

Analysis and Geometry on Groups

Nicholas Th. Varopoulos

Mathématiques, Université de Paris VI, 4 place Jussieu, F-75230 Paris Cedex 05, France

The results that I shall survey here can be seen from several different angles. There is a discrete point of view related to discrete finitely generated groups; there is also a C^∞ point of view related to connected Lie groups. One can, to a certain extent, unify the above two settings by considering general compactly generated locally compact groups but I shall not do so here. Both in the discrete and in the C^∞ case we can put forward either the Geometric formulation, such as Sobolev inequalities, or the analytical formulation that examines the behaviour of natural semigroups of operators on $L^2(G)$. What makes the theory hold together, in a final analysis, is that equivalence of all these different aspects. To explain how this comes about I have to start with some definitions.

1. Distance and Volume Growth

Let G be a discrete group generated by a finite number of generators $\gamma_1, \dots, \gamma_k \in G$. One defines then a distance $d(\cdot, \cdot)$ on G by requiring that $d(gx, gy) = d(x, y)$ ($x, y, g \in G$) and that $d(e, x)$, the distance of $x \in G$ from the neutral point $e \in G$ is, by definition, the smallest $n \geq 0$ for which we can write $x = \gamma_{i_1}^{\varepsilon_1} \dots \gamma_{i_n}^{\varepsilon_n}$ ($i_1, \dots, i_n = 1, \dots, k; \varepsilon_j = 0, \pm 1$).

Let G be a connected Lie group and let $X_1, \dots, X_k \in \mathcal{L}(G)$ be a finite number of generators of the Lie algebra of G ; in other words X_1, \dots, X_k are left invariant C^∞ fields on G that together with their successive brackets $[X_{i_1}[X_{i_2}, \dots, X_{i_s}] \dots]$ generate the tangent space. We say that an absolutely continuous path $l(t) \in G$ ($0 \leq t \leq T$) is of length less or equal to T if its speed vector $\dot{l}(t) = dl\left(\frac{\partial}{\partial t}\right)$ (with respect to X_1, \dots, X_k) is almost everywhere of length ≤ 1 : This means that $\dot{l}(t) = \sum_{j=1}^k a_j X_j$ (p.p. $t \sum a_j^2 \leq 1$). We then say that $d(x, y) \leq T$ ($x, y \in G$) if we can join x to y with a path of length $\leq T$.

The growth function $\gamma(t)$ ($t > 0$) of G is in either of the above two cases defined to be $\gamma(t) =$ The Haar measure of a ball of radius t . For large t ($t \geq 1$) the above function $\gamma(t)$ is essentially independent of the particular choice of the generators used: $\gamma(t)$ ($t \geq 1$) is thus a group invariant. For Lie groups and $0 < t < 1$ the behaviour of $\gamma(t)$ does depend on the choice of X_1, \dots, X_k but we always have

$\gamma(t) \approx t^\delta$ with $\delta = \delta(G, X_1, X_2, \dots, X_k) = 1, 2, \dots$ (This is a theorem of Nagel-Stein-Wainger). For $t \geq 1$ and a Lie group we have either $\gamma(t) \approx t^D$ with $D = D(G) = 0, 1, \dots$ or $\gamma(t) \geq Ce^{ct}$ (This is a theorem of Guivarc'h). In the discrete case we have either $\gamma(t) \approx t^D$ ($D(G) = 0, 1, 2, \dots$) if G is a finite extension of a nilpotent group or $\gamma(t)t^{-A} \rightarrow \infty$ for all $A \geq 1$ in all other cases (this is a theorem of Gromov).

2. The Diffusion and the Random Walks

Let G be a unimodular Lie group with $X_1, \dots, X_k \in \mathcal{L}(G)$ as above, we can then consider $\Delta = -\sum X_j^2$ which can be identified with a self adjoint (positive) operator on $L^2(G)$ and we can also consider $T_t = \exp(-t\Delta)$ the corresponding submarkovian semigroup. The kernel of that semigroup will be denoted by $p_t(x, y) = p_t(x^{-1}y)$ ($t > 0$; $x, y \in G$). The discrete analogue of the above diffusion is of course the random walk defined on a discrete group by the transition matrix $M(x, y) = \mu(x^{-1}y)$ ($x, y \in G$) where $\mu \in \mathbb{P}(G)$ is a symmetric probability measure on G . We shall consider in what follows, essentially, only random walks that are defined by symmetric measures that have *generating* supports ($: Gp(\text{supp } \mu) = G$). What we shall examine then is the convolution powers μ^n of that measure or equivalently $T_t = \exp(-t(\delta - \mu))$ the continuous time Markov semigroup that it generates.

3. Analytic and Probabilistic Formulation

One of the main accomplishments of the present methods is that it allows us to study the convolution powers of a finitely supported symmetric measure as considered in the previous section.

Theorem 1. *Let G be a discrete finitely generated group and let μ be a measure as above. Let us also assume that $\gamma(t) \geq ct^D$ for some $c, D > 0$. We then have $\mu^n(\{e\}) = O(n^{-D/2})$.*

The above theorem allows us in particular to classify the discrete groups for which the series $\sum \mu^n(\{e\}) = +\infty$. Such groups are called recurrent groups, the reason being that the random walk with transition matrix $M(x, y) = \mu(x^{-1}y)$ is a recurrent random walk (and this fact is independent of the particular choice of μ):

Corollary. *The only recurrent groups are the finite extensions of the following three groups: $\{0\}, \mathbb{Z}, \mathbb{Z}^2$.*

Theorem 1 easily generalizes to convolution products $\mu_1 * \dots * \mu_n$ provided of course that the measures μ_j satisfy the appropriate conditions uniformly in j . Theorem 1 is a typical result of the discrete version of our theory. The continuous variant of the same result is the following.

Theorem 2. *Let G be a unimodular Lie group and let $X_1, \dots, X_k \in \mathcal{L}(G)$ be as before. Let us assume that the induced growth function satisfies $\gamma(t) \approx t^\delta$ ($t \rightarrow 0$) and $\gamma(t) \geq ct^D$ ($t \geq 1$) for some $\delta, D = 0, 1, \dots$.*

We then have $\|p_t\|_\infty = O(t^{-\delta/2})$ ($t \rightarrow 0$) and $\|p_t\|_\infty = O(t^{-D/2})$ ($t \rightarrow \infty$).

The small time behaviour of $\|p_t\|_\infty$ described in the Theorem is contained in a previous more general result of A. Sanchez-Calle. The group structure is not essential for this small time behaviour of p_t . The above two theorems can of course be unified to a single result on locally compact groups and the methods of the proofs, as we shall see, have very little to do with “real analysis”.

The metric $ds^2 = \varphi(y)(dx^2 + dy^2)$ on \mathbb{R}^2 where $\varphi(y) = y^{-2}$ for $|y| \geq 1$ gives an example of a Riemannian manifold that has exponential volume growth (since for $|y| \geq 1$ it is just the hyperbolic plane) but has “slow” decay for its canonical p_t as $t \rightarrow \infty$. Indeed the above metric is conformal with the Euclidean metric and therefore has no Green’s function i.e. $\int_1^\infty p_t = +\infty$. This shows that the group structure in Theorem 2 is essential for the behaviour of p_t as $t \rightarrow \infty$.

4. Geometric Formulation

Let G be a unimodular Lie group and let $X_1, \dots, X_k \in \mathcal{L}(G)$ be as before. We shall denote the corresponding gradient by: $\nabla f = (X_1 f, \dots, X_k f) \in \mathbb{R}^k$ ($f \in C_0^\infty(G)$). The main Geometric Theorem is

Theorem 3. *Let G and X_1, \dots, X_k be as before and let $\delta, D \geq 0$ be as in Theorem 2. Let also $n \geq 1, \delta \leq n \leq D$ we then have*

$$\|f\|_{n/(n-1)} \leq C \|\nabla f\|_1; \quad f \in C_0^\infty.$$

Conversely if the above Sobolev inequality holds for some n then $n \geq \delta$ and $\gamma(t) \geq ct^n$ ($t \geq 1$).

(All the $\|\cdot\|_p$ norms in what follows are taken in $L^p(G)$ for the Haar measure).

The above Sobolev type estimates are usually reformulated by the Geometers in terms of isoperimetric inequalities of the type $|A|_r^{(n-1)/n} \leq C |\partial A|_{r-1}$ ($A \subset G$) where $|\cdot|_s$ refers to the appropriate s -dimensional Hausdorff measure and r is the topological dimension of G . The discrete analogue of the above theorem states:

Theorem 4. *Let G be a discrete finitely generated group, then the Sobolev inequality $\|f\|_{n/(n-1)} \leq C \|\nabla f\|_1$ ($f \in C_0(G)$) holds for some $1 \leq n \in \mathbb{R}$ if and only if $\gamma(t) \geq ct^n$ ($t \geq 1$).*

In the above theorem the L_1 -norm of the gradient is of course $\|\nabla f\|_1 = \sum_{d(x,y)=1} |f(x) - f(y)|$. Once more the above result can be stated in terms of discrete isoperimetric inequalities.

5. The Connection Between Analysis and Geometry

What unifies the Geometric and the Analytic point of view and what is, in a final analysis, the pivot of the proofs is the following general result of Functional Analysis:

Let A be the generator of an appropriate semigroup $T_t = e^{-tA}$ of operators on $L^p(X; dx)$ (the spaces of p -integrable functions on an abstract measure space $(X; dx)$). Let $n > 2$, the following two conditions are then equivalent:

- (i) $\|f\|_{2n/(n-2)} \leq C(Af, f)^{1/2}; f \in \text{Dom}(A)$.
- (ii) $\|T_t f\|_\infty \leq ct^{-n/2} \|f\|_1; t > 0, f \in L^1$.

For the semigroups associated to our random walks on discrete groups the generator is: $A = \delta - \mu$ and the Dirichlet form satisfies $D_\mu(f) = (Af, f) \approx D_0(f)$ where we denote by $D_0(f) = \sum_{d(x,y)=1} |f(x) - f(y)|^2$ the “standard” Dirichlet form on G . This equivalence $D_\mu \approx D_0$ is trivial to see if μ has *finite* support but what is important is that it remains true for a more general class of measures; namely for all symmetric Probability measures on G with generating support and whose “variance” is finite:

$$E(\mu) = \sum_{x \in G} d^2(e, x) \mu(\{x\}) < +\infty.$$

This observation although not very difficult to prove is absolutely crucial for us.

In the case of a Lie group the Dirichlet form of our semigroup $T_t = e^{-tA}$ is of course the familiar expression

$$(Af, f) = \|\nabla f\|_2^2 = \int_G \sum |X_j f|^2.$$

Observe finally that the $L^1 \rightarrow L^\infty$ operator norm $\|e^{-tA}\|_{1,\infty}$ on a Lie group is $p_t(e)$ and similarly $\|e^{-n(\delta-\mu)}\|_{1,\infty} \sim \mu^n(e)$ for a discrete group (This last \sim has to be interpreted correctly but it certainly implies $\mu^n(e) = O(n^{-\alpha}) \Leftrightarrow \|e^{-t(\delta-\mu)}\|_{1,\infty} = O(t^{-\alpha})$).

With the above facts in mind the connection between the Geometric and the Analytic theory becomes obvious. Another thing that becomes apparent (and this is the single most important feature of all the proofs) is that *changing the measure* μ , say in Theorem 1, *makes no difference* as long as we restrict ourselves to measures of finite variance. Indeed such changes leave invariant (up to equivalence) the Dirichlet form $D_\mu(f)$. What remains to be done to complete the proof of, say Theorem 1, is to produce *one* symmetric probability measure with finite variance and with convolution powers that decay optimally: $\mu^n(\{e\}) = O(n^{-D/2})$.

This last step is done “by hand”. We simply try out a measure of the form: $\mu = \sum \lambda_j \chi_j$ where $\lambda_j \geq 0$, $\sum \lambda_j = 1$ and where χ_j denotes the normalized characteristic function of the j -ball in G . The condition $E(\mu) < +\infty$ is easy to express in terms of the λ 's and the convolution powers μ^n can be estimated by an elementary argument. The above construction does not seem to work if we restrict ourselves to measures of finite support and this is something that to this day I cannot really explain to myself in a satisfactory manner.

6. Further Development and Open Problems

In the interaction between the Geometric and Analytical results there is one point that remains obscure. Indeed what is natural to consider from the semigroup point of view is the norm $\|\Delta^{1/2}f\|_p$ ($f \in C_0^\infty$) (Δ is a self adjoint positive operator) and what occurs naturally in the Geometric formulation is the norm $\|Vf\|_p$. It is only for $p = 2$ that the two norms are obviously equivalent and it is an open problem whether we have in general $\|Vf\|_p \approx \|\Delta^{1/2}f\|_p$. That this is the case in the real variable situation $G = \mathbb{R}^n$ is the content of the classical M. Riesz theorem (for $p \neq 1, \infty$). This equivalence holds when G is a group of polynomial growth (This is a recent theorem of G. Alexopoulos). It also holds when G is non amenable e.g. a classical non compact semi-simple group (this is a result of N. Lohoué). The problem for a general unimodular group remains open and seems difficult. The above problem has an obvious discrete formulation that contains, no doubt, the essence of the difficulty.

When the group G is not unimodular then, as we already pointed out, the geometric aspect of our theory goes through in a very satisfactory fashion. What remains very much open is the analytical theory. Indeed the long time behaviour of the appropriate heat kernel remains untractable by the above methods. The problem is very much connected with the analysis of the canonical heat kernel on symmetric spaces. Indeed any symmetric space of non-compact type can be realized, by the Iwasawa decomposition KAN, as the *non-unimodular* group AN.

The last problem that I shall consider consists in obtaining a finer analysis of the behaviour of p_t as $t \rightarrow \infty$, for Lie group, or μ^n for a discrete group. Assume that G is a unimodular Lie group. If G is not amenable, and only then, we have $p_t(e) = O(e^{-\lambda t})$ where $\lambda > 0$ is the spectral gap of Δ and depends on the particular choice of the fields X_1, \dots, X_k . There are good reasons to suspect that in fact $p_t(e) \sim t^{a/2} e^{-\lambda t}$ where a is some integer or possibly “ $+\infty$ ” that only depends on the group and not on the choice of the fields (just as for amenable groups where we have $\lambda = 0$). The analogous conjecture for discrete groups is false (the counter example is due to D. Cartwright). If G is semi-simple this is, once more, related to the heat kernel on symmetric spaces (Ph. Bougerol has examined this case).

For a Lie group of polynomial growth G , Alexopoulos has proved a “local Central Limit” theorem: $p_t(e)t^{D/2} \xrightarrow{t \rightarrow \infty} \alpha_0 > 0$. The following asymptotic development $p_t(e) \sim t^{-D/2}[\alpha_0 + \alpha_1 t^{-1/2} + \dots]$ should hold, but this is an open problem. Similarly for semi-simple groups and symmetric spaces C. Herz conjectures that $p_t(e) \sim e^{-\lambda t} t^{a/2}[\alpha_0 + \alpha_1 t^{-1/2} + \dots]$ (as $t \rightarrow \infty$). Some logarithms could possibly appear in these asymptotic developments.

Let G be a discrete group and let $\mu, \nu \in \mathbb{P}(G)$ be two symmetric probability measure of finite variance. Let us also assume that $\mu^n(e) = O[\exp(-\alpha(n))]$ where $\alpha(t) \geq 0$ is an increasing positive function of ($t \geq 0$). By a slight variance of the previous methods (here we make essential use of E.B. Davies work in the subject) we can then show that: $\nu^n(e) = 0$ [$\exp(-c\tilde{\alpha}(cn))$] for some $0 < c$, where we denote by

$$\tilde{\alpha}(t) = \frac{1}{t} \int_0^t \alpha(t) dt.$$

The analogous result for unimodular Lie groups also holds. What makes this fact interesting is that for many natural functions e.g. $\alpha(t) = t^\alpha$, $t^\alpha \operatorname{Log}(1+t)$ e.c.t. we have $\alpha \approx \tilde{\alpha}$. This fact is used to analyse the groups that have *superpolynomial* growth:

Assume that G (discrete or Lie, amenable or not, but unimodular) satisfies the growth condition $\gamma(t) \geq \exp(ct^\alpha)$, $t \geq 1$, for some $0 < \alpha \leq 1$ [cf. R. Grigorchuk's paper in these proceedings]. Using our methods then we can easily establish that, say for a discrete group, we have $\gamma(n+1) - \gamma(n) \geq \exp(cn^\alpha)$ with possibly a different $c > 0$ but the same $0 < \alpha \leq 1$. Using this fact and refining our methods further (we use in particular here an idea of L. Saloff-Coste) we can then prove that (again for a discrete group) we have:

$$\mu^n(\{e\}) = O[\exp(-cn^{\alpha/(\alpha+2)})]$$

The analogous result when G is a Lie group and $\alpha = 1$ also holds. The above estimate is optimal. Indeed for any non virtually Nilpotent polycyclic group and every finitely supported symmetric $\mu \in \mathbb{P}(G)$ we have $\mu^{2n}(e) \geq C \exp[-cn^{1/3}]$ (this was shown by G. Alexopoulos) and for all these groups $\alpha = 1$. The details of the above result will appear elsewhere.

A decay of the type $\exp(-cn^\beta)$ for $p_t(e)$ gives rise of course to Orlicz type Sobolev inequalities of the form $\|f\|_{L \log L} \leq C \|Vf\|_1$ where $\gamma = \gamma(\beta)$. In terms of isoperimetric inequalities for discrete groups for instance, we can say that if $\mu^n = O[\exp(-cn^{-\beta})]$ ($0 < \beta \leq 1$) then we have:

$$|\partial\Omega| \geq C |\Omega| (\log |\Omega|)^{\frac{\beta-1}{\beta}}$$

for all finite $\Omega \subset G$ with $|\Omega| \geq 2$ where $|\cdot|$ denotes the cardinality of a finite set.

For exponential groups this gives:

$$|\partial\Omega| \geq C |\Omega| (\log |\Omega|)^{-2}.$$

A final result that I shall mention concerns $p_t(x, y)$ the canonical heat kernel on a Riemannian manifold that covers normally some compact manifold with deck transformation group G . With the present methods we can show that the behaviour of $\|p_t\|_\infty$ (as $t \rightarrow \infty$) is "identical" with the behaviour of $\mu^n(e)$ for $\mu \in \mathbb{P}(G)$ (as in Section 3). The term "identical" means for instance that $\mu^n(e) = O(n^{-a}) \Leftrightarrow \|p_t\|_\infty = O(t^{-a})$ or more generally that we have the:

$$O[\exp(-\alpha(\cdot))] \Leftrightarrow O[\exp(-c\tilde{\alpha}(c.))]$$

correspondence that we considered above. This is one of the very first results that I obtained in the subject and it is this that convinced me of the fundamental connection that existed between the discrete and the continuous theory.

In this survey I have said nothing about the Gaussian estimates of the heat kernels. It would take a different paper to do that. The interested reader could consult the literature below.

Literature

The theory that we surveyed in this paper is the subject matter of a forthcoming book [1]. A preliminary version of this book exists in the form of mimeographed notes: University Paris VI.

Most of the results that I presented were developed by the author in a series of papers the most significant being [2].

For the Functional Analytic tools of Section 5, and the work of E.B. Davies cf. [3, 4, 5, 6, 12].

For further developments in locally compact groups, cf. [7, 8, 9, 13]. Most of the work of G. Alexopoulos has not yet appeared in print, cf. [10, 14]. For the Symmetric space point of view, cf. [11, 15].

1. N. Th. Varopoulos, L. Saloff-Coste, Th. Coulhon: Analysis and geometry on groups. Cambridge University Press (to appear)
2. N. Th. Varopoulos: Analysis on Lie groups. J. Funct. Anal. **76**, no. 2 (1988) 346–410
3. N. Th. Varopoulos: Hardy-Littlewood theory for semigroups. J. Funct. Anal. **63**, no. 2 (1985) 240–260
4. P. B enilan: Op erateurs accr etifs et semi-groupes dans les espaces L^p ($1 \leq p \leq +\infty$). In: Functional analysis and numerical analysis. Japan-France Seminar
5. A. Yoshikawa: Fractional powers of operators, interpolation theory and imbedding theorems. J. Fac. Sci. Univ. Tokyo, I.A. **18** (1971)
6. Th. Coulhon: Semigroup theory and evolution equations, Clement, Mitidien, De Pagter, ed. Marcel Dekker 1991
7. L. Saloff-Coste: In egalit  de Sobolev produite sur les groupes de Lie nilpotents. J. Funct. Anal. **79** (1) (1988) 44–56
8. L. Saloff-Coste: Sur la d croissance des puissances de convolution sur les groupes. Bull. Sci. Math., 2  s rie **113** (1989) 3–21
9. L. Saloff-Coste: Analyse sur les groupes de Lie   croissance polynomiale. Arkov for Mathematik **28** (2) 1990 315–331
10. G. Alexopoulos: C.R. Acad. Sci. Paris **309** (I) (1989) 661–662 and **305** (I) (1987) 777–779
11. N. Lohou : Estim es de type Hardy pour l’op rateur $\Delta + \lambda$ d’un espace sym trie de type non compact. C.R. Acad. Sci. Paris **308** (I) (1989) 11–14
12. E.B. Davies: Heat kernels and spectral theory. Cambridge University Press, 1990
13. N.Th. Varopoulos: ICM-90 Satellite Conference Proceedings Harmonic Analysis (Sendai 1990)
14. G. Alexopoulos: ICM-90 Satellite Conference Proceedings Harmonic Analysis (Sendai 1990) and Canadian J. Math. (to appear)
15. P. Bougerol: Ann. Sci. Ec. Norm. Sup. 4  s rie **14** (1981) 403–432