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Finite Rank Projections on Banach Spaces and a Conjecture of Grothendieck

In this report we discuss several recent results concerning the existence or non-existence of well behaved finite rank projections on a Banach space. We will be interested in projections with large ranks and norms as small as possible.

0. Standard notations

We remind the definition of the Banach–Mazur distance $d(E, F)$ between two Banach spaces:

$$d(E, F) = \inf\{\|T\|\ \|T^{-1}\|\},$$

where the infimum is over all isomorphisms T from E onto F. If E and F are not isomorphic, we set $d(E, F) = +\infty$. As usual, we will denote by l_p^n the space $\boldsymbol{R}^n$ equipped with the norm

$$\|(\alpha_i)\| = \Big(\sum_1^n |\alpha_i|^p\Big)^{1/p}.$$

1. The finite-dimensional basis problem

Since Enflo's example [4], we know that there are Banach spaces which fail the approximation property and, *a fortiori*, fail to have a basis.

By definition, a Banach space X has the *approximation property* (in short A.P.) if the identity is approximable by finite rank operators uniformly on every compact subset of X.

The space X has a basis if (and only if) there is a sequence $\{P_n\}$ of finite rank projections on X such that

$$P_{n-1}(X) \subset P_n(X) \text{ for all } n > 1; \tag{1}$$

$$\operatorname{rank} P_n = n; \tag{2}$$

$$\overline{\bigcup P_n(X)} = X; \tag{3}$$

$$\sup \|P_n\| < \infty. \tag{4}$$

If we then select a sequence $\{x_n\}$ such that $x_n \neq 0$ and $x_n \in P_n(X) \cap \cap \operatorname{Ker} P_{n-1}$, we obtain a basis of X in the usual sense.

One defines the *basis constant* of X as $b(X) = \inf\{\sup \|P_n\|\}$ where the infimum runs over all possible sequences $\{P_n\}$ as above. Clearly, this makes sense also when X is of finite dimension d, by restricting (1) and (2) to all $n \leqslant d$ (and $P_n = \mathrm{Id}_X$ for all $n \geqslant d$).

We now examine $b(X)$ for a finite-dimensional space X. Although this is surprising at first glance, the result of Enflo does *not* imply (and is *not* implied by !) the existence of a sequence $\{X_n\}$ of finite-dimensional spaces with basis constants $b(X_n)$ tending to infinity with n. In fact, until recently, the following question was still open:

Question 1. Is there a universal bound for $b(X)$ when X runs over all finite-dimensional (in short f.d.) spaces ?

The only known upper bound for $b(X)$ is $b(X) \leqslant (\dim X)^{1/2}$, which follows immediately from a classical result of F. John: on every n-dimensional space there is an inner product norm which is $\sqrt{n}$ equivalent to the original one (cf. [14]). Recently, Gluskin and Szarek gave the expected negative answer to Question 1, (cf. [8], [32]). More precisely, we have

THEOREM 1 ([32]). *There is an absolute constant $\delta > 0$ with the following property: for each integer n, there is an n-dimensional space X_n such that, for every projection $P: X_n \to X_n$ with rank $\left[\frac{n}{2}\right]$, we have $\|P\| \geqslant \delta\sqrt{n}$. In particular, we have $b(X_n) \geqslant \delta\sqrt{n}$.*

We refer to [32] for the rather long and delicate proof of this result. It should be mentioned that Szarek's construction relies on probabilistic ideas: the spaces X_n are selected "at random" in a clever way so that the property in Theorem 1 occurs actually with large probability. Szarek's methods were directly inspired by a previous (quite remarkable) paper of Gluskin [7], where the latter proved the existence (with positive probability, in some sense) for each n of two n-dimensional spaces X_n, Y_n

satisfying

$$\inf_n n^{-1} d(X_n, Y_n) > 0.$$

Later, Gluskin pushed his methods to answer Question 1 with a weaker form of Theorem 1 (cf. [8]), while Szarek proved independently the essentially sharp version of Theorem 1 quoted above.

2. The complemented l_p^n problem

Enflo's example tells us that there exist Banach spaces which do not have "enough" finite rank projections of uniformly bounded norms to approximate the identity operator in the pointwise topology.

This leaves open the following question, which can be roughly formulated as follows: are there non-trivial finite rank projections *at all* on a general space?

Question 2. Let X be an infinite-dimensional Banach space. Does there exist a sequence $\{P_n\}$ of finite rank projections on X with uniformly bounded norms and unbounded ranks?

Of course, this is much weaker than saying that X has a basis, since $\{P_n\}$ is not required to satisfy (1) or (3); and, quite obviously, there are spaces without the A.P. which possess the above property. In [20], Lindenstrauss reformulated the preceding question, in a stronger formulation:

Question 3. Let $k_n = \operatorname{rank} P_n$. Can one find P_n's as in Question 2 with the additional property that, for some p in $[1, \infty]$, we have $\sup_n d(P_n(X), l_p^{k_n}) < \infty$?

In that case, we can as well assume that $k_n = n$ and we then say that X contains uniformly complemented l_p^n's. Moreover, it is easy to see that the problem reduces to the cases $p = 1, 2$ and ∞.

In several special cases, positive answers were given in [34], [15] and in [35]. To motivate Question 3, let us recall a fundamental theorem of Dvoretzky (cf. [5]): for every infinite-dimensional space X, for every n and $\varepsilon > 0$, we can find a subspace X_n of X such that $d(X_n, l_2^n) < 1+\varepsilon$. Roughly, this means that X reproduces somewhere the structure of Euclidean spaces almost isometrically. For various reasons, mainly in operator theoretic considerations, it is of interest to decide when we can find subspaces X_n as above together with projections $P_n: X \to X_n$ such that $\sup \|P_n\| < \infty$. It is rather simple to check that L^p spaces or l_p spaces have this property for $1 < p < \infty$ and not for $p = 1$ or ∞. In L^p, we even have complemented *infinite*-dimensional Hilbertian subspaces (take the span of Gaussian variables, or Rademacher functions); but, in l_p, this is no longer true, so that, in general, we must restrict our attention to the finite-dimensional subspaces.

Unfortunately, in general, the answer to Question 2 is negative:

THEOREM 2 ([30]). *There is an infinite-dimensional Banach space X and a constant $\delta > 0$ such that all finite rank projections P: $X \to X$ satisfy*

$$\|P\| \geqslant \delta \ (\operatorname{rank} P)^{1/2}.$$

In particular, X is a counterexample to the property in Question 1.

Remark. Let E be an n-dimensional subspace of a general space X. Then there is a projection P: $X \to E$ with $\|P\| \leqslant \sqrt{n}$. This is a classical result (originally due to Kadeč–Snobar). Therefore, in the above space X, this general upper bound cannot be improved (at least asymptotically)-regardless of how E is chosen in X.

The space constructed for Theorem 2 also fails the A. P. We will rel turn to this in the next section.

There is however a major difference here from the approximation problem. Indeed, now we know that there are extremely "nice" spaces, e.g. uniformly convex spaces, which fail the A.P. (see [31] for examples of subspaces of $l_p, p \neq 2$, failing the A.P.). However, it turns out that, in all uniformly convex spaces, Question 3 (and a fortiori Question 2) has a positive answer, so that the "optimistic" conjecture is correct for these spaces. (See the corollary of Theorem 5). To state this in full generality, we will need some terminology. We will say that a Banach space X contains l_p^n's *uniformly* if, for some $\lambda > 1$, there is a sequence of subspaces X_n of X such that $d(X_n, l_p^n) \leqslant \lambda$. It is known (cf. [13] [17]) that, if this property holds for some $\lambda > 1$, it also holds for all $\lambda > 1$. In particular, a uniformly convex space cannot contain l_1^n's uniformly (consider the case $n = 2$ and let λ tend to 1).

With this terminology, Dvoretzky's theorem says that any infinite-dimensional space contains l_2^n's uniformly.

The study of the l_p^n-subspaces of a Banach space is intimately connected with the notions of type and cotype, which are defined as follows. Let $D = \{-1, 1\}^N$, let μ be the uniform probability on D and let ε_n: $D \to \{-1, +1\}$ be the n-th coordinate on D. We will denote the space $L^2(D, \mu; X)$ simply by $L^2(X)$.

DEFINITION. Let $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. A space X *is of type* p (resp. *of cotype* q) if there is a constant C such that, for all finite sequences $(x_1, \ldots, x_n)$ in X, we have

$$\Big\| \sum_1^n \varepsilon_i x_i \Big\|_{L^2(X)} \leqslant C\Big(\sum \|x_i\|^p\Big)^{1/p}$$

$$(\text{resp.} \quad \Big\| \sum_1^n \varepsilon_i x_i \Big\|_{L^2(X)} \geqslant \frac{1}{C}\Big(\sum \|x_i\|^q\Big)^{1/q}).$$

We will denote by $T_p(X)$ (resp. $C_q(X)$) the smallest constant C satisfying this. Every normed space is of type 1 and of cotype ∞. We refer to [22] for more details. We should mention that these notions are used frequently in the current study of probability on Banach spaces (cf. e.g. [12]). In the latter area, the spaces which do not contain l_1^n's uniformly are called *B-convex*; this class of spaces was introduced by A. Beck, in the early sixties, to investigate the strong law of large numbers for vector valued random variables.

The results of [22] and [17] combined together, yield the following theorem, which relates these analytic notions with the more geometric concept of "containing l_p^n's".

THEOREM 3 ([22] [17]). *For an infinite-dimensional space X, let*

$$p(X) = \sup\{p | X \text{ is of type } p\}$$

and

$$q(X) = \inf\{q | X \text{ is of cotype } q\}.$$

Then, X contains l_p^n's uniformly for $p = p(X)$ and also for $p = q(X)$.

This implies that $p(X)$ (resp. $q(X)$) coincides with the smallest (resp. largest) p such that X contains l_p^n's uniformly.

In particular, $p(X)$ is non-trivial, i.e. $p(X) > 1$, iff X does not contain l_1^n's uniformly; while $q(X)$ is non-trivial, i.e. $q(X) < \infty$, iff X does not contain l_∞^n's uniformly.

In the concluding remarks of [22], it was asked whether there is a "duality" between $p(X)$ and $q(X^*)$, when $p(X) > 1$.

The key to solve this problem is the notion of "K-convexity"; a Banach space X is called K-convex if the orthogonal projection R from $L^2(D, \mu)$ onto the span of the sequence $\{\varepsilon_n\}$ induces a bounded operator, denoted by $\tilde{R}$, on $L^2(X)$. Whenever it is bounded, $\tilde{R}$ is a projection onto the space of all series $\sum_{n=1}^{\infty} \varepsilon_n x_n$, with x_n in X, which converge in $L^2(X)$.

It is rather striking that the boundedness of this single projection on $L^2(X)$ implies the boundedness of many projections on X, as the following result of Figiel and Tomczak-Jaegermann shows:

THEOREM 4 ([6]). *Any K-convex space X is locally π-Euclidean, which means that there is a constant C and, for each n and $\varepsilon > 0$, there is an integer $N = N(\varepsilon, n)$ satisfying the following property:*

for every subspace $E \subset X$ with $\dim E \geqslant N$, there is a subspace $F \subset E$ of dimension n and a projection P: $X \to F$ such that $d(F, l_2^n) < 1+\varepsilon$ and

$$\|P\| \leqslant C.$$

The proof uses the same isoperimetric inequality as in [5]; the space F and the projection P are obtained by a suitable random choice. For a different approach using random matrices, see [1].

Fortunately, it turns out that K-convexity admits a simple "geometric" characterization:

THEOREM 5 ([28]). *A Banach space is K-convex iff it does not contain l_1^n's uniformly.*

It is the "if" part which is non-trivial. This shows that the converse to Theorem 4 is true:

COROLLARY. *The properties "X does not contain l_1^n's uniformly", "$p(X) > 1$", "X is K-convex" and "X is locally π-Euclidean" are all equivalent.*

Several special cases were already known, in particular, for Banach lattices, cf. [34] [15]. Moreover, if X is K-convex, then X is of type p iff X^* is of cotype p' (cf. [22], remark 2.10); hence if $p(X) > 1$, we have:

$$\frac{1}{p(X)} + \frac{1}{q(X^*)} = \frac{1}{p(X^*)} + \frac{1}{q(X)} = 1.$$

We should mention that the proof of Theorem 5 relies heavily on some results from the theory of holomorphic semi-groups.

We refer to [28] for more details.

Remark. The results of [29] suggest the following conjecture.

CONJECTURE. *In any space X in which Question 2 has an affirmative answer, the same is true for Question 3.*

Indeed, this is verified in [29] if $q(X) = 2$, and there is still some hope that the approach of [29] will prove the conjecture in general (cf. [29], p. 143).

3. A conjecture of Grothendieck

Let X, Y be Banach spaces and let $u = \sum_{i=1}^{n} x_i \otimes y_i$ be an element of the algebraic tensor product $X \otimes Y$ ($x_i \in X$, $y_i \in Y$). Let B_X be the unit ball of X. Grothendieck defined the injective and projective norms as

$$\|u\|_\vee = \sup\left\{ \sum_1^n x^*(x_i) y^*(y_i) | \, x^* \in B_{X^*}; \, y^* \in B_{Y^*}\right\}$$

and

$$\|u\|_\wedge = \inf\left\{ \sum_1^n \|x_i\| \|y_i\| \right\},$$

where the infimum runs over all possible representations of u. He denoted by $X \check{\otimes} Y$ and $X \hat{\otimes} Y$ the completions of $X \otimes Y$ with respect to the corresponding norms (cf. [10] [11]).

Obviously, $\|u\|_\vee \leqslant \|u\|_\wedge$, so that there is a natural norm decreasing map from $X \hat{\otimes} Y$ into $X \check{\otimes} Y$. At the end of [10], Grothendieck listed six open questions, which are now (essentially) all solved. The first (and main) one was the A.P. problem. The last one was the following:

Question 4 If $X \hat{\otimes} Y = X \check{\otimes} Y$, is it true that either X or Y must be finite-dimensional?

In view of the fact that $\| \ \|_\vee$ and $\| \ \|_\wedge$ are, respectively, the smallest and the greatest reasonable tensor norm, it is natural to ask if they can happen to be equivalent on $X \otimes Y$ in any other case than the trivial one when one of the dimensions is finite. This is precisely the content of Question 4. Let us consider the case when X and Y are in duality. Then $X^* \check{\otimes} X$ can be identified with the closure of the finite rank operators in the space $\mathscr{L}(X, X)$ of all bounded operators on X.

On the other hand, the elements u in $X^* \check{\otimes} X$ which are in the image of the natural map J: $X^* \hat{\otimes} X \to X \check{\otimes} X$ are exactly those which can be written as

$$u(x) = \sum_{n=1}^{\infty} x_n^*(x) x_n \tag{5}$$

for all x in X, with x_n^* in X^* and x_n in X, such that $\sum_1^\infty \|x_n^*\|\,\|x_n\| < \infty$. These are called *nuclear operators* and their "nuclear norm" is defined as $N(u) = \inf \sum_1^\infty \|x_n^*\|\,\|x_n\|$ where the infimum runs over all representations satisfying (5). Grothendieck showed that X has the A. P. iff $J: X^* \hat{\otimes} X \to X \check{\otimes} X$ is *injective*; in that case the trace of a nuclear operator is well defined, and it is then easy to prove that J is an *isomorphism* only if the dimension of X is finite. However, until recently it was not known whether X must be f.d. when J is merely assumed to be *surjective*. This question belongs to the same family as Question 2. Roughly formulated, it reads: if the dimension of X is infinite, is there any non-trivial operator at all in $X^* \check{\otimes} X$? Indeed, the nuclear operators are trivial in the sense that they are just absolutely convergent series of *rank one* operators.

We should mention that a positive answer to each of the preceding questions was given in [3] if X does not contain l_1^n's uniformly (this can now be derived easily from the more recent Theorems 4 and 5). Moreover, the following finite-dimensional version of Grothendieck's conjecture was proved in [25].

THEOREM 6. *Let $\{X_n\}$ be a sequence of f.d. Banach spaces and let Y be a Banach space. Assume that for some constant C, we have $\|u\|_\wedge \leqslant C\|u\|_\vee$, for all u in $X_n \otimes Y$ and for all n. Then either* $\sup_n \dim X_n < \infty$ *or* $\dim Y < \infty$.

In particular, if either X or Y has a basis (or merely the property in Question 2), then the answer to Question 4 in "yes". However, in general, the answer to both of these questions is negative:

THEOREM 7. *Let E be any Banach space of cotype 2. (For instance $E = l_1$ or $E = l_2$.) Then there exists a Banach space X which contains E isometrically and is such that:*

(i) $X \hat{\otimes} X = X \check{\otimes} X$

(ii) *The map J from $X^* \hat{\otimes} X$ into $X^* \check{\otimes} X$ is surjective. Equivalently, there is a constant C such that every finite rank operator u on X satisfies $N(u) \leqslant C\|u\|$.*

Moreover, if E is separable, we can obtain a separable space X as above.

We do not know, however, if there is a reflexive space X (or merely not containing l_1) which possesses any of the properties (i) and (ii). Similarly, Question 2 is still open for reflexive spaces. Also, we could not construct a space X such that every *compact* operator on X is nuclear.

Note that if $P\colon X\to X$ is a finite rank projection, then it is well known that $N(P)\geqslant(\operatorname{rank}P)^{1/2}$, therefore Theorem 2 is a corollary of Theorem 7 with $\delta=\frac{1}{C}$. To describe the proof of Theorem 7, we focus on (i) ((ii) is obtained as a consequence of (i) by the results of [25]).

The basic idea is to construct a sequence of Banach spaces $E_0\subset E_1\subset E_2\subset\ldots E_n\subset E_{n+1}\subset\ldots$, with E_n isometrically embedded in E_{n+1} with $E_0=E$, and such that, for some constant K, we have for all n and all u in $E_n\otimes E_n$

$$\|u\|_{E_{n+1}\hat{\otimes}E_{n+1}}\leqslant K\|u\|_{E_n\check{\otimes}E_n}. \tag{6}$$

Once the sequence $\{E_n\}$ is obtained, it is quite easy to check that $X=\overline{\bigcup E_n}$ satisfies the above property (i).

The difficulty in the construction of the sequence $\{E_n\}$ lies in the fact that (6) can hold for some E_{n+1} containing E_n, *only if* E_n satisfies a certain restrictive condition; therefore, to carry on the construction, we must make sure, at each step, that E_{n+1} satisfies not only (6) but also this condition, which we now make more explicit.

Let $u\colon E\to F$ be an operator between Banach spaces, we say that u factors through a Hilbert space H if there are operators $A\colon E\to H$ and $B\colon H\to F$ such that $u=BA$; this property is "controlled" by the following norm: $\gamma_2(u)=\inf(\|B\|\,\|A\|)$, where the infimum is over all possible factorizations of u. If u is an element of $E\otimes E$, we will denote by $\gamma_2(u)$ the above norm computed for the operator from E^* into E associated to u. It is then easy to see that $\gamma_2(u)\leqslant\|u\|_{E\hat{\otimes}E}$ for any u in $E\otimes E$. Therefore, if (6) holds, then the space E_n must satisfy

$$\forall u\in E_n\otimes E_n,\qquad \gamma_2(u)\leqslant K\|u\|_{\vee}. \tag{7}$$

This strongly indicates that, in order to prove Theorem 7, we must first investigate this condition (7). This was done in [25].

THEOREM 8. [25] *Let E and F be Banach spaces such that both E^* and F are of cotype 2. Then there exists a constant K (depending only on the cotype 2 constants of E^* and F) such that every finite rank operator $u\colon E\to F$ satisfies*

$$\gamma_2(u)\leqslant K\|u\|.$$

COROLLARY. *If moreover E or F has the A.P., then any bounded operator $u\colon E\to F$ factors through a Hilbert space.*

Applied to the identity operator, this yields

COROLLARY. *If a Banach space E and its dual E^* are of cotype 2, and if E possesses the A.P., then E is isomorphic to a Hilbert space.*

These results were conjectured in [21] (without the A.P.). Up to now, they cover all the known couples of Banach spaces E and F such that every bounded operator $u: E \to F$ factors through a Hilbert space. They can be viewed as an "abstract" form of a classical theorem of Grothendieck, who proved this for $E = L^\infty$ and $F = L^1$. His result was extended in many ways. Maurey (cf. [21]) discovered the relation with the notion of cotype and proved this result for $E = L_\infty$ and F any space of cotype 2. The main examples of cotype 2 spaces are L^1 spaces and their subspaces. More generally, the dual or the predual of a C^*-algebra is of cotype 2 [33], as well as the quotients L^1/R when R is a reflexive subspace of L^1 ([16], [24]). Recently, Bourgain [2] proved that L^1/H^1 is of cotype 2. Actually, the last two examples play an important role in the proof of Theorem 7. It is conceivable that the assumptions of Theorem 8 are necessary if neither E nor F is isomorphic to a Hilbert space (see [25], remark 2.4). However, the A.P. cannot be removed from the preceding two corollaries. Indeed, in the proof of Theorem 7, we actually construct a sequence $\{E_n\}$ verifying (6) and also such that

$$\sup_n C_2(E_n) < \infty. \tag{8}$$

This last property implies by Theorem 8 that for some constant K (independent of n) we have (7) and this enables us to carry on the inductive process. Finally, the space X constructed for Theorem 7 is of cotype 2, as well as its dual, but it cannot be isomorphic to a Hilbert space; in fact, this space X fails the A.P. and this shows that both corollaries would be false without the A.P.

4. Upper bounds for the projection constants

Let X be a Banach space.

In this section we estimate the projection constant of an n-dimensional subspace E of X when n tends to infinity. We can define

$$\lambda_X(E) = \inf \|P\| \quad \text{and} \quad \mu_X(E) = \inf\{\gamma_2(P)\}$$

where the infimum runs over all possible projections $P: X \to E$. $\lambda_X(E)$ is called the *projection constant* of E relative to X. We have clearly $\lambda_X(E) \leqslant \mu_X(E)$.

We then let

$$e_n(X) = \sup \mu_X(E)$$

where the supremum runs over all n-dimensional subspaces $E \subset X$. For such an E, we have $d(E, l_2^n) \leqslant e_n(X)$ and there exists a projection P: $X \to E$ such that $\|P\| \leqslant e_n(X)$. The asymptotic behaviour of $e_n(X)$ when $n \to \infty$ has attracted a lot of attention in recent years. For a general space, we have $e_n(X) \leqslant \sqrt{n}$, and the "worst" cases are attained (at least asymptotically) for $X = L^1$ or $X = L^\infty$. But if a space is "far" from these extreme cases, this can be improved. In [19], Lewis proved that

$$e_n(L^p) \leqslant n^{\left|\frac{1}{p} - \frac{1}{2}\right|}$$

Following Lewis, this was generalized by many authors (Lewis, Tomczak-Jaegermann, ...). For instance, it was proved in [18] (cf. also [37] for a better proof and other results) that if X is of type $p > 1$ and of cotype $q < \infty$, then $e_n(X) \leqslant Cn^a$ for some constant C and $a = 1/p - 1/q$. The question whether this can be improved to $a = \max(1/p - \frac{1}{2}, \frac{1}{2} - 1/q)$ (or any $a < \frac{1}{2}$, when $1/p - 1/q \geqslant \frac{1}{2}$) is still open. It was (essentially) verified for Banach lattices in [27].

Although the "right" exponent is still in doubt, we *do* know that $n^{-1/2} e_n(X) \to 0$ when $n \to \infty$ iff X does not contain l_1^n's uniformly (cf. [23] and [26]), which means that $p(X) > 1$ and $q(X) < \infty$.

5. Open problems

In this section, we mention two important open questions. First, the infinite-dimensional analogue of Theorem 1 or 2 is not known:

Problem 1. Let X be an arbitrary infinite-dimensional space. Is there a bounded projection P: $X \to X$ such that both P and $I - P$ have infinite-dimensional ranges? In other words, can any X be split into a non-trivial direct sum?

For an interesting particular case, see [9], page 226. More generally, although there are spaces with few finite rank operators (cf. Section 3), it is not known whether there is a space which admits few bounded operators. Precisely, the following is open:

Problem 2. Is there an infinite-dimensional space X such that every bounded operator u: $X \to X$ is of the form $\lambda Id_X + v$ with λ scalar and v nuclear?

Actually, this is unknown even if we only ask for a compact v. A related example (X non-separable and v's of separable ranges) is constructed in [36], using special axioms.

Of course, a positive answer to Problem 2 implies a negative one to Problem 1. Moreover, a separable space X as in Problem 2 would be the first example of a separable Banach space on which every bounded operator has a non-trivial invariant subspace.

References

[1] Benyamini Y. and Gordon Y., Random factorizations of operators between Banach spaces, *J. d'Analyse Jerusalem* **39** (1981), pp. 45–75.

[2] Bourgain J., New Banach Space Properties of the Disc Algebra and H^∞, to appear in *Acta Math.*

[3] Davis W. and Johnson W., Compact, Non-Nuclear Operators, *Studia Math.* **51** (1974), pp. 81–85.

[4] Enflo P., A Counterexample to the Approximation Problem in Banach Spaces, *Acta Math.* **130** (1973), pp. 309–317.

[5] Figiel T., Lindenstrauss J. and Milman V., The Dimensions of Almost Spherical Sections of Convex Bodies, *Acta Math.* **139** (1977), pp. 53–94.

[6] Figiel T. and Tomczak-Jaegermann N., Projections onto Hilbertian Subspaces of Banach Spaces, *Israel J. Math.* **33** (1979), pp. 155–171.

[7] Gluskin E., The Diameter of the Minkowski Compactum is Roughly Equal to n, *Funct. Anal. Appl.* **15** (1981), pp. 72–73.

[8] Gluskin E., to appear.

[9] Graham C. and Mc. Gehee C., *Essays in Commutative Harmonic Analysis*, Springer Verlag, 1979.

[10] Grothendieck A., Résumé de la théorie métrique des produits tensoriels topologiques, *Bol. Soc. Matem.* Sao-Paulo **8** (1956), pp. 1–79.

[11] Grothendieck A., *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the A.M.S. **16** (1955).

[12] Hoffmann-Jørgensen J., *Probability in Banach Spaces*, Ecole d'Eté de St. Flour. VI – 1976, Springer Lecture Notes **598**.

[13] James R. C., Uniformly Non Square Banach Spaces, *Annals of Maths.* **80** (1964), pp. 542–550.

[14] John F., Extremum Problems with Inequalities as Subsidiary Conditions, *Courant Anniversary Volume*, pp. 187–204, Interscience, New-York 1948.

[15] Johnson W. and Tzafriri L., On the Local Structure of Subspaces of Banach Lattices., *Israel J. Math.* **20** (1975), pp. 292–299.

[16] Kisliakov S. V., On Spaces with "Small" Annihilators, Zap. Naucn. Sem. Leningrad. Otdel. Math. Institute Steklov (LOMI) 65 (1976), pp. 192–195 (in Russian).

[17] Krivine J. L., Sous-espaces de dimension finie des espaces de Banach réticulés, *Annals of Math.* **104** (1976), pp. 1–29.

[18] König H., Retherford J. and Tomczak-Jaegermann N., On the Eigenvalues of $(p, 2)$-Summing Operators and Constants Associated to Normed Spaces, *Journal of Funct. Analysis* **37** (1980), pp. 88–126.

[19] Lewis D., Finite Dimensional Subspaces of L_p, *Studia Math.* **63** (1978), pp. 207–212.
[20] Lindenstrauss J., The Geometric Theory of the Classical Banach Spaces, *Actes Congr. Internat. Math.* Nice, 1970, Vol. 2, pp. 365–372.
[21] Maurey B., Quelques problémes de factorisation opérateurs linéaires, *Actes. Congrès Intern. Math.* Vancouver, 1974, Vol. 2, p. 75.
[22] Maurey B. and Pisier G., Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach, *Studia Math.* **58** (1976), pp. 45–90.
[23] Milman V. and Wolfson H., Minkowski Spaces with Extremal Distance from the Euclidian Space, *Israel J. Math.* **29** (1978), pp. 113–131.
[24] Pisier G., Une nouvelle classe d'espaces vérifiant le théorème de Grothendieck, *Annales de l'Inst. Fourier,* **28** (1978), pp. 69–80.
[25] Pisier G., Un théorème sur les opérateurs linéaires entre espaces de Banach qui se factorisent par un espace de Hilbert, *Annales de l'E.N.S.,* **13** (1980), pp. 23–43.
[26] Pisier G., *Sur les espaces de Banach de dimension finie à distance extrémale d'un espace euclidien, d'après V. D. Milman et H. Wolfson,* Exposé n° 16, Séminaire d'Analyse Fonctionnelle 78/79, Ecole Polytechnique–Palaiseau.
[27] Pisier G., Some applications of the complex interpolation method to Banach lattices, *Juornal d'Anal. Math. Jerusalem* **35** (1979), pp. 264–281.
[28] Pisier G., Holomorphic Semi-Groups and the Geometry of Banach Spaces, *Annals of Math.* **115**, (1982), pp. 375–392.
[29] Pisier G., On the Duality Between Type and Cotype, Proceedings of a conference, *Martingale Theory in Harmonic Analysis and Banach Spaces,* Cleveland, July 1981, Lecture Notes 939, Springer.
[30] Pisier G., Counterexamples to a Conjecture of Grothendieck, *Acta Math.* **151** (1983), pp. 181–208.
[31] Szankowski S., Subspaces Without the Approximation Property, *Israel J. Math.* **30** (1978), pp. 123–129.
[32] Szarek S., The Finite Demensional Basis Problem with an Appendix on Nets of Grassmann Manifold. *Acta Math.* **151** (1983), pp. 153–180.
[33] Tomczak-Jaegermann N., On the Moduli of Smoothness and Convexity and the Rademacher Averages of the Trace Classes S_p $(1 \leqslant p < \infty)$, *Studia Math.* **50** (1974), pp. 163–182.
[34] Tzafriri L., On Banach Spaces with Unconditional Basis, *Israel J. Math.* **17** (1974), pp. 84–93.
[35] Bellenot S., Uniformly Complemented l_p^n's in Quasi-Reflexive Spaces, *Israel J. Math.* **39** (1981), pp. 234–246.
[36] Shelah S., A Banach Space with Few Operators, *Israel J. Math.* **30** (1978), pp. 181–191.
[37] Tomczak-Jaegermann N., Computing 2-Summing Norms with few Vectors, *Ark. Math.* **17** (1979), pp. 173–177.