Sheaf Theory for Partial Differential Equations

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0. Introduction

In order to analyze the singularities of hyperfunction solutions of systems of partial differential equations, M. Sato introduced in 1969 the microlocalization functor and, more fundamentally, the microlocal point of view. Then began an intense activity, in what is now called "microlocal analysis", and in the field of analytical partial differential equations the main tools, microdifferential operators and quantized contact transformations, were developed in Sato-Kawai-Kashiwara's paper [S-K-K]. However, after their study of micro-hyperbolic systems [K-S1], M. Kashiwara and the author realized that for many problems these analytical tools were not really necessary on the condition to work with the complex of holomorphic solutions of the system, $R \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$, and only to keep in mind the codirections of non-propagation of this complex, here the characteristic variety of \mathscr{M} . In other words one simply works with a complex of sheaves F on a real manifold X, and what we defined as its micro-support, SS(F), a closed conic involutive subset of T^*X .

This was the starting point of the "microlocal study of sheaves", developed in [K-S3, K-S4].

It is not our purpose to discuss this theory here, but we need to recall a few basic facts in order to introduce the new notion of an elliptic pair (obtained in collaboration with J.-P. Schneiders), a generalization of that of an elliptic system, the real manifold M on which the system is elliptic being replaced by an **R**-constructible sheaf. We construct a characteristic class associated with an elliptic pair, and prove that when the pair has compact support, the complex of its holomorphic solutions has finite dimensional cohomology and the index is calculated as the integral of the characteristic class.

1. Microlocal Study of Sheaves

In this section we fix some notations and recall a few results of [K-S3, K-S4].

Let X be a real C^{∞} -manifold. One denotes by $\tau : TX \to X$ and by $\pi : T^*X \to X$ its tangent and cotangent bundles, respectively. If M is a submanifold,

one denotes by $T_M X$ and $T_M^* X$ the normal and conormal bundles to M in X, respectively. In particular $T_X^* X$ is the zero-section of $T^* X$.

One denotes by $\delta : \Delta \hookrightarrow X \times X$ the diagonal embedding and we identify X with Δ and T^*X with $T^*_{\Delta}(X \times X)$ by the first projection defined on $X \times X$ and on $T^*(X \times X) \simeq T^*X \times T^*X$, respectively. If Δ is a subset of T^*X , Λ^a will denote its image by the antipodal map on T^*X .

If X and Y are two manifolds, one denotes by q_1 and q_2 the first and second projection, defined on $X \times Y$.

Let A be a commutative unitary ring with finite global homological dimension (e.g. $A = \mathbb{Z}$). One denotes by D(X) the derived category of the category of sheaves of A-modules on X, and by $D^b(X)$ the full subcategory consisting of objects with bounded cohomology. If Z is a locally closed subset of X, one denotes by A_Z the sheaf on X which is constant with stalk A on Z and zero on $X \setminus Z$. One denotes by or_X the orientation sheaf on X, and by ω_X the dualizing complex on X. Hence:

$$\omega_X \simeq or_X[\dim X],$$

where dim X is the dimension of X.

The "six operations" (as says Grothendieck), that is, the operations \otimes^L , *R* Hom, Rf_* , Rf_1 , f^{-1} , f^1 , are now classical tools that we shall not recall. We simply introduce some notations. For $F \in Ob(D^b(X))$ and $G \in Ob(D^b(Y))$, one sets:

$$F \boxtimes^{L} G = q_{1}^{-1}F \otimes^{L} q_{2}^{-1}G,$$
$$D'F = R \operatorname{Hom}(F, A_{X}),$$
$$DF = R \operatorname{Hom}(F, \omega_{X}).$$

There are other operations of interest on sheaves. If M is a closed submanifold of X and $F \in Ob(D^b(X))$, the specialization of F along M, $v_M(F)$, is an object of $D^b(T_M X)$ and the microlocalization of F along M, $\mu_M(F)$, an object of $D^b(T_M^* X)$. Sato's functor μ_M has been generalized in [K-S3] as follows. For F and G in $D^b(X)$ on sets:

$$\mu \hom(G, F) = \mu_{\Delta} R \, \mathscr{H}_{om}(q_2^{-1}G, q_1^{!}F).$$

Then:

$$R\pi_*\mu \hom(G, F) \simeq R \mathscr{H}_{om}(G, F)$$
,
 $\mu \hom(A_M, F) \simeq \mu_M(F)$.

After the introduction of the functor μ_M it became natural to work in T^*X , and M. Kashiwara and the author introduced in 1982 (cf. [K-S2]) the micro-support SS(F) of an object F of $D^b(X)$. This is a closed conic subset of T^*X which roughly speaking describes the set of codirections of non-propagation of F. More precisely:

Definition 1.1. We say that an open subset U of T^*X does not meet SS(F) if for any real C^1 -function φ on X and any $x_0 \in X$ such that $(x_0; d\varphi(x_0)) \in U$, one has:

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$$\left(R\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}(F)\right)_{x_0}=0.$$

An important property of the micro-support is given by:

Theorem 1.2. Let $F \in Ob(D^b(X))$. Then SS(F) is an involutive subset of T^*X .

(For the precise definition of "involutive", cf. [K-S4, Ch. VI].)

One can evaluate the micro-support of sheaves after the main operations described above. For example one proves that for F and G in $D^b(X)$:

$$SS(\mu \hom(G, F)) \subset C(SS(F), SS(G)), \tag{1.1}$$

where $C(\Lambda_1, \Lambda_2)$ is the normal cone of Λ_1 along Λ_2 , a closed subset of TT^*X that we identify with a subset of T^*T^*X by the Hamiltonian isomorphism. In particular:

$$\operatorname{supp}(\mu \operatorname{hom}(G, F)) \subset SS(G) \cap SS(F).$$
(1.2)

Let $f: Y \to X$ be a morphism of manifolds. One associates the maps:

$$T^*Y \xleftarrow{_{f'}} Y \underset{X}{\times} T^*X \xrightarrow{_{f_{\pi}}} T^*X \tag{1.3}$$

and one sets:

$$T_Y^* X = {}^{t} f'^{-1} (T_Y^* Y) . (1.4)$$

Using (1.1), one can evaluate the micro-support of $f^{-1}F$ or $f^{\dagger}F$ (cf. [K-S3]). In particular if f is non-characteristic for F, that is

$$T_Y^*X \cap f_\pi^{-1}(SS(F)) \subset Y \underset{X}{\times} T_X^*X, \qquad (1.5)$$

one gets:

$$SS(f^{-1}F) \subset {}^{t}f'_{\pi} (SS(F)).$$
 (1.6)

Similarly, if $G \in Ob(D^b(Y))$ and f is proper on supp(G), one proves:

$$SS(Rf_*G) \subset f_{\pi} f'^{-1}(SS(G)).$$
 (1.7)

Remark that formulas (1.6) and (1.7) are similar to classical formulas obtained when calculating the wave front set of distributions or hyperfunctions or when calculating the characteristic variety of \mathcal{D} -modules, after non characteristic inverse images of proper direct images.

2. Constructible Sheaves (cf. [K-S3, K-S4])

In this section we assume all manifolds are real analytic and the base ring A is noetherian. An object F of $D^b(X)$ is called weakly **R**-constructible(w-**R**-

constructible for short) if there exists a subanalytic stratification $X = \bigsqcup_{\alpha} X_{\alpha}$ such that for all α , all $j \in \mathbb{Z}$, the sheaves $H^{j}(F) \mid_{X_{\alpha}}$ are locally constant. If moreover for each $x \in X$, each $j \in \mathbb{Z}$, the stalk $H^{j}(F)_{x}$ is finitely generated, one says F is **R**-constructible. One denotes by $D^{b}_{w-R-c}(X)$ (resp. $D^{b}_{R-c}(X)$) the full subcategory of $D^{b}(X)$ consisting of w-**R**-constructible (resp. **R**-constructible) objects.

The involutivity Theorem 1.2 allows us to characterize microlocally w-R-constructible objects.

Theorem 2.1. Let $F \in Ob(D^b(X))$. The following conditions are equivalent.

(a) F is w-**R**-constructible.

(b) SS(F) is contained in a closed conic subanalytic isotropic subset of T^*X .

(c) SS(F) is a closed conic subanalytic Lagrangian subset of T^*X .

By this result one proves easily that the category of w-R-constructible (resp. **R**-constructible) sheaves is stable by the main operations on sheaves (Rf_* when f is proper, f^{-1} , $f^!$, \otimes^L , R Hom, μ_M , ν_M , μ hom).

If X is a complex manifold one defines similarly the notions of w-Cconstructible and C-constructible sheaves, by assuming that the stratas of the subanalytic stratification $X = \bigsqcup_{\alpha} X_{\alpha}$ are complex analytic submanifolds. Then the link between **R**- and C-constructibility is given by:

Theorem 2.2. Let $F \in Ob(D^b_{w-\mathbf{R}-c}(X))$. Then F is w-C-constructible if and only if SS(F) is conic for the action of \mathbb{C}^{\times} on T^*X .

Remark 2.3. Note that the microlocal study of constructible sheaves was initiated by Kashiwara [K1].

3. *D*-Modules

We shall not review this theory here and refer to [S-K-K, K1, S1] for detailed expositions. We shall only fix a few notations and make the link with the micro-support.

From now on the base ring A is the field C of complex numbers.

Let (X, \mathcal{O}_X) be a complex manifold of complex dimension *n*. One denotes by \mathscr{D}_X (resp. \mathscr{D}_X^{∞}) the sheaf on X of finite order (resp. infinite order) holomorphic differential operators and one sets $\Omega_X = \mathscr{O}_X^{(n)} \otimes or_X$, where $\mathscr{O}_X^{(n)}$ is the sheaf of holomorphic *n*-forms. One denotes by $D(\mathscr{D}_X)$ (resp. $D(\mathscr{D}_X^{op})$) the derived category of the abelian category of left (resp. right) \mathscr{D}_X -modules, and by $D^b(\mathscr{D}_X)$ (resp. $D_{\mathrm{coh}}^b(\mathscr{D}_X)$) the full triangulated subcategory of $D(\mathscr{D}_X)$ consisting of objects with bounded (resp. bounded and coherent) cohomology. One defines similarly $D^b(\mathscr{D}_X^{op})$ and $D_{\mathrm{coh}}^b(\mathscr{D}_X^{op})$.

If \mathscr{M} is an object of $D^b_{\rm coh}(\mathscr{D}_X)$, its characteristic variety denoted char(\mathscr{M}), is a closed conic analytic subset of T^*X , which is involutive ([S-K-K]). In fact one has:

Theorem 3.1 ([K-S2]). Let $\mathcal{M} \in Ob(D^b_{coh}(\mathcal{D}_X))$. Then:

 $SS(R \operatorname{Hom}_{\mathscr{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \operatorname{char}(\mathcal{M}).$

Note that the inclusion \subset in Theorem 3.1, which is the most useful for applications, is easily deduced from the Cauchy-Kowalevski theorem, in its precised form due to Leray [L]. The converse inclusion makes use of the sheaf of rings $\mathscr{E}_X^{\mathbf{R}}$ of [S-K-K]. Also note that this result, combined with Theorem 1.2, gives a new proof of the involutivity of the characteristic variety of \mathscr{D} -modules.

Let $f: Y \to X$ be a morphism of complex manifolds. One denotes by $\mathscr{D}_{Y \to X}$ the sheaf $\mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_X} f^{-1}\mathscr{D}_X$ endowed with its natural structure of a $(\mathscr{D}_Y, f^{-1}\mathscr{D}_X)$ -bimodule.

Let $\mathcal{M} \in \operatorname{Ob}(D^b(\mathcal{D}_X))$. One sets:

 $\underline{f}^{-1}\mathcal{M} = \mathcal{D}_{\mathbf{Y} \to \mathbf{X}} \otimes_{f^{-1}\mathcal{D}_{\mathbf{X}}}^{L} f^{-1}\mathcal{M}.$

Let $\mathcal{N} \in \mathrm{Ob}(D^b(\mathcal{D}_Y^{op}))$. One sets

$$\underline{f}_*\mathcal{N} = Rf_*(\mathcal{N} \otimes^L_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}).$$

If $\mathcal{M} \in \mathrm{Ob}(D^b(\mathcal{D}_X))$ and $\mathcal{N} \in \mathrm{Ob}(D^b(\mathcal{D}_Y))$ one sets:

$$\mathscr{M}\underline{\boxtimes}\mathscr{N}=\mathscr{D}_{X\times Y}\otimes_{\mathscr{D}_X\underline{\boxtimes}\mathscr{D}_Y}(\mathscr{M}\boxtimes\mathscr{N}),$$

and there is a similar formula for right modules.

Finally one sets:

$$\begin{split} \underline{D}'\mathcal{M} &= R \, \mathcal{H}\!om_{\mathcal{D}_{X}}(\mathcal{M}, \mathcal{D}_{X}), \\ D\mathcal{M} &= R \, \mathcal{H}\!om_{\mathcal{D}_{X}}(\mathcal{M}, \Omega_{X} \otimes_{\mathcal{O}_{Y}} \mathcal{D}_{X}[n]). \end{split}$$

(In this last formula, \mathcal{M} is a right module.)

4. Microfunction Solutions of *D*-Modules

Let M be a real analytic manifold, X a complexification of M, \mathcal{M} a left coherent \mathcal{D}_X -module. By considering the complex $R \mathscr{H}_{\mathcal{M}\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and using Theorem 3.1, one may recover many classical results. For example, applying (1.2) with $G = D'(\mathbb{C}_M)$ one gets:

$$\operatorname{supp}(R \operatorname{\mathscr{H}om}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_M)) \subset T^*_M X \cap \operatorname{char}(\mathscr{M}),$$

where \mathscr{C}_M is the sheaf of Sato microfunctions. In particular this shows that the analytic wave front set of a hyperfunction solution of a system of linear differential equations is contained in the characteristic variety of the system ("Sato's principle"). More generally, the inclusion (1.1) immediately implies that the microfunction solutions of the system \mathscr{M} extend in the micro-hyperbolic directions, and one recovers the results of [K-S1] in the differential case. Microdifferential

systems can be treated similarly, once the microlocal action of $\mathscr{E}_X^{\mathbb{R}}$ on \mathscr{O}_X is defined as in [K-S3].

These techniques can also be applied to the study of boundary value problems and diffraction, including the case of non smooth obstacles. It is then useful to introduce new sheaves of microfunctions (using the functor μ hom) and new wave front sets. We refer to [S2] for details.

5. Elliptic Pairs

In this section we expose new results obtained in collaboration with J.-P. Schneiders. Let X be a complex manifold of complex dimension n. If there is no risk of confusion, we identify X with the real underlying manifold.

Definition 5.1. An elliptic pair (\mathcal{M}, F) on X is the data of $\mathcal{M} \in Ob(D^b_{coh}(\mathcal{D}_X))$ and $F \in Ob(D^b_{\mathbf{R}-c}(X))$ satisfying: char $(\mathcal{M}) \cap SS(F) \subset T^*_X X$.

We use the same terminology for objects of $D^{b}_{\rm coh}(\mathscr{D}^{op}_{X})$.

Theorem 5.2 (cf. [S-Sc]). Let (\mathcal{M}, F) be an elliptic pair. (i) Regularity. The natural morphism:

$$R \operatorname{Hom}_{\mathfrak{D}_X}(\mathcal{M}, D'F \otimes \mathcal{O}_X) \to R \operatorname{Hom}_{\mathfrak{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$$

is an isomorphism.

(ii) Assume $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(F)$ is compact.

(a) Finiteness. For all $j \in \mathbb{Z}$, the C-vector spaces

$$H^{j}R\Gamma(X; R \operatorname{Hom}_{\mathscr{D}_{X}}(\mathcal{M} \otimes F, \mathcal{O}_{X}))$$

are finite dimensional.

(b) Duality. The pairing

$$R \operatorname{Hom}_{\mathscr{D}_X}(\mathscr{M} \otimes F, \mathscr{O}_X) \otimes (\Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M} \otimes F) \longrightarrow \Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{O}_X$$

and the integration morphism $H^n_c(X; \Omega_X \otimes_{\mathscr{D}_X}^L \mathscr{O}_X) \simeq H^{2n}_c(X; or_X) \longrightarrow \mathbb{C}$ induce a perfect duality on the spaces $H^j R\Gamma(X; R \mathscr{H}_{om_{\mathscr{D}_X}}(\mathcal{M} \otimes F, \mathscr{O}_X))$ and $H^{n-j} R\Gamma(X; \Omega_X \otimes_{\mathscr{D}_X}^L \mathscr{M} \otimes F).$

(c) Parameters. Let Y be another complex manifold. Then the natural morphism

$$(Rq_{2*}q_1^{-1}R \operatorname{\mathscr{H}om}_{\mathscr{D}_X}(\mathscr{M} \otimes F, \mathscr{O}_X)) \otimes \mathscr{O}_Y \longrightarrow Rq_{2*}R \operatorname{\mathscr{H}om}_{q_1^{-1}\mathscr{D}_X}(q_1^{-1}(\mathscr{M} \otimes F), \mathscr{O}_{X \times Y})$$

is an isomorphism.

Sketch of Proof. (i) follows from a general result of [K-S4] which asserts that the natural morphism

$$R \operatorname{Hom}(F, A_X) \otimes^L G \to R \operatorname{Hom}(F, G)$$

is an isomorphism as soon as $SS(F) \cap SS(G)$ is contained in T_X^*X .

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(ii) Using techniques of [Sc] one can reduce the problem to the case where \mathcal{M} admits a free presentation. Next by adding the Cauchy-Riemann system to \mathcal{M} , one can assume F is supported by a real analytic manifold \mathcal{M} whose complexification is X. Then one represents F by a bounded complex whose components are direct sums of sheaves C_U , U open, relatively compact, subanalytic in \mathcal{M} and such that $D'_{\mathcal{M}} C_U = C_{\overline{U}}$, (here $D'_{\mathcal{M}}$ is the duality functor on \mathcal{M}). Then (ii) is proved by similar arguments as those in [B-S] or [R-R].

By adapting Kashiwara's construction of the characteristic cycle of \mathbf{R} constructible sheaves (cf. [K-S4, Ch. IX]) we shall now construct a characteristic
class associated with an elliptic pair.

Let (\mathcal{M}, F) be an elliptic pair and assume \mathcal{M} is a right module. Sato's isomorphism $\mathscr{D}_X^{\infty} \simeq \delta^{\dagger} \mathscr{O}_{X \times X}^{(0,n)}[n]$ induces a morphism:

$$R \operatorname{Hom}_{\mathscr{D}_{X}}(\mathscr{M} \otimes F, \mathscr{M} \otimes F) \to \delta^{!}(\underline{D}(\mathscr{M} \otimes F) \underline{\boxtimes}(\mathscr{M} \otimes F) \otimes^{L}_{\mathscr{D}_{X \times X}} \mathcal{O}_{X \times X}).$$
(5.1)

Moreover the natural morphism:

$$\Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X[n] \otimes^L_{\mathscr{D}_X \otimes \mathscr{D}_X} \delta^{-1} \mathscr{O}_{X \times X} \to \Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{O}_X[n]$$

induces a morphism:

$$\delta^{-1}(\underline{D}(\mathscr{M} \otimes F) \boxtimes (\mathscr{M} \otimes F) \otimes^{L}_{\mathscr{D}_{X \times X}} \mathscr{O}_{X \times X}) \to \omega_{X}.$$
(5.2)

Set for short:

$$H = \underline{D}(\mathscr{M} \otimes F) \boxtimes (\mathscr{M} \otimes F) \otimes_{\mathscr{D}_{X \times X}}^{L} \mathscr{O}_{X \times X}.$$
(5.3)

We get the chain of morphisms:

$$\begin{array}{l} R \operatorname{\mathscr{H}om}_{\mathscr{D}_{X}}(\mathscr{M} \otimes F, \mathscr{M} \otimes F) \to \delta^{1} H \\ \to \delta^{1} \delta_{*} \delta^{-1} H \cdot \simeq & \delta^{-1} H \\ \to \omega_{X} \, . \end{array}$$

Hence setting:

$$S = \operatorname{char}(\mathcal{M}) \cap \operatorname{supp}(F), \qquad (5.4)$$

we get a morphism:

$$\operatorname{Hom}_{\mathscr{D}_{X}}(\mathscr{M} \otimes F, \mathscr{M} \otimes F) \to H^{0}_{\mathcal{S}}(X; \omega_{X}).$$
(5.5)

In fact this construction can be made "microlocal". Set:

$$\Lambda = \operatorname{char}(\mathcal{M}) + SS(F)^a.$$
(5.6)

Since the micro-support of H is contained in $\Lambda \times \Lambda^a$, we have the isomorphisms:

$$\delta^{1}H \simeq R\pi_{*}\mu_{\Delta}H$$
$$\longleftrightarrow R\pi_{*}R\Gamma_{\Lambda}\mu_{\Delta}H,$$

and we get the morphism:

$$\operatorname{Hom}_{\mathscr{D}_{X}}(\mathscr{M}\otimes F,\mathscr{M}\otimes F)\to H^{0}_{A}(T^{*}X;\pi^{-1}\omega_{X}).$$
(5.7)

Definition 5.3. We call the image of 1 in $H^0_S(X; \omega_X)$ (resp. in $H^0_A(T^*X; \pi^{-1}\omega_X)$) by the morphism (5.5) (resp. (5.7)) the Euler class (resp. the microlocal Euler class) of the elliptic pair (\mathcal{M}, F) and we denote it by $eu(\mathcal{M}, F)$ (resp. $\mu eu(\mathcal{M}, F)$).

Of course $eu(\mathcal{M}, F)$ is the restriction of $\mu eu(\mathcal{M}, F)$ to the zero-section of T^*X .

Definition 5.4. Let (\mathcal{M}, F) be an elliptic pair such that $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(F)$ is compact and assume \mathcal{M} is a right module. One sets:

$$\chi(X;\mathscr{M},F) = \sum_{j} (-1)^{j} \dim(H^{j}(R\Gamma(X;F\otimes \mathscr{M}\otimes^{L}_{\mathscr{D}_{X}}\mathcal{O}_{X}))).$$

By adapting to the case of \mathcal{D} -modules Kashiwara's proof of the index theorem for constructible sheaves (cf. [K-S4, Ch. IX]), we can prove:

Theorem 5.5. In the situation of Definition 5.4, one has:

$$\chi(X;\mathcal{M},F) = \int_X \operatorname{eu}(\mathcal{M},F).$$
(5.8)

Examples and Comments 5.6. (a) Assume X is the complexification of a real analytic manifold M, and let \mathcal{M} be a coherent \mathcal{D}_X -module. Then \mathcal{M} is elliptic on M in the classical sense if an only if (\mathcal{M}, C_M) is an elliptic pair, which simply means that:

$$\operatorname{char}(\mathcal{M}) \cap T^*_{\mathcal{M}} X \subset T^*_{\mathcal{X}} X.$$

In this case the isomorphism of Theorem 5.2 applied to the elliptic pair $(\mathcal{M}, D'C_M)$ gives the isomorphism:

$$R \operatorname{Hom}_{\mathfrak{D}_X}(\mathcal{M}, \mathcal{A}_M) \simeq R \operatorname{Hom}_{\mathfrak{D}_X}(\mathcal{M}, \mathcal{B}_M)$$

where \mathscr{A}_M (resp. \mathscr{B}_M) denotes the sheaf of real analytic functions (resp. hyperfunctions) on M. If M is compact, one recovers the classical finiteness theorem for elliptic systems, and the index is calculated by the Atiyah-Singer theorem [A-S].

(b) Let Ω be an open subset of X with real analytic boundary. Then $(\mathcal{M}, \mathbb{C}_{\Omega})$ is an elliptic pair if and only if $\partial \Omega$ is non-characteristic for \mathcal{M} , that is:

$$\operatorname{char}(\mathscr{M}) \cap T^*_{\partial\Omega} X \subset T^*_X X.$$

If Ω is relatively compact, one gets the finiteness of the spaces $\operatorname{Ext}_{\mathscr{D}_X}^j(\Omega, \mathscr{M}, \mathscr{O}_X)$, a result of Bony and Schapira [B-S] (in case $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X.P$, cf. Kashiwara [K1] and Kawai [Ka] for various generalizations), extended to the relative case by Houzel and Schapira [H-S], the index being calculated by Boutet de Monvel and Malgrange [B-M].

(c) For any $\mathcal{M} \in \operatorname{Ob}(\mathcal{D}^b_{\operatorname{coh}}(\mathcal{D}_X))$, $(\mathcal{M}, \mathbf{C}_X)$ is an elliptic pair. In this case the duality theorem is due to Mebkhout [M]. If \mathcal{M} has compact support one recovers many classical results. In particular if \mathcal{G} is a coherent \mathcal{O}_X -module with compact support, one can apply the theorem with $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{G}$ and recover theorems of Cartan and Serre (cf. [C-S, Se]). Concerning the index, let us recall that O'Brian, Toledo and Tong [O-T-T], generalizing the Hirzebruch-Riemann-Roch formula [H], constructed the Chern class of coherent \mathcal{O}_X -modules with compact support, and proved an index theorem in this case. For the case of \mathcal{D}_X -modules with compact support, cf. Angéniol-Lejeune [A-L].

(d) For any $F \in Ob(D^b_{\mathbf{R}-c}(X))$, (\mathcal{O}_X, F) is an elliptic pair and its microlocal Euler class coincides with the Lagrangian cycle of F defined by Kashiwara in [K2]. Remark that if G is an **R**-constructible object on a real manifold M, one can associate to it an elliptic pair, namely (\mathcal{O}_X, i, G) where $i : M \hookrightarrow X$ is a complexification of M. In this case the index is calculated by Kashiwara (loc. cit.) (cf. also Dubson [D] and Ginsburg [G] in the complex case).

(e) Let $B(x_0; \varepsilon)$ denote the open ball (in a local chart at x_0) with center x_0 and radius ε and let \mathscr{M} be an holonomic \mathscr{D}_X -module. Then for ε small enough, $(\mathscr{M}, C_{B(x_0;\varepsilon)})$ is an elliptic pair (cf. [K1]).

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