Variations of Hodge Structure and Algebraic Cycles

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0 Introduction

0.1. Let X be a complex projective variety. Then each cohomology group of X admits a Hodge structure, that is a decomposition of $H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C}$ into a direct sum $\bigoplus_{p+q=k} H^{p,q}(X)$, where $H^{p,q}(X) \simeq H^q(\Omega_X^p) \subset H^k(X,\mathbb{C})$ is the set of classes that can be represented by a closed k-form everywhere of type (p,q). We will be concerned in this paper with the relations between Hodge structures and Chow groups $CH_{.}(X)$, where $CH_\ell(X)$ is the group of ℓ-cycles (= arbitrary integral combinations of ℓ-dimensional subvarieties) modulo rational equivalence [5].

0.2. The simplest way to go from Chow groups to Hodge structures is to use the cycle class map $c : CH_k(X) \to H^{2n-2k}(X)$, which to a cycle $\Gamma = \Sigma n_i W_i$ associates $c(\Gamma) = \Sigma n_i c(W_i)$, where $c(W_i)$ is the Poincaré dual of the current of integration over W_i. The cycle class $c(\Gamma)$ is easily seen to be a Hodge class; that is, to belong to $H^{2k}(X,\mathbb{Z}) \cap H^{k,k}(X)$. The famous Hodge conjecture asserts that $H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$ is equal to $\operatorname{Im} c \otimes \mathbb{Q}$. Not much is known except for the case $k = 1$ (due to Lefschctz) and particular cases for $k > 1$ (see e.g. [35], [36], [37]). But recently an important theoretical evidence for it was given by Cattani, Deligne, and Kaplan, who proved:

0.3. THEOREM [10]. *Let $\mathcal{X} \to B$ be an algebraic family of smooth algebraic varieties $\mathcal{X}_b$ parametrized by a quasi-projective variety B. Then the set $\{(b,\lambda),\ b \in B,\ \lambda \in H^{2k}(\mathcal{X}_b,\mathbb{Z}) \cap H^{k,k}(\mathcal{X}_b)\}$ is a countable union of finite covers of algebraic subvarieties of B.*

These sets are called Hodge loci or Noether-Lefschetz loci and were studied in [4], [34], [IVHS,II].

The class of a cycle is sometimes a very poor invariant: for example the class of a zero-cycle $\Sigma n_i p_i$ is just its degree $\Sigma n_i \in \mathbb{Z}$. Of course a much deeper relation between $CH(X)$ and Hodge structures on X is expected (see [5], [28], [30]); however, for Z a cycle in a family of varieties $(\mathcal{X}_b)_{b\in B}$, the Hodge class of Z carries very much information on the family of cycles $Z_{|\mathcal{X}_b} \in CH(\mathcal{X}_b)$, and this will be the main topic of Section 1.

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0.4. One way to refine the cycle class map is to consider the Deligne cycle class c_D : $CH^p(X) \to H_D^{2p}(X, \mathbb{Z}(p))$ (Deligne cohomology) where $H_D^{2p}(X, \mathbb{Z}(p)) = \mathbb{H}^{2p}(0 \to \mathbb{Z} \to \mathcal{O}_X \to \Omega_X \to \cdots \to \Omega_X^{p-1} \to 0)$ (see [20], [17]). Its restriction to the set of codimension p-cycles homologous to zero was first defined by Griffiths [26] and called the Abel-Jacobi map. It takes values into the pth intermediate jacobian:

$$J^{2p-1}(X) = H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1} \oplus H^{2p-1}(X, \mathbb{Z}).$$

(Here and in the sequel we use the notation $F^k H^\ell(X) := \bigoplus_{p \geq k} H^{p,\ell-p}(X)$.)

0.5. Deligne cohomology groups also appear as the targets of regulator maps, which are defined on higher Chow groups ([6], [27]). Regulators have the same formal properties, from the point of view of infinitesimal variations of Hodge structure, as Abel-Jacobi maps, and we will see in the next section that the result of [23] holds as well for them. To give an idea of what they are, consider for simplicity the case of $K_1(X)^{(p+1)} \simeq CH^{p+1}(X,1) \simeq H^p_{\mathrm{Zar}}(X, \mathcal{K}_{p+1})$. Using Bloch's definition or using the Gersten resolution of the sheaf $\mathcal{K}_{p+1}$ [5], this group is generated by sums $\alpha = \Sigma(Z_i, \varphi_i)$, where $Z_i \subset X$ is irreducible of codimension p and φ_i is a non-zero rational function on Z_i, subject to the condition: $\Sigma \mathrm{div}(\varphi_i) = 0$ as a cycle of codimension $p+1$ on X. The regulator map R will send it to an element of the partial torus

$$H_D^{2p+1}(X, \mathbb{Z}(p+1)) \simeq H^{2p}(X, \mathbb{C})/F^{p+1}H^{2p}(X, \mathbb{C}) \oplus H^{2p}(X, \mathbb{Z}).$$

Modulo the image of $\langle [Z_i] \rangle \otimes \mathbb{C}$ in this torus, $R(\alpha)$ is constructed as follows: let $Z = \bigcup_i Z_i$, $U = X \backslash Z$. Because $\Sigma \mathrm{div}(\varphi_i) = 0$ it follows that the one forms $w_i = \frac{1}{2i\pi} \frac{d\varphi_i}{\varphi_i}$ on Z_i satisfy: $\mathrm{Res}_{Z_i \cap Z_j} w_i + \mathrm{Res}_{Z_j \cap Z_i} w_j = 0$, hence determine an element w_α of

$$H_Z^{2p+1}(X) \subset \bigoplus_i H^1(Z_i \backslash \bigcup_{j \neq i} Z_j \cap Z_i).$$

$H_Z^{2p+1}(X)$ carries a mixed Hodge structure [14], induced by the mixed Hodge structure on $\bigoplus_i H^1(Z_i \backslash \bigcup_{j \neq i} Z_i \cap Z_j)$, and because w_i have a class in $H^1(Z_i \backslash \bigcup_{j \neq i} Z_i \cap Z_j, \mathbb{Z}) \cap F^1 H^1(Z_i \backslash \bigcup_{j \neq i} Z_i \cap Z_j)$, it follows that

$$w_\alpha \in F^{p+1} H_Z^{2p+1}(X) \cap H_Z^{2p+1}(X, \mathbb{Z}).$$

Consider the exact sequence:

$$0 \longrightarrow H^{2p}(X)/ < Z_i > \longrightarrow H^{2p}(U) \longrightarrow H_Z^{2p+1}(X) \longrightarrow H^{2p+1}(X).$$

Clearly w_α vanishes in $H^{2p+1}(X)$ because $F^{p+1}H^{2p+1}(X) \cap H^{2p+1}(X, \mathbb{Z}) = 0$. So w_α admits liftings in $F^{p+1}H^{2p}(U)$ and in $H^{2p}(U, \mathbb{Z})$, whose difference will give $R(\alpha) \in H^{2p}(X, \mathbb{C})/F^{p+1}H^{2p}(X) \oplus H^{2p}(X, \mathbb{Z}) \oplus \mathbb{C}[Z_i]$. (We have made abstraction here of singularities but the construction works in general [27].)

0.6. One way to study the objects described above is to look at their variation when X varies in a family: suppose $\mathcal{X} \xrightarrow{\pi} B$ is a smooth family of complex projective varieties parametrized by a smooth complex variety B ; then the inclusions $F^pH^k(\mathcal{X}_b) \subset H^k(\mathcal{X}_b,\mathbb{C})$ determine a $\mathcal{C}^\infty$-subbundle $F^p\mathcal{H}^k_\infty \subset \mathcal{H}^k_\infty$ of the bundle $\mathcal{H}^k_\infty$ with fiber $H^k(\mathcal{X}_b,\mathbb{C})$. $\mathcal{H}^k_\infty$ is a flat bundle w.r.t. the Gauss-Manin connection ∇, so in particular it has a natural holomorphic structure, and we will denote by $\mathcal{H}^k$ the sheaf of its holomorphic sections. We have $\mathcal{H}^k = R^k\pi_*\mathbb{C} \otimes \mathcal{O}_B$. The most important results of Griffiths are the following [25]:

0.7. THEOREM.

(i) *$F^p\mathcal{H}^k_\infty$ is a holomorphic subbundle of $\mathcal{H}^k_\infty$; we will denote by $F^p\mathcal{H}^k \subset \mathcal{H}^k$ its sheaf of holomorphic sections.*

(ii) *(Transversality) The Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_B$ satisfies:*

$$\nabla F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k \otimes \Omega_B\,.$$

(iii) *(Description of the differential of the period map): The $\mathcal{O}_B$-linear map*

$$\begin{array}{cccc} \overline{\nabla}: & F^p/F^{p+1}\mathcal{H}^k & \longrightarrow & F^{p-1}/F^p\mathcal{H}^k \otimes \Omega_B \\ & \| & & \| \\ & \mathcal{H}^{p,k-p} & \longrightarrow & \mathcal{H}^{p-1,k-p+1} \otimes \Omega_B \end{array},$$

obtained from ∇ by passing to the quotient, gives for any $b \in B$ a map: $TB_b \to \mathrm{Hom}(H^{k-p}(\Omega^p_{\mathcal{X}_b}), H^{k-p+1}(\Omega^{p-1}_{\mathcal{X}_b}))$, which identifies to the composite:

$$TB_{(b)} \xrightarrow{\text{Kodaira-Spencer}} H^1(T\mathcal{X}_b) \longrightarrow \mathrm{Hom}(H^{k-p}(\Omega^p_{\mathcal{X}_b}),\ H^{k-p+1}(\Omega^{p-1}_{\mathcal{X}_b}))\,,$$

where the last map is given by the interior product.

0.8. To deduce consequences of this theorem, one needs to know much about the structure of the couplings $H^1(T_{\mathcal{X}_b}) \otimes H^{k-p}(\Omega^p_{\mathcal{X}_b}) \to H^{k-p+1}(\Omega^{p-1}_{\mathcal{X}_b})$. Their description is especially beautiful in the case of hypersurfaces $\{F=0\}$ in projective space $\mathbb{P}^n$ (and more generally sufficiently ample hypersurfaces in any variety [22]). In this case, the spaces considered (modulo the cohomology of $\mathbb{P}^n$) are homogeneous pieces of the jacobian ring $R(F) = \mathbb{C}[X_0,\ldots,X_n]/ < \partial F/\partial X_i >_{i=0,\ldots,n}$, of F and the coupling is just multiplication [9]. [16], [21] provide a thorough study of the algebraic properties of these rings.

0.9. The Transversality Theorem 0.7 (ii) has its analog for the Abel-Jacobi maps or regulators, known as "quasi-horizontality of normal functions" [44], [IVHS,III], which follows in fact from 0.7 (ii) for variations of mixed Hodge structures, if one constructs the Abel-Jacobi invariants as extension classes [8], [17] (see also 0.5). Concretely it says the following: let $\mathcal{X} \xrightarrow[\pi]{} B$ be a smooth family and let

$\mathcal{Z} \subset \mathcal{X}$ be a codimension p cycle, whose support is flat over B, and such that $\mathcal{Z}_b \subset \mathcal{X}_b$ is homologous to zero, $\forall b \in B$. The family of intermediate jacobians $(J^{2p-1}(\mathcal{X}_b))_{b \in B}$ has a natural complex structure, for which the sheaf of holomorphic sections is $\mathcal{J}^{2p-1} = \mathcal{H}^{2p-1}/F^p\mathcal{H}^{2p-1} \oplus H_{\mathbb{Z}}^{2p-1}$. The cycle $\mathcal{Z}$ gives a normal function $\nu_{\mathcal{Z}} \in \mathcal{J}^{2p-1}$ defined by $\nu_{\mathcal{Z}}(b) = \Phi_{\mathcal{X}_b}(\mathcal{Z}_b)$. (The analog of 0.7 (i) is that $\nu_{\mathcal{Z}}$ is holomorphic.)

0.9.1. The horizontality property is the following: *let $\tilde{\nu}_{\mathcal{Z}} \in \mathcal{H}^{2p-1}$ be a local lifting of $\nu_{\mathcal{Z}}$. Then $\nabla\tilde{\nu}_{\mathcal{Z}} \in F^{p-1}\mathcal{H}^{2p-1} \otimes \Omega_B$.* (Note that this is independent of the choice of the lifting by 0.7 (ii).) A similar statement holds for the regulator.

0.10. In Section 1 we will explain how to exploit this property to study the Abel-Jacobi map in families.

In Section 2, we will state a criterion due to Green for the density of the Noether-Lefschetz locus (0.3), and describe its consequences on the Abel-Jacobi map of certain threefolds. In Section 3, we describe briefly Nori's work, which is the most important recent contribution in the field.

1 Infinitesimal invariants

1.1. Let $\mathcal{X} \xrightarrow{\pi} B$ be a family of smooth complex projective varieties. Let $\mathcal{H}^{p,q} = F^p\mathcal{H}^{p+q}/F^{p+1}\mathcal{H}^{p+q}$ be the Hodge bundles and let $\overline{\nabla} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega_B$ be the map of 0.7 (iii). Define $\overline{\nabla}_{(s)} : \mathcal{H}^{p,q} \otimes \Omega_B^{s-1} \to \mathcal{H}^{p-1,q+1} \otimes \Omega_B^s$, by $\overline{\nabla}_{(s)}(\sigma \otimes \alpha) = \overline{\nabla}(\sigma) \wedge \alpha$. Using the fact that $\overline{\nabla}$ is obtained from ∇ by passing to the quotient, and the integrability of ∇, one finds that $\overline{\nabla}_{(s+1)} \circ \overline{\nabla}_{(s)} = 0$. So for fixed (p,q) we get a complex on B:

$$K^{p,q} : O \to \mathcal{H}^{p,q} \xrightarrow{\overline{\nabla}} \mathcal{H}^{p-1,q+1} \otimes \Omega_B \xrightarrow{\overline{\nabla}_{(2)}} \mathcal{H}^{p-2,q+2} \otimes \Omega_B^2 \to \cdots \to \mathcal{H}^{0,p+q} \otimes \Omega_B^p \to 0\,.$$

This complex is in fact the pth graded piece of the De Rham complex of $(\mathcal{H}^{p+q}, \nabla)$ for the decreasing filtration (introduced by Deligne and Zucker [44]):

$$K^p(DR\mathcal{H}^{p+q}) :=$$
$$0 \to F^p\mathcal{H}^{p+q} \xrightarrow{\nabla} F^{p-1}\mathcal{H}^{p+q} \otimes \Omega_B \xrightarrow{\nabla_{(2)}} F^{p-2}\mathcal{H}^{p+q} \otimes \Omega_B^2 \to \cdots \to F^0\mathcal{H}^{p+q} \otimes \Omega_B^p \to 0.$$

Now, by the degeneracy of the Leray spectral sequence of π [14], one has (non canonically): $H^n(\mathcal{X}, \mathbb{C}) = \bigoplus_{r+s=n} H^r(B, R^s\pi_*\mathbb{C})$ and the Hodge filtration [15] on $H^n(\mathcal{X}, \mathbb{C})$ induces on $H^r(B, R^s\pi_*\mathbb{C}) = \mathbb{H}^r(B, DR(\mathcal{H}^s))$ a filtration that is the one induced by K^p, if one imposes "logarithmic growth at infinity", that is if one works with the subcomplex $DR(\mathcal{H}^s)(\log \partial B)$.

1.2. The first infinitesimal invariant associated to a Hodge class on $\mathcal{X}$ is a holomorphic section of one of the cohomology sheaves of the complexes $\mathcal{K}^{p,q}$. Precisely let $\alpha \in F^n\mathcal{H}^{2n}(\mathcal{X})$; (integrality of α does not play any rule here). Assume

$\alpha \in H^k(B, R^{2n-k}\pi_*\mathbb{C})$; so $\alpha \in F^n H^k(B, R^{2n-k}\pi_*\mathbb{C}) = \mathbb{H}^k(K^n DR\mathcal{H}^{2n-k}(\log \partial B))$. Then the infinitesimal invariant $\delta\alpha \in H^0(B, \mathcal{H}^k(Gr^n_K(DR\mathcal{H}^{2n-k})))$ is just the image of α under the composite map:

$$\mathbb{H}^k(K^n DR\mathcal{H}^{2n-k}(\log \partial B)) \to \mathbb{H}^k(Gr^n_K DR\mathcal{H}^{2n-k}(\log \partial B)) \\ \to H^0(\mathcal{H}^k(Gr^n_K DR\mathcal{H}^{2n-k})).$$

This is a local invariant of α, which can be as well obtained by looking at the image of α in $H^0(R^n\pi_*\Omega^n_{\mathcal{X}})$, and by studying the spectral sequence associated to the filtration of $\Omega^n_{\mathcal{X}}$ by the subbundles $\pi^*\Omega^p_B \wedge \Omega^{n-p}_{\mathcal{X}}$.

Now we want to describe more concretely these invariants and explain how to use them:

(A) Infinitesimal invariants of normal functions ([23], [44], [IVHS,III]):

1.3. Let $\mathcal{X} \xrightarrow{\pi} B$ be as before and let $\alpha \in H^{2p}(\mathcal{X}, \mathbb{Z}) \cap F^p H^{2p}(\mathcal{X})$; assuming $H^{2p-1}(\mathcal{X}) = 0$, α determines $\alpha_D \in H^{2p}_D(\mathcal{X}, \mathbb{Z}(p))$, and if $\alpha_{|\mathcal{X}_b} = 0$ in $H^{2p}(\mathcal{X}_b, \mathbb{Z})$, $\alpha_{D|\mathcal{X}_b} \in J^{2p-1}(\mathcal{X}_b) \subset H^{2p}_D(\mathcal{X}_b, \mathbb{Z}(p))$, and we get a section ν_α of $\mathcal{J}^{2p-1}$, (cf. 0.9), defined by $\nu_\alpha(b) = \alpha_{D|\mathcal{X}_b}$.

When α is the class of a cycle $\mathcal{Z}$, one has $\nu_\alpha = \nu_{\mathcal{Z}}$. The infinitesimal invariant of α is in the cohomology at the middle of the sequence:

$$\mathcal{H}^{p,p-1} \xrightarrow{\overline{\nabla}} \mathcal{H}^{p-1,p} \otimes \Omega_B \xrightarrow{\overline{\nabla}_{(2)}} \mathcal{H}^{p-2,p+1} \otimes \Omega^2_B ,$$

and we construct now the infinitesimal invariant $\delta\nu_\alpha$ of ν_α, which lies in the same sheaf, as follows. Let $\tilde{\nu}_\alpha$ be a local lifting of ν_α in $\mathcal{H}^{2p-1}$; then by 0.9.1 $\nabla\tilde{\nu}_\alpha \in F^{p-1}\mathcal{H}^{2p-1} \otimes \Omega_B$. It is then easily seen that the projection of $\nabla\tilde{\nu}_\alpha$ in $\mathcal{H}^{p-1,p} \otimes \Omega_B$ is in $\operatorname{Ker} \overline{\nabla}_{(2)}$ and well-defined modulo $\operatorname{Im} \overline{\nabla}$. It is shown in [38] that $\delta\nu_\alpha = \delta\alpha$.

1.4. Clearly the vanishing of $\delta\nu_\alpha$ is equivalent to the fact that ν_α has a local lifting $\tilde{\nu}_\alpha \in \mathcal{H}^{2p-1}$ satisfying the stronger horizontality condition: $\nabla\tilde{\nu}_\alpha \in F^p\mathcal{H}^{2p-1} \otimes \Omega_B$. One can then construct a second infinitesimal invariant [23] living in

$$\frac{\operatorname{Ker} \overline{\nabla}_{(2)} : \mathcal{H}^{p,p-1} \otimes \Omega_B \to \mathcal{H}^{p-1,p} \otimes \Omega^2_B}{\operatorname{Im} \overline{\nabla} : \mathcal{H}^{p+1,p-2} \to \mathcal{H}^{p,p-1} \otimes \Omega_B},$$

which measures the obstruction to the existence of a lifting that satisfies: $\nabla\tilde{\nu}_\alpha \in F^{p+1}\mathcal{H}^{2p-1} \otimes \Omega_B$. Finally, if all the cohomology sheaves involved vanish, one can continue this process to get a flat lifting of ν_α in $\mathcal{H}^{2p-1}$. Under mild assumptions on the IVHS, this flat lifting will be unique up to a section of $H^{2p-1}_{\mathbb{Z}}$. Now the necessary vanishing assumptions are true for the universal family of hypersurfaces of degree ≥ 6 in $\mathbb{P}^4$, modulo isomorphisms (one uses there 0.8 and the symmetrizer lemma [16]), and a standard monodromy argument shows that flatness of normal functions implies their triviality mod. torsion, hence we get:

1.5. Theorem (Green [23], Voisin, unpublished). *Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree ≥ 6. Then the Abel-Jacobi map of X is of torsion.*

Green proved in fact the analogous result for all dimensions.

Green and Müller-Stach have generalized this result to any sufficiently ample linear system in any even dimensional variety [24]. To be precise, they show that for $X \subset Y$, $\dim Y = 2n$, X a general member of a sufficiently ample linear system on Y, the image of the Deligne-Abel-Jacobi map $c_D : CH^n(X) \to H_D^{2n}(X, \mathbb{Z}(n))$ is equal, up to the torsion, to the image of the composite map $CH^n(Y) \to H_D^{2n}(Y, \mathbb{Z}(n)) \to H_D^{2n}(X, \mathbb{Z}(n))$ — and that the last restriction map is injective.

As Bloch and Nori mentioned to me, the same argument applies as well to the regulator map (0.5). This gives the following:

1.6. THEOREM. *Let S be a general surface of degree at least five in $\mathbb{P}^3$; then the image of $R : H^1(\mathcal{K}_2(S)) \to H^2(S, \mathbb{C})/F^2H^2(S) \oplus H^2(S, \mathbb{Z})$ is of torsion modulo* $\mathrm{Pic}\, S \otimes \mathbb{C}^* = \langle c_1(\mathcal{O}_S(1)) \rangle \otimes \mathbb{C}^*$.

As in the previous theorem, the assumption $d \geq 5$ is necessary. In the case $d = 4$ ($K3$-surfaces), Oliva (work in progress) shows the nontriviality of $R(S)$ mod. torsion, using the method of [39].

Theorem 1.6 disproves a conjecture of Beilinson [27], stating that the real Deligne cohomology is generated by the regulator.

As for the geometric content of the infinitesimal invariant δ_ν, we mention the following result of Collino and Pirola:

1.7. Let $\mathcal{M}_3$ be the moduli space of curves of genus three and let $J \xrightarrow{\pi} \mathcal{M}_3$ be the associated jacobian fibration. For $C \in \mathcal{M}_3$, one can choose an Abel-Jacobi embedding $C \subset J_C$, and the Abel-Jacobi image of the one-cycle $C - (-C)$ in the primitive part of the intermediate jacobian of J_C does not depend on the embedding. The normal function so obtained on $\mathcal{M}_3$ has an infinitesimal invariant defined as in 1.3, and one has:

1.8. THEOREM [13]. *This infinitesimal invariant at C lives in a space naturally isomorphic to $S^4H^0(K_C)$, and for C non-hyperelliptic, it is non-zero and gives the equation of C in its canonical embedding.*

(B) Infinitesimal invariants for families of zero-cycles on surfaces:

1.9. Let $\mathcal{S} \xrightarrow{\pi} B$ be a family of smooth regular projective surfaces, and let $Z \subset \mathcal{S}$ be a codimension two cycle, $Z = \Sigma n_i Z_i$, with $Z_i \to B$ flat and $\Sigma n_i d^0 Z_{i/B} = 0$. The class $[Z]$ of Z has then an infinitesimal invariant $\delta[Z]$ in $\mathcal{H}^{0,2} \otimes \Omega_B^2 / \overline{\nabla}_{(2)}(\mathcal{H}^{1,1} \otimes \Omega_B)$. If Z satisfies the assumption: $\forall b \in B$, Z_b is rationally equivalent to zero in $\mathcal{S}_b$, a multiple of Z is homologous to a cycle supported over a proper Zariski closed subset of B, and we conclude that $\delta[Z]$ vanishes on a Zariski open set of B.

1.10. Now, using Serre's duality one finds an isomorphism:

$$(\mathcal{H}^{0,2} \otimes \Omega_B^2 / \operatorname{Im} \overline{\nabla}_{(2)})_{(b)} \simeq (H^0(\Omega^N_{\mathcal{S}|S_b} \otimes \pi^* K_B^{-1} / \mathcal{O}_{S_b}))^* ,$$

where $N = \dim B$. The geometric content of $\delta[Z]$ is then the following. Suppose $Z = \Sigma n_i \sigma_i(B)$, where $\sigma_i : B \to S$ are sections, and $\Sigma n_i = 0$. At $b \in B$ one has $\sigma_i^* : (\Omega^N_{S|S_b})_{\sigma_i(b)} \to \Omega^N_{B(b)} = K_{B(b)}$, and $\delta[Z]$, as an element of $H^0(\Omega^N_{S|S_b} \otimes \pi^* K_B^{-1})^*$, is given by $\delta[Z] = \Sigma n_i \sigma_i^*$, which factors through the quotient $H^0(\Omega^N_{S|S_b} \otimes \pi^* K_B^{-1} / \mathcal{O}_{S_b})$ by the assumption $\Sigma n_i = 0$.

1.11. In [43] it is shown that if $\mathcal{S}_\pi \to B$ is the family of smooth hypersurfaces of $\mathbb{P}^3$ of degree ≥ 7, modulo isomorphism, the bundle $\Omega^N_{S|S_b} / \pi^* K_{B(b)}$ is very ample on S_b, $\forall b \in B$. From 1.9, 1.10, 1.11 one deduces:

1.12. THEOREM [43]. *Let $S \subset \mathbb{P}^3$ be general of degree ≥ 7. Then two distinct points of S are not rationally equivalent.*

2 Green's infinitesimal criterion and the nontriviality of the Abel-Jacobi map

2.1. Consider a family of surfaces $\mathcal{S} \to B$. Inside B, we have the Noether-Lefschetz loci, characterized by the existence of a certain Hodge class in H^2 of the fiber; that is, by the Lefschetz theorem, by the presence of an "extra" line bundle on the fiber. It is better to consider as in 0.3, the NL loci as contained in the C^∞ vector bundle $\mathcal{H}^{1,1}_{\mathbb{R}}$, with fiber $H^{1,1}(S_b) \cap H^2(S_b, \mathbb{R})$ at $b \in B$. The NL locus will be then defined as the set $\{(\lambda, b) / \lambda \in H^{1,1}(S_b) \cap H^2(S_b, \mathbb{Q})\}$. Green's lemma gives the following purely algebraic criterion for the density of this locus:

2.2. LEMMA (Green, [29]). *Suppose that for some $b \in B$, $\lambda \in H^1(\Omega_{S_b})$, the map $\overline{\nabla}(\lambda) : TB_{(b)} \to H^2(\mathcal{O}_{S_b})$ is surjective. Then the Noether-Lefschetz locus is dense in $\mathcal{H}^{1,1}_{\mathbb{R}}$.*

2.3. In [40], the criterion was checked for sufficiently ample hypersurfaces in Calabi-Yau threefolds.

2.4. Now this lemma gives a way to produce interesting cycles in threefolds: if $S \underset{j}{\hookrightarrow} X$ and $\lambda \in H^{1,1}(S) \cap \operatorname{Ker}(H^2(S, \mathbb{Z}) \xrightarrow{j_*} H^4(X, \mathbb{Z}))$, λ determines an element of $\operatorname{Pic} S$ (assuming S regular), hence a one-cycle on S, which will be homologous to zero in X. The next question is to decide whether the cycles Z_λ so obtained have non-trivial Abel-Jacobi invariants. If the expected dimension of the components $\mathcal{S}_\lambda$ of the NL locus is strictly positive, it is possible to study formally the differential of the Abel-Jacobi map $\Phi : \mathcal{S}_\lambda \to JX$, $\Phi(S, \lambda) = \Phi_X(Z_\lambda)$, and to show that it is nonzero. This method was used in [41] to solve the generalized Hodge-Grothendieck conjecture for certain sub-Hodge structures on certain threefolds. (See [2] for a more geometric solution of a similar example.)

2.5. In the case of a Calabi-Yau threefold, the expected dimension of the NL locus is zero, but one can deform X together with the zero-dimensional components of this NL locus. Using the same construction as above, this will now give normal functions on the family of deformations of X, and the nonvanishing of their infinitesimal invariants gives:

2.6. THEOREM [40]. *Let X be a Calabi-Yau threefold that is nonrigid; then a general deformation of X has a non-torsion Abel-Jacobi map.*

This theorem was known previously for the quintic threefold (see [26] and [12] for a much stronger statement) but the cycles in [26] were easy to get. They are the lines on X.

3 Nori's theorem

3.1. The essential point in 1.4, 1.5 was the vanishing of some cohomology sheaves of the complexes $K^{p,q}$, on the family of all hypersurfaces of sufficiently large degree. Nori realized that these vanishing statements and their generalizations to the case of complete intersections of large degree in any variety are partial aspects of a deep vanishing theorem for the cohomology of the universal hypersurfaces or complete intersections, which is the following:

3.2. CONNECTIVITY THEOREM [32]. *Let X be projective of dimension $n+k$. Let $L_1, \ldots, L_k$ be ample line bundles, and for $n_1, \ldots, n_k \in \mathbb{N}$, let $S := \prod_1^k H^0(X, L_i^{n_i})$. Let $Y_S \subset X \times S$ be the universal complete intersection. Then for n_i large enough, and for any submersive map $T \to S$, one has $H^k(X \times T, Y_T) = 0$, $k = 0, \ldots, 2n$.*

The most striking application of this theorem is the proof of the existence of cycles homologous and Abel-Jacobi equivalent to zero but not algebraically equivalent to zero:

3.3. THEOREM [32]. *Using notation as above, let Z be a cycle on X of codimension $d < n$: suppose that $[Z] \neq 0$ in $H^{2d}(X, \mathbb{Q})$, or that the Abel-Jacobi image of Z is not contained in the algebraic part of JX. Then for n_i such that the conclusion of 3.2 holds, $Z_{|Y_S}$ is not algebraically equivalent to zero, for general s.*

3.4. Griffiths in [26] proved the existence of cycles homologous to zero but not algebraically equivalent to zero, but he used the Abel-Jacobi invariant, which vanishes on cycles algebraically equivalent to zero when the intermediate jacobians do not contain a nontrivial algebraic part.

Albano and Collino [1] have even shown that the kernel of the Abel-Jacobi map can be nonfinitely generated modulo algebraic equivalence. This was obtained as a consequence of 3.3, and of the following result (an analog of Clemens' theorem [12]):

3.5. THEOREM [1]. *Let $X \subset \mathbb{P}^8$ be a general cubic sevenfold; then $J^7(X)$ has no algebraic part and the image of the Abel-Jacobi map $\Phi_X : CH_3(X)_{\mathrm{hom}} \to J^7(X)$ is a countable infinitely generated group.*

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