$ of permutations of $\mathbb{N}$ on the space of models with domain $\mathbb{N}$).

Orbit equivalence relations of Borel actions of Polish groups are analytic (and again not always Borel). The natural generalization in descriptive set theory of Vaught's conjecture is the so-called topological Vaught conjecture: "For any Borel action of a Polish group on a Polish space, the orbit equivalence relation has either countably or perfectly many classes."

A lot of work has been done on the two conjectures, both in model theory and in descriptive set theory, and it is impossible to give here a fair account of the known results.

In model theory, Vaught's conjecture has been established for many theories, in particular among the theories for which the classification tools apply. The best results at present are that Vaught's conjecture holds for the so-called ω-stable theories (Harrington-Makkai-Shelah [HMS]), and the superstable theories of finite rank (Buechler [Bu]). The conjecture is also known for some specific theories, trees for example (Steel [St]).

In descriptive set theory, the topological Vaught conjecture is known for various acting groups (the locally compact ones, the abelian ones), and for orbit equivalence relations with special (rather technical) properties (see Steel [St]). Still, both conjectures are open.

2 An alternative notion of size for quotients

DEFINITION. *Let E and F be two equivalence relations on the Polish spaces X and Y, respectively. A reduction of E to F is a map $f : X \to Y$ that satisfies, for all x and y in the space X,*

$$xEy \leftrightarrow f(x)Ff(y).$$

We say that E is Borel reducible to F, or F Borel reduces E, written $E \preceq F$, if there exists a Borel reduction of E to F. The relations E and F are Borel bireducible, written $E \cong F$, if both $E \preceq F$ and $F \preceq E$.

Note that a reduction f of E to F induces a quotient map $f^* : X/E \to Y/F$, which is one-to-one. So the idea of Borel reducibility is to compare the sizes of quotients not by arbitrary one-to-one maps, as is done in cardinality theory, but by one-to-one maps that are induced by Borel functions. And heuristically the class of equivalence relations that are Borel bireducible with a given E is a measure of the "Borel size" of the quotient space X/E.

The notion of Borel reducibility was first introduced, in the context of model theory, by Friedman and Stanley [FS]. It corresponds there to the search, for countable models of a given theory, of invariants that describe the models up to isomorphism, and is used in [FS] as a way of comparing theories.

What are the possible "Borel sizes" of quotients? Let us first consider the case of Borel equivalence relations.

First, Silver's dichotomy result can be interpreted in terms of the ordering $\preceq$. Denote by $\Delta(A)$, for any set A, the relation of equality on A. Then Silver's result asserts that for any Borel E, either $E \preceq \Delta(\mathbb{N})$, or else $\Delta(C) \preceq E$, where C is the Cantor space. So among uncountable Borel quotients, C has the smallest "Borel size".

The equivalence relations that are Borel reducible to equality on C are called *smooth*. They can alternatively be characterized as those Borel equivalence relations that admit a countable Borel separating family, i.e. a sequence (B_n) of Borel E-invariant sets such that

$$xEy \leftrightarrow \forall n(x \in B_n \leftrightarrow y \in B_n)$$

and also as those for which the σ-algebra of E-invariant Borel sets is countably generated.

Moreover, using a selection theorem of Jankov and von Neumann, smooth equivalence relations admit C-measurable transversals. (In a Polish space, a set is *C-measurable* if it is in the smallest σ-algebra containing the open sets and closed under Suslin operation. These sets are more general than the analytic and coanalytic ones, but still well behaved, in particular, universally measurable.)

Closed, and even G_δ equivalence relations are smooth. But it is well known that there are simple F_σ equivalence relations with no universally measurable transversals, hence that are not smooth, e.g. Vitali's famous example of equality mod $\mathbb{Q}$ on the space $\mathbb{R}$. Here we will consider another example (which happens to be Borel bireducible to Vitali's example), the relation E_0 of eventual equality on the Cantor space C, given by

$$\alpha E_0 \beta \leftrightarrow \exists n \forall m \geq n \ \alpha(m) = \beta(m).$$

This F_σ equivalence relation is not smooth, hence C/E_0 represents a new quotient, of "Borel size" bigger than C. The next dichotomy result, which extends earlier results of Glimm [G] and Effros [E1], [E2], says that it is indeed the smallest Borel quotient above C.

THEOREM (HARRINGTON-KECHRIS-LOUVEAU [HKL]). *Let E be a Borel equivalence relation on the Polish space X. Then either*

(i) E is smooth, i.e. E is Borel reducible to $\Delta(C)$,

or

(ii) $E_0 \preceq E$, in fact there is a one-to-one and continuous reduction of E_0 to E.

A consequence of this result is the following: Say that a probability measure μ on X is *ergodic* for E if every Borel E-invariant set has μ-measure 0 or 1, and *nonatomic* for E if every E-class has μ-measure 0. For example, $\Delta(C)$ admits no ergodic nonatomic measures, but the usual Lebesgue measure on C is both ergodic and nonatomic for E_0. It is easy to check that ergodicity and nonatomicity are preserved under images by Borel (or even universally measurable) reductions. Hence the previous result implies that, for a Borel equivalence relation, smoothness is indeed equivalent to the existence of a C-measurable transversal (this is the way smoothness is proved for G_δ equivalence relations), and also that the relation $E_0 \preceq E$ is equivalent to the existence of an ergodic and nonatomic probability measure for E.

Equivalence relations that are Borel reducible to E_0 are called *hyperfinite*. Among the Borel equivalence relations with countable equivalence classes, they can be characterized as those induced by a Borel automorphism of X, i.e. as the orbit equivalence relations of Borel $\mathbb{Z}$-actions. So they are the ones considered in ergodic theory (although in a different context, as the domain is a measure space, and the automorphism is usually assumed to be nonsingular with respect to the measure). Hyperfinite equivalence relations with countable equivalence classes have been extensively studied by Dougherty, Jackson, and Kechris, in a paper [DJK] that contains in particular a complete classification up to Borel isomorphism.

However, there is no known dichotomy result, similar to the Harrington-Kechris-Louveau result, that would separate the hyperfinite equivalence relations from the other ones, and some basic questions about them are open. For example, it is not known exactly which countable groups have the property that all their Borel actions have hyperfinite orbit equivalence relations. It holds when the group is $\mathbb{Z}^n$ (Weiss) or more generally is a finitely generated group of polynomial growth (Jackson-Kechris-Louveau), and it does not hold if the group is not amenable (in analogy with the measure case, amenability should be the right conjecture).

Above E_0, the partial order of Borel reducibility between Borel equivalence relations is not very well understood, and the known results indicate that it has a rather complicated structure.

First, it has a cofinal sequence of length ω_1, but no maximum element (Friedman). In fact, if one defines, for a Borel equivalence relation E on X, the Borel relations E^+ and E^* on $X^{\mathbb{N}}$ by

$$(x_n)_{n\in\mathbb{N}}\ E^+\ (y_n)_{n\in\mathbb{N}} \leftrightarrow \{x_n/E : n \in \mathbb{N}\} = \{y_n/E : n \in \mathbb{N}\},$$

where x/E denotes the E-class of x, and

$$(x_n)_{n\in\mathbb{N}}\ E^*\ (y_n)_{n\in\mathbb{N}} \leftrightarrow \exists n\ \forall m \geq n\ x_n E y_n,$$

then for E with at least two classes, both E^+ (Friedman) and E^* (Louveau) are strictly bigger than E in the ordering $\preceq$. These two "jump operators" are also incomparable (e.g. the relations $\Delta(C)^+$ and $\Delta(C)^*$ are $\preceq$-incomparable).

Another result (Kechris-Louveau) is that for any Borel E strictly above E_0, there is always another Borel E' that is $\preceq$-incomparable to it. It follows that there can be no similar dichotomy results above E_0, at least for the whole class of Borel equivalence relations. Finally (in this list of negative results), Louveau and Velickovic [LV] have shown that the partial order of almost inclusion between subsets of $\mathbb{N}$ can be embedded, as a partial order, in the order $\preceq$ between Borel equivalence relations. Hence so does any partial order of size at most $\aleph_1$, and there is no hope to describe in a reasonable way the various possible "Borel sizes" of Borel quotients.

The ongoing research on this partial ordering focuses on some important subclasses of Borel equivalence relations, like the ones that have countable equivalence classes. For more on the subject, we refer the reader to the forthcoming monograph of Kechris [K].

Let us consider now the analytic equivalence relations. First, one should relax a bit the notion of definable reducibility used to compare the quotients, and consider, say, C-measurable reductions instead of Borel ones. Even then, the Harrington-Kechris-Louveau result does not extend to the analytic case.

Consider for example, in model theory, the relation E of isomorphism between countable abelian torsion groups. These groups can be classified, up to isomorphism, by invariants called the Ulm invariants. These invariants can be considered as transfinite sequences of 0's and 1's of countable length, i.e. elements of $\{0,1\}^{<\omega_1}$. If one considers only groups with Ulm invariant of fixed bounded length, isomorphism is smooth on them. This prevents E_0 from being reducible to the isomorphism in any reasonable way. But the length of the Ulm invariants cannot be bounded below ω_1, and this prevents E from being reducible to $\Delta(C)$ in any reasonable way.

The following very recent and unpublished result says that the situation above is in fact typical. To state the result, note that, like the set ω_1, the set $\{0,1\}^{<\omega_1}$ can be realized as the quotient of the Cantor space C by some analytic equivalence relation, call it E_1. One then has

THEOREM (BECKER, HJORTH-KECHRIS). *Let E be the orbit equivalence relation of the Borel action of a Polish group on the Polish space X. Then either*

(i) E is reducible, via a C-measurable reduction, to E_1,

or

(ii) E_0 is continuously reducible to E.

This result could have interesting consequences in model theory. It suggests that some of the theories with continuum many nonisomorphic models, which are usually considered unclassifiable, might still admit some kind of classification, analogous to the Ulm invariants for abelian torsion groups, when E_0 is not reducible to the isomorphism relation. But work has to be done, first to understand what it means for a theory that E_0 is not reducible to its isomorphism relation, and secondly in order to replace the abstract invariants of case (i) of the theorem by concrete ones in specific situations.

Let us mention finally that a weak version of the above theorem, valid for arbitrary analytic equivalence relations, has also been established by Hjorth and Kechris, using a strong set-theoretic assumption.

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