

Double affine Hecke algebras and Hecke algebras associated with quivers

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Abstract. This is a short survey of some geometrical and categorical approaches to the representation theory of several algebras related to Hecke algebras, including cyclotomic Hecke algebras, double affine Hecke algebras and quiver-Hecke algebras.

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1. Introduction

Affine Hecke algebras are very important in representation theory and have been studied extensively over the past few decades, along with their degenerate version introduced by Drinfeld and Lusztig. About twenty years ago, Cherednik introduced the notion of double affine Hecke algebra, abbreviated as DAHA, which he used to prove the Macdonald's constant term conjecture for Macdonald polynomials. This algebra also admits degenerate versions, the rational one, which is also called Cherednik algebra, having been introduced by Etingof and Ginzburg in 2002.

A rational DAHA is defined for any complex reflection group W . Its representation theory yields a new approach to the representation theory of the Hecke algebra of W . Remarkably, this representation theory is also similar to the representation theory of semi-simple Lie algebras. In particular, it admits a highest weight category which is analogous to the BGG category \mathcal{O} . Highest weight representations are infinite dimensional in general, but they admit a character. An important question is to determine the characters of simple modules.

One of the most important family of rational DAHA's is the cyclotomic one. One reason is that their representation theory is closely related to the representation theory of cyclotomic Hecke algebras, which are relevant in group theory. Another reason is that their highest weight category is closely related to the representation theory of affine Kac-Moody algebras. This was one important motivation for the development of categorical representations (in representation theory).

Categorical representations of Kac-Moody algebras is a relatively young subject that arises in Representation theory and in Knot theory. The first formal definition appeared in a paper of Chuang and Rouquier. The general case was treated independently by Rouquier

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and by Khovanov and Lauda. The ideas leading to categorical representations were around for some two decades. One of the most remarkable application is the work of Ariki, inspired by a conjecture of Lascoux-Leclerc-Thibon, on cyclotomic Hecke algebras. It was observed there that the module category of cyclotomic Hecke algebras has endofunctors that on the level of the Grothendieck group give actions of Kac-Moody Lie algebras of (affine) type A .

This structure appears at several other places in Representation theory, such as the representations of symmetric groups, of the general linear groups or of Lie algebras of type A . An important fact is that the endofunctors come equipped with some natural transformations which satisfy the relations of a new algebra called quiver-Hecke algebra.

Our aim is not to give a general introduction to the subject. There are a lot of them available in the literature, both on DAHA's and on categorical representations. We'll simply focus on some recent results concerning the representation theory of these algebras.

2. Double affine Hecke algebras

2.1. Rational double affine Hecke algebras.

2.1.1. Definition. A *complex reflection group* W is a group acting on a finite dimensional complex vector space \mathfrak{h} that is generated by complex reflections, i.e., non-trivial elements that fix a complex hyperplane in \mathfrak{h} pointwise.

Given a complex reflection group W , let S be its set of complex reflections. For each $s \in S$ let $\alpha_s \in \mathfrak{h}^*$ be a generator of $\text{Im}(s|_{\mathfrak{h}^*} - 1)$, and α_s^\vee be the generator of $\text{Im}(s|_{\mathfrak{h}} - 1)$ such that $(\alpha_s, \alpha_s^\vee) = 2$. Let $c : S \rightarrow \mathbb{C}$, $s \mapsto c_s$ be a W -invariant function.

Definition 2.1 ([18]). The *rational DAHA*, abbreviated RDAHA, associated with W , \mathfrak{h} and c is the quotient $H_c(W)$ of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the ideal generated by the relations $[x, x'] = [y, y'] = 0$ and $[y, x] = (y, x) - \sum_{s \in S} c_s (y, \alpha_s) (\alpha_s^\vee, x)$, for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

The algebra $H_c(W)$ may as well be defined as the subalgebra of $\text{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$ generated by the action of $w \in W$, the multiplication by all elements of $\mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$, and the *Dunkl-Opdam operators* $\partial_y + \sum_{s \in S} c_s (y, \alpha_s) \alpha_s^{-1}(s - 1)$ where $y \in \mathfrak{h}$. The (faithful) representation of $H_c(W)$ on $\mathbb{C}[\mathfrak{h}]$ is called the *polynomial representation*.

2.1.2. The highest weight category $\mathcal{O}_c(W)$. The algebra $H_c(W)$ contains commutative subalgebras $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$. We define the category $\mathcal{O}_c(W)$ to be the category of $H_c(W)$ -modules which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and locally nilpotent under the action of \mathfrak{h} . It is discussed in details in [21]. This is an analogue of the BGG category \mathcal{O} for semisimple Lie algebras.

The algebra $H_c(W)$ admits a *triangular decomposition*. More precisely, the multiplication yields an isomorphism $H_c(W) \simeq \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}^*]$. The most important objects in the category $\mathcal{O}_c(W)$ are the *standard modules* $\Delta_c(\tau) = \text{Ind}_{W \otimes \mathbb{C}[\mathfrak{h}^*]}^{H_c(W)} \tau$, where τ is an irreducible representation of W with the zero action of \mathfrak{h} , and their irreducible quotients $L_c(\tau)$.

It is easy to see that $\mathcal{O}_c(W)$ contains all finite dimensional modules and that the standard module $\Delta_c(\text{triv})$ is isomorphic to the polynomial representation, where triv is the trivial one-dimensional representation of W .

Definition 2.2. A *highest weight category* is a pair (\mathcal{C}, Λ) where \mathcal{C} artinian abelian category with enough projectives and injectives, such that the endomorphism algebra of the irreducible objects are one dimensional, and Λ is an interval-finite poset indexing a set of pairwise non-isomorphic irreducible object $\{L(\lambda) ; \lambda \in \Lambda\}$ of \mathcal{C} .

Further, the following axioms hold. Let $P(\lambda)$ be the projective cover of $L(\lambda)$ in \mathcal{C} . Define the *standard object* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = \delta_{\lambda, \mu}$ for $\mu \not\leq \lambda$. Then $P(\lambda)$ has a finite filtration with top section isomorphic to $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\mu > \lambda$.

The BGG category \mathcal{O} for semisimple Lie algebras is an highest weight category. The category $\mathcal{O}_c(W)$ is also an an highest weight category.

Let $\mathcal{H}_t(W)$ be the *Hecke algebra* of W at the parameter $t = \exp(2\pi ic)$, see [7] for a definition. According to [21], there is a functor $KZ_c : \mathcal{O}_c(W) \rightarrow \mathcal{H}_t(W)\text{-mod}$, which is a *quotient functor* in the general sense of Gabriel. This functor has many good properties. In particular, by [41], this functor determines the highest weight category $\mathcal{O}_c(W)$ up to an equivalence.

More precisely, let R be a commutative local \mathbb{C} -algebra which is a domain and let \mathcal{C} be a highest weight category over R . Let H be a finite projective R -algebra. An R -linear functor $F : \mathcal{C} \rightarrow H\text{-mod}$ is a *highest weight cover* if it is a quotient functor which is fully faithful on projective modules. It is a *d-faithful highest weight cover* if it is a quotient functor which induces an isomorphism $\text{Ext}_{\mathcal{C}}^i(M, N) \rightarrow \text{Ext}_H^i(FM, FN)$ for all $i \leq d$ and all $M, N \in \mathcal{C}$ admitting a finite filtration whose sections are standard modules.

For any R -algebra R' and any R -linear category \mathcal{C} , let $\mathcal{C} \otimes_R R'$ be the R' -linear category with the same objects as \mathcal{C} and with $\text{Hom}_{\mathcal{C} \otimes_R R'}(M, N) = \text{Hom}_{\mathcal{C}}(M, N) \otimes_R R'$ for each objects M, N .

Now, let K be the fraction field of R . We have the following.

Theorem 2.3 ([40]). *Assume that the K -algebra $H \otimes_R K$ is split semisimple and that $F_i : \mathcal{C}_i \rightarrow H\text{-mod}$ is a 1-faithful highest weight cover for $i = 1, 2$. Then the category $\mathcal{C}_i \otimes_R K$ is semisimple and the functor $F_i \otimes_R K$ induces a bijection $\text{Irr}(\mathcal{C}_i \otimes_R K) \simeq \text{Irr}(H \otimes_R K)$. Let \leq_i be the partial orders on $\text{Irr}(H \otimes_R K)$ induced by the poset of \mathcal{C}_i . If \leq_1 is a refinement of \leq_2 , then there is an equivalence of highest weight categories $\mathcal{C}_1 \simeq \mathcal{C}_2$.*

2.1.3. Support of modules in $\mathcal{O}_c(W)$. The functor KZ_c is not generally a category equivalence, since the restriction from \mathfrak{h} to \mathfrak{h}_{reg} kills any object of $\mathcal{O}_c(W)$ supported on $\mathfrak{h} \setminus \mathfrak{h}_{reg}$, the union of all reflecting hyperplanes of W . The support of an irreducible object is always a W -orbit of an intersection of reflecting hyperplanes by [20]. So it has, up to conjugacy, a parabolic subgroup W' attached to it by taking the stabilizer of a generic point in the intersection of these hyperplanes. Despite there usually being no non-trivial homomorphism $H_c(W') \rightarrow H_c(W)$, Bezrukavnikov and Etingof have constructed in [2] an induction functor and a restriction functor between the categories $\mathcal{O}_c(W)$ and $\mathcal{O}_c(W')$, for each $x \in \mathfrak{h}$ with stabilizer W' . Up to isomorphism, these functors are independent of the choice of the element x . Therefore, it is important to know the support of representations in $\mathcal{O}_c(W)$.

A module is supported at 0 if and only if it is finite dimensional. The values of the parameter c for which the module $L_c(\text{triv})$ is finite dimensional has been determined in [52] by geometric methods (with some restrictions on W and c), see Section 2.2.4 below. More generally, the support of $L_c(\text{triv})$ has been completely determined by Etingof in [16], using the Macdonald-Mehta integral for Weyl groups.

Example 2.4. The complex reflection groups have been classified in [57]. One infinite family appears, labelled $G(d, p, n)$, where d, e, n are positive integers such that p divides d . The subfamily $G(d, 1, n)$ takes an important place. We have $G(d, 1, n) = S_n \ltimes (\mathbb{Z}/d)^n$, the *wreath product* of the symmetric group S_n and the cyclic group \mathbb{Z}/d . We'll abbreviate $H_c(d, n) = H_c(G(d, 1, n))$ and $\mathcal{O}_c(d, n) = \mathcal{O}_c(G(d, 1, n))$. The algebra's $H_c(d, n)$ are called the *cyclotomic RDAHA*, and abbreviated CRDAHA. The Hecke algebra $\mathcal{H}_t(G(d, 1, n))$ at the parameter $t = \exp(2\pi ic)$ associated with $H_c(d, n)$ is an important algebra in representation theory. It is called the *cyclotomic Hecke algebra*. We'll write $\mathcal{H}_t(d, n) = \mathcal{H}_t(G(d, 1, n))$, hence the KZ -functor is a functor $KZ_c : \mathcal{O}_c(d, n) \rightarrow \mathcal{H}_t(d, n)\text{-mod}$.

To each tuple of integers e, s_1, \dots, s_d with $e \geq 0$, one associates the *level d Fock space of multicharge* $s = (s_1, \dots, s_d)$. It is a semisimple $\widehat{\mathfrak{sl}}_e$ -module $F(s)$ defined in a combinatorial way and equipped with a (dual) canonical basis, defined also in a combinatorial manner, see [50] and Section 4.2. The dimension of the support of all simple object in $\mathcal{O}_c(d, n)$ has been characterized in [48] via the representation theory of $F(s)$, using categorical representations, answering positively to a conjecture of Etingof in [17]. See Section 4.2.

2.2. Affine and double affine Hecke algebras.

2.2.1. Cartan data and braid groups. A *Cartan datum* consists of a finite-rank free abelian group X whose dual lattice is denoted X^\vee , a finite set of vectors $\Phi = \{\alpha_1, \dots, \alpha_n\} \subset X$ called *simple roots* and a finite set of vectors $\Phi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset X^\vee$ called *simple coroots*. Set $I = \{1, \dots, n\}$. The $I \times I$ matrix A with entries $a_{ij} = (\alpha_j, \alpha_i^\vee)$ is assumed to be a *generalized Cartan matrix*.

Let $\alpha \in X$ and $\alpha^\vee \in X^\vee$ satisfy $(\alpha, \alpha^\vee) = 2$. The linear automorphism $s_{\alpha, \alpha^\vee}(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$ of X is a reflection. If α^\vee is implicitly associated to α we write s_α for both s_{α, α^\vee} and $s_{\alpha^\vee, \alpha}$. When $\alpha = \alpha_i$ and $\alpha^\vee = \alpha_i^\vee$ are a simple root and the corresponding coroot, we write $s_i = s_{\alpha_i}$. The s_i are called the *simple reflections*.

We'll assume that the Cartan datum is *non-degenerate*, i.e., the simple roots are linearly independent. The *Weyl group* W is the group of automorphisms of X (and of X^\vee) generated by the simple reflections s_i . The sets of *roots* and *coroots* are $R = \bigcup_i W(\alpha_i)$, $R^\vee = \bigcup_i W(\alpha_i^\vee)$. The *root* and *coroot lattices* are $Q = \mathbb{Z}\Phi \subset X$ and $Q^\vee = \mathbb{Z}\Phi^\vee \subset X^\vee$. The set of *positive roots* is $R_+ = R \cap Q_+$, where $Q_+ = \mathbb{N}\Phi$. For each element $\alpha = \sum_{i \in I} a_i \alpha_i$ in Q_+ , let $|\alpha| = \sum_{i \in I} a_i$ be the *height* of α . The *dominant weights* are the elements of the cone $X_+ = \{\lambda \in X; (\lambda, \alpha_i^\vee) \geq 0 \text{ for all } i\}$.

The Cartan datum is *finite* if W is a finite group, or equivalently, R is a finite set. The finite Cartan data classify connected reductive algebraic groups G over any algebraically closed field. Then X is the character group $X^*(T)$ of a maximal torus T in G , called the *weight lattice* of G , and X^\vee is the group $X_*(T)$ of one-parameter subgroups of T , called the *coweight lattice* of G . An element $\omega_i \in \mathbb{R} \otimes X$ is called a *i -th fundamental weights* if we have $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ for all j .

The Cartan datum is *affine* if its Cartan matrix A is singular, and for every proper subset $J \subset I$, the Cartan datum $(X, (\alpha_i)_{i \in J}, X^\vee, (\alpha_i^\vee)_{i \in J})$ is finite. This definition implies that the nullspace of A is one-dimensional. Since X is non-degenerate, then $\{\lambda \in Q; (\lambda, \alpha_i^\vee) = 0 \text{ for all } i\}$ is a sublattice of rank 1. It has a unique generator $\delta \in Q_+$, called the *nullroot*. The affine Cartan matrices are classified in [23] and [37].

The Weyl group W is a *Coxeter group* with defining relations $s_i^2 = 1$ and $s_j s_i s_i \dots = s_j s_i s_j \dots$ (m_{ij} factors on each side) where if $a_{ij} a_{ji} = 0, 1, 2, 3$ then $m_{ij} = 2, 3, 4, 6$ re-

spectively, and if $a_{ij}a_{ji} \geq 4$ there is no relation between s_i, s_j . The *length* $l(w)$ of $w \in W$ is the minimal l such that $w = s_{i_1} \cdots s_{i_l}$. Such an expression is called a *reduced factorization*. The *braid group* $B(W)$ is the group with generators T_i and the braid relations $T_j T_i T_j \cdots = T_j T_i T_j \cdots$ (m_{ij} factors on each side). If $w = s_{i_1} \cdots s_{i_l}$ is a reduced factorization, we set $T_w = T_{i_1} \cdots T_{i_l}$. There is a canonical homomorphism $B(W) \rightarrow W$, $T_i \mapsto s_i$.

The *affine Weyl group* is the semidirect product $W \ltimes X$. We use multiplicative notation for the group X , denoting $\lambda \in X$ by x^λ . So $W \ltimes X$ is generated by its subgroups W and X with the additional relations $s_i x^\lambda s_i = x^{s_i(\lambda)}$. For any finite Cartan datum X with Weyl group W and set of simple roots $\{\alpha_i\}$, there is an affine Cartan datum with weight lattice $\tilde{X} = X \oplus \mathbb{Z}\delta$, Weyl group $\tilde{W} = W \ltimes Q^\vee$ and set of simple roots $\tilde{\Phi} = \{\tilde{\alpha}_0, \dots, \tilde{\alpha}_n\}$. Here, we set $\tilde{\alpha}_0 = \delta - \theta$ where θ the highest root in R , and $\tilde{\alpha}_i = \alpha_i$ if $i \neq 0$. Let $\tilde{\alpha}_0^\vee, \dots, \tilde{\alpha}_n^\vee$ be the affine simple coroots in the dual lattice \tilde{X}^\vee . The canonical pairing $\tilde{X} \times \tilde{X}^\vee \rightarrow \mathbb{Z}$ is such that $(\delta, \tilde{\alpha}_i^\vee) = 0$ for all i . There is also an affine Cartan datum with weight lattice $X_{\text{aff}} = \tilde{X} \oplus \mathbb{Z}\tilde{\omega}_0$, Weyl group \tilde{W} and set of simple roots $\tilde{\Phi}$ such that $(\tilde{\omega}_0, \tilde{\alpha}_i^\vee) = \delta_{i,0}$.

2.2.2. Affine and double affine Hecke algebras. Consider a non-degenerate Cartan datum with weight lattice X , Weyl group W and root system R . To simplify, we'll assume that $\alpha_i^\vee \notin 2X^\vee$ for each i . Fix a commutative ground ring A and a W -invariant function $t : R \rightarrow A^\times$. We abbreviate $t_i = t_{\alpha_i}$.

Definition 2.5. The *affine Hecke algebra* $\mathcal{H}_t(W, X)$ is the A -algebra generated by elements T_i satisfying the braid relations of $B(W)$, the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$, and elements x^λ , with $\lambda \in X$, satisfying the relations of the group algebra AX and the relation

$$T_i x^\lambda - x^{s_i(\lambda)} T_i = (t_i - t_i^{-1})(x^\lambda - x^{s_i(\lambda)})(1 - x^{\alpha_i})^{-1}.$$

The subalgebra of $\mathcal{H}_t(W, X)$ generated by the elements T_i is isomorphic to the ordinary Hecke algebra $\mathcal{H}_t(W)$. The induced representation $\text{Ind}_{\mathcal{H}_t(W)}^{\mathcal{H}_t(W, X)}(\text{triv})$ is called the *polynomial representation*.

Definition 2.6. Let W be finite. The *double affine Hecke algebra* associated with X is the A -algebra $\mathcal{H}_t(\tilde{W}, \tilde{X})$.

2.2.3. Geometric realization of double affine Hecke algebras. Let G be a *universal Chevalley group*, i.e., G is a connected, simple and simply connected algebraic group over \mathbb{C} . Let (X, R, X^\vee, R^\vee) be the *root datum* of G . Consider the corresponding affine Cartan data with Weyl group \tilde{W} and weight lattices \tilde{X} or X_{aff} . The affine Hecke algebra $\mathcal{H}_t(\tilde{W}, X_{\text{aff}})$ associated with X_{aff} contains $\mathcal{H}_t(\tilde{W}, \tilde{X})$ as a subalgebra, and we have a semidirect decomposition $\mathcal{H}_t(\tilde{W}, X_{\text{aff}}) = A[x^{\pm\tilde{\omega}_0}] \ltimes \mathcal{H}_t(\tilde{W}, \tilde{X})$. Thus, the representation theory of $\mathcal{H}_t(\tilde{W}, \tilde{X})$ may be deduced from the representation theory of $\mathcal{H}_t(\tilde{W}, X_{\text{aff}})$ by Clifford theory. The element $q = x^\delta$ in $\mathcal{H}_t(\tilde{W}, X_{\text{aff}})$ is central.

Set $\mathbf{F} = \mathbb{C}((\varpi))$ and $\mathbf{O} = \mathbb{C}[[\varpi]]$. Let $G(\mathbf{F})$ be the *loop group* of G (this is an infinite-dimensional group ind-scheme whose set of \mathbb{C} -points is equal to the set of \mathbf{F} -points of G). Since G is simply connected, the isomorphism classes of central extensions of $G(\mathbf{F})$ by \mathbb{G}_m are naturally in bijection with the W -invariant even, negative-definite symmetric bilinear forms $X^\vee \times X^\vee \rightarrow \mathbb{Z}$, see e.g., [39]. Let \tilde{G} be the central extension associated with the minimal such pairing. The multiplicative group \mathbb{G}_m acts naturally on $G(\mathbf{F})$ by ‘rotation of

the loop' and this action lifts to \tilde{G} . We denote the corresponding semi-direct product by G_{aff} . The weight lattice of G_{aff} is X_{aff} , the weight lattice of \tilde{G} is \tilde{X} .

The affine flag manifold \mathcal{B} is an ind-scheme equal to the fpqc quotient G_{aff}/I , where $I \subset G$ is the *Iwahori subgroup*. The set of \mathbb{C} -points of \mathcal{B} is canonically identified with the set of all conjugates of the Lie algebra \mathfrak{i} of I , under the adjoint action of G_{aff} on its Lie algebra. For each $\mathfrak{b} \in \mathcal{B}$, let $\mathfrak{b}_{\text{nil}}$ denote its pro-nilpotent radical. Set $\mathcal{N} = \{(x, \mathfrak{b}) \in \mathfrak{i}_{\text{nil}} \times \mathcal{B}; x \in \mathfrak{b}_{\text{nil}}\}$, an ind-coherent ind-scheme, see [53]. The ind-scheme \mathcal{N} admits a natural action of $G_{\text{aff}} \times \mathbb{G}_m$, where \mathbb{G}_m acts by dilatations on $\mathfrak{i}_{\text{nil}}$.

Let $K^{I \times \mathbb{G}_m}(\mathcal{N})$ be the Grothendieck group of the abelian category of $I \times \mathbb{G}_m$ -equivariant coherent sheaves on \mathcal{N} . From now on, we assume that the function t on R is constant, i.e., the Hecke algebra depends on a single parameter t . Set $A = \mathbb{Z}[t^{-1}, t]$. Using correspondences on \mathcal{N} we prove the following, see [19, 53].

Theorem 2.7. *There is an A -algebra structure on $K^{I \times \mathbb{G}_m}(\mathcal{N})$, and $\mathcal{H}_t(\tilde{W}, X_{\text{aff}})$ is isomorphic to $K^{I \times \mathbb{G}_m}(\mathcal{N})$ as A -algebras.*

Consider the tori $\tilde{T} = \text{Spec}(\mathbb{C}\tilde{X})$ and $T_{\text{aff}} = \text{Spec}(\mathbb{C}X_{\text{aff}})$ in \tilde{G} and G_{aff} . A character $\chi : A[X_{\text{aff}}] \rightarrow \mathbb{C}$ is a triple (s, τ, ζ) where $\zeta = \chi(t)$, $\tau = \chi(q)$ and s is an element of \tilde{T} . The pair (s, τ) can be viewed as an element of the group T_{aff} . It acts on the ind-scheme \mathcal{B} by left multiplication. Let $\mathcal{B}^{s, \tau}$ be the fixed points subset.

For each $x \in \mathfrak{g}_{\text{aff}}$, the *affine Springer fiber* \mathcal{B}_x is the ind-scheme $\mathcal{B}_x = \{\mathfrak{b} \in \mathcal{B}; x \in \mathfrak{b}_{\text{nil}}\}$. Set $\mathcal{B}_x^{s, \tau} = \mathcal{B}^{s, \tau} \cap \mathcal{B}_x$.

Let $G(s, \tau, x) \subset G_{\text{aff}}$ be the subgroup of elements commuting with x and (s, τ) , and let $A(s, \tau, x)$ be the group of connected components of $G(s, \tau, x)$. An element of \tilde{G} is called *semisimple* if it is conjugate to an element of \tilde{T} . Let $\Sigma_{\tau, \zeta}$ be the set of triples (s, x, π) where $s \in \tilde{G}$ is semisimple, $x \in \mathfrak{g}_{\text{aff}}$ is *topologically nilpotent* in the sense of [32] with $\text{ad}_{(s, \tau)}(x) = \zeta^{-1}x$, and π is an irreducible representation of $A(s, \tau, x)$ which is a constituent of the natural representation of $A(s, \tau, x)$ in $H_*(\mathcal{B}_x^{s, \tau}, \mathbb{C})$. Two triples in $\Sigma_{\tau, \zeta}$ are *equivalent* if they are conjugated by an element of G_{aff} .

The A -algebra $\mathcal{H}_t(\tilde{W}, X_{\text{aff}})$ has a triangular decomposition $\mathcal{H}_t(\tilde{W}, X_{\text{aff}}) \simeq AX_{\text{aff}} \otimes_A \mathcal{H}_t(W) \otimes_A AX_{\text{aff}}^\vee$. Let $\mathcal{O}_{\tau, \zeta}(\tilde{W}, X_{\text{aff}})$ be the category of all finitely generated modules over \mathbb{C} which are locally finite over $\mathbb{C}X_{\text{aff}}$ and such that q, t act by scalar multiplication by τ, ζ . Using the theorem above, we get the following.

Theorem 2.8 ([53]). *Assume that τ is not a root of 1 and that $\tau^k \neq \zeta^{2m}$ for each $k, m > 0$. The isomorphism classes of simple objects in $\mathcal{O}_{\tau, \zeta}(\tilde{W}, X_{\text{aff}})$ are in bijection with the equivalence classes of triples (s, x, π) in $\Sigma_{\tau, \zeta}$.*

Remark 2.9.

- (a) Theorem 2.7 is an affine version of the Kazhdan-Lusztig classification of the simple modules of affine Hecke algebras in [31], see also Ginzburg's proof in [10].
- (b) Let $\tau^{\mathbb{Z}} \subset \mathbb{C}^\times$ be the subgroup generated by τ . By [5], there is a bijection from the set of all $\tau^{\mathbb{Z}} \ltimes G(\mathbb{F})$ -conjugacy classes in $G(\mathbb{F})$ containing a point in $G(\mathbb{O})$ onto the set $\mathcal{M}(G)$ of isomorphism classes of *topologically trivial semistable principal G -bundles* over the elliptic curve $E = \mathbb{C}^\times / \tau^{\mathbb{Z}}$. We deduce that the set of equivalence classes in $\Sigma_{\tau, \zeta}$ can be described in terms of isomorphism classes of *Higgs bundles* over E , see [4] for details.

- (c) The theorem above admits a *global version* which yields a representation of an analogue of the double affine Hecke algebra in the cohomology groups of some fibers of the Hitchin map associated with a smooth projective curve C , see [60]. These fibers are closed subschemes of the moduli space of parabolic Higgs bundles over C . If $C = \mathbb{P}^1$, equipped with its natural \mathbb{G}_m -action, the algebra above is closely related to the graded version of $\mathcal{H}_t(\widetilde{W}, X_{\text{aff}})$ introduced by Cherednik, see Section 2.2.4 below.

2.2.4. Application to finite dimensional representations. Fix a Cartan datum with weight lattice X , Weyl group W , root system R and set of simple roots $\Phi = \{\alpha_i\}$. Fix a commutative ground ring A . We'll assume that $\alpha_i^\vee \notin 2X^\vee$ for each i . Let $\kappa : R \rightarrow A^\times$ be a W -invariant function. We abbreviate $\kappa_i = \kappa_{\alpha_i}$.

The affine Hecke algebra $\mathcal{H}_t(W, X)$ admits a *graded version*, which is the A -algebra $\mathcal{H}'_\kappa(W, X)$ generated by elements σ_w , with $w \in W$, satisfying the relations of the group algebra AW and elements ξ^λ , with $\lambda \in X$, satisfying the relations of the group algebra AX and the relation $\sigma_i \xi^\lambda - \xi^{s_i(\lambda)} \sigma_i = \kappa_i(\alpha_i^\vee, \lambda)$.

The induced representation $\text{Ind}_{AW}^{\mathcal{H}'_\kappa(W, X)}(\text{triv})$ is called the *polynomial representation*. It is faithful, which permits to view $\mathcal{H}'_\kappa(W, X)$ as a subalgebra of the semi-direct product $AW \ltimes D(T)_{\text{rat}}$, where $D(T)_{\text{rat}}$ is the ring of differential operators with rational coefficients on the torus T associated with the lattice X , see [34].

Assume that the Cartan datum is of finite type. By [34] the irreducible representations of the affine Hecke algebra $\mathcal{H}_t(W, X)$ may be described in terms of irreducible representations of graded affine Hecke algebras associated with root subsystems of R . Similarly, by [52] the irreducible representations of the double affine Hecke algebra $\mathcal{H}_t(\widetilde{W}, \widetilde{X})$ may be described in terms of irreducible representations of some graded double affine Hecke algebras.

Assume also that the Cartan datum is associated with a universal Chevalley group G . According to Cherednik, there is an exact fully faithful functor which embeds the category of finite dimensional $H_c(W)$ -modules into the category of finite dimensional $\mathcal{H}'_\kappa(\widetilde{W}, \widetilde{X})$ for a good choice of the parameters, see e.g., [52, sec. 2.3]. By [2, sec. 5.4], all finite dimensional representations of $\mathcal{H}'_\kappa(\widetilde{W}, \widetilde{X})$ may indeed be described in terms of representations of rational DAHA's associated with root subsystems of maximal rank via a version of the Borel-de Siebenthal algorithm.

Using this, it is proved in [52] that $H_c(W)$ acts on the homology groups $H_*(\mathcal{B}_x^{s, \tau}, \mathbb{C})$, whenever the affine Springer fiber \mathcal{B}_x has a finite dimensional cohomology, yielding a classification of all finite dimensional modules which are quotient of the polynomial representation.

3. Quiver-Hecke algebras

3.1. Quantum groups.

3.1.1. Definition. Fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with a *symmetrizable* generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$, i.e., there exist non-zero integers d_i such that $d_i a_{ij} = d_j a_{ji}$ for all i, j . The integers d_i are unique up to an overall common factor. They can be assumed positive. Then d_i is the *length* of the root α_i . Note that the generalized Cartan matrix of finite and affine type are all symmetrizable.

Assume that for each $i \in I$, there exists $\omega_i \in X$, a fundamental weight, such that $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ for all $j \in I$. Let q be an indeterminate and set $q_i = q^{d_i}$. For $m, n \in \mathbb{N}$ we set $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i!$ and $\begin{bmatrix} m \\ n \end{bmatrix}_i = [m]_i!/[m-n]_i![n]_i!$.

Definition 3.1. The *quantum group* associated with $(X, \Phi, X^\vee, \Phi^\vee)$ is the associative algebra U_q over $\mathbb{Q}(q)$ with 1 generated by $e_i, f_i, i \in I$, and $l_h, h \in X^\vee$, satisfying the following relations

$$\begin{aligned} l_0 &= 1, l_h l_{h'} = l_{h+h'}, \\ l_h e_i l_{-h} &= q^{(h, \alpha_i)} e_i, l_h f_i l_{-h} = q^{-(h, \alpha_i)} f_i, \\ e_i f_j - f_j e_i &= \delta_{i,j} (k_i - k_i^{-1}) (q_i - q_i^{-1}) \text{ where } k_i = l_{d_i \alpha_i^\vee}, \\ \sum_{r=0}^{1-a_{i,j}} \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_i e_i^{1-a_{i,j}-r} e_j e_i^r &= 0 \text{ if } i \neq j, \\ \sum_{r=0}^{1-a_{i,j}} \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_i e_i^{1-a_{i,j}-r} e_j e_i^r &= 0 \text{ if } i \neq j. \end{aligned}$$

Let U_q^+, U_q^- be the subalgebra of U_q generated by e_i 's, f_i 's respectively, and let U_q^0 be the subalgebra of U_q generated by l_h with $h \in X^\vee$. Then we have a triangular decomposition $U_q = U_q^- \otimes U_q^0 \otimes U_q^+$, and the *weight space decomposition* $U_q^- = \bigoplus_{\alpha \in Q_+} U_{q, \alpha}^-$ where $U_{q, \alpha}^- = \{x \in U_q^-; l_h x l_{-h} = q^{-(h, \alpha)} x \text{ for any } h \in X^\vee\}$.

For each $\lambda \in X$ there exists a unique irreducible *highest weight module* $L_q(\lambda)$ with *highest weight* λ , i.e., a U_q -module $L_q(\lambda) = M$ with a *weight space decomposition* $M = \bigoplus_{\mu \in X} M_\mu$, where $M_\mu = \{v \in M; l_h v = q^{(h, \mu)} v \text{ for all } h \in X^\vee\}$, such that there is a non-zero vector $v_\lambda \in M_\lambda$ with $e_i v_\lambda = 0$ for all $i \in I$ and $M = U_q v_\lambda$.

Let $A = \mathbb{Z}[q, q^{-1}]$ and set $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$ for all $n \in \mathbb{N}$. We define the *A-form* U_A to be the A -subalgebra of U_q generated by $e_i^{(n)}, f_i^{(n)}, l_h$ with $i \in I, n \in \mathbb{N}$ and $h \in X^\vee$. We define the *A-form* $L_A(\lambda)$ to be the A -submodule of $L_q(\lambda)$ given by $L_A(\lambda) = U_A v_\lambda$.

According to Lusztig and Kashiwara, see [35], [28], the quantum group U_q^- admits a *canonical basis*, which is an A -basis \mathbf{B} of the A -module $U_A^- = U_A \cap U_q^-$ such that, for each integrable dominant weight $\lambda \in X_+$ the set $\{b v_\lambda; b v_\lambda \neq 0\}$ is an A -basis of $L_A(\lambda)$.

3.2. Quiver-Hecke algebras.

3.2.1. Definition. Fix a symmetrizable generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$ and a commutative graded ring $k = \bigoplus_{n \in \mathbb{Z}} k_n$ such that k_0 is a field and $k_n = 0$ if $n < 0$. Let $c_{i,j,p,q} \in k$ be of degree $-2d_i(a_{ij} + p) - 2d_j q$. Assume that $c_{i,j,-a_{i,j},0}$ is invertible. For $i, j \in I$ let $Q_{i,j} \in k[u, v]$ be such that $Q_{ij}(u, v) = Q_{ji}(v, u)$, $Q_{ij}(u, v) = 0$ if $i = j$ and $Q_{ij}(u, v) = \sum_{p,q \geq 0} c_{i,j,p,q} u^p v^q$ if $i \neq j$.

Definition 3.2 ([32, 42]). The *quiver-Hecke algebra* of degree $n \geq 0$ associated with A and $(Q_{i,j})_{i,j \in I}$ is the associative algebra $R(n)$ over k generated by $e(\mathbf{i}), x_k, \sigma_l$ with $\mathbf{i} \in I^n$, $k \in [1, n], l \in [1, n]$ satisfying the following defining relations

$$\begin{aligned} e(\mathbf{i}) e(\mathbf{i}') &= \delta_{\mathbf{i}, \mathbf{i}'} e(\mathbf{i}), \sum_{\mathbf{i}} e(\mathbf{i}) = 1, \\ x_k x_l &= x_l x_k, x_k e(\mathbf{i}) = e(\mathbf{i}) x_k, \\ \sigma_l e(\mathbf{i}) &= e(s_l \mathbf{i}) \sigma_l, \sigma_k \sigma_l = \sigma_l \sigma_k \text{ if } |k - l| > 1, \\ \sigma_l^2 e(\mathbf{i}) &= Q_{i_l, i_{l+1}}(x_l, x_{l+1}) e(\mathbf{i}), \end{aligned}$$

$$\begin{aligned}
(\sigma_k x_l - x_{s_k(l)} \sigma_k) e(\mathbf{i}) &= \begin{cases} -e(\mathbf{i}) & \text{if } l = k, i_k = i_{k+1}, \\ e(\mathbf{i}) & \text{if } l = k+1, i_k = i_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\
(\sigma_{k+1} \sigma_k \sigma_{k+1} - \sigma_k \sigma_{k+1} \sigma_k) e(\mathbf{i}) &= \begin{cases} a_k(\mathbf{i}) e(\mathbf{i}) & \text{if } i_k = i_{k+2}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where $a_k(\mathbf{i}) = (Q_{i_k, i_{k+1}}(x_k, x_{k+1}) - Q_{i_{k+2}, i_{k+1}}(x_{k+2}, x_{k+1})) / (x_k - x_{k+2})$. The algebra $R(n)$ admits a \mathbb{Z} -grading given by $\deg e(\mathbf{i}) = 0$, $\deg x_k e(\mathbf{i}) = 2d_{i_k}$ and $\deg \sigma_l e(\mathbf{i}) = -d_{i_k} a_{i_k, i_{k+1}}$.

Now, fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with generalized Cartan matrix A . Fix a dominant integral weight $\lambda \in X_+$. Given $i \in I$, set $s = (\lambda, \alpha_i^\vee)$ and fix a monic polynomial $a_i^\lambda(u) = \sum_{r=0}^s c_{i,r} u^{s-r}$ in $k[u]$ of degree s such that the element $c_{i,r} \in k$ has the degree $2rd_i$.

Definition 3.3. The *cyclotomic quiver-Hecke algebra* of degree $n \geq 0$ associated with $R(n)$, the weight $\lambda \in X_+$ and the polynomials a_i^λ is the quotient $R^\lambda(n)$ of the \mathbb{Z} -graded algebra $R(n)$ by the homogeneous two-sided ideal generated by the elements $a_{i_1}^\lambda(x_1)e(\mathbf{i})$ for all $\mathbf{i} \in I^n$.

Let $\text{proj}(R(n))$, $\text{proj}(R^\lambda(n))$ be the categories of finitely generated projective graded modules over $R(n)$, $R^\lambda(n)$ respectively. Let $[\text{proj}(R(n))]$, $[\text{proj}(R^\lambda(n))]$ be their Grothendieck groups. They are A -modules, where the action of q is given by the grade shift functor. There are natural embeddings $R(m) \otimes R(n) \subset R(m+n)$. The induction and restriction functors equip the A -module $[\text{proj}(R)] = \bigoplus_{n \geq 0} [\text{proj}(R(n))]$ with the structure of a bialgebra.

Theorem 3.4 ([32]). *The A -module $[\text{proj}(R)]$ is isomorphic to U_A^- as a bialgebra.*

Composing the induction and restriction functors with the functor

$$\text{proj}(R(n)) \rightarrow \text{proj}(R^\lambda(n)), M \mapsto R^\lambda(n) \otimes_{R(n)} M,$$

Kang and Kashiwara proved the following, see also [58].

Theorem 3.5 ([24]). *There is a natural structure of U_A -module on*

$$[\text{proj}(R^\lambda)] = \bigoplus_{n \geq 0} [\text{proj}(R^\lambda(n))]$$

such that it is isomorphic to $L_A(\lambda)$.

Remark 3.6. For each $\alpha \in Q_+$ of height n we write $I^\alpha = \{\mathbf{i} = (i_1, \dots, i_n) \in I^n; \sum_{k=1}^n \alpha_{i_k} = \alpha\}$, $e(\alpha) = \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i})$, $R(\alpha) = e(\alpha) R(n) e(\alpha)$ and $R(\alpha)^\lambda = e(\alpha) R(n)^\lambda e(\alpha)$. Then, the isomorphisms in Theorems 3.4, 3.5 map $[\text{proj}(R(\alpha))]$ and $[\text{proj}(R(\alpha)^\lambda)]$ to the weight subspaces $U_{A,\alpha}^- = U_A^- \cap U_{q,\alpha}^-$ and $L_A(\lambda)_{\lambda-\alpha} = L_A(\lambda) \cap L_q(\lambda)_{\lambda-\alpha}$.

3.2.2. Geometric realization of quiver-Hecke algebras. Let $\Gamma = (I, \Omega)$ be a locally finite quiver without loops, with a vertex set I and an oriented edge set Ω . For each arrow $h \in \Omega$ let h', h'' denote the incoming and outgoing vertex. For $i, j \in \Omega$ with $i \neq j$, let $\Omega_{ij} = \{h \in \Omega; h' = i, h'' = j\}$ and $h_{ij} = \# \Omega_{ij}$. The matrix A given by $a_{ii} = 2$ and $a_{ij} = -h_{ij} - h_{ji}$ is a symmetric generalized Cartan matrix, and any symmetric generalized Cartan matrix can be realized in this way via a quiver.

Fix a finite dimensional I -graded \mathbb{C} -vector space $V = \bigoplus_{i \in I} V_i$. A *representation* of Γ in V is an element of $E_V = \bigoplus_{h \in \Omega} \text{Hom}(V_{h'}, V_{h''})$. The groups $G_V = \prod_{i \in I} \text{GL}(V_i)$ and $T_\Omega = (\mathbb{G}_m)^\Omega$ act on the space of representations E_V by $(g, t) \cdot (x_h)_{h \in \Omega} = (t_h g_{h''} x_h g_{h'}^{-1})_{h \in \Omega}$. We'll abbreviate $E = E_V$.

For $\mathbf{i} = (i_1, \dots, i_m) \in I^n$, the variety of *complete flags of type \mathbf{i}* is a \mathbb{C} -scheme whose set of \mathbb{C} -points is the set $\mathcal{F}_{\mathbf{i}}$ of tuples $\phi = (0 = \phi_0 \subset \phi_1 \subset \dots \subset \phi_m = V)$ where ϕ_k is an I -graded subspace such that $\dim(\phi_k / \phi_{k-1}) = \alpha_{i_k}$ for $k \in [1, m]$. The group G_V acts transitively on $\mathcal{F}_{\mathbf{i}}$ and T_Ω acts trivially.

For $x \in E$, a flag $\phi \in \mathcal{F}_{\mathbf{i}}$ is *x-stable* if $x_h(\phi_k \cap V_{h'}) \subset \phi_{k-1} \cap V_{h''}$ for each $h \in \Omega$, $k \in [1, m]$. Let $\tilde{\mathcal{F}}_{\mathbf{i}}$ be the set of pairs $(x, \phi) \in E \times \mathcal{F}_{\mathbf{i}}$ such that ϕ is *x-stable*. The group $G_V \times T_\Omega$ acts diagonally on $\tilde{\mathcal{F}}_{\mathbf{i}}$. Let $\pi_{\mathbf{i}} : \tilde{\mathcal{F}}_{\mathbf{i}} \rightarrow E$ be the obvious projection. We write $\mathcal{L}(\mathbf{i}) = R\pi_{\mathbf{i}!}(\mathbb{C}_{\tilde{\mathcal{F}}_{\mathbf{i}}}[2 \dim \tilde{\mathcal{F}}_{\mathbf{i}}])$, a semisimple complex in the bounded $G_V \times T_\Omega$ -equivariant derived category $\mathbf{D}_{G_V \times T_\Omega}^b(E)$ of sheaves of \mathbb{C} -vector spaces on E . Set $\mathcal{L}(n) = \bigoplus_{\mathbf{i} \in I^n} \mathcal{L}(\mathbf{i})$.

The \mathbb{Z} -graded module $\text{Ext}(\mathcal{L}(n), \mathcal{L}(n)) = \bigoplus_i \text{Ext}^i(\mathcal{L}(n), \mathcal{L}(n))$ is a \mathbb{Z} -graded k -algebra for the Yoneda multiplication. We call it the *Yoneda algebra* of $\mathcal{L}(n)$. Here the extension groups are computed in the triangulated category $\mathbf{D}_{G_V \times T_\Omega}^b(E)$.

Now, take $k = H_{T_\Omega}^*(\bullet, \mathbb{C})$ as the base ring. We have $k = \mathbb{C}[\chi_h; h \in \Omega]$, where χ_h is the equivariant Chern class of the 1-dimensional representation of the h -th factor \mathbb{C}^\times in T_Ω . Set $Q_{ij}(u, v) = \prod_{h \in \Omega_{ij}} (v - u + \chi_h) \prod_{h \in \Omega_{ji}} (u - v + \chi_h)$ if $i \neq j$ and $Q_{ij}(u, v) = 0$ if $i = j$. Let $R(n)$ be the quiver-Hecke algebra of degree $n \geq 0$ associated with the generalized Cartan matrix A and the matrix $(Q_{i,j})_{i,j \in I}$.

Theorem 3.7 ([42, 55]). *There is a \mathbb{Z} -graded k -algebra isomorphism*

$$R(n) \simeq \text{Ext}(\mathcal{L}(n), \mathcal{L}(n))$$

which identifies the idempotent $e(\mathbf{i})$ with the projection to the direct summand $\mathcal{L}(\mathbf{i}) \subset \mathcal{L}(n)$.

Now, set $k = \mathbb{C}$, viewed as the quotient of $\mathbb{C}[\chi_h; h \in \Omega]$ by the maximal ideal generated by all elements χ_h . Fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with generalized Cartan matrix equal to the matrix A above. Let $\lambda \in X_+$ be a dominant weight and U_q , $R(n)$, $R(n)^\lambda$ be the corresponding quantum group, quiver-Hecke algebra and cyclotomic quiver-Hecke algebra. Using the previous theorem and Lusztig's geometric realization of the canonical bases, see [35], we obtain the following refinement of Theorem 3.4, 3.5.

Corollary 3.8.

- (a) *There is a bialgebra isomorphism $[\text{proj}(R)] \simeq U_A^-$ which identifies the canonical basis in the right hand side with the set of projective indecomposable self-dual modules in the left hand side.*
- (b) *There is a U_A -module isomorphism $[\text{proj}(R^\lambda)] \simeq L_A(\lambda)$ which identifies the canonical basis in the right hand side with the set of projective indecomposable self-dual modules in the left hand side.*

Remark 3.9.

- (a) The construction above can be generalized to allow quiver with loops, arbitrary partial flags of a quiver representation, or sheaves of vector spaces over a field of positive characteristic, see [25, 36, 49] for details. Taking a more general version of flags in representations of the quiver yields a more general version of quiver-Hecke algebras called *weighted KLR algebras* by Webster in [59].
- (b) If the polynomial $Q_{ij}(u, v)$ satisfies the conditions in Section 3.2.1, but does not satisfy the conditions in Section 3.2.2, then Corollary 3.8 may not hold, see [26] for details.
- (c) It is not known how to construct the canonical basis of U_q^- using quiver-Hecke algebras when the Cartan matrix A is not symmetric. However, one can construct the canonical basis of U_q^- for any non symmetric A of finite or affine type by mimicking the construction in [35]. More precisely, let Γ be a quiver with a *compatible automorphism* γ , i.e., a pair of automorphisms $\gamma : I \rightarrow I, \gamma : \Omega \rightarrow \Omega$ such that $\gamma(h)' = \gamma(h')$, $\gamma(h)'' = \gamma(h'')$ for each $h \in \Omega$. Assume that γ is of finite order ℓ and that h', h'' are not in the same γ -orbit for each h .

Put $[I] = I/\gamma$, and for each $i \in I$ let $[i] \in [I]$ be its γ -orbit. Let $h_{[i],[j]}$ be the number of γ -orbits in the set $\Omega_{[i],[j]} = \{h \in \Omega; h' \in [i], h'' \in [j]\}$. Put $d_{[i]} = \sharp[i]$. The matrix A given by $a_{[i],[i]} = 2$ and $a_{[i],[j]} = -(h_{ij} + h_{ji})/d_{[i]}$ is a symmetrizable generalized Cartan matrix, and any generalized Cartan matrix of finite or affine type can be realized in this way. Let U_q be the corresponding quantum group.

For any element $\alpha = \sum_{i \in I} a_i \alpha_i$ in Q_+ such that $a_{\gamma(i)} = a_i$ for all i , the quiver-Hecke algebra $R(\alpha)$ admits a natural action of γ . This yields a *periodic functor* on the category $\text{proj}(R(\alpha))$, with the terminology of [35, chap. 11]. Let $\mathcal{K}(\text{proj}(R(\alpha)))$ be the corresponding *twisted Grothendieck group*, as defined in [35, sec. 11.1.5].

Let $\mathcal{O} \subset \mathbb{C}$ be the subring consisting of all \mathbb{Z} -linear combinations of ℓ -th roots of 1. There is a bialgebra isomorphism $\mathcal{K}(\text{proj}(R)) \simeq U_A^- \otimes_{\mathbb{Z}} \mathcal{O}$ which identifies the canonical basis of U^- with the set of projective indecomposable self-dual modules in $\mathcal{K}(\text{proj}(R)) = \bigoplus_{\alpha} \mathcal{K}(\text{proj}(R(\alpha)))$. A similar construction gives a realization of the canonical basis of all integrable simple modules of U_q .

3.2.3. Affine Hecke algebras of type A : Ariki's theorem. Consider the Cartan datum of type A_{n-1} with weight lattice $X = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \simeq \mathbb{Z}^n$ and simple roots given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ with $i \in [1, n)$. The Weyl group is the symmetric group $W = S_n$.

Set $A = \mathbb{Z}[t^{-1}, t]$. The *affine Hecke algebra* of $\text{GL}(n)$ is the A -algebra $H_t^A(n) = \mathcal{H}_t(W, X)$ which is generated by elements T_1, \dots, T_{n-1} satisfying the braid relations of $B(S_n)$, the quadratic relations $(T_i - t)(T_i + t^{-1}) = 0$, and commuting elements $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ satisfying the relation $T_i X_i T_i = X_{i+1}, T_i X_j = X_j T_i$ if $i \in [1, n)$ and $j \neq i, i+1$.

Fix an element $\zeta \in \mathbb{C}^\times$ such that $\zeta^2 \neq 1$. Set $H_\zeta^A(n) = H_t^A(n) \otimes_A \mathbb{C}$, where $\chi : A \rightarrow \mathbb{C}$ is the character such that $t \mapsto \zeta$. The group \mathbb{Z} acts on \mathbb{C}^\times by $\mathbb{Z} \ni n : i \mapsto i\zeta^{2n}$. Let I be a \mathbb{Z} -invariant subset in \mathbb{C}^\times .

Let $\text{mod}(H_\zeta^A(n))$ be the category of all finitely generated $H_\zeta^A(n)$ -modules. Let $\text{mod}_I^{\text{fd}}(H_\zeta^A(n))$ be the full subcategory of all finite dimensional modules of type I , i.e., the finite dimensional modules M such that $M = \bigoplus_{i \in I^n} M_i$ where $M_i = \{v \in M; (X_k - i_k)^r v = 0 \text{ for any } k \text{ and for some } r \gg 0\}$. Let $[\text{mod}_I^{\text{fd}}(H_\zeta^A(n))]$ be the Grothendieck group

of $\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A(n))$, and set $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A)] = \bigoplus_{n \geq 0} [\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A(n))]$. The group $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A)]$ is a bialgebra where the product and coproduct are given by the induction and restriction with respect to the obvious inclusion $\mathcal{H}_\zeta^A(m) \otimes \mathcal{H}_\zeta^A(n) \subset \mathcal{H}_\zeta^A(m+n)$.

We can view I as a quiver without loops, with vertex set I and with an arrow $i \rightarrow i\zeta^2$ for each $i \in \mathbb{C}^\times$. Let U_q be the quantum group associated with this quiver and let U be its specialization at $q = 1$. We define U^- , $L(\lambda)$ in the obvious way. The following was observed by Grojnowski. It follows from the Kazhdan-Lusztig and Ginzburg works [32], [10].

Theorem 3.10 ([22]). *The group $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A)]$ is isomorphic to U^- as a bialgebra. Under this isomorphism, the classes of the simple modules is identified with the dual canonical basis of U^- .*

If J is another \mathbb{Z} -invariant subset in \mathbb{C}^\times such that $I \cap J = \emptyset$, then the induction yields an equivalence of categories $\text{mod}_{I \cup J}^{\text{fd}}(\mathcal{H}_\zeta^A(n)) \simeq \text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A(n)) \times \text{mod}_J^{\text{fd}}(\mathcal{H}_\zeta^A(n))$. Hence it is enough to assume that I is a \mathbb{Z} -orbit. Then, the Cartan datum associated with the quiver I above, see Section 3.2.2, is either of type A_∞ or of type $A_e^{(1)}$ for some integer $e > 0$. We deduce that $U \otimes_{\mathbb{Z}} \mathbb{C}$ is either the enveloping algebra of \mathfrak{gl}_∞ , if ζ is not a root of 1, or the enveloping algebra of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_e$, if ζ is a e -th primitive root of 1.

The Hecke algebra of the complex reflection group $G(d, 1, n)$ with parameters ζ, u_1, \dots, u_d is isomorphic to the quotient $\mathcal{H}_\zeta^{A,u}(n)$ of the affine Hecke algebra $\mathcal{H}_\zeta^A(n)$ by the *cyclotomic relation* $(X_1 - u_1) \cdots (X_1 - u_d) = 0$.

Let $\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,u}(n))$ be the category of all finite dimensional $\mathcal{H}_\zeta^{A,u}(n)$ -modules of type I . Let $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,u}(n))]$ be the Grothendieck group of $\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,u}(n))$, and set $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,u})] = \bigoplus_{n \geq 0} [\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,u}(n))]$.

Assume that $u_p = t^{2s_p}$, with $s_p \in \mathbb{Z}$ for each $p \in [1, d]$. Let $\lambda = \sum_{p=1}^d \omega_{u_p}$, where ω_i is the i -th fundamental weight of the Cartan datum associated with I . Let $\mathcal{H}_\zeta^{A,\lambda}(n)$ denote the corresponding cyclotomic Hecke algebra. Composing the induction and restriction with the functor $\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,\lambda}) \rightarrow \text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^A)$ induced by the obvious surjective algebra homomorphism $\mathcal{H}_\zeta^A \rightarrow \mathcal{H}_\zeta^{A,\lambda}$, Ariki has obtained the following, yielding a proof of a conjecture of Lascoux-Leclerc-Thibon.

Theorem 3.11 ([1]). *There is a natural structure of U -module on $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,\lambda})]$ such that $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^{A,\lambda})]$ is isomorphic to the dual of the integrable highest weight U -module $L(\lambda)$ with highest weight λ . Under this isomorphism, the classes of the simple modules is identified with the dual canonical basis of $L(\lambda)$.*

Let $R(n)$, $R^\lambda(n)$ be the quiver-Hecke algebra and the cyclotomic quiver-Hecke algebra of degree n associated with the Cartan datum of the quiver I and the dominant integral weight λ .

Now, we specialize χ_h to 0 for all $h \in \Omega$. Let $\text{mod}^0(R(n))$ be the category of finitely generated modules over $R(n)$ such that $x_k e(\mathbf{i})$ acts locally nilpotently for each $k \in [1, n]$ and $\mathbf{i} \in I^n$.

The relation between Theorems 3.10, 3.11 and Theorems 3.4, 3.5 is the following. It is a consequence of the theory of *intertwiners* of affine Hecke algebras developed in [34] to prove that affine Hecke algebras and their graded versions are Morita equivalent, see Section 2.2.4.

Theorem 3.12 ([9, 42]). *For each λ , n the following hold*

- (a) *there is an equivalence of categories $\text{mod}_I(\mathcal{H}_\zeta^A(n)) \simeq \text{mod}^0(R(n))$,*
- (b) *there is an algebra isomorphism $\mathcal{H}_\zeta^{A,\lambda}(n) \simeq R^\lambda(n)$.*

Remark 3.13.

- (a) Historically, Theorems 3.10, 3.11 have been proved before Theorems 3.4, 3.5 and have been one of the major motivation for the discovery of quiver-Hecke algebras.
- (b) For each λ as above, Dipper-James-Mathas have defined in [12] some *cyclotomic ζ -Schur algebras* $\mathcal{S}_\zeta^{A,\lambda}(n)$ with a Schur functor $\mathcal{S}_\zeta^{A,\lambda}(n) \rightarrow \mathcal{H}_\zeta^{A,\lambda}(n)$ which is a highest weight cover in the sense of Section 2.1.2, see also [38], [41]. Theorem 3.11 has been extended conjecturally by Yvonne in [61] in the following way.

Assume that ζ is a e -th primitive root of 1. Then, there should be a natural structure of U -module on $[\text{mod}_I^{\text{fd}}(\mathcal{S}_\zeta^{A,\lambda})] = \bigoplus_{n \geq 0} [\text{mod}_I^{\text{fd}}(\mathcal{S}_\zeta^{A,\lambda}(n))]$ such that $[\text{mod}_I^{\text{fd}}(\mathcal{S}_\zeta^{A,\lambda})]$ is isomorphic to the level d Fock space $F(s)$ of multicharge $s = (s_1, \dots, s_d)$. Under this isomorphism, the classes of the simple modules is identified with the dual canonical basis of $F(s)$. For $d = 1$ this conjecture was formulated previously by Lascoux-Leclerc-Thibon and proved in [51]. For arbitrary d it follows from the results in Section 4.2 below.

3.2.4. Affine Hecke algebras of types B, C : the conjecture of Enomoto-Kashiwara. Fix a non-degenerate Cartan datum with a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Fix an involution θ of the set I such that $a_{\theta(i),\theta(j)} = a_{ij}$ for all $i, j \in I$.

Definition 3.14 ([15]). Let \mathcal{B}_θ be the associative $\mathbb{Q}(q)$ -algebra with 1 generated by $e_i, f_i, i \in I$, satisfying the usual Serre relations and by commuting invertible elements $l_i, i \in I$, satisfying the relations $l_{\theta(i)} = l_i$ and

$$l_j e_i l_j^{-1} = q^{a_{ij} + a_{i\theta(j)}} e_i, \quad l_j f_i l_j^{-1} = q^{-a_{ij} - a_{i\theta(j)}} f_i, \quad e_i f_j = q^{-a_{ij}} f_j e_i + \delta_{ij} + \delta_{\theta(i),j} l_i.$$

Lemma 3.15 ([15]). *For each dominant integral weight $\lambda = \sum_{i \in I} \lambda_i \omega_i$ in X_+ , there is a unique irreducible \mathcal{B}_θ -module $V_\theta(\lambda)$ generated by a vector v_λ such that $\{v \in V_\theta(\lambda); e_i v = 0\} = \mathbb{Q}(q) v_\lambda$ and $l_i v_\lambda = q^{\lambda_i + \lambda_{\theta(i)}} v_\lambda$ for all $i \in I$.*

Now, set $A = \mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$. The *affine Hecke algebra* of type C_n is the A -algebra $\mathcal{H}_t^C(n)$ which is generated by elements T_0, \dots, T_{n-1} satisfying the braid relations of type B_n , i.e., the elements T_i, \dots, T_{n-1} satisfy the braid relations of $B(S_n)$ and $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$, the quadratic relations $(T_0 - t_0)(T_0 + t_1^{-1}) = 0$ and $(T_i - t_2)(T_i + t_2^{-1}) = 0$ if $i \neq 0$, and commuting elements $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ satisfying the relations $T_0 X_1^{-1} - X_1 T_0 = (t_1^{-1} - t_0) X_1 + t_0 t_1^{-1} - 1$, $T_i X_i T_i = X_{i+1}$ and $T_i X_j = X_j T_i$ if $i \neq 0, j \neq i, i+1$.

Fix $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{C}^\times$ with $\zeta_2^2 \neq 1$. Set $\mathcal{H}_\zeta^C(n) = \mathcal{H}_t^C(n) \otimes_A \mathbb{C}$, where $\chi : A \rightarrow \mathbb{C}$ is the character such that $t_i \mapsto \zeta_i$ for $i = 0, 1, 2$. The semi-direct product $\{1, -1\} \times \mathbb{Z}$ acts on \mathbb{C}^\times by $(\epsilon, n) : i \mapsto i^\epsilon \zeta_2^{2n}$. Let I be a $\{1, -1\} \times \mathbb{Z}$ invariant subset in \mathbb{C}^\times . As above, we may assume that I is a $\{1, -1\} \times \mathbb{Z}$ -orbit. Let $\text{mod}_I(\mathcal{H}_\zeta^C(n))$ be the category of all finitely generated $\mathcal{H}_\zeta^C(n)$ -modules and let $\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^C(n))$ be the full subcategory of all finite dimensional modules of type I (as above). Let $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^C(n))]$ be the Grothendieck group of $\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^C(n))$, and set $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^C)] = \bigoplus_{n \geq 0} [\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^C(n))]$. The group $[\text{mod}_I^{\text{fd}}(\mathcal{H}_\zeta^C)]$

is a module over the bialgebra $[\text{mod}_I^{\text{fd}}(H_\zeta^A)]$. The product and coproduct are given by the induction and restriction with respect to the obvious inclusion $H_\zeta^C(m) \otimes H_\zeta^A(n) \subset H_\zeta^A(m+n)$.

Now, recall the following standard definition.

Definition 3.16 ([13]). A *quiver with involution* (or *symmetric quiver*) is a pair (Γ, θ) , where Γ is a quiver and θ is an *involution* of Γ , i.e., θ consists of a pair of involutions of the sets I, Ω such that $\theta(h)'' = \theta(h')$, $\theta(h)' = \theta(h'')$ for each $h \in \Omega$.

We can view the subset $I \subset \mathbb{C}^\times$ as a quiver with involution without loops, with vertex set I , with an arrow $i \rightarrow i\zeta^2$ for each $i \in \mathbb{C}^\times$ and with the involution $\theta : i \mapsto i^{-1}$. Assume that A is the generalized Cartan matrix associated with I , see Section 3.2.2, and set $\lambda_i = \delta_{i, \zeta_1} + \delta_{i, -\zeta_0}$. The following was conjectured in [15] and proved in [56].

Theorem 3.17 ([56]). Assume that $1, -1 \notin I$. Then, the \mathcal{B}_θ -module $V_\theta(\lambda)$ has a canonical basis, the vector space $[\text{mod}_I^{\text{fd}}(H_\zeta^{B, \lambda})] \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to a specialization of $V_\theta(\lambda)$ at $q = 1$, and the classes of the simple modules are identified with the dual canonical basis of $V_\theta(\lambda)$ at $q = 1$.

Remark 3.18.

- (a) An analogous construction in type D has been given in [29]. The corresponding conjectures are proved in [46].
- (b) If $1 \in I$ or $-1 \in I$ then $[\text{mod}_I^{\text{fd}}(H_\zeta^{B, \lambda})] \otimes_{\mathbb{Z}} \mathbb{Q}$ is no longer irreducible as a \mathcal{B}_θ -module.

3.2.5. Quiver-Hecke algebras of types B, C. The main ingredient in the proof of Theorem 3.17 is a \mathbb{Z} -graded algebra which is an analogue, for affine Hecke algebras of type B, C , of quiver-Hecke algebras.

The Weyl group of type C is the semidirect product $W = S_n \ltimes \{-1, 1\}^n$. For $k \in [1, n]$ let $\varepsilon_k \in W$ be -1 placed at the k -th spot.

Fix a set I with an involution θ . The group W acts on a tuple $\mathbf{i} = (i_{1-n}, \dots, i_{n-1}, i_n)$ of I^{2n} in the obvious way : the reflection $s_l \in S_n$ switches the entries i_l, i_{l+1} and the entries i_{1-l}, i_{-l} , while ε_k switches the entries i_k, i_{1-k} . This action preserves the subset $I^{\theta, n} = \{\mathbf{i} \in I^{2n} ; \theta(i_k) = i_{1-k} \text{ for all } k\}$. The group W also acts on algebra $P = k[x_1, \dots, x_n]$ so that s_l switches x_l and x_{l+1} , while ε_k switches x_k and $-x_k$.

Now, fix $\zeta_2 \in \mathbb{C}^\times \setminus \{-1, 1\}$. Let $I \subset \mathbb{C}^\times$ be a $\{1, -1\} \ltimes \mathbb{Z}$ invariant subset. We view it as a quiver with involution without loops, with vertex set I , with an arrow $i \rightarrow i\zeta_2^2$ for each $i \in \mathbb{C}^\times$, and with the involution given by $\theta : i \mapsto i^{-1}$, compare Section 3.2.4. Assume that A is the generalized Cartan matrix associated with I , see Section 3.2.2. Fix a dominant integral weight $\lambda = \sum_{i \in I} \lambda_i \omega_i$ in X_+ .

Definition 3.19. The *quiver-Hecke algebra* of degree n associated with Γ, θ, λ is the subalgebra $R(n)^{\theta, \lambda} \subset \text{End}_k(\bigoplus_{\mathbf{i} \in I^{\theta, n}} Pe(\mathbf{i}))$ generated by the linear operators $e(\mathbf{i}), x_k, s_l$ with $\mathbf{i} \in I^{\theta, n}, k \in [1, n], l \in [0, n)$ such that $x_k(fe(\mathbf{i})) = x_k fe(\mathbf{i})$ and

$$\sigma_0(fe(\mathbf{i})) = \begin{cases} (2x_1)^{-1}(\varepsilon_1 f - f)e(\mathbf{i}) & \text{if } i_1 = i_0, \\ (x_1)^{\lambda_{i_0}} \varepsilon_1 f e(\varepsilon_1 \mathbf{i}) & \text{if } i_l = \zeta_2^2 i_{l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma_l(fe(\mathbf{i})) = \begin{cases} (x_{l+1} - x_l)^{-1}(s_l f - f)e(\mathbf{i}) & \text{if } i_l = i_{l+1}, \\ (x_{l+1} - x_l) f e(s_l \mathbf{i}) & \text{if } i_l = \zeta_2^2 i_{l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where $l \neq 0$. The k -algebra $R(n)^{\theta, \lambda}$ is \mathbb{Z} -graded, the grading being given by $\deg e(\mathbf{i}) = 0$, $\deg x_k e(\mathbf{i}) = 2d_{i_k}$, $\deg \sigma_0 e(\mathbf{i}) = \lambda_{i_0} + \lambda_{i_1} - 2\delta_{i_1, i_0}$ and $\deg \sigma_l e(\mathbf{i}) = -d_{i_k} a_{i_k, i_{k+1}}$.

The algebra $R(n)^{\theta, \lambda}$ has a presentation similar to the one in [56]. It also admits a geometric realization. More precisely, fix $\sigma \in \{-1, 1\}$. For any representation $x \in E_V$ of Γ , let x° be the representation on the I -graded vector space V° such that $V_i^\circ = V_{\theta(i)}^*$ and $x_h^\circ = \sigma x_{\theta(h)}^*$.

Definition 3.20 ([13]). A σ -orthogonal (resp. a σ -symplectic) representation of (Γ, θ) in V is the datum of a representation $x \in E_V$ with an isomorphism $x \rightarrow x^\circ$ such that the underlying isomorphism $V \rightarrow V^\circ$ defines a symmetric (resp. antisymmetric) non-degenerate bilinear form $V \times V \rightarrow \mathbb{C}$.

Then, the \mathbb{Z} -graded algebra $R(n)^{\theta, \lambda}$ is isomorphic to the Yoneda algebra of a complex of sheaves on the space $E_V^{\theta, \lambda}$ consisting of 1-symplectic representations of Γ which admit a λ -framing in the sense of [56, sec. 4.4].

The relation between $R(n)^{\theta, \lambda}$ and affine Hecke algebras is the following. Fix elements $\zeta_0, \zeta_1 \in \mathbb{C}^\times$. Let $H_\zeta^C(n)$ be the corresponding affine Hecke algebra of type C_n . Set $\lambda_i = \delta_{i, \zeta_1} + \delta_{i, -\zeta_0}$.

We specialize χ_h to 0 for all $h \in \Omega$. Let $\text{mod}^0(R(n)^{\theta, \lambda})$ be the category of finitely generated modules over $R(n)^{\theta, \lambda}$ such that $x_k e(\mathbf{i})$ acts locally nilpotently for each $k \in [1, n]$ and $\mathbf{i} \in I^{\theta, n}$. We have the following analogue of Theorem 3.12.

Theorem 3.21 ([56]). *There is an equivalence of categories*

$$\text{mod}_I(H_\zeta^C(n)) \simeq \text{mod}^0(R(n)^{\theta, \lambda}).$$

Remark 3.22.

- (a) Elements of the 2-exotic nilpotent cone in [30] can be identified with nilpotent λ -framed 1-symplectic representations as above. Theorem 3.21 and the geometric realization of $R(n)^{\theta, \lambda}$ yield another proof of Kato's theorem which parametrizes the simple $H_\zeta^C(n)$ -modules via the 2-exotic nilpotent cone.
- (b) In [56] the \mathbb{Z} -graded algebra $R(n)^{\theta, \lambda}$ is realized as the Yoneda algebra of a complex of sheaves on the space of (-1) -orthogonal representations of (Γ, θ) with a λ -framing. The space of (-1) -orthogonal representations of (Γ, θ) is also used in [14] and yields a geometric construction of some simple $H_\zeta^C(n)$ -modules.

4. Categorical representations and rational DAHA's

4.1. Definition. Fix a non-degenerate Cartan datum $(X, \Phi, X^\vee, \Phi^\vee)$ with a symmetrizable generalized Cartan matrix. Let U_q be the quantum group associated with $(X, \Phi, X^\vee, \Phi^\vee)$.

Let $\{c_{i,j,p,q}\}$ be a family of indeterminates with $i \neq j \in I$ and $p, q \in [0, -a_{ij}]$ such that $c_{i,j,p,q} = c_{j,i,q,p}$. Set $k = \mathbb{Z}[c_{i,j,p,q}][c_{i,j,-a_{ij},0}]^{-1}$ and consider the polynomials given by $Q_{ij}(u, v) = \sum_{p,q \geq 0} c_{i,j,p,q} u^p v^q$ if $i \neq j$, and $Q_{ij}(u, v) = 0$ if $i = j$. Let $R(n)$ be the quiver-Hecke k -algebra associated with $Q_{ij}(u, v)$.

Finally, let Z be a noetherian commutative k -algebra and \mathcal{C} be a Z -linear category whose Hom 's are finitely generated Z -modules. We'll abbreviate $R(n)$ for the Z -algebra $Z \otimes_k R(n)$.

Definition 4.1. An *integrable representation of U on \mathcal{C}* is the datum of a decomposition $\mathcal{C} = \bigoplus_{\mu \in X} \mathcal{C}_\mu$, an adjoint pair of \mathbb{Z} -linear functors (F_i, E_i) with $E_i : \mathcal{C}_\mu \rightarrow \mathcal{C}_{\mu+\alpha_i}$, $F_i : \mathcal{C}_\mu \rightarrow \mathcal{C}_{\mu-\alpha_i}$ and elements $x_i \in \text{End}(F_i)$, $\sigma_{ij} \in \text{Hom}(F_i F_j, F_j F_i)$ satisfying the following conditions

- (a) E_i is isomorphic to a left adjoint of F_i ,
- (b) E_i, F_i are locally nilpotent,
- (c) the relations of the quiver-Hecke algebra $R(n)$ hold for x_i and σ_{ij} ,
- (d) given $\mu \in X$, there are isomorphisms of functors $(E_i F_i)|_{\mathcal{C}_\mu} \simeq (F_i E_i)|_{\mathcal{C}_\mu} \oplus \text{Id}_{\mathcal{C}_\mu}^{(\mu, \alpha_i^\vee)}$ if $(\mu, \alpha_i^\vee) \geq 0$, and $(F_i E_i)|_{\mathcal{C}_\mu} \simeq (E_i F_i)|_{\mathcal{C}_\mu} \oplus \text{Id}_{\mathcal{C}_\mu}^{-(\mu, \alpha_i^\vee)}$ if $(\mu, \alpha_i^\vee) \leq 0$.

An integrable representation of U on \mathcal{C} yields a representation of U on the Grothendieck group $[\mathcal{C}]$ of \mathcal{C} . We'll say that the representation of U on \mathcal{C} *categorifies* the representation of U on $[\mathcal{C}]$.

Now, fix a dominant integral weight $\lambda \in X_+$. Let c_{ir} be a family of indeterminates with $i \in I$ and $r \in (0, s]$, where $s = (\lambda, \alpha_i^\vee)$. Set $Z = \mathbb{k}[c_{ir}]$ and $c_{i0} = 1$. Consider the monic polynomial in $Z[u]$ given by $a_i^\lambda(u) = \sum_{r=0}^n c_{ir} u^{s-r}$. Let $R(n)^\lambda$ be the cyclotomic quiver-Hecke \mathbb{Z} -algebra associated with the quiver-Hecke \mathbb{Z} -algebra $R(n)$, the dominant weight λ and the polynomials a_i^λ .

Let $\text{proj}^0(R^\lambda(n))$ be the \mathbb{Z} -linear category of finitely generated projective modules over $R^\lambda(n)$. We abbreviate $\mathcal{L}(\lambda) = \bigoplus_{n \geq 0} \text{proj}^0(R^\lambda(n))$. Then it is proved in [24], [27], [58] that the induction and restriction yield functors E_i, F_i on $\mathcal{L}(\lambda)$ which satisfy the axioms above. Hence, Theorem 3.5 can be rephrased as follows.

Theorem 4.2. *The induction and restriction functors yield a categorification of the integrable U -module $L(\lambda)$ on $\mathcal{L}(\lambda)$.*

We have the following unicity result.

Theorem 4.3 ([42]). *Given an integrable categorical representation of U on a \mathbb{Z} -linear category \mathcal{C} which is idempotent-closed, and an object $M \in \mathcal{C}_\lambda$ such that $\text{End}(M) = Z$ and $E_i(M) = 0$ for all i , there is a fully faithful functor $\mathcal{L}(\lambda) \otimes_{Z_\lambda} Z \rightarrow \mathcal{C}$ taking the module Z_λ over $R^\lambda(0) \simeq Z_\lambda$ to M .*

Remark 4.4.

- (a) If \mathcal{C} is indeed an abelian category, then the notion of a categorical representation on \mathcal{C} can be formulated in a simpler way, see e.g., [43].
- (b) Using the A -algebra U_A instead of the ring U , and using a \mathbb{Z} -graded category (i.e., a category enriched in \mathbb{Z} -graded modules) instead of the abelian category \mathcal{C} , we define in a similar way a notion of *categorification* of the integrable U_A -module $L_A(\lambda)$ such that the action of q is given by the grade shift functor.
- (c) A proof of the bi-adjointness of the functors E_i, F_i is given in [27, 58].

In the next section we consider two remarkable applications of categorical representations for RDAHA's.

4.2. Categorical representations and CRDAHA's. We fix the integer $d \geq 1$ and we allow n to vary in \mathbb{N} . Consider the categories $\mathcal{O}_c(d, n)$'s introduced in Section 2.1.3. Since the set S/W has exactly d elements, we can view the parameter c of the algebra $H_c(d, n)$ as a d -tuple. We'll assume that this parameter c is *integral*, which means that the parameter $t = \exp(2i\pi c)$ of the cyclotomic Hecke algebra $\mathcal{H}_t(d, n)$ is a tuple (q, q_1, \dots, q_d) where q is a primitive e -th root of 1 and $q_p = q^{s_p}$ for some integers e, s_1, \dots, s_d with $e > 0$. Here (q_1, \dots, q_d) is determined modulo the diagonal action of \mathbb{C}^\times .

Let $F(s)$ be Fock space of multicharge $s = (s_1, \dots, s_d)$, which was introduced in Section 2.1.3. It is a level d integrable module over the affine Kac-Moody algebra of $\widehat{\mathfrak{sl}}_e$ which can be defined as follows.

Set $N = s_1 + \dots + s_d$. Let $\ell \in [0, d]$ be the residue class of N modulo d . Let $L(\omega_\ell)$ be the ℓ -th *fundamental module* of the Lie algebra $\widehat{\mathfrak{gl}}_d$, i.e., the simple integrable module with highest weight the ℓ -th fundamental weight ω_ℓ . Recall that $\widehat{\mathfrak{gl}}_d$ is a central extension of the Lie algebra $\mathfrak{gl}_d[\varpi, \varpi^{-1}]$. The assignment $\varpi \mapsto \varpi^e$ yields a Lie algebra endomorphism of $\widehat{\mathfrak{gl}}_d$ which multiplies the central element by e . Pulling back $L(\omega_\ell)$ by this endomorphism we get a level e integrable representation of $\widehat{\mathfrak{gl}}_d$ on $L(\omega_\ell)$, which is no longer simple but only semisimple. This level e representation admits a commuting level d action of the affine Kac-Moody algebra of $\widehat{\mathfrak{sl}}_e$. The Fock space $F(s)$ is the weight space of $L(\omega_\ell)$ associated with some weight γ_s of the level e action of $\widehat{\mathfrak{sl}}_e \subset \widehat{\mathfrak{gl}}_d$ which depends on the d -tuple s . Hence, it is a level d module of $\widehat{\mathfrak{sl}}_e$.

Theorem 4.5 ([45]). *The induction and restriction functors yield a categorification of the integrable module $F(s)$ on $\bigoplus_{n \geq 0} \mathcal{O}_c(d, n)$.*

The next step is to identify the simple modules in $\mathcal{O}_c(d, n)$ with some *canonical basis* in $F(s)$ and to compute their dimension, for which a conjecture was formulated in [41]. This follows from theorem 4.7 below.

Another remarkable example of categorical representation, inspired by [6, 11], is the following. Assume that s_1, \dots, s_d are non negative. We can consider the *parabolic category* \mathcal{O} of the affine Lie algebra $\widehat{\mathfrak{gl}}_N$, denoted by $\mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$, which consists of modules of level $-e - N$ in the usual category \mathcal{O} of $\widehat{\mathfrak{gl}}_N$ which are integrable with respect to the parabolic subalgebra associated with the blocks decomposition $N = s_1 + \dots + s_d$.

Theorem 4.6 ([54]). *The Kazhdan-Lusztig fusion product of $\widehat{\mathfrak{gl}}_N$ -modules yields a categorical representation of $\widehat{\mathfrak{sl}}_e$ on $\mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$.*

The categorical representations of $\widehat{\mathfrak{sl}}_e$ on $\bigoplus_{n \geq 0} \mathcal{O}_c(d, n)$ and $\mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$ are different : the first one categorify an integrable module of level d and the second one an integrable module of level 0. However, using them one proves the following (see also [33] for a closely related result), which was conjectured in [55].

Theorem 4.7 ([44]). *Assume that $s_p \geq n$ for each $p \in [1, d]$. Then there is a fully faithful exact functor $\mathcal{O}_c(d, n) \subset \mathcal{O}(s)^{\widehat{\mathfrak{gl}}_N}$.*

Note that this theorem implies that the category $\mathcal{O}_c(d, n)$ is Koszul by [47], and it describes its Koszul dual via the *level-rank* duality of I. Frenkel.

Another remarkable application of categorical representations is the following. Let \mathfrak{H} be the Heisenberg algebra. It is an infinite dimensional Lie algebra. The inclusion of the center

$\mathbb{C} \subset \mathfrak{gl}_d$ yields an inclusion $\mathfrak{H} \subset \widehat{\mathfrak{gl}}_d$. The Lie algebra \mathfrak{H} acts on $F(s)$. This action lifts to an action of \mathfrak{H} on the category $\bigoplus_{n \geq 0} \mathcal{O}_c(d, n)$. Using the latter and the level-rank duality, which yields an explicit description of the decomposition of $L(\omega_\ell)$ as a $\widehat{\mathfrak{sl}}_d \times \mathfrak{H} \times \widehat{\mathfrak{sl}}_e$ -module of level (e, de, d) , one proves the theorem below which was conjecture by Etingof [17].

Since the parameter c is a d -uple of complex numbers, it can be identified with a weight of $\widehat{\mathfrak{sl}}_d$. One defines a Lie subalgebra $\mathfrak{a} \subset \widehat{\mathfrak{sl}}_d$ which is generated by the weight vectors of $\widehat{\mathfrak{sl}}_d$ which are *integral with respect to the weight c* , see [17] for details. Let $L^\mathfrak{a}$ be the \mathfrak{a} -submodule of the fundamental module $L(\omega_0)$ of $\widehat{\mathfrak{gl}}_d$ which is generated by the sum of all *extremal* weight subspaces of $L(\omega_0)$. Let δ be the smallest positive imaginary root.

Theorem 4.8 ([48]). *The number of isomorphism classes of finite dimensional irreducible $H_c(d, n)$ -modules is equal to the dimension of the weight subspace of $L^\mathfrak{a}$ associated with the weight $\omega_0 - n\delta$.*

The Etingof conjecture is more general and yields indeed a characterization of the whole filtration of the category $\mathcal{O}_c(d, n)$ by the dimension of the support of the modules, see [48] for the proof. It extends also to a larger family of algebras than the $H_c(d, n)$'s, see [3]. These algebras are not associated in any natural way to Hecke algebras any more. They are called *symplectic reflection algebras* and have been introduced in [18].

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