Operator Algebras and Duality

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The Pontrjagyn duality associates to an abelian locally compact group a dual group and studies the properties of this correspondence. A natural idea is to try and generalize this duality to nonabelian groups, in particular to define an object dual to a group. Such dual objects were defined, first for compact groups [45, 19] then for locally compact groups ([38], [46]). Up to a large extent, it was the search of such a dual object for general locally compact groups that led to the theory of C^*-algebras. Then appeared the need of objects generalizing groups as well as their dual objects. These general objects can be called in a modern language "quantum groups". These "groups" can be studied as abstract groups, Lie groups, deformations of true groups .... It is certainly beyond our goals to review all aspects of the theory of "quantum groups" (see [3] and references therein). We will in fact concentrate our attention to the operator algebra approach, in other words to the study of the "locally compact quantum groups".

In terms of operator algebras Pontrjaygn duality takes the form of Takesaki-Takai duality [43, 40] based on the construction of W^*- and C^*-crossed-products. Along the years, under conjugated efforts of many specialists a set of axioms was built [11, 12, 42, 47, 16, 5] and duality was obtained [43, 20, 21, 30, 39, 4, 10, 6] for von Neumann algebras obeying these axioms called Kac von Neumann algebras. In a recent fundamental work [51–54], Woronowicz defined some objects that he called "compact matrix pseudogroups". Although they aren't Kac algebras, Woronowicz' "pseudo-groups" enjoy duality properties. Further examples with the same properties were given by Majid [26] and Podles-Woronowicz [32]. One of the motivations of this report is to describe a setting including both the Kac von Neumann algebras and these new examples, in which the duality results still hold.

Let H be a Hilbert space and $V \in L(H \otimes H)$ a unitary operator. Let us say V is multiplicative if it satisfies the pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$. This relation appears in the framework of categories with associative tensor product (cf. [24, 25]); it is the one satisfied by the fusion operator (cf. [29]). It is also very similar to the Yang-Baxter equation and in some sense more primitive. In many papers concerned with operator algebras possessing duality properties, a multiplicative unitary plays a fundamental role (e.g. [12, 42, 16, 5, 18, 10]...); it is clear and more or less explicit in these papers that this unitary describes the whole situation.

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It is therefore very natural to look for additional conditions that a mutliplicative unitary should satisfy in order to correspond to a "locally compact quantum group". Studying this problem, we were led to consider two conditions that we call *regularity* and *irreducibility*. We show how two pairwise dual Hopf C^*-algebras can be associated to a regular multiplicative unitary. When moreover this unitary is irreducible, we establish Takesaki-Takai duality results generalizing the previous ones.

An advantage of our approach is that a "quantum group" and its dual play completely symmetric roles. Also, it treats simultaneously the C^*- and W^*-algebra point of view. In fact, in our approach it is clear that for "locally compact quantum groups" the measure theory determines the topology. In some sense this can be thought of as a generalization of the famous theorem of Weil ([50], see also [23]): a "measurable quantum group" with an invariant (class of) measure(s) carries a unique structure of "locally compact quantum group".

Let us also mention that many algebraic constructions can be performed in our setting. In particular, we may associate a "quantum double" to any (irreducible) multiplicative unitary, and together with it, comes a solution of the quantum Yang Baxter equation (cf. [3]).

The question of the minimality of our axioms remains still unanswered: is a multiplicative unitary automatically regular? irreducible? Does one of these properties imply the other? Partial solutions to these questions were obtained: when the Hilbert space H is finite dimensional and when the unitary V satisfies a commutativity condition, regularity and irreducibility are both automatic; if the unitary V is of compact or discrete type (in other words if the associated quantum group is compact or discrete) its regularity implies its irreducibility.

In this report, we will first present some examples of occurrence of multiplicative unitaries, then explain the conditions of regularity and irreducibility and their consequences; we will finally construct the multiplicative unitaries associated with the examples of [26, 32] and discuss possible future developments. All the proofs, as well as more precise statements of the results given here can be found in [2].

1. Multiplicative Unitaries and Hopf Algebras

Let H be a Hilbert space. We will say that a unitary operator V acting on the tensor square $H \otimes H$ is *multiplicative* if it satisfies the pentagon equation:

$$V_{23} V_{12} = V_{12} V_{13} V_{23}$$

Here, by V_{12}, V_{23} and V_{13} we denote the operators $V \otimes 1_H$, $1_H \otimes V$ and $(1_H \otimes \Sigma)(V \otimes 1_H)(1_H \otimes \Sigma)$ acting on $H \otimes H \otimes H$, where Σ is the "flip" operator defined by $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ $(\xi, \eta \in H)$.

Note that the identity operator $1_{H \otimes H}$ is a multiplicative unitary. The importance of multiplicative unitaries in connection with operator algebras possessing duality properties was shown by many authors [12, 42, 16, 5, 10]. The multiplicative unitary associated with a locally compact group is constructed as follows:

Let G be a locally compact group and let m denote its right Haar measure. Let $H = L^2(G; m)$ be the Hilbert space of square integrable functions on G with respect with the measure m. Identify $H \otimes H$ with the space $L^2(G \times G; m \times m)$. Let then V be the operator acting on $H \otimes H$ by the formula $V(f)(s, t) = f(st, t)$ for every square integrable function f on $G \times G$ and $s, t \in G$. This operator is clearly unitary and its "multiplicativity" follows from the associativity of the composition law of G.

Operators satisfying this pentagon equation are naturally associated with Hopf algebras. Recall that a Hopf C^*-Algebra is a C^*-algebra A endowed with a coproduct which is a $*$-homomorphism $\delta : A \to A \otimes A$[1] satisfying the associativity condition: $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta$. Let us describe three different ways for interpreting the pentagon relation:

a) Haar States and GNS Representations

A Haar state on a Hopf C^*-algebra is a state $\phi \in A^*$ such that for any form $\psi \in A^*$ we have $(\phi \otimes \psi) \circ \delta = (\psi \otimes \phi) \circ \delta = \psi(1)\phi$. Let then $(H_\phi, \pi_\phi, \xi_\phi)$ be the GNS construction associated with ϕ. Then the operator V_ϕ defined by $V_\phi(\pi_\phi(x)\xi_\phi \otimes \eta) = (\pi_\phi \otimes \pi_\phi)(\delta(x))(\xi_\phi \otimes \eta)$ is an isometry of $H_\phi \otimes H_\phi$ satisfying the pentagon equation. In particular, if V_ϕ is surjective, it is a multiplicative unitary.

b) Covariant Representations

Let (A, δ) be a Hopf C^*-algebra. A corepresentation of A in a Hilbert space H is a unitary $u \in L(H \otimes A)$ of the Hilbert A-module $H \otimes A$ satisfying the relation: $(\mathrm{id} \otimes \delta)(u) = u_{12}u_{13}$.

A coaction of A on some C^*-algebra B is a $*$-homomorphism $\delta_B : B \to B \otimes A$ satisfying the associativity condition: $(\delta_B \otimes \mathrm{id}) \circ \delta_B = (\mathrm{id} \otimes \delta) \circ \delta_B$. A covariant representation of A, B on a Hilbert space H is a pair (π, u) where $\pi : B \to L(H)$ is a $*$-representation and $u \in L(H \otimes A)$ is a corepresentation of A such that $\forall b \in B$, $(\pi \otimes \mathrm{id}) \circ \delta_B(b) = u(\pi(b) \otimes 1)u^*$.

The coproduct δ is a coaction of A on itself. Let (π, u) be a covariant representation on the Hilbert space H. Then $V = (\mathrm{id} \otimes \pi)(u)$ is a multiplicative unitary.

c) The Canonical Element

Let A be a finite dimensional Hopf algebra. Let E be the algebra of endomorphisms of the vector space A. Let us denote by v the canonical element of $A^* \otimes A$: through the identification of $A^* \otimes A$ with E, v is the identity of A. Denote by L the action of A on A by left multiplication. If $x \in A^*$ and $a \in A$, set $\varrho(x)a = (\mathrm{id} \otimes x)\delta(a)$. Consider L and ϱ as homomorphisms from A and A^* into the algebra E. Simple computations then show:

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[1] This is the C^*-algebraic "min" tensor product. If A has no unit, δ takes its values in the multiplier algebra $M(A \otimes A)$ of $A \otimes A$.

– For $a \in A$ we have in $E \otimes A$, $(\varrho \otimes \mathrm{id})(v)(L(a) \otimes 1) = (L \otimes \mathrm{id})(\delta(a))(\varrho \otimes \mathrm{id})(v)$.
– We have the equality $(\mathrm{id} \otimes \delta)(v) = v_{12}v_{13}$.

Therefore, the operator $V = (\varrho \otimes L)(v)$ satisfies the pentagon equation.

2. Algebras Associated with Multiplicative Unitaries

Let $V \in L(H \otimes H)$ be a multiplicative unitary. If ω is a continuous linear form on $L(H)$, we may form the operators $L(\omega) = (\omega \otimes \mathrm{id})(V) \in L(H)$ and $\varrho(\omega) = (\mathrm{id} \otimes \omega)(V) \in L(H)$. Let $L(H)_*$ denote the predual of $L(H)$ i.e. the set of linear mappings of the form $x \to \mathrm{Tr}(xT)$ where T spans the space of trace class operators.

2.1. Proposition. *The sets $A(V) = \{L(\omega)/\omega \in L(H)_*\}$ and $\hat{A}(V) = \{\varrho(\omega)/\omega \in L(H)_*\}$ are subalgebras of $L(H)$.*

Indeed, $L(\omega)L(\omega') = (\omega \otimes \omega' \otimes \mathrm{id})(V_{13}V_{23}) = L(\psi)$ where $\psi(x) = (\omega \otimes \omega') \times (V^*(1 \otimes x)V)$

since $V_{12}V_{13} = V_{23}V_{12}V_{23}^*$; in the same way, $\varrho(\omega)\varrho(\omega') = (\mathrm{id} \otimes \omega \otimes \omega')(V_{12}V_{13}) = \varrho(\psi')$ where $\psi'(x) = (\omega \otimes \omega')(V(x \otimes 1)V^*)$. In fact, all properties which may be proved for $A(V)$ are automatically proved for $\hat{A}(V)$ since $\Sigma V^* \Sigma$ is a multiplicative unitary.

There is a natural duality between $A(V)$ and $\hat{A}(V)$ expressed by the equalities $\langle L(\omega), \varrho(\omega')\rangle = \omega(\varrho(\omega')) = \omega'(L(\omega)) = (\omega \otimes \omega')(V)$.

It is also natural to consider the norm closures of the algebras $A(V)$ and $\hat{A}(V)$ that we denote by S_V and $\hat{S}_V$. It is not clear whether these are always C^*-algebras ie. if they are closed under the involution $x \to x^*$ of $L(H)$. For this reason, we are led to make, in the next sections, some extra assumptions.

In the case of the multiplicative unitary associated with a group, the algebras $A(V)$ and $\hat{A}(V)$ are respectively the Fourier algebra $A(G)$ acting by multiplication on $L^2(G)$ and $L^1(G)$ acting by (right) convolution on $L^2(G)$. Also S_V is the abelian C^*-algebra $C_0(G)$ of continuous functions vanishing at infinity and $\hat{S}_V$ the reduced C^*-algebra of the group G. In particular the Gelfand spectrum of S_V is G: we already have recovered G out of the associated multiplicative unitary. In fact, we get a converse to this statement:

2.2 Theorem. *If the associated algebra $A(V)$ is commutative, the multiplicative unitary V is (up to multiplicity) the multiplicative unitary associated with a locally compact group.*

This theorem is a generalization theorem of [50, 23, 12, 41, 42, 47, 5, 52]. Of course, this theorem also classifies the multiplicative unitaries for which $\hat{A}(V)$ is commutative, since this is equivalent to saying that $A(\Sigma V^* \Sigma)$ is commutative.

Let us mention another case where no extra assumptions are needed:

2.3 Theorem. *A multiplicative unitary acting on a finite dimensional Hilbert space is (up to multiplicity) the multiplicative unitary associated with a finite dimensional Kac von Neumann algebra.*

3. Regularity; the "Compact" Case

Let us begin with a rather easy fact:

3.1 Proposition. *The set* $\mathscr{C}(V) = \{(\mathrm{id} \otimes \omega)(\Sigma\omega)/\omega \in L(H)_*\}$ *is a subalgebra of* $L(H)$.

Studying this algebra in the case of locally compact groups and more generally in the examples to be discussed below, we find that this algebra is formed of compact operators and is norm dense in the algebra of compact operators. This leads to the following definition:

3.2 Definition. *We will say that the multiplicative unitary* V *is* regular *if the norm closure of* $\mathscr{C}(V)$ *coincides with the algebra* $K(H)$ *of compact operators of.* H.

Regularity turns out to be extremely efficient in proving nice properties of the associated algebras:

3.3 Theorem. *Let* V *be a regular multiplicative unitary. Then the algebras* S *and* $\hat{S}$ *are Hopf* C^**-algebras with coproducts given by* $\delta(x) = V(x \otimes 1)V^*$ *and* $\hat{\delta}(y) = V^*(1 \otimes y)V\,(x \in S,\ y \in \hat{S})$. *The operator* V *is a multiplier of the (spatial) tensor product* $\hat{S} \otimes S$.

This last property means that the closed subalgebra of $L(H \otimes H)$ generated by $y \otimes x, x \in S, y \in \hat{S}$ is closed under left and right multiplication by V. It is quite natural and helpful. In particular, it allows us to consider S and $\hat{S}$ as abstract C^*-algebras and still make sense of V in every representation.

We are also in position to form crossed products for algebras with coactions of the Hopf algebra S: if a C^*-algebra A is endowed with a coaction $\delta_A : A \to A \otimes S$ of S, the (reduced) crossed product $A \times \hat{S}$ is the C^*-algebra of operators acting on the Hilbert A-module $A \otimes H$ generated by the products of the form $\delta_A(a)(1 \otimes y)$, $a \in A$, $y \in \hat{S}$. For $a \in A$, $y \in \hat{S}$, $\delta_A(a)$ and $(1 \otimes y)$ are multipliers of $A \times \hat{S}$. We thus get homomorphisms π and $\hat{\theta}$ from A and $\hat{S}$ respectively into the multiplier algebra of $A \times \hat{S}$. Still our set of axioms is not complete in order to allow us to prove the suitable duality. On the other hand, this duality may now be proved in the "compact" case.

3.4 Definition. *A multiplicative unitary is said to be of compact type if the unit operator belongs to the algebra* $A(V)$.

If V is a multiplicative unitary associated with a compact group or with a Haar state of a unital Hopf algebra, it is of compact type. In a recent fundamental work

[51–54], S.L. Woronowicz introduced a set of axioms for "quantum groups" generated by a finite dimensional unitary representation. Woronowicz initially called his objects "compact matrix pseudogroups" but they are referred to as "compact quantum Lie groups". Nice and tractable examples were produced [51, 53]. It is natural to define "compact quantum groups" as projective limits (this corresponds to inductive limits for the Hopf algebra of functions) of "compact quantum Lie groups". It was shown in [52] that "compact quantum groups" possess a Haar state. It is quite clear that the corresponding operator is unitary. Its regularity is also easy. The converse to these facts is true, namely:

3.5 Theorem. *A regular multiplicative unitary of compact type is (up to multiplicity) the multiplicative unitary associated with a "compact quantum group" of Woronowicz.*

4. Irreducibility and Takesaki-Takai Duality

In order to introduce the last condition needed for the duality, let us examine again the case of locally compact groups: we have been able to produce out of the multiplicative unitary associated with a locally compact group, the multiplication operators and the right regular representation. On $L^2(G)$ acts moreover the left regular representation; moreover, left and right regular representations are equivalent and intertwined by a unitary operator U given by $(U\xi)(g) = \Delta(g)^{1/2}\xi(g^{-1})$, where Δ is the module of the group.

This leads us to assume the existence of an operator U satisfying some equations:

4.1 Definition. a) *A multiplicative unitary $V \in L(H \otimes H)$ is said to be irreducible if there exists a unitary $U \in L(H)$ such that $U^2 = 1_H$, $(V(U \otimes 1)\Sigma)^3 = 1_{H\otimes H}$ and such that the unitary $\hat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ is multiplicative.*

b) *A Kac system is a triple (H, V, U) where H is a Hilbert space, $V \in L(H \otimes H)$ is a multiplicative unitary and $U \in L(H)$ satisfies the requirements of* a); *moreover, we require that V and $\hat{V}$ be regular.*

If (H, V, U) is a Kac system, $(H, \hat{V}, U)$ is also a Kac system and $\hat{\hat{V}} = (U \otimes U)V(U \otimes U)$. Taking the dual again, we find $\tilde{V} = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$; a fourth time will give us back V. This is the well noticed but still somewhat mysterious period 4 periodicity. (Note that as $\hat{\hat{V}} = (U \otimes U)V(U \otimes U)$ and $\tilde{V} = (U \otimes U)\hat{V}(U \otimes U)$ they are regular multiplicative unitaries).

We now have two representations of S and $\hat{S}$ in H: we will denote by $L : S \to L(H)$ and $\varrho : \hat{S} \to L(H)$ the inclusions considered up to now as identity representations; we will then set $R(x) = UL(x)U$ and $\lambda(y) = U\varrho(y)U$ $(x \in S,\ y \in \hat{S})$.

Replacing V by $\hat{V}$ we may now form crossed products for algebras with coactions of the Hopf algebra $\hat{S}$: if a C^*-algebra A is endowed with a coaction $\delta_A : A \to A \otimes \hat{S}$ of $\hat{S}$, the (reduced) crossed product $A \times S$ is the C^*-algebra of operators acting on the Hilbert A-module $A \otimes H$ generated by the products of the form

$$(\mathrm{id} \otimes \lambda)\delta_A(a)(1 \otimes L(x)), \qquad a \in A, \qquad x \in S.$$

We still have homomorphisms π and θ from A and S respectively into the multiplier algebra of $A \times S$, and $A \times S$ is spanned by products $\pi(a)\theta(x)$ $a \in A$, $x \in S$.

Let the C^*-algebra A be endowed with a coaction δ_A of S (resp. $\hat{S}$), then the crossed product $A \times \hat{S}$ (resp. $A \times S$) is endowed with a coaction $\hat{\delta}_A$ of $\hat{S}$ (resp. S) given by $\hat{\delta}_A(\pi(a)\hat{\theta}(y)) = (\pi(a) \otimes 1)(\hat{\theta} \otimes \mathrm{id})\hat{\delta}(y)$, $a \in A$, $y \in \hat{S}$ (resp. $\hat{\delta}_A(\pi(a)\theta(x)) = (\pi(a) \otimes 1)(\theta \otimes \mathrm{id})\delta(x)$, $a \in A$, $x \in S$).

4.2 Theorem (Takesaki-Takai Duality Theorem). *Let (H, V, U) be a Kac system. Then for any algebra A endowed with a coaction*[2] *δ_A of S the double crossed product $A \times \hat{S} \times S$ is naturally isomorphic with $A \otimes K(H)$.*

Remark. Replacing (H, V, U) by $(H, \hat{V}, U)$ we may exchange the roles of S and $\hat{S}$.

It is transparent in many papers (cf. eg. [21], [22], [39], [4], [10]) that Takesaki duality only relies on the "fundamental" operator; our proof is therefore just an adaption of methods used by these authors.

A first step to this duality is the case $A = \mathbf{C}$:

4.3 Lemma. *Let (H, V, U) be a Kac system. Then the closed vector span of $\{L(x)\varrho(y)/x \in S, y \in \hat{S}\}$ and the closed vector span of $\{L(x)\lambda(y)/x \in S, y \in \hat{S}\}$ are the algebra $K(H)$.*

This lemma, which can be thought of as a generalization of the famous Weyl-von Neumann theorem, explains the terminology of irreducibility.

Remark. It is also quite easy to prove a Takesaki duality theorem in the von Neumann algebra setting for Kac systems. In fact the regularity can be replaced by the weaker condition: the weak closure of $\mathscr{C}(V)$ is $L(H)$. The proof (if not the precise statement) of the main theorem of [10] applies in this context.

Also, one may generalize results of [1] and prove a Takesaki-Takai duality theorem for equivariant KK-theory with respect to the Hopf C^*-algebras S and $\hat{S}$.

5. Examples of Majid and Podles-Woronowicz

In [26] and [32] appeared a series of new constructions of interesting "quantum groups". These "quantum groups" are not in general Kac von Neumann algebras but they can still be expressed by a multiplicative unitary; in this way Takesaki-Takai duality is just an easy check.

The algebraic setting in the examples of [26, 27] and [32] is that of matched pairs of Hopf algebras (cf. [23, 28]). To such a matched pair are associated two new Hopf algebras: the one "generated by the matched pair" and the "bicrossproduct". These constructions were given in [37] and [28] in purely algebraic terms but may be performed in the multiplicative unitary setting. Examples of such matched pairs

[2] Provided a technical assumption called non-degeneracy in [21] is fulfilled by δ_A.

are given by any (locally compact) group G with two (closed) subgroups H and K such that every element g of G admits a unique decomposition $g = hk$ ($h \in H$, $k \in K$ – [44, 26]). Other examples are given by the "quantum double" construction of Drinfeld [3]. The examples of Podles and Woronowicz [32] are in fact based on this "quantum double" construction.

Let (A, δ_A) and (B, δ_B) be two Hopf C^*-algebras. Consider the *-homomorphism $\delta_A \otimes \delta_B : A \otimes B \to A \otimes A \otimes B \otimes B$. In order to put a Hopf C^*-algebra structure on $A \otimes B$, we use a *-isomorphism $\tau : A \otimes B \to B \otimes A$ and wish to put $\delta = (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)(\delta_A \otimes \delta_B)$. For δ to be coassociative it is enough that the following condition be satisfied:

$$(C)(\tau \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \tau)(\delta_A \otimes \mathrm{id}_B) = (\mathrm{id}_B \otimes \delta_A)\tau \qquad \text{and}$$

$$(\mathrm{id}_B \otimes \tau)(\tau \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \delta_B) = (\delta_B \otimes \mathrm{id}_A)\tau$$

Condition (C) is stated in [32] and, from a dual point of view, in [37] and [28] (in purely algebraic terms).

5.1 Definition. *Let (A, δ_A) and (B, δ_B) be two Hopf C^*-algebras. An inversion on A, B is a *-isomorphism $\tau : A \otimes B \to B \otimes A$ satisfying the conditions (C).*

Let (A, B, τ) be as in definition 5.1. Note that $b \to \tau(1 \otimes b)$ is a (right) coaction, called β, of the Hopf C^*-algebra A on the C^*-algebra B and $a \to \tau(a \otimes 1)$ is a (left) coaction, called α, of the Hopf C^*-algebra B on the C^*-algebra A.

Let $X \in L(H \otimes H)$ and $Y \in L(K \otimes K)$ be two regular multiplicative unitaries. Denote by S_X, $\hat{S}_X$, S_Y and $\hat{S}_Y$ the associated Hopf C^*-algebras associated with X and Y and by δ_X, $\hat{\delta}_Y$ the coproducts of S_X and $\hat{S}_Y$. Let $\tau : S_X \otimes \hat{S}_Y \to \hat{S}_Y \otimes S_X$ be an inversion on $(S_X, \hat{S}_Y)$.

5.2 Proposition. *The unitary operator $T = (\tau \otimes \mathrm{id})(Y_{23})(\mathrm{id} \otimes \tau)(X_{23})$ acting on $K \otimes H \otimes K \otimes H$ is multiplicative. It is called the bicrossproduct of X and Y with respect to τ.*

Let (H, X, u) and (K, Y, v) be two Kac systems. It is more natural to assume that τ is an inversion on (S_X, S_Y). Of course, in this case, we may form the bicrossproduct of X and $\hat{Y}$. In order to form the twisted and bicrossproducts of the Kac systems, we need the inversion τ to be suitably implemented.

5.3 Definition. *A matched pair of Kac systems is given by two Kac systems (H, X, u) and (K, Y, v) together with a unitary operator $Z \in L(H \otimes K)$ such that*

a) *There exists an inversion $\tau : S_X \otimes S_Y \to S_Y \otimes S_X$ such that for all $x \in S_X$, $y \in S_Y$ we have $Z\tau^{-1}(y \otimes x)Z^* = x \otimes y$.*

b) *$(H \otimes K, V, U)$ is a Kac system where $V = (Z_{12}^* X_{13} Z_{12}) Y_{24}$ and $U = (u \otimes v)Z$.*

5.4 Theorem. *Let $((H, X, u); (K, Y, v); Z)$ be a matched pair of Kac systems. Define the unitary operator $W = (Z_{34}{}^* \hat{Y}_{24} Z_{34})(Z_{12}{}^* X_{13} Z_{12})$ acting on $H \otimes K \otimes H \otimes K$.*

Then $(H \otimes K, W, U)$ *is a Kac system. The Hopf algebra* S *associated with* V *is* $(S_X \otimes S_Y, \delta_\tau)$. *The algebras* S *and* $\hat{S}$ *associated with* W *are isomorphic respectively to* $S_X \times_\alpha \hat{S}_Y$ *and* $S_Y \times_\beta S_X$.

Moreover, the multiplicative unitary operator T of proposition 5.2 is equivalent to W.

5.5 Definition. *With the notation of the above theorem, the Kac system* $(H \otimes K, V, U)$ *is called the product of* (H, X, u) *and* (K, Y, v) *twisted by* Z; *the Kac system* $(H \otimes K, W, U)$ *is called the bicrossproduct of* (H, X, u) *and* (K, Y, v) *relative to* Z.

5.6 Examples

a) Let (H, X, u) be a Kac system and G be a locally compact group acting by Hopf C^*-algebra automorphisms on S_X. Let $\tau : C_0(G) \otimes S_X \to S_X \otimes C_0(G)$ be given by $\tau(f)(x) = \alpha_x(f(x))$ $x \in G$, $f \in C_0(G; S_X)$ where we have identified $C_0(G) \otimes S_X$ and $S_X \otimes C_0(G)$ with the C^*-algebra $C_0(G; S_X)$ of continuous S_X-valued functions vanishing at ∞ on G. In this case, G acts naturally on the Hopf C^*-algebra $\hat{S}_X$ and the twisted and bicrossproducts are both obtained by the well known crossed-product constructions.

b) Let G_1 and G_2 be two locally compact groups. An inversion on $(C_0(G_1), C_0(G_2))$ is given by a homeomorphism $\tau : G_2 \times G_1 \to G_1 \times G_2$; then the product $(x_1, x_2)(y_1, y_2) = (x_1 z_1, z_2 x_2)$ where $(z_1, z_2) = \tau(x_2, y_1)$ is associative on $G_1 \times G_2$ and, endowed with this product, $G_1 \times G_2$ is a locally compact group G. Then, the twisted product of the associated Kac systems is the Kac system of the group G. The bicrossproduct construction gives new examples of Kac systems. In general, these examples are not associated with Kac von Neumann algebras [26, 27] and the antipode κ is unbounded. However, many computations may still be performed in this context.

Another way of understanding this example, is to start with a locally compact group G and assume that it has two closed subgroups G_1 and G_2 such that the map $(x_1, x_2) \to x_1 x_2$ is a homeomorphism from $G_1 \times G_2$ onto G. In this case, the actions of G_1 on G_2 and of G_2 on G_1 are the restrictions of the actions of G on $G_2 = G_1 \backslash G$ and on $G_1 = G/G_2$ and it is easy to compute the corresponding crossed products and thus the algebras S and $\hat{S}$ associated with W. Also, it is quite easy to construct groups with these properties:

- the Iwasawa decomposition $G = KP$ $(P = AN)$ of semisimple Lie groups;
- let G be a locally compact group acting by homeomorphisms on a locally compact group G_2 and containing the right translations of G_2; let then G_1 be the set of elements of G fixing the neutral element of G_2;
- in the above example, we may take G_2 to be any finite group and G be the group of all permutations of the set G_2 ...

A third way of interpreting this example (cf. [2] Appendix C) is the search of measure spaces X and transformations of $X \times X$ satisfying the pentagon relation.

It is then natural to add cocycles and form new multiplicative unitaries. In this way one recovers examples of Kac and Paliutkin [14, 15].

c) Let (H, X, u) be a Kac system and set $Y = sX^*s$ where $s \in L(H \otimes H)$ is the flip operator; then (H, Y, u) is a Kac system and $S_Y = \hat{S}_X$, $\hat{S}_Y = S_X$[3]. Let then $\tau : S_X \otimes S_Y \to S_Y \otimes S_X$ be given by $\tau(x) = XsxsX^*$. Since X is a multiplier of $S_Y \otimes S_X$ this is well defined. It turns out that τ is a non degenerate inversion and that $((H, X, u); (K, Y, v); Z)$ is a matched pair of Kac systems, where $Z = sX(u \otimes u)X^*(u \otimes u)s$. The corresponding twisted product $(H \otimes H, V, U)$ is the *quantum double* of the Kac system (H, X, u). Let S_V and $\hat{S}_V$ denote the corresponding Hopf C^*-algebras. There is a unitary operator R which is a multiplier of $\hat{S}_V \otimes \hat{S}_V$ which satisfies the algebraic properties of [3] and in particular R is a solution of the quantum Yang-Baxter equation.

Note that the construction of this twisted product was used by Podles and Woronowicz to build the "quantum Sl(2, C)" out of the "quantum SU(2)" ([32]).

In this case, the bicrossproduct is just a direct product.

6. Concluding Remarks

We developed here one point of view: find conditions easy to check on the "fundamental operator" that ensure Takesaki-Takai duality. However, we do not know if these conditions may turn out to be automatic.

Maybe one should look for a counterexample to regularity in transformations satisfying the pentagon equation.

The operator U defining irreducibility, is usually the product $\hat{J}J$ of the Tomita operators associated with Haar measures. Therefore, to prove irreducibility one would need to prove the existence of these Haar measures. Note that, in our context, this problem doesn't seem too difficult since we are given the regular representations and therefore the class of the Haar measures.

Once the Haar measures are found one needs to perform modular theory on them. Concerning this, we may formulate the following conjecture:

Call $\phi, \psi, \hat{\phi}$ and $\hat{\psi}$ the left and right Haar measures of S and $\hat{S}$. Then there should exist positive unbounded operators F and $\hat{F}$ affiliated with the centralizers of ϕ and $\hat{\phi}$ such that for all $x \in S$ and $y \in \hat{S}$, $\psi(x) = \phi(FxF)\,\hat{\psi}(y) = \hat{\phi}(\hat{F}y\hat{F})$. The Hilbert spaces H_ϕ, H_ψ, $H_{\hat{\phi}}$ and $H_{\hat{\psi}}$ are naturally identified. The weights ϕ, ψ, $\hat{\phi}$ and $\hat{\psi}$ are faithful when extended to the bicommutants, therefore Tomita theory can be performed. Call J and $\hat{J}$ the Tomita operators associated with ϕ and $\hat{\phi}$, and put $U = \hat{J}J = J\hat{J}$. Let L and λ be the GNS representations associated with ϕ and $\hat{\phi}$; then form R and ϱ using L, λ and U. Since V is a multiplier of $\hat{S} \otimes S$, $(\varrho \otimes L)(V)$ acts naturally on $H_\phi \otimes H_\phi$. Then (H_ϕ, V, U) is a Kac system. Moreover, the operators F and $\hat{F}$ are representations of the Hopf algebras, therefore they are unbounded multipliers of S and $\hat{S}$. Moreover, the operators $L(F)$, $R(F)$, $\lambda(\hat{F})$ and $\varrho(\hat{F})$ commute pairwise. The modular operators are computed in terms of F and $\hat{F}$. We find:

[3] Note however that the coproducts of S_Y and $\hat{S}_Y$ differ from the ones of $\hat{S}_X$ and S_X by the flip.

16. Kac, G.I., Vainerman, L.I.: Nonunimodular ring-groups and Hopf-von Neumann algebras. Math. USSR Sb. **23** (1974) 185–214. Translated from Mat. Sb. **94** (136) (1974), 2 194–225
17. Katayama, Y.: Takesaki's duality for a non degenerate coaction. Math. Scand. **55** (1985) 141–151
18. Kirchberg, E.: Representation of coinvolutive Hopf-W^*-algebras and non abelian duality. Bull. Acad. Pol. Sc. **25** (1977) 117–122
19. Krein, M.G.: Hermitian-positive kernels in homogeneous spaces. Amer. Math. Soc. Transl. (2) **34** (1963) 109–164. Translated from Ukr. Mat. Z. **2**, no. 1 (1950) 10–59
20. Landstad, M.B.: Duality theory for covariant systems. Trans. AMS **248** (1979) 223–267
21. Landstad, M.B.: Duality for dual covariance algebras. Comm. Math. Phys. **52** (1977) 191–202
22. Landstad, M.B., Phillips, J., Raeburn, I., Sutherland, C.E.: Representations of crossed products by coactions and principal bundles. Trans. AMS **299**, no. 2 (1987) 747–784
23. Mackey, G.W.: Borel structures in groups and their duals. Trans. AMS **85** (1957) 134–165
24. Mac Lane, S.: Categories for the working mathematician. (Graduate Texts in Mathematics, vol. 5). Springer, New York Berlin Heidelberg 1974
25. Mac Lane, S.: Natural associativity and commutativity. Rice Univ. Studies **49** (1963) 4–28
26. Majid, S.H.: Non-commutative geometric groups by a bicrossproduct construction: Hopf algebras at the Planck scale. Thesis, Harvard Univ. 1988
27. Majid, S.H.: Hopf von Neumann algebra Bicrossproducts, Kac Algebras Bicrossproducts, and the classical Yang-Baxter equation. To appear in J. Funct. Anal.
28. Majid, S.H.: Physics for algebraists: Non-commutative and Non-cocommutative Hopf algebras by a Bicrossproduct construction. J. Algebra **130**, no. 1 (1990) 17–64
29. Moore, G., Seiberg, N.: Classical and quantum conformal field theory. Comm. Math. Phys. **123** (1989) 177–254
30. Nakagami, Y.: Dual action on a von Neumann algebra and Takesaki's duality for a locally compact group. Publ. RIMS Kyoto Univ. **12** (1977) 727–775
31. Nakagami, Y., Takesaki, M.: Duality for crossed products of von Neumann algebras. (Lecture Notes in Mathematics, vol. 731.) Springer, Berlin Heidelberg New York 1979
32. Podles, P., Woronowicz, S.L.: Quantum deformation of Lorentz group. Preprint
33. Rosso, M.: Comparaison des groupes SU(2) quantiques de Drinfeld et Woronowicz. Note aux C. R. Acad. Sci. **304** (1987) 323–326
34. Rosso, M.: Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif. Prépublication
35. Schwartz, J.M.: Sur la structure des algèbres de Kac I. J. Funct. Anal. **34** (1979) 370–406
36. Schwartz, J.M.: Sur la structure des algèbres de Kac II. Proc. London Math. Soc. **41** (1980) 465–480
37. Singer, W.M.: Extension theory for connected Hopf algebras. J. Algebra **21** (1972) 1–16
38. Stinespring, W.F.: Integration theorems for gages and duality for unimodular groups. Trans. AMS **90** (1959) 15–56
39. Stratila, S., Voiculescu, D. and Zsido, L.: On crossed products. I and II. Rev. Roumaine Math. Pures Appl. **21** (1976) 1411–1449 and **22** (1977) 83–117
40. Takai, H.: On a duality for crossed products of C^*-algebras. J. Funct. Anal. **19** (1975) 25–39
41. Takesaki, M.: A characterization of group algebras as a converse of Tannaka-Stinespring-Tatsuuma duality theorem. Amer. J. Math. **91** (1969) 529–564
42. Takesaki, M.: Duality and von Neumann algebras. (Lecture Notes in Mathematics, vol. 427.) Springer, Berlin Heidelberg New York 1972, pp. 665–779
43. Takesaki, M.: Duality for crossed products and the structure of von Neumann algebras of type III. Acta Math. **131** (1973) 249–310

44. Takeuchi, M.: Matched pairs of groups and bismash products of Hopf algebras. Comm. Algebra **9**, no. 8 (1981) 841–882
45. Tannaka, T.: Über den Dualität der nicht-kommutativen topologischen Gruppen. Tôhoku Math. J. **45** (1938) 1–12
46. Tatsuuma, N.: A duality theorem for locally compact groups. J. Math. of Kyoto Univ. **6** (1967) 187–293
47. Vainerman, L.I.: Characterization of dual objects for locally compact groups. Funct. Anal. Appl. **8** (1974) 66–67. Translated from Funkt. Anal. Prilozen. **8**, no. 1 (1974) 75–76
48. Vallin, J.M.: C^*-algèbres de Hopf et C^*-algèbres de Kac. Proc. London Math. Soc. (3) **50** (1985) 131–174
49. Voiculescu, D.: Amenability and Katz algebras. Algèbres d'opérateurs et leurs applications en physique mathématique. Colloques Internationaux C.N.R.S. no. **274** (1977) 451–457
50. Weil, A.: L'intégration dans les groupes topologiques et ses applications. Act. Sc. Ind. no. **1145**. Hermann, Paris (1953)
51. Woronowicz, S.L.: Twisted SU(2) group. An example of a non commutative differential calculus. Publ. RIMS **23** (1987) 117–181
52. Woronowicz, S.L.: Compact matrix pseudogroups. Comm. Math. Phys. **111** (1987) 613–665
53. Woronowicz, S.L.: Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) group. Inv. math. **93** (1988)
54. Woronowicz, S.L.: Differential calculus on compact matrix pseudogroups (quantum groups). Comm. Math. Phys. **122** (1989) 125–170