Applications of equivariant cohomology

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Abstract. We will discuss the equivariant cohomology of a manifold endowed with the action of a Lie group. Localization formulae for equivariant integrals are explained by a vanishing theorem for equivariant cohomology with generalized coefficients. We then give applications to integration of characteristic classes on symplectic quotients and to indices of transversally elliptic operators. In particular, we state a conjecture for the index of a transversally elliptic operator linked to a Hamiltonian action. In the last part, we describe algorithms for numerical computations of values of multivariate spline functions and of vector-partition functions of classical root systems.

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1. Introduction

The aim of this article is to show how theorems of localization in equivariant cohomology not only provide beautiful mathematical formulae, but also stimulated progress in algorithmic computations. I will focus on my favorite themes: quantization of symplectic manifolds and algorithms for polytopes, and neglect many other applications. Many mathematicians have shared their ideas with me, notably Welleda Baldoni, Nicole Berline, Michel Brion, Michel Duflo, Shrawan Kumar, Paul-Emile Paradan and Andras Szenes. I will therefore often employ a collective "we", instead of anxiously weighing my own contribution.

I will describe here the theory of equivariant cohomology with generalized coefficients of a manifold M on which a Lie group K acts. The integral of such a cohomology class is a generalized function $I(\phi)$ on $\mathfrak{k}$, with ϕ in $\mathfrak{k}$, the Lie algebra of K. We wish to solve two problems. The first is to give a "localization formula" for $I(\phi)$ as a "short" expression. The second is: given such a short formula for $I(\phi)$, compute the value $\hat{I}(\xi)$ of the Fourier transform of I at a point $\xi \in \mathfrak{k}^*$ in terms of the initial geometric data. Let me give the motivation for such questions.

By integrating de Rham cohomology classes on a manifold, one obtains certain numerical quantities. For example, the symplectic volume vol_M of a compact symplectic manifold M is the integral of the Liouville form, and the Atiyah–Singer cohomological formula for the index of an elliptic operator D on M is an integral of a cohomology

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class with compact support on T^*M. In the interplay between toric varieties and polytopes, these numerical quantities correspond respectively to the volume of a polytope and to the number of integral points inside a rational polytope. Moreover, the volume is the classical limit of the discrete version, the number of integral points in dilated polytopes.

When the manifold is provided with the action of a compact Lie group K, similar objects are described by integrals of equivariant cohomology classes. The equivariant volume $\mathrm{vol}_M(\phi)$ of a compact Hamiltonian manifold M is a C^∞ function of $\phi \in \mathfrak{k}$, obtained by integrating a particular equivariant cohomology class on M. More generally, if M is a non-compact Hamiltonian manifold with proper moment map, and additional convergence conditions, its equivariant volume $\mathrm{vol}_M(\phi)$ is a generalized function on $\mathfrak{k}$. As shown by Duistermaat–Heckman, the value at $\xi \in \mathfrak{k}^*$ of the inverse Fourier transform of $\mathrm{vol}_M(\phi)$ is the symplectic volume of the Marsden–Weinstein reduction of M at ξ. If a K-invariant operator D is elliptic in the directions transverse to the orbits of K, its index $\mathrm{Index}(D)$ is a generalized function on K, that is, a series of characters of K. It can be described in terms of integrals of equivariant cohomology classes on T^*M. The discrete inversion problem is to determine each Fourier coefficient of $\mathrm{Index}(D)$. When D is an operator linked to the symplectic structure, we think of $\mathrm{Index}(D)$ as the quantum version of the equivariant volume. The Guillemin–Sternberg conjecture, now established by Meinrenken–Sjamaar for any compact Hamiltonian manifold, is an example where such an inversion problem has a beautiful answer in geometric terms.

In the case of a manifold with a circular symmetry, we proved a localization formula for integrals of equivariant cohomology classes as a sum of local contributions from the fixed points. This formula is similar to the Atiyah–Bott Lefschetz fixed point formula for the equivariant index of an elliptic operator on M. A drawback of such formulae is that each individual term has poles, and the Fourier transform of an individual term is meaningless. We will describe here a more general principle of localization for integrals of equivariant cohomology classes. Let κ be a K-invariant vector field tangent to the orbits of K. Witten showed that equivariant integrals on M can be computed in terms of local data near the set C of zeroes of κ. Furthermore, for each connected component C_F of C, the local contribution of C_F is a generalized function on $\mathfrak{k}$. Witten's localization theorem can be best understood through Paradan's identity: $1 = 0$ on $M - C$, in equivariant cohomology with generalized coefficients. Basic definitions and Paradan's identity are explained in Section 3.

The identity $1 = 0$ on $M - C$ has many independent applications that we describe in Section 4. When M is a Hamiltonian manifold with moment map μ, the set of zeroes of the Kirwan vector field is the set of critical points of the function $\|\mu\|^2$. According to Witten's theorem, integrals on reduced spaces of M can be related to equivariant integrals on M. Using a similar localization argument for transversally elliptic operators, Paradan was able to extend the proof of the Guillemin–Sternberg conjecture to some non-compact Hamiltonian spaces linked to representation theory of real semi-simple Lie groups via Kirillov's orbit method. We will state a generalization of the Guillemin–Sternberg conjecture for a transversally elliptic operator canonically attached to a Hamiltonian action in Section 4.

From the localization formulae, one is led to study generalized functions which are regular outside a union of hyperplanes. This will be the topic of Section 5. In particular, we will relate the cohomology ring of toric manifolds to cycles in the complement of an arrangement of hyperplanes.

As there are some relations between Hamiltonian geometry and convex polytopes, these localization theorems have an analogue for polytopes. Such an analogue is the local Euler–Maclaurin formula for polytopes, which was conjectured by Barvinok–Pommersheim. We will indicate in Section 6 how some theoretical results on intersection rings can be turned into effective tools for numerical computations. We implemented algorithms for various problems such as computing the value of the convolution of a large number of Heaviside distributions, the number of integral points in network polytopes and Kostant partitions functions, with applications to the tensor multiplicities formulae. This last section can be read independently. Indeed, these applications to polytopes have elementary proofs, but it was through interaction with Hamiltonian geometry that some of these tools were discovered.

For lack of space, I was only able to include central references to the topics discussed in this text. For more bibliographical comments, references and motivations, one might consult [13], [25], [27], [49] and my home page (notably, the text called "Exégèse") at math.polytechnique.fr/cmat/vergne/. The texts [50] and [48] are introductory and hopefully reader-friendly.

2. Simple examples

In this section, I will give simple examples of sums which can be represented by short formulae, and a simple example of the inverse problem we have in mind. I will also give a sketch of the proof of the stationary phase formula as similar stationary phase arguments will be our fundamental tools.

2.1. Geometric series. Some formulae in mathematics condense a large amount of information in short expressions. The most striking formula perhaps is the one that sums a very long geometric series:

$$\sum_{i=0}^{10000} q^i = \frac{1}{1-q} + \frac{q^{10000}}{1-q^{-1}}.$$

For a straightforward calculation of the left hand side for a given value q, one needs to know the value of the function q^i at all the 10001 integral points of the interval [0, 10000], while for the right hand side one needs only the value of this function at the end points 0, 10000. Note that each term of the right hand side has a pole at $q = 1$.

The short formula (here A, B, i are integers)

$$\sum_{i=A}^{B} q^i = \frac{q^A}{1-q} + \frac{q^B}{1-q^{-1}} = -\frac{q^{A-1}}{1-q^{-1}} - \frac{q^{B+1}}{1-q} \tag{1}$$

is related to the following equalities of characteristic functions:

$$\begin{aligned}\chi([A,B]) &= \chi([A,\infty[) + \chi(]-\infty,B]) - \chi(\mathbb{R})\\ &= \chi(\mathbb{R}) - \chi(]-\infty,A[) - \chi(]B,\infty[).\end{aligned}$$

We draw the picture of the last equality.

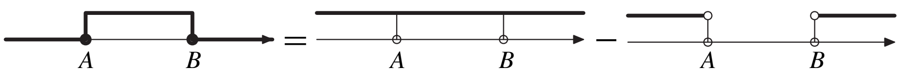


Figure 1. Decomposition of an interval.

Then to sum q^i from A to B, we first sum q^i from $-\infty$ to ∞ and subtract the two sums over the integers strictly less than A and over the integers strictly greater than B. Thus, if

$$S_0 := \sum_{i=-\infty}^{\infty} q^i, \quad S_A := \sum_{-\infty}^{A-1} q^i, \quad S_B := \sum_{B+1}^{\infty} q^i,$$

we obtain formally, or, setting $q = e^{i\phi}$, in the sense of generalized functions on the unit circle,

$$S = S_0 - S_A - S_B. \tag{2}$$

For a value $q \neq 1$, the first sum S_0 is 0 as follows from $(1-q)S_0 = 0$, while S_A, S_B are just geometric progressions and we come back to the short formula (1).

The reader may recognize in Formula (1) a very simple instance of the Atiyah–Bott Lefschetz fixed point formula on the Riemann sphere. Formula (2) illustrates Paradan's localization of elliptic operators, which we describe in Section 4.2. Indeed, Formula (2) is an example of the decomposition of the equivariant index of an elliptic operator on the Riemann sphere in a sum of indices of three transversally elliptic operators (see Example 13).

2.2. Inverse problem. The inverse problem may be described as follows: given a short expression for a sum, compute an individual term of the sum.

Here is an example. Consider the following product of geometric series $G := \big(\sum_{i=0}^{\infty} q_1^i\big)^3 \big(\sum_{j=0}^{\infty} q_2^j\big)^3 \big(\sum_{k=0}^{\infty} q_1^k q_2^k\big)^3$ given by the short expression:

$$S(q_1,q_2) := \frac{1}{(1-q_1)^3}\frac{1}{(1-q_2)^3}\frac{1}{(1-q_1q_2)^3}.$$

Let us compute the coefficient $c(a,b)$ of $q_1^a q_2^b$ in G. If $a \geq b$, an iterated application of the residue theorem in one variable leads to

$$c(a,b) = \mathrm{res}_{x_2=0}\left(\mathrm{res}_{x_1=0}\,\frac{e^{ax_1}e^{bx_2}\,dx_1\,dx_2}{(1-e^{-x_1})^3(1-e^{-x_2})^3(1-e^{-(x_1+x_2)})^3}\right).$$

If we set

$$g(a,b) = \frac{(b+1)(b+2)(b+3)(b+4)(b+5)(7a^2-7ab+2b^2+21a-9b+14)}{14\cdot 5!},$$

we obtain the following equalities.

$$\text{If } a \geq b, \text{ then } c(a,b) = g(a,b). \tag{3}$$

$$\text{If } a \leq b, \text{ then } c(a,b) = g(b,a). \tag{4}$$

We will discuss in Section 5.1 a residue theorem (Theorem 18) in several variables, which gives an algorithmic solution to this type of inversion problem.

The Guillemin–Sternberg conjecture (see Section 4.3) gives a geometric interpretation of the Fourier coefficients of series for similar inversion problems.

2.3. Stationary phase. Let M be a compact manifold of dimension n, f a smooth function on M and dm a smooth density. Consider the function

$$F(t) := \int_M e^{itf(m)}dm.$$

The dominant contribution to the value of this integral as t tends to infinity arises from the neighborhood of the set C of critical points of f. We indicate a proof of this fact, as similar arguments will be employed later on. Consider the image of M by the map $x = f(m)$ and the push-forward of the density dm. Then $F(t) = \int_{\mathbb{R}} e^{itx} f_*(dm)$. Choose a smooth function χ on M, equal to 1 in a neighborhood of the set C and supported near C. Then $F(t) = F_C(t) + R(t)$, where

$$F_C(t) := \int_{\mathbb{R}} e^{itx} f_*(\chi dm), \quad R(t) = \int_{\mathbb{R}} e^{itx} f_*((1-\chi)dm).$$

$R(t)$ is the Fourier transform of a smooth compactly supported function, and thus decreases rapidly at infinity. It is not hard to show that, if f has a finite number of non-degenerate critical points, then

$$F(t) \sim F_C(t) \sim \sum_{p\in C} e^{itf(p)} \sum_{k\geq 0}^{\infty} a_{p,k} t^{-\frac{n}{2}+k},$$

where the constants $a_{p,k}$ can be computed in terms of f and dm near $p \in C$. We can say that asymptotically, the integral "localizes" at a finite number of points p.

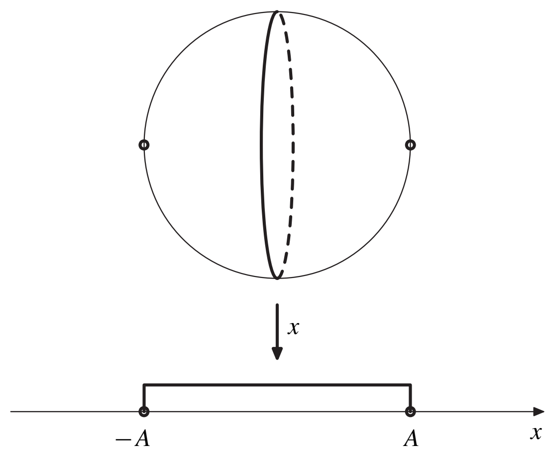


Figure 2. Projecting the sphere $x^2 + y^2 + z^2 = A^2$.

Example 1. Let M be the sphere $\{x^2 + y^2 + z^2 = A^2\}$ of radius A endowed with the Liouville volume form $dm := \frac{dy \wedge dz}{2\pi x}$. Let our function f be the projection onto the x-axis: $f = x$. We immediately see that $f_*(dm)$ is the characteristic function of the interval $[-A, A]$. Thus we obtain the formula

$$F(t) = \int_{-A}^{A} e^{itx} dx = \frac{e^{-iAt}}{-it} + \frac{e^{iAt}}{it}.$$

Observe that here $F(t)$ is not just asymptotically, but exactly equal to the local expression. The reason is that in this example the function f is the Hamiltonian of an action of the circle group $S^1 := \{e^{i\phi}\}$ on a compact symplectic manifold, and dm is the Liouville measure. In such a case, the Duistermaat–Heckman exact stationary phase formula [26] implies that $f_*(dm)$ is locally polynomial on $f(M)$ and that

$$F(t) = \sum_{p \in C} e^{itf(p)} a_{p,0} t^{-\frac{n}{2}}. \tag{5}$$

We will interpret the Duistermaat–Heckman formula as an example of the abelian localization formula (Theorem 7) of integrals of equivariant forms in Section 3.5.

3. Equivariant differential forms

Our motivation to study equivariant differential forms came from representation theory.

Let M be a manifold with an action of the circle group S^1. The Atiyah–Bott fixed point formula [3] describes the equivariant index of an elliptic operator on M in terms of local data near the fixed points of the action. One of the applications of the formula was a geometric interpretation of the Weyl formula for the characters of irreducible representations of compact Lie groups.

The character formula has continuous analogues: the formulae for the Fourier transforms of coadjoint orbits, which are linked to representation theory via Kirillov's orbit method. For compact groups, this is the Harish-Chandra formula; for non-compact semi-simple groups, Rossmann gave a fixed point formula in the case of discrete series characters.

In joint work with Nicole Berline, I found a geometric interpretation of Rossmann's formula using equivariant forms [14]. The cohomological tool behind our computation was a deformation of the de Rham complex with the use of vector fields. A similar formalism was described by Witten [51] with different motivation. There were earlier results which condensed certain integrals on M in short formulae localized near "fixed points", such as Bott's residue formulae [19], its generalization by Baum–Cheeger [11] and the Duistermaat–Heckman exact stationary phase formula [26]. As explained by Atiyah and Bott [4], our result was related to localization in topological equivariant cohomology. However, this revival of "de Rham" theory of equivariant cohomology in terms of differential forms turned out to be very fruitful, especially in applications to non-compact spaces and stationary phase type arguments.

3.1. Equivariant de Rham complex. Notation. I keep the notation N for not necessarily compact manifolds, and M for compact manifolds. Similarly a compact group will be denoted by the letter K, while G will be an arbitrary real Lie group. The letters T, H will be reserved for tori, which are compact connected abelian Lie groups, and therefore are just products of circle groups $\{e^{i\theta_a}\}$. In this case, I take as basis of the Lie algebra $\mathfrak{t}$, elements J_a such that $\exp(\theta_a J_a) := e^{i\theta_a}$ ($\theta_a \in \mathbb{R}$). The gothic german letters $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$, $\mathfrak{h}$ denote the corresponding Lie algebras, $\mathfrak{g}^*$, $\mathfrak{k}^*$, $\mathfrak{t}^*$, $\mathfrak{h}^*$ the dual vector spaces, J^a the dual basis to a basis J_a. If $s \in G$, I denote by N_s the set of fixed points of the action of s on the G-manifold N. The letter ϕ denotes an element of $\mathfrak{g}$. If $\mathfrak{g} = \mathbb{R}J$ is the Lie algebra of S^1, I identify $\mathfrak{g}$ and $\mathbb{R}$. I denote by $S(\mathfrak{g}^*)$ the algebra of polynomial functions on $\mathfrak{g}$, by $C^\infty(\mathfrak{g})$ the space of C^∞ functions on $\mathfrak{g}$ and by $C^{-\infty}(\mathfrak{g})$ the space of generalized functions on $\mathfrak{g}$. An element $v \in C^{-\infty}(\mathfrak{g})$ is denoted by $v(\phi)$ although the value at $\phi \in \mathfrak{g}$ of v may not be defined. By definition, it is always defined in the distributional sense: if $F(\phi)$ is a C^∞ function on $\mathfrak{g}$ with compact support (a test function), then $\langle v, F d\phi\rangle$, denoted by $\int_{\mathfrak{g}} v(\phi)F(\phi)d\phi$, is well defined.

Let us first define the equivariant cohomology algebra with C^∞ coefficients of a G-manifold N.

Let G be a Lie group acting on a manifold N. For $\phi \in \mathfrak{g}$, we denote by $V\phi$ the vector field on N generated by the infinitesimal action of $-\phi$: for $x \in N$, $V_x\phi := \frac{d}{d\varepsilon}\exp(-\varepsilon\phi)\cdot x|_{\varepsilon=0}$. If N is provided with an action of S^1, we simply denote by J the vector field VJ. Let $\mathcal{A}(N)$ be the algebra of differential forms on N with complex coefficients, and denote by d the exterior derivative. If V is a vector field, let $\iota(V)$

be the contraction by V. If $\nu := \sum_{i=0}^{\dim N} \nu_{[i]}$ is a differential form on an oriented manifold N, then the integral of ν over N is by definition the integral of the top degree term of ν: $\int_N \nu := \int_N \nu_{[\dim N]}$, provided that this last integral is convergent.

A smooth map $\alpha\colon \mathfrak{g} \to \mathcal{A}(N)$ is called an *equivariant form*, if α commutes with the action of G on both sides. The equivariant de Rham operator D ([14], [51]) may be viewed as a deformation of the de Rham operator d with the help of the vector field $V\phi$. It is defined on equivariant forms by the formula

$$(D(\alpha))(\phi) := d(\alpha(\phi)) - \iota(V\phi)\alpha(\phi).$$

Then $D^2 = 0$. An equivariant form α is equivariantly closed if $D\alpha = 0$. The *cohomology space*, denoted by $\mathcal{H}^\infty(\mathfrak{g}, N)$, is, as usual, the kernel of D modulo its image. This is an algebra, $\mathbb{Z}/2\mathbb{Z}$-graded in even and odd classes. If $G := \{1\}$, this is the usual cohomology algebra $\mathcal{H}(N)$.

The integral of an equivariant differential form may be defined as a generalized function. Indeed, let $F(\phi)$ be a test function on $\mathfrak{g}$; then $\int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi$ is a differential form on N. If this differential form is integrable on N for all test functions F, then $\int_N \alpha$ is defined by

$$\left\langle \int_N \alpha, F d\phi \right\rangle := \int_N \int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi.$$

Of course if N is compact oriented, $\int_N \alpha(\phi)$ is a C^∞ function.

3.2. Hamiltonian spaces. Examples of equivariantly closed forms arise in Hamiltonian geometry.

Let N be a symplectic manifold with symplectic form Ω. We say that the action of G on N is Hamiltonian with moment map $\mu\colon N \to \mathfrak{g}^*$ if, for every $\phi \in \mathfrak{g}$, $d(\langle \phi, \mu \rangle) = \iota(V\phi) \cdot \Omega$. Thus the zeroes of the vector field $V\phi$ (that is, the *fixed points* of the one parameter group $\exp(t\phi)$) are the critical points of $\langle \phi, \mu \rangle$.

The equivariant symplectic form $\Omega(\phi) := \langle \phi, \mu \rangle + \Omega$ is a closed equivariant form. Indeed,

$$(d - \iota(V\phi))(\langle \phi, \mu \rangle + \Omega) = d(\langle \phi, \mu \rangle) - \iota(V\phi) \cdot \Omega + d(\Omega)$$

and this is equal to 0 as both equations

$$d\Omega = 0, \quad d(\langle \phi, \mu \rangle) = \iota(V\phi) \cdot \Omega$$

hold.

The two basic examples of Hamiltonian spaces with an Hamiltonian action of S^1 are:

(1) $\mathbb{R}^2$ if the action of S^1 has a fixed point.

(2) The cotangent bundle T^*S^1 if the action of S^1 is free.

(1) Let $N := \mathbb{R}^2$ with coordinates $[x, y]$. The circle group S^1 acts by rotations with isolated fixed point $[0, 0]$. The symplectic form is $\Omega := dx \wedge dy$. The function $\frac{x^2+y^2}{2}$ is the Hamiltonian function for the vector field $J := y\partial_x - x\partial_y$. Thus the equivariant symplectic form is

$$\Omega(\phi) = \phi\left(\frac{x^2+y^2}{2}\right) + dx \wedge dy.$$

(2) Let $N := T^*S^1 = S^1 \times \mathbb{R}$. The circle group S^1 acts freely by rotations on S^1. If $[e^{i\theta}, t]$ is a point of T^*S^1 with $t \in \mathbb{R}$, the symplectic form is $\Omega := dt \wedge d\theta$. The function t is the Hamiltonian function for the vector field $J := -\partial_\theta$. Thus the equivariant symplectic form is

$$\Omega(\phi) = \phi\, t + dt \wedge d\theta.$$

A particularly important closed equivariant form is $e^{i\Omega(\phi)}$. If $\dim N := 2\ell$, then

$$e^{i\Omega(\phi)} = e^{i\langle\phi,\mu\rangle}\left(1 + i\Omega + \frac{(i\Omega)^2}{2!} + \cdots + \frac{(i\Omega)^\ell}{\ell!}\right).$$

3.3. Equivariant volumes. Let M be a compact K-Hamiltonian manifold of dimension 2ℓ. By definition, the equivariant symplectic volume of M is the function of $\phi \in \mathfrak{k}$ given by

$$\mathrm{vol}_M(\phi) := \frac{1}{(2i\pi)^\ell}\int_M e^{i\Omega(\phi)} = \int_M e^{i\langle\phi,\mu(m)\rangle}\frac{\Omega^\ell}{\ell!(2\pi)^\ell}.$$

Note that $\mathrm{vol}_M(0)$ is the symplectic volume of M. The last integral, according to the Duistermaat–Heckman-formula [26], localizes as a sum of integrals on the connected components of the set of zeroes of $V\phi$. If this set of zeroes is finite,

$$\mathrm{vol}_M(\phi) = \sum_{p\in \text{ zeroes of } \mathrm{V}\phi} \frac{e^{i\langle\phi,\mu(p)\rangle}}{i^\ell\sqrt{\det_{T_pM} L_p(\phi)}}, \tag{6}$$

where $L_p(\phi)$ is the endomorphism of T_pM determined by the infinitesimal action of ϕ at p. As K is compact, there is a well-defined polynomial square root of the function $\phi \mapsto \det_{T_pM} L_p(\phi)$, the sign being determined by the orientation.

Example 2. Consider, as in Example 1, Section 2.3, the sphere M with S^1-action given by rotation around the x-axis and $\Omega := \frac{dy\wedge dz}{x}$. Then $f := x$ is the Hamiltonian function of the vector field $J := (y\partial_z - z\partial_y)$. The equivariant volume is the C^∞ function

$$\mathrm{vol}_M(\phi) = \int_M e^{i\phi x} dm = \frac{e^{-iA\phi}}{-i\phi} + \frac{e^{iA\phi}}{i\phi}.$$

Let us point out some examples of non-compact manifolds N where the equivariant symplectic volume exists in the sense of generalized functions. We will use the following generalized functions:

$$Y^+(\phi) := \int_0^\infty e^{i\phi t} dt, \quad Y^-(\phi) := \int_{-\infty}^0 e^{i\phi t} dt, \quad \delta_0(\phi) := \int_{-\infty}^\infty e^{i\phi t} dt.$$

Note that the generalized function $Y^+(\phi)$ is the boundary value of the holomorphic function $\frac{1}{-i\phi}$ defined on the upper-half plane, so that it satisfies the relation $(-i\phi)Y^+(\phi) = 1$. The generalized function $\delta_0(\phi)$ satisfies the relation $\phi\, \delta_0(\phi) = 0$.

Return to our two basic examples $\mathbb{R}^2$ and T^*S^1 with action of S^1.

(1) $N := \mathbb{R}^2$. We have

$$\mathrm{vol}_N(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\phi \frac{(x^2+y^2)}{2}} dx dy = \int_0^\infty e^{i\phi r} dr = Y^+(\phi). \tag{7}$$

When $\phi \neq 0$, we have $\mathrm{vol}_N(\phi) = \frac{1}{-i\phi}$. This coincides with what would be the Duistermaat–Heckman formula in the non-compact case: there is just one fixed point $[0, 0]$ for the action.

(2) $N := T^*S^1$. We have

$$\mathrm{vol}_N(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}\times S^1} e^{i\phi t} dt d\theta = \int_{\mathbb{R}} e^{i\phi t} dt = \delta_0(\phi).$$

Thus $\mathrm{vol}_N(\phi)$ is always 0 when $\phi \neq 0$. This is consistent with the fixed point philosophy: the action of S^1 on T^*S^1 is free, thus the set of zeroes of $V\phi$ is empty when $\phi \neq 0$.

The next example illustrates our original motivation to introduce the equivariant differential complex.

Coadjoint orbits. Let G be a real Lie group. Recall [30] that when $N := G\lambda$ is the orbit of an element $\lambda \in \mathfrak{g}^*$ by the coadjoint representation, then N has a G-Hamiltonian structure, such that the moment map is the inclusion $N \to \mathfrak{g}^*$. The equivariant volume $\mathrm{vol}_N(\phi)$ is defined as a generalized function on $\mathfrak{g}$, if the orbit $G\lambda$ is tempered. This is just the Fourier transform of the G-invariant measure supported on $G\lambda \subset \mathfrak{g}^*$.

When N is a coadjoint orbit of a compact Lie group K, Harish-Chandra gave a fixed point formula for $\mathrm{vol}_N(\phi)$. Now this is seen as a special case of the Duistermaat–Heckman formula (6). Rossmann [41] and Libine [32] extended the Harish-Chandra formula to the case of closed coadjoint orbits of reductive non-compact Lie groups, involving delicate constants at fixed points at "infinity" defined combinatorially by Harish-Chandra and Hirai and topologically by Kashiwara.

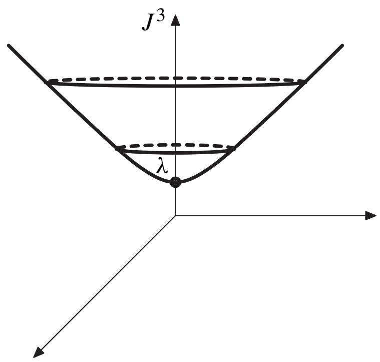
Here is an example. Consider the group $\mathrm{SL}(2, \mathbb{R})$ with Lie algebra $\mathfrak{g}$ with basis

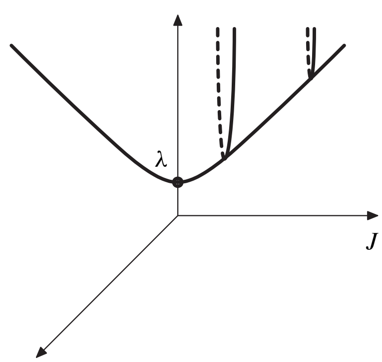
$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The one-parameter group generated by J_3 is compact, while those generated by J_1 and J_2 are non-compact. Let $\lambda > 0$. The manifold

$$N := \{\xi_1 J^1 + \xi_2 J^2 + \xi_3 J^3;\, \xi_3^2 - \xi_1^2 - \xi_2^2 = \lambda^2, \xi_3 > 0\}$$

is a coadjoint orbit. Then the generalized function $\mathrm{vol}_N(\phi_1 J_1 + \phi_2 J_2 + \phi_3 J_3)$ is given by an invariant locally L_1-function, analytic outside $\phi_1^2 + \phi_2^2 - \phi_3^3 = 0$.





$$\mathrm{vol}_N(\phi_3 J_3) = -\frac{e^{i\lambda\phi_3}}{2i\phi_3}, \qquad \mathrm{vol}_N(\phi_1 J_1) = \frac{e^{-|\lambda\phi_1|}}{2|\phi_1|}.$$

The formula for the generator J_3 of a compact group action is in agreement with the "fixed point formula philosophy". The formula for J_1 is difficult to explain within a general framework. Indeed, the non-compact group $\exp(\phi_1 J_1)$ acts freely on N; however, the value of the function $\mathrm{vol}_N(\phi_1 J_1)$ is non-zero even though there are no fixed points on N. In [32], N is embedded in the cotangent bundle of the Riemann sphere $M := P_1(\mathbb{C})$, and a subtle argument of deformation to fixed points of J_1 in M "explains" the formula for $\mathrm{vol}_N(\phi_1 J_1)$.

3.4. Equivariant cohomology groups. After having defined $\mathcal{H}^\infty(\mathfrak{g}, N)$, I will move on to the definition of two other equivariant cohomology groups.

Cartan's complex. Here we consider, for a K-manifold N, the space $\mathcal{A}^{\mathrm{pol}}(\mathfrak{k}, N) := (S(\mathfrak{k}^*) \otimes \mathcal{A}(N))^K$ of equivariant forms $\alpha(\phi)$ depending polynomially on ϕ. The corresponding cohomology space $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N)$ is a $\mathbb{Z}$-graded algebra, where elements of $\mathfrak{k}^*$ have degree two, and differential forms their exterior degree. If N is a vector space with linear action of K, then $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, N) = S(\mathfrak{k}^*)^K$. A basic theorem of H. Cartan says: if K acts on a compact manifold M with finite stabilizers, then $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, M) = \mathcal{H}^*(M/K)$.

If N is non-compact, we can also consider the space $\mathcal{A}^{\mathrm{pol,cpt}}(\mathfrak{k}, N) := (S(\mathfrak{k}^*) \otimes \mathcal{A}^{\mathrm{cpt}}(N))^K$ of equivariant forms $\alpha(\phi)$ which are compactly supported on N. We denote by $\mathcal{H}^{\mathrm{pol,cpt}}(\mathfrak{k}, N)$ the corresponding cohomology space. Integration is well defined on it if N is oriented and the result of integration $\int_N \alpha(\phi)$ is a polynomial function on $\mathfrak{k}$, invariant under the adjoint action of K on $\mathfrak{k}$.

If N is a vector space, there exists a unique element $\mathrm{Thom}(\phi) \in \mathcal{H}^{\mathrm{pol,cpt}}(\mathfrak{k}, N)$ with integral equals to 1.

Let us give the formula for $N := \mathbb{R}^2$ with action of S^1.

• $N := \mathbb{R}^2$. Let χ be any smooth compactly supported function on $\mathbb{R}$ such that $\chi(0) = 1$. Then

$$\mathrm{Thom}_\chi(\phi) := \frac{-1}{2\pi}(\phi\, \chi(x^2 + y^2) + 2\chi'(x^2 + y^2)dx \wedge dy) \tag{8}$$

is a representative of $\mathrm{Thom}(\phi)$.

If N is a vector space, a representative of $\mathrm{Thom}(\phi)$ with "Gaussian look" is given by Mathai–Quillen in [33].

Details on Cartan's theory and further developments can be found in the stern monograph (which contains treasures) [25], or in the attractive book [27]. This de Rham point of view for topological equivariant cohomology seems to be adapted only to smooth spaces. However, the use of equivariant Poincaré dual allows us to work on algebraic varieties, where the Joseph polynomials and the Rossmann localization formula (see [42]) are important tools. For lack of space, I will not pursue this topic. Let me also mention the theory of equivariant Chow groups for algebraic actions on algebraic varieties defined over any field, initiated by Totaro and developed by Edidin–Graham and Brion.

Generalized coefficients ([25]). An equivariant form $\alpha(\phi)$ with $C^{-\infty}$ coefficients is a generalized function on $\mathfrak{g}$ with values in $\mathcal{A}(N)$. Thus for any smooth function F on $\mathfrak{g}$ with compact support, the integral $\int_{\mathfrak{g}} \alpha(\phi)F(\phi)d\phi$ is a differential form on N. We denote by $\mathcal{A}^{-\infty}(\mathfrak{g}, N)$ the space of such forms. If $N := \bullet$ is a point, an equivariant form with $C^{-\infty}$ coefficients is just an element of $(C^{-\infty}(\mathfrak{g}))^G$, that is, an invariant generalized function on $\mathfrak{g}$. The operator D is well defined on $\mathcal{A}^{-\infty}(\mathfrak{g}, N)$, and we denote the corresponding cohomology space by $\mathcal{H}^{-\infty}(\mathfrak{g}, N)$. It is a module over $\mathcal{H}^{\infty}(\mathfrak{g}, N)$. If K acts freely on N, the natural image of $\mathcal{H}^{\infty}(\mathfrak{k}, N)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$ is equal to 0.

Example 3. Let $M := S^1 = \{e^{i\theta}\}$. The group $S^1 := \{e^{i\phi}\}$ acts freely on M by rotations. Let $\mathfrak{g} := \mathbb{R}J$ be the Lie algebra of S^1. Then $\mathcal{H}^{-\infty}(\mathfrak{g}, M) = \mathbb{C}v$, where $\phi v = 0$. A representative of v, still denoted by v, is the closed equivariant form

$$v(\phi) := \delta_0(\phi)d\theta.$$

Note that $\int_M v(\phi) = (2\pi)\delta_0(\phi)$.

On the other hand, we have $1 = -i\, D(Y^+(\phi)d\theta)$ so that

$$1 = 0 \quad \text{in } \mathcal{H}^{-\infty}(\mathfrak{g}, M).$$

Thus the image of $\mathcal{H}^{\infty}(\mathfrak{g}, M) = \mathbb{C} \cdot 1$ in $\mathcal{H}^{-\infty}(\mathfrak{g}, M)$ vanishes.

3.5. Localization or 1 = 0. Let N be a K-manifold and let κ be a K-invariant vector field, tangent to K-orbits. If $\nu\colon N \to \mathfrak{k}$ is a K-invariant map, then κ, defined by

$$\kappa_m := \frac{d}{d\varepsilon} \exp(\varepsilon \nu(m)).m|_{\varepsilon=0} \tag{9}$$

is such a vector field.

Let C be the set of zeroes of κ. Via a K-invariant Riemannian structure $(\bullet, \bullet)$ on N, identify κ with the K-invariant 1-form on N: $\langle \kappa, \bullet \rangle := (\kappa, \bullet)$. Witten considers the exact equivariant form $D\kappa(\phi) = -\langle \kappa, V\phi \rangle + d\kappa$. From our tangential hypothesis, $\phi \mapsto \langle \kappa_m, V\phi \rangle$ is a non-zero element of $\mathfrak{k}^*$ when m is not in C.

Let $\alpha(\phi) \in \mathcal{H}^\infty(\mathfrak{k}, N)$, compactly supported on N. For any test function $F(\phi)$ on $\mathfrak{k}$ and any a in $\mathbb{R}$, we have the equality

$$\iint_{N \times \mathfrak{k}} \alpha(\phi) F(\phi) d\phi = \int_N \int_{\mathfrak{k}} e^{-iaD\kappa(\phi)} \alpha(\phi) F(\phi) d\phi. \tag{10}$$

When a tends to infinity, standard estimates on Fourier transforms shows that the differential form $\int_{\mathfrak{k}} e^{-iaD\kappa(\phi)} \alpha(\phi) F(\phi) d\phi$ becomes very small outside C.

Inspired by Witten's deformation argument, Paradan proves that outside C, the constant 1 is equal to 0 in $\mathcal{H}^{-\infty}(\mathfrak{k}, N - C)$.

Theorem 4 (Paradan, [37]). *On $N - C$, the integral*

$$B(\phi) := i \int_0^\infty e^{-iaD\kappa(\phi)} \kappa \, da$$

is a well defined element of $\mathcal{A}^{-\infty}(\mathfrak{k}, N - C)$ and we have $1 = D(B(\phi))$. Thus

$$1 = 0 \quad \text{in } \mathcal{H}^{-\infty}(\mathfrak{k}, N - C).$$

Indeed, intuitively $B(\phi) = \frac{\kappa}{D\kappa(\phi)}$, so that $DB(\phi) = \frac{D\kappa(\phi)}{D\kappa(\phi)} = 1$.

Multiplying an element $\alpha(\phi)$ of $\mathcal{H}^\infty(\mathfrak{k}, N)$ by 1, we see that $\alpha(\phi)$ vanishes on $N - C$. In the next proposition, we give an explicit representative of α with support near C.

Proposition 5 ([37]). *Let χ be a K-invariant function on N supported on a small neighborhood of C and such that $\chi = 1$ on a smaller neighborhood of C. Let*

$$P(\phi) := \chi + d\chi \wedge B(\phi).$$

Then $P(\phi)$ is a closed equivariant form in $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$ supported near C. Furthermore, we have the equation in $\mathcal{A}^{-\infty}(\mathfrak{k}, N)$:

$$P = 1 + D((\chi - 1)B).$$

Thus, if $\alpha(\phi) \in \mathcal{H}^\infty(\mathfrak{k}, N)$, then $P(\phi)\alpha(\phi)$ is supported near C and equal to $\alpha(\phi)$ in $\mathcal{H}^{-\infty}(\mathfrak{k}, N)$.

In the basic examples $\mathbb{R}^2$ or T^*S^1 with action of S^1, and κ appropriately chosen, the forms $B(\phi)$ and $P(\phi)$ are easy to calculate.

• Consider $N := \mathbb{R}^2$ with $\kappa := y\partial_x - x\partial_y$. On $\mathbb{R}^2 - \{[0,0]\}$, in polar coordinates r, θ, we compute that $B(\phi) = -iY^+(\phi)d\theta$. Thus, if χ is a smooth function with compact support on $\mathbb{R}$ and equal to 1 in a neighborhood of 0, then

$$P(\phi) = (2i\pi)Y^+(\phi)\mathrm{Thom}_\chi(\phi),$$

where $\mathrm{Thom}_\chi(\phi)$ is defined by Formula (8). Note that the integral of $P(\phi)$ on N is $(2i\pi)Y^+(\phi)$.

• Consider $N := T^*S^1$ with $\kappa := -t\partial_\theta$. Then in coordinates t, θ,

$$\begin{aligned} B(\phi) &= -iY^+(\phi)d\theta \quad \text{if } t > 0,\\ B(\phi) &= iY^-(\phi)d\theta \qquad \text{if } t < 0. \end{aligned}$$

If χ is a smooth function with compact support on $\mathbb{R}$ and equal to 1 in a neighborhood of 0, then

$$P(\phi) = \chi(t) + \chi'(t)dt \wedge B(\phi).$$

Note that the integral of $P(\phi)$ on N is $(2i\pi)(Y^+(\phi) + Y^-(\phi)) = (2i\pi)\delta_0(\phi)$.

For the sake of simplicity, assume that N is compact. Consider the form $P \in \mathcal{H}^{-\infty}(\mathfrak{k}, N)$ constructed in Proposition 5 and supported near the set C of zeroes of κ. We write $C = \cup C_F$ where C_F are the connected components of the set C. Write $P = \sum_F P_F$ where P_F is compactly supported on a small neighborhood U_F of C_F. Proposition 6 reduces the calculation of the integral of $\alpha(\phi)$ on N to calculations near C. We obtain the following localization theorem.

Theorem 6 ([37]). *Consider an equivariant class $\alpha(\phi) \in \mathcal{H}^\infty(\mathfrak{k}, N)$. For any component C_F of the set C, let $\alpha_F(\phi) \in \mathcal{H}^\infty(\mathfrak{k}, U_F)$ an equivariant class equal to $\alpha(\phi)$ on U_F. Then*

$$\int_N \alpha(\phi) = \sum_{C_F} \int_{U_F} P_F(\phi)\alpha_F(\phi).$$

In this localization theorem, each local contribution $\int_{U_F} P_F(\phi)\alpha_F(\phi)$ is a generalized function on $\mathfrak{k}^*$. Thus the Fourier transform of each local contribution has a meaning, under a moderate growth condition for α.

As an application, we recover the exact stationary phase, and more generally the "abelian" localization formula, with the following tool. For a S^1-action with generator J, we choose $\kappa := J$, so that C is the set of fixed points of the one parameter group $\exp(\phi J)$. We obtain the following result that we state in the case of isolated fixed points.

Theorem 7 ([14], [51], [4]). *Let S^1 acting on a compact manifold M with isolated fixed points. Let $\alpha(\phi)$ be a closed equivariant form with C^∞ coefficients. Then*

$$(2\pi)^{-\frac{\dim M}{2}} \int_M \alpha(\phi) = \sum_{p\in\{\text{fixed points}\}} \frac{i_p^*\alpha(\phi)}{\sqrt{\det_{T_pM} L_p(\phi)}}.$$

4. Applications and conjectures

4.1. Integrals on reduced spaces

4.1.1. Reduced spaces. Let N be a Hamiltonian K-manifold. Assume that $\xi \in \mathfrak{k}^*$ is a regular value of the moment map μ and let K_ξ be the stabilizer of ξ. Then K_ξ acts with finite stabilizers in $\mu^{-1}(\xi)$ so that $\mu^{-1}(\xi)/K_\xi$ is a symplectic orbifold, called the reduced space at ξ and denoted by N_ξ. We denote by s_ξ the number of elements of the stabilizer of a generic point in $\mu^{-1}(\xi)$. If $\xi = 0$, we also denote $N_0 = \mu^{-1}(0)/K$ by $N//K$. When N is a projective manifold, then $N//K$ is the quotient in the sense of Mumford's geometric invariant theory (see chapter 8.2 [36]). By considering the symplectic manifold $N \times (K \cdot (-\xi))$ (the shifting trick), we may always consider reduction at 0.

If 0 is a regular value, Kirwan associates to an equivariant closed form $\alpha(\phi)$ on N a cohomology class α_{red} on $N//K$: $\alpha(\phi)|_{\mu^{-1}(0)}$ is equivalent to the pull-back of α_{red}. The Kirwan map $\chi : \mathcal{H}^*_K(N) \to \mathcal{H}^*(N//K)$ is surjective, at least when N is compact.

The following result relates the equivariant volume of M to volumes of reduced spaces.

Proposition 8 ([26]). *If M is a K-Hamiltonian manifold, then*

$$\mathrm{vol}_M(\phi) = \int_{\mathfrak{k}^*} e^{i\langle \xi, \phi\rangle}\, \mathrm{vol}(M_\xi) d\xi.$$

This theorem holds also if N is a K-Hamiltonian manifold with proper moment map, under some convergence conditions. As shown by Formula (7) (Section 3.3), if a torus T acts on a vector space N with weights $\beta_a \in \mathfrak{t}^*$, all contained in a half-space, then the equivariant volume $\mathrm{vol}_N(\phi)$ is the boundary value of $\frac{1}{\prod_a(-i\beta_a(\phi))}$. Its Fourier transform is the convolution H of the Heaviside distributions supported on the half-lines $\mathbb{R}^+\beta_a$. Computing volumes of the reduced manifolds N_ξ is the same as computing the value of H at a point $\xi \in \mathfrak{t}^*$. In Section 5.1, we will explain how to do it using iterated residues.

In the next section, we explain Witten's generalization of Proposition 8.

4.1.2. Witten's localization theorem. Assume that M is a compact K-Hamil-tonian manifold with moment map $\mu\colon M \to \mathfrak{k}^*$. We choose a K-invariant identification $\mathfrak{k}^* \to \mathfrak{k}$ given by a K-invariant inner product. The vector field κ defined by $\kappa_m := \frac{d}{d\varepsilon}\exp(-\varepsilon\mu(m))\cdot m|_{\varepsilon=0}$ is K-invariant. We refer to this particularly important vector field as the Kirwan vector field. In this case, the set C of zeroes of κ is the set of critical points of the invariant function $\|\mu\|^2$ on M. One connected component of C is the set $\mu^{-1}(0)$ of zeroes of the moment map (if not empty). The following theorem follows from Witten's deformation argument: Formula (10) in Section 3.5.

Theorem 9 (Witten [52]). *Let M be a compact Hamiltonian K-manifold and $p(\phi)$ an equivariantly closed form with polynomial coefficients. Assume that* 0 *is a regular value of the moment map. Then*

$$\int_{\mathfrak{k}} \left(\int_M e^{i\Omega(\phi)} p(\phi) \right) d\phi = s_0 (2i\pi)^{\dim \mathfrak{k}} \operatorname{vol}(K) \int_{M//K} e^{i\Omega_{\mathrm{red}}} p_{\mathrm{red}}.$$

Let me explain the meaning of the integral on the left. Let $I_M(\phi) := \int_M e^{i\Omega(\phi)} p(\phi)$. This is an analytic function on $\mathfrak{k}$ with at most polynomial growth. We compute $\int_{\mathfrak{k}} e^{i\langle \xi, \phi\rangle} I_M(\phi) d\phi$ in the sense of Fourier transform. This Fourier transform is a polynomial near $\xi = 0$ (this is part of the theorem). The left-hand side $\int_{\mathfrak{k}} I_M(\phi) d\phi$ is by definition the value of this polynomial at $\xi = 0$.

The theorem above is used to compute integrals on reduced spaces. Indeed, the right hand side of the equality is the integral of a cohomology class over the reduced space $M//K$ of M, which is difficult to compute. Instead, we first compute an equivariant integral on the original space M (easy to do thanks to the usual reduction to the maximal torus T and the abelian localization formula). Then we have to compute the value of the Fourier transform of $I_M(\phi)$ at the point 0. This in turn demands the computation of the value of the convolution of Heaviside distributions at some explicit points of $\mathfrak{t}^*$: the images by μ of the fixed points of the action of T on M.

Using different methods, other proofs and refinements to Witten's theorem have been given ([28], [47], [37], [43]). Let us recall Paradan's method. We apply Theorem 6 to the form $\alpha(\phi) = e^{i\Omega(\phi)} p(\phi)$. Here C_F varies over the connected components of the set of critical points of $\|\mu\|^2$. The image of a connected component C_F by the moment map μ is a K-orbit $K\beta$. The set $C_0 := \mu^{-1}(0)$ projecting on 0 is one connected component of C (if non-empty). The Fourier transform of $\int_M e^{i\Omega(\phi)} p(\phi) P_F(\phi)$ when C_F projects on $K\beta$ with $\beta \neq 0$ is supported on $\|\xi\| \geq \|\beta\|$. Thus the value of the Fourier transform of $\int_M e^{i\Omega(\phi)} p(\phi)$ at 0 comes only from $\int_M e^{i\Omega(\phi)} p(\phi) P_0(\phi)$ and requires only local knowledge of our data near $\mu^{-1}(0)$. To summarize, in Witten's localization formula, the Fourier transform of the local terms arising from components different from C_0 are moved away from our focus of attention: the point 0 in $\mathfrak{k}^*$.

These facts are illustrated in the example below. This also shows that local calculations near critical points essentially reduce to $\mathbb{R}^2$ or T^*S^1.

Example 10. Return to Example 2 of the sphere $M := \{x^2 + y^2 + z^2 = A^2\}$, with moment map $\mu(x, y, z) = x$. The critical values of x^2 are $0, A, -A$. The set of critical points has three connected components: the circle C_0 drawn in black in Figure 3, $\{p^+\}$ and $\{p^-\}$. The normal bundle to C_0 is identified with T^*S^1 and the normal bundles to p^+, p^- with $\mathbb{R}^2$. Let

$$\operatorname{vol}_M(\phi) = \frac{1}{2i\pi} \int_M e^{i\Omega(\phi)}.$$

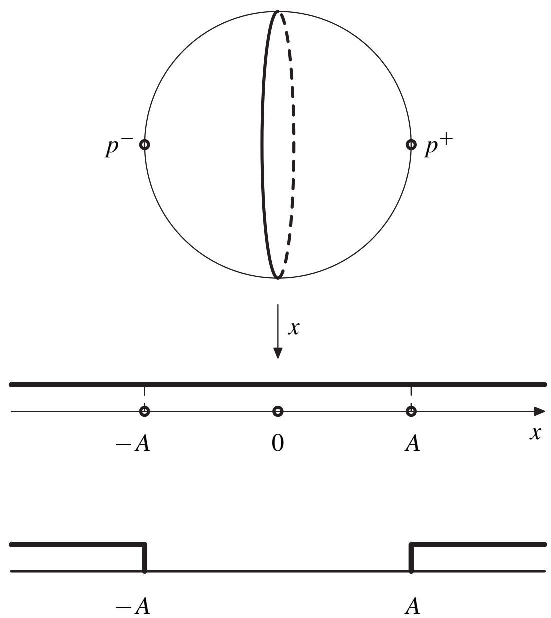


Figure 3. Decomposition of equivariant volumes.

Using the Kirwan vector field, we obtain a decomposition $\mathrm{vol}_M(\phi) = v_0(\phi) + v_{p^-}(\phi) + v_{p^+}(\phi)$ with

$$\begin{aligned}
v_0(\phi) &:= \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)} P_0(\phi) = \delta_0(\phi),\\
v_{p^-}(\phi) &:= \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)} P_{p^-}(\phi) = -e^{-i\phi A} Y^-(\phi),\\
v_{p^+}(\phi) &:= \frac{1}{2i\pi}\int_M e^{i\Omega(\phi)} P_{p^+}(\phi) = -e^{i\phi A} Y^+(\phi).
\end{aligned}$$

This decomposition corresponds to the cone decomposition of the interval $[-A, A]$ described in Figure 1 in Section 2.1.

4.2. Index of transversally elliptic operators. Consider a compact even-dimensional oriented manifold M. For the sake of simplicity, we assume M provided with an almost complex structure. We choose an Hermitian metric $\|\xi\|^2$ on T^*M. For $[x, \xi] \in T^*M$, the symbol of the Dolbeault–Dirac operator $\overline{\partial} + \overline{\partial}^*$ is the Clifford multiplication $c(\xi)$ on the complex vector bundle $\Lambda T_x^* M$. It is invertible for $\xi \neq 0$, since $c(\xi)^2 = -\|\xi\|^2$. Let $\mathcal{E}$ be an auxiliary vector bundle over M, then $c_{\mathcal{E}}([x, \xi]) := c(\xi) \otimes \mathrm{Id}_{\mathcal{E}_x}$ defines an element of $\boldsymbol{K}(T^*M)$, the $\boldsymbol{K}$-theory group of T^*M. Assume that a compact group K acts on M and $\mathcal{E}$. Now the topological index $\mathrm{Index}(c_{\mathcal{E}})$

of $c_{\mathcal{E}} \in \boldsymbol{K}_K(T^*M)$ is an invariant function on K (which computes the equivariant index of the K-invariant operator $\overline{\partial}_{\mathcal{E}} + \overline{\partial}_{\mathcal{E}}^*$). The index theorem of Atiyah–Segal–Singer expresses $\mathrm{Index}(c_{\mathcal{E}})(k)$ $(k \in K)$ in terms of the fixed points of k on M. We constructed (see [13]) the equivariant Chern character $\mathrm{ch}(\phi, \mathcal{E})$ of the vector bundle $\mathcal{E}$ and the equivariant Todd class $\mathrm{Todd}(\phi, M)$ such that (for ϕ small)

$$\mathrm{Index}(c_{\mathcal{E}})(\exp \phi) = (2i\pi)^{-(\dim M)/2} \int_M \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, M). \tag{11}$$

For $\phi = 0$, this is the Atiyah–Singer formula. Formula (11) is a "delocalization" of the Atiyah–Segal–Singer formula. The delocalized index formula (11) can be adapted to new cases such as:

- Index of transversally elliptic operators.
- L^2-index of some elliptic operators on some non-compact manifolds (as in Narasimhan–Okamoto, Parthasarathy, Atiyah–Schmid, Connes–Moscovici).

Indeed, in these two contexts, the index exists in the sense of generalized functions but cannot be always computed in terms of fixed point formulae.

Recall Atiyah–Singer's definition of transversally elliptic operators (see [2]). Let N be a K-manifold and $T^*_K N$ be the conormal bundle to K-orbits. A transversally elliptic pseudo-differential operator S is elliptic in the directions normal to the K-orbits. Thus S together with the action of the Casimir of $\mathfrak{k}$ defines an elliptic system, and the space of solutions of S decomposes as a Hilbert direct sum of finite-dimensional spaces of K-finite solutions. The symbol of S defines an element $\sigma(S)$ of $\boldsymbol{K}_K(T^*_K N)$. The index of the operator S is the character of K in the virtual vector space obtained as difference of K-finite solutions of S and its adjoint. This is an invariant generalized function on K. In [16], we gave a cohomological formula for the index of S in terms of $\sigma(S) \in \boldsymbol{K}_K(T^*_K N)$, as an equivariant integral on T^*N in the spirit of the delocalized formula (11). This result was inspired by Bismut's ideas on delocalizations [18] and Quillen's superconnection formalism.

The following example shows that, contrary to the melancholy remark of Atiyah about his work on transversally elliptic operators (page 6, vol. 4, [1]), there are many transversally elliptic bundle maps of great interest.

Consider a K-manifold N with a K-invariant vector field κ tangent to orbits. As before, we assume that N is provided with a K-invariant almost complex structure and Hermitian metric. We still denote by $c(\xi)$ the Clifford action of $\xi \in T^*_x N$ on the complex space $\Lambda T^*_x N$. The analogue in K-theory of Witten's deformation is the bundle map

$$c_{\kappa,\mathcal{E}}([x, \xi]) := c(\xi - \kappa_x) \otimes \mathrm{Id}_{\mathcal{E}_x}, \tag{12}$$

defined by Paradan [38]. Note that $c_{\kappa,\mathcal{E}}([x, \xi])$ is invertible except if $\xi = \kappa_x$. If furthermore $[x, \xi] \in T^*_K N$, then $\xi = 0$ and $\kappa_x = 0$. Indeed, by our hypothesis, under identification of T^*N with TN, κ_x is tangent to Kx while ξ is normal to Kx.

When N is compact, $c_{\kappa,\mathcal{E}}$ is transversally elliptic and equal in $\boldsymbol{K}$-theory to the elliptic symbol $c_{\mathcal{E}}$, via the deformation $c(\xi - a\kappa_x) \otimes 1_{\mathcal{E}}$, for $a \in [0,1]$. Under the conditions stated below, Paradan's construction defines a transversally elliptic element even if N is a non-compact manifold. See also the construction by M. Braverman [21] of a related operator.

Proposition 11 ([38]). *Assume that the set C of zeroes of κ is compact. Then $c_{\kappa,\mathcal{E}}$ is transversally elliptic on T^*N with support the zero section* $[C, 0]$.

Recall the closed equivariant form P on N supported on a neighborhood of C constructed with the help of κ in Proposition 5. Then

Theorem 12 ([40]). *Near the identity element* 1 *of K, the index of $c_{\kappa,\mathcal{E}}$ is given by the formula*

$$\mathrm{Index}(c_{\kappa,\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim N)/2} \int_N \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, N)P(\phi) \tag{13}$$

and by similar integral formulae over N_s near any point $s \in K$.

When M is compact, Formula (13) reduces to Formula (11) since $P(\phi)$ is equal to 1 in cohomology. But even in this case, Formula (13) has important implications, as the symbol $c_{\mathcal{E}}$ is broken into several parts according to the connected components of C: $c_{\mathcal{E}} = \sum_F c_{\mathcal{E},F}$ where $c_{\mathcal{E},F}$ is supported on $[C_F, 0]$. Thus

$$\mathrm{Index}(c_{\mathcal{E}}) = \sum_F \mathrm{Index}(c_{\mathcal{E},F}).$$

Each local contribution $\mathrm{Index}(c_{\mathcal{E},F})$ is well defined as a character of an infinite-dimensional representation of K. This was one of the motivations of Atiyah and Singer for introducing transversally elliptic operators.

As in the Witten localization formula, this allows in important cases to compute the invariant part $\mathrm{Index}(c_{\mathcal{E}})^K$ through considering only the contribution of C_0. The Fourier series attached to the other components do not interfere with our focus of attention: the multiplicity of the trivial representation. This fact is illustrated in the example below.

Example 13. Return to Example 10. Let A be a positive integer. We identify $P_1(\mathbb{C})$ with $M_A := \{x^2 + y^2 + z^2 = A^2\}$ through the map

$$[z_1, z_2] \mapsto \left(A\frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}, 2A\frac{\Re(z_1\overline{z_2})}{|z_1|^2 + |z_2|^2}, 2A\frac{\Im(z_1\overline{z_2})}{|z_1|^2 + |z_2|^2} \right),$$

the action $(e^{i\phi}z_1, z_2)$ becoming the rotation around the x-axis. We consider the Dolbeault–Dirac operator D_{2A} on $P_1(\mathbb{C})$ with solution space $\oplus_{j+k=2A}\mathbb{C}z_1^j z_2^k$. Twisting the action by $e^{i\phi A}$, its equivariant index is $\sum_{k=-A}^{A} q^k$ with $q := e^{i\phi}$. Using the

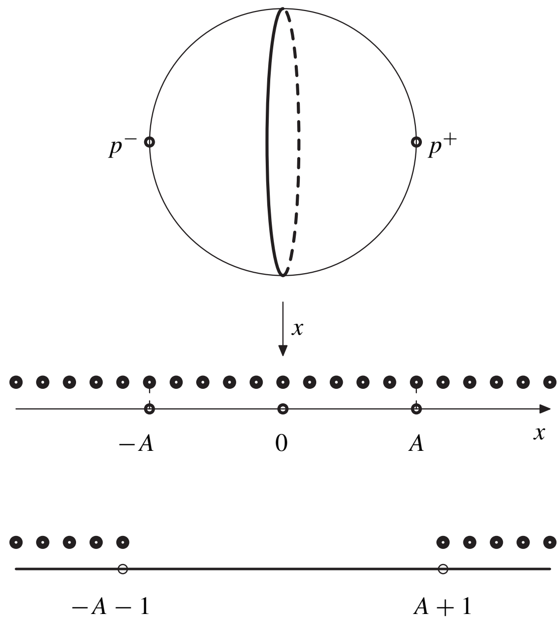


Figure 4. Decomposition of equivariant indices.

Kirwan vector field, we decompose $D_{2A} = D_0 + D_{p^+} + D_{p^-}$ into the sum of three transversally elliptic operators with support $[C_0, 0]$, $[p^+, 0]$, $[p^-, 0]$, respectively. To compute the index of D_0, we are led to compute the set of solutions of the Dolbeault operator on the complex manifold $\mathbb{C}/\mathbb{Z} = S^1 \times \mathbb{R}$, the action of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ being by translations, and we obtain all functions e^{ikz} for any $k \in \mathbb{Z}$. Thus

$$\mathrm{Index}(D_0) = \sum_{k=-\infty}^{\infty} q^k.$$

Near the fixed points p^+, p^-, we obtain the index of the lift of the operators $\overline{\partial}^{\pm}$ (see [2]) on $\mathbb{C}$ (shifted):

$$\mathrm{Index}(D_{p^+}) = -\sum_{k=A+1}^{\infty} q^k, \quad \mathrm{Index}(D_{p^-}) = -\sum_{k=-\infty}^{-A-1} q^k.$$

The equality

$$\mathrm{Index}(D_{2A}) = \mathrm{Index}(D_0) + \mathrm{Index}(D_{p^+}) + \mathrm{Index}(D_{p^-})$$

is Formula (2) in Section 2.1.

It might happen that the integral $\int_N \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, N)$ over our non-compact manifold N is already convergent in the distributional sense, and as $P = 1$ in cohomology, it might happen, modulo the convergence of the boundary term, that the following equality holds

$$\mathrm{Index}(c_{\kappa,\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim N)/2}\int_N \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, N).$$

This is indeed the case for discrete series. To state the result, we rephrase the preceding constructions in the spin context. If N is an even-dimensional oriented spin manifold, and $\mathcal{E}$ a twisting vector bundle, we denote by $\sigma(\xi)$ the Clifford action of $\xi \in T_x^*N$ on spinors, and by $\sigma_{\mathcal{E}}$ the symbol of the twisted Dirac operator $D_{\mathcal{E}}$. If M is a compact K-manifold, the equivariant index of $D_{\mathcal{E}}$ is given by a formula similar to (11):

$$\mathrm{Index}(\sigma_{\mathcal{E}})(\exp\phi) = (2i\pi)^{-(\dim M)/2}\int_M \mathrm{ch}(\phi, \mathcal{E})\hat{A}(\phi, M), \tag{14}$$

where the equivariant class $\hat{A}$ replaces the equivariant Todd class.

Under the same hypothesis as in Proposition 11, the bundle map

$$\sigma_{\kappa,\mathcal{E}}([x,\xi]) = \sigma(\xi - \kappa_x) \otimes \mathrm{I}_{\mathcal{E}_x}$$

is transversally elliptic and its equivariant index is a generalized function on K.

Let G be a real reductive Lie group with maximal compact subgroup K. We assume that the maximal torus T of K is a maximal torus in G. Let $N := G\lambda$ be the orbit of a regular admissible element $\lambda \in \mathfrak{t}^*$. Harish-Chandra associates to λ a representation of G, realized as the L^2-index of the twisted Dirac operator D_λ. The moment map μ for the K-action on N is the projection $G\lambda \to \mathfrak{k}^*$ and the set C of zeroes of the Kirwan vector field κ is easy to compute in this case: it consists of the compact orbit $K \cdot \lambda$.

Theorem 14 (Paradan [39]). *The character of the discrete series $\Theta^G(\lambda)$ restricted to K is the index of the transversally elliptic element $\sigma_{\kappa,\mathcal{L}_\lambda}$ on N.*

Here $\mathcal{L}_\lambda$ is the Kostant line bundle $G \times_{G(\lambda)} \mathbb{C}_\lambda$ on $N = G/G(\lambda)$. A calculation of the index of $\sigma_{\kappa,\mathcal{L}_\lambda}$ (which is supported on $K \cdot \lambda$) leads immediately to Blattner's formula for $\Theta^G(\lambda)|_K$.

4.3. Quantization and symplectic quotients. Let N be a G-manifold (N, G non-necessarily compact), and $\mathcal{E}$ a G-equivariant vector bundle on N with G-invariant connection ∇. We can then construct the closed equivariant form $\mathrm{ch}(\phi, \mathcal{E})$ ([15], [20]). For the sake of simplicity, I assume the existence of a G-invariant complex structure. Then I conjectured (under additional conditions that I do not know how to formulate exactly, see attempts in [46])

Conjecture. There exists a representation $Q(N, \mathcal{E})$ of G such that the character $\mathrm{Tr}_{Q(N,\mathcal{E})}(g)$ is given by the formula

$$\mathrm{Tr}_{Q(N,\mathcal{E})}(\exp\phi) = (2i\pi)^{-(\dim N)/2} \int_N \mathrm{ch}(\phi, \mathcal{E})\mathrm{Todd}(\phi, N) \tag{15}$$

near $1 \in G$ and by a similar integral formula over N_s near any elliptic point s of G.

Thus, via integration of equivariant cohomology classes, it should be possible to define a push-forward map from a generalized $\boldsymbol{K}$-theory of vector bundles with connections on G-manifolds to invariant generalized functions on G, under some convergence conditions, and assuming the existence of a suitable equivariant Todd class.

Remark 15. When N is a coadjoint admissible regular orbit of any real algebraic Lie group G and $\mathcal{E}$ the Kostant half-line bundle, Formula (15), with the $\hat{A}$ class instead of the Todd class, becomes Kirillov's universal formula [29] for characters (proved by Kirillov for compact and nilpotent groups, by Duflo, Rossmann, Bouaziz, Khalgui, Vergne,... for any real algebraic group). If N, G are compact, Formula (15), with $\hat{A}$ instead of Todd, is the equivariant index formula for the Dirac operator twisted by $\mathcal{E}$. Thus Formula (15), modified as in [46], is a fusion of the Kirillov universal character formula and of the formulae of Atiyah–Segal–Singer for indices of twisted Dirac operators.

Now let (M, Ω) be a compact symplectic manifold with Hamiltonian action of a compact group K. We assume the existence of a K-equivariant line bundle $\mathcal{L}$ on M with connection ∇ of curvature equal to $i\Omega$. In other words, M is prequantizable in the sense of [30] and we call $\mathcal{L}$ the Kostant line bundle. We take an almost complex structure compatible with Ω (see [35]). Then we denote $Q(M, \mathcal{L})$ simply by $Q(M)$. This is a canonical finite-dimensional virtual representation $Q(M)$ of K, the quantization of the symplectic manifold M. The spectrum of the action of $\phi \in \mathfrak{k}$ in $Q(M)$ should be the "quantum" version of the levels of energy of the Hamiltonian function $\langle \mu, \phi \rangle$ on M (see [49] for survey). Guillemin and Sternberg conjectured in 1982 that the multiplicity of the irreducible representation V_ξ of K (of highest weight $\xi \in \mathfrak{t}^*_+ \subset \mathfrak{k}^*$) in the representation $Q(M)$ is equal to $Q(M_\xi)$ and proved it for the case of Kähler manifolds. This is summarized by the slogan: "Quantization commutes with Reduction". In other words, when $\xi = 0$, we should have the equality

$$\int_K \mathrm{Tr}_{Q(M)}(k)dk = \int_{M//K} \mathrm{ch}(\mathcal{L}//K)\mathrm{Todd}(M//K).$$

Although a fixed point formula exists for $\mathrm{Tr}_{Q(M)}(k)$, it is difficult to extract the Guillemin–Sternberg conjecture directly from the Atiyah–Bott Lefschetz formula. Thus this conjecture (fundamental for the credo of quantum mechanics) remained unproved for years. Witten's inversion formula [52]

$$\int_{\mathfrak{k}} \left(\int_M e^{i\Omega(\phi)} p(\phi) \right) d\phi = s_0 (2i\pi)^{\dim \mathfrak{k}} \mathrm{vol}(K) \int_{M//K} e^{i\Omega_{\mathrm{red}}} p_{\mathrm{red}}$$

is in strong analogy with this conjecture. In particular, apart from factors of $2i\pi$, the form $e^{i\Omega_{\mathrm{red}}}$ is equal to $\mathrm{ch}(\mathcal{L}//K)$. Meinrenken [34] used the Atiyah–Bott Lefschetz formula and symplectic cutting in a subtle way to give a proof of Guillemin–Sternberg conjecture for any compact K-Hamiltonian manifold. This result was extended further to singular symplectic quotients in Meinrenken–Sjamaar [35].

Definition 16. Let N be a prequantizable Hamiltonian K-manifold with Kostant line bundle $\mathcal{L}$ such that the moment map is proper and the set of zeroes of the Kirwan vector field κ is compact. Define

$$Q(N) := \mathrm{Index}(c_{\kappa,\mathcal{L}}).$$

Thus $Q(N)$ is a Fourier series of characters $\mathrm{Tr}(V_\xi)$.

Conjecture. The multiplicity m_ξ of the irreducible representation V_ξ in $Q(N)$ is equal to $Q(N_\xi)$.

When N is compact, this is the Guillemin–Sternberg conjecture.

Paradan [39] proved this conjecture (in the spin context) when $N := G\lambda$ is an admissible regular elliptic coadjoint orbit of a reductive real Lie group G and K the maximal compact subgroup of G. Together with Theorem 14, this implies that irreducible representations $\Theta^K(\xi)$ (of highest weight $\xi - \rho_{\mathfrak{k}}$) of K occurring in Harish-Chandra's discrete series $\Theta^G(\lambda)|_K$ are such that ξ lies in the interior of the Kirwan polytope $\mu(N) \cap \mathfrak{t}^*_+$. This is a strong constraint on representations appearing in $\Theta^G(\lambda)|_K$.

Example 17. Figure 5 is the drawing for the restriction of the representation $\Theta^G(\lambda)$ of SO(4, 1) to SO(4). The black dots are the ξ such that $\Theta^K(\xi)$ occurs in $\Theta^G(\lambda)$ (they all occur with multiplicity 1). The horizontal strip is the Kirwan polytope $\mu(G\lambda)\cap\mathfrak{t}^*_+$.

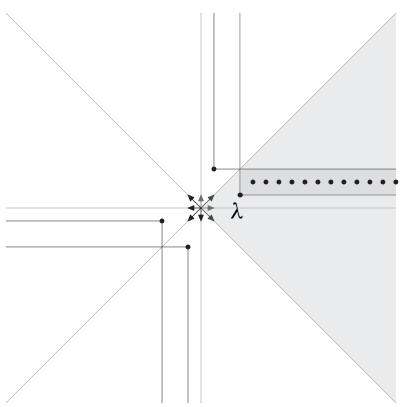


Figure 5. Restriction of discrete series and the Kirwan polytope.

5. Arrangement of hyperplanes

5.1. Convolution of Heaviside distributions and cycles in the complement of a set of hyperplanes. Let us consider a set $\mathcal{B} := \{\beta_1, \dots, \beta_n\}$ of linear forms β_a on a vector space V of dimension r, all in an open half-space of V^*. We assume that the set $\mathcal{B}$ spans V^*. By definition, an element $\xi \in V^*$ is regular if it does not lie in a cone spanned by $(r-1)$ elements of $\mathcal{B}$. A connected component of the set of regular elements is called a chamber.

The convolution H of the Heaviside distributions of the half-lines $\mathbb{R}^+\beta_a$ is a multivariate spline function on V^*, that is, a locally polynomial function continuous on the cone $\mathrm{Cone}(\mathcal{B})$ spanned by $\mathcal{B}$. Our problem is to compute $H(\xi)$ at a particular point $\xi \in V^*$. In principle, $H(\xi)$ is given by the following limit of integrals (on the non-compact "cycle" V of dimension r, and in the sense of Fourier transforms):

$$H(\xi) = \lim_{\varepsilon \to 0} (2i\pi)^{-r} \int_V e^{-i\langle \xi, v\rangle} \frac{1}{\prod_{a=1}^n \langle \beta_a, v + i\varepsilon\rangle} dv$$

where ε is in the dual cone to $\mathrm{Cone}(\mathcal{B})$.

Consider the complement of the hyperplanes defined by $\mathcal{B}$ in the complexified space $V_{\mathbb{C}}$:

$$V(\mathcal{B}) := \{v \in V_{\mathbb{C}};\, \langle v, \beta\rangle \neq 0 \text{ for all } \beta \in \mathcal{B}\}.$$

Jeffrey and Kirwan [28] introduced a residue calculus on the space of functions defined on $V(\mathcal{B})$. A rational function on $V(\mathcal{B})$ is of the form $R(v) = \frac{L(v)}{\prod_{a=1}^n \langle \beta_a, v\rangle^{n_a}}$ where $L(v)$ is a polynomial. The following theorem results from Jeffrey–Kirwan ideas, further refined in [22] and [45]. We still denote by dv the holomorphic r-form $dv_1 \wedge \cdots \wedge dv_r$ on $V_{\mathbb{C}}$.

Theorem 18. *Let $\mathfrak{c} \subset \mathrm{Cone}(\mathcal{B})$ be a chamber. There exists a compact oriented cycle $Z(\mathfrak{c})$ of dimension r contained in $V(\mathcal{B})$ such that for any rational function R on $V(\mathcal{B})$ and any $\xi \in \mathfrak{c}$*

$$\lim_{\varepsilon \to 0} \int_V e^{-i\langle \xi, v\rangle} R(v + i\varepsilon) dv = \int_{Z(\mathfrak{c})} e^{-i\langle \xi, v\rangle} R(v) dv.$$

We gave in [45] a representative for the r-dimensional cycle $Z(\mathfrak{c})$ in $\mathbb{C}^r$ as the set of solutions of r real analytic equations related to quantum cohomology. Furthermore, we gave a simple algorithm, further simplified by De Concini–Procesi [24], to compute the homology class of $Z(\mathfrak{c})$ as a disjoint union of tori, so that integration on $Z(\mathfrak{c})$ is simply the algebraic operation of taking ordinary iterated residues. Indeed, if $T(\boldsymbol{\varepsilon}) \subset V(\mathcal{B})$ is a compact torus of the form in some coordinates $(v_1, v_2, \dots, v_r) \in \mathbb{C}^r := V_{\mathbb{C}}$

$$T(\boldsymbol{\varepsilon}) := \{v \in V(\mathcal{B});\, |v_k| = \varepsilon_k, \text{ for } k = 1, \dots, r\},$$

with $\boldsymbol{\varepsilon} := [\varepsilon_1 \ll \varepsilon_2 \ll \cdots \ll \varepsilon_r]$ a sequence of increasing real numbers (here $\varepsilon_1 \ll \varepsilon_2$ meaning that ε_2 is significantly greater than ε_1, see [45] for precise definitions), then the integration on $T(\boldsymbol{\varepsilon})$ of a function $F(v_1, v_2, \ldots, v_r)$ with poles on the hyperplanes defined by $\mathcal{B}$ is

$$\frac{1}{(2i\pi)^r}\int_{T(\boldsymbol{\varepsilon})} F(v_1, v_2, \ldots, v_r)dv = \operatorname{res}_{v_r=0}\operatorname{res}_{v_{r-1}=0}\cdots\operatorname{res}_{v_1=0} F(v_1, v_2, \ldots, v_r),$$

where each residue is taken assuming that the variables with higher indices have a fixed, non-zero value.

Let me explain why this algorithm is efficient for computing the convolution $H(\xi)$ of a large number of Heaviside distributions in a vector space of small dimension. The usual way to compute $H(\xi)$ is by induction on the cardinal of $\mathcal{B}$. Here we fix ξ in a chamber $\mathfrak{c}$ and we compute the cycle $Z(\mathfrak{c})$ (depending on $\mathfrak{c}$) by induction on the dimension of V. It can be done quite quickly using the maximal nested sets of De Concini–Procesi, at least for classical root systems [5].

5.2. Intersection numbers on toric manifolds. Let T be a torus of dimension r acting diagonally on $N := \mathbb{C}^n$ with weights $\mathcal{B} := [\beta_1, \beta_2, \ldots, \beta_n]$. We assume that the cone $\mathrm{Cone}(\mathcal{B})$ spanned by the vectors β_a is an acute cone in $\mathfrak{t}^*$ with non-empty interior. The moment map $\mu\colon \mathbb{C}^n \to \mathfrak{t}^*$ for the action of T is $\mu(z_1, \ldots, z_n) = \sum_{a=1}^n |z_a|^2\beta_a$. The reduced space $N_\xi = \mu^{-1}(\xi)/T$ at a point $\xi \in \mathrm{Cone}(\mathcal{B})$ is *a toric variety*. It is an orbifold if ξ is regular. The space N_ξ is still provided with a Hamiltonian action of the full diagonal group $H := (S^1)^n$ with Lie algebra $\mathfrak{h} := \{\sum_{a=1}^n \nu_a J_a\}$. The image of N_ξ under the moment map for H is the convex polytope

$$P(\xi) := \left\{ \textstyle\sum_{a=1}^n x_a J^a \in \mathfrak{h}^*;\ x_a \geq 0;\ \sum_{a=1}^n x_a\beta_a = \xi\right\}.$$

Computing the volume of the polytope $P(\xi)$ is the same as computing the symplectic volume of N_ξ. All manifolds N_ξ when ξ varies in a chamber $\mathfrak{c}$ are the same toric manifold $N_\mathfrak{c}$, the additional data $\xi \in \mathfrak{c}$ being in one-to-one correspondence with the Hamiltonian structure on $N_\mathfrak{c}$ coming from its identification with the reduced space N_ξ.

The T-equivariant cohomology of N is $S(\mathfrak{t}^*)$. For each chamber $\mathfrak{c}$, the Kirwan map gives a surjective map $\chi(p) := p_{\mathrm{red}}$ from $S(\mathfrak{t}^*)$ to $\mathcal{H}^*(N_\mathfrak{c})$. The following theorem allows us to compute integrals on toric manifolds.

Theorem 19 ([45]). *Let $p \in S(\mathfrak{t}^*)$, then*

$$\int_{N_\mathfrak{c}} \chi(p) = (2i\pi)^{-r}\int_{Z(\mathfrak{c})} \frac{p(\phi)}{\prod_{a=1}^n\langle\beta_a, \phi\rangle}d\phi.$$

Let $\xi \in \mathfrak{c}$ and let $p(\phi) := \langle\phi, \xi\rangle$. Then the cohomology class p_{red} is the symplectic form of $N_\mathfrak{c}$ determined by ξ. This way we obtain the formula:

Corollary 20. *Let $\xi \in \mathfrak{c}$, then*

$$\mathrm{vol}(N_\xi) = \frac{1}{(2\pi)^r}\int_{Z(\mathfrak{c})} \frac{e^{-i\langle\xi,\phi\rangle}}{\prod_{a=1}^n\langle\beta_a, \phi\rangle}d\phi.$$

We recall that the homology class of the cycle $Z(\mathfrak{c})$ is computed recursively so that the preceding integral is easily calculated using iterated residues.

6. Polytopes and computations

It is well known that many theorems on toric varieties have analogues in the world of polytopes. With Brion, Szenes, Baldoni, Berline, we carefully gave elementary proofs of the corresponding theorems on polytopes, even if our inspiration came from equivariant cohomology on Hamiltonian manifolds.

Let $\mathcal{B} := [\beta_1, \ldots, \beta_n]$ be a sequence of linear forms on a vector space V of dimension r strictly contained in a half-space of V^*. If $\xi \in V^*$, the partition polytope is

$$P_{\mathcal{B}}(\xi) := \left\{ \boldsymbol{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n;\ x_a \geq 0;\ \textstyle\sum_{a=1}^n x_a \beta_a = \xi \right\}.$$

Any polytope can be realized as a partition polytope.

Example 21 (Transportation polytopes). Consider two sequences $[r_1, r_2, \ldots, r_k]$, $[c_1, c_2, \ldots, c_\ell]$ of positive numbers with $\sum_i r_i = \sum_j c_j$. Then Transport(k, ℓ, r, c) is the polytope consisting of all real matrices with k rows and n columns, with non-negative entries, and with sums of entries in row i equal to r_i and in column j equal to c_j. This is a special case of a network polytope (see [6], [7]).

The volume of $P_{\mathcal{B}}(\xi)$ is equal to the value at ξ of the convolution of the Heaviside distributions supported on the half-lines $\mathbb{R}^+\beta_a$. This becomes computationally hard if there is a large number of convolutions. The volume of Transport(k, ℓ, r, c) necessitates the convolution of $k\ell$ Heaviside distributions in a space of dimension $k+\ell-1$. For example, Beck–Pixton [12] could compute, on parallel computers, the volume of Transport(k, ℓ, r, c) for $k = 10$, $\ell = 10$, for special values $r_i = c_j = 1$ in 17 years of computation time (scaled on 1 Ghz processor).

Theorem 22. *Let $\mathfrak{c}$ be a chamber of* Cone$(\mathcal{B})$ *and let $\xi \in \bar{\mathfrak{c}}$. Then*

$$\operatorname{vol}(P_{\mathcal{B}}(\xi)) = (2i\pi)^{-r} \frac{1}{(n-r)!} \int_{Z(\mathfrak{c})} \frac{\langle \xi, v \rangle^{n-r}}{\prod_{a=1}^n \langle \beta_a, v \rangle} dv.$$

Using De Concini–Procesi recursive determination of $Z(\mathfrak{c})$, this formula is expressed as a specific sum of iterated residues.

Assume the β_a span a lattice Λ in V^*, and that ξ is in Λ. The discrete analogue of the volume of $P_{\mathcal{B}}(\xi)$ is the number $N_{\mathcal{B}}(\xi)$ of integral points in the rational polytope $P_{\mathcal{B}}(\xi)$. A fundamental result of Barvinok [8] asserts that $N_{\mathcal{B}}(\xi)$ can be computed in polynomial time, when n is fixed.

The function $N_{\mathcal{B}}(\xi)$ associates to the vector ξ the number of ways to represent the vector ξ as a sum of a certain number of vectors β_a. This is called the vector-partition function of $\mathcal{B}$. There is also a formula [44] for $N_{\mathcal{B}}(\xi)$ as an integral on the cycle $Z(\mathfrak{c})$. This integral formula has interesting theoretical applications, such as information on the jumps of the partition function from chamber to chamber. For example, the appearance of the five linear factors in $g(a, b)$ (Formula (3) of Section 2.2) follows from [44]. However, except for relatively good systems $\mathcal{B}$, this formula does not allow polynomial time computations. A program for the counting of number of points in any rational polytope following Barvinok's algorithm is done in Latte [31]. For systems not too far from unimodularity, our programs based on integration on $Z(\mathfrak{c})$, that is, on iterated residues, are more efficient. It leads to the fastest computation of number of integral points in network polytopes [6], Kostant partition functions, weight multiplicities c_μ^λ and tensor product multiplicities $c_{\lambda,\mu}^\nu$ of classical Lie algebras (the bit size of the weights λ, μ, ν can be very large [5], [23]).

Finally, let me describe the local Euler–Maclaurin formula which was conjectured by Barvinok–Pommersheim [10]. It was after observing the analogy of this conjecture with the localization theorem (Theorem 6) that I fully realized the beauty of this conjecture. Nicole Berline and I proved it by using elementary means, based on the study of some valuations on rational cones in an Euclidean space,

Let P be a convex polytope in $\mathbb{R}^d$. For the sake of simplicity we assume that P has *integral vertices*. Let $\mathcal{F}$ be the set of faces of P. For each face F of P, the transverse cone of P along F is a cone of dimension equal to the codimension of F.

Theorem 23 (Local Euler–Maclaurin formula). *For each face F, there exists a constant coefficients differential operator D_F (of infinite order), depending only on the transverse cone of P along F, such that, for any polynomial function Φ on $\mathbb{R}^d$,*

$$\sum_{\xi \in P \cap \mathbb{Z}^d} \Phi(\xi) = \sum_{F \in \mathcal{F}} \int_F D_F(\Phi).$$

The detailed statement for any rational convex polytope and what we really mean by "depending only on" is in [17].

The operators D_F have rational coefficients and can be computed in polynomial time when d and the order of the expansion are fixed, with the help of the Barvinok signed decomposition of cones and the LLL short vector algorithm. The local property of D_F means that if two polytopes P and P' are the same in a neighborhood of a generic point of F, then the operators D_F for P and P' coincide.

The local Euler–Maclaurin formula gives in particular a local formula for the number of integral points in P or in the dilated polytopes tP. The Ehrhart polynomial $E(P)(t)$ is defined as the number of integral points in tP, for t a non-negative integer. Then $E(P)(t) = \sum_{i=0}^n e_i t^{n-i}$, with $e_0 = \mathrm{vol}(P)$. Barvinok [9] recently showed that the (periodic) coefficients e_i with $i \leq k$ can be computed in polynomial time, when P is a rational simplex. We hope to implement soon another polynomial time algorithm for the same problem based on our local formula.

Even though time often prevails, in numerical computations as in life, it was rewarding for us to see that our theoretical results could help in effective computations.

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